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Frictional contact of piezoelectric shallow shell

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Dedication

I dedicate this wark to my ♡ parent ♡
My twa brother's ♡ Salim and Oualid ♡
And sisters, ♡ Khadidja, Zahra, Souad, Fatima and Souhila ♡
al ♡ Ilye ♡
My friends ♡ gherbi and nadji ♡
I'am greatly gratful as well to teacher ♡ Mr. Ghezal Abderrezak and Med El-hadi-Mezabia.♡
And all tho

♡ Miloud Baoua.♡

Acknowledgement

Ithank god who gave me the power and the will to finish this pro
I thank my parent
light of this life .
I offer my hone
Mr. Ghezal Abderrezak and Med El-hadi-Mezabia for his precious
time,his kindne
I thank all my teachers that were patient
this work.
I thank all my classmate

Contents

Notations and Conventions

Latin indices take their values in the set $\{1,2,3\}$ and Greek indices take their values in the set $\{1,2\}$; the repeated index convention for summation is systematically used in conjunction with the above rules.

- $\widehat{\Omega}^{\varepsilon}$:Is a domain in $\mathbb{R}^3$.
- $\partial^{\varepsilon}\widehat{\Omega}^{\varepsilon}$: Boundary of $\widehat{\Omega}^{\varepsilon}$.
- $\widehat{\partial}_i^{\varepsilon}$: $=\frac{\widehat{\partial}^{\varepsilon}}{\partial^{\varepsilon}\widehat{x}_i^{\varepsilon}}$ derived part of $\widehat{x}_i^{\varepsilon}$.
- ∂_i^{ε}:=$\frac{\partial^{\varepsilon}}{\partial^{\varepsilon}x_i^{\varepsilon}}$ derived part of x_i^{ε}.
- ∂_i: $=\frac{\partial}{\partial x_i}$ derived part of x_i.
- θ^{ε}: A mapping for defining the middle surface $\widehat{\omega}^{\varepsilon}$ of the shallow shell.
- α^{ε} :Continuously varying unit vector normal to the middle.
- Θ^{ε} :A mapping for defining the reference configuration of the shallow shell.
- $(.,.)$: The inner product in $L_2(\Omega)$.
- $\widehat{u}^{\varepsilon}$: $\widehat{\Omega}^{\varepsilon}\longrightarrow R^3$:] displacement vector.
- $\widehat{\Gamma}_+^{\varepsilon}$: Upper face of the set $\widehat{\Omega}^{\varepsilon}$.
- $\widehat{\Gamma}_-^{\varepsilon}$: Lower face of the set $\widehat{\Omega}^{\varepsilon}$.
- $d\Gamma$: Area element along $\partial\Omega$.
- $\pi : \Omega \longrightarrow \Omega^{\varepsilon}$: Bijection from Ω onto Ω^{ε} .
- $H^1(\Omega)$: $\{u/u \in L^2(\Omega), \dfrac{\partial u}{\partial x_i} \in L^2(\Omega)\}$.
- $\|u\|_{H^1(\Omega)} = (|u|^2 + \sum_{i=1}^{n} |\dfrac{\partial u}{\partial x_i}|^2)^{1/2}$.
- $\widehat{u}_t^{\varepsilon}$: Tangential components.

- $\widehat{u}_n^\varepsilon$: Normal components.
- $\widehat{\omega}^\varepsilon$: Middle surface of the shallow shell.
- $\longrightarrow$: Strong convergence .
- $\rightharpoonup$: Weak convergence.

Introduction

Piezoelectrics are materials that can create electricity when subjected to a mechanical stress. They will also work in reverse, generating a strain by the application of an electric field.

The phenomenon was first discovered in 1880 when Pierre and Jacques Curie [1]demonstrated that when specially prepared crystals (such as quartz, topaz and Rochelle salt) were subjected to a mechanical stress they could measure a surface charge. A year later, Gabriel Lippmann deduced from thermodynamics that they would also exhibit a strain in an applied electric field. The Curies later experimentally confirmed this effect and provided proof of the linear and reversible nature of piezoelectricity. The continued development of piezoelectric materials has led to a huge market of products ranging from those for everyday use to more specialised devices. Some typical applications can be seen below,

1. Automotive : Air bag sensor, air flow sensor, audible alarms, fuel atomiser, keyless door entry, seat belt buzzers, knock senso rs.

2. Computer : Disc drives, inkjet printers.

3. Medical : Disposable patient monitors, foetal heart monitors, ultrasonic imaging.

Influence of piezoelectric is ditided into two part:

Direct influence : there are some matrials that are influenced ley applying a mechnical force: force $\Longrightarrow$ deform $\Longrightarrow$ electric volume charge

[1] Pierre and Jacques Curie :(1859,1906) paris France Almamater UN of Pariss-(1856,1941), Paris France, fields phsiccs

Indirect influence : there are some other materials that are deformed ley applying a electric volume charge: electric volume $\Longrightarrow$ deformation.

Léger and Miara [2] justified the obstacle problem for shallow shells. Next, Figueiredo and stadler [1] studied the asymptotic analysie of frictional contact for anisotropic piezoelectric plates. Recently, in 2016, yan and Miara [3] studied the asymptotic analysis of the obstacle problem for piezoelectric plates.

The objective of this thesis is to study the asymptotic modeling of three-dimensional problems of linearly elastic shallow shells, in unilateral contact with friction .

The first chapter presents an overview of the three-dimensional piezoelectric preblem for shallow shells.

The second chapter concerns asymptotic analysis.We change the domain Ω^ε having thickness 2ε into a fixed domain Ω independent of ε via the simple geometrical transformation.

In chapter 3, we prove the convergence of the solution when the thickness of the Shallow Shells tends to zero and establish the limit problem of a piezoelectric preblem for shallow shells in unilateral contact and friction.

Chapter 1

The three-dimensional piezoelectric preblem for shallow shells.

1.1 Geometry of shallow shells.

Let $\omega \subset R^2$ be a bounded domain with a Lipschitz continuous boundary $\widehat{\gamma}^\varepsilon$. Let $\widehat{\gamma}_0^\varepsilon \subset \widehat{\partial\omega}^\varepsilon$ with meas $\widehat{\gamma}_0^\varepsilon > 0$. For each $\varepsilon > 0$,we define the sets

$$\widehat{\Omega}^\varepsilon = \widehat{\omega}^\varepsilon \times]-\varepsilon, \varepsilon[$$

$$\widehat{\Gamma}_-^\varepsilon = \widehat{\omega}^\varepsilon \times \{-\varepsilon\} \quad ; \quad \widehat{\Gamma}_+^\varepsilon = \widehat{\omega}^\varepsilon \times \{\varepsilon\} \quad ; \quad \widehat{\Gamma}_0^\varepsilon = \widehat{\gamma}_0 \times]-\varepsilon, \varepsilon]$$

$$(\widehat{\gamma}_0^\varepsilon \subset \partial\widehat{\omega}^\varepsilon)$$

for each $\varepsilon > 0$, we are given a function $\theta^\varepsilon \in C^3(\overline{\omega}^\varepsilon)$;

$$\widehat{\omega}^\varepsilon = \{(x_1, x_2, \theta^\varepsilon(x_1, x_2)) \in \mathbb{R}^3; (x_1, x_2) \in \widehat{\omega}^\varepsilon\}.$$

At each point of the surface $\{\widehat{\omega}^\varepsilon\}^-$,we define the normal vector,

$$\widehat{a}_3^\varepsilon = \{\alpha^\varepsilon\}^{\frac{-1}{2}}(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1).$$

Where,

$\alpha^\varepsilon = |-\partial_1\theta^\varepsilon|^2 + |\partial_2\theta^\varepsilon|^2 + 1$,

$$\left(\widehat{a}_3^\varepsilon = \frac{(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1)}{\|-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1\|}\right)$$

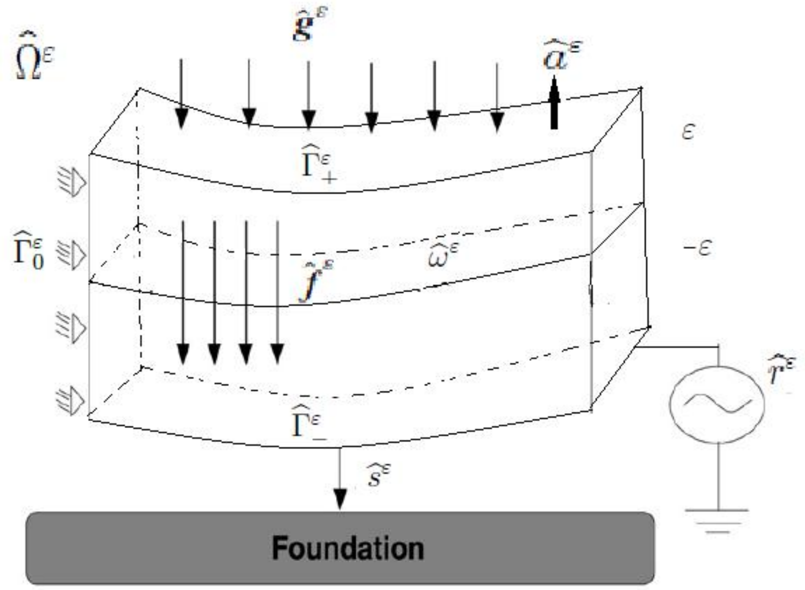

Figure 1.1:

For each $\varepsilon > 0$, we define the mapping $\Theta^\varepsilon : \overline{\Omega}^\varepsilon \longrightarrow (\overline{\widehat{\Omega}^\varepsilon})$ is a C^1-diffeomorfhismine,

$$\Theta^\varepsilon(\widehat{x}^\varepsilon) = (x_1, x_2, \theta^\varepsilon(x_1, x_2)) + x_3^\varepsilon \widehat{a}_3^\varepsilon(x_1, x_2),$$

the set $\Theta^\varepsilon(\Omega^\varepsilon) = \widehat{\Omega}^\varepsilon$ is the shell.

$$\Theta^\varepsilon(\Gamma_0^\varepsilon) = \widehat{\Gamma}_0^\varepsilon,$$

Vector field $\widehat{u}^\varepsilon$ the displacement($\widehat{u}^\varepsilon = 0$ on $\widehat{\Gamma}_0^\varepsilon$).

1.2 Classical problem

A piezoelectric material is described by three tensors: the fourth order symmetric positive definite stiffness tensor $\widehat{C}^\varepsilon_{ijkl}$, the third order piezoelectric tensor $\widehat{P}^\varepsilon_{kij}$ and the second order

symmetric positive definite dielectric tensor $\widehat{d}^{\varepsilon}_{ij}$.
There exists positive c such that, for every second order 3×3 symmetric tensor $M=(M_{ij})$ and for every vector of dimension 3, $\epsilon=(\epsilon_k)$ we have,

$$\begin{cases} \widehat{C}^{\varepsilon}_{ijkl}=\widehat{C}^{\varepsilon}_{ijlk}=\widehat{C}^{\varepsilon}_{klij}, \qquad \widehat{C}^{\varepsilon}_{ijkl}\ M_{kl}M_{ij}\geqslant c\Sigma^3_{i,j=1}M^2_{ij},\\ \widehat{P}^{\varepsilon}_{kij}=\widehat{P}^{\varepsilon}_{kji},\\ \widehat{d}^{\varepsilon}_{ij}=\widehat{d}^{\varepsilon}_{ji}, \qquad \widehat{d}^{\varepsilon}_{kl}\ \epsilon_k\epsilon_l\geqslant c\Sigma^3_{k=1}\epsilon^2_k.\end{cases}$$

Mechanical equilibrium equations and boundary conditions

We suppose that $\widehat{f}^{\varepsilon}_i\in L^2(\widehat{\Omega}^{\varepsilon})$ is the density of the applied body forces acting on $\widehat{\Omega}^{\varepsilon}$.and $\widehat{g}^{\varepsilon}_i\in L^2(\widehat{\Gamma}^{\varepsilon}_+)$ is the density of applied surface force on $\widehat{\Gamma}^{\varepsilon}_+$.

$$\begin{cases} -\widehat{div}^{\varepsilon}\widehat{\sigma}=\widehat{f}^{\varepsilon}(\text{i.e..}-\widehat{\partial}^{\varepsilon}_j\widehat{\sigma}^{\varepsilon}_{ij}=\widehat{f}^{\varepsilon}_i), & \text{on} \ \ \widehat{\Omega}^{\varepsilon}, \quad (1,a)\\ \widehat{\sigma}^{\varepsilon}\widehat{n}^{\varepsilon}=\widehat{g}^{\varepsilon}(i.e..\widehat{\sigma}^{\varepsilon}_{ij}\widehat{n}^{\varepsilon}_j=\widehat{g}^{\varepsilon}_i), & \text{on} \ \ \widehat{\Gamma}^{\varepsilon}_+, \quad (1,b)\\ \widehat{u}^{\varepsilon}=0, & \text{on} \ \ \widehat{\Gamma}^{\varepsilon}_0, \quad (1,c)\end{cases} \tag{1.1}$$

Maxwell-Gauss equations and electric boundary conditions

We assume $\widehat{\Omega}^{\varepsilon}$ is subject to an electric volume charge of density $(\widehat{r}^{\varepsilon})\in L^2(\widehat{\Omega}^{\varepsilon})$, and suppose given an electric surface charge of density $\widehat{\phi}^{\varepsilon}\in L^2(\widehat{\Gamma}^{\varepsilon}_+\cup\widehat{\Gamma}^{\varepsilon}_-)$.

$$\begin{cases} div\widehat{D}^{\varepsilon}=\widehat{r}^{\varepsilon}(\text{i.e..}-\partial_j\widehat{D}^{\varepsilon}_{ij}=\widehat{r}^{\varepsilon}_i), & on \ \ \widehat{\Omega}^{\varepsilon},\\ \widehat{D}^{\varepsilon}\widehat{n}^{\varepsilon}=\widehat{\phi}^{\varepsilon}(i.e..\widehat{D}^{\varepsilon}_{ij}\widehat{n}^{\varepsilon}_j=\widehat{\phi}^{\varepsilon}_i), & on \ \ \widehat{\Gamma}^{\varepsilon}_+\cup\widehat{\Gamma}^{\varepsilon}_-,\\ \widehat{\varphi}^{\varepsilon}=0, & on \ \ \widehat{\Gamma}^{\varepsilon}_0.\end{cases} \tag{1.2}$$

Constitutive equations

the linear strain tensor $\widehat{e}^{\varepsilon}_{ij}(\widehat{v}^{\varepsilon})=\frac{1}{2}(\widehat{\partial}^{\varepsilon}_i\widehat{v}^{\varepsilon}_j+\widehat{\partial}^{\varepsilon}_j\widehat{v}^{\varepsilon}_i)$,
the electric field vector $\widehat{E}^{\varepsilon}(\widehat{\varphi}^{\varepsilon})=-\nabla\widehat{\varphi}^{\varepsilon})$, $\widehat{E}^{\varepsilon}_l(\widehat{\varphi}^{\varepsilon})=-\widehat{\partial}^{\varepsilon}_l\widehat{\varphi}^{\varepsilon}$,

$$\begin{cases} \widehat{\sigma}^{\varepsilon}_{ij}=\widehat{C}^{\varepsilon}_{ijkl}\widehat{e}^{\varepsilon}_{kl}(\widehat{u}^{\varepsilon})-\widehat{P}^{\varepsilon}_{kij}\widehat{E}^{\varepsilon}_k(\widehat{\varphi}^{\varepsilon}), & \text{in} \ \ \widehat{\Omega}^{\varepsilon},\\ \widehat{D}^{\varepsilon}_k=\widehat{P}^{\varepsilon}_{kij}\widehat{e}^{\varepsilon}_{ij}(\widehat{u}^{\varepsilon})+\widehat{d}^{\varepsilon}_{kl}\widehat{E}^{\varepsilon}_l(\widehat{\varphi}^{\varepsilon}), & \text{in} \ \ \widehat{\Omega}^{\varepsilon}.\end{cases} \tag{1.3}$$

