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UNIVERSAL ENVELOPING ALGEBRAS OF LIE ALGEBRAS AND APPLICATIONS.

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In the Name of Allah, the Most Merciful, the Most Compassionate all praise be to Allah, the Lord of the worlds; and prayers and peace be upon Mohamed His servant and messenger.

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Notations and conventions

1. Let A, B be arbitary sets and $f : A \to B$
   - $imf = \{y \in B \quad /\exists x \in A, f(x) = y\}$.
   - $\ker f = \{x \in A \quad /f(x) = 0\}$.
   - $cokerf = B/imf$.
   - $End(A) =$ the set of the endomorphims of A.
2. $\otimes$: Tensor Product.
3. $\oplus$: Direct Sum .
4. $\mathcal{AB}$: The abelian groups category.
5. Λ-moudule: The set of all modules of the ring Λ.
6. I_{ab}: $I/[I, I]$. Such that I is an ideal.
7. $\mathcal{M}_{\Lambda}^{\mathbb{Z}}$: The category of graded (left) modules

Contents

Introduction

The enveloping algebras seem to be very interesting objects for several reasons.
First,these algebras are important for statisyical mechanics, conformal field theory and passion geometry. In addition, a number of things about algebric groups, Lie algebras and their representation are now seen to be shadows from the corresponding quntized objects, and this gives new insight about them. For example,the finite dimensional irreductible representation of the quantized enveloping algebra. This fact has led to a costruction of a canonical basis of these representation with very favourable propereties.
As is well known, every associative algebra A can be turned into a Lie algebra $L(A)$ by replacing its multiplication $(a, b) \mapsto ab$ by the commutator $[a, b] = ab - ba$. We consider a functor U, where a Lie algebra L is embedded into the corresponding associative algebra $U(L)$. The algebra $U(L)$ is called the universal enveloping algebra of the Lie algebra L ; It was first considered in the year 1899.
This work divided into three chapters:
In the first chapter, we intoduced some basic defintions and notations which we will need later. In chapter two, we present the Poincare-Brikhoff-Witt theorem, that Poincare was the first in (1900) who published a fundamental result in this theorem. In 1977, Brikhoff and Witt indepedently formulated and proved the version of the theorem that we use today. Finally, in the last chapter we discuss some aspects of the the cohomology of Lie algebras. Universal enveloping algebras permite us to move into the universe of associative algebras, in which the the theory of derived functors applies, and hence the cohomology of Lie algebras can be defined in a natural way. Some applications of Lie algebras cohomology was mentioned, like Weyl theorem and Levi-Malcev theorem are given in the last section.

Chapter 1

Preliminaires

Modules will be needed later (at least) to treat the cohomology of Lie algebras in its natural framework: the theory of derived functors. For this reason, a reminder on categories and functors seems necessary.
The definiton of a Lie algebra does not need advanced tools, so we decided to treat it in this chapter.

1.1 Modules

Throughout this section, Λ denotes an associative unital ring, which is not necessary commutative.

1.1.1 Modules

Definition 1.1.1 *A* ***left*** $\boldsymbol{\Lambda}$***-module*** *is an abelian group* A *together with an operation* $\Lambda \times A \to A$*, satisfying for all* $a, a_1, a_2 \in A,\ \lambda, \lambda_1, \lambda_2 \in \Lambda$*:*

M1: $(\lambda_1 + \lambda_2)a = \lambda_1 a + \lambda_2 a.$

M2: $(\lambda_1 \lambda_2)a = \lambda_1(\lambda_2 a).$

M3: $1_\Lambda a = a.$

M4: $\lambda(a_1 + a_2) = \lambda a_1 + \lambda a_2.$

Denote by Λ^{opp} the opposite ring of Λ. The elements $\lambda^{opp} \in \Lambda^{opp}$ are in one-to-one correspondance with the elements $\lambda \in \Lambda$. As abelian groups Λ and $\Lambda^{opp} \times A \to A$ are isomorphic under this correspondence. The product in Λ^{opp} is given by $\lambda_1^{opp}\lambda_2^{opp} = (\lambda_1\lambda_2)^{opp}$. We naturally identify the underlying sets of Λ and Λ^{opp}.

A **right module** over Λ or right Λ-module is simply a left Λ^{opp}-module, where Λ^{opp} is the opposite ring of Λ. that is, an abelian group A together with an operation Λ^{opp}. Such that the following rules are satisfied for all lowing rules are satisfied for all $a, a_1, a_2 \in A,\ \lambda, \lambda_1, \lambda_2 \in \Lambda$:

M'1: $a(\lambda_1 + \lambda_2) = a\lambda_1 + a\lambda_2.$

M'2: $a(\lambda_1\lambda_2) = (a\lambda_1)\lambda_2$.

M'3: $a1_\Lambda = a$.

M'4: $(a_1 + a_2)\lambda = a_1\lambda + a_2\lambda$.

Clearly, if Λ is commutative, the notions of a left and a right module over Λ coincide.
For *convenience*, we shall use the term "*module*" always to mean "*a left module*".

Examples

1. The multiplication in Λ defines a left operation of Λ on the underlying abelian group of Λ, which satisfies $\mathbf{M1}, ..., \mathbf{M4}$. Thus Λ is a left module over itself; and similarly Λ can be viewed as a right module over itself.

   Analogously, any left ideal of Λ becomes a left module over Λ, and any right ideal of Λ becomes a right module over Λ.

2. Let $\Lambda = \mathbb{Z}$, the ring of integers. Every abelian group A possedes the structure of a $\mathbb{Z}$-module: for $a \in A, n \in \mathbb{Z}$ define

$$\begin{cases} na = 0 & \text{if } \ n = 0 \\ na = a + ... + a \ \ \text{(n times)} & \text{if } n > 0 \\ na = -(-na) & \text{if } n < 0. \end{cases}$$

3. Let $\Lambda = K$, a field. A K-module is a vector space over K.

Definition 1.1.2 *Let A, B two Λ-modules. A* ***homomorphism (or map) of Λ-modules*** *$\varphi : A \to B$ is a homomorphism of abelian groups which satisfies*

$$\varphi(\lambda a) = \lambda\varphi(a) \text{ for all } a \in A, \lambda \in \Lambda.$$

Definition 1.1.3 *The* ***identity*** *map of A is a homomorphism of Λ-modules; we denote it by $1_A : A \to A$.*
If φ is ***surjective****, we call it also an epimorphism, and we use the symbol $\varphi : A \twoheadrightarrow B$ to denote it.*
If φ is ***injective****, or a monomorphism, we write $\varphi : A \rightarrowtail B$.*
We call $\varphi : A \to B$ an ***isomorphism****, and write $\varphi : A \xrightarrow{\sim} B$, if φ is surjective and injective.*

Let $Hom_\Lambda(A, B)$ denote the set of all Λ-module homomorphisms from A to B. Clearly, this set has the structure of an abelian group; if $\varphi : A \to B$ and $\psi : A \to B$ are Λ-module homomorphisms, then $\varphi + \psi : A \to B$ is defined as $(\varphi + \psi)(a) = \varphi(a) + \psi(a)$ for all $a \in A$.

Remark. $\varphi + \psi$ is a Λ-module homomorphism.

Definition 1.1.4 *A* ***submodule*** *of A is a subgroup A' of A with $\lambda a' \in A'$ for all $\lambda \in A$ and all $a' \in A'$.*

Let A' be a submodule of A. Then the quotient group A/A' may be given the structure of a Λ-module by defining $\lambda(a + A') = \lambda a + A'$ for all $\lambda \in \Lambda, a \in A$.
We have an injective homomorphism $\mu : A' \rightarrowtail A$ and a surjective homomorphism $\pi : A \twoheadrightarrow A/A'$.

Definition 1.1.5 *Let $\varphi : A \to B$ and $\psi : B \to C$ be homomorphisms of Λ-modules. The sequence $A \overset{\varphi}{\to} B \overset{\psi}{\to} C$ is called* ***exact*** *(at B) if $ker\psi = im\varphi$.*

If a sequence $A_0 \to A_1 \to ... \to A_n \to A_{n+1}$ is exact at $A_1, ..., A_n$, then the sequence is simply called exact.

Theorem 1 *Let $A' \overset{\mu}{\rightarrowtail} A \overset{\varepsilon}{\twoheadrightarrow} A''$ be an exact sequence of Λ-modules. For every Λ-module B the induced sequence*

$$0 \to Hom_\Lambda(A'', B) \overset{\varepsilon^*}{\rightarrowtail} Hom_\Lambda(A, B) \overset{\mu^*}{\twoheadrightarrow} Hom_\Lambda(A', B)$$

is exact.

