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by : Merabet Ali Djamel Addine

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The Restricted Burnside Problem

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Before the Jury composed of :

Mr. Mohammed BOUSSAID	M.A. university of KASDI Merbah - Ouargla	Examiner
Mr. Mohammed A. BAHAYOU	M.A. university of KASDI Merbah - Ouargla	Examiner
Mr. Mohammed L. YOUMBAI	M.A. university of KASDI Merbah - Ouargla	Examiner
Mr. Mohammed T. BENMOUSSA	M.A. university of KASDI Merbah - Ouargla	Examiner
Mr. Yassine GUERBOUSSA	M.A. university of KASDI Merbah - Ouargla	Rapporteur

Dedication

I dedicate this work to my parents :

My mother, who has helped me with her love, her support, sacrifices and her precious and her presence in my life, the expression of my feeling and of me eternal gratify.

My father, who can be proud to find here the result of lengthy year of sacrifices and privation to help me advance in life can make sure that the work, thanks for the values nobies, education and ongoing support come from you.

My brother and my friendes who have ceased to be for me examples of preservance and courage.

All my teachers from primary to university, especially professors of UKMO who must see in this work pride of a knowledge well acquired.

INTRODUCTION

The Burnside Problem (1902) asks if whether every finitely generated group of a given exponent n is finite. This problem turned out to be one of the most difficult problems in Group Theory, and its influence on it is much like the influence of Fermat's Last Theorem on the development of Number Theory, specifically its algebraic aspect. While that problem has a positive solution for $n = 2, 3, 4, 6$, it was shown by Adjan and Novikov (1967) in an extraordinary paper that it has a negative solution for n odd and greater than 4381. More recently (2014) Adyan improved this bound to $n \geq 100$ with n odd ; it is also known that the problem has a negative solution for large enough even n, and it is still open in the remaining cases. Note here that the cases $n = 5$, and $7 \leq n \leq 72$ are of special interest since the Adyan-Novikov approach could not adopted to deal with them.

The Restricted Burnside Problem (**RBP** for short) first raised by Magnus (1940') can be seen as weakened version of the Burnside Problem and it can be stated as :

Is every finitely generated residually finite group of a given exponent n finite ?

It turned out that this problem is very related to the theory of Lie algebras ; it can be reduced to a Kurosh's like problem on Lie algebras. There were a well developed theory of associative algebras related to the Kurosh problem, namely the theory of PI identity algebras developed manely by Kaplanski, Jacobson, Levitzky and others. This development suggested that similar results could be obtained for Lie algebras. Finally, E. Zelmanov obtained the desired results for Lie algebras in the two papers [30],[31]. The remaining part of this thesis is devoted to discussing the work of Zelmanov on the subject and some later developments. Perhaps, it is worth mentioning that Zelmanov earned the Fields Medal for his solution of the RBP.

This work is organized as follows :

The first chapter begins with some basic facts on groups. The second section contains a definition of the free Burnside groups and their relevance to the Burnside Problem. Some commutator calculus that is needed in the sequel is discussed in the third section. The last section contains a survey on the Burnside Problem.

The second chapter is about the reduction of the RBP to Lie algebras ; the main notions discussed therein are filtrations of groups and their associated Lie algebras, the dimension subgroups, and identities in groups and in Lie algebras.

The last chapter discusses Zelmanov's main results on the subject, and some of their applications beside the Restricted Burnside Problem.

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NOTATIONS

Let G be a group.

- $\mathrm{Aut}(G)$ the automorphism group of the group G.
- $\mathrm{Inn}(G)$ the group of inner automorphims of G.
- $Out(G) = Aut(G)/Inn(G)$ the group of outer automorphisms (the outer automorphisms are not automorphisms but classes of automorphisms).
- For $N \leq G$, $Aut_N(G)$ is the group of the automorphisms σ of G which satisfies $x^{-1}\sigma(x) \in N$, for all $x \in G$.
- For $x, y \in G$, $[x, y]$ is equal to $x^{-1}y^{-1}xy$.
- For $x_1, x_2, \ldots x_n \in G$, $n \in \mathbb{N}^*$, the left normed commutator $[x_1, x_2, \ldots, x_n]$ is defined by induction : $[x_1] = x_1$ and $[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$.
- $[x,_n y] = [x, y, \ldots, y]$, where y appears n times.
- $\langle S \rangle$ subgroup generated by S.
- G' derived subgoup $\langle [x, y] | x, y \in G \rangle$.
- $\gamma_i(G)$ the terms of the lower central series of G.
- F_d the free group of rank d.
- $B(d, n)$ the free Burnside group ; by definition it coincides with the quotient F_d/N, where $N = \langle g^n | g \in F_d \rangle$.

- $R(d,n)$ the restricted Burnside group, which is by definition equals to the quotient $B(d,n)/R$, where R is the intersection of all normal subgroups of finite index in $B(d,n)$.
- I_G the augementation ideal.
- $D_n(G)$ the nth dimension subgroup.
- E_n the linearized Engel identities.
- $N \lhd_\circ G$ means that N is an open subgroup of the topological group G.

Chapitre 1

Preliminaries

1.1 Basic facts on groups

Definition 1.1 *A group is a set G equipped with an operation $G \times G \to G$ which satisfies the following properties :*

1) Associativity : $a(bc) = (ab)c$, for all a, b, c in G.