Signorini's contact conditions

We denote by $\widehat{v}_n=\widehat{v}\cdot\widehat{n}$ is the normal components, and $\widehat{v}_t=\widehat{v}-\widehat{v}_n n$ the tangential components of v.

and

$$\widehat{\sigma}_{ij}^{\varepsilon}\widehat{n}^{\varepsilon} = \widehat{\sigma}_t^{\varepsilon} + \widehat{\sigma}_n^{\varepsilon}.$$

If $(\widehat{u}_n^{\varepsilon} < \widehat{s}^{\varepsilon})$, $\widehat{\Omega}^{\varepsilon}$ is not in contact with the rigid foundatinon $(\widehat{\sigma}^{\varepsilon} = 0)$.In order $\widehat{u}_n^{\varepsilon} = \widehat{s}^{\varepsilon}$, $\widehat{\Omega}^{\varepsilon}$ is in contact with a rigid foundatino.

$$\widehat{u}_n^{\varepsilon} \leq \widehat{s}^{\varepsilon} \quad , \quad \widehat{\sigma}_n^{\varepsilon} \leq 0 \quad , \quad \widehat{\sigma}_n^{\varepsilon}(\widehat{u}_n^{\varepsilon} - \widehat{s}^{\varepsilon}) = 0 \quad \text{on} \quad \widehat{\Gamma}_-^{\varepsilon}. \tag{1.4}$$

Tresca's law of friction

$\widehat{q}^{\varepsilon}$ represents a friction function $(\widehat{q}^{\varepsilon} \geq 0)$.

$$\begin{cases} |\widehat{\sigma}_t^{\varepsilon}| \leq \widehat{q}^{\varepsilon}, & \text{on} \quad \widehat{\Gamma}_-^{\varepsilon}, \\ |\widehat{\sigma}_t^{\varepsilon}| < \widehat{q}^{\varepsilon} \Longrightarrow \widehat{u}_t = 0, & \text{on} \quad \widehat{\Gamma}_-^{\varepsilon}, \\ |\widehat{\sigma}_t^{\varepsilon}| = \widehat{q}^{\varepsilon} \Longrightarrow \exists c \geq 0 : \widehat{u}_t = -c\widehat{\sigma}_t, & \text{on} \quad \widehat{\Gamma}_-^{\varepsilon}. \end{cases} \tag{1.5}$$

1.3 Variationl formulation

We define the spaces of admissible mechannical displacementes :

$$\widehat{V}^{\varepsilon}(\widehat{\Omega}^{\varepsilon}) = \{\widehat{v}^{\varepsilon} \in H^1(\widehat{\Omega}^{\varepsilon}), \widehat{v}^{\varepsilon} = 0 \quad \text{on} \quad \widehat{\Gamma}_0^{\varepsilon}\},$$

The closed and convex,

$$\widehat{K}^{\varepsilon}(\widehat{\Omega}^{\varepsilon}) = \{\widehat{v}^{\varepsilon} \in \widehat{V}^{\varepsilon}(\widehat{\Omega}^{\varepsilon}), \widehat{v}_n^{\varepsilon} \leq \widehat{s}^{\varepsilon} \quad \text{on} \widehat{\Gamma}_-^{\varepsilon}\},$$

and the space of adnissible electric potentials :

$$\widehat{\Psi}^{\varepsilon}(\widehat{\Omega}^{\varepsilon}) = \{\widehat{\psi}^{\varepsilon} \in H^1(\widehat{\Omega}^{\varepsilon}), \widehat{\psi}^{\varepsilon} = 0 \quad \text{on} \quad \widehat{\Gamma}_0^{\varepsilon}\},$$

using the Green formula in $(1, a)$ of the equation(1.1),

$$\int_{\widehat{\Omega}^{\varepsilon}} \widehat{\sigma}_{ij}^{\varepsilon}\widehat{e}_{ij}^{\varepsilon}(\widehat{v}^{\varepsilon} - \widehat{u}^{\varepsilon})d\widehat{x}^{\varepsilon} - \int_{\widehat{\partial^{\varepsilon}}\widehat{\Omega}^{\varepsilon}} \widehat{\sigma}_{ij}^{\varepsilon}\widehat{n}^{\varepsilon}(\widehat{v}^{\varepsilon} - \widehat{u}^{\varepsilon})d\widehat{\Gamma}^{\varepsilon} = \int_{\widehat{\Omega}^{\varepsilon}} \widehat{f}_i^{\varepsilon}(\widehat{v}^{\varepsilon} - \widehat{u}^{\varepsilon})d\widehat{x}^{\varepsilon}, \qquad \forall(\widehat{v}^{\varepsilon} - \widehat{u}^{\varepsilon}) \in \widehat{V}^{\varepsilon}(\widehat{\Omega}^{\varepsilon}),$$

from the equation (1.1) and $\partial\widehat{\Omega}^{\varepsilon} = \widehat{\Gamma}_0^{\varepsilon} \cup \widehat{\Gamma}_+^{\varepsilon} \cup \widehat{\Gamma}_-$:

$$\int_{\widehat{\Gamma}_+^{\varepsilon}} \widehat{\sigma}_{ij}^{\varepsilon}\widehat{n}_j^{\varepsilon}(\widehat{v}^{\varepsilon} - \widehat{u}^{\varepsilon})d\widehat{\Gamma}^{\varepsilon} = \int_{\widehat{\Gamma}_+^{\varepsilon}} \widehat{g}_i^{\varepsilon}(\widehat{v}^{\varepsilon} - \widehat{u}^{\varepsilon})d\widehat{\Gamma}^{\varepsilon} \qquad \text{on} \quad \widehat{\Gamma}_+^{\varepsilon},$$

$$\int_{\widehat{\Gamma}_0^\varepsilon} \widehat{\sigma}_{ij}^\varepsilon \widehat{n}_j^\varepsilon (\widehat{v}^\varepsilon - \widehat{u}^\varepsilon) d\widehat{\Gamma}^\varepsilon = 0 \qquad \text{on } \widehat{\Gamma}_0^\varepsilon,$$

$\widehat{v}_n^\varepsilon \leq \widehat{s}^\varepsilon,$

$$\begin{aligned} \widehat{\sigma}_{ij}^\varepsilon \widehat{n}_i^\varepsilon (\widehat{v}_i^\varepsilon - \widehat{u}_i^\varepsilon) &= \widehat{\sigma}_t^\varepsilon (\widehat{v}_t^\varepsilon - \widehat{u}_t^\varepsilon) + \widehat{\sigma}_n^\varepsilon (\widehat{v}_n^\varepsilon - \widehat{u}_n^\varepsilon) \\ &= \widehat{\sigma}_t^\varepsilon (\widehat{v}_t^\varepsilon - \widehat{u}_t^\varepsilon) + \widehat{\sigma}_n^\varepsilon (\widehat{v}_n^\varepsilon - \widehat{s}^\varepsilon) - \overbrace{\widehat{\sigma}_n^\varepsilon (\widehat{u}_n^\varepsilon - \widehat{s}^\varepsilon)}^{0}, \end{aligned}$$

$\widehat{\sigma}_n^\varepsilon (\widehat{v}_n^\varepsilon - \widehat{s}^\varepsilon) \geq 0,$

$$\int_{\widehat{\Omega}^\varepsilon} \widehat{\sigma}_{ij}^\varepsilon \widehat{e}_{ij}^\varepsilon (\widehat{v}^\varepsilon - \widehat{u}^\varepsilon) d\widehat{x}^\varepsilon - \int_{\widehat{\Omega}^\varepsilon} \widehat{f}_i^\varepsilon (\widehat{v}_i^\varepsilon - \widehat{u}_i^\varepsilon) d\widehat{x}^\varepsilon - \int_{\widehat{\Gamma}_+^\varepsilon} \widehat{g}_i^\varepsilon (\widehat{v}_i^\varepsilon - \widehat{u}_i^\varepsilon) d\widehat{\Gamma}^\varepsilon - \int_{\widehat{\Gamma}_-^\varepsilon} (\widehat{\sigma}_t^\varepsilon (\widehat{v}_t^\varepsilon - \widehat{u}_t^\varepsilon) - \widehat{\sigma}_n^\varepsilon (\widehat{v}_n^\varepsilon - \widehat{s}^\varepsilon)) d\widehat{\Gamma}^\varepsilon = 0,$$

$$\int_{\widehat{\Omega}^\varepsilon} \widehat{\sigma}_{ij}^\varepsilon \widehat{e}_{ij}^\varepsilon (\widehat{v}^\varepsilon - \widehat{u}^\varepsilon) d\widehat{x}^\varepsilon - \int_{\widehat{\Omega}^\varepsilon} \widehat{f}_i^\varepsilon (\widehat{v}_i^\varepsilon - \widehat{u}_i^\varepsilon) d\widehat{x}^\varepsilon - \int_{\widehat{\Gamma}_+^\varepsilon} \widehat{g}_i^\varepsilon (\widehat{v}_i^\varepsilon - \widehat{u}_i^\varepsilon) d\widehat{\Gamma}^\varepsilon \geq \int_{\widehat{\Gamma}_-^\varepsilon} \widehat{\sigma}_t^\varepsilon (\widehat{v}_t^\varepsilon - \widehat{u}_t^\varepsilon), \quad (1.6)$$

an contact conditions and friction that,

$$\begin{aligned} -\int_{\widehat{\Gamma}_-^\varepsilon} \widehat{\sigma}_t \widehat{v}_t^\varepsilon d\Gamma &\leqslant \int_{\widehat{\Gamma}_-^\varepsilon} |\widehat{\sigma}_t^\varepsilon| |\widehat{v}_t^\varepsilon| d\widehat{\Gamma}^\varepsilon \\ &\leqslant \int_{\widehat{\Gamma}_-^\varepsilon} \widehat{q}^\varepsilon |\widehat{v}_t^\varepsilon| d\widehat{\Gamma}^\varepsilon, \end{aligned}$$

$$\int_{\widehat{\Gamma}_-^\varepsilon} (\widehat{\sigma}_t \widehat{v}_t^\varepsilon + \widehat{q}^\varepsilon |\widehat{v}_t^\varepsilon|) d\widehat{\Gamma}^\varepsilon \geqslant 0,$$

$$\int_{\widehat{\Omega}^\varepsilon} \widehat{\sigma}_t^\varepsilon \widehat{u}_t^\varepsilon d\widehat{\Gamma}^\varepsilon = -\int_{\widehat{\Omega}^\varepsilon} c \widehat{\sigma}_t^{\varepsilon 2} d\widehat{\Gamma}^\varepsilon = -\int_{\widehat{\Omega}^\varepsilon} |\widehat{\sigma}_t^\varepsilon| |\widehat{u}_t^\varepsilon| d\widehat{\Gamma}^\varepsilon = -\int_{\widehat{\Omega}^\varepsilon} \widehat{q}^\varepsilon |\widehat{u}_t^\varepsilon| \widehat{\Gamma}^\varepsilon,$$

$$\int_{\widehat{\Omega}^\varepsilon} \widehat{\sigma}_t^\varepsilon \widehat{u}_t^\varepsilon d\widehat{\Gamma}^\varepsilon + \int_{\widehat{\Omega}^\varepsilon} \widehat{q}^\varepsilon |\widehat{u}_t^\varepsilon| d\widehat{\Gamma}^\varepsilon = 0 \Rightarrow \int_{\widehat{\Omega}^\varepsilon} (\widehat{\sigma}_t^\varepsilon \widehat{u}_t^\varepsilon + \widehat{q}^\varepsilon |\widehat{u}_t^\varepsilon|) d\widehat{\Gamma}^\varepsilon = 0,$$

we find,

$$\int_{\widehat{\Omega}^\varepsilon} (\widehat{\sigma}_t^\varepsilon (\widehat{v}_t^\varepsilon - \widehat{u}_t^\varepsilon) + \widehat{q}^\varepsilon |\widehat{v}_t^\varepsilon| - \widehat{q}^\varepsilon |\widehat{u}_t^\varepsilon|) \geq 0.$$

Adding $\widehat{j}^\varepsilon(\widehat{v}^\varepsilon) - \widehat{j}^\varepsilon(\widehat{u}^\varepsilon)$ to both sides of the equation (1.6) :

$$\int_{\widehat{\Omega}^\varepsilon} \widehat{\sigma}^\varepsilon_{ij}\widehat{e}^\varepsilon_{ij}(\widehat{v}^\varepsilon - \widehat{u}^\varepsilon)d\widehat{x}^\varepsilon + \widehat{j}^\varepsilon(\widehat{v}^\varepsilon) - \widehat{j}^\varepsilon(\widehat{u}^\varepsilon) - \int_{\widehat{\Omega}^\varepsilon} \widehat{f}^\varepsilon_i(\widehat{v}^\varepsilon_i - \widehat{u}^\varepsilon_i)d\widehat{x}^\varepsilon - \int_{\widehat{\Gamma}^\varepsilon_+} \widehat{g}^\varepsilon_i(\widehat{v}^\varepsilon_i - \widehat{u}^\varepsilon_i)d\widehat{\Gamma}^\varepsilon \geq 0. \quad (1.7)$$

where,

$$\widehat{j}^\varepsilon(\widehat{v}^\varepsilon) = \int_{\widehat{\Gamma}^\varepsilon_-} \widehat{q}^\varepsilon|\widehat{v}^\varepsilon_t|d\widehat{\Gamma}^\varepsilon,$$

in forme the equation(1.2), we find,

$$\int_{\widehat{\Omega}^\varepsilon} \widehat{div}^\varepsilon \widehat{D}^\varepsilon\widehat{\psi}^\varepsilon d\widehat{x}^\varepsilon = \int_{\widehat{\Omega}^\varepsilon} \widehat{r}^\varepsilon\widehat{\psi}^\varepsilon d\widehat{x}^\varepsilon, \quad (1.8)$$

using the Green formula in (1.8),

$$\int_{\widehat{\Gamma}^\varepsilon_- \cup \widehat{\Gamma}^\varepsilon_+} \widehat{\phi}^\varepsilon\widehat{\psi}^\varepsilon d\widehat{\Gamma}^\varepsilon - \int_{\widehat{\Omega}^\varepsilon} \widehat{D}^\varepsilon\widehat{\partial}^\varepsilon_i\widehat{\psi}^\varepsilon d\widehat{x}^\varepsilon = \int_{\widehat{\Omega}^\varepsilon} \widehat{r}^\varepsilon\widehat{\psi}^\varepsilon d\widehat{x}^\varepsilon. \quad (1.9)$$

We collect equations (1.7)(1.9) to obtain the following form variational inequality:

$$\begin{cases} \text{Find } (\widehat{u}^\varepsilon, \widehat{\varphi}^\varepsilon) \in \widehat{K}^\varepsilon \times \widehat{\Psi}^\varepsilon \text{ such that :} \\ \widehat{b}^\varepsilon((\widehat{u}^\varepsilon, \widehat{\varphi}^\varepsilon), (\widehat{v}^\varepsilon - \widehat{u}^\varepsilon, \widehat{\psi}^\varepsilon)) + \widehat{j}^\varepsilon(\widehat{v}^\varepsilon) - \widehat{j}^\varepsilon(\widehat{u}^\varepsilon) \geq \widehat{l}^\varepsilon((\widehat{v}^\varepsilon - \widehat{u}^\varepsilon), \widehat{\psi}^\varepsilon), \\ \quad \forall\ (\widehat{v}^\varepsilon, \widehat{\psi}^\varepsilon) \in \widehat{K}^\varepsilon \times \widehat{\Psi}^\varepsilon. \end{cases} \quad (1.10)$$

