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Présenté par

BASLI MAHDI

ULAM-HYERS-RASSIAS STABILITY OF SOME FUNCTIONAL DIFFERENTIAL EQUATIONS

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devant le Jury composé de

Président:	Pr CHACHA Djamal Ahmed	Univ Kasdi Merbah (Ouargla)
Directeur de mémoire :	Dr. HAMMOUCHE Hadda	Univ Ghardaia (Ghardaia)
Examinateur:	Dr.MEFLAH Mabrouk	Univ Kasdi Merbah (Ouargla)
Examinateur:	Dr.GAZAL Abd Razak	Univ Kasdi Merbah (Ouargla)

Dedication

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Contents

Notations and conventions

$\mathbb{R}$: the real numbers
$\mathbb{C}$: the complex numbers
Ω : usually denotes an open set in a topological space
$D(A)$: the domain of A
$R(A)$: the image of A
$\rho(A)$: the resolvent set of A
A^{-1} : the inverse of A
$C([0,T]:X)$: the space of continuous functions defined on $0 \leq t \leq T$ with value in X
L^p : the usual space of measurable whose p th power is Lebesgue integrable
I : Identity operator
Δ : the Laplace operator
∇ : gradient operator
$R(\lambda, A)$: the resolvent operator of A
$\frac{\partial u}{\partial \eta}$: the outward normal derivative
$\|u\|_p$: the norm of u in L^p
$\|u\|_\infty$: the norm of u in L^∞

Introduction

The stability of functional equation was originally raised by Ulam in 1940 in a talk given at Wisconsin University.The problem posed by Ulam was the following :Under what conditions does there exist an additive mapping near an approximately additive mapping[13] .The first answer to Ulam's question was given by Hyers in 1941 in the case of Banach spaces in [4].thereafter ,this type of stability is called the Ulam-Hyers stability in 1978,Rassias[10]provieded a remarkable generalization of Ulam-Hyers stability of mappings by considering variables .The concept of stability for a functional equation arises when we replace the functional equation by an inquality which acts as a perturbation of the equation.Thus,the stability question of functional equations is how do the solutions of the inequality differ from those of functional equations ?,or equivalently for every solution of the perturbed equation there exists a solution of the equation that is close to it.Consiserable attention has been given to the study of the Ulam-Hyers and Ulam-Hyers-Rrassias stability of all kinds of functional equations; one can see the mongraphs of[[3] ,[6]] . Bota-Boriceanu and Petrusel[1], Petru et al [8] ,[9] . and Rus[11] ,[12] , discussed the Ulam-Hyers stability for operatorial equations and inclusions. Castro and ramos [2], and Jung [5] , considered the Ulam-Hyers-Rrassias stability for a class of Volterra integral equations. Motivated by recent works in stability, we will study the Ulam-Hyers stability and the generalized Ulam-Hyers-Rassias stability of the following impulsive evolution equations:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad \text{if} t \in J_k, k = 0, \dots m \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), \text{if} t \in [0, b], k, k = 0, \dots m \\ u(0) = u_0 \end{cases} \tag{1}$$

This memory will be organized as follows. In Chapter 1, we will recall some basic definitions and preliminaries facts which will be used throughout this work. In Chapter 2, we give some sufficient conditions for the Ulam-Hyers stability of the mild solution of the considered problem and we illustrate our theory with example. The last Chapter is devoted the generalized Ulam-Hyers-Rassias stability, an example is given illustrating the abstract theory.

Chapter 1

Preliminaries

In this chapter, we introduce notations, definitions, lemmas and fixed point theorems which are used throughout this thesis. Let $J := [0, b]$ be an interval of $\mathbb{R}$. Let $(X, |.|)$ be a real Banach space.

$C(J, X)$ is the Banach space of all continuous functions from $[0, b]$ into X with the norm

$$\|y\|_\infty = \sup\{|y(t)| : 0 \leq t \leq b\}.$$

$B(X)$ denotes the Banach space of bounded linear operators from X into X, with norm

$$\|N\|_{B(X)} = \sup\{|N(y)| : |y| = 1\}.$$

A measurable function $y : J \longrightarrow X$ is Bochner integrable if and only if $|y|$ is Lebesgue integrable.

$L^1(J, X)$ denotes the Banach space of functions $y : J \longrightarrow X$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)| dt.$$

Let $L^1_{loc}([0, +\infty); \mathbb{R}_+)$ be the Banach space of measurable functions which are locally Bochner integrable. we shall consider the following space

$$PC = \Big\{y : [0, b] \to X : \quad y_k \in C[J_k, X], k = 0, ..., m \quad \text{such that} \\ y(t_k^-), y(t_k^+) \text{ exist with } y(t_k) = y(t_k^-), k = 1, ..., m\Big\},$$

which is a Banach space with the norm

$$\|y\|_{PC} := \max\{\|y_k\|_\infty : k = 0, ., ., ., m\},$$

where y_k is the restriction of y to $J_k = (t_k, t_{k+1}], k = 0, ..., m$
The following definitions are used in the sequel.

Definition 1.1 *A map $f : [a,b] \times E \longrightarrow E$ is said to be Carathéodory if*

(i) $t \longmapsto f(t,y)$ is measurable for all $y \in E$,

(ii) $y \longmapsto f(t,y)$ is continuous for almost each $t \in [a,b]$,

Definition 1.2 *A map $f : [a,b] \times E \longrightarrow E$ is said to be L^1-Carathéodory if*

(i) f is Carathéodory

(ii) $y \longmapsto f(t,y)$ is continuous for almost each $t \in [0,b]$,

(iii) For each $q > 0$, there exists $h_q \in L^1([0,b], \mathbb{R}_+)$ such that

$$|f(t,y)| \le h_q(t) \quad \textit{for all} \quad |y| \le q \quad \textit{and almost each} \quad t \in [0,b].$$

We remark here as well that conditions (i) and (ii) imply for $t \in [0,b]$ $f(t, u(t)$ is measurable for any measurable and almost every where finite function $u(.)$. this is a result of Carathédory. Also, (iii) implies that $f(t, u(t))$ is L^1-Carathédory.