Proof. See [2]. ■

1.1.2 Free, Projective and Injective Modules

Free Module

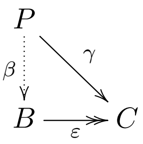
Let A be a Λ-module and let S be a subset of A.
We consider the set A_0 of all elements $a \in A$ of the form $a = \sum \lambda_s s$ where $\lambda_s \in A$ and $\lambda_s \neq 0$ for only a finite number of elements $s \in S$. It is trivially seen that A_0 is a submodule of A; hence it is the smallest submodule of A containing S. If for the set S the submodule A_0 is the whole of A, we shall say that S is a set of generators of A. If A admits a finite set of generators it is said to be **finitely generated**. A set S of generators of A is called a basis of A if every element $a \in A$ may be expressed *uniquely* in the form $a = \sum \lambda_s s$ with $\lambda_s \in \Lambda$ and $\lambda_s \neq 0$ for only a finite number of elements $s \in S$.
It is readily seen that a set S of generators is a basis if and only if it is *linearly independent*, that is, if $\sum \lambda_s s = 0$ implies $\lambda_s = 0$ for all $s \in S$.
We should note that not every module possesses a basis.

Definition 1.1.6 *A Λ-module P is* ***free*** *on some subset $S \subseteq P$ If S is form a basis of P.*

Projective Module

Definition 1.1.7 *A Λ-module P is* ***projective*** *if to every surjective homomorphism $\varepsilon : B \twoheadrightarrow C$ of Λ-modules and to every homomorphism $\gamma : P \to C$ there exists a homomorphism $\beta : P \to B$ with $\varepsilon\beta = \gamma$.*

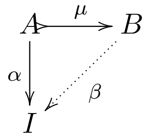
Equivalently, to any homomorphisms ε, γ with ε surjective in the diagram below there exists β such that the triangle



is commutative.

Injective modules

Definition 1.1.8 *A Λ-module I is called **injective** if for every homomorphism $\alpha : A \to I$ and every monomorphism $\mu : A \rightarrowtail B$ there exists a homomorphism $\beta : B \to I$ such that $\beta\mu = \alpha$, i.e. such that the diagram*



is commutative.

1.2 The tensor product

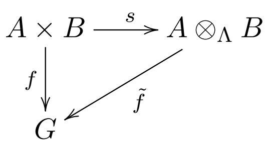
Definition 1.2.1 *Let A and B be two Λ-modules. The tensor product of A and B over Λ is the abelian group, $A \otimes_\Lambda B$, endowed by a map: $s : A \times B \to A \otimes_\Lambda B$ verify: For each map $f : A \times B \to G$ (with G an abelian group) wich satisfy:*

1. *bi-additif:*

$$\begin{cases} f(a_1 + a_2, b) = f(a_1, b) + f(a_2, b), & a_1, a_2 \in A, b \in B; \\ f(a, b_1 + b_2) = f(a, b_1) + f(a, b_2), & a \in A, b_1, b_2 \in B. \end{cases}$$

2. $f(a\lambda, b) = f(a, \lambda b), \quad a \in A, b \in B, \lambda \in \Lambda$.

Exists a unique group homomorphism $\tilde{f} : A \otimes_\Lambda B \to G$ such that the diagram:



is commutative.

1.3 Categories and Functors

1.3.1 Categories

Definition 1.3.1 *To define a category $\mathcal{C}$ we must give three pieces of data:*

1. *A class of objects $A, B, C, ...$*
2. *To each pair of objects (A, B) of $\mathcal{C}$, a set $\mathcal{C}(A, B)$ of morphisms from A to B,*
3. *To each triple of objects (A, B, C) of $\mathcal{C}$, a law of composition*

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C).$$

Before giving the axioms which a category must satisfy we introduce some auxiliary notations: this should also serve to relate our terminology and notation with ideas which are already very familiar.
If $f \in \mathcal{C}(A, B)$ we may think of the morphism f as a generalized "function" from A to B and write:

$$f : A \to B \quad \text{or} \quad A \xrightarrow{f} B$$

we call f a morphism from the domain A to the codomain (or range) B.
The set $\mathcal{C}(A, B) \times \mathcal{C}(B, C)$ consists, of course, of pairs (f, g) where $f : A \to B$, $g : B \to C$ and we will write the composition of f and g as $g \circ f$ or, simply, gf . The rational for this notation lies in the fact that if A, B, C are sets and f, g are functions then the **composite function** from A to C is the function h given by:

$$h(a) = g(f(a)) \quad a \in A.$$

We are now ready to state the axioms.

A1: The sets $\mathcal{C}(A_1, B_1), \mathcal{C}(A_2, B_2)$ are disjoint unless $A_1 = A_2, B_1 = B_2$.

A2: Given $f : A \to B, g : B \to C, h : C \to D$, then

$$h(gf) = (hg)f \quad \text{(Associativity of composition).}$$

A3: To each object A there is a morphism $1_A : A \to A$ such that, for any f: $A \to B, g : C \to A$,

$$f1_A = f, 1_Ag = g \quad \text{(Existence of identities).}$$

It is easy to see that the morphism 1_A is uniquely determined by Axiom **A3**.
We call 1_A the **identity morphism** of A, and we will often suppress the suffix A, writing simply

$$f1 = f \qquad 1g = g.$$

Examples

1. The category $\mathcal{M}$ of Λ**-modules** and **module-homomorphisms**.
2. The category $\mathcal{C}$ of **sets** and **functions**.
3. The category $\mathcal{I}$ of **topological spaces** and **continuous functions**.
4. The category $\mathcal{G}$ of **groups** and **their homomorphisms**.
5. The category $\mathcal{R}$ of **rings** and **ring-homomorphisms**.

1.3.2 Functors

Within a category $\mathcal{C}$ we have the morphism sets $\mathcal{C}(X, Y)$ which serve to establish connections between different objects of the category.
Now the language of categories has been developed to delineate the various areas of mathematical theory; thus it is natural that we should wish to be able to describe connections between different categories.
We now formulate the notion of transformation from one category to another. Such a transformation is called a **functor**.

Definition 1.3.2 *A* ***functor*** $F : \mathcal{C} \to \mathcal{D}$ *is a rule which associates with:*

a1: every object X *of* $\mathcal{C}$ *an object* $F(X)$ *of* $\mathcal{D}$ *.*

a2: every morphism $f \in \mathcal{C}(X, Y)$ *a morphism* $F(f) \in \mathcal{D}(F(X), F(Y))$*, subject to the rules:*

$$F(fg) = F(f)F(g), \qquad F(1_A) = 1_{F(A)}.$$

Such a functor is called **covariant functor**; There also exist **contravariant functors** wich changing the direction of the Arrows i.e. If $f \in \mathcal{C}(X, Y)$ than $F(f) \in \mathcal{D}(F(Y), F(X))$ and $F(fg) = F(g)F(f)$.

Note. We have evidently the notion of an *identity functor* and of the *composition* of functors. Composition is associative and we may thus pass to invertible functors and isomorphic categories.

Examples

1. The embedding of a subcategory $\mathcal{C}_0$ in a category $\mathcal{C}$; is a functor.
2. Let S be a set and let $F(S)$ be the free abelian group on S as basis. This construction yields the free functor $F : \mathcal{C} \to \mathcal{AB}$. Similarly there are free functors $F : \mathcal{C} \to \mathcal{G}, F : \mathcal{C} \to \mathcal{B}_F, ...$ etc.
3. The fundamental group may be regarded as a functor $\pi : \mathcal{I}^0 \to \mathcal{G}$, where $\mathcal{I}^0$ is the category of spaces-with-base-point .
   It may also be regarded as a functor $\bar{\pi} : \mathcal{I}^0_h \to \mathcal{G}$, where the subscript h indicates that the morphisms are to be regarded as (based) homotopy classes of (based) continuous functions. Indeed there is an evident *classifying* functor $Q : \mathcal{I}^0 \to \mathcal{I}^0_h$ and then π factors as $\pi = \bar{\pi}Q$.

1.4 Lie Algebras

In this section Λ denotes a commutative ring with unity.

Lie algebras are algebraic structures which arose first in Sophus Lie's theory of continuous groups. These objects form now an independent major area of research, and have important applications to many branchs of pure mathematics and physique.
We start by the definition of an algebra.