Where,

$$\begin{aligned}
\widehat{b}^\varepsilon((\widehat{u}^\varepsilon, \widehat{\varphi}^\varepsilon), (\widehat{v}^\varepsilon, \widehat{\psi}^\varepsilon)) &= \int_{\widehat{\Omega}^\varepsilon} \widehat{\sigma}^\varepsilon_{ij}\widehat{e}^\varepsilon_{ij}(\widehat{v}^\varepsilon)d\widehat{x}^\varepsilon - \int_{\widehat{\Omega}^\varepsilon} \widehat{D}^\varepsilon\widehat{\partial}^\varepsilon_i\widehat{\psi}^\varepsilon d\widehat{x}^\varepsilon \\
&= \int_{\widehat{\Omega}^\varepsilon} [\widehat{C}^\varepsilon_{ijkl}\widehat{e}^\varepsilon_{kl}(\widehat{u}^\varepsilon) - \widehat{P}^\varepsilon_{kij}\widehat{E}^\varepsilon_k(\widehat{\varphi}^\varepsilon)]\widehat{e}^\varepsilon_{ij}(\widehat{v}^\varepsilon)d\widehat{x}^\varepsilon \\
&\quad - \int_{\widehat{\Omega}^\varepsilon} [\widehat{P}^\varepsilon_{kij}\widehat{e}^\varepsilon_{ij}(\widehat{u}^\varepsilon) + \widehat{d}^\varepsilon_{kl}\widehat{E}^\varepsilon_l(\widehat{\varphi}^\varepsilon)]\widehat{\partial}^\varepsilon_i\widehat{\psi}^\varepsilon d\widehat{x}^\varepsilon \\
&= \int_{\widehat{\Omega}^\varepsilon} \widehat{C}^\varepsilon_{ijkl}\widehat{e}^\varepsilon_{ij}(\widehat{v}^\varepsilon)\widehat{e}^\varepsilon_{kl}(\widehat{u}^\varepsilon)d\widehat{x}^\varepsilon - \int_{\widehat{\Omega}^\varepsilon} \widehat{d}^\varepsilon_{kl}\widehat{E}^\varepsilon_l(\widehat{\varphi}^\varepsilon)\widehat{\partial}^\varepsilon_i\widehat{\psi}^\varepsilon d\widehat{x}^\varepsilon \\
&\quad - \int_{\widehat{\Omega}^\varepsilon} \widehat{P}^\varepsilon_{ijk}[\widehat{E}^\varepsilon_k(\widehat{\varphi}^\varepsilon)\widehat{e}^\varepsilon_{ij}(\widehat{v}^\varepsilon) + \widehat{e}^\varepsilon_{ij}(\widehat{u}^\varepsilon)\widehat{\partial}^\varepsilon_i\widehat{\psi}^\varepsilon]d\widehat{x}^\varepsilon \\
&= \int_{\widehat{\Omega}^\varepsilon} \widehat{C}^\varepsilon_{ijkl}\widehat{e}^\varepsilon_{ij}(\widehat{v}^\varepsilon)\widehat{e}^\varepsilon_{kl}(\widehat{u}^\varepsilon)d\widehat{x}^\varepsilon + \int_{\widehat{\Omega}^\varepsilon} \widehat{d}^\varepsilon_{kl}\widehat{\partial}^\varepsilon_l(\widehat{\varphi}^\varepsilon)\widehat{\partial}^\varepsilon_i\widehat{\psi}^\varepsilon d\widehat{x}^\varepsilon \\
&\quad + \int_{\widehat{\Omega}^\varepsilon} \widehat{P}^\varepsilon_{ijk}\widehat{\partial}^\varepsilon_k(\widehat{\varphi}^\varepsilon)\widehat{e}^\varepsilon_{ij}(\widehat{v}^\varepsilon)d\widehat{x}^\varepsilon - \int_{\widehat{\Omega}^\varepsilon} \widehat{P}^\varepsilon_{ijk}\widehat{e}^\varepsilon_{ij}(\widehat{u}^\varepsilon)\widehat{\partial}^\varepsilon_i\widehat{\psi}^\varepsilon d\widehat{x}^\varepsilon
\end{aligned}$$

and

$$\begin{aligned}\widehat{l}^{\varepsilon}((\widehat{v}^{\varepsilon}),\widehat{\psi}^{\varepsilon}) &= \int_{\widehat{\Omega}^{\varepsilon}} \widehat{r}\widehat{\psi}^{\varepsilon} d\widehat{x}^{\varepsilon} - \int_{\widehat{\Gamma}^{\varepsilon}_{-}\cup\widehat{\Gamma}^{\varepsilon}_{+}} \widehat{\phi}^{\varepsilon}\widehat{\psi}^{\varepsilon} d\widehat{\Gamma}^{\varepsilon} + \int_{\widehat{\Omega}^{\varepsilon}} \widehat{f}^{\varepsilon}_{i}(\widehat{v}^{\varepsilon}_{i}) d\widehat{x}^{\varepsilon} \\ &+ \int_{\widehat{\Gamma}^{\varepsilon}_{+}} \widehat{g}^{\varepsilon}_{i}(\widehat{v}^{\varepsilon}_{i}) d\widehat{\Gamma}^{\varepsilon}.\end{aligned}$$

Theorem There exists a unique solution $(\widehat{u}^{\varepsilon},\widehat{v}^{\varepsilon})$ of (1.10).

Proof. The bilinear form $\widehat{b}^{\varepsilon}((.,.),(.,.))$ is $\widehat{V}^{\varepsilon}\times\widehat{\Psi}^{\varepsilon}$-coercive and linear form $\widehat{l}^{\varepsilon}(.,.)$ are continuous $\widehat{V}^{\varepsilon}\times\widehat{\Psi}^{\varepsilon}$ .The set $\widehat{K}^{\varepsilon}(\widehat{\Omega}^{\varepsilon})$ is a nonempty, closed and convex subset of $\widehat{V}^{\varepsilon}(\widehat{\Omega}^{\varepsilon})$ and the functional $\widehat{j}^{\varepsilon}(.)$ is proper, convex and continuous on $\widehat{V}^{\varepsilon}(\widehat{\Omega}^{\varepsilon})$,There exists a unique solution $(\widehat{u}^{\varepsilon},\widehat{v}^{\varepsilon})$ of (1.10). for more details[1].

■

Chapter 2

Asymptotic analysis

2.1 Transformation into a problem posede over a domain Ω^ε

Since the mappings $\Theta^\varepsilon : \overline{\Omega}^\varepsilon \longrightarrow \Theta^\varepsilon(\widehat{\Omega}^\varepsilon)$ is a C^1-diffeomorfhismine then the field,

$$\widehat{u}^\varepsilon(\widehat{x}^\varepsilon) = u^\varepsilon(x^\varepsilon) \quad , \quad \forall \widehat{x}^\varepsilon = \Theta^\varepsilon(x^\varepsilon) \in \overline{\{\widehat{\Omega}^\varepsilon\}}.$$

induces a bijection between the spaces $H^1(\widehat{\Omega}^\varepsilon)$ and $H^1(\Omega^\varepsilon)$, consequence a bijection between the spaces $\widehat{V}^\varepsilon(\widehat{\Omega}^\varepsilon)$ and $V^\varepsilon(\Omega^\varepsilon)$ $\forall x^\varepsilon \in \Omega^\varepsilon$.

Let $\nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon)$ denote the Jacobian matrix $(\partial_j^\varepsilon \Theta_j^\varepsilon(x^\varepsilon))$, see [4].

$$b_{ij}^\varepsilon(x^\varepsilon) = (\{\nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon)\}^{-1})_{ij}, \qquad \forall\ x^\varepsilon \in \Omega^\varepsilon,$$

$$\delta^\varepsilon(x^\varepsilon) = det\{\nabla^\varepsilon \Theta^\varepsilon(x^\varepsilon)\}, \qquad \forall\ x^\varepsilon \in \Omega^\varepsilon,$$

$\forall \varepsilon > 0, \quad \theta^\varepsilon = \varepsilon\theta.$

then using the formulas,

$\widehat{\partial}_j^\varepsilon \widehat{v}_i^\varepsilon(\widehat{x}^\varepsilon) = b_{kj}^\varepsilon(x^\varepsilon)\partial_k^\varepsilon v_k^\varepsilon(x^\varepsilon).$

We obtain the expression for the linearized strain tensor,

$\widehat{e}_{ij}^\varepsilon(\widehat{v}^\varepsilon) = e_{ij}^\varepsilon(v^\varepsilon) = \frac{1}{2}(b_{ki}^\varepsilon \partial_k^\varepsilon(v_j^\varepsilon) + b_{lj}^\varepsilon \partial_l^\varepsilon(v_i^\varepsilon)),$

$$d\widehat{\Gamma}^\varepsilon = \delta^\varepsilon\{b_{3i}^\varepsilon b_{3i}^\varepsilon\}^{1/2} d\Gamma^\varepsilon,$$

$$d\widehat{x}^\varepsilon = \delta^\varepsilon dx^\varepsilon,$$

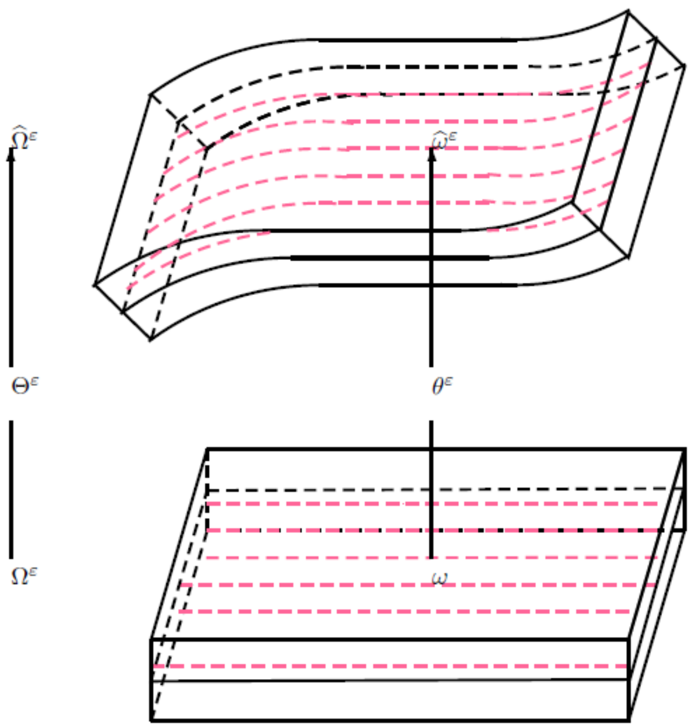

Figure 2.1:

The functions $\widehat{b}^\varepsilon_{ij}$ and $\widehat{\delta}^\varepsilon$:

$$\widehat{b}^\varepsilon_{ij} : \widehat{\Omega}^\varepsilon \longrightarrow \mathbb{R} \quad , \quad \widehat{\delta}^\varepsilon : \widehat{\Omega}^\varepsilon \longrightarrow \mathbb{R}.$$

$$b^\varepsilon_{\alpha\beta} = \delta^\varepsilon_{\alpha\beta} + \varepsilon^2 b^\sharp_{\alpha\beta},$$

$$b^\varepsilon_{\alpha 3} = \varepsilon\{\partial^\varepsilon_\alpha \theta^\varepsilon + \varepsilon^2 b^\sharp_{\alpha 3}\},$$

$$b^\varepsilon_{3\beta} = -\varepsilon\{\partial^\varepsilon_\beta \theta^\varepsilon + \varepsilon^2 b^\sharp_{3\beta}\},$$

$$b^\varepsilon_{33} = 1 + \varepsilon^2 b^\sharp_{33},$$

$$\delta^\varepsilon = 1 + \varepsilon^2 \delta^\sharp.$$

And there exists a constant C_0 shuch that :

$$\sup_{0<\varepsilon\leqslant\varepsilon_0} \max_{ij} \max_{x\in\Omega} |b^{\#}_{ij}(\varepsilon,\theta)(x)| \leqslant C_0(\theta),$$

$$\sup_{0<\varepsilon\leqslant\varepsilon_0}\max_{x\in\Omega}|\delta^{\#}_{ij}(\varepsilon,\theta)(x)|\leqslant C_0(\theta).$$

Proof. See [4, p 336]. ■

Here, the applied volume and surface forces f^ε and g^ε are defined by $g^\varepsilon = \widehat{g}^\varepsilon o\Theta^\varepsilon$ and $f^\varepsilon = \widehat{f}^\varepsilon o\Theta^\varepsilon$, the electric volume charge and surface charge are defined by $r^\varepsilon = \widehat{r}^\varepsilon o\Theta^\varepsilon$ and $\phi^\varepsilon = \widehat{\phi}^\varepsilon o\Theta^\varepsilon$.

Replacing $(\widehat{u}^\varepsilon, \widehat{\varphi}^\varepsilon)$ by $(u^\varepsilon, \varphi^\varepsilon)$ end compensation, the variationl equation a domain Ω^ε the following form variational inequality:

$$\left\{\begin{array}{l}\text{Find } \ (u^\varepsilon,\varphi^\varepsilon)\in K^\varepsilon\times\Psi^\varepsilon \ \text{ such that :}\\ b^\varepsilon((u^\varepsilon,\varphi^\varepsilon),(v^\varepsilon-u^\varepsilon,\psi^\varepsilon))+j^\varepsilon(v^\varepsilon)-j^\varepsilon(u^\varepsilon)\geq l^\varepsilon((v^\varepsilon-u^\varepsilon),\psi^\varepsilon),\\ \qquad\forall\ (v^\varepsilon,\psi^\varepsilon)\in K^\varepsilon\times\Psi^\varepsilon.\end{array}\right. \tag{2.1}$$

$$\begin{aligned} j^\varepsilon(v^\varepsilon) &= \int_{\Gamma^\varepsilon_-} q^\varepsilon|v^\varepsilon_t|(1+\varepsilon^2\delta^\sharp)\{b^\varepsilon_{3i}b^\varepsilon_{3i}\}^{1/2}d\Gamma^\varepsilon\\ &= \int_{\Gamma^\varepsilon_-} q^\varepsilon|v^\varepsilon_t|(1+\varepsilon^2\delta^\sharp)\{\varepsilon\{\partial^\varepsilon_\beta\theta^\varepsilon+\varepsilon^2 b^\sharp_{3\beta}\}\varepsilon\{\partial^\varepsilon_\beta\theta^\varepsilon+\varepsilon^2 b^\sharp_{3\beta}\}\}^{1/2}d\Gamma^\varepsilon\\ &= \int_{\Gamma^\varepsilon_-} \varepsilon q^\varepsilon|v^\varepsilon_t|(1+\varepsilon^2\delta^\sharp)\{\partial^\varepsilon_\beta\theta^\varepsilon+\varepsilon^2 b^\sharp_{3\beta}\}d\Gamma^\varepsilon\end{aligned}$$

$$\begin{aligned} b^\varepsilon((u^\varepsilon,\varphi^\varepsilon),(v^\varepsilon,\psi^\varepsilon)) \ &= \ \int_{\Omega^\varepsilon} C^\varepsilon_{ijkl}e^\varepsilon_{ij}(v^\varepsilon)e^\varepsilon_{kl}(u^\varepsilon)(1+\varepsilon^2\delta^\sharp)dx^\varepsilon+\int_{\Omega^\varepsilon} d^\varepsilon_{kl}\partial^\varepsilon_l(\varphi^\varepsilon)\partial^\varepsilon_i\psi^\varepsilon(1+\varepsilon^2\delta^\sharp)dx^\varepsilon\\ &+ \ \int_{\Omega^\varepsilon} P^\varepsilon_{ijk}\partial^\varepsilon_k(\varphi^\varepsilon)e^\varepsilon_{ij}(v^\varepsilon)(1+\varepsilon^2\delta^\sharp)dx^\varepsilon-\int_{\Omega^\varepsilon} P^\varepsilon_{ijk}e^\varepsilon_{ij}(u^\varepsilon)\partial^\varepsilon_i\psi^\varepsilon(1+\varepsilon^2\delta^\sharp)dx^\varepsilon\end{aligned}$$

$$\begin{aligned} l^\varepsilon(v^\varepsilon,\psi^\varepsilon) \ &= \ \int_{\Omega^\varepsilon} r^\varepsilon\psi^\varepsilon(1+\varepsilon^2\delta^\sharp)dx^\varepsilon-\int_{\Gamma^\varepsilon_-\cup\Gamma^\varepsilon_+}\phi^\varepsilon\psi^\varepsilon(1+\varepsilon^2\delta^\sharp)\{\partial^\varepsilon_\beta\theta^\varepsilon+\varepsilon^2 b^\sharp_{3\beta}\}d\Gamma^\varepsilon\\ &+ \ \int_{\Omega^\varepsilon} f^\varepsilon_i v^\varepsilon_i(1+\varepsilon^2\delta^\sharp)dx^\varepsilon+\int_{\Gamma^\varepsilon_+} g^\varepsilon_i v^\varepsilon_i(1+\varepsilon^2\delta^\sharp)\{\partial^\varepsilon_\beta\theta^\varepsilon+\varepsilon^2 b^\sharp_{3\beta}\}d\Gamma^\varepsilon\end{aligned}$$

Keep space experimental functions,

$$V^\varepsilon(\Omega^\varepsilon)=\{v^\varepsilon\in H^1(\Omega^\varepsilon), v^\varepsilon=0 \ \text{ on } \ \Gamma^\varepsilon_0\},$$

$$\Psi^\varepsilon(\Omega^\varepsilon)=\{\psi^\varepsilon\in H^1(\Omega^\varepsilon), \psi^\varepsilon=0 \ \text{ on } \ \Gamma^\varepsilon_0\},$$

and convex,

$K^\varepsilon(\Omega^\varepsilon)=\{v^\varepsilon\in V^\varepsilon(\Omega^\varepsilon), v^\varepsilon_n\leq s^\varepsilon \quad \text{on } \ \Gamma^\varepsilon_0\}$.