Definition 1.3 *A map f is said compact if its image is relatively compact. f is said completely continuous if it's continuous and the image of every bounded set is relatively compact.*

Lemma 1.4 *(Gronwall's)*
for all $t_0 \le t$, Let $\phi(t) \ge 0$ and $\psi(t) \ge 0$ be locally integrable functions. If there are constants $K > 0$ and $L > 0$ such that

$$\phi(t) \le K + L \int_{t_0}^{t} \psi(s)\phi(s)\,\mathrm{d}s,$$

then

$$\phi(t) \le K \exp\left(L \int_{t_0}^{t} \psi(s)\,\mathrm{d}s\right) \quad \textit{pour} \quad t_0 \le t$$

1.1 Semigroups

Let E be a Banach space and $B(E)$ be the Banach space of linear bounded operators.

Definition 1.5 *A semigroup of class C_0 is a one parameter family $\{T(t) \mid t \geq 0\} \subset B(E)$ satisfying the conditions:*

(i) $T(t) \circ T(s) = T(t+s)$, *for* $t, s \geq 0$,

(ii) $T(0) = I$,

(iii) the map $t \to T(t)(x)$ is strongly continuous, for each $x \in E$, i.e;

$$\lim_{t \to 0} T(t)x = x, \ \forall x \in E.$$

A semigroup of bounded linear operators $T(t)$, is uniformly continuous if

$$\lim_{t \to 0} \|T(t) - I\| = 0.$$

Where I denotes the identity operator in E.

We note that if a semigroup $T(t)$ is of class (C_0) then the following growth condition is satified

$\|T(t)\|_{B(E)} \leq M \cdot \exp(\beta t)$, for $0 \leq t < \infty$, with some constants $M > 0$ and β.

In particular, if $M = 1$ and $\beta = 0$, i.e; $\|T(t)\|_{B(E)} \leq 1$, for $t \geq 0$, then the semigroup $T(t)$ is called a contraction C_0-semigroup .

Definition 1.6 *Let $T(t)$ be a (C_0)-semigroup defined on E. The infinitesimal generator A of $T(t)$ is the linear operator defined by*

$$A(x) = \lim_{h \to 0} \frac{T(h)x - x}{h}, \quad \textit{for } x \in D(A),$$

where $D(A) = \{x \in E \mid \lim_{h \to 0} \frac{T(h)(x) - x}{h}$ exists in $E\}$.

Let us recall the following property:

Proposition 1.7 *The infinitesimal generator A is closed linear and densely defined operator in E. If $x \in D(A)$, then $T(t)(x)$ is a C^1-map and*

$$\frac{d}{dt}T(t)(x) = A(T(t)(x)) = T(t)(A(x)) \quad \text{on } [0, \infty).$$

Theorem 1.8 (Hille and Yosida). *Let A be a densely defined linear operator with domain and range in a Banach space E. Then A is the infinitesimal generator of uniquely determined semigroup $T(t)$ of class (C_0) satisfying*

$$\|T(t)\|_{B(E)} \leq M \exp(\omega t), \quad t \geq 0,$$

where $M > 0$ and $\omega \in \mathbb{R}$ if $(\lambda I - A)^{-1} \in B(E)$ and $\|(\lambda I - A)^{-n}\| \leq M/(\lambda - \omega)^n$, $n = 1, 2, \ldots$, for all $\lambda \in \mathbb{R}$.

1.2 Fixed point theorem

1.3 Stability

Now, we consider the problem of impulsive functional differential equations of the form :

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & \text{if} \quad t \in J_k, k = 0, \ldots m \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), & \text{if} \quad y \in [0, b], k, k = 0, \ldots m \end{cases} \tag{1.1}$$

Let ϵ be a positive real number and $\Phi : J \to [0, \infty)$ be a continuous function.We consider the following inequalities

$$\begin{cases} ||u'(t) - Au(t) - f(t, u(t))||_{\mathbb{B}} \leq \epsilon, & \text{if} \quad t \in J_k, k = 0, \ldots m \\ ||u(t_k^+) - u(t_k^-) - I_k(u(t_k^-))||_{\mathbb{B}} \leq \epsilon, & \text{if} \quad t = t_k, k = 0, \ldots m \end{cases} \tag{1.2}$$

$$\begin{cases} ||u'(t) - Au(t) - f(t, u(t))||_{\mathbb{B}} \leq \Phi(t), & \text{if} \quad t \in J_k, k = 0, \ldots m \\ ||u(t_k^+) - u(t_k^-) - I_k(u(t_k^-))||_{\mathbb{B}} \leq \Phi(t), & \text{if} \quad t = t_k, k = 0, \ldots m \end{cases} \tag{1.3}$$

$$\begin{cases} ||u'(t) - Au(t) - f(t, u(t))||_{\mathbb{B}} \leq \epsilon\Phi(t), & \text{if} \quad t \in J_k, k = 0, \ldots m \\ ||u(t_k^+) - u(t_k^-) - I_k(u(t_k^-))||_{\mathbb{B}} \leq \epsilon\Phi(t), & \text{if} \quad t = t_k, k = 0, \ldots m \end{cases} \tag{1.4}$$

let us give the meaning of different stabilities.

Definition 1.9 *the equation(1.1) is Ulam-Hyers stable if there exists a real number $c > 0$ such that for each $\varepsilon > 0$ and for each solution $u \in PC$ of(1.2) there exists a mild solution $v \in PC$ of problem with*

$$|v(t) - u(t)|_{PC} \leq c\varepsilon, \quad \forall t \in J$$

Definition 1.10 *The equation(1.1) is generalized Ulam-Hyers stable exists theta $\in C(\mathbb{R}_+, \mathbb{R}_+), \theta(0) = 0$, such that for each solution $v \in Pc$ of (1.2) there exists a mild solution $u \in Pc$ of (1.1) with*

$$||v(t) - u(t)||_{Pc} \leq \theta(\varepsilon), if \quad y \in Pc, t \in J$$

Definition 1.11 *The equation (1.1) is Ulam-Hyers-Rassias stable, with respect to φ, if there exists $c_\varphi > 0$ such that for each $\varepsilon > 0$ and for each solution $v \in C^1(J, \mathbb{R})$ of (1.4) there exists a mild solution $u \in C(J, \mathbb{R})$ of (1.1) with*

$$|v(t) - u(t)| \leqslant \varepsilon c_\varphi \varphi(t), \quad \forall t \in J.$$

Definition 1.12 *The equation (1.1) is generalized Ulam-Hyers-Rassias stable, with respect to φ, if there exists $c_\varphi > 0$ such that for each solution $v \in Pc$ of (1.3) with*

$$|v(t) - u(t)| \leqslant c_\varphi \varphi(t), \quad \forall t \in Pc.$$

Remark 1.13 *It is clear that :*

(1) Definition 1.9 $\Rightarrow$ Definition 1.10.

