KASDI MERBAH UNIVERSITY OUARGLA

Faculty of Mathmatics and Sciences Material

Departement of: Mathematics

MASTER

Path: Mathematics

Speciality: Analysis

Present by: Marouf Nour el imene

Theme

Study of global existence of reaction-diffusion system modeling a spread of epidemic

Represented by:19 /06/2017

Limbfrom jury:

Mr A.Bensayah	.KASDI Merbah University - Ouargla	chairman
Mr M.T. Ben moussa	.KASDI Merbah University - Ouargla	E xamininer
H. Daddiouissa	.KASDI Merbah University - Ouargla	Supervisor

1.5

This thesis is dedicated to:
The sake of Allah, my creator and my Master
My great teacher and messenger, Mohammed who, taught us the purpose of life
My great parents, who never stop giving of themselves in countless ways
My beloved brother and sisters
to stop loving. To all my familly, the symbol of love and giving,
My friends who encourage and support me,
All people in my life who touch my heat,
Idedicat this research. .

I would like to thank all the people who contributed in some way to the work described in this thesis. Firstly thanks go to Allah before and after who enabled me to complet this work. A special thanks to express my deep gratitude to prof. H. Daddioussa,for supervisor contructive criticism, help and gvidance, providing me access, data and informations, to the writing process until the completion of this thesis.
I would like to acknowledge and thank my for allowing me to conduct my research and providing any assistance requested. Special thank goes to the members of staff development and human resources departement for their continued support
Finally Iwould like to thank the beginning teachers, mentor-teachers and administractors in our school division that assisted me with this project. Their excitement and willingness to provide feedback made the completion of this research an enjoyable experience.

CONTENTS

Notations

SETS

$\mathbb{R}$	The real numbers.
$\mathbb{C}$	The complex numbers.
X	The Banch spaces.
Ω	usually denotes an open set in a topologie space.
$[0, T[$	intervalle $0 \leq t < T$.
$[0, t_{max}[$	intervalle $0 \leq t < t_{max}$

Operators

A	Operator.
$R(\lambda, A)$	The resolvent operator of A.
∇	gradient operator .
$\triangle$	The laplace operator .
I	Identity operator.

FONCTIONNELS SPACE

$C([0;T] : X)$	The space of continuous functions defined on $0 \leq t \leq T$ with value in X.
$\mathcal{D}(A)$	The domain of A.
$R(A)$	The image of A.
$\rho(A)$	The image of A.

L^p	The usual space of measurable whose P the power is lebesgue integrable.
$\|.\|$	The norm in X.
$\|.\|_p$	The norm of u in L^p.
$\|.\|_\infty$	The norm of u in L^∞.
$< .,. >$	inner producte

ANOTHER SYMBOLS

$E_\alpha(x)$	Mittag-Leffler function
$E_{\alpha,\beta}(x)$	Generalized Mittag-Leffler function

INTRODUCTION

A reaction-diffusion equation comprises a reaction term and a diffusion term, ie the tybical form is as follows:

$$u_t = D\Delta u + f(u),$$

$u = u(x,t)$ is a state variable and describes density /concertraction of a substance, a population....at position $x \in \Omega \subset \mathbb{R}^n$ at time $t(\Omega$ is a open set). Δ denotes the Laplace operator. So the first term on the right hand side describes the diffusion, including D as diffusion coefficient.

The second term, $f(u)$ is a smooth function $f : \mathbb{R} \longrightarrow \mathbb{R}$ and describes process with really "change" the present u i.e. Somthing happens to it (birth,death ,chemical reaction...) not just diffuse in the space.

It is also possible, that the reaction term depends not only on the first derivative of u, i.e.∇u, and /or explicity on x.

Instead of a scalar equation, one can also introduce systems of reactions diffusion equations ,which of the form

$$u_t = D\Delta u + f(x, u, \nabla u)$$

Where $u(x,t) \in \mathbb{R}^n$

The study of the fractional differential equations found place in several different topics, already discussed and solved for the used differential equations.

In this work we have the time fractional reaction-diffusion system with a blance law

$$\begin{cases} {}^cD_t^\beta u - d\Delta u = -uf(v) & \text{in} \qquad \Omega \times \mathbb{R}^+, \\ {}^cD_t^\beta v - \Delta v = uf(v) & \text{in} \qquad \Omega \times \mathbb{R}^+, \end{cases}$$

Supplemented with the boundary and initial conditions

$$\frac{\partial u}{\partial \eta}(x,t) = \frac{\partial v}{\partial \eta}(x,t) = 0 \qquad on \qquad \partial\Omega \times \mathbb{R}^+,$$

$$u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \qquad in \qquad \Omega.$$

Where Ω is regular bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial\eta}$ denote the normal derivative on $\partial\Omega$,
Δstands for the Laplacian operator, d is the diffusion constant, $u_0$andv_0 are nonnegative functions, ${}^cD_t^\beta$, for $\beta \in (0,1)$, is the Caputo fractional derivative of order β.
Concerning the nonlinearity f, we assume that there exist positive constant M_1 and M_2 a real number $p \geq 1$ such that

$$0 \leq f(v) \leq M_1|v|^p + M_2,$$

and for all $|v|, |\tilde{v}| \leq R$,there exist a positive number L such that

$$|f(v) - f(\tilde{v})| \leq L|v - \tilde{v}|$$

Our main purpose in this work is to see the influence of thr fractional time derivatives on the behavior of the solution .
In **chapter1** we begin presenting the basic of the thoery that we want to study, we diccuss the theory of bounded operators semigroups. This chapter is fundamental to the basic estimates and constructions that will be recurrently used during all this work.
In **chapter2** the emphases are the fractional calculs and the Mittag-Leffler function, that plays an important role in this theory. Among other things, we study some proprerties of the gamma function. In particular we study the local existence and the theorems of analytic semigroup, and finaly the theory of fractional powe of closed operators
In **chapter3** we finally cosider the results to study existence global solutions, we also derive the large time behavior of bunded solutions.

CHAPTER 1

GENERALITY

1.1 SEMIGROUPS OF BOUNDED LINEAR OPERATORS

[9]:

Let X be a banach space. A one parameter $T(t), 0 \leq t < \infty$ of bounded linear from X into X is a semigroup of bounded linear operators on X if :

1. $T(0) = I$
2. $T(t+s) = T(t)T(s) \quad$ for every $\quad s, t \geq 0$ (semigroup property)

$T(t)$ is called a **uniformly continuous** if:

$$\lim_{t \to 0} \|T(t) - I\| = 0$$

Let a linear operator A defined by :

$$\mathcal{D}(A) = \left\{ x \in X; \lim_{t \to 0} \frac{T(t)x - x}{t} \quad exists \right\}$$

and

$$Ax = \lim_{t \to 0} \frac{T(t)x - x}{t} = \frac{d^+ T(t)}{dt} x|_{t=0} \qquad \text{for every} \quad x \in \mathcal{D}(A)$$

is the infinitesimal generator of the semigroup $T(t)$
$D(A)$ is called the domain of A

[9]:

Let X be a banach space

A semigroup $T(t)$, $0 \le t < \infty$ of bounded linear operations on X, is a **strongly continuous semigroup**of bounded linear operators if :

$$\lim_{t \to 0} T(t)x = x \qquad \text{for every } x \in X$$

we usally call it C_0semigroup.

1.1.1 Properties of C_0 semigroup

[4]:
Let $T(t)$ be a $c_0$semigroup, then there exist constants $w \ge 0$ and $M \ge 1$ such that

$$\|T(t)\| \le Me^{wt}, \qquad 0 < t < \infty.$$

:
First, there exist a constant $\eta > 0$ such that $\|T(t)\|$ is bounded for $t \in [0, \eta]$. Suppose this is false, then there exists a sequence $\{t_n\}, t_n \ge 0$ and $\lim_{n \to \infty} t_n = 0$ such that

$$\|T(t_n)\| \ge n.$$

From uniform boundedness theorem, there exist some $x \in X$ such that $\|T(t_n)x\|$is unbounded, which contradicts with definition(1.1).
Thus

$$\|T(t)\| \le M \quad \text{for} \quad t \in [0, \eta]$$

since $\|T(0)\| = 1, M \ge 1$. Let $w = \eta^{-1} log M \ge 0$. Given $t \ge 0$ we have $t = n\eta + \delta$ with $0 \le \delta < \eta$.
Therefore, by semigroup property

$$\|T(t)\| = \|T(\delta)T(\eta)^n\| \le M^{n+1} \le MM^{t/n} = Me^{wt}$$

The proof is complete. **[9]**:
If $T(t)$ is a C_0semigroup then for every $x \in X, t \longrightarrow T(t)x$ is a continuous function from R_0^+ into X. Let $t, h \ge 0$

$$\|T(t+h)x - T(t)x\| \le \|T(t)\|\|T(h)x - x\| \le Me^{wt}\|T(h)x - x\|$$

The proof is complete.

1.1.2 Main theorem

[9] :
Let $T(t)$ be a C_0semigroup, A be its infinitesimal generator then :

1. For $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x. \tag{1.1}$$

2. For $x \in X$, $\int_0^t T(s)xds \in D(A)$ and
$$A\left(\int_0^t T(s)x\right)ds = T(t)x - x. \tag{1.2}$$
3. For $x \in D(A), T(t)x \in D(A)$ and
$$\frac{d}{dt}T(t)x = AT(t)x = T(t)Ax. \tag{1.3}$$
4. For $x \in D(A)$,
$$T(t)x - T(s)x = \int_s^t T(\tau)Axd(\tau) = \int_s^t AT(\tau)xd(\tau). \tag{1.4}$$

1.1.3 Proof of Main theorem

1. It follows from the continuity of $T(t)$.
2. Let $x \in X$ and $h > 0$, then
$$\begin{aligned}\frac{T(h-I)}{h}\int_0^t T(t)xds &= \frac{1}{h}\int_0^t (T(s+h)x - T(s)x)ds \\ &= \frac{1}{h}\int_t^{t+h} T(s)xds - \frac{1}{h}\int_0^h T(s)xds\end{aligned}$$
and as $h \longrightarrow 0$ by property(1), the right side tends to $T(t)x - x$
3. Let $x \in D(A)$ and $h > 0$, then
$$\frac{T(h)-I}{h}T(t)x = T(t)x\left(\frac{T(h)-I}{h}\right),$$
$\longrightarrow T(t)Ax$ as $h \to 0$ This $T(t)x \in D(A)$ and $AT(t)x = T(t)Ax$, we have
$$\frac{d^+}{dt}T(t)x = AT(t)x = T(t)Ax$$
To prove (3)we need to show that for $t > 0$, the left derivative of $T(t)x$ exist and equals $T(t)Ax$.
This follows from
$$\lim_{h\to 0}\left[\frac{T(t)x - T(t-h)x}{h} - T(t)Ax\right]$$
$$= \lim_{h\to 0}T(t-h)\left[\frac{T(h)x - x}{h} - Ax\right] + \lim_{h\to 0}[T(t-h)Ax - T(t)Ax]$$
The first limit vanishes because $T(t-h)$ is bounded and $x \in D(A)$, the second limit is zero because of the continuity of $T(t)Ax$.

4. It is obvious by taking integration of (3)

[8](second fundamental limit theorem):
Let $T(t):t\geq 0$ be a C_0semigroup on X,then $A:D(A)\subset X\longrightarrow X$is its infinitesimal generator,then

$$T(t)x=\lim_{n\to+\infty}\left(I-\frac{t}{n}A\right)^{-n}x=\lim_{n\to+\infty}\left[\frac{n}{t}\left(\frac{n}{t}-A\right)^{-1}A\right]^{n}x$$

[9]:
If A is the infinitesimal generator of a C_0semigroup $T(t)$, then $D(A)$ is dense in X and X is a closed operator . :
For every $x\in X$, set $X_t=\frac{1}{t}\int_0^t T(t)xds$, by (2) $x_t\in D(A)$, by(1)
$x_t\longrightarrow x$ as $t\longrightarrow 0$ thus $\overline{D(A)}=X$
Linearity of A follows from its definition.
closedness: Let $x_n\in D(A)$ such that $x_n\longrightarrow x$ and $Ax_n\to y$ as $n\to\infty$, by (4), we have

$$T(t)x_n-x_n=\int_0^t T(s)Ax_nds$$

The integrand of (2) converges to $T(t)y$ on bounded intervals .
Let $n\to\infty$ in (2), then

$$T(t)x-x=\int_0^t T(s)yds$$

Divide by t and $t\to 0$, from (1)we have

$$x\in D(A)\quad and\quad Ax=y$$

The proof is complete.

