Multiobjective programming under generalized V-type I invexity

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Abstract. In this paper, we are concerned with a differentiable multiobjective programming problem with inequality constraints. We introduce new concepts of generalized V-type I invexity problems in which each component of the objective and constraint functions is considered with respect to its own function η_i or θ_j. In the setting of these definitions, we establish new Karush-Kuhn-Tucker type necessary and sufficient optimality conditions for a feasible point to be efficient or properly efficient. Furthermore, we show, with examples, that the obtained results allow to prove that a feasible point is an efficient or properly efficient solution even if it is not an usual vector Karush-Kuhn-Tucker point for a multiobjective programming problem.
Keywords: Multiobjective programming; Generalized V-type I problem; Generalized Karush-Kuhn-Tucker condition; Optimality; (Properly) efficient point

1 Introduction

In optimization theory, convexity plays a very important role especially in the construction of sufficient conditions of optimality and duality theory see, for example, Mangasarian [22] and Bazaraa et al. [6]. Several generalizations were introduced in the literature in order to weaken the hypothesis of convexity in mathematical programming and multiobjective problems. Hanson [14] introduced the concept of invexity for the differentiable functions, generalizing the difference $(x - x_0)$ in the definition of convex function to any function $\eta(x, x_0)$. He proved that if, in a mathematical programming problem, instead of the convexity assumption, the objective and constraint functions are invex with respect to a same vector function η, then both the sufficiency of Karush-Kuhn-Tucker conditions and weak and strong Wolfe duality still hold. Later, Hanson and Mond [15] introduced two new classes of functions called type I and type II functions, which are not only sufficient but are also necessary for optimality in primal and dual problems, respectively. In [28], Rueda and Hanson extended type I functions to pseudo-type I and quasi-type I functions and have obtained sufficient optimality criteria for a nonlinear programming problem involving these functions. Rueda et al. [29] obtained optimality and duality results for several mathematical programs by combining the concepts of type I and univex functions defined by Bector et al. [7]. For other generalizations of invexity, see [2, 8, 11, 19, 23, 27, 30] and the references cited therein.

On the other hand, Kaul et al. [18] considered a multiobjective problem involving generalized type-I functions, with scalarization, and obtained some results on optimality and duality, where the Wolfe and Mond-Weir duals are considered. Mishra [24] considered a multiple objective nonlinear programming problem and obtained optimality, duality and saddle point results of a vector valued Lagrangian by combining the concepts of generalized type-I and univex functions. Aghezzaf and Hachimi [1] introduced new classes of generalized type-I vector-valued functions and, without scalarization, derived various duality results for a nonlinear multiobjective programming problem. Following Jeyakumar and Mond [17] and Kaul et al. [18], Hanson et al. [16] introduced the V-type I problem with respect to η, including positive real-valued functions α_i and β_j in their definition, and they obtained optimality conditions and duality results under various types of generalized V-type I requirements. For other optimality conditions and approaches to duality for multiobjective optimization problems, the reader can refer to the references [3–5, 9, 10, 12, 13, 20, 25, 26, 33].

However, in the literature, the type I functions and the V-type I problems (the invex problems in general) are considered with respect to a same function η. Jeyakumar and Mond [17] have observed that one major difficulty in all of these extensions of convexity is that invex problems require a same function η for the objective and constraint functions. This requirement turns out to be a major restriction in applications. In [31], a nonlinear programming is considered and KT-invex, weakly KT-pseudo-invex and type I problems with respect to different η_i are defined. A new Karush-Kuhn-Tucker type necessary condition is introduced and duality results are obtained, for Wolfe and Mond-Weir type dual programs, under generalized invexity assumptions. In [32], the invexity with respect to different η_i is used in the nondifferentiable case.

Motivated and inspired by work in [31, 32], in this paper, we define new classes of generalized V-type I invexity problems in which each component of the objective and constraint functions is considered with respect to its own function η_i or θ_j. These multiobjective programming problems preserve the sufficient optimality conditions under a generalized Karush-Kuhn-Tucker condition, and avoid the major difficulty of verifying that the inequality holds for a same function η for invex functions. This relaxation widens the area of application and allows to get results which are applicable to prove that a feasible point is an efficient or properly efficient solution even if it is not an usual vector Karush-Kuhn-Tucker point for a multiobjective programming problem. Further, we illustrate the obtained results by some examples where we have a large choice to take the different functions η_i and θ_j with respect to which the objective and constraint functions are considered.

2 Preliminaries and definitions

The following conventions for inequalities will be used. If $x = (x_1, ..., x_n)$, $y = (y_1, ..., y_n) \in \mathbb{R}^n$, then: $x \leqq y \Leftrightarrow x_i \leqq y_i, \ \forall \ i = 1, ..., n$; $x \leq y \Leftrightarrow x \leqq y$ and $x \neq y$; $x < y \Leftrightarrow x_i < y_i, \ \forall \ i = 1, ..., n$. We also note $\mathbb{R}^q_{\geqq}$ (resp. $\mathbb{R}^q_{\geq}$ or $\mathbb{R}^q_{>}$) the set of vectors $y \in \mathbb{R}^q$ with $y \geqq 0$ (resp. $y \geq 0$ or $y > 0$).