Definition 1.4.1 *An* ***algebra*** *A over Λ, or Λ-algebra A is a ring on which we define the vector product $\cdot : A \times A \to A$, such that for all $x, y, z \in A$ and $\lambda \in \Lambda$:*

1. *$x.(y + z) = x.y + x.z$ and $(y + z).x = y.x + z.x$.*
2. *$\lambda(x.y) = (\lambda x).y = x.(\lambda y)$.*

Definition 1.4.2 *A Lie algebra L is a module over a ring Λ, endowed with a bilinear map*

$$\begin{array}{ccc} L \times L & \to & L \\ (x, y) & \longmapsto & [x, y] \end{array}$$

satisfying the following two axioms:

1. *$[x, x] = 0$, for all $x \in L$.*
2. *For all $x, y, z \in L$, we have $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.* [1]

Remark. The condition (1) implies [x,y] =-[y,x]. Indeed, we have $0 = [x+y, x+y] = [x, y]+[y, x]$.

Examples

1. Let $\mathcal{G}$ be any Λ-module. Define $[x, y] = 0$ for all $x, y \in \mathcal{G}$; such a $\mathcal{G}$ is called a **commutative Lie algebra**.

2. Let A be an associative algebra over Λ, then A has a canonical structure of Lie algebra with the commutator $[x, y] = xy - yx, x, y \in A$.

3. Let V be any vector space. The space of $End(V)$ forms an associative algebra under function composition. It is also a Lie algebra with the commutator $[f, g] = fg - gf, f, g \in End(V)$. Whenever we think of it as a Lie algebra we denote it by $\mathfrak{gl}(V)$. This is the General Linear Lie algebra.

4. Let V be a finite dimentional vector space over a ring Λ. Then we identify the Lie algebra $\mathfrak{gl}(V)$ with set of $n \times n$ matrices $\mathfrak{gl}_\mathfrak{n}(V)$, where n is the dimension of V.
   The set of all matrices with the trace zero $\mathfrak{sl}_\mathfrak{n}(V)$ is a subalgebra of $\mathfrak{gl}_\mathfrak{n}(V)$.

[1] This axiome is called **Jacobi identity**.

5. **Heisenberg algebra:** We look at the vector space H generated over Λ by the matrices: $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ This is a linear subspace of $\mathfrak{gl}_3(V)$ and becomes a Lie algebra under the commutator bracket.
   The fact that H is closed under the commutator bracket follows from the well-known commutator identity on the standard basis of $n \times n$ matrices:
   $$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}$$
   where δ_{ij} is the **Kronecker delta**.

6. **Definition 1.4.3** *Let A be an algebra over Λ. A derivation $D : A \to A$ is a Λ-linear map with the property $D(x.y) = D(x).y + x.D(y)$.*

   The set $Der(A)$ of all derivations of an algebra A is a Lie subalgebra (we will define it next) of $End(A)$ with the product
   $$[D, D'] = DD' - D'D.$$

Definition 1.4.4 *A **Lie algebra homomorphism** $f : L \to H$ is a Λ-linear map with $f([x, y]) = [f(x), f(y)], x, y \in L$.*

Definition 1.4.5 *Let L be a Lie algebra over Λ. A Lie **subalgebra** S is a submodule $S \subset L$ such that: $[x, y] \in S$ for all $x, y \in S$.*

Definition 1.4.6 *Let I be a submodule of a Lie algebra L. We say that I is an **ideal** of L if $[x, y] \in I$ whenever $x \in I$ and $y \in L$.*

Definition 1.4.7 *Let U a non empty subset of L, and let:*

$$\langle U \rangle = \bigcap \{I \subset L : I \text{ is a Lie subalgebra (ideal) containing } U\}$$

*we call $\langle U \rangle$ a **Lie subalgebra (ideal) generated by** U.*

Definition 1.4.8 *A Lie algebra L is called **semi-simple** if $\{0\}$ is the only abelian ideal of L.*

Definition 1.4.9 *A **representation** of a Lie algebra L is a Lie algebra homomorphism from L to the Lie algebra $\mathfrak{gl}(\Lambda)$:*

$$\rho : L \to \mathfrak{gl}(\Lambda).$$

For a Lie algebra L and any $x \in L$ we define a map:

$$\begin{aligned} ad_x : L &\to L \\ y &\mapsto [x, y] \end{aligned}$$

*which is the **adjoint action**.*
Every Lie algebra has a representation on itself, the adjoint representation defined via the map:

$$\begin{aligned} ad : L &\to \mathfrak{gl}(L) \\ x &\mapsto ad_x. \end{aligned}$$

Chapter 2

Poincare-Brikhoff-Witt Theorem

Now, we shall talk about the main subject of our thesis wich is the universal envelopping algebra, and a very important result of it, the Poincare-Brikhoff-witt theory (often abbreviated to PBW theorem).
Before passing to this theorem, we shall speak,a little, about what's a symmetric algebra of a module and the construction of the gradued algebra.

2.1 Universal Algebra of a Lie Algebra

For any associative algebra we construct a Lie algebra by taking the commutator as the Lie bracket. Now let us think in the reverse direction. We want to see if we can construct an associative algebra from a given Lie algebra and its consequences. With this construction, instead of non-associative scructures;Lie algebras, we can work with nicer and better developed structures: Unital associative algebras that captures the important properties of our Lie algebra.

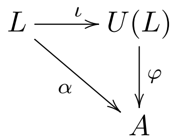
2.1.1 Universal algebra of a Lie Algebra

Let Λ be a commutative ring and let L be a Lie algebra over Λ.

Definition 2.1.1 *A **universal enveloping algebra** of L is a pair $(U(L), \iota)$, where $U(L)$ is an associative algebra with 1 over Λ, $\iota : L \to U(L)$ is a map satisfying:*

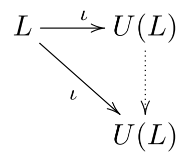
1. *ι is a Lie algebra homomorphism, and i.e., ι is Λ-linear and $\iota([x, y]) = \iota(x).\iota(y) - \iota(y).\iota(x)$.*

2. *If A is any associative algebra with a unit and $\alpha : L \to A$ is any Lie algebra homomorphism, there is a unique homomorphism of associative algebras $\varphi : U(L) \to A$ such that the diagram*



is commutative.

The uniqueness of such a pair $(U(L), \iota)$ is easy to prove. Indeed, given another pair $(U'(L), \iota')$ satisfying the same hypotheses, we get homomorphisms $\varphi : U(L) \to U'(L), \psi : U'(L) \to U(L)$. By definition, there is a unique dotted map making the following diagram commute:



But $1_{U(L)}$ and $\psi \circ \varphi$ both do the trick, so $\psi \circ \varphi = 1_{U(L)}$. Similarly, $\varphi \circ \psi = 1_{U'(L)}$.
To prove its existence, we use the **tensor algebra** $T(L)$ of L, ie.,

$$T(L) = \bigoplus_{k=0}^{\infty} T^k(L) \quad \text{where} \quad T^k(L) = \underbrace{L \otimes ... \otimes L}_{k-times}.$$

Now let I be the two-sided ideal of $T(L)$ generated by the elements of the form

$$x \otimes y - y \otimes x - [x, y].$$

Take

$$U(L) = T(L)/I$$

Let α be a Lie homomorphism of L into an associative algebra A. Since α is Λ-linear, it extends to a unique homomorphism $\psi : T(L) \to A$. It is clear that $\psi(I) = 0$, hence ψ defines $\varphi : U(L) \to A$, and we have checked the universal property of $U(L)$.

2.1.2 Symmetric algebra of a module

Definition 2.1.2 *Let L be a Λ-module and define $[x, y] = 0$ for all $x, y \in L$.*
In this case, the universal algebra $U(L)$ of L is called the symmetric algebra of the Λ-module L and it is denoted by $S(L)$.