2) Identity element : there exists an element $e \in G$ such that $ge = eg = e$, for all $g \in G$.

3) Inverses : $\forall g \in G$, there exists $g' \in G$ such that $gg' = g'g = e$.

Throughout, assume that we have a group G.

One checks easily that the identity element and the inverse of each element of G are uniquely determined. As usual, we denote the identity element of G by 1, and the inverse of $g \in G$ by g^{-1}. If G is abelian, i.e., $ab = ba$ for all $a, b \in G$, then the operation in G will be denoted by $+$, the identity and the inverse of $g \in G$ will be denoted instead by 0 and $-g$.

Examples.

1. The usual sets of numbers $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{C}$ form (abelian) groups under the usual additions.
2. For any set X, the set S_X of the bijective maps from X to it self, forms a group under the compositions of maps. The group S_X is known as the group of permutations of X. For $X = \{1, 2, \ldots, n\}$, the permutation group on X is denoted by S_n and called *the symmetric group* of degree n.
3. For a field K, the set $\mathrm{GL}(n, K)$ of the $n \times n$ invertible matrices with coefficients in K is a group under the multiplication of matrices. The groups $\mathrm{GL}(n, K)$ are known as *the general linear groups* over K.

Definition 1.2 *A subgroup of G is a non empty subset H of G which satisfies $xy^{-1} \in H$, for all $x, y \in H$.*

We write $H \leq G$ if H is a subgroup of G. In this case, the restriction of the operation in G on the subset H defines a group structure on H. We have for instance :

1. For any integer $n \in \mathbb{Z}$, the subset $n\mathbb{Z} = \{nx \,|\, x \in \mathbb{Z}\}$ is a subgroup of $(\mathbb{Z}, +)$. Conversely, one can show easily that every subgroup of $(\mathbb{Z}, +)$ has the form $n\mathbb{Z}$, for some $n \in \mathbb{N}$.
2. If we consider the additive group $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ of integers modulo 4, then $H = \{0, 2\}$ is a subgroup of $\mathbb{Z}_4$.
3. The set of $n \times n$ matrices with coefficients in a field K and with determinant 1 is a subgroup of the general linear group $\mathrm{GL}(n, K)$, denoted by $\mathrm{SL}(n, K)$ and called *the special linear group* of dimension n over K.

The intersection of any family of subgroups of G is again a subgroup of G. Hence, if we have $S \subseteq G$, then the intersection of the subgroups of G containing S is the smallest subgroup which contains S ; this subgroup is denoted by $\langle S \rangle$ and called the subgroup generated by S.

Proposition 1.3 *Let G be a group and $S \subseteq G$. Then $\langle S\rangle$ is formed by all the finite products $s_1 s_2 \dots s_n$, where $s_i \in S \cup S^{-1}$ (S^{-1} is the set of the inverses of the elements of S).*

The group G is said to be *finitely generated* if $G = \langle S\rangle$, for some finite subset S of G. More precisely, we say that G is d-generated if it can be generated by a subset having d elements. A well-known result in group theory asserts that if G is a d-generated group and $H \leq G$ has finite index m, then H is finitely generated ; more precisely H can be generated by $m(d-1)+1$ elements. We recall that $H \leq G$ has finite index if the set $\{xH \subseteq G \,|\, x \in G\}$ is finite.

The cardinality of G, denoted by $|G|$, is usually called the order of G. We say that G is finite if $|G|$ is finite. An element $g \in G$ has finite order if the subgroup $\langle g\rangle$ is finite. If every element of G has finite order, we say that G is periodic, or G is a torsion group.

A subgroup $H \leq G$ is said to be normal if we have

$$gHg^{-1} \subseteq H, \text{ for all } \; g \in G.$$

We write in this case $H \trianglelefteq G$, and $H \triangleleft G$ when we want to emphasize that H is proper in G.

The intersection of any family of normal subgroups of G is again a normal subgroup of G. Hence, if we have $S \subseteq G$, then the intersection of the normal subgroups of G containing S is the smallest normal subgroup which contains S ; this subgroup is denoted by $\langle\langle S\rangle\rangle$ and called the normal subgroup generated by S. It is readily seen that $\langle\langle S\rangle\rangle$ is the subgroup generated by all the elements of the form $s^g = g^{-1}sg$, where $s \in S$ and $g \in G$.

Having a normal subgroup $H \trianglelefteq G$, the set G/H of the left cosets xH of G modulo H can be endowed with a group structure by setting :

$$(xH)(yH) = (xy)H \text{ for } \; x, y \in G.$$

The resulting group G/H is called the quotient of G by H.

A map $f : G \to G'$ between the two groups G and G' is called a group homomorphism if it satisfies :

$$f(xy) = f(x)f(y) \text{ for all } \; x, y \in G.$$

For example, the canonical map $\pi : x \mapsto xH$ defines a group homomorphism from G to the quotient group G/H, whenever $H \trianglelefteq G$.

The kernel of a homomorphism $f : G \to G'$ is defined by

$$\ker f = \{x \in G \,|\, f(x) = 1\}.$$

It follows immediately that $\ker f$ is a normal subgroup of G. By the just above example, the normal subgroups of G can be characterized by the property of being the kernels of homomorphisms starting from G.