Remark 2.1 *For any given ε we then have a pair which consists of a shell of thickness 2ε and a plane horizontal obstacle at the level $-\varepsilon$, see[2, p 245].*

We know so

$K^\varepsilon(\Omega^\varepsilon) = \{v^\varepsilon \in V^\varepsilon(\Omega^\varepsilon), v_n^\varepsilon \leq -\theta^\varepsilon - \varepsilon + \frac{\varepsilon}{\sqrt{\alpha}} \quad \text{on} \;\; \Gamma_0^\varepsilon\}$.

2.2 Scaling and equilibrium equations in the fixed domain

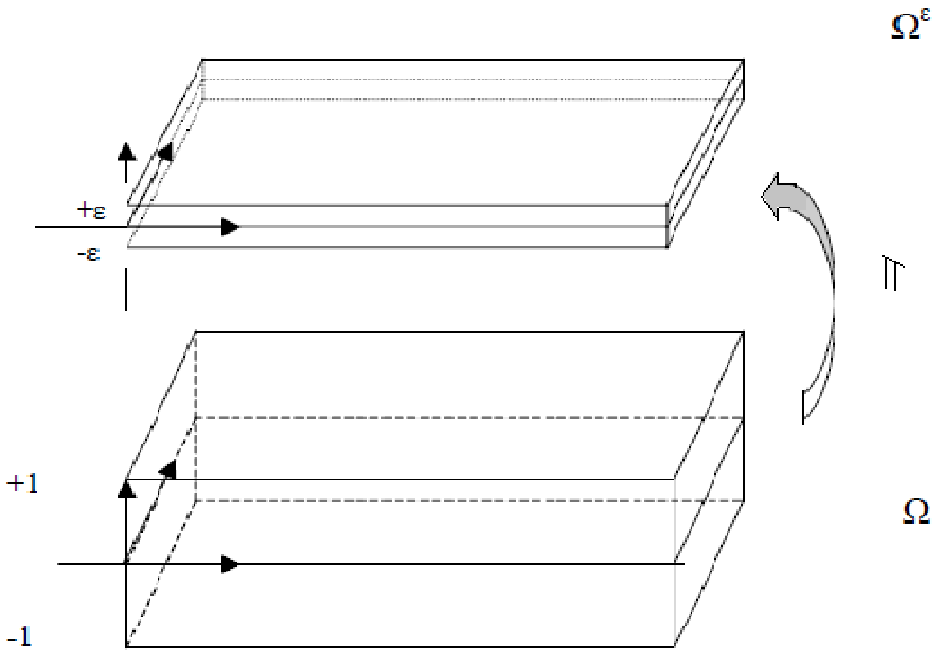

Figure 2.2:

For convenience, we consider a reference domain independent of the small parameter ε. Hence, let us define the three-dimensional domain $\Gamma = \omega \times (-1, 1)$, and its boundary $\Gamma = \partial\Omega$.

We also define the following parts of the boundary,

$$\Omega = \omega \times]-1, 1[\;\;,\;\; \Gamma_+ = \omega \times \{1\} \;\;,\;\; \Gamma_- = \omega \times \{-1\} \;\;,\;\; \Gamma_0 = \gamma_0 \times [-1, 1].$$

Let $x = (x_1, x_2, x_3)$ be a generic point in and we consider the notation ∂_i for the partial derivative with respect to x_i.
We define the bijektion map $\pi : \Omega^\varepsilon \longmapsto \Omega$, such that:

$$\begin{cases} \pi : (x_1, x_2, x_3) \in \Omega^\varepsilon \longrightarrow (x_1, x_2, x_3^\varepsilon) \in \Omega, \\ x_3^\varepsilon = \varepsilon x_3. \end{cases}$$

This induces $\partial_\alpha^\varepsilon = \frac{\partial}{\partial x_\alpha^\varepsilon} = \frac{\partial}{\partial x_\alpha}$ and $\partial_3^\varepsilon = \frac{\partial}{\partial x_3^\varepsilon} = \frac{1}{\varepsilon}\frac{\partial}{\partial x_3}$.

Scalings of the unknowns and test functions

We introduce the scaled displacement $u(\varepsilon)$ and scaled test functions v defined as

$$\begin{cases} u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 u_\alpha(\varepsilon)(x), & v_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^2 v_\alpha(x). \\ u_3^\varepsilon(x^\varepsilon) = \varepsilon u_3(\varepsilon)(x), & v_3^\varepsilon(x^\varepsilon) = \varepsilon v_3(x). \\ \varphi^\varepsilon = \varepsilon^2 \varphi(\varepsilon), & \psi^\varepsilon = \varepsilon^2 \psi. \end{cases}$$

We add the following notations,

$$\begin{cases} b_{ij}^\varepsilon(x^\varepsilon) = b_{ij}(\varepsilon)(x), \\ \delta^\varepsilon(x^\varepsilon) = \delta(\varepsilon)(x) = 1 + \varepsilon^2 \delta^\sharp. \end{cases}$$

Along with this scaling procedure, we take $e(\varepsilon) = (e_{ij}(\varepsilon))$ to denote the scaled linearized strain tensor, the components,

$$\begin{cases} e_{\alpha\beta}^\varepsilon(v^\varepsilon) = e_{\alpha\beta}(\varepsilon)(v) = \varepsilon^2 \{e_{\alpha\beta}^\theta(v) + \varepsilon^2 e_{\alpha\beta}^\sharp(v)\}, \\ e_{\alpha 3}^\varepsilon(v^\varepsilon) = e_{\alpha 3}(\varepsilon)(v) = \varepsilon \{e_{\alpha 3}^\theta(v) + \varepsilon^2 e_{\alpha 3}^\sharp(v)\}, \\ e_{33}^\varepsilon(v^\varepsilon) = e_{33}(\varepsilon)(v) + \varepsilon^2(\partial_\alpha \partial_\alpha v_3 + b_{33}^\sharp(\varepsilon)\partial_3 v_3) + \varepsilon^4 e_{33}^\sharp(\varepsilon)(v), \end{cases} \tag{2.2}$$

where
$e_{ij}(v) = \frac{1}{2}(\partial_j v_i + \partial_i v_j)$.

And there existe constant C_1 such that :

$$\sup_{0<\varepsilon\leqslant\varepsilon_0} \max_{ij} \|e_{ij}^\sharp(\varepsilon)(u)\|_{L^2} \leqslant C_1 \|v\|_{H^1(\Omega)}, \quad \text{for all} \quad v \in H^1(\Omega),$$

$$\sup_{0<\varepsilon\leqslant\varepsilon_0} \|e_{ij}^\sharp(\varepsilon)(u)\|_{L^2} \leqslant C_1 \|v\|_{H^1(\Omega)}, \quad \text{for all} \quad v \in H^1(\Omega).$$

Proof. See [5, $p340$] ■

Assumptions about the Data

More precisely we assume that there exist functions $f \in L^2(\Omega)$, $g \in L^2(\Gamma_+), r \in L^2(\omega), \phi \in L^2(\Gamma_+ \cup \Gamma_-)$ not depending on ε such that,

$$\begin{cases} f_\alpha^\varepsilon(x^\varepsilon) = f_\alpha(\varepsilon)(x) = \varepsilon^2 f_\alpha(x), & f_3^\varepsilon(x^\varepsilon) = f_3(\varepsilon)(x) = \varepsilon^3 f_3(x), & \forall\ x \in \Omega, \\ g_\alpha^\varepsilon(x^\varepsilon) = g_\alpha(\varepsilon)(x) = \varepsilon^3 g_\alpha(x), & g_3^\varepsilon(x^\varepsilon) = g_3(\varepsilon)(x) = \varepsilon^4 g_3(x), & \forall\ \ x \in \Gamma_+, \\ r^\varepsilon(x^\varepsilon) = r(\varepsilon)(x) = \varepsilon^2 r(x), & & \forall\ \ x \in \Omega, \\ \phi^\varepsilon(x^\varepsilon) = \phi(\varepsilon)(x) = \varepsilon^2 \phi(x), & & \forall\ \ x \in \Gamma_+ \cup \Gamma_-, \end{cases}$$

Replacing $(u^\varepsilon, \varphi^\varepsilon)$ by (u, φ) end Compensation, the variationl equation a domain Ω^ε the following form variational inequality:

$$\begin{cases} \text{Find}(u(\varepsilon), \varphi(\varepsilon)) \in K \times \Psi \ \ \text{such that :} \\ b((u, \varphi), (v - u, \psi)) + j(v) - j(u) \geq l((v - u), \psi), \\ \qquad \forall\ (v, \psi) \in K \times \Psi. \end{cases} \tag{2.3}$$

$$j(v) = \int_{\Gamma_-} q|v_t(\varepsilon)|(1 + \varepsilon^2 \delta^\sharp)\{b_{3i}(\varepsilon) b_{3i}(\varepsilon)\}^{1/2} d\Gamma.$$

Moreover we have the symmetries,

$$\begin{cases} C_{ijkl} = C_{klij} = C_{jikl}, \\ d_{kl} = d_{lk}, \\ P_{mkl} = P_{mlk}. \end{cases}$$

$$
\begin{aligned}
&b((u(\varepsilon),\varphi(\varepsilon)),(v,\psi)) = 2\varepsilon\int_\Omega C_{\alpha\beta\sigma\tau}e_{\alpha\beta}(v)e_{\sigma\gamma}(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx + \varepsilon\int_\Omega C_{3333}e_{33}(v)\\
&e_{33}(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx + 2\varepsilon\int_\Omega C_{\alpha\beta\sigma3}(e_{\alpha\beta}(v)e_{\sigma3}(u(\varepsilon)) + e_{\sigma3}(v)e_{\alpha\beta}(u(\varepsilon)))\\
&(1+\varepsilon^2\delta^\sharp)dx + 2\varepsilon\int_\Omega C_{\alpha333}(e_{\alpha3}ve_{33}(u(\varepsilon)) + e_{33}(v)e_{\alpha3}(u(\varepsilon)))(1+\varepsilon^2\delta^\sharp)dx\\
&+2\varepsilon\int_\Omega C_{\alpha\beta33}(e_{\alpha\beta}(v)e_{33}(u(\varepsilon)) + e_{33}(v)e_{\alpha\beta}(u(\varepsilon)))(1+\varepsilon^2\delta^\sharp)dx\\
&+4\varepsilon\int_\Omega C_{\alpha3\sigma3}e_{\alpha3}(v)e_{\sigma3}(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx + \varepsilon^5\int_\Omega d_{\alpha\beta}\partial_\beta(\varphi(\varepsilon))\partial_\alpha\psi(1+\varepsilon^2\delta^\sharp)dx\\
&+\varepsilon^4\int_\Omega d_{\alpha3}(\partial_3(\varphi(\varepsilon))\partial_\alpha\psi + \partial_\alpha(\varphi(\varepsilon))\partial_3\psi)(1+\varepsilon^2\delta^\sharp)dx + \varepsilon^3\int_\Omega d_{33}\partial_3(\varphi(\varepsilon))\partial_3\psi(1+\varepsilon^2\delta^\sharp)dx\\
&+\varepsilon^3\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma(\varphi(\varepsilon))e_{\alpha\beta}(\varepsilon)(v)(1+\varepsilon^2\delta^\sharp)dx - \varepsilon^3\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma(\psi)e_{\alpha\beta}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
&+2\varepsilon^3\int_\Omega P_{\gamma\alpha3}\partial_\gamma(\varphi(\varepsilon))e_{\alpha3}(\varepsilon)(v)(1+\varepsilon^2\delta^\sharp)dx - 2\varepsilon^3\int_\Omega P_{\gamma\alpha3}\partial_\gamma\psi e_{\alpha3}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
&+\varepsilon^2\int_\Omega P_{3\alpha\beta}\partial_3(\varphi(\varepsilon))e_{\alpha\beta}(\varepsilon)(v)(1+\varepsilon^2\delta^\sharp)dx - \varepsilon^2\int_\Omega P_{3\alpha\beta}\partial_3(\psi)e_{\alpha\beta}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
&+2\varepsilon^2\int_\Omega P_{3\alpha3}\partial_3(\varphi(\varepsilon))e_{\alpha3}(\varepsilon)(v)(1+\varepsilon^2\delta^\sharp)dx - 2\varepsilon^2\int_\Omega P_{3\alpha3}\partial_3(\psi)e_{\alpha3}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
&+\varepsilon^2\int_\Omega P_{\gamma33}\partial_\gamma(\varphi(\varepsilon))e_{33}(\varepsilon)(v)(1+\varepsilon^2\delta^\sharp)dx - \varepsilon^2\int_\Omega P_{\gamma33}\partial_3(\psi)e_{33}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
&+\varepsilon^2\int_\Omega P_{333}\partial_3(\varphi(\varepsilon))e_{33}(\varepsilon)(v)(1+\varepsilon^2\delta^\sharp)dx - \varepsilon^2\int_\Omega P_{333}\partial_3(\psi)e_{33}(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx
\end{aligned}
$$

$$
\begin{aligned}
l(v,\psi) &= \varepsilon\int_\Omega r(\varepsilon)\psi(1+\varepsilon^2\delta^\sharp)dx - \int_{\Gamma_-\cup\Gamma_+}\phi(\varepsilon)\psi\{\partial_\beta\theta + \varepsilon^2 b^\sharp_{3\beta}\}(1+\varepsilon^2\delta^\sharp)d\Gamma\\
&+ \varepsilon\int_\Omega f(\varepsilon)v_\alpha(1+\varepsilon^2\delta^\sharp)dx + \int_{\Gamma_+} g(\varepsilon)v_\alpha(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta + \varepsilon^2 b^\sharp_{3\beta}\}d\Gamma
\end{aligned}
$$