We have the following strong result on A comparing to the previous corollary. **[9]**:
Let A be the infinitesimal generator of a C_0semigroup $T(t)$. If $D(A^n)$ is the domain of A^n then $\bigcap_{n=1}^{\infty}D(A^n)$ is dense in X.

A few concepts

Let $T(t)$ be a C_0semigroup. By theorem, it follows that there exist $W\geq 0$ and $M\geq 1$ such that

$$\|T(t)\|\leq Me^{wt},\qquad t\geq .0$$

If $w=0$, then $T(t)$ is called **uniformily bounded**.
If $w=0, M=1$, then $T(t)$is called a C_0**semigroup of contractions**.

1.1.4 Resolvent

[9]:
If A is linear operator in X, the resolvent set $\rho(A)$ of A is the set of all complex number

$$\rho(A)=\Big\{\ \lambda\in\mathbb{C}:(\lambda I-A)^{-1}:R(\lambda I-A)\subset X\to X\quad\text{is injective, bounded and}\quad\overline{R(\lambda I-A)}=X\Big\}$$

λ for wich $\lambda I - A$ is invertible ,i.e.

$$\lambda \in \rho(A) \subset \mathbb{C} \Longleftrightarrow \lambda - A : D(A) \longrightarrow X(bijection)$$

and $R(\lambda, A) = (\lambda I - A)^{-1}$ where, is a bounded linear operator in X.
of bounded linear operators is called **the resolvent** of A
Then the spectrum is the complement of the resolvent set

$$\sigma(A) = \rho(A)^c = C \backslash \rho(A)$$

1.2 THE HILLE-YOSIDA THEOREM

[9]:
A linear (unbounded) operator A is the infinitesimal generator of a $c_0$semigroup of contractions $T(t), t \geq 0$ if and only if

1. A is closed and $\overline{D(A)} = X$
2. The resolvent set $\rho(A)$ of A contains $\mathbb{R}^+$ and for every $\lambda > 0$

$$\| R(\lambda; A) \| \leq \frac{1}{\lambda}. \tag{1.5}$$

[9]:
Let A satisfy conditions of the above theorem, then

$$\lim_{n \longrightarrow \infty} \lambda R(\lambda; A)x = x \qquad for \qquad x \in X$$

1.2.1 Hille-Yosida for uniformly bounded semigroups

[9]:
A linear operator A is the finitesimal generator of a C_0semigroup $T(t)$, satisfying $\|T(t)\| \leq M(M \geq 1)$,if and only if :

1. A is closed and $D(A)$ is dense in X.
2. The resolvent set $\rho(A)$ of A contains $\mathbf{R}^+$ and

$$\|R(\lambda; A)^n\| \leq \frac{M}{\lambda^n}, \qquad for \quad \lambda > 0, n = 1, 2, ...$$

1.2.2 Hille-Yosida for C_0semigroup

[9]:
A linear operator A is the infinitesimal generator of a C_0semigroup $T(t)$ satisfing $\|T(t)\| \leq Me^{wt}$ if and only if

1. A is closed and $D(A)$ is dense in X.
2. $]w, \infty] \subset \rho(A)$and

$$\|R(\lambda; A)^n\| \leq \frac{M}{(\lambda - w)^n} \quad \text{for} \quad \lambda > w, n = 1, 2, ...$$

[9]:
Let A be the infinitesimal generator of a C_0semigroup $T(t)$ on X, if A_λ is the Yosida approximation of A, ie.

$$A_\lambda = \lambda AR(\lambda; A), \quad then \quad T(t)x = \lim_{\lambda \to 0} e^{wA_\lambda}x \quad where \quad e^{tA_\lambda} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}$$

1.3 A SUFFCIENT CONDITION FOR C_0SEMIGROUP

An easier to use theorem showing A is the infinitesimal generator of a C_0semigroup is given below

[9]:
Let A be a densly defined operator in X satisfying the following contradictions:

1. For some $0 < \delta < \frac{\pi}{2}$, $\rho(A) \supset \sum_\delta = \{\lambda :\mid arg\lambda \mid< \frac{\pi}{2} + \delta\} \cup \{0\}$
2. There exists a constant M such that

$$\|R(\lambda; A)\| \leq \frac{M}{\mid \lambda \mid} \quad for \quad \lambda \in \Sigma_\delta, \lambda \neq 0 \tag{1.6}$$

Then A is the infinitesimal generator of a C_0semigroup $T(t)$ satisfying $\|T(t)\| \leq C$ for some constant C. Moreover

$$T(t) = \frac{1}{2\pi i} \int_\Gamma e^{\lambda t} R(\lambda; A) d\lambda \tag{1.7}$$

where $\Gamma$is a smooth curve in Σ_δ from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\frac{\pi}{2} < \theta < \frac{\pi}{2}+\delta$ the integral(1.7) converges for $t < 0$ in the uniform operator topologie.

1.4 DIFFERENTIABILITY

[9]:
Let $T(t)$ be a C_0semigroup on $X, T(t)$ is called differentiable for $t > t_0$, if for every $x \in X$, $t \longrightarrow T(t)x$ is differentiable for $t > t_0$. **[9]**:
Let $T(t)$ be a C_0 semigroup and let A be its inifinitesimal generator. If $\|T(t)\| \leq Me^{wt}$ then the following two assertions are equivalent :

1. There exists a $t_0 > 0$ such that $T(t)$ is differentiable for $t > t_0$
2. There exist real contants a, b and c such that $b > 0, c > 0$

$$\rho(A) \supset \Sigma = \{\lambda : Re\lambda \geq a - log \mid Im\lambda \mid\} \tag{1.8}$$

and

$$\|R(\lambda; A)\| \leq\mid Im\lambda \mid \quad \text{for} \qquad \lambda \in \Sigma, Re\lambda \leq w \tag{1.9}$$

1.5 GREEN FORMULA

:
Let Ω be a open bounded with regularly boundary; $\frac{\partial}{\partial\eta}$ the outward normal derivative let u, v are tow function such that $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ then we have

$$\int_\Omega \Delta uv = \int_{\partial\Omega} \frac{\partial u}{\partial\eta} vdx - \int_\Omega \nabla u \nabla vdx$$

1.6 LYAPUNOV FUNCTION

:
We say that a function L is lyapunov functional if $L : [0, \infty) \longrightarrow [0, \infty)$ is continuously differentiable function such that

$$\frac{d}{dt}L(t) \leq -sL(t) + r \quad s, r \geq 0$$

1.7 MAXIMUM PRINCIPALE

We use the technique of maximum principales to show a positivity and boundedness of solution

1.7.1 The Signal of solution:

We defined the function

$$u^+ = sup(u,0) = \begin{cases} u & \text{if} \quad 0 \leq u \\ 0 & \text{if} \quad 0 \geq u \end{cases}$$

$$u^- = sup(-u,0) = \begin{cases} -u & \text{if} \quad 0 \geq u \\ 0 & \text{if} \quad 0 \leq u \end{cases}$$

We have

$$\begin{aligned} u^+ \geq 0, u^- &\geq 0 \\ u =& u^+ - u^-, \\ |u| =& u^+ + u^-. \\ u^+.u^- &= 0; \\ u \geq 0 \Leftrightarrow u^- &= 0 \end{aligned}$$

[10]:
Let the following initial value problem :

$$\begin{cases} \frac{dw}{dt} - a\Delta w = -u\psi(v) \\ \frac{dz}{dt} - d\Delta z = (\dfrac{c}{a-b} + 1)u\psi(v) \end{cases} \tag{1.10}$$

supplement with the boundary and initial conditions

$$\frac{\partial w}{\partial \eta} = \frac{\partial z}{\partial \eta} = 0 \quad \text{sur} \qquad \mathbb{R}^+ \times \Omega$$

$$w(0;x) = w_0(x) \qquad z(0,x) = z_0(x) \qquad \text{sur} \quad \Omega$$

Where Δ stands for the Laplacian operator in $\mathbb{R}^n$, Ω is a regular bounded domain in $\mathbb{R}^n$with smooth boundary $\Gamma = \partial\Omega$, a, b, c is the diffusion constant, $a > 0, d > 0$, $c \geq 0$ et $c^2 > 4ad, a > d$. $\psi(v)$ function class $c^1(\mathbb{R} \times \mathbb{R}, \mathbb{R}^+)$ verifying

$$\lim_{v \to \infty} \frac{log(1 + u\psi(v))}{v} = 0$$

Multiplying the first equations by $w^-$and integrating over Ω, we obtain

$$\frac{d}{dt} \int_\Omega ww^- dx = a \int_\Omega \Delta ww^- dx - \int_\Omega ww^- \psi(v) dx$$

$$(w^+ - w^-)w^- = -w^{-2}$$

Applying the Green formula

$$\int_\Omega \Delta ww^- dx = -\int_\Omega \nabla w \nabla w^- dx = \int_\Omega |\nabla w^-|^2$$

Then

$$\frac{1}{2}\frac{d}{dt} \int_\Omega w^{-2} = -a \int_\Omega |\nabla w^-|^2 dx + \int_\Omega ww^- \psi(v) dx \leq C(T) \int_\Omega (w^-)^2 dx$$

We use the first Gronwall formula we get
$\int_\Omega w^{-2} dx = 0 \Longrightarrow \|w^-\|_{L^2(\Omega)} = 0 \Longrightarrow w^- = 0 \Longrightarrow w \geq 0 \quad \text{on} \quad (0,T) \times \Omega$

1.7.2 The Boundedness of solution

In this case we use the function $(u-M)^+$ because $u(t,x)$ is bounded function meaning there exist a positive constant $M>0$ such that $u(t,x)<M$ for all $(t,x)\in[0,T]\times\Omega$ equivalent to $u(t,x)-M\leq 0$ equivalent to $(u-M)^+=0$ **[5]**:
Let the reaction-diffusion system of from:

$$\begin{cases} u_t-d_1\Delta u=f(u,v) & \text{on } \mathbb{Q}\\ v_t-d_2\Delta v=g(u;v) & \text{on } \mathbb{Q}\\ u(0,.)=u_0(.)\geq 0; v(0,.)=v_0(.)\geq 0 & \\ u,v\text{satisfy some good boundary conditions on } \partial\Omega & \end{cases}$$

Where $\mathbb{Q}\times\Omega,\Omega$ is a regular bounded opedn subset of $\mathbb{R}^N$,$d_1,d_2>0$ and $f,g$are regular function
With for good bondary conditions on $\partial\Omega$ like for instance $u=v=0$ or $\partial_n u=\partial_n v=0$
We have
$F=f+g$
$(u-M)^+=sup(u-M,0)$
Mutiply the first equations by $(u-M)^+$

$$\frac{\partial u}{\partial t}(u-M)^+-d\Delta u(u-M)^+=F(u-M)^+$$

Integration over Ωwe get

$$\int_\Omega (u-M)^+dx-d\int_\Omega \Delta w(u-M)^+dx=\int_\Omega F(u-M)^+dx$$

$$\frac{1}{2}\int_\Omega 2\frac{\partial u}{\partial t}(u-M)^+dx=\frac{1}{2}\int_\Omega\frac{\partial (u-M)^{+^2}}{\partial t}dx$$

Applying the Green formula

$$\int_\Omega \Delta u(u-M)^+dx=\int_{\partial\Omega}\frac{\partial u}{\partial\eta}(u-M)^+ds-\int_\Omega\nabla u\nabla(u-M)^+dx=-\int_\Omega|\nabla(u-M)^+|^2dx$$

Where

$$\nabla u=\nabla(u-M)^+$$

Then

$$\frac{1}{2}\frac{\partial}{\partial t}\int_\Omega (u-M)^{+^2}dx+d\int_\Omega|\nabla(u-M)^+|^2dx=\int_\Omega F(u-M)^+dx\leq 0$$

By integration in $[0,t]$

$$\frac{1}{2}\int_\Omega(w-M)^{+^2}dx-\frac{1}{2}\int_\Omega(u_0-M)^{+^2}+d\int_0^t\int_\Omega|\nabla(u-M)^+|^2dxds=\int_0^t\int_\Omega F(u-M)^+dxds\leq 0$$

$$\frac{1}{2}\int_\Omega(u-M)^{+^2}dx+d\int_0^t\int_\Omega|\nabla(u-M)^+|^2dxds\leq 0$$

$\Rightarrow$

$$\int_\Omega(u-M)^{+^2}dx\leq\int_\Omega(u_0-M)^{+^2}dx$$

We choossing M such that $(u(t,x)-M)\leq 0$ for all $x\in X$.
Let $M\geq\|u_0\|_\infty\Longrightarrow u(t,x)\leq M$.