1.2 The free Burnside groups

Definition 1.4 *Let S be a set. A free group on S is a group F_S together with a map $i : S \longrightarrow F_S$, such that whenever G is a group and $\phi : S \longrightarrow G$ is a map, there exists a unique group homomorphism $\tilde{\phi} : F_S \longrightarrow G$ which satisfies $\tilde{\phi} \circ i = \phi$*

$$\begin{array}{ccc} S & \xrightarrow{i} & F_S \\ \phi \searrow & & \swarrow \tilde{\phi} \\ & G & \end{array}$$

The above universal propriety gurantees that F_S is unique up to isomorphism.

Theorem 1.5 *There exists a free group on every non-empty set S.*

Proof. Let us give a sketch of the proof :

— Consider the manoid M of the words on $S \cup S^{-1}$, where $S^{-1} = \{s^{-1} \mid s \in S\}$ is just a set which does not encounter S (one can call the elements of S^{-1} the formal inverses of the elements of S). Recall that a word on $S \cup S^{-1}$ of length n is a finite sequence $w = x_1 x_2 \ldots x_n$ of elements of $S \cup S^{-1}$. We denote the unique word of length 0 by 1 and we call it the empty word. The operation on M is defined by concatenation of words, that is for two words $u = x_1 x_2 \ldots x_n$ and $v = y_1 y_2 \ldots y_m$ in M, the product uv is defined as

$$uv = x_1 x_2 \ldots x_n y_1 \ldots y_m.$$

Note that 1 is the identity element for this operation, that is $u1 = 1u = u$, for all $u \in M$.

— Define an equivalence relation on M by setting : $w \sim w'$ if w can be obtained from w' by adding or deleting subwords of the form ss^{-1} or $s^{-1}s$, with $s \in S$.

— We define F_S to be the quotient of M by the relation defined above. If we have two classes $[u], [v]$ of words, then we define their product as usual by $[u][v] = [uv]$. Th canonical map from S to F_S is defined by $s \mapsto [s]$.

■

One can show that if F is a group which is free on a subset X, and free on a subset Y then $|X| = |Y|$. So we can speak about the rank of a free group, by which we mean the cardinality of a set on which this group is free.

We shall focus on the free groups of finite rank (free on a finite generating set) ; we denote by F_d the free group on d generators, say, $\{x_1, \cdots, x_d\}$.

Proposition 1.6 *Let G be a finitely generated group then there exists an epimorphism $G \twoheadrightarrow F_d$. In particular, G is isomorphic to a quotient of F_d*

Proof. Assume that $\{g_1, ..., g_d\}$ is a generating set of G and consider the free group F_d on gunerators $x_1, ..., x_n$. The map $x_i \mapsto g_i$, $i = \overline{1, d}$, extends by the universal property of F_d to a group homomorphism

$$\tilde{\phi} : F_n \longrightarrow G.$$

The morphism $\tilde{\phi}$ is surjective because $g_1, ..., g_d$ generate G. It follows that $F_n/\ker\tilde{\phi} \cong G$. ■

Definition 1.7 *The free Burnside group $B(d,n)$ is the quotient of F_d by the subgroup generated by the set $\{x^n \,|\, x \in F_d\}$*

Proposition 1.8 *(The universal property of the free Burnside groups) Let n and d be two positive integers. If G is a group which can be generated by d elements and satisfies $g^n = 1$, for all $g \in G$, then G is a quotient of $B(d,n)$.*

The above proposition can be stated as : for every d-generated group G which satisfies the identity $x^n = 1$, there exists an epimorphism $B(d,n) \twoheadrightarrow G$.

Proof of proposition 1.8. Assume that $\{g_1, \cdots, g_d\}$ is a generating set of G ; and let F_d be the free group on $X = \{x_1, \cdots, x_n\}$. Then we have a map $f : X \longrightarrow G, f(x_i) = g_i$ with $i = \overline{1,d}$. So f extends to an epimorphism $F_d \to G$. An element of F_d of the from x^n satisfies $\widehat{f}(x^n) = \widehat{f}(x)^n = 1$, hence $x^n \in \ker f$, for all $x \in G$, that is $\{x^n|x \in F_d\}$ Lies in $\ker\widehat{f}$ and so $F_d^n = \langle x^n|x \in F_d\rangle$ also lies in $\ker\widehat{f}$. Therefore

$$B(d,n) = F/F_d^n \to G$$
$$\overline{x} \mapsto \widehat{f}(x)$$

is a well defined epimorphism. ■

1.3 Commutators

Preliminaries

Let G be a group. For every $x, y \in G$, the commutator $[x,y]$ is defined by

$$[x,y] = x^{-1}y^{-1}xy.$$

It is straightforward to see that for all $x, y, z \in G$, the following identities hold :

$$[xy, z] = [x,z]^y[y,z] \tag{1.1}$$

$$[x, yz] = [x, z][x, y]^z \tag{1.2}$$

$$[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1. \tag{1.3}$$

The last identity is known as the *Hall-Witt identity*. Note the similarity between this identity and the Jacoby identity for Lie algebras (the first two identities are similiar to bilinearity !).

If X, Y are two subsets of G, we define $[X, Y]$ to be the subgroup generated by all the commutators $[x, y]$, with $x \in X$ and $y \in Y$. More generally, If $X_1, \ldots, X_n$ are subsets of G, then the subgroup $[X_1, \ldots, X_n]$ can be defined by induction :

$$[X_1] = \langle X_1 \rangle \text{ and } [X_1, \ldots, X_n] = [[X_1, \ldots, X_{n-1}], X_n].$$

The following result is known as the *Three Subgroups Lemma*, and it is a direct consequence of the Hall-Witt identity.