The equilibrium problem now consists in looking for a solution to a problem set over Ω which reads

$$
\begin{cases}
\text{Find}(u(\varepsilon),\varphi(\varepsilon)) \in K(\Omega)\times\Psi(\Omega) \ \text{ such that :}\\
\varepsilon\int_\Omega C_{\alpha\beta\sigma\tau}e_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))e_{\sigma\gamma}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx+\varepsilon\int_\Omega C_{3333}e_{33}(\varepsilon)(v-u(\varepsilon))e_{33}(\varepsilon)(u(\varepsilon))(1\\
+\varepsilon^2\delta^\sharp)dx+2\varepsilon\int_\Omega C_{\alpha\beta\sigma3}(e_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))e_{\sigma3}(\varepsilon)(u(\varepsilon))+e_{\sigma3}(\varepsilon)(v)e_{\alpha\beta}(\varepsilon)(u(\varepsilon)))(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon\int_\Omega C_{\alpha333}(e_{\alpha3}(\varepsilon)(v-u(\varepsilon))e_{33}(\varepsilon)(u(\varepsilon))+e_{33}(\varepsilon)(v-u(\varepsilon))e_{\alpha3}(\varepsilon)(u(\varepsilon)))(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon\int_\Omega C_{\alpha\beta33}(e_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))e_{33}(\varepsilon)(u(\varepsilon))+e_{33}(v-u(\varepsilon))e_{\alpha\beta}(\varepsilon)(u(\varepsilon)))(1+\varepsilon^2\delta^\sharp)dx\\
+4\varepsilon\int_\Omega C_{\alpha3\sigma3}e_{\alpha3}(\varepsilon)(v-u(\varepsilon))e_{\sigma3}(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx+\varepsilon\int_\Omega C_{3333}e_{33}(\varepsilon)(v-u(\varepsilon))e_{33}(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^5\int_\Omega d_{\alpha\beta}\partial_\beta(\varphi(\varepsilon))\partial_\alpha\psi(1+\varepsilon^2\delta^\sharp)dx+\varepsilon^4\int_\Omega d_{\alpha3}(\partial_3(\varphi(\varepsilon))\partial_\alpha\psi\\
+\partial_\alpha(\varphi(\varepsilon))\partial_3\psi)(1+\varepsilon^2\delta^\sharp)dx+\varepsilon^3\int_\Omega d_{33}\partial_3(\varphi(\varepsilon))\partial_3\psi(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^3\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma(\varphi(\varepsilon))e_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx-\varepsilon^3\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma(\psi)e_{\alpha\beta}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^3\int_\Omega P_{\gamma\alpha3}\partial_\gamma(\varphi(\varepsilon))e_{\alpha3}(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx-2\varepsilon^3\int_\Omega P_{\gamma\alpha3}\partial_\gamma\psi e_{\alpha3}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^2\int_\Omega P_{3\alpha\beta}\partial_3(\varphi(\varepsilon))e_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx-\varepsilon^2\int_\Omega P_{3\alpha\beta}\partial_3(\psi)e_{\alpha\beta}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^2\int_\Omega P_{3\alpha3}\partial_3(\varphi(\varepsilon))e_{\alpha3}(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx-2\varepsilon^2\int_\Omega P_{3\alpha3}\partial_3(\psi)e_{\alpha3}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^2\int_\Omega P_{\gamma33}\partial_\gamma(\varphi(\varepsilon))e_{33}(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx-\varepsilon^2\int_\Omega P_{\gamma33}\partial_3(\psi)e_{33}(\varepsilon)(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^2\int_\Omega P_{333}\partial_3(\varphi(\varepsilon))e_{33}(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx-\varepsilon^2\int_\Omega P_{333}\partial_3(\psi)e_{33}(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
\ +\int_{\Gamma_-}q|v_t(\varepsilon)|(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2b^\sharp_{3\beta}\}d\Gamma-\int_{\Gamma_-}q|u_t(\varepsilon)|(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2b^\sharp_{3\beta}\}d\Gamma\\
\geqslant\varepsilon\int_\Omega r(\varepsilon)\psi(1+\varepsilon^2\delta^\sharp)dx-\int_{\Gamma_-\cup\Gamma_+}\phi(\varepsilon)\psi(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2b^\sharp_{3\beta}\}d\Gamma\\
+\varepsilon\int_\Omega f(\varepsilon)(v_\alpha-u_\alpha(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx+\int_{\Gamma_+}g(\varepsilon)(v_\alpha-u_\alpha(\varepsilon))(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2b^\sharp_{3\beta}\}d\Gamma\\
\forall(v,\psi)\in V(\Omega)\times\Psi(\Omega),
\end{cases}
\tag{2.4}
$$

where,

$$V(\Omega)=\{v\in H^1(\Omega), v=0 \ \text{ on } \ \Gamma_0\},$$

$$\Psi(\Omega) = \{\psi \in H^1(\Omega), \psi = 0 \ \text{ on } \ \Gamma_0\},$$

and the convex,
$K(\Omega) = \{v \in V(\Omega), v_n \leq -\theta - 1 + \frac{1}{\sqrt{\alpha}} \ , \quad \text{ on } \ \Gamma_0\}.$

Chapter 3

Two-dimensional limit scaled solution in the shallow shells

In this section, the principal idea is letting the shallow shells thickness ε tend to zero, after rescaling the $3D$ variational inequality (2.4)to a fixed reference domain that does not depend on ε. We investigate the convergence of the unknowns as $\varepsilon \longrightarrow 0$ and analyze the resulting system of equations.

Lemma 3.1 *Let $\theta \in C^3(\omega)$ be a given function, and let the functions $e^\theta_{ij}(v)$ be defined as, for all $v \in H^1(\Omega)$,*

$$e^\theta_{\alpha\beta}(v) = e_{\alpha\beta}(v) - \tfrac{1}{2}(\partial_\beta\theta\partial_3 v_\alpha + \partial_\alpha\theta\partial_3 v_\beta),$$
$$e^\theta_{\alpha 3}(v) = e^\theta_{3\alpha}(v) = e_{\alpha 3}(v) - \tfrac{1}{2}\partial_\alpha\theta\partial_3 v_3,$$
$$e^\theta_{33}(v) = e_{33}(v),$$

for all $v \in H^1(\Omega)$,the mapping ,

$$v \longrightarrow \{\Sigma_{ij}|e_{ij}(v)|^2_{0,\Omega}\}^{\frac{1}{2}},$$

is a norm over the set $K(\Omega)$, which is equivalent to the norm $\|.\|_{H^1(\Omega)}$.

Theorem 3.2 *Assume,*

$$f \in L^2(\Omega), g \in L^2(\Gamma_+), r \in L^2(\Omega), \phi \in L^2(\Gamma_+ \cup \Gamma_-).$$

1. As ε tends to 0, the family $\{(u(\varepsilon), \varphi(\varepsilon))\}_{\varepsilon>0}$ converges strongly$\{(u, \varphi)\}$, in the set $K(\Omega) \times \Psi(\Omega)$, where,

$$K(\Omega) = \{v \in H^1(\Omega), v = 0 \text{ on } \Gamma_0, v_3 \geqslant -\theta \text{ on } \Gamma_-\},$$

$$\Psi(\Omega) = \{\psi \in H^2(\Omega),\ \psi = 0 \text{ on } \gamma_0\}.$$

2. Then the limit of $u(\varepsilon)$ as ε tends to 0 is Kirchhoff-Love displacement field, namely

$$u_\alpha = \zeta_\alpha - x_3\partial_\alpha\zeta_3, \qquad u_3 = \zeta_3.$$

3. The strong limit solution (u, φ) solves the following coupled problem:

$$\begin{cases} \text{Find } (u, \varphi) \in V_{KL} \cap K \times \Psi_l \text{ such that:} \\ b^*((u, \varphi), (v - u, \psi)) + j(v) - j(u) \geqslant l((v - u), \psi), \\ \forall (v, \psi) \in V_K \cap K \times \Psi. \end{cases} \tag{3.1}$$

where,

$$\begin{aligned} b^*((u, \varphi), (v, \psi)) &= \int_\Omega C_{ijkl} R^\theta_{ij}(v) R^\theta_{kl}(u) dx + \int_\Omega d_{ij} \tau_l(\varphi) \tau_i(\psi) dx \\ &+ \int_\Omega P_{ijk} \tau_k(\varphi) R^\theta_{ij}(v) dx - \int_\Omega P_{ijk} R^\theta_{ij}(u) \tau_i(\psi) dx, \end{aligned}$$

$$V_{KL}(\Omega) = \{v \in H^1(\Omega),\ e_{i3}(v) = 0\}.$$

$$\Psi_l = \{\psi \in L^2(\Omega), \partial_3\psi \in L^2(\Omega)\}.$$

Step 1. We define the tensor $R^\theta(\varepsilon) = (R^\theta_i(\varepsilon)) \in L^2(\Omega)$, and the scaled vector $\tau = (\tau_{ij}(\varepsilon))$.

$$\begin{cases} e^\theta_{\alpha\beta} = R^\theta_{\alpha\beta}(\varepsilon)(v), \\ e^\theta_{\alpha 3} = \varepsilon R^\theta_{\alpha 3}(\varepsilon)(v), \\ e^\theta_{33} = \varepsilon^2 (R^\theta_{33}(\varepsilon)(v) - \partial_\alpha\theta\partial_\alpha u_3(\varepsilon)), \end{cases} \tag{3.2}$$

and,

$$\begin{cases} \partial_\alpha\psi = \tau_\alpha(\varepsilon)(\psi), \\ \partial_3\psi = \varepsilon\tau_3(\varepsilon)(\psi), \end{cases} \tag{3.3}$$

In (2.4), we use (2.2), we obtain,

$$
\left\{
\begin{array}{l}
\text{Find } \ (u(\varepsilon),\varphi(\varepsilon)) \in K(\Omega)\times\Psi(\Omega) \ \text{ such that :}\\
\varepsilon^5\int_\Omega C_{\alpha\beta\sigma\tau}\{e^\theta_{\alpha\beta}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}\{e^\theta_{\sigma\tau}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\sigma\tau}(u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon\int_\Omega C_{3333}\{e^\theta_{33}(\varepsilon)(v-u(\varepsilon))+\varepsilon^2(\partial_\alpha\theta\partial_\alpha(v_3-u_3(\varepsilon))+b^\sharp_{33}(\varepsilon)\partial_3(v_3-u_3(\varepsilon))+\varepsilon e^{\#}_{33}(\varepsilon)(u(\varepsilon)))\}\\
\{e^\theta_{33}(\varepsilon)(u(\varepsilon))+\varepsilon^2(\partial_\alpha\theta\partial_\alpha u_3(\varepsilon)+b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^4\int_\Omega C_{\alpha\beta\sigma3}[\{e^\theta_{\alpha\beta}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}\{e^\theta_{\sigma3}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\sigma3}(u(\varepsilon))\}\\
+\{e^\theta_{\alpha\beta}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(u(\varepsilon))\}\{e^\theta_{\sigma3}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\sigma3}(v)\}](1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^2\int_\Omega C_{\alpha333}[\{e^\theta_{\alpha3}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha3}(v)\}\{e^\theta_{33}(\varepsilon)(u(\varepsilon))+\varepsilon^2(\partial_\alpha\theta\partial_\alpha u_3(\varepsilon)\\
+b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(u(\varepsilon))\}+\{e^\theta_{33}(\varepsilon)(v-u(\varepsilon))+\varepsilon^2(\partial_\alpha\theta\partial_\alpha(v-u(\varepsilon))\\
+b^\sharp_{33}(\varepsilon)\partial_3(v_3-u_3(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))\}\{e^\theta_{\alpha3}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha3}(u(\varepsilon))\}](1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^3\int_\Omega C_{\alpha\beta33}[\{e^\theta_{\alpha\beta}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}\{e^\theta_{33}(\varepsilon)(u(\varepsilon))+\varepsilon^2(\partial_\alpha\theta\partial_\alpha u_3(\varepsilon)\\
+b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(u(\varepsilon))\}+\{e^\theta_{33}(\varepsilon)(v-u(\varepsilon))+\varepsilon^2(\partial_\alpha\theta\partial_\alpha(v_3-u_3(\varepsilon))\\
+b^\sharp_{33}(\varepsilon)\partial_3(v_3-u_3(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))\}\{e^\theta_{\alpha3}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha3}(u(\varepsilon))\}](1+\varepsilon^2\delta^\sharp)dx\\
+4\varepsilon^3\int_\Omega C_{\alpha3\sigma3}[\{e^\theta_{\alpha\beta}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}\{e^\theta_{\sigma3}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\sigma3}(u(\varepsilon))\}](1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^5\int_\Omega d_{\alpha\beta}\partial_\beta(\varphi)\partial_\alpha\psi(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^4\int_\Omega d_{\alpha3}(\partial_3(\varphi(\varepsilon))\partial_\alpha\psi+\partial_\alpha(\varphi(\varepsilon))\partial_3\psi)(1+\varepsilon^2\delta^\sharp)dx+\varepsilon^3\int_\Omega d_{33}\partial_3(\varphi(\varepsilon))\partial_3\psi(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^5\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma(\varphi(\varepsilon))\{e^\theta_{\alpha\beta}(v)+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-\varepsilon^5\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma(\psi)\{e^\theta_{\alpha\beta}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(u(\varepsilon))\}(u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^4\int_\Omega P_{\gamma\alpha3}\partial_\gamma(\varphi(\varepsilon))\{e^\theta_{\alpha\beta}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha3}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-2\varepsilon^4\int_\Omega P_{\gamma\alpha3}\partial_\gamma\psi\{e^\theta_{\alpha\beta}u(\varepsilon)+\varepsilon^2 e^\sharp_{\alpha3}u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^4\int_\Omega P_{3\alpha\beta}\partial_3(\varphi(\varepsilon))\{e^\theta_{\alpha\beta}(v)+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-\varepsilon^4\int_\Omega P_{3\alpha\beta}\partial_3(\psi)\{e^\theta_{\alpha\beta}u(\varepsilon)+\varepsilon^2 e^\sharp_{\alpha\beta}u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^4\int_\Omega P_{3\alpha3}\partial_3(\varphi(\varepsilon))\{e^\theta_{\alpha3}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha3}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-2\varepsilon^4\int_\Omega P_{3\alpha3}\partial_3(\psi)\{e^\theta_{\alpha3}u(\varepsilon)+\varepsilon^2 e^\sharp_{\alpha3}u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^3\int_\Omega P_{\gamma33}\partial_\gamma(\varphi(\varepsilon))\{e^\theta_{33}(\varepsilon)(v-u(\varepsilon))+\varepsilon^2(\partial_\alpha\theta\partial_\alpha(v_3-u_3(\varepsilon))+b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon))\\
+\varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx-\varepsilon^3\int_\Omega P_{\gamma33}\partial_3(\psi)\{e^\theta_{33}(\varepsilon)u(\varepsilon)+\varepsilon^2(\partial_\alpha\theta\partial_\alpha u_3(\varepsilon)\\
+b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon)+\varepsilon^4 e^\sharp_{33}(\varepsilon)u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx+\varepsilon^2\int_\Omega P_{333}\partial_3(\varphi(\varepsilon))\{e^\theta_{33}(\varepsilon)(v-u(\varepsilon))\\
+\varepsilon^2(\partial_\alpha\theta\partial_\alpha(v_3-u_3(\varepsilon))+b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon))\\
+\varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx-\varepsilon^2\int_\Omega P_{333}\partial_3(\psi)\{e^\theta_{33}(\varepsilon)u(\varepsilon)+\varepsilon^2(\partial_\alpha\theta\partial_\alpha u_3(\varepsilon)\\
+b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon)+\varepsilon^4 e^\sharp_{33}(\varepsilon)u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx\quad+\int_{\Gamma_-} q|v_t(\varepsilon)|(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2 b^\sharp_{3\beta}\}d\Gamma\\
-\int_{\Gamma_-} q|u_t(\varepsilon)|(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2 b^\sharp_{3\beta}\}(1+\varepsilon^2\delta^\sharp)d\Gamma\ \geqslant \varepsilon\int_\Omega r(\varepsilon)\psi(1+\varepsilon^2\delta^\sharp)dx\\
-\int_{\Gamma_-\cup\Gamma_+}\phi(\varepsilon)\psi(1+\varepsilon^2\delta^\sharp)d\Gamma+\varepsilon\int_\Omega f(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
+\int_{\Gamma_+} g(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2 b^\sharp_{3\beta}\}d\Gamma\\
\forall(v,\psi)\in V(\Omega)\times\Psi(\Omega),
\end{array}
\right.
\tag{3.4}
$$