1.8 EIGENVALUES OF THE LAPLACIAN

We consider the following general eigenvalue problem for the laplacian.

$$\begin{cases} -\Delta v = \lambda v & x \in \Omega \\ v & \text{satisfies symmetric BCs } x \in \partial\Omega \end{cases}$$

To say that the boundary conditions are symmetric for an open, bounded set $\Omega \in \mathbf{R}^n$ means that

$$\langle u, \Delta v \rangle = \langle \Delta u, v \rangle$$

For all functions u and v which satisfy the boundary conditions, where $\langle ., . \rangle$denotes the L^2inner product on Ω ; that is, for any real-valued function $f$and g on Ω,

$$\langle f, g \rangle = \int_{\Omega} f(x)g(x)dx$$

We not that this definition is equivalent to the definition given earlier for all case when Ω is an interval in $\mathbf{R}$.
The most common symmetric boundary conditions are the follwing**[3]**:

1. Dirichlet :$v = 0$.
2. Neumann :$\frac{\partial v}{\partial \nu} = 0$.
3. Robin : $\frac{\partial v}{\partial \nu} + a(x)v = 0$.

1.8.1 Application to heat equation

[3] :
Heat equation on abounded domain $\Omega \in \mathbb{R}^n$,

$$\begin{cases} u_t = K\Delta u & x \in \Omega, t > 0 \\ u(x;0) = \phi(x) & \\ u(0;t) = 0 & x \in \partial\Omega, t \geq 0 \end{cases} \tag{1.11}$$

Using separation of variable, we look for a solution of the form $u(x,t) = v(x)T(t)$, which leads to the following eigenvalue problem

$$\begin{cases} -\Delta v = \lambda v & x \in \Omega \\ v = 0 & x \in \partial\Omega \end{cases}$$

1.8.2 Facts on eigenvalues

[3]:
For any of the boundary conditions listed abouve,

1. All eigenvalues are real.
2. All eigenfunction can be chosen to be real-valued.
3. Eigenfunctions corresponding to distinct eigenvalues are orthogonal.

[3]:
For the eigenvalue problem above,

1. All eigenvalues are positive in the Dirichlet case.
2. All eigenvalues are zero or positive in The Neumann case and the Robin if $a \geq 0$.

:
We prove this result for the Dirichlet case.The another proofs can be handled similarly. Let v be an eigenfunction with corresponding eigenvalue λ. Then

$$\begin{aligned}\lambda \int_\Omega v^2 dx &= -\int_\Omega (\Delta v) v dx \\ &= \int_\Omega |\nabla v|^2 dx - \int_{\partial\Omega} v \frac{\partial v}{\partial \nu} ds(x) \\ &= \int_\Omega |\nabla v|^2 dx\end{aligned}$$

Therefore

$$\lambda \int_\Omega v^2 dx = \int_\Omega |\nabla \lambda|^2 dx \geq 0$$

Further , we claim that

$$\int_\Omega |\nabla v|^2 dx \geq 0$$

. We prove this claim as follows. Suppose $\int_\Omega |\nabla v|^2 = 0$, then $|\nabla v|$with implies v on $\partial\Omega$. Therefore, if v is constant on Ω and $v = 0$ on $\partial\Omega$, then $v = 0$. Howerover, the zero function is not an eigenfunction.There fore

$$\lambda \int_\Omega v^2 dx = \int_\Omega |\nabla v|^2 dx > 0, \text{ wich implies } \lambda > 0$$

1.9 FRACTIONAL CALCULUS

One of the essential Knowledege for the study that follows, is notation of fraction calculs. Therefore is concerned with the study of some concepts and results. We start studying with the usal notations and the basic definitions.
It is also important to understand that in this chapter we will study two fractional differential operators (the Riemann-Liouville and the caputo).

1.9.1 The Gamma function

This transcendental function function, represent by $\Gamma(z)$ has caught the interest of some of the most prominent mathematicians of all time. On 19th century, that rewrote Euler's results as an infinite product, that allowed him to discover new properties of the gamma a function, been the first to consider complex variables, this formulation was

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!nz}{z(1+z)(2+z).....(n+z)} \quad forall \quad z \in \mathbb{C}\backslash 0,-1,-2,....$$

[4]-[8]:
The Gamma function , denoted by $\Gamma(z)$, is a generalization of fractional function $n!$ i.e., $\Gamma(z) = (n-1)!$ for $n \in \mathbf{N}$. For complex arguments with positive real part it is defined as

$$\Gamma(z) = \int_0^\infty t^{z-t}e^{-t}dt, \quad Rez > 0.$$

By analytic continuation the function is extended to the whole complex plane except for simple poles. Thus $\Gamma : \mathbf{C}\ 0,-1,-2,... \longrightarrow \mathbf{C}$.
Somme of the most important properties are

$$\begin{aligned} \Gamma(1) &= \Gamma(2) = 1, \\ \Gamma(z+1) &= z\Gamma(z), \\ \Gamma(n) &= (n-1)!, n \in \mathbf{N}, \\ \Gamma(1/2) &= \sqrt{\pi}, \\ \Gamma(n+1/2) &= \frac{\sqrt{\pi}}{2^n}(2n-1)!n \in \mathbf{N}. \end{aligned} \tag{1.12}$$

1.9.2 The Mittag-Leffler Function

Motivated essentially by the success of the applications of the Mittag-Leffler Function in many areas of science and engineering,we discuss this subject in a brief survey of their interesting and useful priperties. During the last two decades this functions has come into prominence after about nine decade of its discovery by the Swedish mathematician Magnus Gustaf (Gosta)Mittag-Leffler(1846-1927) **[4]**:

While the Gamma function is a generalization of the factorial function, the Mittag-leffler function is a generalization of the expontial function , first introduced as a one-parameter function by the series

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+1)} \qquad , \alpha > 0, \alpha \in \mathbb{R}, z \in \mathbb{C}.$$

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \qquad , \alpha, \beta > 0, \beta \in \mathbb{R}, z \in \mathbb{C}.$$

Which is of great important for fractional calculus,it is called two-parameter function of Mittag-leffler type .
Some of its interesting properties

$$\begin{aligned} E_{1,1}(z) =& e^z, \\ E_{2,1}(z^2) =& cosh(z), \\ E_{2,2}(z^2) =& \frac{sinh(z)}{z}, \\ E_{\alpha,1}(z) =& E_\alpha(z), \\ E_{1/2,1}(z) =& e^{z^2} erfc(-2) \end{aligned}$$

Where erfc(z)is the complementary error function. :
There are many important functions related to the general Mittag-Leffler function . **[4]**:

$$E_{1,2} = \frac{e^z - 1}{z}$$

$$\begin{aligned} E_{1,2} =& \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} \\ =& \sum_{k=0}^{\infty} \frac{z^k}{(k+1)!} \\ =& \frac{1}{z} \sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} \\ =& \frac{1}{z}(e^z - 1) \end{aligned}$$

1.9.3 The writte-type function

Denot by $\phi_\beta(z), \beta \in (0,1)$ the following function of the wright type , also called The Wright M-function or Minardi function

$$\phi_\beta(\theta) = \sum_{k=0}^{+\infty} \frac{(-\theta)^k}{k!\Gamma(-\beta k + 1 - \beta)}.$$

In particular , this indentity implies that $\phi_\beta(t)$is a probability density function :

$$\phi_\beta(t) \geq 0, t > 0; \int_0^\infty \phi(\beta)(t)dt = 1$$

with $0 < \beta < 1$. For $-1 < r < \infty$,the following result hold**[13]**

1. $\phi_\beta(\theta) \geq 0, \theta > 0$
2. $\int_0^\infty (\frac{\beta}{\theta^{\beta+1}})e^{-\lambda\theta}d\theta = e^{-\lambda^\beta}$
3. $\int_0^\infty \phi_\beta(\theta)\theta^r d\theta = \dfrac{\Gamma(1+r)}{\Gamma(1+\beta r)}$
4. $\int_0^\infty \phi_\beta(\theta)e^{-z\theta}d\theta = E_\alpha(-z), \quad z \in \mathbb{C}$
5. $\int_0^\infty \beta\theta\phi_\beta(\theta)e^{-z\theta}d\theta = E_{\alpha,\beta}(-z), \quad z \in \mathbb{C}$

1.9.4 Fractional integration and derivation

Our intial goal in this section is to introduce an extension of the operations of integration and differentions to the case of fractional powers. This area, wasgiven by Caputo-Riemann-Liouville. **[1]**:
For an integrable function f, the Riemann-Liouville integration of order $\beta \in (0,1)$ is defined by

$$J_t^\beta f(t) := \frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}f(s)ds, \quad t > 0, \tag{1.13}$$

Let $J^\beta = J_t^\beta$
by convention $J^0 f(t) = f(t)$ i.e.,$J^0 := I$is the idendity operation
Semigroup perperties $J^\alpha J^\beta = J^{\beta+\beta}$.
Commutative property $J^\beta J^\alpha = J^\alpha J^\beta$.
Effect on powerfunctions

$$J^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1+\alpha)}t^{\gamma+\alpha}, \qquad \alpha > 0, \gamma > -1, t > 0.$$

(Natural generalization of the positive integer properties)
Another property is linearity

$$J^\beta(\lambda f(t) + g(t)) = \lambda J^\beta f(t) + J^\beta g(t), \qquad \alpha \in \mathbb{R}_+, \lambda \in \mathbb{C}.$$

[1]:
For an absolututely continous function $f$the Caputo fractional derivative of order $\beta \in (0,1)$ is

$$^cD_t^\beta f(t) := D_t^\beta(f(t) - f(0)), \quad t > 0. \tag{1.14}$$

Where D_t^β is the Riemann-Liouville fractional derivative of order β given by

$$D_t^\beta f(t) := \frac{d}{dt} J_t^{1-\beta} f(t). \tag{1.15}$$

In particular, if $f(0) = 0$ we have

$${}^c D_t^\beta f(t) = D_t^\beta f(t), \qquad t > 0 \tag{1.16}$$

Where $D_t^m = \frac{d^m}{dt^m}$, $D^\alpha J^\alpha = I$. By convention it is defined

$$D^0 f(t) = f(t), \text{i.e.;} D^0 := I$$

[1]:
It holds

$$J_t^\beta \quad {}^c D_t^\beta f(t) = f(t) - f(0), \quad t > 0 \tag{1.17}$$

and

$${}^c D_t^\beta J_t^\beta f(t) = f(t), \qquad t > 0 \tag{1.18}$$

:

1. observe that if $F(t) = f(t) - f(0)$

$$J_t^{\beta c} D_t^\beta f(t) = J_t^\beta D_t^\beta F(t) = F(t) - \frac{1}{\Gamma(\beta)} t^{1-\beta} (J_s^\beta F(s))|_{s=0} = f(t) - f(0)$$

2. ${}^c D_t^\beta J_t^\beta f(t) = D_t^\beta (J_t^\beta f(t) - J_s^\beta f(s)|_{s=0}) = D_t^\beta J_t^\beta f(t) = f(t)$

[1]:
We denote by A the realization of $-\Delta$ with homogeneous Neumann condictions in $L^2(\Omega)$. Let $0 = \lambda_0 \leq \lambda_1 \leq \lambda_2$,be the eigenvalues of A and let $\varphi_{n\{n \geq 0\}}$ be the orthonormal eigenfunction system corresponding to $\{\lambda_n\}_{n \geq 0}$; $A\varphi_n = \lambda \varphi_n$ and

$$D(A) = \left\{ u \in L^2(\Omega) / \frac{\partial u}{\partial \eta}; \|Au\|^2_{L^2(\Omega)} = \sum_{k=1}^{+\infty} |\lambda_k (u, \varphi_k)|^2 < +\infty \right\}$$

1.10 The Banach fixed point theorem

:
Let K be a nonempty closed subset of a Banach space $(X, \|.\|_X)$. Assume that $T : K \to K$ is a contraction, i.e., there exists a constant $\alpha \in [0, 1)$ such that

$$\|Tu - Tv\|_X \leq \alpha \|u - v\|_X \;\; \forall u, v \in k$$

Then there exists a unique $u \in K$such that $Tu = u$. A solution $u \in K$ of the operator equation $Tu = u$ is called a fixed point of T in K.