We consider the following multiobjective optimization problem

$$(VP) \qquad \begin{array}{l} \text{Minimize } f(x) = (f_1(x), ..., f_N(x)), \\ \text{subject to } \ g(x) \leqq 0, \end{array}$$

where $f : D \to \mathbb{R}^N$ and $g : D \to \mathbb{R}^k$ are differentiable functions on the open set $D \subseteq \mathbb{R}^n$. Let $X = \{x \in D : g(x) \leqq 0\}$ the set of feasible solutions of (VP). For $x_0 \in D$, we denote by $J(x_0)$ the set $\{j \in \{1, ..., k\} : g_j(x_0) = 0\}$, $J = |J(x_0)|$ and by $\tilde{J}(x_0)$ (resp. $\bar{J}(x_0)$) the set $\{j \in \{1, ..., k\} : g_j(x_0) < 0 \text{ (resp. } g_j(x_0) > 0)\}$. We have $J(x_0) \cup \tilde{J}(x_0) \cup \bar{J}(x_0) = \{1, ..., k\}$ and if $x_0 \in X, \bar{J}(x_0) = \varnothing$.

We recall some optimality concepts, the most often studied in the literature, for the problem (VP). For other notions and their connections, see [34].

Definition 1. *A point $x_0 \in X$ is said to be a weakly efficient (an efficient) solution of the problem (VP), if there exists no $x \in X$ such that*

$$f(x) < f(x_0) \ (f(x) \leq f(x_0)). \tag{1}$$

Definition 2. *An efficient solution $x_0 \in X$ of (VP) is said to be properly efficient, if there exists a positive real number M such that the inequality*

$$f_i(x_0) - f_i(x) \leqq M[f_j(x) - f_j(x_0)], \tag{2}$$

is verified for all $i \in \{1, ..., N\}$ and $x \in X$ such that $f_i(x) < f_i(x_0)$, and for a certain $j \in \{1, ..., N\}$ such that $f_j(x) > f_j(x_0)$.

Kaul et al. [18] and Hanson et al. [16] defined the invex type I functions and the invex V-type I problem respectively, by taking a same η for the objective and constraint functions. In what follows, we define vector type I problems, where each component of the objective and constraint functions is considered with respect to its own function η_i or θ_j.

Definition 3. *We say that the problem (VP) is of V-type I at $x_0 \in D$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exist $(N + k)$ vector functions $\eta_i : X \times D \to \mathbb{R}^n, \ i = \overline{1, N}$ and $\theta_j : X \times D \to \mathbb{R}^n, \ j = \overline{1, k}$ such that for all $x \in X$:*

$$f_i(x) - f_i(x_0) \geqq [\nabla f_i(x_0)]^t \eta_i(x, x_0), \ \forall \ i = 1, ..., N, \tag{3}$$

$$-g_j(x_0) \geqq [\nabla g_j(x_0)]^t \theta_j(x, x_0), \ \forall \ j = 1, ..., k. \tag{4}$$

If the inequalities in (3) are strict (whenever $x \neq x_0$), we say that (VP) is of semi strictly V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$.

Example 1. We consider the following multiobjective optimization problem

$$\text{Minimize } f(x) = (x + \sin x, \cos x),$$
$$\text{subject to } \; g(x) = x - \tfrac{\pi}{6} \leqq 0,$$

where $f :]0, \frac{\pi}{2}[\to \mathbb{R}^2$ and $g :]0, \frac{\pi}{2}[\to \mathbb{R}$. The set of feasible solutions of problem is $X = \{x \in]0, \frac{\pi}{2}[\,:\, g(x) \leqq 0\} =]0, \frac{\pi}{6}]$. The problem is V-type I at $x_0 = \frac{\pi}{6} \in X$ with respect to $(\eta_i)_{i=1,2}$ and θ defined as follows: $\eta_1(x, x_0) = (\sin x - \sin x_0)/\cos x_0$, $\eta_2(x, x_0) = (\cos x_0 - \cos x)/\sin x_0$ and $\theta(x, x_0) = \sin(x - \frac{\pi}{6})$ ($\theta(x, x_0)$ may be any negative scalar function on X).

Definition 4. *We say that the problem (VP) is of quasi V-type I at $x_0 \in D$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exist $(N + k)$ vector functions $\eta_i : X \times D \to \mathbb{R}^n$, $i = \overline{1, N}$ and $\theta_j : X \times D \to \mathbb{R}^n$, $j = \overline{1, k}$ such that for some vectors $\mu \in \mathbb{R}^N_{\geqq}$ and $\lambda \in \mathbb{R}^k_{\geqq}$:*

$$\sum_{i=1}^{N} \mu_i [f_i(x) - f_i(x_0)] \leqq 0 \quad \Rightarrow \quad \sum_{i=1}^{N} \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) \leqq 0, \; \forall \, x \in X, \tag{5}$$

$$\sum_{j=1}^{k} \lambda_j g_j(x_0) \geqq 0 \quad \Rightarrow \quad \sum_{j=1}^{k} \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \leqq 0, \; \forall \, x \in X. \tag{6}$$

If the second (implied) inequality in (5) is strict ($x \neq x_0$), we say that (VP) is of semi strictly-quasi V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$.

Example 2. We consider the following multiobjective optimization problem

$$\text{Minimize } f(x) = (\sin x, \cos x),$$
$$\text{subject to } \; g(x) = x - \tfrac{\pi}{3} \leqq 0,$$

where $f :]0, \frac{\pi}{2}[\to \mathbb{R}^2$ and $g :]0, \frac{\pi}{2}[\to \mathbb{R}$. The set of feasible solutions of problem is $X =]0, \frac{\pi}{3}]$. The problem is semi strictly-quasi V-type I at $x_0 = \frac{\pi}{3} \in X$ with respect to $(\eta_i)_{i=1,2}$ and θ defined as follows: $\eta_1(x, x_0) = x_0 - x$, $\eta_2(x, x_0) = \sin(x - x_0)$ and $\theta(x, x_0) = -\cos(x + \frac{\pi}{6})$ (as it can be seen by taking $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{3}{4}$ and $\lambda = \frac{1}{2}$).