(2) Definition 1.11 $\Rightarrow$ Definition1.12

Remark 1.14 *A function $v \in PC$ is mild solution of inequality 1.2: and only if ther exists a function $g \in PC$ and a sequence $g_k; k = 1, \ldots, m$(which depend on v) such that*

1) $||g(t)|| \leq \varepsilon$ *and* $||g_k|| \leq \varepsilon, \quad k = 1, \ldots, m$

2) $v'(t) = A(v(t)) + f(t, v(t)) + g(t), t \in (t_k, t_{k+1}, \quad k = 1, \ldots, m$

3) $v(t_k^+) = v(t_k^-) + I_k(v(t_k^+)) + g_k, \quad k = 1, \ldots, m$

Remark 1.15 *A function $v \in Pc$ is solution of inequality 1.3: and only of there exists a function $g \in PC$ and a sequence $g_k; k = 1, \ldots, m$(which depend on v) such that*

1) $||g(t)|| \leq \varphi(t)$ *and* $||g_k|| \leq \varphi, \quad k = 1, \ldots, m$

2) $v'(t) = A(v(t)) + f(t, v(t)) + g(t), t \in (t_k, t_{k+1}, \quad k = 1, \ldots, m$

3) $v(t_k^+) = v(t_k^-) + I_k(v(t_k^+)) + g_k, \quad k = 1, \ldots, m$

1.4 Existence Results

Ouahab in [7] has studied the existence if mild solutions of the problem the form 1.1 where $f : J \to X$ is a given function, A is the infinitesimal generator of a family of C_0-semigroup $T(t : t \geqslant 0$, A is a bounded linear operator from X into $X, u_0 \quad \in X, \quad I_k \in C(X, X)(k = 1, \ldots\ldots, m)$,and

$$\Delta_u|_{t=t_k} = u(t_k^+) - u(t_k^-), \quad u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$$

and $u(t_k^-) = \lim_{h\to 0^+} u(t_k - h)$ represent the right and left of $u(t)$ at $t = t_k$, respectively,$k = 1, \ldots.., m$

Definition 1.16 *A function $u \in \quad PC(J, X)$ is said to be a mild solution of (1.1)if u is the solutions of the impulsive integral equation*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds) + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k)). \quad (1.5)$$

The existence result was upon Schaefer's theorem

Theorem 1.17 *[7] Let $f : J \times X \to X$ be an L^1-Carathéodory function. Assume that:*
There exist constants c_k, such that $|I_k(u)| \leqslant c_k, \quad k = 1, \ldots., m$ for each $u \in PC$
There exist constants M, such that $||T(t)||_{B(X)} \leqslant M, \quad$ for each $t \geq 0$
and $A : D(A) \subset E \to E$ is the infinitesimal generator of compact semigroup $T(t) : t > 0$ There exist continuous nondecreasing function $\varphi : [0, \infty) \to (0, \infty)$ and $p \in L^1(J, \mathbb{R}+)$ such that

$$|f(t,u)| \leqslant p(t)\varphi(|u|)$$

for a.e $t \in J$*and each* $u \in PC$
with
$$\int_0^t m(s)ds < \int_0^\infty \frac{du}{u+\varphi(u)}$$

where

$$m(s) = maxM||B||_{B(X)}, Mp(s) \quad and \quad c = M\left[|u_0| + \sum_{k=1}^{m} c_k\right]$$

For each bounded $B \subseteq C(J_k, X)$ *and* $t \in J$ *the set*

$$\left\{T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds) + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k^-)) : u \in B\right\}$$

is relatively compact inE
Then the impulsive initial value problem (IVP for short)(1.1)has at least one mild solution

proof Ttransform the problem (1.5) into a fixed point problem . Consider the operator $N : PC(J,X) \to PC(J,X)$ defined by :

$$N(u)(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds) + \sum_{0<t_k<t} T(t-t_k)I_k(u(t_k^-))$$

Clearly the fixed points of N are mild solutions to(1.5)
We shall show N is completely continous. That proof will be in several steps.
step 1:N is continuous
Let u_n be a sequence in $PC(J,X)$ such that $u_n \to u$. We shall prove that $N(u_n) \to N(u)$. For each $t \in J$ we have

$$N(u_n)(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u_n(s))ds) + \sum_{0<t_k<t} |T(t-t_k)I_k(u_n(t_k^-))|$$

Then

$$|N(u_n)(t) - N(u)(t)| \leqslant \int_0^t |T(t-s)||f(s,u_n(s)) - f(s,u(s))ds|$$

$$+ \sum_{0<t_k<t} |T(t-t_k)||I_k(u_n(t_k^-)) - I_k(u(t_k^-))|$$

$$\leqslant bM||B||_{B(E)}||u_n - u||_PC$$

$$+M \int_0^t |f(s,u_n(s)) - f(s,u(s))ds|$$

$$+M \sum_{0<t_k<t} |I_k(u_n(t_k^-)) - I_k(u(t_k^-))|$$

. Since $I_k, k = 1....,m$ are continous ,B is bounded and $f$is an L^1Carathéodory function we have by the Lebesgue dominated convergence theorem

$$||N(u_n) - N(u)||_PC \leqslant bM||B||_{B(E)}||u_n - u||_PC$$

$$+M \int_0^t |f(s,u_n(s)) - f(s,u(s))ds|$$

$$+M \sum_{0<t_k<t} |I_k(u_n(t_k^-)) - I_k(u(t_k^-))| \to 0 \quad as \quad n \to \infty$$

Thus N is continuous.