1.11 INTEGRAL DEPENDENT ON PARAMATER

[7]:
Let $f : I \times E \longrightarrow \mathbb{R}$ for all $t \in I, f(t,x) : (E, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$is mesurable. Then

1. For $t_0 \in I$, for μ-pp$x$, $t \longrightarrow f(t,x)$ is continuous an t_0
   and there exist $g : (E, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ integrable
   for all $t \in I$, $|f(t,x)| \leq g(x)$, then $h : t \longrightarrow \int_E f(t,x)d\mu(x)$ is continuons an t_0
2. if for all $t \in I$ is integrable,if
   for μ-pp$x$, $t \longrightarrow f(t,x)$ is derivable on all interval I,
   and exist $g_1; (E, \mathcal{A}) \longrightarrow (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ integrable
   for μ-pp$x$, for all $t \in I$, $|\dfrac{\partial f}{\partial t}(t,x)| \leq g_1(x)$,
   Then h is derivable on all I,

$$h^{'} = \int_E \frac{\partial f}{\partial t}(t,x)d\mu(x), \quad t \in I.$$

1.12 COMPARISON THEOREM

[11]:
We consider first nonlinear parabolic equations :

$$Pu = f(x,t,u) \qquad (x,t) \in \Omega) \times [0,T]$$

Where $Pu = u_t - Au$ and $f$is continuously differentiable in u,and Holder continuous in $x$and t.
$\Omega$is a boundary domain $\mathbb{R}^n$. Let u and $v$each be $\mathbb{C}^2$ function of $x$in $\Omega, \mathbb{C}^1$functions of t on $[0,T]$, and consider the following conditions:

$$\begin{cases} Pu - f(t,x,u) \leq Pv - f(t,x,u), \\ u(0,x) \leq v(0,x) \\ Bu \mid_{\partial\Omega} = Bv \mid_{\partial\Omega} = 0 \end{cases} \tag{1.19}$$

B:Dirchle or Neumann condtion.
under the above conditions on P and f, if (1.19)hold, then $u(x,t) \leq v(x,t)$
for all $(x,t) \in \Omega \times [0,T]$.

CHAPTER 2

LOCAL EXISTENCE

2.1 ANALYTIC SEMIGROUPS

[9]:
Let $\Delta = \{z : \varphi_1 < argz < \varphi_2, \varphi_1 < 0 < \varphi_2\}$ and for $z \in \Delta$ let $T(z)$ be a bounded linear operator. The family $T(t), z \in \Delta$ is an analytic semigroup in Δ if

1. $z \to T(z)$ is analytic in Δ
2. $T(0) = I$ and $\lim_{z \to 0} T(z)x = x$ for every $x \in X$.
3. $T(z_1 + z_2) = T(z_1)T(z_2)$ for $z_1, z_2 \in \Delta$

A semigroup $T(z)$ will be called **analytic** if it is analytic in some sector Δ containing the nononegative real axis.

2.1.1 Main theorem

[9]:
Let $T(t)$ be a uniformly bounded C_0semigroup. Let A be the infinitesimal generator of $T(t)$ and assume $0 \in \rho(A)$. The following statement are equivalent:

1. $T(t)$ can be extended to an analytic semigroup in a sector $\Delta_\delta = \{z :\mid argz \mid < \delta\}$ and $\parallel T(z) \parallel$ is uniformly bounded in every closed subsector $\bar{\Delta}_{\delta}, \delta' < \delta,$ of Δ_δ.

2. There exist a constant C such that for every $\sigma > 0, \tau \neq 0$

$$\parallel R(\sigma + i\tau : A) \parallel \leq \frac{C}{\mid \tau \mid}. \tag{2.1}$$

3. There exist $0 < \delta < \pi/2$ and $M > 0$ such that

$$\rho(A) \supset \Sigma = \{\lambda : |arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}. \tag{2.2}$$

$$\|R(\lambda; A)\| \leq \frac{M}{\lambda} \quad for \quad \lambda \in \Sigma, \lambda \neq 0. \tag{2.3}$$

4. $T(t)$ is differntiable for $t > 0$ and there is a constant C such that

$$\|AT(t)\| \leq \frac{C}{t} \quad for \quad t > 0 \tag{2.4}$$

:
Let A be a densily operator in X satisfying

$$\rho(A) \supset \Sigma = \{\lambda : |arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\} \quad 0 < \delta < \pi/2 \quad and \quad M > 0$$

$$\|R(\lambda; A)\| \leq \frac{M}{\lambda} \quad for \quad \lambda \in \Sigma; \lambda \neq 0.$$

Can be extended to an analytic semigroup.

2.1.2 Characterisation of analytic semigroup

[9]:
Let A be the infinitesimal generator of a C_0semigroup $T(t)$ satisfing $\|T(t)\| \leq Me^{wt}$, then $T(t)$is analytic if and only if there are constant $C > 0$ and $\Lambda \geq 0$ such that

$$\parallel AR(\lambda; A)^{n+1} \parallel \leq \frac{C}{n\lambda^n} \quad for \quad \lambda > n\Lambda, \quad n = 1, 2, ...$$

:
We not first that from the theorem it folows easily that $T(t)$ is analytic if and only if it is differentiable for $t > 0$ and there are constants $C_1 > 0$and w_1 such that

$$\|AT(t)\| < \frac{C_1}{t} e^{w_1 t}, \quad for \quad t > 0.$$

$\Rightarrow$ For $\lambda > n\Lambda$ and $x \in D(A)$ we have

$$\|AR(\lambda : A)^{n+1} x\| = \|R(\lambda : A)^{n+1} Ax\| \leq \frac{c}{n\lambda^n} \|x\|,$$

Choosing $t < 1/\Lambda$ and substituting $\lambda = n/t$ we find

$$\|A(\frac{n}{t} R(\frac{n}{t} : A))^{n+1} x\| = \|(\frac{n}{t} R(\frac{n}{t} : A))^{n+1} Ax\| < \frac{c}{t} \|x\|,$$

Letting $n \longrightarrow \infty$and closedness of A

$$\|AT(t)x\| \leq \frac{c}{t}\|x\|\text{for} \quad x \in D(A) \quad 0 < t < 1/\Lambda$$

Since $D(A)$is dense in X and $AT(t)$is closed then

$$\|AT(t)x\| \leq \frac{c}{t}\|x\|, \quad \text{for every} \qquad x \in X.$$

$\Leftarrow$ We have

$$R(\lambda : A) = \int_0^\infty e^{-\lambda t}T(t)xdt$$

n times with respect to λ and find

$$\begin{aligned} R(\lambda : A)^n x =&(-1)^n n!(\lambda : A)^{n+1}x \\ =&(-1)^n \int_0^\infty e^{-\lambda t}T(t)xdt \end{aligned}$$

We have

$$\begin{aligned} n!\|AR(\lambda : A)^{n+1}\| \leq& C_1 \left(\int_0^\infty t^{n-1}e^{-(\lambda - w_1)t}T(t)x\right)\|x\| \\ =&\frac{C_1}{(\lambda - w_1)^n}(n-1)!\|x\| \end{aligned}$$

And therefore, for $\lambda > n\Lambda$

$$\|AR(\lambda : A)^{n+1}\| \leq \frac{c}{n\lambda^n}(\frac{1}{1 - \frac{w_1}{n\Lambda}}^n) \leq \frac{C_2}{n\lambda^n}.$$

[9]:

For a uniformly bounded C_0semigroup $T(t)$. The following statements are equivalent:

1. $T(t)$is analytic in a sector around the nom negative real axis .
2. For every complex $\xi, \xi \neq 1, |\xi| \geq 1$ there exist positive constants δ and K such that $\delta \in \rho(T(t))$ and $\|(\xi I - T(t))^{-1}\| \leq K$ and positive constants δ and K such that for $0 < t < \delta$
3. There exist a complex number $\xi, |\xi| = 1$ and a positive constants δ and K such that
$$\|(\xi I - T(t))\| \leq K\|x\| \quad \text{for every} \quad x \in X, \quad 0 < t < \delta.$$

[9]:

Let $T(t)$ be a C_0semigroup. If

$$\lim_{t \to 0} sup\|I - T(t)\| < 2.$$

Then $T(t)$is an analytic in a sector aroud the nonnegative real axis. It is follos that there exist $\delta > 0$ and $\varepsilon > 0$ such that

$$\|I - T(t)\| \leq 2 - \varepsilon, \qquad 0 < t < \delta,$$

But then

$$\|(I - T(t))x\| = \|(-I + I + I - T(t))x\| \geq 2\|x\|\|(I - T(t))x\| \geq \varepsilon\|x\|.$$

This implies by theorem (2.1.2) with $\varepsilon = -1$ that $T(t)$ is analytic.

2.2 Fractional power of closed operator

Assumption(*):
Let A be a densly defined closed linear operator for with

$$\rho(A) \supset \Sigma^{+} = \{\lambda : 0 < w < |arg\lambda| \leq \pi\} \cup \{V\} \tag{2.5}$$

where V is neighbourhood of zero and

$$\|R(\lambda, A)\| \leq \frac{M}{1+|\lambda|} \qquad for \quad \lambda \in \Sigma^{+}, \tag{2.6}$$

If $M = 1$ and $w = \pi/2$ then by theorem $-A$ is infinitesimal generator of C_0semigroup, if $w < \pi/2$ then $-A$ is the infinitesimal generator of an analytic semigroup. The assumption that $0 \in \rho(A)$ and there for whole neighbourhood $V$of zero is in $\rho(A)$ was made result on fractional powers that we will obtain in this section remain true even $0 \notin \rho(A)$ from an operation satisfying assumption and $\alpha > 0$ we define

$$A^{-\alpha} = \frac{1}{2\pi i}\int_C z^{-\alpha}(A - zI)dz, \tag{2.7}$$

where the path C runs in the resolvent set of A from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$,
$w < \theta < \pi$ for $1 > \alpha > 0$, we can deform the path of integeration C in to the upper and lower sides of the negative real axis and obtain

$$A^{-\alpha} = \frac{sin\pi\alpha}{\pi}\int_0^{\infty} t^{-\alpha}(A - tI)^{-1}dt, \qquad 0 < \alpha < 1. \tag{2.8}$$

[9]:

1. For $\alpha, \beta \geq 0$
$$A^{-(\alpha+\beta)} = A^{-\alpha}A^{-\beta},$$
2. There exist a constant c such that :
$$\|A^{-\alpha}\| < c \quad \text{for} \quad 0 \leq \alpha \leq 1,$$

[9]:
Let A satisfy assumption with $w < \pi/2$. For every $\alpha > 0$ we define

$$A^{\alpha} = (A^{-\alpha})^{-1} \qquad \text{for} \quad \alpha = 0, \quad A^{\alpha} = I.$$

[9]:
Let A^{α} be defined by definition then:

1. A^{α} is closed operator with domain $D(A) = R(A^{-\alpha})$ the rang of $A^{-\alpha}$.

2. $0 < \alpha \leq \beta$ implies $D(A^{\alpha}) \subset D(A^{\beta})$.
3. $D(A^{\alpha}) = X$ for every $\alpha \geq 0$.
4. if $\alpha, \beta$are real then
$$A^{\alpha+\beta}x = A^{\alpha}.A^{\beta}x,$$
for every $x \in D(A^{\gamma})$ where $\gamma = max(\alpha, \beta, \alpha+\beta)$.

[9]:
Let $0 < \alpha < 1$ if $x \in D(A)$then

$$A^{\alpha}x = \frac{sin\pi\alpha}{\pi}\int_0^{\infty} t^{\alpha-1}(tI + A)^{-1}xdt.$$

[9]:
Let $-A$ be the infinitesimal generator of an analytic semigroup $T(A)$. If $0 \in \rho(A)$then :

1. $T(t) : X \longrightarrow D(A)$ for every $t > 0$ and $\alpha > 0$
2. For every $x \in D(A^{\alpha})$ we have $T(t)A^{\alpha}x = A^{\alpha}T(t)x$.
3. For every $t > 0$ the operator$A^{\alpha}T(t)$ is bounded and $\|AT(t)\| \leq M_{\alpha}t^{-\alpha}e^{-\delta t}$.
4. Let $0 < \alpha \leq 1$ and $x \in D(A^{\alpha})$ then $\|T(t)x - x\| \leq c_{\alpha}t^{-\alpha}\|A^{\alpha}x\|$.