In (3.4), we use (3.2) and (3.3), we obtain,

$$
\left\{
\begin{array}{l}
\text{Find } \ (u(\varepsilon),\varphi(\varepsilon)) \in K(\Omega)\times\Psi(\Omega) \ \text{ such that :}\\
\varepsilon^5\int_\Omega C_{\alpha\beta\sigma\tau}\{R^\theta_{\alpha\beta}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}\{R^\theta_{\sigma\tau}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\sigma\tau}(u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon\int_\Omega C_{3333}\{\varepsilon^2 R^\theta_{33}(\varepsilon)(v-u(\varepsilon))+b^\sharp_{33}(\varepsilon)\partial_3(v_3 v-u_3(\varepsilon)))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(u(\varepsilon))\}\\
\{\varepsilon^2 R^\theta_{33}(\varepsilon)(u(\varepsilon))+b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^4\int_\Omega C_{\alpha\beta\sigma3}\{R^\theta_{\alpha\beta}(vv-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}\{\varepsilon R^\theta_{\sigma3}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\sigma3}(u(\varepsilon))\}\\
+\{R^\theta_{\alpha\beta}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(u(\varepsilon))\}\{\varepsilon R^\theta_{\sigma3}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\sigma3}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^2\int_\Omega C_{\alpha333}[\{\varepsilon R^\theta_{\alpha3}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha3}(v-u(\varepsilon))\}\{\varepsilon^2 R^\theta_{33}(\varepsilon)(u(\varepsilon))\\
+b^\sharp_{33}(\varepsilon)\partial_3(u_3(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(u(\varepsilon))\}+\{\varepsilon^2 R^\theta_{33}(\varepsilon)(v-u(\varepsilon))\\
+b^\sharp_{33}(\varepsilon)\partial_3(v)+\varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))\}\{\varepsilon R^\theta_{\alpha3}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha3}(u(\varepsilon))\}](1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^3\int_\Omega C_{\alpha\beta33}[\{R^\theta_{\alpha\beta}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}\{\varepsilon^2 R^\theta_{33}(\varepsilon)(u(\varepsilon))\\
+b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(u(\varepsilon))\}+\{\varepsilon^2 R^\theta_{33}(\varepsilon)(v-u(\varepsilon))\\
+b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))\}\{R^\theta_{\alpha\beta}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(u(\varepsilon))\}](1+\varepsilon^2\delta^\sharp)dx\\
+4\varepsilon^3\int_\Omega C_{\alpha3\sigma3}[\{\varepsilon R^\theta_{\alpha3}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha3}(v-u(\varepsilon))\}\{\varepsilon R^\theta_{\sigma3}(u(\varepsilon))+\varepsilon^2 e^\sharp_{\sigma3}(u(\varepsilon))\}](1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^5\int_\Omega d_{\alpha\beta}\partial_\beta(\varphi)\partial_\alpha\tau_\alpha(\varepsilon)(\psi)(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^4\int_\Omega d_{\alpha3}(\partial_3(\varphi(\varepsilon))\partial_\alpha\tau_\alpha(\varepsilon)(\psi)+\varepsilon\partial_\alpha(\varphi(\varepsilon))\partial_3\tau_3(psi)(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^4\int_\Omega d_{33}\partial_3(\varphi(\varepsilon))\partial_3\tau_3(\varepsilon)(\psi)(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^5\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma(\varphi(\varepsilon))\{R^\theta_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))+\varepsilon^2 e^\#_{\alpha\beta}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-\varepsilon^5\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma\tau_\gamma(\varepsilon)(\psi)\{R^\theta_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))+\varepsilon^2 e^\#_{\alpha\beta}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^5\int_\Omega P_{\gamma\alpha3}\partial_\gamma(\varphi(\varepsilon))\{R^\theta_{\alpha\beta}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-2\varepsilon^5\int_\Omega P_{\gamma\alpha3}\partial_\gamma\tau_\gamma(\varepsilon)(\psi)\{R^\theta_{\alpha\beta}u(\varepsilon)+\varepsilon^2 e^\sharp_{\alpha\beta}u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^4\int_\Omega P_{3\alpha\beta}\partial_3(\varphi(\varepsilon))\{R^\theta_{\alpha\beta}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-\varepsilon^4\int_\Omega P_{3\alpha\beta}\partial_3\tau_3(\varepsilon)(\psi)\{R^\theta_{\alpha\beta}u(\varepsilon)+\varepsilon^2 e^\sharp_{\alpha\beta}u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx\\
+2\varepsilon^4\int_\Omega P_{3\alpha3}\partial_3(\varphi(\varepsilon))\{\varepsilon R^\theta_{\alpha3}(v-u(\varepsilon))+\varepsilon^2 e^\sharp_{\alpha3}(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-2\varepsilon^4\int_\Omega P_{3\alpha3}\partial_3\tau_3(\varepsilon)(\psi)\{\varepsilon R^\theta_{\alpha3}u(\varepsilon)+\varepsilon^2 e^\sharp_{\alpha3}u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^3\int_\Omega P_{\gamma33}\partial_\gamma(\varphi(\varepsilon))\{\varepsilon^2 R^\theta_{33}(\varepsilon)(v-u(\varepsilon))+b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-\varepsilon^3\int_\Omega P_{\gamma33}\partial_3\tau_3(\varepsilon)(\psi)\{\varepsilon^2 R^\theta_{33}(\varepsilon)u(\varepsilon)+b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon)+\varepsilon^4 e^\sharp_{33}(\varepsilon)u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx\\
+\varepsilon^2\int_\Omega P_{333}\partial_3(\varphi(\varepsilon))\{\varepsilon^2 R^\theta_{33}(\varepsilon)(v-u(\varepsilon))+b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon))+\varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx\\
-\varepsilon^2\int_\Omega P_{333}\partial_3\tau_3(\varepsilon)(\psi)\{\varepsilon^2 R^\theta_{33}(\varepsilon)u(\varepsilon)+b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon)+\varepsilon^4 e^\sharp_{33}(\varepsilon)u(\varepsilon)\}(1+\varepsilon^2\delta^\sharp)dx\\
\ \ +\int_{\Gamma_-} q|v_t(\varepsilon)|(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2 b^\sharp_{3\beta}\}d\Gamma-\int_{\Gamma_-} q|u_t(\varepsilon)|(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2 b^\sharp_{3\beta}\}(1+\varepsilon^2\delta^\sharp)d\Gamma\\
\geqslant \varepsilon\int_\Omega r(\varepsilon)\psi(1+\varepsilon^2\delta^\sharp)dx-\int_{\Gamma_-\cup\Gamma_+}\phi(\varepsilon)\psi(1+\varepsilon^2\delta^\sharp)d\Gamma+\varepsilon\int_\Omega f(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx\\
+\int_{\Gamma_+} g(\varepsilon)(v-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta+\varepsilon^2 b^\sharp_{3\beta}\}d\Gamma\\
\forall (v,\psi)\in V(\Omega)\times\Psi(\Omega),
\end{array}
\right.
\tag{3.5}
$$

Step 2. Since α^ε, is such that $\lim_{\varepsilon\to 0}\alpha^\varepsilon = 1$,
and $\alpha^\varepsilon > 0$ we can introduce the following subset of $K(\Omega)$ defined by

$$K(\Omega)=\{v\in H^1(\Omega),\quad v=0 \ \text{ on } \ \Gamma_0, v_3(x_1,x_2,-1)\geqslant-\theta\},$$

Let us insert the scaled strain tensor $R^\theta(\varepsilon)$ into problem (3.5).

After tedious but straight for ward computations.

$$\left\{\begin{array}{l}
\text{Find } \ (u(\varepsilon),\varphi(\varepsilon))\in K(\Omega)\times\Psi(\Omega) \ \text{ such that :}\\
\varepsilon^5\int_\Omega C_{\alpha\beta\sigma\tau}\{R^\theta_{\alpha\beta}(v-u(\varepsilon))\}\{R^\theta_{\sigma\tau}(u(\varepsilon))\}dx\\
+\varepsilon^3\int_\Omega C_{3333}\{R^\theta_{33}(\varepsilon)(v-u(\varepsilon))\}\{R^\theta_{33}(\varepsilon)(u(\varepsilon))\}dx\\
+2\varepsilon^5\int_\Omega C_{\alpha\beta\sigma3}\{R^\theta_{\alpha\beta}(v-u(\varepsilon))\}\{R^\theta_{\sigma3}(u(\varepsilon))\}+\{R^\theta_{\alpha\beta}(u(\varepsilon))\}\{R^\theta_{\sigma3}(v-u(\varepsilon))\}dx\\
+2\varepsilon^5\int_\Omega C_{\alpha333}[\{R^\theta_{\alpha3}(v-u(\varepsilon))\}\{R^\theta_{33}(\varepsilon)(u(\varepsilon))\}+\{R^\theta_{33}(\varepsilon)(v-u(\varepsilon))\}\{R^\theta_{\alpha3}(u(\varepsilon))\}]dx\\
+2\varepsilon^5\int_\Omega C_{\alpha\beta33}[\{R^\theta_{\alpha\beta}(v-u(\varepsilon))\}\{R^\theta_{33}(\varepsilon)(u(\varepsilon))\}+\{R^\theta_{33}(\varepsilon)(v-u(\varepsilon))\}\{R^\theta_{\alpha3}(u(\varepsilon))\}]dx\\
+4\varepsilon^5\int_\Omega C_{\alpha3\sigma3}[\{R^\theta_{\alpha3}(v-u(\varepsilon))\}\{R^\theta_{\sigma3}(u(\varepsilon))\}]dx\\
+\varepsilon^5\int_\Omega d_{\alpha\beta}\partial_\beta(\varphi)\partial_\alpha\tau_\alpha(\varepsilon)(\psi)dx\\
+\varepsilon^4\int_\Omega d_{\alpha3}(\partial_3(\varphi(\varepsilon))\partial_\alpha\tau_\alpha(\varepsilon)(\psi)+\varepsilon\partial_\alpha(\varphi(\varepsilon))\partial_3\tau_3(\psi)dx\\
+\varepsilon^4\int_\Omega d_{33}\partial_3(\varphi(\varepsilon))\partial_3\tau_3(\varepsilon)(\psi)dx\\
+\varepsilon^5\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma(\varphi(\varepsilon))\{R^\theta_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))\}dx\\
-\varepsilon^5\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma\tau_\gamma(\varepsilon)(\psi)\{R^\theta_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))\}dx\\
+2\varepsilon^4\int_\Omega P_{\gamma\alpha3}\partial_\gamma(\varphi(\varepsilon))\{R^\theta_{\alpha\beta}(v-u(\varepsilon))\}dx\\
-2\varepsilon^5\int_\Omega P_{\gamma\alpha3}\partial_\gamma\tau_\gamma(\varepsilon)(\psi)\{R^\theta_{\alpha\beta}u(\varepsilon\}dx\\
+\varepsilon^4\int_\Omega P_{3\alpha\beta}\partial_3(\varphi(\varepsilon))\{R^\theta_{\alpha\beta}(v-u(\varepsilon))\}dx\\
-\varepsilon^5\int_\Omega P_{3\alpha\beta}\partial_3\tau_3(\varepsilon)(\psi)\{R^\theta_{\alpha\beta}u(\varepsilon)\}dx\\
+2\varepsilon^5\int_\Omega P_{3\alpha3}\partial_3(\varphi(\varepsilon))\{R^\theta_{\alpha3}(v-u(\varepsilon))\}dx\\
-2\varepsilon^5\int_\Omega P_{3\alpha3}\partial_3\tau_3(\varepsilon)(\psi)\{R^\theta_{\alpha3}u(\varepsilon)\}dx\\
+\varepsilon^5\int_\Omega P_{\gamma33}\partial_\gamma(\varphi(\varepsilon))\{R^\theta_{33}(\varepsilon)(v-u(\varepsilon))\}dx\\
-\varepsilon^5\int_\Omega P_{\gamma33}\partial_3\tau_3(\varepsilon)(\psi)\{R^\theta_{33}(\varepsilon)u(\varepsilon)\}dx\\
+\varepsilon^5\int_\Omega P_{333}\partial_3(\varphi(\varepsilon))\{R^\theta_{33}(\varepsilon)(v-u(\varepsilon))\}dx\\
-\varepsilon^5\int_\Omega P_{333}\partial_3\tau_3(\varepsilon)(\psi)\{R^\theta_{33}(\varepsilon)u(\varepsilon)\}dx\\
\ \ +\int_{\Gamma_-}q|v_t(\varepsilon)|\{\partial_\beta\theta\}d\Gamma-\int_{\Gamma_-}q|u_t(\varepsilon)|\{\partial_\beta\theta\}d\Gamma\ +B^\sharp(\varepsilon,\theta,u(\varepsilon),R^\sharp,v,\psi)\\
\geqslant\varepsilon^5\int_\Omega r\psi dx-\varepsilon^5\int_{\Gamma_-\cup\Gamma_+}\phi(\varepsilon)\psi d\Gamma+\varepsilon^5\int_\Omega f(v-u(\varepsilon))dx+L^\sharp(\varepsilon,\theta,v,\psi)\\
+\varepsilon^5\int_{\Gamma_+}g(v-u(\varepsilon))\{\partial_\beta\theta\}d\Gamma\\
\forall(v,\psi)\in V(\Omega)\times\Psi(\Omega),
\end{array}\right.\tag{3.6}$$

then,

$$\begin{cases} \text{Find } \ (u(\varepsilon), \varphi(\varepsilon)) \in K(\Omega) \times \Psi(\Omega) \ \text{ such that :} \\ \int_\Omega C_{ijkl} R^\theta_{ij}(\varepsilon)(v - u(\varepsilon)) R^\theta_{kl}(u(\varepsilon)) dx + \int_\Omega d_{ij} \tau_l(\varepsilon)(\varphi(\varepsilon)) \tau_i(\varepsilon)(\psi) dx \\ \ + \ \int_\Omega P_{ijk} \tau_k(\varepsilon)(\varphi(\varepsilon)) R^\theta_{ij}(\varepsilon)(v - u(\varepsilon)) dx - \int_\Omega P_{ijk} R^\theta_{ij}(\varepsilon)(u(\varepsilon)) \tau_i(\varepsilon)(\psi) dx \\ \ + \int_{\Gamma_-} q|v_t| \partial_\beta \theta d\Gamma - \int_{\Gamma_-} q|u_t(\varepsilon)| \partial_\beta \theta d\Gamma \\ B^\sharp(\varepsilon, \theta, u(\varepsilon), R^\sharp, v, \psi) \geqslant \int_\Omega r\psi dx - \int_{\Gamma_- \cup \Gamma_+} \phi\psi\delta d\Gamma + \int_\Omega f(v - u(\varepsilon)) dx \\ + \int_{\Gamma_+} g(v - u(\varepsilon)) \{\partial_\beta\} d\Gamma + L^\sharp(\varepsilon, \theta, u, v, \psi), \\ \forall (v, \psi) \in V(\Omega) \times \Psi(\Omega), \end{cases} \tag{3.7}$$

$B^\sharp(\varepsilon, \theta, u(\varepsilon), R^\sharp, v)$is uniformly bounded, i.e., there exists a positive constant $C(\theta)$ independent of ε such that, for all $u \in K(\Omega)$. $v \in K(\Omega)$. $R^\theta \in L^2(\Omega)$ we have,

$$\sup_{x<\varepsilon<\varepsilon_0} B^\sharp(\varepsilon, \theta, u(\varepsilon), R^\sharp, v, \psi) \leqslant C(\theta)(|R^\theta|_{L^2(\Omega)} + |u|_{H^2(\Omega)})|v|_{H^2(\Omega)}$$

We are now in a position to examine quantity $B^\#$.