Proposition 1.9 *Let X, Y, Z be three subgroups of a group G, and N be a normal subgroup of G. If N contains two of the subgroups $[X, Y, Z]$, $[Y, Z, X]$ and $[Z, X, Y]$, then it contains the third.*

We shall discuss now the notion of *formal commutators*. Let S be an arbitrary set we define a commutator of weight 1 on S to be just an element of S, and by induction if c_1 is a commutator on S of weight n_1 and c_2 is a commutator of weight n_2, then the expression $[c_1, c_2]$ is a formal commutator on S of weight $n_1 + n_2$.
Now, assume that S is a group (the same definition makes sense for Lie algebras !) if $x_1, \cdots, x_n \in S$ and c is a formal commutator or on S of weight n, then we can compute $c(x_1, \cdots, x_n)$ the value of c on $(x_1, \cdots, x_n)$ as follows :

1) If $n = 1$, then $c(x_1) = x_1 \in S$.
2) By induction, if every formal commutator of weight $< n$, has been given a value, then for $c = [c_1, c_2]$, where c_1, c_2 are from can on sat weight 1 and n respectively , we set Then if $c = [c_1, c_2]$, we let $c(x_1, \cdots, x_n) = [c_1(x_1, \cdots, x_i), c_2(x_{i+1}, \cdots, x_n)]$ where c_1, c_2 are formal commutator, on S of weight i and $n - i$ respectively.

1.4 A survey on the Burnside Problem

In 1902 W. Burnside formulated his general problem :

Is every finitely generated torsion group finite ?

Let G be a finitely generated abelian group. Assume that G can be generated by elements $g_1, ..., g_d$ of finite order, and denote their orders by $n_1, ..., n_d$ respectively. Therefore, every elements $g \in G$ can be written as a product :

$$g = g_1^{i_1} g_2^{i_2} \dots g_d^{i_d},$$

where each i_k lies between 0 and n_k. Therefore G contains at most n elements, where n is the product of the n_i's. We have established.

Proposition 1.10 *Every abelian finitely generated torsion group is finite. Effectively, if G is an abelian group which can be generated by elements $g_1, ..., g_d$ of finite orders $n_1, ..., n_d$ respectively, then G is finite of order not exceeding $\prod_{k=1}^{d} n_k$.*

Hence, the General Burnside Problem has a positive answer in the class of abelian groups. The later result can be extended easily to the class of soluble groups. Recall that a group G is soluble if it has a finite series of subgroups

$$1 = G_n \lhd G_n \lhd \dots G_0 = G$$

such that each factor G_i/G_{i+1} is abelian. By assuming that G is finitely generated and periodic, it follows at once from the last proposition that G/G_1 is finite ; and as we mentioned, it that G_1 is also finitely generated. By the same argument we see that G_1/G_2 is finite, and G_2 is finitely generated. By repeating this argument for the remaining factors one sees that each factor G_i/G_{i+1} is finite, from which it follows that G is finite. In conclusion we have.

Corollary 1.11 *Every soluble finitely generated torsion group is finite.*

Another interesting class in which the General Burnside Problem has a positive solution is the class of linear groups. Recall that a group G is linear if it can be embedded in $\mathrm{GL}(n, K)$ for some field K.

Theorem 1.12 *Every linear finitely generated torsion group is finite.*

This theorem was established in 1911 by I. Schur. It is worth noting that the Tits alternative, which forms the subject of the dissertation of my colleague A. Belmazouzi, implies [Theorem (1.12)] in a straightforward manner. The Tits alternative asserts that every finitely generated linear group G contains a free subgroup on two generators or a normal soluble subgroup of finite index (we say in the later case G is virtually solvable). Hence, if G is periodic, then we are certainly in the second case as the free groups are torsion free. It follows at once from that last corollary that G is finite.

The General Burnside Problem remained open in full generality for about 60 years. While the mentioned results support the fact that the problem has a positive solution, E. Golod constructed a counter example in 1964 (see [15]). Golod's ideas were inspired by his results with Shafarevich (see [14]) which settled another longstanding problem in number theory (the class field tower problem). Other counter examples to the General Burnside Problem were constructed later by N. Gupta jointly with S. Sidki, and by R.I. Grigorchuk (the groups constructed by Grigorchuk are of great interest in geometric group theory.)

A weaker form of the General Burnside problem asks the following

Is every group which satifies the identity $x^n = 1$ finite ?

This problem is known as the *(ordinary) Burnside problem.*

Proposition 1.13 *Let d and n be two positive integers. Then the (ordinary) Burnside problem has a positive answer for d and n, if and only if $B(d, n)$ is finite.*

Proof. Assume that every d-generated group which satisfies the identity $x^n = 1$ is finite ; since $B(d,n)$ is d-generated and has exponent n, it follows at once that $B(d,n)$ is finite. Conversely, assume that $B(d,n)$ is finite, By Proposition 1.8, if G is a d-generated group of exponent dividing, then G is a quotient of $B(d,n)$, so G is finite. ■

For every n, the groups $\mathrm{B}(1,n)$ is cyclic, and so abelian. Thus $\mathrm{B}(1,n)$ is finite.