:
Our assumption on A imply that it satisfies Assumption (*)with $w < \pi/2$ and therefore we have the existence of A for $\alpha \geq 0$ since $T(t)x$ is analytic we have $T(t)x$:

$$X \longrightarrow \cap_{n=0}^{\infty} D(A^{n}) \subset D(A^{\alpha}) \quad \text{for every} \quad \alpha \geq 0$$

Which proves (1).
Let $x \in D(A^{\alpha})$ then $x = A^{-\alpha}y$ for some $y \in Y$and

$$\begin{aligned} T(t)x =& T(t)A^{-\alpha}y \\ =& \frac{1}{\Gamma(\alpha)}\int_0^{\infty} s^{\alpha-1}T(s)T(t)xds \\ =& A^{-\alpha}T(t)y = A^{-\alpha}T(t)A^{\alpha}x. \end{aligned}$$

and (2) follows.
Since $A^{-\alpha}$ is closed so is $A^{\alpha}T(t)$. By part $A^{\alpha}T(t)$ is every where defined and therefore by the closed graph theorem $A^{\alpha}T(t)$ is bounded. Let $n-1 < \alpha < M$ then using

$$\|A^{m}T(t)\| \leq M_{n}t^{-m}e^{-\delta t},$$

we have

$$\begin{aligned}\|A^{\alpha}T(t)\| =&\|A^{\alpha-n}A^{n}T(t)\|\\ \leq&\frac{1}{\Gamma(n-\alpha)}\int_{0}^{\infty}s^{n-\alpha-1}T(t+s)\|ds\\ \leq&\frac{M_n}{\Gamma(n-\alpha)}\int_{0}^{\infty}s^{n-\alpha-1}(t+s)^{-n}e^{-\delta(t+s)}ds\\ \leq&\frac{M_n e^{-\delta t}}{\Gamma(n-\alpha)t^{n}}\int_{0}^{\infty}u^{n-\alpha-1}(1+u)^{-n}du=\frac{M_{\alpha}}{t^{\alpha}}e^{-\delta t},\end{aligned}$$

Finally

$$\begin{aligned}\|T(t)x-x\| =&\|\int_{0}^{t}AT(s)xds\|\\ =&\|A^{1-\alpha}T(s)A^{\alpha}xds\|\\ \leq&C\int_{0}^{t}s^{\alpha-1}\|A^{\alpha}x\|ds\\ =&C_{\alpha}t^{\alpha}\|A^{\alpha}x\|.\end{aligned}$$

2.3 The Homongeneous Initial value problem

We consider the homongeneous initial value problem

$$\begin{cases}\dfrac{du(t)}{du}=Au(t) & ,t\geq 0,\\ u(0)=x.\end{cases}\tag{2.9}$$

[9]:
An X-valued function $U(t)$ is called a solution of above problem if: $U(t)$ is continuous and continuously differentiable for $t\geqslant 0, U(t)\in D(A)$ for $t>0$ and (2.9) is satisfied.

If A is the infinitesimal generator of a C_0semigroup $T(t)$, then (2.9) has a solution $U(t)=T(t)x$ for every $x\in D(A)$. **[9]**:
Let A be a densly defined linear operator, If $R(\lambda;A)$ exist for all real $\lambda\geq\lambda_0$ and

$$\lim_{\lambda\to\infty} sup\lambda^{-1}log\|R(\lambda;A)\|=0.$$

Then the initial value problem (2.9) has at most one solution for every $x\in X$.
[9] **[9]**:
Let A be a densly defined linear operator with a nonempty resolvent set $\rho(A)$. The initial value problem (2.9) has a unique solution $u(t)$, which is continuously differentiale on $[0,\infty[$ for every initial value $x\in D(A)$ if and anly if A the infinitesimal generator of a $c_0$semigroup $T(t)$. **[9]** **[9]**:
If A is the infinitesimal generator of a differentiable semigroup then for every $x\in X$ the

initial value problem (2.9) has a unique solution . **[9]** **[9]**:
If A is the infinitesimal generator of an analytic semigroup then for every $x \in X$ the initial problem(2.9) has unique solution . **[9]**:
If A is the infinitesimal generator of a C_0semigroup which is not differentiable, then in general; if $x \in D(A)$, (2.9) does not have a solution. The function $t \longrightarrow T(t)x$ is called a **mild solution**.

2.4 THE NONHOMOGENEOUS INITIAL VALUE PROBLEM

[9]:
Let $f : [0, \infty[\times X \longrightarrow X$ be continuous in t for $t \geq 0$ and locally lipschitz continuous in u, uniformly in t on bounded intervals if $-A$ is the infinitesimal generator of a C_0semigroup $T(t)$ on X then for every $u_0 \in X$ there is a $t_{max} \leq \infty$ such that the initial value problem

$$\begin{cases} \dfrac{du(t)}{du} + AU(t) = f(t, U(t)) \quad , t \geq 0 \\ U(0) = X. \end{cases} \tag{2.10}$$

has a unique mild solution U on $[0, t_{max}]$, Moreover, if $t_{max} < \infty$ then

$$\lim_{t \to t_{max}} \|U(t)\| = \infty.$$

[9]:
Let A be the infinitesimal generator of a C_0semigroup $T(t)$. Let $x \in X$ and
$f \in L^1(0, T : X)$ given by

$$U(t) = T(t)x + \int_0^t T(t-s)f(s)ds, \quad 0 \leq t \leq T. \tag{2.11}$$

is the mild solution to the initial value problem (2.10) on $[0, T]$
[9]:
A function $u$which is differentiale almost every where on $[0, T]$ such that $u^{'} \in L^1(0, T : X)$ is called a **strong solution** of the initial problem
$u(0) = x$ and $u^{'} = Au(t) + f(t)$ a.e on $[0, T]$.

2.5 SEMILINEAR EQUATION WITH ANALYTIC SEMIGROUP

Let the following value problem

$$\begin{cases} \frac{du(t)}{dt} + Au(t) = f(t, u(t)) \quad , \qquad t > t_0, \\ u(t_0) = x_0. \end{cases} \tag{2.12}$$

Assumption (F)
Let U an open subset of $\mathbb{R}^+ \times X_\alpha$, the function $f : U \longrightarrow X$ satisfies the assumption(F)

if for every $(t,x) \in U$ there is a neighorbood $V \subset U$ and constants $L \geq 0, 0 < \theta < 1$ such that

$$\|f(t_1,x_1) - f(t_2,x_2)\| \leq L\left(\mid t_1 - t_2 \mid^{\theta} + \|x_1 - x_2\|_{\alpha}\right) \tag{2.13}$$

for$(t_1;x_1) \in V, (t_2,x_2) \in V$, where X is a real or complex banach space with norm $\|.\|$, A^{α} is a closed linear, invertible operator with domain $D(A^{\alpha})$ endowed with the graph norm$\|.\|_{\alpha}$ of A^{α} with

$$\|x\|_{\alpha} = (\|x\|^2 + \|A^{\alpha}x\|^2)^{1/2} \qquad x \in D(A),$$

[9]:

Let $-A$ is the infinitesimal generator of an analytic semigroup $T(t)$sayisfing $\|T(t)\| \leq M$ and assume further that $0 \in \rho(A)$ if $f$satisfies the assumption (F) then for every intial data $(t_0,x_0) \in U$ the initial value problem has a unique lacal solution $u \in C([t_0,t_1] : X) \cap C^1([t_0,t_I] : X)$ where $t_I = t_1(t_0,x_0) > t_0$:
From our assumption on the operator A it follows(Theorem (2.2)) that

$$\|A^{\alpha}T(t)\| \leq C_{\alpha}t^{-\alpha} \qquad for \quad t > 0, \tag{2.14}$$

For the rest of the proof, we fix $(t_0,x_0) \in U$ and choose $t_1^{'} > t_0, \delta > 0$ such that the estimate (3.13)with some fixed constant L and ϑ holds in the set
$V = \{(t,x) : t_0 \leq t \leq t_1^{'}, \|x - x_0\|_{\alpha} \leq \delta\}$. Let

$$B = \max_{t_0 \leq t \leq t_1^{'}} \|f(t,x_0)\|, \tag{2.15}$$

and choose t_1 such that

$$\|T(t-t_0)A^{\alpha}x_0 - A^{\alpha}x_0\| < \delta/2 \qquad for \quad t_0 \leq t < t_1, \tag{2.16}$$

and

$$0 < t_1 - t_0 < \min\left\{t_1' - t_0, \left[\frac{\delta}{2}(1-\alpha)C_{\alpha}^{-1}(B + \delta L)^{-1}\right]^{1/1-\alpha}\right\}, \tag{2.17}$$

Let Y be the Banch space $C([t_0,t_1] : X)$ with the usal supremum norm which we denote by $\|.\|_{\gamma}$. On Y we define a maping F by

$$F_y(t) = T(t - t_0A^{\alpha}x_0 + \int_{t_0}^{t} T(t-s)f(s, A^{-\alpha}y(s))ds, \tag{2.18}$$

clearly, $F : Y \longrightarrow Y$ and for every $y \in Y, F_y(t_0) = A^{\alpha}x_0$. Let S be the nonempty closed and bounded subset of Y defined by

$$S = \{y : y \in Y, y(t_0) = A^{\alpha}x_0, \|y(t) - A^{\alpha}x_0\| \leq \delta\}, \tag{2.19}$$

For $y \in S$ we have

$$\begin{aligned}\|Fy(t) - A^{\alpha}x_0\| \leq & \|T(t-t_0)A^{\alpha}x_0 - A^{\alpha}x_0\| \\ & + \|\int_{t_0}^{t} A^{\alpha}T(t-s)[f(s,A^{-\alpha}y(s)) - f(s,x_0)]ds\| \\ & + \|\int_{t_0}^{t} A^{\alpha}T(t-s)f(s,x_0)ds\| \\ \leq & \frac{\delta}{2} + C_{\alpha}(1-\alpha)^{-1}(L\delta + B)(t_1-t_0)^{1-\alpha} \\ = & \frac{\delta}{2} + C_{\alpha}(1-\alpha)^{-1}(L\delta + B)(t_1-t_0)^{1-\alpha} \leq \delta,\end{aligned}$$

Where we used (3.13),(3.14),(2.17)and(3.19). Therfore $F : S \to S$. Further-more, if $y_1, y_2 \in S$ then

$$\begin{aligned}& \|Fy_1(t) - Fy_2(t)\| \\ & \leq \int_{t_0}^{t} \|A^{\alpha}T(t-s)\|\|f(s,A^{-\alpha}y_1(s)) - f(s,A^{-\alpha}y_2(s))\|ds \\ & \leq LC_{\alpha}(1-\alpha)^{-1}(t_1-t_0)^{1-\alpha}\|y_1-y_2\|_{\gamma} \leq \frac{1}{2}\|y_1-y_2\|_{\gamma},\end{aligned}$$

Which implies

$$\|Fy_1(t) - Fy_2(t)\|_{\gamma} \leq \frac{1}{2}\|y_1 - y_2\|, \tag{2.20}$$

By the contraction mapping theorem the maping F has a unique fixed point $y \in S$. This fixed point satisfies the integral equation

$$y(t) = T(t-t_0)A^{\alpha}x_0 + \int_{t_0}^{t} A^{\alpha}T(t-s)f(s,A^{-\alpha}y(s))ds \qquad \text{for} \quad t_0 \leq t \leq t_1 x \tag{2.21}$$

From (3.13) and the continuity of y it follows that $t \to f(t, A^{-\alpha}y(t))$ is continuous on $[t_0, t_1]$and a fortiori bounded on this interval. Let

$$\|f(t, A^{-\alpha}y(t))\| \leq N, \qquad \text{for} \quad t_0 \leq t \leq t_1 \tag{2.22}$$

Next we want to show that $t \to f(t, A^{-\alpha}y(t))$ is lacally Holder continuous on $[t_0, t_1]$. To this end we show first that the solution y of (3.20) is lacally Holder continious on $]t_0, t_1]$. We note that for every β satisfying $0 < \beta < 1 - \alpha$ and every $0 < h < 1$ we have by Theorem(2.2)

$$\|(T(h) - I)A^{\alpha}T(t-s)\| \leq C_{\beta}h^{\beta}\|A^{\alpha+\beta}T(t-s)\| \leq Ch^{\beta}(t-s)^{-(\alpha+\beta)} \tag{2.23}$$