$$B^\sharp(\varepsilon,\theta,u(\varepsilon),R^\sharp,v,\psi)=\varepsilon^5\int_\Omega C_{\alpha\beta\sigma\tau}\{R^\theta_{\alpha\beta}(v-u(\varepsilon))\varepsilon^2e^\sharp_{\sigma\tau}(u(\varepsilon))+R^\theta_{\sigma\tau}(u(\varepsilon))\varepsilon^2e^\sharp_{\alpha\beta}(v$$
$$-u(\varepsilon))(1+\varepsilon^2\delta^\sharp)dx+\varepsilon\int_\Omega C_{3333}\{\varepsilon^2R^\theta_{33}(\varepsilon)(v-u(\varepsilon))(b^\sharp_{33}(\varepsilon)\partial_3(v_3-u_3(\varepsilon))+\varepsilon^4e^\#(u(\varepsilon)))$$
$$+\varepsilon^2R^\theta_{33}(\varepsilon)(v-u(\varepsilon))(b^\#_{33}(\varepsilon)\partial_3(u_3(\varepsilon))+\varepsilon^4e^\sharp_{33}(\varepsilon)(u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx$$
$$+2\varepsilon^4\int_\Omega C_{\alpha\beta\sigma3}\{R^\theta_{\alpha\beta}(v-u(\varepsilon))\varepsilon^2e^\sharp_{\sigma3}(u(\varepsilon))+\varepsilon R^\theta_{\sigma3}(u(\varepsilon))\varepsilon^2e^\sharp_{\alpha\beta}(v)$$
$$+R^\theta_{\alpha\beta}(u(\varepsilon))\varepsilon^2e^\sharp_{\sigma3}(v)+\varepsilon R^\theta_{\sigma3}(v-u(\varepsilon))\varepsilon^2e^\sharp_{\alpha\beta}(u(\varepsilon))\}(1+\varepsilon^2\delta^\sharp)dx$$
$$+2\varepsilon^2\int_\Omega C_{\alpha333}[\varepsilon R^\theta_{\alpha3}(v-u(\varepsilon))(b^\sharp_{33}(\varepsilon)\partial_3u_3(\varepsilon))+\varepsilon^4e^\sharp_{33}(\varepsilon)(u(\varepsilon)(\varepsilon^2R^\theta_{33}(\varepsilon)(u(\varepsilon))(\varepsilon^2e^\sharp_{\alpha3}(v-u(\varepsilon))$$
$$+b^\sharp_{33}(\varepsilon)\partial_3v_3+\varepsilon^4e^\sharp_{33}(\varepsilon)(v-u(\varepsilon)))+\varepsilon^2R^\theta_{33}(\varepsilon)(v-u(\varepsilon))\varepsilon^2e^\sharp_{\alpha3}(u(\varepsilon))$$
$$+(b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon))+\varepsilon^4e^\sharp_{33}(\varepsilon)(v-u(\varepsilon)))(\varepsilon R^\theta_{\alpha3}(u(\varepsilon))+\varepsilon^2e^\sharp_{\alpha3}(u(\varepsilon)))](1+\varepsilon^2\delta^\sharp)dx$$
$$+2\varepsilon^3\int_\Omega C_{\alpha\beta33}[R^\theta_{\alpha\beta}(v-u(\varepsilon))(b^\#_{33}(\varepsilon)\partial(u_3(\varepsilon))+\varepsilon^2e^\sharp_{\alpha\beta}(v-u(\varepsilon)))+\varepsilon^2e^\#_{\alpha\beta}(\varepsilon)(v$$
$$-u(\varepsilon))(\varepsilon^2R^\theta_{33}(\varepsilon)(u(\varepsilon))+(b^\#_{33}(\varepsilon)\partial_3(u_3(\varepsilon))+\varepsilon^4e^\#_{33}u(\varepsilon)+\varepsilon^2R^\theta_{33}(v-u(\varepsilon))\varepsilon^2e^\#_{\sigma3}(u(\varepsilon)))$$
$$+\varepsilon^4e^\#_{33}(u(\varepsilon))(\varepsilon^2R^\theta_{33}(\varepsilon)(u(\varepsilon))+b^\#_{33}(\varepsilon)\partial_3(u_3(\varepsilon))+\varepsilon^4e^\#_{33}u(\varepsilon))](1+\varepsilon^2\delta^\sharp)dx$$
$$+4\varepsilon^3\int_\Omega C_{\alpha3\sigma3}[\varepsilon R^\theta_{\alpha3}(v-u(\varepsilon))+\varepsilon^2e^\sharp_{\sigma3}(u(\varepsilon))+\varepsilon R^\theta_{\sigma3}(u(\varepsilon))\varepsilon^2e^\sharp_{\alpha3}(v-u(\varepsilon))](1+\varepsilon^2\delta^\sharp)dx$$
$$+\varepsilon^5\int_\Omega d_{\alpha\beta}\partial_\beta(\varphi)\partial_\alpha\tau_\alpha(\varepsilon)(\psi)\varepsilon^2\delta^\sharp dx+\varepsilon^4\int_\Omega d_{\alpha3}(\partial_3(\varphi(\varepsilon))\partial_\alpha\tau_\alpha(\varepsilon)(\psi)+\varepsilon\partial_\alpha(\varphi(\varepsilon))\partial_3\tau_3(\psi))\varepsilon^2\delta^\sharp dx$$
$$+\varepsilon^4\int_\Omega d_{33}\partial_3(\varphi(\varepsilon))\partial_3\tau_3(\varepsilon)(\psi)\varepsilon^2\delta^\sharp dx+\varepsilon^5\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma(\varphi(\varepsilon))[\{R^\theta_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))$$
$$+\varepsilon^2e^\#_{\alpha\beta}(v-u(\varepsilon))\}\varepsilon^2\delta^\sharp+\varepsilon^2e^\#_{\alpha\beta}(v-u(\varepsilon))]dx-\varepsilon^5\int_\Omega P_{\gamma\alpha\beta}\partial_\gamma\tau_\gamma(\varepsilon)(\psi)[\{R^\theta_{\alpha\beta}(\varepsilon)(v-u(\varepsilon))$$
$$+\varepsilon^2e^\#_{\alpha\beta}(v-u(\varepsilon))\}(\varepsilon^2\delta^\sharp)+\varepsilon^2e^\#_{\alpha\beta}(v-u(\varepsilon))]dx+2\varepsilon^4\int_\Omega P_{\gamma\alpha3}\partial_\gamma(\varphi(\varepsilon))[\{R^\theta_{\alpha\beta}(v-u(\varepsilon))$$
$$+\varepsilon^2e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}(\varepsilon^2\delta^\sharp)+\varepsilon^2e^\sharp_{\alpha\beta}(v-u(\varepsilon))]dx-2\varepsilon^4\int_\Omega P_{\gamma\alpha3}\partial_\gamma\tau_\gamma(\varepsilon)(\psi)[\{R^\theta_{\alpha\beta}u(\varepsilon)$$
$$+\varepsilon^2e^\sharp_{\alpha\beta}u(\varepsilon)\}(\varepsilon^2\delta^\sharp)+\varepsilon^2e^\sharp_{\alpha\beta}u(\varepsilon)]dx-\varepsilon^4\int_\Omega P_{3\alpha\beta}\partial_3(\varphi(\varepsilon))[\{R^\theta_{\alpha\beta}(v-u(\varepsilon))$$
$$+\varepsilon^2e^\sharp_{\alpha\beta}(v-u(\varepsilon))\}(\varepsilon^2\delta^\sharp)+\varepsilon^2e^\sharp_{\alpha\beta}(v-u(\varepsilon))]dx-\varepsilon^4\int_\Omega P_{3\alpha\beta}\partial_3\tau_3(\varepsilon)(\psi)[\{R^\theta_{\alpha\beta}u(\varepsilon)$$
$$+\varepsilon^2e^\sharp_{\alpha\beta}u(\varepsilon)\}(\varepsilon^2\delta^\sharp)+\varepsilon^2e^\sharp_{\alpha\beta}u(\varepsilon)]dx+2\varepsilon^4\int_\Omega P_{3\alpha3}\partial_3(\varphi(\varepsilon))[\{\varepsilon R^\theta_{\alpha3}(v-u(\varepsilon))$$
$$+\varepsilon^2e^\sharp_{\alpha3}(v-u(\varepsilon))\}(\varepsilon^2\delta^\sharp)+\varepsilon^2e^\sharp_{\alpha3}(v-u(\varepsilon))]dx$$

$$
\begin{aligned}
&- \quad 2\varepsilon^4 \int_\Omega P_{3\alpha 3}\partial_3\tau_3(\varepsilon)(\psi)[\{\varepsilon R^\theta_{\alpha 3}u(\varepsilon) + \varepsilon^2 e^\sharp_{\alpha 3}u(\varepsilon)\}(\varepsilon^2\delta^\sharp) + \varepsilon^2 e^\sharp_{\alpha 3}u(\varepsilon)]dx \\
&+ \quad \varepsilon^3 \int_\Omega P_{\gamma 33}\partial_\gamma(\varphi(\varepsilon))[\{\varepsilon^2 R^\theta_{33}(\varepsilon)(v-u(\varepsilon)) + b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon)) + \varepsilon^4 e^\sharp_{33}(\varepsilon)(v \\
&- \quad u(\varepsilon))\}(\varepsilon^2\delta^\sharp)b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon)) + \varepsilon^4 e^\sharp_{33}(\varepsilon)(v)]dx - \varepsilon^3 \int_\Omega P_{\gamma 33}\partial_3\tau_3(\varepsilon)(\psi)[\{\varepsilon^2 R^\theta_{33}(\varepsilon)u(\varepsilon) \\
&+ \quad b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon) + \varepsilon^4 e^\sharp_{33}(\varepsilon)u(\varepsilon)\}(\varepsilon^2\delta^\sharp) + b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon) + \varepsilon^4 e^\sharp_{33}(\varepsilon)u(\varepsilon)]dx \\
&+ \quad \varepsilon^2 \int_\Omega P_{333}\partial_3(\varphi(\varepsilon))[\{\varepsilon^2 R^\theta_{33}(\varepsilon)(v-u(\varepsilon)) + b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon)) + \varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))\}(\varepsilon^2\delta^\sharp) \\
&+ \quad b^\sharp_{33}(\varepsilon)\partial_3(v-u(\varepsilon)) + \varepsilon^4 e^\sharp_{33}(\varepsilon)(v-u(\varepsilon))]dx - \varepsilon^2 \int_\Omega P_{333}\partial_3\tau_3(\varepsilon)(\psi)[\{\varepsilon^2 R^\theta_{33}(\varepsilon)u(\varepsilon) \\
&+ \quad b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon) + \varepsilon^4 e^\sharp_{33}(\varepsilon)u(\varepsilon)\}(\varepsilon^2\delta^\sharp) + b^\sharp_{33}(\varepsilon)\partial_3 u_3(\varepsilon) + \varepsilon^4 e^\sharp_{33}(\varepsilon)u(\varepsilon)]dx \\
&+ \quad \int_{\Gamma_-} q|v_t(\varepsilon)|\varepsilon^2\delta^\sharp\{\partial_\beta\theta + \varepsilon^2 b^\sharp_{3\beta}\}d\Gamma - \int_{\Gamma_-} q|u_t(\varepsilon)|(1+\varepsilon^2\delta^\sharp)\{\partial_\beta\theta + \varepsilon^2 b^\sharp_{3\beta}\}\varepsilon^2\delta^\sharp d\Gamma\ ,
\end{aligned}
$$

Finally, it can easily be shown that,

$$
\begin{aligned}
L^\#(\varepsilon,\theta,u(\varepsilon),v,\psi) \;&=\; \varepsilon^5 \int_\Omega r\psi\varepsilon^2\delta^\sharp dx - \varepsilon^5 \int_{\Gamma_-\cup\Gamma_+} \phi\psi\varepsilon^2\delta^\sharp d\Gamma \\
&+\; \varepsilon^5 \int_\Omega f(v-u(\varepsilon))\varepsilon^2\delta^\sharp dx + \varepsilon^5 \int_{\Gamma_+} g(v-u(\varepsilon))\varepsilon^2\delta^\sharp\{\partial_\beta\theta\varepsilon^2 b^\sharp_{3\beta}\}d\Gamma
\end{aligned}
$$

Lemma 3.3 *The norms $|u(\varepsilon)|_{L^2(\Omega)}$ and $|\varphi(\varepsilon)|_{L^2(\Omega)}$ are bounded uniformly in ε.*

Step 3. Next we let $v = 2u(\varepsilon)$ and $\psi = 0$ in inequality (3.7),

$$
\begin{aligned}
&-\int_\Omega C_{ijkl}R_{ij}u(\varepsilon)R_{kl}(u(\varepsilon))\delta dx \;-\; \int_\Omega P_{ijk}\tau_k(\varepsilon)(\varphi)R_{ij}u(\varepsilon)\delta dx \\
&+\int_{\Gamma_-} q|u_t|d\Gamma \geqslant -\int_\Omega f(u)\delta dx - \int_{\Gamma_+} g(u)\delta\{\partial_\beta\theta + \varepsilon^2 b^\sharp_{3\beta}\}d\Gamma,
\end{aligned}
$$

Next we let $v = 0$ and $\psi = -\varphi(\varepsilon)$ in inequality (3.7),

$$
\int_\Omega d_{ij}\tau_l(\varepsilon)(\varphi)\tau_i(\varepsilon)(\varphi(\varepsilon))\delta dx + \int_\Omega P_{ijk}R_{ij}(u(\varepsilon))\tau_i(\varepsilon)(\varphi(\varepsilon))\delta dx
$$

$$-\int_{\Gamma_-} q|u_t|d\Gamma \geqslant \int_\Omega r\varphi(\varepsilon)\delta dx - \int_{\Gamma_-\cup\Gamma_+} \phi\varphi(\varepsilon)\delta d\Gamma,$$

then,

$$\begin{aligned}
&\int_\Omega C_{ijkl}R_{ij}u(\varepsilon)R_{kl}(u(\varepsilon))\delta dx \ + \ \int_\Omega P_{ijk}\tau_k(\varepsilon)(\varphi)R_{ij}u(\varepsilon)\delta dx \\
&-\int_\Omega d_{ij}\tau_l(\varepsilon)(\varphi)\tau_i(\varepsilon)(\varphi(\varepsilon))\delta dx - \int_\Omega P_{ijk}R_{ij}(u(\varepsilon))\tau_i(\varepsilon)(\varphi(\varepsilon))\delta dx \\
&\leqslant \int_\Omega f(u)\delta dx + \int_{\Gamma_+} g(u)\delta\{\partial_\beta\theta\}d\Gamma - \int_\Omega r\varphi(\varepsilon)\delta dx + \int_{\Gamma_-\cup\Gamma_+} \phi\varphi(\varepsilon)\delta d\Gamma
\end{aligned}$$

we find,

$$\int_\Omega C_{ijkl}R^\theta_{ij}(\varepsilon)u(\varepsilon)R^\theta_{kl}(\varepsilon)(u(\varepsilon))\delta dx - \int_\Omega d_{ij}\tau_l(\varepsilon)(\varphi)\tau_i(\varepsilon)(\varphi(\varepsilon))dx$$

$$\leqslant \int_\Omega [f(u) - r\varphi(\varepsilon)]dx + \int_{\Gamma_+} [g(u)\partial_\beta\theta + \phi\varphi(\varepsilon)]\delta d\Gamma,$$

From this inequality, using the coerciveness properties of tensors C and d, we get,

$$\begin{aligned}
|R(\varepsilon)(u(\varepsilon))| + |\tau(\varepsilon)(\varphi(\varepsilon))| &\leqslant c(\|u(\varepsilon)\|_{L^2(\Omega)} - \|\varphi(\varepsilon)\|_{L^2(\Omega)}) \\
&\leqslant c(\|u(\varepsilon)\|_{H^1(\Omega)} - \|\varphi(\varepsilon)\|_{H^1(\Omega)}),
\end{aligned}$$

We recall korn ineqality and poincare inequality :
there exists $c > 0$ such that,

$$\|u(\varepsilon)\|^2_{H^1(\Omega)} \leqslant c|e^\theta(u(\varepsilon))|^2_{L^2(\Omega)}, \qquad \|\phi(\varepsilon)\|^2_{H^2(\Omega)} \leqslant c|\tau(\varepsilon)(\varphi(\varepsilon))|^2_{L^2(\Omega)},$$

Therefore for $\varepsilon \leqslant 1$, there exist $c > 0$:

$$\begin{aligned}
\|u(\varepsilon)\|^2_{H^1(\Omega)} + \|\phi(\varepsilon)\|^2_{H^2(\Omega)} &\leqslant c(|e^\theta(u(\varepsilon))|^2_{L^2(\Omega)} + \|\tau(\varepsilon)(\varphi(\varepsilon))\|^2_{L^2(\Omega)}) \\
&\leqslant \ c(|R^\theta(u(\varepsilon))|^2_{L^2(\Omega)} + \|\tau(\varepsilon)(\varphi(\varepsilon))\|^2_{L^2(\Omega)}) \\
&\leqslant \ c(\|u(\varepsilon))\|_{H^1(\Omega)} + \|\varphi(\varepsilon)\|_{H^1(\Omega)})
\end{aligned}$$

Lemma 3.4 *The tensor $R^\theta(\varepsilon) = (R^\theta_{ij}(\varepsilon)) \in L^2(\Omega)$ is bounded uniformly in ε in the space $L^2(\Omega)$.*

these inequalities imply that the norms $\|u(\varepsilon)\|_{H^1(\Omega)}, \|\varphi(\varepsilon)\|_{H^1(\Omega)}, |R(\varepsilon)(u(\varepsilon))|_{L^1(\Omega)}$, $|\tau(\varepsilon)(\varphi(\varepsilon))|_{L^1(\varepsilon)}$ are uniformly bounded.