A group which satisfies the identity $x^2 = 1$ is abelian. Thus $\mathrm{B}(d,2)$ is abelian, and so finite as we have seen. The order of $\mathrm{B}(d,2)$ is at most 2^d, and since a vector space of dimension d over the field with two elements has cardinality 2^d, it follows that $|\,\mathrm{B}(d,2)| = 2^d$.

Proposition 1.14 (Burnside 1903) *The group* $\mathrm{B}(d,3)$ *is finite, for all* d*.*

The proof of the last result proceeds as follows : one shows that every element of $\mathrm{B}(d,3)$ commute with all its conjugates, and then deduce that $\mathrm{B}(d,3)$ satisfies the Engel identity $[x,y,y] = 1$. It follows that $\mathrm{B}(d,3)$ is soluble and hence finite as we have mentioned.

Theorem 1.15 (Sanov 1940) *The group* $\mathrm{B}(d,4)$ *is finite, for all* d*.*

For a proof we refer the reader to [25].

The last known positive result on the groups $\mathrm{B}(d,n)$ is due to Marshal Hall.

Theorem 1.16 (M. Hall 1957) *The group* $\mathrm{B}(d,6)$ *is finite, for all* d*.*

Hence the Burnside Problem has positive solution for exponents $n = 2, 3, 4, 6$, and naturally the case $n = 5$ is of particular interest. According to M. Sapir, a leading specialist in the area, a solution to the later problem certainly deserves a Fields Medal !

Three years after Golod's construction, S.I.Adian and P.S. Novikov proved the following.

Theorem 1.17 *The groups* $B(d,n)$ *are infinite for* n *odd and* $n \geq 4381$*.*

In 1975, Adian reduced the above bound to 665, and to 100 in 2014. The result of Adian and Novikov was published in a series of three long papers [21, 22, 23]. Quoting from Magnus and Chandler in their book "The history of combinatorial group theory" (page. 147) :

"this paper is possibly the most difficult paper to read that has ever been written on mathematics."

Adian-Novikov theory was simplified later in Adian's [1] ; but still the machinery is very complicated and the ideas therein weren't followed up extensively.

The unsuccessful attempts to solve the Burnside Problem led in the 40th to considering the following problem, which we call those days the *Restricted Burnside problem* :

Is there a bound on the sizes of the finite quotients of $B(d, n)$?

Obviously, the above problem is equivalent to the following : Is there an integer $f(d, n)$ such that for every d-generated finite group G satisfying the identity $x^n = 1$, we have $|G| \leq f(d, n)$?

Let R be the intersection of all the normal subgroups of finite index in $B(d, n)$, and let $R(d, n) = B(d, n)/R$.

Proposition 1.18 *The Restricted Burnside Problem has a positive answer for the positive integers d and n if, and only if, $R(d, n)$ is finite.*

Proof. Assume that $R(d, n)$ is finite, and let $f(d, n)$ be its order. Let G be a finite d-generated group which satisfies the identity $x^n = 1$. By Proposition 1.8, there is an epimorphism, $\phi : B(d, n) \to G$ and $B(d, n)/\ker\phi \simeq G$ is finite.
The residual of $B(d, n)$, is $R = \cap\{\ N \triangleleft G \mid G/N\} \subseteq \ker\phi$. Hence

$$B(d, n)/\ker\phi \simeq (B(d, n)/R)/(\ker\phi/R).$$

thus $G = R(d,n)/(\ker \phi/R)$. That is G is a quotient of $R(d,n)$, so

$$|G| \leq |R(d,n)| = f(d,n)$$

Conversely, assume that there is a bound $f(d,n)$ on the orders of the finite d-generated group satisfying the identity $x^n = 1$, and let G be a group of maximal order among these groups. As we have seen $G \simeq R(d,n)/N$, we claim that every $N \triangleleft R(d,n)$ of finite index contains N. In deed if $N \nsubseteq M$ then $N \cap M$ has finite index in $B(d,n)$ and $N \cap M < N$, so

$$|B(d,n)/(N \cap M)| > |R(d,n)/N| = f(d,n)$$

So $N \subseteq M$ contradiction $N \subseteq \cap\{M \ntriangleleft G \mid G/M\}$ and $N \subseteq R$ requires $|R(d,n)| \leqslant f(d,n)$. ■

Contrary, to the Burnside Problem, there were some evidences to think that the answer is likely positive. One may start with the work of Hall and Higman [7] in which they obtained the following reduction.

Theorem 1.19 *If the following assertions hold :*

(i) *the restricted Burnside groups $R(d,p^n)$ are finite, for all positive integrs n, and all primes p,*

(ii) *for every positive intger m, there are only finitely many simple groups which satisfy the identity x^m,*

(iii) *the Schreier conjecture holds ; in other word, for every finite simple groups S, the group* $\mathrm{Out}(S)$ *is soluble,*

then the Restricted Burnside Problem has a positive solution.

So, the attention is directed to p-groups. One can reduce the problem in that case to a problem about the nilpotency of some Lie algebras with PI (see the next chapter). In 1959, Kostriking proved the claim for the groups $R(d,p)$ (G. Higman proved this before for $p = 5$), and his work suggests that a similar approach would solve the problems for all prime powers. On the other hand, a problem similar to the Burnside Problem was formulated by Kurosh (1941).