Definition 5. *We say that the problem (VP) is of pseudo V-type I at $x_0 \in D$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exist $(N + k)$ vector functions $\eta_i : X \times D \to \mathbb{R}^n$, $i = \overline{1, N}$ and $\theta_j : X \times D \to \mathbb{R}^n$, $j = \overline{1, k}$ such that for some vectors $\mu \in \mathbb{R}^N_{\geqq}$ and $\lambda \in \mathbb{R}^k_{\geqq}$:*

$$\sum_{i=1}^{N} \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) \geqq 0 \quad \Rightarrow \quad \sum_{i=1}^{N} \mu_i [f_i(x) - f_i(x_0)] \geqq 0, \; \forall \, x \in X, \tag{7}$$

$$\sum_{j=1}^{k} \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \geqq 0 \quad \Rightarrow \quad \sum_{j=1}^{k} \lambda_j g_j(x_0) \leqq 0, \; \forall \, x \in X. \tag{8}$$

If the second (implied) inequality in (7) (resp. (8)) is strict ($x \neq x_0$), we say that (VP) is of semi strictly-pseudo V-type I in f (resp. in g) at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$. If the second (implied) inequalities in (7) and (8) are both strict ($x \neq x_0$), we say that (VP) is of strictly-pseudo V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$.

Example 3. We consider the following multiobjective optimization problem

$$\begin{gathered}\text{Minimize } f(x) = (-x, -\cos^2 x),\\ \text{subject to } \ g(x) = x - \tfrac{\pi}{3} \leqq 0,\end{gathered}$$

where $f :]0, \frac{\pi}{2}[\to \mathbb{R}^2$ and $g :]0, \frac{\pi}{2}[\to \mathbb{R}$. The set of feasible solutions of problem is $X =]0, \frac{\pi}{3}]$. The problem is strictly-pseudo V-type I at $x_0 = \frac{\pi}{3} \in X$ with respect to $(\eta_i)_{i=1,2}$ and θ defined as follows: $\eta_1(x, x_0) = x - x_0$, $\eta_2(x, x_0) = \sin x_0(\cos x_0 - \cos x)$ and $\theta(x, x_0) = \sin(x - x_0)$ (as it can be seen by taking $\mu_1 = \frac{3}{4}$ and $\mu_2 = \lambda = \frac{1}{4}$), but the problem is not V-type I at x_0 with respect to the same $(\eta_i)_{i=1,2}$ and θ because f_2 is not invex at $x_0 = \frac{\pi}{3}$ with respect to η_2 (take $x = \frac{\pi}{6}$).

Definition 6. *We say that the problem (VP) is of quasi pseudo V-type I at $x_0 \in D$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exist $(N + k)$ vector functions $\eta_i : X \times D \to \mathbb{R}^n, i = \overline{1, N}$ and $\theta_j : X \times D \to \mathbb{R}^n$, $j = \overline{1, k}$ such that for some vectors $\mu \in \mathbb{R}^N_{\geqq}$ and $\lambda \in \mathbb{R}^k_{\geqq}$ the relations (5) and (8) are satisfied.*
If the second (implied) inequality in (8) is strict ($x \neq x_0$), we say that (VP) is of quasi strictly-pseudo V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$.

Example 4. We consider the following multiobjective optimization problem

$$\begin{gathered}\text{Minimize } f(x) = (\tfrac{1}{x}, x),\\ \text{subject to } \ g(x) = x - 1 \leqq 0,\end{gathered}$$

where $f :]0, +\infty[\to \mathbb{R}^2$ and $g :]0, +\infty[\to \mathbb{R}$. The set of feasible solutions of problem is $X =]0, 1]$. The problem is quasi strictly-pseudo V-type I at $x_0 = 1 \in X$ with respect to $(\eta_i)_{i=1,2}$ and θ defined as follows: $\eta_1(x, x_0) = x^2 - x_0^2$, $\eta_2(x, x_0) = x_0 - x$, and $\theta(x, x_0) = x - x_0$ (as it can be seen by taking $\mu_1 = \frac{3}{4}$ and $\mu_2 = \lambda = \frac{1}{4}$).

Definition 7. *We say that the problem (VP) is of pseudo quasi V-type I at $x_0 \in D$ with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$, if there exist $(N + k)$ vector functions $\eta_i : X \times D \to \mathbb{R}^n, i = \overline{1, N}$ and $\theta_j : X \times D \to \mathbb{R}^n$, $j = \overline{1, k}$ such that for some vectors $\mu \in \mathbb{R}^N_{\geqq}$ and $\lambda \in \mathbb{R}^k_{\geqq}$ the relations (7) and (6) are satisfied.*
If the second (implied) inequality in (7) is strict ($x \neq x_0$), we say that (VP) is of strictly-pseudo quasi V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j=\overline{1,k}}$.