Step 2: N map bounded sets into bounded sets in $PC(J,X)$

Indeed, it is enough to show that for any $q > 0$, there exists a positive canstant l such that for each $u \in B_q = u \in PC(J,X) : ||u||_{PC} \leqslant q$ One has $||N(u)||_{PC} \leqslant l$ let $u \in B_q$ By the fact that f is anL^1Carathéodory function we have for each $t \in J$

$$|N(u)(t)| \leqslant M|u_0| + M \int_0^t \varphi_q(s)ds + M \sum_{k=1}^m c_k$$

$$\leqslant M|y_0| + M||\varphi_q||_{L^1} + M \sum_{(k=1)}^m c_k := l$$

Step 3: N map bounded sets into equicontinuous of $PC(J,X)$ Let $\tau_1, \tau_2 \in J, \tau_1 < \tau_2$ and β_q be a bounded sat of $PC(J,X)$ as in Step 2 . Let $u \in B_q$, then for each $t \in J$ we have

$$|(Nu)(\tau_2) - (Nu)(\tau_1)| \leqslant |T(\tau_2)u_0 - T(\tau_1)u_0|$$

$$+ \int_0^{\tau_1-\epsilon} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(X)} |Bu(s)| ds$$

$$+ \int_{\tau_1}^{\tau_1-\epsilon} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(X)} |Bu(s)| ds$$

$$+ \int_{\tau_1}^{\tau_1-\epsilon} \|T(\tau_2 - s)\|_{B(X)} |Bu(s)| ds$$

$$+ \int_0^{\tau_1-\epsilon} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(X)} |f(s, u(s))| ds$$

$$+ \int_{\tau_1}^{\tau_1-\epsilon} \|T(\tau_2 - s) - T(\tau_1 - s)\|_{B(X)} |f(s, u(s))| ds$$

$$+ \int_{\tau_1}^{\tau_1-\epsilon} \|T(\tau_2 - s)\|_{B(X)} |f(s, u(s))| ds + M \sum_{k=1}^{m} c_k(\tau_2 - \tau_1)$$

$$+M \sum_{0<t_k<\tau_1} c_k \|T(\tau_2 - t_k) - T(\tau_2 - t_k)\|$$

The right-hand side tends to zero as andϵ sufficiently mall, since $T(t)$ is a strongly continous operator and the compactness of $T(t)$for $t > 0$ implies the continuity in the uniform operator topology. This proves the equicontinuity for the case where $t \neq t_i \quad i = 1, ..., m$. It remains to examine the equicontinuity$t = t_i$

$$h_1(t) = T(t)\Phi(0) + \sum_{0<t_k<t} T(t - t_k) I_k(u(t_k))$$

and

$$h_2(t) = \int_0^t T(t - s) f(s, u(s)) ds$$

First we prove equicontinuity at $t = t_i^- Fix \quad \delta_1 > 0$ such that
$t_k : k \neq i \cap [t_i - \delta_1, t_i - \delta_1] = \emptyset$

$$h_1(t) = T(t_i)u_0 + \sum_{0<t_k<t} T(t - t_k)I_k(u(t_k))$$

$$= T(t_i)u_0 + \sum^{i} -1_k = 1T(t - t_k)I_k(u(t_k))$$

For $0 < h < \delta_1$ we have that

$$|h_1(t_i - h) - h_1(t_i)| \leqslant |(T(t_i - h) - T(t_i))\Phi|$$

$$+\sum_{k=1}^{i-1} |[T(t_i - h - t_k) - T(t_i - t_k)]I_k(u(t_k^-))|$$

. The right-hand side tends to zero as $h \to 0$
Moreover

$$|h_2(t_i - h) - h_2(t_i)| \leqslant \Phi_q(s) \int_0^{t_i+h} |[(T(t_i - h - s) - T(t_i - s)]p(s)|$$

$$+M\Phi_q(s) \int_{t_i-h}^{t_i} p(s)ds$$

$$+q \int_0^{ti-h} |[(T(t_i - h - s) - T(t_i - s)]|||B|||_B(X)ds$$

$$q \int_{ti-h}^{t_i} M||B||_B(X)ds$$

wich tends to zero as $h \to$ Define

$$h_0(t) = h(t) \quad t \in [0, t_1]$$

and $h_i(t) =$

$$\begin{cases} (Nu)(t), \quad t \in (t_i, t_i + 1] \\ (Nu)(t_i^+), \quad t = t_i \end{cases}$$

Next we prove equicontinuity at$t = t_i^+$.Fix $\delta_2 > 0$ such that
$t_k : k \neq i \cap [t_i - \delta_2, t_i + \delta_2] = \emptyset$. Then

$$h(t_i) = T(t_i)u_0 + \int_0^{t_i} T(t_i - s)f(s, u(s))ds) + \sum_{k=1}^{i} |T(t_i - t_k)I_k(u(t_k))|$$

For $0 < h < \delta$ we have that

$$|h(t_i + h) - h(t_i)| \leqslant |(T(t_i + h) - T(t_i))u_0|$$

$$+ \int_0^{t_i} |[(T(t_i - h - s) - T(t_i - s)]f(s, u(s))|ds$$

$$+ \int_{t_i}^{t_i+h} M\Phi_q(s)ds$$

$$+ \sum_{k=1}^{i} |[T(t_i + h - t_k) - T(t_i - t_k)]I_k(u(t_k^-))|.$$

The right-hand side tends to zero as $h \to 0$. As a consequence of Steps 1 to 3, together with the Arzela-Ascoli theorem we can conclude that $N : PC(J, X) \to PC(J, X)$ is a completely continuous operator.

Step 4:

Now it remains to show that the set

$$\varepsilon(N) := u \in PC(J, X) : u = \lambda N(u, for some \quad 0 < \lambda < 1)$$

is bounded.Let $u \in \varepsilon(N)$.Then $u = \lambda N(u)$forsome $0 < \lambda < 1$. Thus for each $t \in J$

$$u(t) = \lambda[T(t)u_0 + \int_0^t (T(t_-s)f(s, u(s))ds$$

$$+ \sum_{0<t_k<t} |[T(t - t_k) - I_k(u(t_k^-))].$$

then for each $t \in J$ we have

$$|u(t)| \leqslant M|u_0| + \int_0^t m(s)\Phi(|u(s)|)ds + M\sum_{k=1}^{m} c_k$$

Let us denote the right- hand side of the above inequaliti as $v(t)$. Then we have

$$|u(t)| \leqslant v(t) \quad \forall t \in J$$

, and$v(0) = M[|u_0| + \sum_{k=1}^{m} c_k], v'(t) = m(t)\Phi(|u(t)|)$for a.e $t \in J$ Using the increasing character of Φ we get

$$v'(t) \leqslant m(t)\Phi|v(t))$$

for a.e $t \in J$ Then for each$t \in J$ we have

$$\int_{v(0)}^{v(t)} \frac{du}{u + \Phi(u)}$$

$$\leqslant \int_0^t m(s)ds < \int_{v(0)}^{\infty} \frac{du}{u + \Phi(u)}$$

Consequently, there exists a canstant $\bar{d}$ such that $v(t) \leqslant \bar{d}, t \in J$, and hence $||u||_{PC} \leqslant \bar{d}$ where $\bar{d}$ there only on the functions p and Φ .This show $\varepsilon(N)$ is bounded.