The General Kurosh Problem. *Is every finitely generated algebra in which every element is nilpotent, nilpotent ?*

A weaker version of the Kurosh Problem similar to the (ordinary) Burnside Problem can be stated as follows : *Is every finitely generated nilpotent algebra which satisfies the identity* $x^n = 0$, *nilpotent ?*

The paper [15] in which Golod established a counter-example to the GBP, contains also a counter-example to the General Kurosh Problem. Though, it turned out before that the Kurosh Problem has a positive answer in a very interesting class of nil algebras (nil algebra stands for an algebra in which every element is nilpotent).

Theorem 1.20 *A finitely generated nil algebra which satisfies a polynomial identity is nilpotent.*

An immediate corollary of Theorem 1.20 is that the (ordinary) Kurosh Problem has a positive solution.

The later results were established by I. Kaplanski in 1948. The theory of associative algebras with polynomial identities (PI for short) is now an essential part of Non-Commutative Algebra, and it owes much of its development to the Kurosh Problem as well as the Burnside Problem.

This development in the theory of associative algebras, suggested that similar results could be obtained for Lie algebras. Finally, E. Zelmanov obtained the desired result for Lie algebras in the two papers [30, 31]. The remaining part of this thesis is devoted to disscussing the work of Zelmanov on the subject and some later developments. Perhaps, it is worth mentioning that Zelmanov earned the Fields Medal for his solution of Restricted Burnside Problem.

Chapitre 2

Reduction to Lie algebras

2.1 Lie algebras

Let K be a commutative unital ring. Recall that we call (left) K-module every abelian group A together with a law $(k, a) \mapsto ka$ defined from $K \times A$ into A which satisfies the following axioms :

1. $(k + k')a = ka + k'a$
2. $k(a + a') = ka + ka'$
3. $(kk')a = k(k'a)$
4. $1a = a$

for all $a, a' \in A$ and all $k, k' \in K$.

A map $f : A \to B$ between two K-modules A and B is said to be K-linear if it is a morphism of abelian groups and $f(ka) = kf(a)$, for all $k \in K$ and $a \in A$.

Let A, B and C be three K-modules. A map $f : A \times B \to C$ is K-bilinear (or simply bilinear) if the maps $y \mapsto f(a,y)$ and $x \mapsto f(x,b)$ are K-linear, whenever we fix $a \in A$ or $b \in B$.

Definition 2.1 *1. We call a K-algebra every K-module A together with a bilinear map $(a,b) \mapsto ab$ from $A \times A$ into A.*

2. A map $f : A \to B$ between the two K-algebras A and B is said to be a morphism of K-algebras if it is K-linear and $f(a_1 a_2) = f(a_1)f(a_2)$, for all $a, a' \in A$.

There are two main classes of algebras which were extensively studied in the litterature :

(1) **Associative algebras.** A K-algebras A is associative if the product in A is associative, that is to say $(xy)z = x(yz)$, for all $x, y, z \in A$.

(2) **Lie algebras.** A K-algebras A is said to be a Lie algebra if the following two axioms hold :

i) $x.x = 0$, for all $x \in A$.

ii) $(xy)z + (yz)x + (zx)y = 0$, for all $x, y, z \in A$.

Usually, the product in a Lie algebra is denoted by bracket $[x, y]$; so the above definition reads as follows.

Definition 2.2 *A Lie algebra over K is a K-module L with a bilinear product $(x,y) \mapsto [x,y]$ such that :*

i) $[x,x] = 0$, for all $x \in L$.

ii) $[[x,y]z] + [[y,z]x] + [[z,x]y] = 0$, for all $x, y, z \in L$.

It follows from the first axiom that $[x,y] = -[y,x]$, for all $x, y \in L$; that is to say that the product in a Lie algebra is anti-commutative. The second axiom is known as the Jacobi identity.

Typical examples of associative K-algebras are provided by the K-endomorphisms $\mathrm{End}_k(M)$ of a K-module M. The K-module structure in $\mathrm{End}_k(M)$ is defined by

$$(f+g)(x) = f(x) + g(x)$$

$$(kf)(x) = kf(x)$$

where $x \in M$, $k \in K$, and $f, g \in \mathrm{End}_k(M)$. The product in $\mathrm{End}_k(M)$ is defined by the usual composition of maps.

To each associative algebra A, we can associate a Lie algebra by taking the same underlying K-module and defining the bracket by $[x, y] = xy - yx$, for $x, y \in A$. It is worth noting that, conversely, every Lie algebra L (over a field) can be embedded in the Lie algebra associated to some associative algebra, which is known as *the universal enveloping algebra of* L ; this can be done by means of the Poincaré-Birkhoff-Witt theorem.

For two subsets X, Y of a K-algebra A, we define $X \cdot Y$ to be the submodule generated by all the products $x.y$, with $x \in X$ and $y \in Y$. In a Lie algebra the product $X \cdot Y$ may be denoted by $[X, Y]$.

Definition 2.3 *Let X be a submodule of a K-algebra A.*

(1) We say that X is a subalgebra of A if $X \cdot X \subseteq X$.

(2) We say that X is an ideal of A if $X \cdot A \subseteq X$ and $A \cdot X \subseteq X$.

For instance, in a Lie algebra L, a subalgebra is a submodule X which satisfies $[X, X] \subseteq X$. The submodule X is an ideal of L if and only if $[X, L] \subseteq X$ (the second condition $[L, X] \subseteq X$ follows from the first by anti-commutativity).

A noteworthy is that every ideal of a K-algebra A is also a subalgebra. For a morphism of K-algebras $f : A \mapsto B$, one sees easily that $\ker f$ is an ideal of A, and that the image of f is a subalgebra of B.