Example 5. We consider the following multiobjective optimization problem

$$\begin{gathered}\text{Minimize } f(x) = (-x^2, -x^4),\\ \text{subject to } \ g(x) = (x - 1)^3 \leqq 0,\end{gathered}$$

where $f :]0, +\infty[\to \mathbb{R}^2$ and $g :]0, +\infty[\to \mathbb{R}$. The set of feasible solutions of problem is $X =]0, 1]$. The problem is pseudo quasi V-type I at $x_0 = 1 \in X$ with respect to $(\eta_i)_{i=1,2}$ and θ defined as follows: $\eta_1(x, x_0) = x^2 - x_0^2$, $\eta_2(x, x_0) = x^4 - x_0^4$ and $\theta(x, x_0) = x^2$ ($\theta(x, x_0)$ may be any scalar function). The problem is not V-type I at x_0 with respect to the same $(\eta_i)_{i=1,2}$ and θ but it is with respect to other functions $\eta'_1(x, x_0) = \frac{1}{2}\eta_1(x, x_0)$, $\eta'_2(x, x_0) = \frac{1}{4}\eta_2(x, x_0)$ and $\theta'(x, x_0) = \theta(x, x_0)$.

Figure 1. summarizes the interconnection between the different concepts of problems defined above.

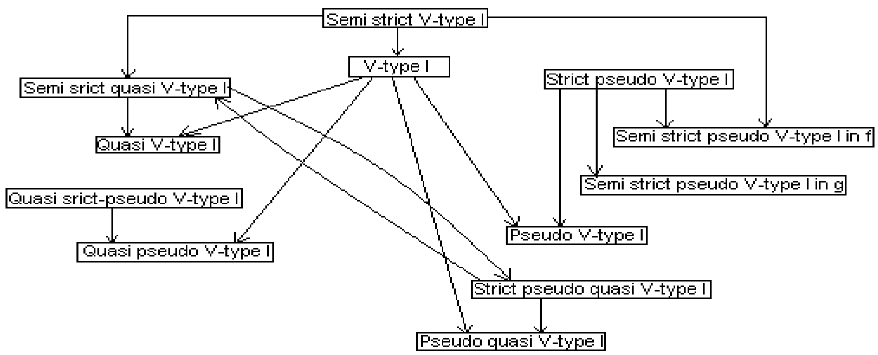

Fig.1. Connections between the different concepts of problems

In the figure 1., "concept $c_1 \to$ concept c_2" means that: if the problem (VP) is of "concept c_1" at x_0 with respect to $(\eta_i)_i$ and $(\theta_j)_j$, then (VP) is of "concept c_2" at x_0 with respect to the same functions $(\eta_i)_i$ and $(\theta_j)_j$.
However, the problem (VP) can be, furthermore, of "concept c_2" at x_0 with respect to other functions $(\bar{\eta}_i)_i$ and $(\bar{\theta}_j)_j$ without it be of "concept c_1" at x_0 with respect to the same functions $(\bar{\eta}_i)_i$ and $(\bar{\theta}_j)_j$.
For example: "V-type I $\to$ pseudo quasi V-type I" means that if the problem (VP) is V-type I at x_0 with respect to $(\eta_i)_i$ and $(\theta_j)_j$, then it is pseudo quasi V-type I at x_0 with respect to the same functions $(\eta_i)_i$ and $(\theta_j)_j$ but the converse is not true in general, see the example 5.

3 Optimality conditions

Weir [33], Kaul et al. [18] and Hanson et al. [16] have given Karush-Kuhn-Tucker type necessary conditions for x_0 to be properly efficient for (VP). Maeda [20, 21] has given the same necessary conditions for x_0 to be efficient for (VP). Osuna-Gómez et al. [26] (resp. Arana-Jiménez et al. [5]) have given Fritz-John and Karush-Kuhn-Tucker type necessary conditions for x_0 to be weakly efficient (resp. efficient) for (VP). Now, in the setting of the new concepts of generalized invexity with respect to different η_i, we give a new Karush-Kuhn-Tucker type necessary optimality condition for x_0 to be efficient for (VP) and then we establish sufficient conditions for a feasible solution to be efficient or properly efficient for (VP).

In the following theorem, we extend the Karush-Kuhn-Tucker type necessary condition established for nonlinear programming programs in [31], to the case of multiobjective programs.

Theorem 1. *(Karush-Kuhn-Tucker type necessary optimality condition) Suppose that x_0 is an efficient solution for (VP) and the functions $f_i,\ i=\overline{1,N}$, $g_j,\ j\in J(x_0)$ are differentiable at x_0. Then there exist vector functions $\eta_i : X\times D\to\mathbb{R}^n,\ i=\overline{1,N},\ \theta_j : X\times D\to\mathbb{R}^n,\ j\in J(x_0)$, $(\eta_i\not\equiv 0,\ \forall\ i=\overline{1,N},\ \theta_j\not\equiv 0,\ \forall\ j\in J(x_0))$ and vectors $\mu\in\mathbb{R}^N_>$ and $\lambda\in\mathbb{R}^J_>$ such that $(x_0,\mu,\lambda,(\eta_i)_{i=\overline{1,N}},(\theta_j)_{j\in J(x_0)})$ satisfies the following generalized Karush-Kuhn-Tucker condition*

$$\sum_{i=1}^{N}\mu_i[\nabla f_i(x_0)]^t\eta_i(x,x_0)+\sum_{j\in J(x_0)}\lambda_j[\nabla g_j(x_0)]^t\theta_j(x,x_0)\geqq 0,\ \forall\ x\in X. \tag{9}$$

Proof. It suffices to take $\eta_i,\ i=1,...,N;\ \theta_j,\ j\in J(x_0);\ \mu$ and λ as follows:

- If f is a constant on X, then $\eta_i(x,x_0)$ can be any nonzero function.
  If f is not a constant on X, then there exists $\bar{x}\in X,\ f(\bar{x})\neq f(x_0)$, it follows that there exists $i_0\in\{1,...,N\},\ f_{i_0}(\bar{x})>f_{i_0}(x_0)$ because x_0 is efficient for (VP1). For all $x\in X$, consider the set $I_x=\{i\in\{1,...,N\}: f_i(x)-f_i(x_0)>0\}$. Note that I_x can be empty. Thus, $\eta_i(x,x_0)=\phi_i(x,x_0)[\nabla f_i(x_0)]$ with $\phi_i(x,x_0)=\begin{cases} f_{i_x}(x)-f_{i_x}(x_0), \text{ if } I_x\neq\emptyset \text{ (with } i_x=\min I_x\text{)};\\ f_{i_0}(\bar{x})-f_{i_0}(x_0), \text{ otherwise.}\end{cases}$
- $\theta_j(x,x_0)=-g_j(x)[\nabla g_j(x_0)]$;
- $\mu_i=\frac{1}{N}$, for all $i=1,...,N$; $\lambda_j=\frac{1}{J}$, for all $j\in J(x_0)$.

Now, we present some Karush-Kuhn-Tucker type sufficient optimality conditions for (VP) under various types of generalized V-type I assumptions. We give several examples to illustrate the obtained results.

Theorem 2. *(Karush-Kuhn-Tucker type sufficient optimality conditions) Let x_0 be a feasible solution for (VP) and suppose that there exist $(N+J)$ vector functions $\eta_i : X\times X\to\mathbb{R}^n,\ i=\overline{1,N},\ \theta_j : X\times X\to\mathbb{R}^n,\ j\in J(x_0)$ and scalars $\mu_i\geqq 0,\ i=\overline{1,N},\ \sum_{i=1}^{N}\mu_i=1,\ \lambda_j\geqq 0,\ j\in J(x_0)$ such that the generalized Karush-Kuhn-Tucker condition (9) is satisfied. Moreover, assume that one of the following conditions is verified:*

(a) the problem (VP) is quasi strictly-pseudo V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$, $(\theta_j)_{j\in J(x_0)}$ and for μ and λ;
(b) the problem (VP) is semi strictly-quasi V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$, $(\theta_j)_{j\in J(x_0)}$ and for μ and λ;
(c) the problem (VP) is strictly-pseudo V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$, $(\theta_j)_{j\in J(x_0)}$ and for μ and λ.

Then x_0 is an efficient solution for (VP).

Proof. Suppose that x_0 is not an efficient solution of (VP). Then there exists a feasible solution $x \in X$ such that $f(x) \leq f(x_0)$, which implies that

$$\sum_{i=1}^{N} \mu_i [f_i(x) - f_i(x_0)] \leqq 0. \tag{10}$$

From the above inequality and the condition (a), we obtain

$$\sum_{i=1}^{N} \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) \leqq 0. \tag{11}$$

By using the condition (9), we deduce that

$$\sum_{j \in J(x_0)} \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) \geqq 0, \tag{12}$$

which implies, from the condition (a) (in view of definition 6), that

$$\sum_{j \in J(x_0)} \lambda_j g_j(x_0) < 0.$$

The last inequality contradicts the fact that $g_j(x_0) = 0, \ \forall \ j \in J(x_0)$ and hence the conclusion follows.
The proof of the part (b) is very similar to the proof of part (a), except that for this case the inequality (11) becomes strict ($<$), it follows that the inequality (12) becomes strict ($>$) and, using the reverse implication in (6), we get the contradiction again.
By condition (c), from $g_j(x_0) = 0, \ \lambda_j \geqq 0, \ \forall \ j \in J(x_0)$, in view of the reverse implication in (8), we obtain $\sum_{j \in J(x_0)} \lambda_j [\nabla g_j(x_0)]^t \theta_j(x, x_0) < 0, \ \forall \ x \in X \setminus \{x_0\}$.
By using (9), we deduce $\sum_{i=1}^{N} \mu_i [\nabla f_i(x_0)]^t \eta_i(x, x_0) > 0, \ \forall \ x \in X \setminus \{x_0\}$,
which implies, according to the relation (7) (for strictly-pseudo V-type I problem), that

$$\sum_{i=1}^{N} \mu_i [f_i(x) - f_i(x_0)] > 0, \ \forall \ x \in X \setminus \{x_0\}. \tag{13}$$

Thus (10) and (13) contradict each other, hence x_0 is an efficient solution of (VP). This completes the proof.

In order to illustrate the obtained result, we shall give an example of multiobjective optimization problem in which an efficient solution will be obtained by the application of theorem 2, whereas it will be impossible to apply for this purpose the sufficient optimality conditions using the usual Karush-Kuhn-Tucker condition.

Example 6. We consider the following multiobjective optimization problem

$$\begin{array}{rl} \text{Minimize} & f(x) = (x_1^3 - x_3, x_2^2 - x_1 - x_3), \\ \text{subject to} & g_1(x) = x_2 \leqq 0 \\ & g_2(x) = x_3^3 - x_2 \leqq 0, \\ & g_3(x) = x_1 \leqq 0, \end{array} \tag{14}$$

where $f : \mathbb{R}^3 \to \mathbb{R}^2$ and $g = (g_1, g_2, g_3) : \mathbb{R}^3 \to \mathbb{R}^3$. The set of feasible solutions of problem is $X = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 \leqq 0,\ x_3^3 - x_2 \leqq 0 \text{ and } x_1 \leqq 0\}$.