Set $E := PC(J, X)$. As a consequence of Schefer's fixed point theorem we deduce that N has a fixed point which is a mild solution of1.1

Chapter 2

Ulam-Hyers stability of first order semilinear differential equations

2.1 Introduction

In this chapter, we study the Ulam-Hyers stability of the mild solution of the problem

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), & \text{if} \quad t \in J_k, k = 0, \dots m \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), \text{if} & t \in [0, b], k, k = 0, \dots m \end{cases} \tag{2.1}$$

where $f : J \to X$ is a given function, A is the infinitesimal generator of a family of C_0-semigroup $T(t : t \leqslant 0$, A is a bounded linear operator from X into $X, u_0 \quad \in X, \quad I_k \in C(X, X)(k = 1, ..., m)$,and

$$\Delta_u|_{t=t_k} = u(t_k^+) - u(t_k^-), \quad u(t_k^+) = \lim_{h \to 0^+} u(t_k + h)$$

and $u(t_k^-) = \lim_{h \to 0^+} u(t_k - h)$ represent the right and left of $u(t)$ at $t = t_k$, respectively,$k = 1, ..., m$

2.2 existence

In order to establish the existence result, we shall give the meaning of the mild solution of the problem 1.1

Definition 2.1 *a function u in PC is a mild solution of the problem 1.1 if and only if u satisfies*

$$\begin{cases} u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds, \quad t \in [0,t_1], \\ u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k, \quad t \in (t_k, t_{k+1}] \end{cases} \tag{2.2}$$

Theorem 2.2 *Assume the following hypotheses:*

(H_1) there exists a constant $l_f > 0$ such that

$$||f(t,u) - f(t,v)||_X \leq l_f ||u-v||_X$$

, for each $t \in [0,b]$,and $u, v \in PC$

(H_2) there exists a canstant $l^* > 0$ such that

$$||I_k(u(t_k^-)) - I_k(v(t_k^-))||_X \leq l^* ||u-v||_X$$

,for each $u, v \in X, k = 1....m$

with

$$\begin{cases} Ml_f| \int_0^t e^{\omega(t-s)}ds < 1, \quad t \in [0,t_1], \\ Ml_f| \int_0^t e^{\omega(t-s)}ds + ML^* \sum_{k=1}^{m} e^{\omega(b-t_k)} < 1, \quad t \in (t_k, t_{k+1}] \end{cases} \tag{2.3}$$

hold then the problem (1.1) has a unique mild solution.

proof First we proove the unique existence.

Let u the mild solution of 1.1, then u has the following integral form:

$$\begin{cases} u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds, \quad t \in [0,t_1], \\ u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k, \quad t \in (t_k, t_{k+1}] \end{cases}$$

transform this problem in a fixed point problem by defining the operator N

$N : \quad PC \to PC$ such that

$$\begin{cases} N(u)(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds, \quad t \in [0,t_1], \\ N(u)(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k, \quad t \in (t_k, t_{k+1}] \end{cases} \tag{2.4}$$

It is clear that u is a fixed point of the operator N.
Let $u \in PC$ and $v \in PC$, We have

$$\begin{cases} |N(u)(t) - N(v)(t)| = |\int_0^t T(t-s)\left((s,u(s)) - f(s,v(s))\right) ds|, \quad t \in [0,t_1], \\ |N(u)(t) - N(v)(t)| = |\int_0^t T(t-s)\left((s,u(s)) - f(s,v(s))\right) ds \\ +\sum_{0<t_k<t} T(t-t_k)\left(I_k(u(t_k^-)) - I_k(v(t_k^-))\right), \quad t \in (t_k, t_{k+1}] \end{cases} \tag{2.5}$$

using hypotheses (H$_1$)and (H$_2$), we obtain

$$\begin{cases} ||N(u) - N(v)|| \leq Ml_f|\int_0^t e^{\omega(t-s)}ds||u-v||, \quad t \in [0,t_1], \\ ||N(u) - N(v)|| \leq \left(Ml_f|\int_0^t e^{\omega(t-s)}ds + ML^* \sum_{k=1}^{m} e^{\omega(b-t_k)}\right)||u-v||, \quad t \in (t_k, t_{k+1}] \end{cases} \tag{2.6}$$

from 2.3, the operator N is a contraction, hence as a consequence of Banach theorem, N has a unique mild solution. **proof**

2.3 Ulam-Hyers Stability

In this section, we present some conditions for Ulam-Hyers stability of the mild solution of problem(1.1) In order to establish our results, we need the following auxiliary lemmas

Lemma 2.3 *A function $u \in PC$ is a solution of the function integral equations*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds + \sum_{0<t_k<t} T(t-t_k)I_k, \quad k = 1, \dots, m$$

if and only if uis a mild solution of the problem

$$|u'(t) - Au(t) - f(t,u(t))| \leq \varepsilon$$

$$||u(t_k^+) - u(t_k^-) - I_k(u(t_k^-))|| \leq \varepsilon$$

Lemma 2.4 *if u is a mild solutions of inequality 1.2 then u is a solution the following*

$$|u(t) - T(t)u_0 - \int_0^t T(t-s)f(s,u(s))ds| \leq \varepsilon \int_0^t ||T(t-s)||ds, \quad t \in [0,t_1]$$

$$|u(t) - T(t)u_0 - \int_0^t T(t-s)f(s,u(s))ds| - \sum_{0<t_k<t} T(t-t_k)I_k \leq 2\varepsilon \int_0^t ||T(t-s)||ds, \quad t \in (t_k, t_k+1]$$

proof Let $u \in PC$ mild solution of inequality 1.2 from remark 1.14 u is a mild solution solution of

$$\begin{cases} u'(t) = Au(t) + f(t,u) + g(t) \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-) + g_k \end{cases} \tag{2.7}$$

wich means that

$$\begin{cases} u(t) = T(t)u_0 + \int_0^t T(t-s)[f(s,u(s)) + g(s)]ds, \quad t \in [0,t_1], \\ u(t) = T(t)u_0 + \int_0^t T(t-s)[f(s,u(s)) + g(s)]ds + \sum_{0<t_k<t} T(t-t_k)(I_k + g_k) \quad t \in (t_k, t_{k+1}] \end{cases}$$

using lemme 2.3

$$\begin{cases} |u(t) - T(t)u_0 - \int_0^t T(t-s)[f(s,u(s))| = |\int_0^t T(t-s)g(s)]ds|, \quad t \in [0,t_1], \\ |u(t) - T(t)u_0 - \int_0^t T(t-s)f(s,u(s))ds - \sum_{0<t_k<t} T(t-t_k)I_k| = \\ |\int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T(t-t_k)g_k| \quad t \in (t_k, t_{k+1}] \end{cases}$$