Having a subset X of an algebra A, we can consider $\langle\langle X \rangle\rangle$ the intersection of all the subalgebras of A containing X. Obviously, $\langle\langle X \rangle\rangle$ is the smallest (with respect to the order defined by inclusion) subalgebra containing X ; we say that $\langle\langle X \rangle\rangle$ is the subalgebra generated by X.

Definition 2.4 *A K-algebra A is finitely generated if $A = \langle\langle X \rangle\rangle$ for some finite subset X*

of A. If X can be chosen to have d-elements, we say more precisely that A is a d-generated algebra.

The last definition applies in particular for associative algebras as well as for Lie algebras. For instance, the associative K-algebra $K[X]$ of polynomials over K is one generated, though it is not finitely generated as a K-module.

Let A be a K-algebra and S be a commutative monoid. We say that A is S-*greded* if there exists a family of submodules $(A_s)_{s\in S}$ (index by the monoid S) such that :

$$A = \bigoplus_{s\in S} A_s$$

and

$$A_s \cdot A_t \subseteq A_{s+t}, \quad \text{for all} s, t \in A.$$

For $S = (\mathbb{N}, +)$, an $\mathbb{N}$-graded algebra will be called for short a graded algebra.

For instance, a graded Lie algebra is a Lie algebra L which can be written as

$$L = \bigoplus_{n\geq 0} L_n$$

for some family of submodules $(L_n)_{n\geq 0}$ such that $[L_n, L_m] \subseteq L_{n+m}$, for all $n, m \in \mathbb{N}$.

An ideal I of a graded Lie algebra $L = \oplus_{n\geq 0} L_n$ is said to be a *graded ideal* if $I = \oplus_n I_n$, where $I_n = I \cap L_n$.

2.1.1 Free Lie algebras

Let X be a set. A free Lie algebra on X is a Lie algebra L_X together with a map $i : X \mapsto L_X$, which satisfies the following universal property :

For every mapping $f : X \to L$, where L is a Lie algebra, there exists a unique morphism of Lie algebras $\widehat{f} : L_X \to L$ such that $\widehat{f} o i = f$, that is to say that the following diagram is *commutative*

$$\begin{array}{ccc} X & \stackrel{i}{\rightarrow} & L_X \\ f \searrow & & \swarrow \tilde{f} \\ & L & \end{array}$$

The above universal property guarantees that a free Lie algebra on X if it exists, is unique up to isomorphism.

Theorem 2.5 *There exists a free Lie algebra on every non-empty set X.*

For a proof, we refer the reader to Serre's excellent book [27, Chapter IV].

For $X = \{x_1, \cdots, x_d\}$, the free Lie algebra L_d on X will be called the free Lie algebra on the generators $x_1, \cdots, x_d$. An elements of that algabra will be called a *Lie polynomial* in the indeterminates $x_1, \cdots, x_d$.

It follows immediately from the universal property of L_d that every d-generated Lie algebra is a quotient of L_d. More explicitely if $L = \langle\langle a_1, \cdots, a_d \rangle\rangle$ is a Lie algebra on the generators $a_1, \cdots, a_d$, then the map $x_i \mapsto a_i$, extends to a unique Lie algebra morphism $\phi : L_d \to L$ which is surjective. If ϕ is a Lie polynomial in $x_1, \cdots, x_d$ and $f = f(x_1, \cdots, x_d)$, then we shall denote $\phi(f)$ by $f(a_1, \cdots, a_d)$ and call it the value of the polynomial f on $(a_1, \cdots, a_d)$. It is worth noting that the subalgebra generated by a subset A in a Lie algebra L coincides with the set of the elements $f(a_1, \cdots, a_d)$, where $a_i \in A$ and f runs over all the polynomials in L_A.

2.2 Filtrations and their associated Lie algebras

Filtrations

Definition 2.6 *Let G be a group we call a filtration (or integral filtration) of G every descending sequence $(G_n)_{n\geq 1}$ of subgroups of G, which satisfies $G_1 = G$ and $[G_n, G_m] \subseteq G_{n+m}$ for all $n, m \geq 1$.*

Recall that the lower central series of G is defined recursively by :

$\gamma_1(G) = G$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$ for $n \geq 1$. It follous that $\gamma_n(G) \triangleleft G$ and $\gamma_{n+1}(G) \subseteq \gamma_n(G)$ for all $n \geq 1$.

Proposition 2.7 *The lower central series $(\gamma_n(G))_{n\geq 1}$ is filtration of G.*

Proof. We have to show that $[\gamma_n(G), \gamma_m(G)] \subseteq \gamma_{n+m}(G)$ for all $n, m \geq 1$. Let us prove the claim by induction m.

If $m = 1$ then $[\gamma_n(G), \gamma_1(G)] = [\gamma_n(G), G]$ which equals to $\gamma_{n+1}(G)$ by definition for all $n \geq 1$. Assume that we have proved the claim for $m-1$ so $[\gamma_n(G), \gamma_{m-1}(G)] \subseteq \gamma_{n+m-1}(G)$ for all $n \geq 1$ we have

$$[\gamma_n(G), \gamma_{m-1}(G), G] \subseteq [\gamma_{n+m-1}(G), G] = \gamma_{n+m}(G)$$

and

$$[G, \gamma_n(G), \gamma_{m-1}(G)] = [\gamma_{n+1}(G), \gamma_{m-1}(G)] \subseteq \gamma_{n+m}(G).$$

Hence by the Three subgroups Lemma, we have $[\gamma_{m-1}(G), G, \gamma_n(G)] \subseteq \gamma_{n+m}(G)$, that is to say $[\gamma_n(G), \gamma_m(G)] \subseteq \gamma_{n+m}(G)$. This completes the proof.