- We have $x_0 = (0, 0, 0) \in X$ is not a vector Karush-Kuhn-Tucker point of problem (14), because the condition of Karush-Kuhn-Tucker at x_0 takes a form $\mu_1 \nabla f_1(x_0) + \mu_2 \nabla f_2(x_0) + \lambda_1 \nabla g_1(x_0) + \lambda_2 \nabla g_2(x_0) + \lambda_3 \nabla g_3(x_0) = (-\mu_2 + \lambda_3, \lambda_1 - \lambda_2, -\mu_1 - \mu_2) \neq (0, 0, 0),\ \forall\ (\mu_1, \mu_2) \geq 0,\ \forall\ (\lambda_1, \lambda_2, \lambda_3) \geqq 0$, then the known sufficient optimality conditions using this concept, for example from [4, 5, 13, 16–18, 24–26, 33] are not applicable.
- However, using the theorem 2, we have: there exist vector functions $\eta_1(x, x_0) = (x_1, x_2, x_3)$, $\eta_2(x, x_0) = (x_1 + x_2, x_3, x_3)$, $\theta_1(x, x_0) = (x_1, x_1, x_1)$, $\theta_2(x, x_0) = (x_2, -x_2, x_2)$, $\theta_3(x, x_0) = (x_3, x_3, x_3)$ and scalars $\mu_1 = 0$, $\mu_2 = \frac{1}{2}$, $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{3}$ such that the generalized Karush-Kuhn-Tucker condition (9) is satisfied and the problem (14) is strictly-pseudo V-type I at x_0 with respect to $(\eta_i)_{i=1,2}$, $(\theta_j)_{j=1,2,3}$, $\mu = (\mu_1, \mu_2)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$ (the problem (14) is, in fact, quasi strictly-pseudo V-type I and semi strictly-quasi V-type I at x_0 with respect to the same $(\eta_i)_{i=1,2}$, $(\theta_j)_{j=1,2,3}$, $\mu = (\mu_1, \mu_2)$ and $\lambda = (\lambda_1, \lambda_2, \lambda_3)$). It follows that, by theorem 2, x_0 is an efficient solution for the given multiobjective optimization problem.

In the above example, we show that the hypothesis of x_0 to be a vector Karush-Kuhn-Tucker point is sometimes a strong sufficient condition and it is not indispensable to prove that x_0 is an efficient solution of (VP). In this way, the obtained optimality conditions may be considered as an extension of previously known results.

Example 7. The hypothesis of theorem 2 (with the condition (a)) are satisfied for the problem given in the example 4 at $x_0 = 1$ with respect to the same functions $(\eta_i)_{i=1,2}$, θ and for $\mu_1 = \frac{3}{4}$ and $\mu_2 = \lambda = \frac{1}{4}$. Then x_0 is an efficient solution for this problem.

Example 8. The hypothesis of theorem 2 (with the condition (b)) are satisfied for the problem given in the example 2 at $x_0 = \frac{\pi}{3}$ with respect to the same functions $(\eta_i)_{i=1,2}$, θ and for $\mu_1 = \frac{1}{4}$, $\mu_2 = \frac{3}{4}$ and $\lambda = \frac{1}{2}$. Then x_0 is an efficient solution for this problem.

Remark 1. As particular cases of theorem 2, if the functions $\eta_i,\ i = \overline{1, N}$ and $\theta_j,\ j \in J(x_0)$ are equal to a same function η and by using the usual Karush-Kuhn-Tucker condition:

(i) with the condition (a), we obtain the theorem 3.6 of Kaul et al. [18]. If further there exist $(N + k)$ positive real-valued functions $\alpha_i,\ i = \overline{1, N},\ \beta_j,\ j = \overline{1, k}$

defined on $X \times D$ such that in the definition 6 the implication (5) remains true when multiplying $\mu_i[f_i(x) - f_i(x_0)]$ by $\alpha_i(x, x_0)$ and the implication (8) remains true when multiplying $\lambda_j g_j(x_0)$ by $\beta_j(x, x_0)$, we obtain the theorem 3.1 of Hanson et al. [16].

(ii) with the condition (b), we obtain the theorem 3.4 of Kaul et al. [18]. If further there exist $(N + k)$ positive real-valued functions $\alpha_i,\ i = \overline{1, N},\ \beta_j,\ j = \overline{1, k}$ defined on $X \times D$ such that in the definition 4 the implication (5) remains true when multiplying $\mu_i[f_i(x) - f_i(x_0)]$ by $\alpha_i(x, x_0)$ and the implication (6) remains true when multiplying $\lambda_j g_j(x_0)$ by $\beta_j(x, x_0)$, we obtain the theorem 3.3 of Hanson et al. [16].

(iii) with the condition (c), and if there exist $(N + k)$ positive real-valued functions $\alpha_i,\ i = \overline{1, N},\ \beta_j,\ j = \overline{1, k}$ defined on $X \times D$ such that in the definition 5 the implication (7) remains true when multiplying $\mu_i[f_i(x) - f_i(x_0)]$ by $\alpha_i(x, x_0)$ and the implication (8) remains true when multiplying $\lambda_j g_j(x_0)$ by $\beta_j(x, x_0)$, we obtain the first case of theorem 3.4 of Hanson et al. [16].