We can define $S(L)$ as the largest commutative quotient of $T(L)$, i.e.,

$$S(L) = \sum_{n=0}^{\infty} S^n(L) = \sum_{n=0}^{\infty} (\otimes^n L)/J$$

where J is generated by the elements of the form $x \otimes y - y \otimes x$ and $x, y \in L$.
We will consider the case where L is a free Λ-module with basis $(e_i)_{i \in I}$.
Let $\iota : L \to \Lambda[(X_i)_{i \in I}]$ be the homomorphism given by $\iota(e_i) = X_i$ where $\Lambda[(X_i)_{i \in I}]$ is the polynomial ring in the indeterminates $X_i, i \in I$.
Then $(\iota, \Lambda[(X_i)_{i \in I}])$ has the universal property of (condition 1 in definition 2.1.1 ,p.10), i.e., ι is a Λ-linear map such that $\iota(x)\iota(y) = \iota(y)\iota(x)$ and if $f : L \to A$ is a Λ-linear map with $f(x)f(y) = f(y)f(x)$ for all $x, y \in L$ where A is an associative algebra, then there exists an associative algebra homomorphism $f^* : \Lambda[(X_i)] \to A$ such that $f^* = f \circ \iota$. In fact if $P(X_i) \in \Lambda[(X_i)]$ then $f^*(P) = P(f(e_i))$.
This shows that we can identify $S(L)$, with the polynomial algebra $\Lambda[(X_i)]$.
If we assume that I is totally ordered, then $S(L)$ has for basis the set of monomials $e_{i_1}, ..., e_{i_n}, i_1 \leq i_2 \leq ... \leq i_n, n \geq 0$.

2.1.3 Filtration of U_L

Definition 2.1.3 *A filtration on an algebra $\mathcal{G}$ is an increasing sequence $(\mathcal{G}_n)_{n\in\mathbb{Z}}$ of subalgebras of $\mathcal{G}$ such that $\mathcal{G} = \bigcup_{n\in\mathbb{Z}} \mathcal{G}_n$*

Let L be a Lie algebra over Λ, and let $U(L)$ be the universal algebra of L. We define a **filtration** of $U(L)$ as follows:
Let $U_n(L)$ be the submodule of $U(L)$ generated by the products $\iota(x_1)...\iota(x_m), m \leq n$, where $x_i \in I$.
We have:

$$U_0(L) = \Lambda$$
$$U_1(L) = \Lambda \oplus L$$
$$\text{and } U_0(L) \subset U_1(L) \subset ... \subset U_n(L) \subset U_{n+1}(L) \subset ...$$

Now we define

$$grU(L) = \sum_{n=0}^{\infty} gr_n U(L) \quad \text{where} \quad gr_n U(L) = U_n(L)/U_{n-1}(L)$$

The map

$$\begin{array}{rcl} U_p(L) \times U_q(L) & \rightarrow & U_{p+q}(L) \\ (a,b) & \longmapsto & ab \end{array}$$

defines, by passage to quotient, a bilinear map

$$gr_p(L) \times gr_q(L) \rightarrow gr_{p+q}(L)$$

We then obtain a structure of graded algebra on $grU(L)$; with this structure $grU(L)$ is called the **graded algebra associated to** $U(L)$. It is associative and has a unit.

2.2 Poincare-Brikhoff-Witt Theorem

There is usually no isomorphism of Λ-algebras, but there is often an isomorphism of Λ-modules, sometimes one of L-modules, and often one between the associated graded Λ-algebras. Here we are interested in the latter.

2.2.1 Theorem

For a Lie algebra L, the symmetric algebra of L and graded algebra associated to $U(L)$ are isomorphic ,i.e.,

$$S(L) \simeq grU(L).$$

2.2.2 Proof

Let $\{x_i\}_{i\in\Omega}$ be an ordered basis for L. Let y_i be the image of $x_i \in U(L)$ under the canonical map $\iota : L \to U(L)$. For $I = (i_1, ..., i_n)$, let y_I denote $y_{i_1}...y_{i_n} \in U(L)$. We say $I \le m$ if $i_j \le m$ for all j. Call I **increasing** if $i_1 \le i_2 \le ... \le i_n$.

Lemma 1 *The set of all y_I with I increasing and $I \le n$ generates $U_n(L)$.*

Proof. Let π be a permuation of n elements. We claim that
$\iota(g_1)...\iota(g_n) - \iota(g_{\pi(1)})...\iota(g_{\pi(n)}) \in U_{n-1}(L)$. As the symmetric group is generated by the transpositions flipping i and $i+1$, it suffices to check our claim in this case: Then $\iota(g_1)...\iota(g_i)\iota(g_{i+1})...\iota(g_n) -$ $\iota(g_1)...\iota(g_{i+1})\iota(g_i)...\iota(g_n) = \iota(g_1)...\iota([g_i, g_{i+1}])...\iota(g_n) \in U_{n-1}$.
Now $U_n(L)$ is generated by elements of the form $y_J = \iota(x_{j_1})...\iota(x_{j_n})$ where $J = (j_1, ..., j_n)$ is not necessarily increasing.
Let π be the permutation with $\pi(j_1) \le \pi(j_2)... \le \pi(j_n)$. Then

$$y_J = \iota(x_{j_1})...\iota(x_{j_n}) = \iota(x_{\pi(j_1)})...\iota(x_{\pi j(n)}) + r.$$

where the first term is increasing and the second is in $U_{n-1}(L)$.
Then by induction y_J is expressable in terms of y_I with I increasing and $I \le n$. ■

Now let P be the algebra of polynomials in variables $x_1...x_n....$ To avoid confusion, we'll denote the variables as z_i instead to make clear which algebra the elements lie in. Filter P so P_n is the polynomials of degree at most n. Set $z_I = z_{i_1}...z_{i_n}$ for $I = (i_1...i_n)$.

Lemma 2 *For all n, there exists a unique function $f_n : L \otimes P_n \to P$ such that:*

A_n : $f_n(x_i \otimes z_I) = z_i z_I$ *for* $i \le I, z_I \in P_n$.

B_n : $f_n(x_i \otimes z_I) = z_i z_I$ *mod* P_q *for* $z_I \in P_q$ *and* $q \le n$.

C_n : $f_n(x_i \otimes f_n(x_j \otimes z_J)) = f_n(x_j \otimes f_n(x_i \otimes z_J)) + f_n([x_i, x_j] \otimes z_J)$ *for* $z_J \in P_{n-1}$
Furthermore, the restriction of f_n to $L \otimes P_{n-1}$ is f_{n-1}.

Proof. First, note that Condition C_n is actually well defined because $f_n(x_j \otimes z_j)$ is in P_n by condition B_n. We will proceed by induction. The base case is when $n = 0$, in which case f_0 must map $x_i \otimes 1$ to z_i to satisfy A_0. Then conditions B_0 and C_0 are vacuously satisfied.
Now suppose we have a unique f_{n-1} satisfying A_{n-1}, B_{n-1} and C_{n-1}. We need to define f_n on elements of the form $x_i \otimes z_J$ where J can be of length n. We may as well assume J is increasing since P is commutative. If $i \le J$, then $f_n(x_i \otimes z_J) = z_i z_J$ in order to fulfil A_n.
Now suppose $J = (j, J')$ and $i > j$. Then

$$\begin{aligned} f_n(x_i \otimes z_j z_{J'}) &= f_n(x_i \otimes f_n(x_j \otimes z_{J'})) \\ &= f_n(x_i \otimes f_{n-1}(x_j \otimes z_{J'})) \\ &= f_n(x_j \otimes f_{n-1}(x_i \otimes z_{J'})) + f_{n-1}([x_i, x_j] \otimes z_{J'}) \end{aligned}$$

using the fact that f_n and f_{n-1} agree where they are both defined and trying to satisfy condition C_p. But now $j < i$ and $j \leq J'$ so by property B_{n-1}

$$f_n(x_j \otimes f_{n-1}(x_i \otimes z_{J'})) = f_n(x_j \otimes (z_i z_{J'} + \omega))$$

where $\omega \in P_{n-1}$. By property A_n, this equals $z_j z_i z_{J'} + f_{n-1}(x_j \otimes \omega)$. Thus we should define $f_n(x_i \otimes z_J) = z_i z_J$ when $i \leq J$, and otherwise

$$f_n(x_i \otimes z_j z_{J'}) = z_i z_J + f_{n-1}(x_j \otimes \omega) + f_{n-1}([x_i, x_j] \otimes z_{J'}) \tag{2.1}$$

If this satisfies A_n, B_n and C_n it will be the unique extension of f_{n-1}, for conditions A_n, B_n and C_n when restricted to P_{n-1} are conditions A_{n-1}, B_{n-1} and C_{n-1} which are satisfied by a unique f_{n-1}. Property A_n is obviously satisfied, and so is B_n, since the second and third terms are in P_{n-1} by B_{n-1}. It remains to verify C_n.
Now we need to check $f_n(x_i \otimes f_n(x_j \otimes z_J)) = f_n(x_j \otimes f_n(x_i \otimes z_J)) + f_n([x_i, x_j] \otimes z_J)$ for z_J in P_{n-1}. By the way we constructed f_n, C_n is satisfied if $j < i$ and $j \leq J$ since

$$\begin{aligned} f_n(x_i \otimes f_{n-1}(x_j \otimes z_J)) &= f_n(x_i \otimes z_j z_J) \\ &= z_i z_j z_J + f_{n-1}(x_j \otimes \omega) + f_{n-1}([x_i, x_j] \otimes z_J) \\ &= f_n(x_j \otimes f_{n-1}(x_i \otimes z_J)) + f_{n-1}([x_i, x_j] \otimes z_J). \end{aligned}$$