By remark1.14 we obtain

$$\begin{cases} |u(t) - T(t)u_0 - \int_0^t T(t-s)f(s,u(s))ds| \leq \varepsilon \int_0^t ||T(t-s)||ds, \quad t \in [0,t_1], \\ |u(t) - T(t)u_0 - \int_0^t T(t-s)f(s,u(s))ds - \sum_{0<t_k<t} T(t-t_k)I_k| \leq \\ \varepsilon \int_0^t ||T(t-s)||ds + \varepsilon \sum_{0<t_k<t} ||T(t-t_k)|| \\ \leq 2\varepsilon \int_0^t ||T(t-s)||ds, \quad t \in (t_k, t_{k+1}] \end{cases}$$

Remark 2.5 *We have similar result for the solutions of the inequalities (1.3)and (1.4)*

Now we are able to start our result

Theorem 2.6 *Under the Hypotheses of theorem2.2, the mild solution of problem 1.1 is stable in Ulam-Hyers sense.*

proof First we proove the stability in $[0, t_1]$
Let $t \in [0, t_1]$ Let $v \in PC$ be a mild solution of inequation (1.2). Let us donote by $u \in PC$ the mild solution of the cauchy problem

$$\begin{cases} u'(t) = A(u(t)) + f(t,u(t)), t \in [0,t_1] \\ u(0) = u_0 \end{cases} \tag{2.8}$$

we have

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s,u(s))ds$$

since A is the infinitesimal generator of a C_0-semigroup,hence tere exist $M > 1$,$\omega \in \mathbb{R}$ with $||T(t)|| \leq Me^{\omega t}$ combining with lemma 2.4 we get

$$\begin{aligned} u(t) - T(t)u_0 - \int_0^t T(t-s)f(s,u(s))ds &\leq \varepsilon \int_0^t ||T(t-s)||ds \\ &\leq \varepsilon M \int_0^t e^{\omega(t-s)}ds \\ &\leq \frac{M}{\omega}e^{\omega(t_1)\varepsilon} \end{aligned}$$

then

$$\begin{aligned} |v(t) - u(t)| &\leqslant |v(t) - T(t)u_0 - \int_0^t T(t-s)f(s,v(s))ds| \\ &+ \int_0^t [|T(t-s)|.|f(s,v(s)) - f(s,u(s)|]ds \\ &\leqslant \frac{M}{\omega}e^{\omega(t_1)}\varepsilon + ML_f e^{\omega(t)} \int_0^t |v(s) - u(s)|ds \end{aligned}$$

but from 1.14 v verifies

$$v(t) = T(t)u_0 + \int_0^t T(t-s)[f(s,u(s) + g(s))]ds$$

using the hypothese (H_1) it follows

$$|v(t) - u(t)| \leqslant \frac{M}{\omega}e^{\omega(t_1)}\varepsilon + ML_f e^{\omega(t)} \int_0^t |v(s) - u(s)|ds \tag{2.9}$$

from a Gronwall lemma we have the conclusion of our theorem for $t \in [0, t_1]$.
Now it remains to proove the stability in $(t_k, t_{k+1}]$

Let $t \in [(_k, t_k + 1]$

$$\begin{aligned}
|v(t) - T(t)u_0 - \int_0^t T(t-s)f(s,u(s))ds & \\
& - \sum_{0<t_k<t} T_k(t-t_k)I_k(u(t_k^-))| \\
& = |\int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T_k(t-t_k)g_k| \\
& \leqslant \varepsilon M \int_0^t e^{\omega(t-s)}ds + \varepsilon M \sum_{0<t_k<t} e^{\omega(t-t_k)} \\
& \leqslant \varepsilon M k e^{\omega T} + \varepsilon M \int_0^t e^{\omega(t-s)}ds
\end{aligned}$$

$$\begin{aligned}
|v(t) - u(t)| &= |v(t) - T(t)u_0 - \int_0^t T(t-s)f(s,u(s))ds \\
&- \sum_{0<t_k<t} T_k(t-t_k)(I_k(u(t_k^-))| \\
&\leq \int_0^t [|T(t-s)|.|f(s,v(s)) - f(s,u(s)|]ds \\
&- \sum_{0<t_k<t} T_k(t-t_k)(I_k(v(t_k^-)) - I_k(u(t_k^-)) \\
&\leq \varepsilon M k e^{\omega T} + \varepsilon M \int_0^t e^{\omega(t-s)}ds \\
&+ \int_0^t M e^{\omega(t-s)} l_f |v(s) - u(s)|ds + M \sum_{0<t_k<t} e^{\omega(t-t_k)}||I_k(v(t_k^-) - I_k(u(t_k^-)|| \\
&\leqslant \varepsilon M k e^{\omega T} + \varepsilon M \int_0^t e^{\omega(t-s)}ds + \int_0^t M e^{\omega(t-s)} l_f |u(s) - v(s)|ds \\
&+ M l^* \sum_{0<t_k<t} e^{\omega(t-t_k)}||u - v||_E \\
&\leqslant \varepsilon M k e^{\omega T} + \varepsilon \frac{M}{\omega} e^{\omega t} ds + M l^* k ||u - v||_E + M \int_0^t e^{\omega(t-s)} l_f |u(s) - v(s)|ds
\end{aligned}$$