Theorem 3. *(Karush-Kuhn-Tucker type sufficient optimality conditions) Let x_0 be a feasible solution for (VP) and suppose that there exist $(N + J)$ vector functions $\eta_i : X \times X \to \mathbb{R}^n,\ i = \overline{1, N},\ \theta_j : X \times X \to \mathbb{R}^n,\ j \in J(x_0)$ and scalars $\mu_i > 0,\ i = \overline{1, N},\ \lambda_j \geqq 0,\ j \in J(x_0)$ such that the generalized Karush-Kuhn-Tucker condition (9) is satisfied. Moreover, assume that one of the following conditions is verified:*

(a) the problem (VP) is V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$ and $(\theta_j)_{j\in J(x_0)}$;
(b) the problem (VP) is pseudo quasi V-type I at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$, $(\theta_j)_{j\in J(x_0)}$ and for μ and λ;
(c) the problem (VP) is semi strictly-pseudo V-type I in g at x_0 with respect to $(\eta_i)_{i=\overline{1,N}}$, $(\theta_j)_{j\in J(x_0)}$ and for μ and λ.

Then x_0 is a properly efficient solution for (VP).

Proof. By condition (a), for all $x \in X$, we have

$$\sum_{i=1}^{N}\mu_i f_i(x) - \sum_{i=1}^{N}\mu_i f_i(x_0) \overbrace{\geqq}^{(3)} \sum_{i=1}^{N}\mu_i[\nabla f_i(x_0)]^t\eta_i(x, x_0) \overbrace{\geqq}^{(9)} -\sum_{j\in J(x_0)}\lambda_j[\nabla g_j(x_0)]^t\theta_j(x, x_0)$$
$$\overbrace{\geqq}^{(4)} \sum_{j\in J(x_0)}\lambda_j g_j(x_0) = 0.$$

Thus $\sum_{i=1}^{N}\mu_i f_i(x) \geqq \sum_{i=1}^{N}\mu_i f_i(x_0)$ for all $x \in X$ with $\mu > 0$. Hence, from theorem 1 of Geoffrion [12], x_0 is a properly efficient solution for (VP).

By condition (b), from $g_j(x_0) = 0,\ \lambda_j \geqq 0,\ \forall\ j \in J(x_0)$ (in view of definition 7), we obtain

$$\sum_{j\in J(x_0)}\lambda_j[\nabla g_j(x_0)]^t\theta_j(x, x_0) \leqq 0,\ \forall\ x \in X.$$

From the above inequality and the condition (9), it follows that

$$\sum_{i=1}^{N}\mu_i[\nabla f_i(x_0)]^t\eta_i(x,x_0) \geqq 0,\ \forall\ x \in X.$$

By using the relation (7) (in view of definition 7), we deduce that $\sum_{i=1}^{N}\mu_i f_i(x) \geqq \sum_{i=1}^{N}\mu_i f_i(x_0),\ \forall\ x \in X$, and the conclusion follows.

For the proof of part (c), we proceed as in part (b) and using the reverse implication in (8), we get $\sum_{j\in J(x_0)}\lambda_j[\nabla g_j(x_0)]^t\theta_j(x,x_0) < 0,\ \forall\ x \in X \setminus \{x_0\}$. In the same way as in (b), we get $\sum_{i=1}^{N}\mu_i f_i(x) \geqq \sum_{i=1}^{N}\mu_i f_i(x_0),\ \forall\ x \in X$ and it follows that x_0 is properly efficient for (VP). This completes the proof.

In order to illustrate the obtained result, we shall give an example of multiobjective optimization problem in which the properly efficient solution will be obtained by the application of theorem 3, whereas it will be impossible to apply for this purpose the theorem 3.1 of Kaul et al. [18].

Example 9. We reconsider the multiobjective optimization problem given in example 1.

- We have: the problem is not V-type I at $x_0 = \frac{\pi}{6}$ with respect to a same function η because there exists no a function $\eta :]0,\frac{\pi}{6}]\times\,]0,\frac{\pi}{6}] \to \mathbb{R}$ for which the functions f_1 and f_2 are both invex at x_0, as it can be seen by taking $x = \frac{\pi}{12}$, then the theorem 3.1 of Kaul et al. [18] is not applicable.
- However, the hypothesis of theorem 3 are verified. In fact: the condition (9) is satisfied for $(\eta_i)_{i=1,2}$ and θ given in the example 1 and $\mu_1 = \frac{1}{10},\ \mu_2 = \frac{9}{10},\ \lambda = \frac{1}{50}$; the problem is V-type I at $x_0 = \frac{\pi}{6} \in X$ with respect to the same $(\eta_i)_{i=1,2}$ and θ. It follows that, x_0 is a properly efficient solution for the given multiobjective optimization problem.

Now, we shall give example of multiobjective optimization problem in which a properly efficient solution will be obtained by the application of theorem 3, whereas it will be impossible to apply for this purpose the sufficient optimality conditions using the usual Karush-Kuhn-Tucker condition.

Example 10. We consider the following multiobjective optimization problem

$$\begin{aligned} &\text{Minimize } f(x) = (-x_1, x_2^2 - x_1),\\ &\text{subject to } \ g_1(x) = x_1^3 - x_2 \leqq 0\\ &\qquad\qquad\ \ g_2(x) = x_2 \leqq 0, \end{aligned} \tag{15}$$

where $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g = (g_1, g_2) : \mathbb{R}^2 \to \mathbb{R}^2$. The set of feasible solutions of problem is $X = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^3 - x_2 \leqq 0 \text{ and } x_2 \leqq 0\}$.