If $t_0 < t < t+h \leq t_1$; then

$$\begin{aligned}\|y(t+h) - y(t)\| \leq & \|(T(h) - I)A^{\alpha}T(t - t_0x_0)\| \\ & + \int_{t_0}^{t} \|(T(h) - I)A^{\alpha}T(t-s)f(s,A^{-\alpha}y(s))\|ds \\ & + \int_{t}^{t+h} \|A^{\alpha}T(t+h-s)f(s,A^{-\alpha}y(s)\|ds \\ = & I_1 + I_2 + I_3\end{aligned} \tag{2.24}$$

Using (3.22) and (2.23) we estimate each of the terms of (2.24) separately.

$$I_1 \le C(t-t_0)^{-(\alpha+\beta)} h^\beta \|x_0\| \le M_1 h^\beta \tag{2.25}$$

$$I_2 \le CNh^\beta \int_{t_0}^{t} (t-s)^{-(\alpha+\beta)} ds \le M_2 h^\beta \tag{2.26}$$

$$I_3 \le NC_\alpha \int_{t}^{t+h} (t+h-s)^{-\alpha} = \frac{NC_\alpha}{1-\alpha} h^{1-\alpha} \le M_3 h^\beta \tag{2.27}$$

Note that M_2 and M_3 can be chosen to be independent of $t \in [t_0, t_1]$ while M_1depends on t and blows up at $t \downarrow t_0$. Combining (2.24) wtih these estimate it follows that for every $t_0^{'} > t_0$ there is a constant C such that

$$\|y(t) - y(s)\| \le C|t-s|^\beta \quad \text{for} \quad t_0 < t_0^{'} \le t, s \le t_1 \tag{2.28}$$

and therefore y is lacally Holder continuous on $]t_0, t]$. The lacal Holder continuity of $t \to f(t, A^{\alpha y}(t))$ follows now form

$$\|f(s, A^{-\alpha}y(t)) - f(t, A^{-\alpha}y(t))\| \le L(|t-s|^\vartheta + \|y(t)-y(s)\|) \le C_1(|t-s|^\vartheta + |t-s|^\beta). \tag{2.29}$$

Let y be the solution of (2.21) and condider the inhomogeneous initial value problem

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) &= f(t, A^{-\alpha}y(t)), \\ u(t_0) &= x_0. \end{cases} \tag{2.30}$$

This problem has a unique solution $u \in C^1(]t_0; t_1] : X)$. The solution of(2.30) is given by

$$u(t) = T(t-t_0)x_0 + \int_{t_0}^{t} T(t-s) f(s, A^{-\alpha}y(s)) ds, \tag{2.31}$$

For $t > t_0$ each term of (2.31) with A^α we fined

$$u(t)A^\alpha = T(t-t_0)A^\alpha x_0 + \int_{t_0}^{t} A^\alpha T(t-s) f(s, A^{-\alpha}y(s)) ds. \tag{2.32}$$

But by (2.30) the right-hand side of (2.31) equals $y(t)$ and therefore $u(t) = A^{-\alpha}y(t)$ and by (2.31), u is a $C^1(]t_0, t_1] : X)$ solution of (2.10). The uniqueness of u follows redily from the uniqueness of the solutions of (2.30) the proof is complet **[8]**:
Let $A : D(A) \subset X \longrightarrow X$ be a closed and densly defined operator. The operator A is said to be **a sectorial operator** if there exist constant $a \in \mathbb{R}, N \ge 1$ and $\phi \in (0, \pi/2)$ such that

$$\begin{cases} S_{\phi,a} := \{\lambda \in \mathbb{C} : \phi \le |arg(\lambda - a)| \le \pi\} \subset \rho(A) \|(\lambda I - A)^{-1} \\ \|_{\mathcal{L}(X)} \le \dfrac{N}{|\lambda - a|}, \quad \forall \lambda \in S_{\phi,a} \backslash \{a\} \end{cases}$$

:

If $a = 0$, on the last definition we say that A is **a positive sectorial operator**. **[6]**:
If A is a sectorial operator, then $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA}\}_{t \ge 0}$.

2.6 Fractional powers of sectorial operator

The Fractional powers of sectorial operator play a fundamental role in the theory of existence of solution to non-linear partial differential equations of parabolic type and to analysis of the asymptotic behavior of solution to these problems. **[8]**:
Let A be a positive sectorial operator an X and $\beta > 0$. Then we define

$$A^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty S^{\beta-1} T(s) ds$$

Where $\{T(t) : t \geq 0\}$ is the C_0semigroup generated by $-A$. :
The positive sectorial operator that play the same role that fractional positive powers of bouned opeartors.

2.7 Abstract fractional equations

The abstract fractional cauchy problem has been studied for some time and although recently many relevant results were obtained in this area, even the very basic theory of fractional differential equations is incomplete and there is much that needs to be done. In this chapter we discuss issues that seek to answer some questions outstanding in this area. To that end, we consider the fractional cauchy problem

$$\begin{cases} {}^cD_t^\beta u(t) = -Au(t) + f(t, u(t)), \quad t > 0, \\ u(0) = u_0 \in X. \end{cases} \tag{2.33}$$

Where$X$ is a banach space over $\mathbb{C}, \beta \in (0,1), A : D(A) \subset X \longrightarrow X$ is a positive sectorial operator, ${}^cD_t^\beta$ the caputo fractional derivative and $f : [0,\infty) \times X$ is a continuous function consider $\{E_\beta(-t^\beta A) : t \geq 0\}$ and $\{E_{\alpha,\beta}(-t^\beta A)t \geq 0\}$ the Mittag-Leffler families associated to $-A$, discuusses, as this point, even formally, we want to find an appropriate definition for the concept of of solution to the problem (2.33) one of the approoches follows the idea that:since A is a positive operator,we already know that there exist a C_0 semigroup $\{T(t) : t \geq 0\}$ associated to $-A$ and inspired by the usual case, some research was done in the study of the solution given by the integral representation **[8]**

$$U(t) = T(t)u_0 + \int_0^t T(t-s) f(s, u(s)) ds, \quad t \geq 0. \tag{2.34}$$

[8]:
Let $\tau > 0$ a function $u : [0, T] \longrightarrow X$ is called **a local mild solution** of (2.33) in $[0, T]$ if $u \in C([0.T] : X)$ and

$$U(t) = E_\beta(-t^\beta A)u_0 + \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}(-(t-s)^\beta A) f(s, u(s)) ds, \quad t \in [0, T]$$

in this work we have $X = L^2$. We consider the followiny fractional reaction-difuusion sustem with a blance law

$$\begin{cases} {}^cD_t^\beta u - d\Delta u = -uf(v) & \text{in } \Omega \times \mathbb{R}^+, \\ {}^cD_t^\beta v - \Delta v = uf(v) & \text{in } \Omega \times \mathbb{R}^+, \\ \frac{\partial u}{\partial \eta}(x,t) = \frac{\partial v}{\partial \eta}(x,t) = 0 & \text{on} \partial\Omega \times \mathbb{R}^+, \\ u(x,0) = u_0(x), \qquad v(x,0) = v_0(x) \qquad \text{in } \Omega, \end{cases} \tag{2.35}$$

Where Ω is regular bounded domain in $\mathbb{R}^N (N \geq 1)$ with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial \eta}$ denote the normal derivative on $\partial\Omega$,
Δ stands for the Laplacian operator, $d$is the diffusion constant, $u_0$andv_0 are nonnegative functions, ${}^cD_t^\beta$, for $\beta \in (0,1)$, is the Caputo fractional derivative of order β.
Concerning the nonlinearity f, we assume that there exist positive constant M_1 and M_2 a real number $p \geq 1$ such that

$$0 \leq f(v) \leq M_1 |v|^p + M_2, \tag{2.36}$$

and for all $|v|, |\tilde{v}| \leq R$,there exist a positive number L such that

$$|f(v) - f(\tilde{v})| \leq L|v - \tilde{v}|, \tag{2.37}$$

[1]mild solution:
Let $u_0; v_0 \in \mathcal{C}(\overline{\Omega})$ and $T > 0$. We say $(u,v) \in C([0,T; \mathcal{C}(\overline{\Omega}) \times \mathcal{C}(\overline{\Omega})])$ is a mild solution of the system if it satisfies

$$u(t) = E_\beta(-dt^\beta A)u_0 - \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}(-d(t-s)^\beta A) u(s) f(v(s)) ds, \tag{2.38}$$

$$v(t) = E_\beta(-t^\beta A)v_0 - \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}(-(t-s)^\beta A) u(s) f(v(s)) ds, \tag{2.39}$$

For all $t \in [0,T]$, where $E_\beta(-t^\beta A)$ and $E_{\alpha,\beta}(-t^\beta A)$ are the linear operators .
Now, we givie some important properties of families $E_\beta(-t^\beta A)_{t>0}$ and $E_{\alpha,\beta}(-t^\beta A)_{t>0}$.

[1]:
For $u \in L^\infty(\Omega)$; we have the estimates

$$\|E_\beta(-t^\beta A)u\|_\infty \leq \|u\|_\infty, t > 0, \tag{2.40}$$

$$\|E_{\alpha,\beta}(-t^\beta A)u\|_\infty \leq \frac{1}{\beta\Gamma(\beta)} \|u\|_\infty, \quad t > 0, \tag{2.41}$$

Moreover, there exist $\gamma > 0$such that

$$\|E_\beta(-t^\beta A)u\|_\infty \leq \|u\|_\infty E_\alpha(-\delta t^\beta), \tag{2.42}$$

$$\|E_{\alpha,\beta}(-t^\beta A)u\|_\infty \leq \|u\|_\infty E_{\alpha,\beta}(-\delta t^\beta), \quad t > 0, \tag{2.43}$$

Where $E_{\alpha,\beta}(z)$is the Mittag-Leffler function defines by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},$$

and

$$E_\beta(z) = E_{\alpha,1}(z) \qquad for \qquad z \in \mathbb{C}.$$

Then we have

$${}^cD_t^\beta E_\beta(-t^\beta A) = -AE_\beta(-t^\beta A), \tag{2.44}$$

$$J_t^{1-\beta}(t^{\beta-1}E_\beta(-t^\beta A)) = E_\beta t(-t^\beta A), \tag{2.45}$$

Moreover, $E_\beta(-x)$ is a **completely monotonic function** for $x \geq 0$ and $0 < \beta \leq 1$,i.e.

$$(-1)^n(d^n/dx^n)E_\beta(-x) \geq 0.$$

The following relationship with the semigroup ($T(t) := e^{-tA}$) and the solution operator given by

$$E_\beta(-t^\beta A) = \int_0^\infty \phi_\beta(\theta)T(\theta t^\beta)dt, \qquad t \geq 0, \tag{2.46}$$

and

$$E_{\alpha,\beta}(-t^\beta A) = \int_0^\infty \beta\theta\phi_\beta(\theta)T(\theta t^\beta)dt, \quad t \geq 0, \tag{2.47}$$

[8]:

Let $\beta \in (0,1)$. Consider the families $E_\beta(-t^\beta A) : t \geq 0$ and $E_{\beta,\beta}(-t^\beta A)t \geq 0$ and the C_0semigroup $T(t) : t \geq 0$ associated to $-A$. Then for each $x \in X$ *quad*and $t \geq 0$

$$\lim_{\beta\to1^-} E_\beta(-t^\beta A)x = T(t)x,$$

and

$$\lim_{\beta\to1^-} E_{\alpha,\beta}(-t^\beta A)x = T(t)x,$$

Moreover, the convergence is uniform on bounded subsets of X and on inertvals

$$[a,b] \subset \mathbb{R}^+, for \quad a > 0$$

2.7.1 L^q-regularity

In this chapter we apply maximal L^qregularity to study nonautonomous fractional order equations. More precisely, consider the following problem for the fractional differential equation with Riemann-Lioville derivative of order $\beta \in (0,1)$

$$\begin{cases} D_t^\beta u(t) + Au(t) = \theta \quad , \qquad \text{for} \quad t \in (0,T), \\ J_t^{1-\beta}u(0) = 0. \end{cases} \tag{2.48}$$