Consequently, there are weakly convergent subseqences of $u \in H^1(\Omega)$,$\varphi \in \quad H^1(\Omega)$ and $R \in H^2(\Omega)$, $\tau \in \quad H^2(\Omega)$ with,

$$\begin{aligned} u(\varepsilon) &\rightharpoonup u, && \text{in } \; H^1(\Omega), \\ \varphi(\varepsilon) &\rightharpoonup \phi, && \text{in } \; H^1(\Omega), \\ R^\theta(\varepsilon)(u(\varepsilon)) &\rightharpoonup R^\theta(u), && \text{in } \; L^2(\Omega), \\ \tau(\varepsilon)(\varphi(\varepsilon)) &\rightharpoonup \tau(\varphi), && \text{in } \; L^2(\Omega). \end{aligned}$$

Moreoer, form the definition of $R(\varepsilon)$ and $\tau(\varepsilon)$, we have the bounds:

$$|e^\theta_{\alpha\beta}(u(\varepsilon))|_{L^2(\Omega)} = |e_{\alpha\beta}(u(\varepsilon)) - \frac{1}{2}(\partial_\beta\theta\partial_3 u(\varepsilon)))|_{L^2(\Omega)} \leqslant c\varepsilon,$$

$$|e^\theta_{\alpha 3}(u(\varepsilon))|_{L^2(\Omega)} = |e_{\alpha 3}(u(\varepsilon)) - \frac{1}{2}\partial_\alpha\theta\partial_3 u(\varepsilon))|_{L^2(\Omega)} \leqslant c\varepsilon,$$

$$|e^\theta_{33}(u(\varepsilon))|_{L^2(\Omega)} \leqslant c\varepsilon^2,$$

$$|\partial_3\varphi(\varepsilon)|_{L^2(\Omega)} \leqslant c\varepsilon.$$

Hence $(\varepsilon\partial_1\varphi(\varepsilon), \varepsilon\partial_2\varphi(\varepsilon), \partial_3\phi(\varepsilon)) \rightharpoonup (0, 0, \partial_3\varphi)$,

Hence $e^\theta_{i3}(u(\varepsilon)) \rightharpoonup 0$ in $L^2(\Omega)$, $\partial_3\varphi(\varepsilon) \rightharpoonup 0$ in $L^2(\Omega)$, we have the bounds :

$$|e_{\alpha 3}(u)| \leqslant \liminf_{\varepsilon\to 0} |e_{i3}(u(\varepsilon))| = 0,$$

and,

$$|\partial_3\varphi| \leqslant \liminf_{\varepsilon\to 0} |\partial\varphi(\varepsilon)| = 0,$$

From (u, φ), since we deduce that $e_{i3}(u) = 0$, we deduce that there exist a bi-dimensional field $\zeta = (\zeta_i)$ such that $\zeta_\alpha \in H^1(\omega)$ and $\zeta_3 \in H^2(\omega)$ and u is a Kirchhoff-Love displacement field.

From the other hand, since $\partial_3\varphi = 0$, the limit φ is independent of x_3.

Lemma 3.5 *Let the operator$A(\varepsilon) : u(\varepsilon) \longrightarrow H^1(\Omega)$ which satisfies the weak convergence,*

$$A(\varepsilon) : u(\varepsilon) \rightharpoonup A(u) \in L^2(\Omega).$$

If $u(\varepsilon) \in K(\Omega)$ solves the variational inequality,

$$\int_\Omega A(\varepsilon)(u(\varepsilon)).\partial_3(v - u(\varepsilon)) \geqslant \varepsilon \int_\Omega f.(v - u(\varepsilon)) \quad \forall v \in K(\Omega),$$

then $A(u) = 0$.

Proof. See [3, p 294] ■

Step 4. We now show that the whole family $\{u(\varepsilon), \varphi(\varepsilon)\}$ converges strongly:

Let us introduce the notation $\int_\Omega CA : Adx = \int_\Omega C_{ijkl}A_{kl}A_{ij}$ for all second order symmetric tensor A and let $\int_\Omega dv : vdx = \int_\Omega d_{ijkl}v_{kl}v_{ij}$ for all vector $v \in R^3$,

$c|R^\theta(\varepsilon)(u(\varepsilon)) - R^\theta(u)|^2_{L^2(\Omega)}$

$\leqslant \int_\Omega C(R^\theta(\varepsilon)(u(\varepsilon)) - R^\theta(u) : (R^\theta(\varepsilon)(u(\varepsilon)) - R^\theta(u)))dx$

$\leqslant \int_\Omega CR^\theta(u) : (R^\theta(u) - 2R^\theta(\varepsilon)(u(\varepsilon))dx + \int_\Omega CR^\theta(u) : R^\theta(u) : R^\theta(\varepsilon)(u(\varepsilon))dx,$

Since we have already established the weak convergences $R^\theta(\varepsilon) \rightharpoonup R^\theta$ in $L^2(\Omega)$ as $\varepsilon \longrightarrow 0$, then,

$\lim_{\varepsilon\to 0}|R(\varepsilon)(u(\varepsilon)) - R(u)|^2_{L^2(\Omega)} \leqslant -\int_\Omega CR(u) : R(u)dx + \lim_{\varepsilon\to 0}\int_\Omega CR(\varepsilon)(u(\varepsilon)) : R(\varepsilon)(u(\varepsilon))dx.$

We let $v = u$,$\psi = 0$ in (3.7), and pass to the limit, we get :

Then we have $\lim_{\varepsilon\to 0}|R(\varepsilon)(u(\varepsilon)) - R(u)|^2_{L^2(\Omega)} \leqslant 0$. which implies that the sequence $\{e^\theta(u(\varepsilon))\}$ converges strongly in $L^2(\Omega)$ to $e^\theta(u)$. Therefore by Korn's inequality the sequence $u(\varepsilon)$ converges strongly in $H^1(\Omega)$ to u.

Similarly,

$c|\tau(\varepsilon)(\varphi(\varepsilon)) - \tau(\varphi)|^2_{L^2(\Omega)}$

$\leqslant \int_\Omega d(\tau(\varepsilon) - \tau(\varepsilon)) : (\tau(\varepsilon)(\varphi(\varepsilon)) - \tau(\varphi))dx,$

$\leqslant \int_\Omega d\tau(\varphi) : (\tau(\varphi) - 2\tau(\varepsilon)(\varphi(\varepsilon)))dx + \int_\Omega d\tau(\varepsilon)(\varphi(\varepsilon))) : \tau(\varepsilon)(\varphi(\varepsilon))dx$

Since we have already established the weak convergence $\tau(\varepsilon)(\varphi(\varepsilon)) \rightharpoonup \tau(\varphi)$ in $L^2(\Omega)$ as $\varepsilon \longrightarrow 0$, we get,

$$\lim_{\varepsilon\to 0} c|\tau(\varepsilon)(\varphi(\varepsilon)) - \tau(\varphi)|^2_{L^2(\Omega)} \leqslant -\int_\Omega d\tau(\varphi) : \tau(\varphi)dx + \lim_{\varepsilon\to 0}\int_\Omega d\tau(\varepsilon)(\varphi(\varepsilon))) : \tau(\varepsilon)(\varphi(\varepsilon))dx,$$

$$\lim_{\varepsilon\to 0} c|\tau(\varepsilon)(\varphi(\varepsilon)) - \tau(\varphi)|^2_{L^2(\Omega)} \leqslant 0,$$

We already know that the limit u is a Kirchhoff-Love field. Choosing $v = u(\varepsilon)$, and $\psi = \varphi(\varepsilon)$ as a test function, in(3.7),

$$\int_\Omega d_{ij}\tau_l(\varepsilon)(\varphi(\varepsilon))\tau_i(\varepsilon)(\psi)dx + \int_\Omega P_{ijk}R_{ij}(u(\varepsilon))\tau_i(\varepsilon)(\varphi(\varepsilon))\delta dx$$

$$B^\sharp(\varepsilon, \theta, u(\varepsilon), R^\sharp, v, \psi) \geqslant \int_\Omega r\psi dx - \int_{\Gamma_- \cup \Gamma_+} \phi\psi\delta d\Gamma + L^\sharp(\varepsilon, \theta, u, v, \psi),$$

$$\int_\Omega d\tau(\varphi) : \tau(\varphi)dx - \lim_{\varepsilon\to 0}\int_\Omega d\tau(\varepsilon)(\varphi(\varepsilon))) : \tau(\varepsilon)(\varphi(\varepsilon))dx.$$

So $|\tau(\varepsilon)(\varphi(\varepsilon)) - \tau(\varphi)|^2_{L^2(\Omega)} \leqslant 0$ we get the strong convergence, the sequence $\{\tau(\varphi(\varepsilon))\}$ converges strongly in $L^2(\Omega)$ to $\tau(\varphi)$, then the sequence of $\varphi(\varepsilon)$ converges strongly to φ in $H^1(\Omega)$.

In addition, using the definition of quantities $R^\theta_{ij}(\varepsilon)$ we have:

$$\begin{aligned} |e^\theta(u(\varepsilon)) - e^\theta(u)|^2_{L^1(\Omega)} \leqslant\ & \Sigma_{\alpha,\beta}|R^\theta_{\alpha\beta}(\varepsilon)(u(\varepsilon)) \\ & -R^\theta_{\alpha\beta}(u)|^2_{L^2(\Omega)} \\ & +2\varepsilon^2\Sigma_\alpha|R^\theta_{\alpha 3}(\varepsilon)(u(\varepsilon)) - R^\theta_{\alpha 3}(u)|^2_{L^2(\Omega)} \\ & +\varepsilon^4|R^\theta_{33}(\varepsilon)(u(\varepsilon)) - R^\theta_{33}(u)|^2_{L^2(\Omega)}. \end{aligned}$$

Step 5. Using the same arguments in [1] and [3], an the strong convergences obtain the limit variational inequality:

$$\left\{ \begin{array}{l} \text{Find } (u,\varphi) \in V_K \cap K \times \Psi \text{ such that:} \\ b^*((u,\varphi),(v-u,\psi)) + j(v) - j(u) \geqslant l((v-u),\psi), \\ \forall (v,\psi) \in V_K \cap K \times \Psi. \end{array} \right. \tag{3.8}$$

where,

$$\begin{aligned} b^*((u,\varphi),(v,\psi)) &= \int_\Omega C_{ijkl} R^\theta_{ij}(v) R^\theta_{kl}(u) dx + \int_\Omega d_{ij} \tau_l(\varphi) \tau_i(\psi) dx \\ &+ \int_\Omega P_{ijk} \tau_k(\varphi) R^\theta_{ij}(v) dx - \int_\Omega P_{ijk} R^\theta_{ij}(u) \tau_i(\psi) dx, \end{aligned}$$

$$V_K(\Omega) = \{v \in H^1(\Omega),\ e_{i3}(v) = 0\}.$$

$$\Psi_l = \{\psi \in L^2(\Omega), \partial_3 \psi \in L^2(\Omega)\}.$$

Conclusion

The present study on the case of a piezoelectric shallow shells of Signorini's contact with Tresca's friction on rigid foundation, gives a new two-dimensional limit models problem from three-dimensional elasticity.

This justification of this model obtained by asymptotic analysis with convergence result.

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Title: Frictional contact of piezoelectric Shallow Shell

Abstract

The objective of this thesis is to study the asymptotic modeling of three dimensional problems of linearly elastic shallow shells, with and unilateral contact and friction.

In the first Part, we are placing a three dimensional models for a linearly elastic Shallow Shells. With and unilateral contact and friction. Called equations of piezoelectric Shallow Shells.

In the second Part, Using technics from asymptotic analysis, we justify two dimensional models. Transformation into a problem posed over an ash domain 2.

In the third Part, we prove the convergence of the solution when the thickness of the Shallow Shells tends to zero and establish the limit problem of a piezoelectric problem for Shallow Shells in unilateral contact and friction

Key words:

Linear shallow shell theory, asymptotic analysis, Signori problem.

Titre: Contact par frottement de pour les coques peu Profondes piézoélectrique

Résume

L'objectif de cette thèse est d'étudier la modélisation asymptotique de problèmes tridimensionnels de coques peu-profondes linéairement élastiques, avec friction et contact unilatéraux.

Dans la première partie, nous mettons un modèle tridimensionnel pour des coquilles peu profondes linéairement élastiques, avec contact et friction unilatéraux. Equations appelées des coquilles piézoélectriques peu profondes

Dans la deuxième partie, en utilisant des techniques issues de l'analyse asymptotique, nous justifions les modèles bidimensionnels. La transformation en un problème posé sur un domaine sh 2.

Dans la troisième partie, nous prouvons la convergence de la solution lorsque l'épaisseur des coquilles peu profondes tend à zéro et établit le problème limite d'un problème piézoélectrique pour les coquilles peu profondes en contact et en friction unilatéraux

Mots clés:

théorie de coque peu-profonde linéaire, analyse asymptotique, problème de Signorini, frottement de Coulomb.

العنوان : التماس و الاحتكاك لهياكل ضعيفة الانحناء الكهرواضغطية

الملخص

الهدف من هذه الاطروحة هو دراسة النمذجة المقاربة لهياكل ضعيفة الانحناء ذات مرونة خطية ، مع اتصال والاحتكاك بجانب واحد .

في الجزء الاول ، نماذج ذات 3 ابعاد لهياكل ضعيفة الانحناء ذات مرونة خطية، مع الاحتكاك والاتصال من جانب واحد مع جامد بشروط حدية . تدعى المعادلات لهياكل ضعيفة الانحناء .

في الجزاء الثاني ، تحويل من هياكل ضعيفة الانحناء الى هياكل منعدمة الانحناء ذات سمك 2، باستخدام تقنيات التحليل المقارب بررنا نموذج ثلاثي الابعاد.

في الجزء الثالث اثبات تقارب الحل عندما يؤول سمكها الى الصفر، لنضع نموذجا ثنائي البعد.

الكلمات المفتاحية :

هياكل ضعيفة الانحناء الخطي ، تحليل مقارب ، احتكاك تيريسكا .