Furthermore, if we flip the role of i and j since $[x_i, x_j] = -[x_j, x_i]$ this holds as long as $i \leq J'$ and $i < j$.
If $i = j$, there is nothing to prove.
Thus the only remaining cases are when neither $i \leq J$ or $j \leq J : J = (k, K)$ where $k < i, j$. Then by induction (z_J in P_{n-1})

$$\begin{aligned} f_n(x_j \otimes z_J) &= f_n(x_j \otimes f_n(x_k \otimes z_K)) \\ &= f_n(x_k \otimes f_n(x_j \otimes z_K)) + f_n([x_j, x_k] \otimes z_K). \end{aligned}$$

Now $f_n(x_j \otimes z_K) = z_j z_K + \omega$ where $\omega \in P_{n-2}$ by B_{n-1}. Then

$$f_n(x_k \otimes f_n(x_j \otimes z_J)) = f_p(x_i \otimes f_n(x_k \otimes (z_j z_K + \omega))) + f_n(x_i \otimes f_n([x_j, x_k] \otimes z_K)))$$

Since $i > k$ and $k \leq j; K$ and $\omega \in P_{n-2}, C_n$ holds for the first term. C_n holds for the second term by induction. Thus this expands as

$$\begin{aligned} f_n(x_i \otimes f_n(x_k \otimes f_n(x_j \otimes z_K))) + f_n(x_i \otimes f_n([x_j, x_k] \otimes z_K))) &= f_n(x_k \otimes f_n(x_i \otimes f_n(x_j \otimes z_K))) \\ &+ f_n([x_i, x_k] \otimes f_n(x_j \otimes z_K)) \\ &+ f_n([x_j, x_k] \otimes f_n(x_i \otimes z_K)) \\ &+ f_n([x_i, [x_j, x_k]] \otimes z_K) \end{aligned}$$

A similar statement holds if interchange the role of i and j. Then

$$\begin{aligned} f_n(x_i \otimes f_n(x_j \otimes z_J)) - f_n(x_j \otimes f_n(x_i \otimes z_J)) &= f_n(x_k \otimes [f_n(x_i \otimes f_n(x_j \otimes z_K)) - f_n(x_j \otimes f_n(x_i \otimes z_K))]) \\ &+ f_n([x_i,[x_j,x_k]] \otimes z_K) - f_n([x_j,[x_i,x_k]] \otimes z_K) \\ &= f_n(x_k \otimes f_n([x_i,x_j] \otimes z_K)) + f_n([x_i,[x_j,x_k]] \otimes z_K) \\ &- f_n([x_j,[x_i,x_k]] \otimes z_K) \\ &= f_n([x_i,x_j] \otimes f_n(x_k \otimes z_K)) + f_n([x_k,[x_i,x_j]] \otimes z_K) \\ &- f_n([x_i,[x_j,x_k]] \otimes z_K) - f_n([x_j,[x_i,x_k]] \otimes z_K) \\ &= f_n([x_i,x_j] \otimes z_J) + f_n(([x_k,[x_i,x_j]] + [x_i,[x_j,x_k]] \\ &- [x_j,[x_i,x_k]]) \otimes z_K) \\ &= f_n([x_i,x_j] \otimes z_J) \end{aligned}$$

by the Jacobi identity. Thus C_p holds in general, completing the proof. ∎

Theorem 2 *The y_I for I increasing form a basis for $U(L)$ as a vector space.*

Proof. Combining the maps for all n, we see there is a bilinear mapping $f : L \times P \to P$ such that $f(x_i, z_I) = z_i z_I$ for $i \leq I$ and

$$f(x_i, f(x_j, z_J)) = f(x_j, f(x_i, z_J)) + f([x_i, x_j], z_J).$$

This is a representation ρ of L on P with the property that $\rho(x_i)z_I = z_i z_I$.
By the universal property of $U(L)$, there is a map $\psi : U(L) \to End(P)$ with $\psi(y_i)z_I = z_i z_I$ for $i \leq I$. Then by induction if $I = (i_1, ..., i_n)$ is increasing we see

$$\psi(y_{i_1}...y_{i_n}).1 = z_{i_1}...z_{i_n}.$$

But the polynomials on the right hand side are linearly independent, so the y_I with I increasing are linearly independent as well.
We already showed they generate $U(L)$. ∎

This then implies all the forms of the PBW theorem.

Corollary 1 *The canonical mapping of L to $U(L)$ is injective.*

Proof. Using the construction of the universal enveloping algebra as a quotient of the tensor algebra, there is a natural filtration on U(L) where $U_n(L)$ is generated by products of the form $x_1 \otimes ... \otimes x_p$ where $x_i \in L$ and $p \leq n$.
Remember that $grU(L) = \sum_{n=0}^{\infty} gr_n U(L)$ where $gr_n U(L) = U_n(L)/U_{n-1}(L)$ and $\mathrm{gr}_0 U(L) = \Lambda$.
Note that $gr_1 U(L) \simeq L$.
Multiplication in $U(L)$ makes this into a commutative ring by the first lemma. ∎

Theorem 3 (Main Theorem) $S(L) \simeq grU(L)$.

Proof. Since $grU(L)$ is commutative, by the universal property of the symmetric algebra the map $L \to grU(L)$ extends to a map $S(L) \to grU(L)$. We know that expressions of the form $x_1^{v_1}...x_n^{v_n}...$ with $\sum v_i \leq n$ form a basis for $U_n(L)$.
The ones with sum exactly n form a basis for $\mathrm{g}r_nU(L)$. Thus elements of this form give a basis for $grU(L)$, and the map $S(L) \to grU(L)$ sends the standard basis for $S(L)$ to this.
Thus the map is an isomorphism. ■

Chapter 3

Cohomology of Lie Algebras

In this chapter, we will introduce an other big result of studing of the universal envelopping algebra of a Lie algebra, the cohomology of Lie algebras. And we will conclude our chapter by two theorems (Weyl Theorem and Levi-Malcev Theorem), which are considered as a result of the cohomlogy of Lie algebras.

3.1 Extensions of Modules

3.1.1 Extensions

Let A, B be two Λ-modules. We want to consider all possible Λ-modules E such that B is a submodule of E and $E/B \cong A$. We then have a short exact sequence

$$B \overset{k}{\rightarrowtail} E \overset{v}{\twoheadrightarrow} A$$

of Λ-modules; such a sequence is called an **extension** of A by B.

Proposition 1 *The extension $B \rightarrowtail E_1 \twoheadrightarrow A$ is equivalent to the extension $B \rightarrowtail E_2 \twoheadrightarrow A$ if there is a homomorphism $\xi : E_1 \to E_2$ such that the diagram*

$$\begin{array}{ccccc} B & \rightarrowtail & E_1 & \twoheadrightarrow & A \\ \| & & \downarrow \xi & & \| \\ B & \rightarrowtail & E_2 & \twoheadrightarrow & A \end{array}$$

is commutative.

Proof (See [2] p.89)

3.2 Derived functors

3.2.1 Complexes

Let Λ be a fixed ring with 1. We remind of **the category $\mathcal{M}_\Lambda^{\mathbb{Z}}$ of graded (left) modules** :
An object $M \in \mathcal{M}_\Lambda^{\mathbb{Z}}$ is a family $\{M_n\}_{n\in\mathbb{Z}}$, of Λ-modules, a morphism $\varphi : M \to M'$ of degree p is

a family $\{\varphi_n : M_n \to M'_{n+p}\}_{n\in\mathbb{Z}}$ of module homomorphisms.