We consider some important. A sectorial operator $A \in X$ is said to admit **bounded imarinary powers**, if $A^{is} \in$ B(X) for each $s \in \mathbb{R}$ and there is a constant $C > 0$, such that$|A^{is}| \leq C$ for $|s| \leq 1$. The call of such operators will be denoted by BIP(X) and we will call

$$\theta_A = \overline{\lim_{|s|\to\infty}} \frac{1}{|s|} log|A^{is}|$$

The power angle of A. The class of operators that admit bounded imaginary powers. An important application of the class BIP(X) concerns the fractional power spaces

$$X_\alpha = X_{A^\alpha} = (D(A^\alpha, |.|_\alpha), |x|_\alpha = |x| + |xA^\alpha|, 0 < \alpha < 1,$$

If A belongs to $\mathcal{BIP}$, a characterization of X_α in terms of complex interpolation spaces can be derived .
Where obtained in the special case, when the space X in such that the Hilbert transform defined by

$$(Hf)(t) = \lim_{\varepsilon\to 0} \frac{1}{\pi} \int_{\varepsilon\leq|s|\leq 1/\varepsilon} f(t-s)\frac{ds}{s}, \quad t \in \mathbb{R}$$

is bounded in $L_p(\mathbb{R}; X)$ for some $p \in (1, \infty)$. The class of spaces with this property will be denoted by HT.
There is a well know theorem which says the set of Banach spaces of class HT coincides with the class of UMD spaces, where UMD stand for unconditional martingale difference property.
It is further know that HT-spaces are reflixive. Every Hillbert space belongs to the class HT, and if (Σ, M, μ) is meassure space and $X \in \mathcal{HT} then \mathrm{L}_p(\Sigma, M, \mu, X)$ is an HT-spaces for $(1 < p < \infty)$**[2]**. **[1]**:
Let A be a positive operator in HT space X satisfying $A \in \mathcal{BIP}(X, \varphi(A))$ with $\varphi(A) < \pi(1 - \frac{\beta}{2})$. Then the problem (2.48), for $\theta \in L^q((0,T); X)$, has maximal regularity on $(0,T)$ in X. More precisely ,

$$\|u\|_{L^q((0,T);X)} + \|D_t^\beta\|_{L^q((0,T);X)} + \|Au\|_{L^q((0,T);X)} \leq C\|\theta\|_{L^q((0,T);X)}.$$

2.7.2 Local existence

[1]:
Let $u_0, v_0 \in \mathcal{C}(\overline{\Omega})$, then there exist a maximal time $T_{max} > 0$ and a unique mild solution $(u, v) \in C([0, T_{max}]; \mathcal{C}(\overline{\Omega}) \times \mathcal{C}(\overline{\Omega}))$ to the problem (2.33) with the alternative:
-either $T_{max} = +\infty$;
-or $T_{max} = +\infty$, and in this case

$$\lim_{t\to T_{max}} \|u(t)\|_\infty + \|v(t)\|_\infty = +\infty.$$

:
The existence of a local solution is obtaint by the Banach fixed point theorem .
Even through this is well document part, we present it for the sake of completeness. For arbitray $T > 0$, we define the following Banach space

$$E := \left\{(u, v) \in C([0, T_{max}]; \mathcal{C}(\overline{\Omega}) \times \mathcal{C}(\overline{\Omega})); |||(u, v)||| \leq 2(\|u_0\|_\infty + \|v_0\|_\infty) = R\right\},$$

Where $\|.\|_\infty = \|.\|_{L_\infty(\Omega)}$ and $|||(u, v)||| := \|u\| + \|v\| := \|u\|_{L^\infty([0,T];L_\infty(\Omega))} + \|v\|_{L^\infty([0,T];L_\infty(\Omega))}$.
Next, for every $(u, v) \in E$, we define

$$\psi(u, v) = (\psi_1(u, v), \psi_2(u, v)),$$

Where for $t \in [0,T]$,

$$\psi_1(u,v) = E_\beta(-dt^\beta A)u_0 - \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}(-d(t-s)^\beta A)u(s)f(v(s))ds,$$

and

$$\psi_2(u,v) = E_\beta(-t^\beta A)v_0 - \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}(-(t-s)^\beta A)u(s)f(v(s))ds,$$

We first prove that ψmaps Eonto$E$:Let $(u,v) \in E$, using (2.40)(2.41)and fact that $\|f(v(s))\|_\infty \leq M_1\|v\|_\infty^p + M_2$,we have

$$\begin{aligned} \|\psi_1(u,v)\|_\infty \leq & \|u_0\|_\infty + \frac{1}{\beta\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} \|u(s)f(v(s))\|_\infty ds \\ \leq & \|u_0\| + \frac{T^\beta R}{\beta\Gamma(\beta+1)}(M_1R^p + M_2) \end{aligned} \tag{2.49}$$

Simily, we obtain

$$\|\psi_2(u,v)\|_\infty \leq \|v_0\| + \frac{T^\beta R}{\beta\Gamma(\beta+1)}(M_1R^p + M_2) \tag{2.50}$$

Whereupon, from (2.49) and (2.50) we get

$$\|\psi(u,v)\| = \|\psi_1(u,v)\| + \|\psi_2(u,v)\| \leq \|u_0v_0\| + 2\frac{T^\beta R}{\beta\Gamma(\beta+1)}(M_1R^p + M_2)$$

If we choose Tsuch that $T^\beta \leq \dfrac{\beta\Gamma(\beta+1)}{4(M_1)R^pM_2}$, we conclude that $\psi(u,v) \in E$.
Now, we show thatψ is a contraction map:For $(u,v),(\tilde{u},\tilde{v}) \in E$, we have

$$\|\psi_1(u,v) - \psi_1(\tilde{u},\tilde{v})\| \leq \frac{T^\beta}{\beta}\|\tilde{u}f(\tilde{v}) - uf(v)\|$$

Using

$$|\tilde{u}f(\tilde{v}) - uf(v)| \leq |u||f(v) - f(\tilde{v})| + |f(\tilde{v}||u - \tilde{u}|$$

and the assumptions (2.38)(2.39), we get

$$|\tilde{u}f(\tilde{v}) - uf(v)| \leq L|u||v - \tilde{v}| + (M_1|\tilde{v}|^p + M_2)|u - \tilde{u}| \tag{2.51}$$

hence,

$$\|\psi_1(u,v) - \psi_1(\tilde{u},\tilde{v})\| \leq \frac{(LR + M_1 + R^pM_2)}{\beta\Gamma(\beta+1)}T^\beta|||(u,v) - (\tilde{u},\tilde{v})|||$$

Similarly, we obtain

$$\|\psi_2(u,v) - \psi_1(\tilde{u},\tilde{v})\| \leq \frac{(LR + M_1 + R^pM_2)}{\beta\Gamma(\beta+1)}T^\beta|||(u,v) - (\tilde{u},\tilde{v})|||$$

Whereupon

$$\begin{aligned}\|\psi(u,v)-\psi(\tilde{u},\tilde{v})\| \leq& 2\frac{(LR+M_1+R^pM_2)}{\beta\Gamma(\beta+1)}T^\beta |||(u,v)-(\tilde{u},\tilde{v})|||\\ \leq& \frac{1}{2}|||(u,v)-(\tilde{u},\tilde{v})|||\end{aligned}$$

for $T^\beta \leq \dfrac{\beta\Gamma(\beta+1)}{4(LR+M_1R^p+M_2)}$.

Therefore, in view of banach fixed point theorem (1.10) ψadmits a unique fixed point on E. Thus the system (2.35) has a unique mild solution.

Using the fact that the solution is unique, we conclude that the existence of the solution can be extended on maximal interval $[0,T_{max})$ where

$T_{max} = sup\{T>0, \text{such that} \quad (u,v)\text{is a mild solution to} \quad (2.35) \quad \text{in} \quad E\} \leq +\infty$

We defined the strong solution we present a necessary and sufficient conditions for the existence of strong solution of (2.35). **[1]**strong solution:

A function $(u,v) \in C([0,T];L^2(\Omega)\times L^2(\Omega))$is called a strong solution of (2.35) if $(u,v) \in C([0,T];D(A)\times D(A))$ and $(J^{1-\beta}(u-u_0), J^{1-\beta}(v-v_0) \in C^1([0,T];L^2(\Omega)\times L^2(\Omega))$ and (2.35)$Hold$. **[1]**:

For $u_0, v_0 \in D(A)$, assume that the Lipschitz constant L which is given in (2.33) satisfies $L < \Gamma(1+\beta)/(MT^\beta)$ where $M = sup_{t\geq 0}\|E_{\alpha,\beta}(-t^\beta A)\|$. Then the problem (2.33) has a unique strong solution given by (2.38)(2.39)

CHAPTER 3

GLOBAL EXISTENCE AND ASYMPTOTIC BEHAVIOR

3.1 BOUNDEDNESS OF u

Our goal is to prove the global existence;we proceed by contradiction. Assume that $T_{max} < +\infty$. Combining the fact that $uf(v) \geq 0$ and the estimate (2.38), we get

$$u(t) = E_\beta(-dt^\beta A)u_0 - \int_0^\infty (t-s)^{\beta-1}E_{\alpha,\beta}(-d(t-s)^\beta A)u(s)f(v(s))ds$$

we use (2.47), we obtain

$$\begin{aligned} \|u(.,t)\|_\infty \leq & \|E_\beta(-dt^\beta A)u_0\|_\infty \\ \leq & \|u_0\|_\infty \end{aligned} \tag{3.1}$$

Hence, u is uniformly bounded **[1] [1]**:
Let $u_0, v_0 \in D(A)$ be such that u_0 and v_0 are a positive and bounded functions. Then the system (2.35) admits a unique global strong solution **[1]**:
the system (2.35) admits a unique global strong solution which satisfies

$$\|u(.,t)\|_\infty \leq \|u\|_\infty E_\beta(-\gamma t^\beta), \qquad for \qquad \gamma > 0, t > 0, \tag{3.2}$$

$$\|v(.,t) - \frac{1}{\Omega}\int_\Omega (u_0 + v_0)dx\|_\infty \leq CE_\beta(-\gamma t^\beta), \qquad fort > 0, \tag{3.3}$$

Where C is a positive constant. :

3.2 Existence and uniquness of global solution

From the local existence result, $w = du + v$ is continuous on $[0, T]$ for $T < T_{max}$, hence it is bounded and there exists $t^* \in [0, T]$ such that

$$w(t) \leq w(t^*)$$

Then

$$\begin{cases} {}^cD_t^\beta W - AW = (d-1)u + u_0 + v_0 \\ W(0) = 0 \end{cases} \tag{3.4}$$

we get that $W(0) := J_t^\beta w(t)|_{t=0} = 0$, let us consider two bounded linear operators $\mathcal{M}$ and $\mathcal{P}$ defined by

$$\mathcal{M}w := \langle w \rangle, \qquad \mathcal{P}w := w - \langle w \rangle \quad \text{where} \quad \langle w \rangle := \frac{1}{|\Omega|}\int_\Omega w(x,t)dx$$

Applying $\mathcal{P}$ to (3.4),it following problem

$$\begin{cases} {}^cD_t^\beta \mathcal{P}W + A\mathcal{P}W = (d-1)\mathcal{P}u + \mathcal{P}(u_0 + v_0) \\ \mathcal{P}W(0) = 0 \end{cases}$$

Using (2.44) and the fact that $\mathcal{P}u$ and $\mathcal{P}u_0 + v_0$ are bounded ,we get

$$\mathcal{P}W = E_\beta(-t^\beta A)\mathcal{P}W(0) + \int_0^t (t-s)^{\beta-1}E_{\alpha,\beta}(-(t-s)^\beta A)((d-1)\mathcal{P}u + \mathcal{P}(u_0+v_0))ds$$

$$\|\mathcal{P}W\| \leq \int_0^t (t-s)^{\beta-1}E_{\alpha,\beta}(-(t-s)^\beta A)(\|(d-1)\mathcal{P}u\| + \|\mathcal{P}(u_0+v_0)\|)ds \leq C$$

Where

$$\int_0^t (t-s)^{\beta-1}E_{\alpha,\beta}(-(t-s)^\beta A) < +\infty$$

On the other hand, as $W(0) = 0$ we deduce from(1.16) that ${}^cD_t^\beta \mathcal{P}W = D_t^\beta \mathcal{P}W$, then the function $\mathcal{P}W$ satisfies the equation

$$\begin{cases} D_t^\beta \mathcal{P}W + A\mathcal{P}W = \theta \\ J_t^{1-\beta}W(0) = J_t^1 w(t)|_{t=0} = 0 \end{cases} \tag{3.5}$$