From a Gronwall lemma thera exists$\delta > 0$ such that

$$|v(t) - u(t)| \leqslant C + \delta M \int_0^t e^{\omega(t-s)} l_f |u(s) - v(s)| ds$$

where

$$C = \varepsilon \frac{M}{\omega} e^{\omega t} ds + \varepsilon M k e^{\omega T} + M l^* k ||u - v||_E$$

this complete the proof

2.4 Example

Consider the following impulsive partial differential equation:

$$\begin{cases} \frac{\partial}{\partial t} x(t, y) = -\frac{\partial^2}{\partial y^2} x(t, y), & y \in (0, 1), t \in [0, \frac{1}{3}) \cup (\frac{1}{3}, 1], \\ \frac{\partial}{\partial y} x(t, 0) = \frac{\partial}{\partial y} x(t, 1) = 0, & t \in [0, \frac{1}{3}) \cup (\frac{1}{3}, 1], \\ \Delta x(\frac{1}{3}, y) = \lambda x(\frac{1}{3}, y), & \lambda \in \mathbb{R}, y \in (0, 1). \end{cases} \tag{2.10}$$

Let $J = [0, 1]$, $m = 1$ and $t - 1 = \frac{1}{3}$. Consider the space $X = L^2(0, 1)$..
Define the operator A such that

$$Ax = -\frac{\partial^2}{\partial y^2} x, \text{for} x \in D(A)$$

where

$$D(A) = \left\{ x \in X : \quad \frac{\partial x}{\partial y}, \frac{\partial^2 x}{\partial y^2} \in X, x(0) = x(1) = 0 \right\}$$

The defined operator A generates a C_0-groupe $\{T(t), t \geq 0\}$in X wich satisfies $||T(t)||_{B(X)} \leq 1 \forall t \geq 0$.

Denote $u(.)(y) = x(t, y)$, $\quad f(., u)(y) = 0$ and $I_1(u(\frac{1}{3}^-))(y) = \lambda u(\frac{1}{3}^-)$, then the problem 3.3 is equivalent to the abstract problem

$$\begin{cases} u'(t) = Au(t), & t \in [0, \frac{1}{3}) \cup (\frac{1}{3}, 1], \\ \Delta u(\frac{1}{3}) = I_1(\frac{1}{3}^-) = \lambda u(\frac{1}{3}^-), & \lambda \in \mathbb{R}. \end{cases} \tag{2.11}$$

We can show that the functions $f = 0$ and I_1 satisfy all hypotheses of theorem 2.6, that is;

$$||f(t, u) - f(t, v)|| \leq l_f ||u - v||_X, \quad l_f > 0, \forall u, v \in X$$

$$||I_1(u(t_k^-)) - I_1(v(t_k^-))||_X = ||\lambda u(\frac{1}{3}^-) - \lambda v(\frac{1}{3}^-)|| = |\lambda| ||u(\frac{1}{3}^-) - v(\frac{1}{3}^-)|| \leq l^* ||u - v||_X$$

where $l* = |\lambda|$ Hence the problem 3.3 is Ulam-Hyers stable.

Chapter 3

Generalized Ulam-Hyers-Rassias Stability

3.1 Introduction

In this chapter, we are concerned by study the Generalized Ulam-Hyers-Rassias Stability of the problem 1.1.We establish suficient conditions for the stability of the mild solution of the impulsive semilinear functional differential problem of the form:

$$\begin{cases} u'(t) = Au(t) + f(t, u(t)), \quad \text{if} t \in J_k, k = 0, \dots m \\ u(t_k^+) - u(t_k^-) = I_k(u(t_k^-)), \text{if} y \in [0, b], k, k = 0, \dots, m \end{cases} \tag{3.1}$$

3.2 Generalized Ulam-Hyers-Rassias Stability

Let us consider the equation (1.1)and the inequation (1.2). we have

Theorem 3.1 *Assume that assumptions of theorem* **??** *and the following hypotheses hold*

H_3 *the function* $\Phi \in L^1(J, [0, \infty))$ *is nondecreasing and there exists* $\lambda_\Phi > 0$ *such that*

$$\int_0^t ||T(t-s)||\Phi(s)ds \leqslant \lambda_\Phi \Phi(t), \forall t \in J.$$

H_4

$$|||I_k(u)||_X \leq \Phi(t) \forall \in PC, k = 1, \dots, m.$$

Then the equation (2.1)is generalized Ulam-Hyers-Rassias stable with respect to Φ

proof Let $v \in PC$ be a solution of the inequation (1.3).By theorem 2.2 the problem 2.1 has a unique solution u. Then we have
For $t \in [0, t_1]$ by Remark (1.15) we have

$$\begin{aligned} |v(t) - T(t)u_0 - \int_0^t T(t-s)f(s, v(s))ds &\leqslant \int_0^t ||T(t-s)||\Phi(s)ds \\ &\leq \lambda_\Phi \Phi(t), \forall t \in [0, t_1] \end{aligned}$$

in $[0, t_1]$ the mild solution satisfies

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s, u(s))ds, \quad t \in [0, t_1]$$

From the above relations if follows

$$\begin{aligned} |v(t) - u(t)| &\leqslant |v(t) - T(t)u_0 - \int_0^t T(t-s)f(s, v(s))ds| \\ &+ \int_0^t [|T(t-s)|.|f(s, v(s)) - f(s, u(s)|]ds \\ &\leqslant \lambda_{\Phi(t)} + \int_0^t ||T(t-s)||L_f|v(s) - u(s)|ds \\ &\leqslant \lambda_{\Phi(t)}\Phi(t) + Ml_f \int_0^t e^{\omega(t-s)}||v(s) - u(s)||ds \end{aligned}$$

From a gronwall lemma we have

$$||v(t) - u(t)|| \leqslant \lambda_{\Phi(t)}\Phi(t) + e^{Ml_f \int_0^t e^{\omega(t-s)}|v(s) - u(s)|ds} \quad \forall t \in [0, t_1]$$