Definition 3.2.1 *A* ***cochain complex*** $\boldsymbol{C} = \{C_n, \sigma_n\}$ *over* Λ *is an object in* $\mathcal{M}_\Lambda^{\mathbb{Z}}$ *together with an endomorphism* $\sigma : \boldsymbol{C} \to \boldsymbol{C}$ *of degree* $+1$ *with* $\sigma\sigma = 0$ *,* σ *denotes the differential in* $\boldsymbol{C}$*.*

In other words we are given a family $\{C_n\}_{n\in\mathbb{Z}}$, of Λ-modules and a family of Λ-module homomorphisms $\{\sigma_n : C_n \to C_{n+1}\}_{n\in\mathbb{Z}}$ such that $\sigma_n\sigma_{n-1} = 0$:

$$\mathbf{C} : .... \to C_{n-1} \xrightarrow{\sigma_{n-1}} C_n \xrightarrow{\sigma_n} C_{n+1} \to ...$$

Definition 3.2.2 *A* ***morphism of complexes*** *or a* ***cochain map*** $\varphi : \boldsymbol{C} \to \boldsymbol{D}$ *is a morphism of degree 0 in* $\mathcal{M}_\Lambda^{\mathbb{Z}}$ *such that* $\varphi\sigma = \tilde{\sigma}\varphi$ *where .*
Thus a chain map φ *is a family* $\{\varphi_n : \boldsymbol{C}_n \to \boldsymbol{D}_n\}_{n\in\mathbb{Z}}$ *of homomorphisms such that, for every* n*, the diagram*

$$\begin{array}{ccc} \boldsymbol{C}_n & \xrightarrow{\sigma_n} & \boldsymbol{C}_{n+1} \\ {\scriptstyle\varphi_n}\downarrow & & \downarrow{\scriptstyle\varphi_{n+1}} \\ \boldsymbol{D}_n & \xrightarrow[\tilde{\sigma}_n]{} & \boldsymbol{D}_{n+1} \end{array}$$

is commutative.

We shall now introduce the notion of **cohomology**.
Let $\mathbf{C} = \{C_n, \sigma_n\}$ be a cochain complex. The condition $\sigma\sigma = 0$ implies that $\mathrm{im}\sigma_n \subset \ker \sigma_{n+1}, n \in \mathbb{Z}$. we can associate with $\mathbf{C}$ the graded module.

$$H(\mathbf{C}) = \{H^n(\mathbf{C})\}_{n\in\mathbb{Z}}, \text{where} \quad H^n(\mathbf{C}) = \ker\sigma_{n+1}/\mathrm{im}\sigma_n, n \in \mathbb{Z}.$$

Then $H(\mathbf{C})(H^n(\mathbf{C}))$ is called the (n^{th}) **cohomology module** of $\mathbf{C}$.
By the diagram above a chain map $\varphi : \mathbf{C} \to \mathbf{D}$ induces a well defined morphism, of degree zero, $H(\varphi) = \varphi_* : H(\mathbf{C}) \to H(\mathbf{D})$ of graded modules. It is clear that, with this definition, $H(-)$ becomes a functor, called the **cohomology functor**, from the category of cochain complexes over Λ to the category of graded Λ-modules. Also, each $H^n(-)$ is a functor into $\mathcal{M}_\Lambda$. Often, in particular in applications to topology, elements of C_n are called **n-cochains**; elements of $\ker \sigma_n$ are called **n-cocycles** and $\ker \sigma_n$ is written $\mathbf{Z}_n = \mathbf{Z}_n(\mathbf{C})$; elements of im σ_{n+1} are called **n-coboundaries** and im σ_{n+1} is written $\mathbf{B}_n = \mathbf{B}_n(\mathbf{C})$.

3.2.2 Homotopy

Definition 3.2.3 *A* ***homotopy*** $\Sigma : \varphi \to \psi$ *between two chain maps* $\varphi, \psi : \boldsymbol{C} \to \boldsymbol{D}$ *is a morphism of degree* $+1$ *of graded modules* $\Sigma : \boldsymbol{C} \to \boldsymbol{D}$ *such that:* $\varphi - \psi = \sigma\Sigma + \Sigma\sigma$*, i.e., such that, for* $n \in \mathbb{Z}$*,*

$$\varphi_n - \psi_n = \sigma_{n+1}\Sigma_n + \Sigma_{n-1}\sigma_n.$$

We say that φ, ψ are **homotopic**, and write $\varphi \simeq \psi$ if there exists a homotopy $\Sigma : \varphi \to \psi$.

3.2.3 Resolution

Definition 3.2.4 *Let A be a Λ-module. We call a projective resolotion of A every exact sequence of Λ-module:*

$$\boldsymbol{P} : P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$$

where each P_i is projective.

3.2.4 Derived functors

Let $T : \mathit{\Lambda}$-module $\to \mathcal{AB}$ ($\mathcal{AB}$ is the category of the abelian groups) be an additive contravariant functor. The **right derived funtors** of T are a sequence of functors $R^nT : \mathit{\Lambda}$-module $\to \mathcal{AB}$ defined as follow:

Definition 3.2.5 *For an object A in Λ-module; Every projective resolution: $\boldsymbol{P} : P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to A \to 0$ defines a cochain complex*

$$T(\boldsymbol{P}) : 0 \to T(P_0) \to T(P_1) \to \cdots \to T(P_n) \to \dots .$$

So we have cohomology groups $H^n(T(\boldsymbol{P}))$. We set:

$$R^nT(A) = H^n(T(\boldsymbol{P}))$$

If $f : A \to B$ is a morphism in Λ-module, then f extends to a morphism of A resolutios:

$$\begin{array}{lccccccccccccc}
\boldsymbol{P}: & P_n & \longrightarrow & P_{n-1} & \longrightarrow & \dots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A & \longrightarrow & 0 \\
& \downarrow f_n & & \downarrow f_{n-1} & & & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f & & \\
\boldsymbol{Q}: & Q_n & \longrightarrow & Q_{n-1} & \longrightarrow & \dots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & B & \longrightarrow & 0
\end{array}$$

Hence, we obtain a morphism of cochain complexes:

$$\begin{array}{lccccccccccc}
\boldsymbol{T(P)}: 0 & \longrightarrow & T(P_0) & \longrightarrow & T(P_1) & \longrightarrow & \dots & \longrightarrow & T(P_{n-1}) & \longrightarrow & T(P_n) & \longrightarrow \dots \\
& & \downarrow T(f_0) & & \downarrow T(f_1) & & & & \downarrow T(f_{n-1}) & & \downarrow T(f_n) & \\
\boldsymbol{T(Q)}: 0 & \longrightarrow & T(Q_0) & \longrightarrow & T(Q_1) & \longrightarrow & \dots & \longrightarrow & T(Q_{n-1}) & \longrightarrow & T(Q_n) & \longrightarrow \dots
\end{array}$$

Therfore, we have for each $n \geq 0$ a morphism

$$R^nT(f) : R^nT(A) \to R^nT(B).$$

Note that any two projective resolutions of a Λ-module A are homotopic, so they give the same cohomology groups for the complex:

$$\mathbf{T(P)} : 0 \to T(P_0) \to T(P_1) \to \cdots \to T(P_{n-1}) \to T(P_n) \to \dots$$

That is the definition of R^nT is independant from the choice of the projective resolution.
We shall deal only with the funtors of the form

$$Hom_\Lambda(-, A) : \mathit{\Lambda}\text{-module} \to \mathcal{AB}.$$

Where A is a Λ-module. The right derived functors of $Hom_\Lambda(-, A)$ are denoted by: $Ext^n_\Lambda(-, A)$, that is

$$Ext^n_\Lambda(-, A) = R^nHom_\Lambda(-, A).$$

3.3 Cohomology of Lie Algebras

3.3.1 Definition of Cohomology H^0, H^1

For notational convenience we shall write $Hom_L(-,-), Ext^n_L(-,-)$, etc ...,
for $Hom_{U(L)}(-,-), Ext^n_{U(L)}(-,-)$, etc ....

Definition 3.3.1 *Let L be a Λ-Lie algebra, and A a Λ-module. We say that A is a L-module if we have a homomorphism of Lie Algebra $\rho : L \to \mathfrak{gl}(A)$.*

For short, we denote $\rho(x)(a)$ by $x.a$, for $x \in L, a \in A$. We say that A is a trivial L-module if $\rho = 0$, or equivalently $x.a = 0$, for all $x \in L$ and $a \in A$. We note that every Λ-module A may be considered as a trivial L-module. In particular, we will always consider Λ as a trivial L-module.

Definition 3.3.2 *Given a Lie algebra L over Λ and a L-module A, we define the n^{th} cohomology group of L with coefficients in A by*

$$H^n(L, A) = Ext^n_L(\Lambda, A), \quad n = 0, 1, ...$$

where Λ is regarded as a trivial L-module.