Where $\theta := (d-1)\mathcal{P}u + \mathcal{P}(u_0 + v_0)$ As u, u_0 and v_0 are bounded ,we can assert that $\theta \in L^q((0,T); L^q(\Omega))$ for $q > 1$. In view lemma(2.7.1), it follows that the problem (3.5) has maxcimal L^q regularity on $(0, t)$. More precisely ,there exists a positive constant $C_q(T)$ such that

$$\|D_t^\beta \mathcal{P}W\|_{L^q((0,T);L^q(\Omega))} \leq C_q(T) \tag{3.6}$$

Therefore, as $D_t^\beta \mathcal{P}W = \mathcal{P}w = \mathcal{P}(du + v)$,we obtain

$$\begin{aligned} \|\mathcal{P}v\|_{L^q((0,T);L^q(\Omega))} =& \|D_t^\beta \mathcal{P}W - d\mathcal{P}u\|_{L^q((0,T);L^q(\Omega))} \\ \leq& \|D_t^\beta \mathcal{P}W\|_{L^q((0,T);L^q(\Omega))} \\ \leq& C_q(T) \end{aligned} \tag{3.7}$$

To estimate$\mathcal{M}v$, we add the two equation(1)of problem (2.35)

$$\begin{aligned} {}^{c}D_t^{\beta}u - d\Delta u +^{c} D_t^{\beta}v - \Delta v =& - uf(v) + uf(v) \\ {}^{c}D_t^{\beta}(u+v) - \Delta(du+v) =& 0 \end{aligned}$$

and integration over Ω

$$^{c}D_t^{\beta}\int_{\Omega}(u+v) = 0$$

Operating J_t^{β}, we have

$$J_t^{\beta} \quad {}^{c}D_t^{\beta}\int_{\Omega}(u+v) = \int_{\Omega}(u+v) - \int_{\Omega}(u_0+v_0)$$

It yields $\int_{\Omega}(u+v) = \int_{\Omega}(u_0+v_0)$, whereupon

$$\mathcal{M}v = \mathcal{M}(u_0+v_0) - \mathcal{M}u \tag{3.8}$$

By writting $v = \mathcal{M}v + \mathcal{P}v$, we have from (2.36)

$$f(v) = f(\mathcal{M}v + \mathcal{P}v) \leq M_1|\mathcal{M}v + \mathcal{P}v|^p + M_2$$

Hence,

$$f(v) \leq M_1(|\mathcal{M}v|^p + |\mathcal{P}v|^p) + M_2$$

Gathering (3.7) and (3.8), we can assert that

$$\begin{aligned} \|f(v)\|_{L^{\frac{q}{p}}(0,T);L^{\infty}(\Omega)} \leq& M_1\left(\|\mathcal{M}v\|_{\infty}^{q}T^{\frac{p}{q}} + C_q(T)\right) + M_2T^{\frac{p}{q}} \\ \leq& M\left(1+T^{\frac{p}{q}}\right) \end{aligned} \tag{3.9}$$

By using this result we will give the L^{∞} bounded of v.
In the light of (2.38) and (2.39), we get

$$\|v\|_{\infty} \leq \|v_0\|_{\infty} + \frac{1}{\beta\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|u(s)f(v)\|_{\infty}ds.$$

Consequnetly, as u it bounded,we obtain

$$\|v\|_{\infty} \leq \|v_0\|_{\infty} + \frac{C}{\beta\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}\|f(v)\|_{\infty}ds. \tag{3.10}$$

Using the Holder inegality and (3.9), it holds, for $\beta > \frac{p}{q}$,

$$\begin{aligned} J(t) :=& \int_0^t (t-s)^{\beta-1}\|f(v)\|_{\infty}ds \\ \leq& \left(\int_0^t (t-s)^{\frac{q-p}{q}}\right)\|f(v)\|_{L^{\frac{q}{p}}((0,t);L^{\infty}(\Omega))} \\ \leq& Mt^{\beta}\left(1+t^{-\frac{p}{q}}\right); \end{aligned}$$

hence, for any $t > 0$, we have

$$\|v\|_{\infty} \leq M(1+t^{\beta}). \tag{3.11}$$

Combinig the local exitence result, (3.1) and (3.11), the system (2.35) admits a unique global solution.**[1]**

3.3 Large time behavior of solution

Using the positivty of $f(v)$ and u, we get

$$^{c}D_t^{\beta} v(x,t) - \Delta v(x,t) \geq 0 \tag{3.12}$$

Let $\bar{v}$ be the solution of the problem

$$\begin{cases} ^{c}D_t^{\beta}\bar{v}(x,t) - \Delta\bar{v}(x,t) = 0, \\ \dfrac{\partial \bar{v}}{\partial \eta}(x,t) = 0, \\ \bar{v}(x,0) = v_0 \end{cases} \tag{3.13}$$

Where $\bar{v} = \sum_{n=0}^{+\infty}(v_0,\varphi_n)E_\beta(-\lambda_n t^\beta)\varphi_n(x)$

$$^{c}D_t^{\beta}\bar{v}(x,t) - \Delta\bar{v}(x,t) = ^{c}D_t^{\beta}\sum_{n=0}^{+\infty}(v_0,\varphi_n)E_\beta(-\lambda_n t^\beta)\varphi_n(x) - \Delta\sum_{n=0}^{+\infty}(v_0,\varphi_n)E_\beta(-\lambda_n t^\beta)\varphi_n(x)$$

We use (2.44), we get

$$^{c}D_t^{\beta}E_\beta(-At^\beta) = -AE_\beta(-At^\beta)$$

$$^{c}D_t^{\beta}\bar{v}(x,t) - \Delta\bar{v}(x,t) = \sum_{n=0}^{+\infty}(v_0,\varphi_n)\varphi_n(x)\lambda_n E_\beta(-\lambda_n t^\beta) + \sum_{n=0}^{+\infty}(v_0,\varphi_n))E_\beta(-\lambda_n t^\beta)\lambda_n\varphi_n(x) = 0$$

By the comparison theorem(1.10), we have $v(x,t) \geq \bar{v}(x,t)$, for all $t > 0$, it follows that

$$v(x,t) \geq \sum_{n=0}^{+\infty}(v_0,\varphi_n)E_\beta(-\lambda_n t^\beta)\varphi_n(x) \tag{3.14}$$

Where $(.,.)$is the usual scalar product in $L^2(\Omega)$as

$$\lim_{t\to+\infty} E_\beta(-\lambda_n t^\beta) = \lim_{t\to+\infty}\sum_{k=0}^{+\infty}\frac{(-\lambda_n t^\beta)^k}{\Gamma(\alpha k+1)} = 0, \qquad for \qquad \lambda_n > 0.$$

we obtain

$$\lim_{t\to+\infty}\sum_{n=0}^{+\infty}(v_0,\varphi_n)E_\beta(-\lambda_n t^\beta)\varphi_n(x) = (v_0,\varphi_0)\varphi_0 = \varphi_0\int_\Omega v_0\varphi_0 dx \tag{3.15}$$

$$\varphi_0 = \frac{1}{\sqrt{|\Omega|}}$$

In view of (3.12)(3.14), it holds that for $t >> T > 0$,

$$v(x,t) \geq \frac{1}{|\Omega|}\int_\Omega v_0 dx \tag{3.16}$$

So, there exists a positive constant γ such that

$$f(v) \geq \gamma, \quad \text{for all} \quad t >> T. \tag{3.17}$$

Cosequently, we have

$${}^cD_t^{\beta} u(x,t) - d\Delta u(x,t) \leq -\gamma u(x,t) \tag{3.18}$$

Moreover, using the fact that u is positive and bounded, it follows that

$$0 \leq u(x,T) \leq \|u_0\|_{\infty}.$$

On the other hand, the function $\bar{u}(t) := \|u_0\|_{\infty} E_{\beta}(-\gamma t^{\beta})$satidfies

$$\begin{cases} {}^cD_t^{\beta} \bar{u}(t) = -\gamma \bar{u}, \\ \quad \bar{u}(0) = \|u_0\|_{\infty}. \end{cases}$$

By comparison, it comes that

$$u(x,t) \leq \|u_0\|_{\infty} E_{\beta}(-\gamma t^{\beta}), \quad \text{for all} \quad t >> T.$$

To prove (3.3), we begin by applying P to the second equation of (2.35); we obtain

$$({}^cD_t^{\beta} - \Delta)\mathcal{P}v = \mathcal{P}(uf(v)) \leq CE_{\beta}(-\gamma t^{\beta}), \tag{3.19}$$

Where $C$is a positive constant. So, it follows that

$$\|\mathcal{P}v\|_{\infty} \leq CE_{\beta}(-\gamma t^{\beta}), \qquad t >> T. \tag{3.20}$$

From (3.8) and the definition of $\mathcal{P}v$, we can write

$$\mathcal{P}v = v - \langle u_0 + v_0 \rangle + \langle u \rangle \tag{3.21}$$

Hence,

$$\|v - \langle u_0 + v_0 \rangle\|_{\infty} \leq \| - \mathcal{P}v \langle u \rangle\|_{\infty} \leq \|\mathcal{P}v\|_{\infty}. \tag{3.22}$$

Combinig(3.20) and (3.22), the estimate (3.3) follows.**[1]**

CONCLUSION

In this work we study some new properties and relations between the classical theory and the fractional theory of reaction-diffison system.

Let u be a mild solution to (2.33)

$$u(t) = E_\beta(-t^\beta A)u_0 + \int_0^t (t-s)^{\beta-1} E_{\alpha,\beta}(-(t-s)^{\beta-1}A) f(s,u(s))ds, \quad t \geq 0$$

But in problem (2.10), let $x \in X$ and $f \in L^1(0,T:X)$ given by

$$u(t) = T(t)x + \int_0^\infty T(t-s)f(s,u(s))ds, \quad 0 \leq t \leq T.$$

That is impossible to suppose that the "semigroup property" remain valid to this new solution, for any $\beta \in (0,1)$, such that

$$E_\alpha((t+s)^\beta A) \neq E_\alpha(t^\beta A)E_\alpha(s^\beta A).$$

We have

$$\lim_{\beta \to 1^-} E_\beta(-t^\beta A)x = T(t)x,$$

and

$$\lim_{\beta \to 1^-} E_{\alpha,\beta}(-t^\beta A)x = T(t)x.$$

To show the global existence we abservse in the case of classical equtionn (2.10) the used "Maximum principal" and "Lyaponov function", but in the case of The caputo equation(2.36) we used the property of "Mittag-Leffler Function"

To study the asymptotic behavior of U of our problem (2.33) in the case of classical equation, the asymptotic behavior depends heavily on the sepectral properties of A, and by comparison theorem.

BIBLIOGRAPHY

[1] A. Alsaedi, M.Kirane, R.Lassoued, *Globale existence and asymptotic behavior for a time fractional reaction-diffusion system*, computer and Mathematics with Application 1-8(2016

[2] E. Bashlekova, *Strict L^p Solutions for Nonautonomous Fractional Evolution Equations*,New series Vol.26, 2012, Fasc.1-2

[3] R. Dautry. J.L.Lious *Mthematical analysis and numerical for science and technology* V3 Springer.(2000)

[4] J. Goldstein, *Semigroup of Linear operators and Application*, Oxford U.Press, New York,(1985)

[5] M. Hadji, *sepectral property of semigroupsq*, university Kasdi Merbah Ouargla, (2014)

[6] D. Hanry, *Geometry theory of semilinear prabolic equations*, Spring-Verlag, Berlint Heidelbing, New York, (1981)

[7] A. Lambert, *Théorie de la mesure et integration*, University Pierre et Marie Curie(Paris 6), 12-2011.

[8] P. Mendu *Fractional Differential equation a novel Study of Local Solution in Banach Spaces*, sao carlos,(2013)

[9] A. Pazy, *Semigroup of Linear operator and application to patyial differential equations, Springer-Verlag*, New York(1983).

[10] M. Pierre, *Weak Global Solution For Reaction-Diffusion Systems With Bounded To Total Mass*, France, May, 2008

[11] M. Teva, *Property and applications of the caputo fractional operator*, master theis, University Karlsrushe(TH), February (2005)

[12] B. Rebiai, *Global classical solution for coupled rection-diffusion systems* without,J.math,analysis.vol.5, 2011

[13] R-N. Wang, T-J.Xiao, *Astract fractional cauchy problems with almost sectorial operators* ,J.Differential Equations 252(2012) 202-235.