So, the equation (1.1)is generalized Ulam-Hyers-Rassias stable in $[0, t_1]$
It remains to proove the generalized stability in $(t_k, t_{k+1}]$ Let $v \in PC$ be a solution of the inequation (1.2).by Remark (1.15) for $t \in [t_k, t_k + 1]$ we have

that

$$
\begin{aligned}
|v(t) - T(t)u_0 & \\
& - \int_0^t T(t-s)f(s,v(s))ds \\
& - \sum_{0<t_k<t} T_k(t-t_k)I_k(v(t_k^-))| \\
& = |\int_0^t T(t-s)g(s)ds + \sum_{0<t_k<t} T_k(t-t_k)g_k| \\
& \leq \int_0^t T(t-s)\Phi(s)ds + M \sum_{0<t_k<t} \Phi(t)e^{\omega(t-t_k)} \\
& \leq \lambda_{\Phi(t)}\Phi(t) + M\Phi(t)\sum_{k=0}^{m} e^{\omega(b-t_k)} \\
& \leq \left(\lambda_{\Phi(t)} + M\sum_{k=0}^{m} e^{\omega(b-t_k)}\right)\Phi(t)
\end{aligned}
$$

using this above inequality, we get

$$
\begin{aligned}
|v(t)-u(t)| & \leqslant |v(t) - T(t)(u_0) - \int_0^t T(t-s)f(s,v(s)) \\
& - \sum_{0<t_k<t} T_k(t-t_k)(I_k(v(t_k^-))| \\
& + \int_0^t [|T(t-s)|.|f(s,v(s)) - f(s,u(s)|]ds \\
& + \sum_{0<t_k<t} ||T(t-t_k)||| \left(I_k(u(t_k^-)) - I_k(v(t_k^-))\right)| \\
& \leqslant \left(\lambda_{\Phi(t)} + M\sum_{k=0}^{m} e^{\omega(b-t_k)}\right)\Phi(t)M\int_0^t e^{\omega(t-s)} l_f |u(s)-v(s)|ds \\
& + M \sum_{0<t_k<t} e^{\omega(t-t_k)} l^* |u(t_k^-) - (v(t_k^-)| \\
& \leqslant \left(\lambda_{\Phi(t)} + M\sum_{k=0}^{m} e^{\omega(b-t_k)}\right)\Phi(t) + 2k\Phi(t) + l_f\int_0^t T(t-s)e^{\omega(t-s)}|u(s)-v(s)|ds
\end{aligned}
$$

From Gronwall lemma

$$|v(t)-u(t)| \leqslant \left(\lambda_{\Phi(t)} + M\sum_{k=0}^{m} e^{\omega(b-t_k)} + 2m\right)\Phi(t) + Ml_f\int_0^t e^{\omega(t-s)}|u(s)-v(s)|ds \quad \forall t\in(t_k, t_{k+1}$$

Then there exists $c_{f,m,\Phi} = \left(\lambda_{\Phi(t)} + M\sum_{k=0}^{m} e^{\omega(b-t_k)} + 2m\right)$ such that:

$$|v(t)-u(t)| \leqslant ce^{Ml_f\int_0^t e^{\omega(t-s)ds}}\Phi(t) \quad \forall t\in(t_k, t_{k+1}$$

So ,the equation (2.1)is generalized Ulam-Hyers-Rassias stable

3.3 Example

Consider

$$\begin{cases} \frac{\partial}{\partial t}x(t,y) = (\Delta_y - 2I)x(t,y) + \sin(x(t,y)), \quad y\in\Omega, quad, t>0, t\notin\mathbb{N} \\ \frac{\partial}{\partial y}x(t,y) = 0, \quad y \\ partial\Omega, \quad t>0, t\notin\mathbb{N} \\ \Delta x(i,y) = \frac{1}{i^2}x(i^-,y), \quad y\in\Omega, \quad i\in\mathbb{N}. \end{cases} \tag{3.2}$$

where $\Omega\subset\mathbb{R}^2$ is a bounded domain, Δ_y is the Laplace operator in $\mathbb{R}^2$ and $\partial\Omega\in\mathbb{C}^2$ is the boundary of Ω.
Note $J=\mathbb{R}^+, \quad t_i = i, i\in\mathbb{N}$
Let $X = L^2(\Omega)$, $D(A) = H^2(\Omega)\cap H_0^1(\Omega)$. Define $A(x) = (\Delta_y - 2I)x, x\in D(A)$, hence the operator A is the infinitesimal generator of a contraction C_0-semigroup in $L^2(\Omega)$, that is
$||T(t)||_{B(X)} \leq e^{-2t}, \forall t\geq 0$
transform our problem3.2 in the abstract form

$$\begin{cases} u'(t) = Au(t) + f(t,u), \quad t\in[0,\infty), t\notin\mathbb{N}, \\ \Delta u(i) = I_i(u(i^-)) = \frac{1}{i^2}u(i^-), \quad i\in\mathbb{N}. \end{cases} \tag{3.3}$$

where $u(.)(y) = x(.,y), \quad f(t,u) = \sin(u)$ and $I_i(u(i^-)) = \frac{1}{i^2}x(i^-,y)$,
we shall show that $f(.,.)$ and I_i satisfy all conditions of 2.6. We have

$$||f(t,u)-f(t,v)|| = ||\sin u - \sin v|| = 2||\cos\frac{u+v}{2}\sin\frac{u-v}{2}|| \leq ||u-v||_X, \forall u,v\in X$$

then there exists $l_f > 0, \quad l_f = 1$ such that

$$||f(t,u) - f(t,v)|| =\leq l_f ||u - v||_X, \forall u, v \in X$$

For the functions I_k we have

$$||I_k(u(t_k^-) - I_k(v(t_k^-)|| = ||\frac{1}{k^2}u(k^-) - \frac{1}{k^2}v(k^-)|| \leq \frac{1}{k^2}\sup|u(k^-) - v(k^-)| \leq ||u - v||_X, \forall u, v \in X$$

then there exists $l^* > 0, \quad l^* = 1$ such that

$$||I_k(u(t_k^-) - I_k(v(t_k^-)|| =\leq l^* ||u - v||_X, \forall u, v \in X$$

set $\Phi(t) = e^t$ we have $||I_k(u)|| \leq \Phi(t), \quad \forall k = 1, \ldots, m$ Hence all hypotheses of theorem2.6 are satisfied then the mild solution of problem 3.2 is generalized Ulam-Hyers-Rassias stable.

Conclusion and Perspective

In this memory, we have considered the problem of first order impulsive semilinear differential functional equations with local conditions, where the operator A is densely defined.
Under sufficient conditions, we have proved the existence of mild solutions using fixed point théory, and we have studied the Ulam-Hyers and the generalized Ulam-Hyers-Rassias stability of mild solution of the mild solution of the considered problem.

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