Note that each $H^n(L, A)$ is actually a Λ-module. We shall compute $H^0(L, A)$ and $H^1(L, A)$. As the functors $Ext^n_L(-, A)$ and $Hom_L(-, A)$ are naturally equivalent, we have $H^0(L, A) = Hom_L(\Lambda, A)$.

Proposition 2 *We have $Hom_L(\Lambda, A) \simeq A^L$; where $A^L = \{a \in A | \ \ x.a = 0, \text{ for all } x \in L\}$.*[1]

Proof. Let $f \in Hom_L(\Lambda, A)$; we have $x.f(\lambda) = f(x\lambda) = f(0) = 0$, for all $x \in L$ and $\lambda \in \Lambda$. Hence, $f(\lambda) \in A^L$, and in particular $f(1) \in A^L$. So the map

$$\begin{array}{rcl} \omega : Hom_L(\Lambda, A) & \to & A^L \\ f & \longmapsto & f(1) \end{array}$$

is well defined. As $\omega(f+g) = (f+g)(1) = f(1)+g(1) = \omega(f)+\omega(g)$, ω is a Λ-homomorphism. We claim that ω is an isomorphism. Indeed, if $f(1) = 0$, then for all $\lambda \in \Lambda$, we have $f(\lambda) = f(\lambda.1) = \lambda.f(1) = 0$; from which it follows that $f = 0$, that is ω is injective. Now, let $a \in A^L$. Define:

$$\begin{array}{rcl} f : \Lambda & \to & A \\ \lambda & \longmapsto & \lambda a \end{array}$$

Then, f is a homomorphism of Λ-modules, and moreover $f(x.\lambda) = x(\lambda a) = \lambda(x.a) = 0 = xf(\lambda)$, for all $x \in L$; therefore $f \in Hom_L(\Lambda, A)$, and $\omega(f) = f(1) = a$. Thus ω is surjective as claimed. ■

We conclude that:

$$H^0(L, A) = \{a \in A \,|\, x.a = 0, \text{ for all } x \in L\}. \tag{3.1}$$

[1] we call this the subspace of **invariant** elements in A.

Let A be an algebra graded by $\mathbb{N}$; that is $A = \bigoplus_{n\geq 0} A_n$, where A_n is a submodule of A and $A_n A_m \subseteq A_{n+m}$, for all $n, m \in \mathbb{N}$. For any such a graded algebra we have an algebra epimorphism $\epsilon : A \to A_0$ (note here that A_0 is a subalgebra of A). This epimorphism is defined by sending each elelment of A to its homogeneous component of degree 0; that is if $x = \sum_n x_n$, then $\epsilon(x) = x_0$. Here we are interested in the tensor algebra $T(L)$. The homogeneous component of $T(L)$ of degree 0 is the base ring Λ. Thus we have $\epsilon : T(L) \to \Lambda$. For any element of $T(L)$ of the form $a = x \otimes y - y \otimes x - [x, y]$, we have $\epsilon(a) = 0$, since the 0-component of a is 0. Hence $\epsilon : T(L) \to \Lambda$ induces a morphism of associative algebras $\epsilon : U(L) \to \Lambda$. We call ϵ the **augmentation map** for $U(L)$, and its kernel the **augmentation ideal** of $U(L)$; this augmentation ideal is denoted by I_L.

Note that we have an exact sequence of associative algebras

$$I_L \rightarrowtail U(L) \twoheadrightarrow \Lambda.$$

Let A be a L-module. We call a derivation from L to A every mapping $d : L \longrightarrow A$ which satisfies

$$d([x, y]) = x.d(y) - y.d(x)$$

for all $x, y, z \in L$. These derivations form a Λ-module in the obvious way, which we will denote $Der(L, A)$.

For any $a \in A$, we can assign a derivation

$$\begin{array}{rcl} d_a : L & \to & A \\ x & \longmapsto & x.a \end{array}$$

the d_a is a derivation, because

$$\begin{aligned} d_a[x, y] &= [x, y].a \\ &= x.(y.a) - y(x.a) \\ &= x.d_a(y) - yd_a(x). \end{aligned}$$

Such a derivation is called **inner**. The inner derivations in $Der(L, A)$ form a Λ-submodule, which we denote by $Ider(L, A)$.

For a derivation $d : L \to A$, let us define a Λ-linear map $\phi_d : T(L) \to A$ by sending Λ to 0, and each homogeneous element $x_1 \otimes x_2 \otimes ... \otimes x_n, (n \geq 1)$ to $x_1.(x_2.....(x_{n-1}.dx_n)\dots)$. For an element of $T(L)$ of the form $x \otimes y - y \otimes x - [x, y] \otimes t_2$, where $t_2 \in T(L)$ and $x, y \in L$, we have:

$$\phi_d(x \otimes y - y \otimes x - [x, y] \otimes t_2) = x.(ya) - y.(xa) - [x, y](a) = 0,$$

where a is the element of A obtained from $\phi_d(t_2)$. It follows that ϕ_d vanishes on every element of the form

$$t_1 \otimes (x \otimes y - y \otimes x - [x, y]) \otimes t_2, x, y \in L, t_1, t_2 \in T(L).$$

Theorem 4 *The functor $Der(L, -)$ is represented by the L-module I_L that is, for any L-module A there is a natural isomorphism between the functors $Der(L, -)$ and $Hom_L(I_L, -)$. In particular, $Der(L, A) \cong Hom_L(I_L, A)$, for every L-module A.*

Proof. For a derivation $d : L \to A$, we have seen how to obtain a Λ-homomorphism $\Phi_d : T(L) \to A$, which vanishes on all the elements of the form $t_1 \otimes (x \otimes y - y \otimes x - [x, y]) \otimes t_2, x, y \in L, t_1, t_2 \in T(L)$. It follows that Φ_d can be lifted to a Λ-homomorphism which we denote by the same symbol $\Phi_d : U(L) \to A$. Clearly, Φ_d vanishes on the homogeneous elements of degree 0, hence we have a map $f'_d : I_L \to A$, which is easily seen to be a L-module homomorphism.
On the other hand, if $f : I_L \to A$ is given, we extend f to $U(L)$ by setting $f(\Lambda) = 0$ and then we define a derivation $d_f : L \to A$ by $d_f = fi$, where $i : L \to U(L)$ is the canonical embedding. It is easy to check that $f_{(df)} = f$ and $d_{(fd)} = d$, and also that the map $f \longmapsto d_f$ is Λ-linear.

1. We check that $f_{(df)} = f$. Viewing I_L as an L-module, we have that

$$\begin{aligned} f(x_1 \otimes x_2 \otimes ... \otimes x_n) &= f(x_1 \otimes x_2 \otimes ... \otimes i(x_n) \dots) \\ &= x_1.(x_2.....(x_{n-1}.fi(x_n)) \dots) \\ &= x_1.(x_2.....(x_{n-1}.d_f(x_n)) \dots) \\ &= f_{(df)}(x_1 \otimes x_2 \otimes ... \otimes x_n). \end{aligned}$$

2. We check that $d_{(fd)} = d$. This is easily seen because $d_{(fd)}(x) = f_d i(x) = d(x)$ for all $x \in L$.

3. And finally, we check that the map $f \longmapsto d_f$ is Λ-linear. Just note that $d_{\lambda f_1 + f_2}(x) = (\lambda f_1 + f_2)i(x) = \lambda f_1 i(x) + f_2 i(x) = \lambda d_{f_1}(x) + d_{f_2}(x)$ for all $x \in L$.

■

As we seen we have an exact sequence of L-modules

$$I_L \rightarrowtail U(L) \twoheadrightarrow \Lambda$$

This gives a long exact sequence when we introduce the functor $Ext^n_L(-, A)$; and it follows from it (in low dimension) that

$$H^1(L, A) = coker(Hom_L(U(L), A) \to Hom_L(I_L, A)). \tag{3.2}$$

Hence $H^1(L, A)$ is isomorphic to the module of derivations from L into A modulo those that arise from L-module homomorphisms $f : U(L) \to A$.
If $f(1_{U(L)}) = a$, then clearly $d_f(x) = x.a$, so that these are precisely the inner derivations. We obtain

Proposition 3 *$H^1(L, A) \cong Der(L, A)/Ider(L, A)$. If A is a trivial L-module, $H^1(L, A) \cong Hom_\Lambda(L_{ab}, A)$.*

Proof. Only the second assertion remains to be proved.
Since A is trivial, there are no non-trivial inner derivations, and a derivation $d : L \to A$ is simply a Lie algebra homomorphism, A being regarded as an abelian Lie algebra. ■

3.3.2 H^2 and Extensions

Let A and H be two Lie algebras over a field Λ. We call extension of A by H every Lie algebra L which contains A as an ideal, and L/A is isomorphic to the Lie algebra H. This situation gives rise to an exact sequence of Lie algebra homomorphisms

$$A \overset{i}{\rightarrowtail} L \overset{p}{\twoheadrightarrow} H.$$

Conversely, any such an exact sequence detremines an extension L of $i(A)$ by H. The Lie algebra A will be called the kernel of that extension. We shall deal only with abelian extensions; i.e., extensions with abelian kernels.