- We have $x_0 = (0,0) \in X$ is not a vector Karush-Kuhn-Tucker point of problem (15), because the condition of Karush-Kuhn-Tucker at x_0 takes the form $\mu_1 \nabla f_1(x_0) + \mu_2 \nabla f_2(x_0) + \lambda_1 \nabla g_1(x_0) + \lambda_2 \nabla g_2(x_0) = (-\mu_1 - \mu_2, -\lambda + \lambda) \neq (0,0),\ \forall\ (\mu_1, \mu_2) \geq 0,\ \forall\ (\lambda_1, \lambda_2) \geqq 0$, then the known sufficient optimality conditions using this concept, for example from [4, 5, 13, 16–18, 24–26, 33] are not applicable.
- However, using the theorem 3, we have: there exist vector functions $\eta_1(x, x_0) = (x_1, x_1),\ \eta_2(x, x_0) = (x_1, x_2),\ \theta_1(x, x_0) = (x_1, -x_2),\ \theta_2(x, x_0) = (x_2, x_1)$ and scalars $\mu_1 = \mu_2 = \lambda_2 = \frac{1}{2},\ \lambda_1 = 0$ such that the generalized Karush-Kuhn-Tucker condition (9) is satisfied and the problem (15) is V-type I at x_0 with respect to $(\eta_i)_{i=1,2}$, $(\theta_j)_{j=1,2}$, $\mu = (\mu_1, \mu_2)$ and $\lambda = (\lambda_1, \lambda_2)$ (the problem (15) is, in fact, pseudo quasi V-type I and (semi) strictly-pseudo V-type I (in g) at x_0 with respect to the same $(\eta_i)_{i=1,2}$, $(\theta_j)_{j=1,2}$, μ and λ). It follows that, by theorem 3, x_0 is a properly efficient solution for the given multiobjective optimization problem.

In the example 10,we show that the hypothesis of x_0 to be a vector Karush-Kuhn-Tucker point is sometimes a strong sufficient condition and it is not indispensable to prove that x_0 is a properly efficient solution of (VP). In this way, the obtained optimality conditions may be considered as an extension of previously known results.

Example 11. The hypothesis of theorem 3 (with the condition (c)) are satisfied for the problem given in the example 3 at $x_0 = \frac{\pi}{3}$ with respect to the same functions $(\eta_i)_{i=1,2}$, θ and for $\mu_1 = \frac{3}{4}$ and $\mu_2 = \lambda = \frac{1}{4}$. Then x_0 is a properly efficient solution for problem.

Remark 2. As particular cases of theorem 3, if the functions $\eta_i,\ i = \overline{1,N}$ and $\theta_j,\ j \in J(x_0)$ are equal to a same function η and by using the usual Karush-Kuhn-Tucker condition:

(i) with the condition (a), we obtain the theorem 3.1 of Kaul et al. [18].

(ii) with the condition (b), we obtain the theorem 3.5 of Kaul et al. [18]. If further there exist $(N+k)$ positive real-valued functions $\alpha_i,\ i = \overline{1,N},\ \beta_j,\ j = \overline{1,k}$ defined on $X \times D$ such that in the definition 7 the implication (7) remains true when multiplying $\mu_i[f_i(x) - f_i(x_0)]$ by $\alpha_i(x, x_0)$ and the implication (6) remains true when multiplying $\lambda_j g_j(x_0)$ by $\beta_j(x, x_0)$ and if also there exist positive real numbers n_i and m_i such that $n_i < \alpha_i(x, x_0) < m_i$ for each $x \in X$ and for all $i = \overline{1,N}$, we obtain the second case of theorem 3.2 of Hanson et al. [16].

(iii) with the condition (c), if there exist $(N+k)$ positive real-valued functions $\alpha_i,\ i = \overline{1,N},\ \beta_j,\ j = \overline{1,k}$ defined on $X \times D$ such that in the definition 5 the implication (7) remains true when multiplying $\mu_i[f_i(x) - f_i(x_0)]$ by $\alpha_i(x, x_0)$ and the implication (8) remains true when multiplying $\lambda_j g_j(x_0)$ by $\beta_j(x, x_0)$ and if also there exist positive real numbers n_i and m_i such that $n_i < \alpha_i(x, x_0) < m_i$ for each $x \in X$ and for all $i = \overline{1,N}$, we obtain the second case of theorem 3.4 of Hanson et al. [16].

4 Conclusion

In this paper, we have defined new classes of problems called V-type I, quasi-, pseudo-, pseudo quasi-, quasi pseudo- V-type I with respect to $(\eta_i)_i$ and $(\theta_j)_j$, as a generalization of invex problems with respect to a same function η. In the setting of these definitions, we have established new Karush-Kuhn-Tucker type necessary and sufficient optimality conditions for a feasible point to be efficient or properly efficient. We illustrated these optimality results with some examples and we have shown that the obtained results allow to prove that a feasible point is an efficient or properly efficient solution even if it is not an usual vector Karush-Kuhn-Tucker point for a multiobjective programming problem. Known results in the literature (Hanson et al. 2001; Kaul et al. 1994) can be deduced as particular cases from the obtained results, when the functions $(\eta_i)_i$ and $(\theta_j)_j$ are equal to a same function η. However, the concept of invexity with respect to different functions η_i may be extended in different directions of the field of multiobjective programming. It may be used, with and without differentiability assumption, in the framework of fractional programming, variational problems, symmetric duality, game theory, etc.

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