Two extensions $A \rightarrowtail L \twoheadrightarrow H$ and $A \rightarrowtail L' \twoheadrightarrow H$ are said to be equivalent, if there is a Lie algebra homomorphism $f : L \to L'$ which makes the following diagram commutative

$$\begin{array}{ccccc} A & \rightarrowtail & L & \twoheadrightarrow & H \\ \| & & \downarrow f & & \| \\ A & \rightarrowtail & L' & \twoheadrightarrow & H \end{array}$$

Note that such a f is automatically an isomorphism. This relation is an equivalence relation on the set of all extensions of A by H. We denote the set of equivalence classes of these extensions by $M(H,A)$. Note that $M(H,A)$ contains at least one element: the equivalence class of the semi-direct product $A \overset{i_A}{\rightarrowtail} A \times H \overset{p_H}{\twoheadrightarrow} H$.

Having an abelian extension of Lie algebras $A \overset{i}{\rightarrowtail} L \overset{p}{\twoheadrightarrow} H$, we can define a structure of H-module on A as follows:

Let $s : H \to L$ be a section of p, that is, a Λ-linear map such that $ps = 1_L$ (note that such a section exists because Λ is a field; more precisely because $i(A)$ can be complemented in the Λ-vector space L). Then, we can define in $i(A)$, and hence in A, an action of H by setting $x.i(a) = [s(x), i(a)], a \in A, x \in h$, where the product $[s(x), i(a)]$ is taken in the Lie algebra L.

It follows that, since A is abelian, the H-action thus defined on A does not depend upon the choice of the section s. This H-module structure on A is called the H-module structure induced by the extension. Now, we can talk about the cohomology groups $H^n(H,A)$.

Theorem 5 *There is a one-to-one correspondence between $H^2(H,A)$ and the set $M(H,A)$ of equivalence classes of extensions of H by A.*
The set $M(H,A)$ therefore has a natural Λ-vector space structure and $M(H,A)$ is a (covariant) functor from H-modules to Λ-vector spaces.

For a proof of this result we refer the reader to [2].

3.3.3 Applications

Let L be a finite dimensional Lie algebra and let A be a finite dimensional L-module.

Theorem 6 ***(Weyl)*** *Every (finite-dimensional) module A over a semi-simple Lie algebra L is a direct sum of simple L-modules.*

Proof. Using induction on the Λ-dimension of A, we have only to show that every non-trivial submodule $0 \neq A' \subset A$ is a direct summand in A. To that end we consider the short exact sequence

$$A \rightarrowtail A' \twoheadrightarrow A'' \tag{3.3}$$

and the induced sequence

$$0 \to Hom_\Lambda(A'', A) \to Hom_\Lambda(A, A') \to Hom_\Lambda(A, A) \to 0 \tag{3.4}$$

which is exact since Λ is a ring. We remark that each of the modules in (3.4) is finite-dimensional and can be made into a L-module by the following procedure:
Let B, C be L-modules; then $Hom_\Lambda(B, C)$ is a L-module by

$$(xf)(b) = xf(b) - f(xb), x \in L, b \in B.$$

With this under- standing, (3.4) becomes an exact sequence of L-modules. Note that the invariant elements in $Hom_\Lambda(B, C)$ are precisely the L-module homomorphisms from B to C. Now consider the long exact cohomology sequence arising from (3.4)

$$0 \to H^0(L, Hom_\Lambda(A'', A')) \to H^0(L, Hom_\Lambda(A, A')) \to H^0(L, Hom_\Lambda(A', A')) \to H^0(L, Hom_\Lambda(A'', A')) \to \cdots \tag{3.5}$$

Like L is semi-simple than $H^1(L, Hom_\Lambda(A'', A'))$ is trivial (Proposition 6.1 p.247 in [2]).Passing to the interpretation of H^0 as the group of invariant elements, we obtain an epimorphism

$$Hom_\Lambda(A, A') \twoheadrightarrow Hom_\Lambda(A', A').$$

It follows that there is a L-module homomorphism $A \to A'$ inducing the identity in A'; hence (3.4) splits.

■

Theorem 7 ***(Levi-Malcev).*** *Every (finite-dimensional) Lie algebra L is the split extension of a semi-simple Lie algebra by the radical r of L.*

Proof. We proceed by induction on the derived length of r. If r is abelian, then it is a L/r-module and $H^2(L/r, r) = 0$ (Proposition 6.3 p.249 in [2]). Since H^2 classifies extensions with abelian kernel the extension $r \rightarrowtail L \twoheadrightarrow L/r$ splits. If r is non-abelian with derived length $n \geqslant 2$, we look at the following diagram

$$\begin{array}{ccccc} r & \longrightarrow & L & \twoheadrightarrow & L/r \\ \downarrow & & \downarrow & & \| \\ r/[r,r] & \rightarrowtail & L/[r,r] & \twoheadrightarrow & L/r \end{array}$$

The bottom sequence splits by the first part of the proof, say by $s : L/r \to L/[r,r]$. Let $h/[r,r]$ be the image of L/r under s; clearly $s : L/r \stackrel{\sim}{\to} h/[r,r]$ and $[r,r]$ taust be the radical of h. Now consider the extension $[r,r] \rightarrowtail h \twoheadrightarrow h/[r,r]$. Since $[r,r]$ has derived length $n-1$. it follows, by the inductive hypothesis, that the extension taust split, say by $q : h/[r,r] \to h$. Finally it is easy to see that the top sequence of diagramm splits by $t = qs, t : L/r \stackrel{\sim}{\to} h/[r,r] \to h \subset L$. ■

Conclusion

In this paper we aim to provide the research with an expository look at several important results in the study of finite dimensional Lie algebras. In particallar we discuss the construction of the Universal Enveloping Lie Algebra; The algebraic structure of the universal enveloping algebra $U(L)$ is not only of an associative unital algebra, but it is a Hopf algebra. That is: there is an structure of coalgebra such that the multiplication and comultiplication in $U(L)$ are compatible. This is a much richer structure. Also these structures are used for representation theory of Lie algebras. To give an example, it had used to construct faithful representations of minimal degree by certain quotients of the universal enveloping algebra; This helped us to study affine structures on nilpotent Lie groups. But in this work we just discuss about some of their results such that the Poincare-Birkhoff-Witt Theorem. It was not easy for the biginer to achieve how its proof made.
In order to more discover the cathegory of Lie Algebra it was necessary to pass a Cohomologie where we try to show H^0, H^1 and H^2 using some of theorems. Our aim in future inchaAllah is to continus this research by turning to Lie groups and their representation.

Abstract

Abstract

Our work is devoted to investigating the notion of universal enveloping algebras of Lie algebras, and some relevant results as the Poincare-Birkhoff-Witt theorem and the cohomology of Lie algebras with some applications for this last.

Résumé

Notre travail est consacré à enquêter sur la notion de algèbre enveloppante des algèbres de Lie, et certains résultats pertinents que le théorème de Poincaré-Birkhoff-Witt et la cohomologie des algèbres de Lie avec certaines applications pour ce dernier.

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Abstract

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Cley words: algebra, Lie algebra, cohomology.

ملخص

ناقشنا في هذه المذكرة المتواضعة مفهوم الجبر العامة المغلفة لجبر لي، وبعض من نتائجها كنظرية بوانكاري بريكوف فيت، وكوهومولوحي جبر لي مع بعض التطبيقات .

كلمات مفتاحية: جبر، جبر لي، كوهومولوجي .

Résumé

Notre travail est consacré à enquêter sur la notion de algèbre enveloppante des algèbres de Lie, et certains résultats pertinents que le théorème de Poincaré-Birkhoff-Witt et la cohomologie des algèbres de Lie avec certaines applications pour ce dernier.

Mots clés: algèbre, algèbres de Lie, cohomologie.

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