■

The lower central series $(\gamma_n(G))_{n\geq 1}$ is canonical in the sense that for any filtration $(G_n)_{n\geq 1}$ of G we have

$$\gamma_n(G) \subseteq G_n, \forall n \geq 1. \tag{2.1}$$

Indeed, for $n = 1$ the result is trivial. So assume that it is true for some $n \geq 1$; let $x \in \gamma_n(G)$ and $y \in G$, then we have $[x, y] \in [\gamma_n(G), G_1] \subseteq G_{n+1}$. This shows that G_{n-1} contains all the set :

$$X = \{[x, y] | x \in \gamma_n(G), y \in G\}$$

Thus $\langle X \rangle \subseteq G_{n+1}$, but $\langle X \rangle$ is nothing but $\gamma_{n+1}(G)$. This proves the property (2.1).

We shall see other examples of filtrations in the following sections.

The Lie algebra associated to a filtration

Assume that $(G_n)_{n\geq 1}$ is a filtration of a given group G.

Note in particular that $[G_n, G] = [G_n, G_1] \subseteq G_{n+1} \subseteq G_n$, so every G_n is normal in G, and in particular the groups G_n/G_{n+1} are well defined for all $n \geq 1$.
Define now, the abelian group $L(G)$ by

$$L(G) = \bigoplus_{n \geq 1} G_n/G_{n+1}$$

It is convenient to set $L_n = G_n/G_{n+1}$ so $L(G) = \oplus_{n\geq 1} L_n$. For $\overline{x}_n \in L_n$ and $\overline{x}_m \in L_m$, we associate an element of L_{n+m} that we denote by $[\overline{x}_n, \overline{x}_m]$ as follows

$$[\overline{x}_n, \overline{x}_m] = [x_n, x_m] \mod G_{n+m+1}$$

where $[x_n, x_m]$ is the group commutator of $x_n, x_m \in G$. The last assignment is independent from the choice of the representatives of the classes $\overline{x}_n$ and $\overline{x}_m$, this follows at once from the commutator identity in section 1.3. Now, for arbitrary elements $x, y \in L(G)$ we can write $x = \sum_i \overline{x}_i$ and $y = \sum_j \overline{y}_j$, with $\overline{x}_i, \overline{y}_j \in L_i$; we define the bracket $[x, y]$ by bilinearty as

$$[x, y] = \sum_{i,j} [\overline{x}_i, \overline{y}_j].$$

Proposition 2.8 *With the above assumptions, the bracket $[x, y]$ defines on $L(G)$ a structure of a graded Lie algebra over $K = \mathbb{Z}$.*

Proof. The bilinearity of the product is immediate ; also since $[x_n, x_m]^{-1} = [x_m, x_n]$ it follows that $[\overline{x}_n, \overline{x}_m] = -[\overline{x}_n, \overline{x}_m]$ for $\overline{x}_n \in L_n$ and $\overline{x}_m \in L_m$. This extends obviously by bilinearty to all the elements of $L(G)$, so $[x, y] = -[y, x]$, for all $x, y \in L(G)$.
It remains just to prove the Jacobi identity ; by bilinearity it suffices to prove it for the homogeneous elements, let $\overline{x}_n \in L_n$, $\overline{x}_m \in L_m$ and $\overline{x}_l \in L_p$; we claim that

$$[\overline{x}_n, \overline{x}_m, \overline{x}_l] + [\overline{x}_m, \overline{x}_l, \overline{x}_n] + [\overline{x}_l, \overline{x}_n, \overline{x}_m] = 0$$

or equivalently

$$[x_n, x_m, x_l].[x_m, x_l, x_n].[x_l, x_n, x_m] = 1 \mod G_{n+m+l+1}. \tag{2.2}$$

We know from the Hall-Witt identity that

$$[x_n, x_m^{-1}, x_l]^{x_m}.[x_m, x_l^{-1}, x_n]^{x_l}.[x_l, x_n^{-1}, x_m]^{x_n} = 1$$

By reducing that modulo $G_{n+m+l+1}$ one obtains

$$([x_n, x_m, x_l].[x_m, x_l, x_n].[x_l, x_n, x_m])^{-1} = 1 \mod G_{n+m+l+1}$$

from which (2.2) follows immediately. ■

The following result gives the connection between the evaluation of formal commutators (see Section 1.3) on a group and on its associated Lie algebra.

Proposition 2.9 *Let $L(G)$ be the associated Lie algebra of a group G with respect to a filtration $(G_n)_{n\geq 1}$. For every formal commutator c of weight n, and any $\overline{x}_1, \cdots, \overline{x}_n \in G_1/G_2$, we have*

$$c(\overline{x}_1, \cdots, \overline{x}_n) = \overline{c(x_1, \cdots, x_n)}$$

where $c(\overline{x}_1, \cdots, \overline{x}_n)$ is the commutator computed by the means of the Lie bracket in $L(G)$ and $c(x_1, \cdots, x_n)$ is the value of c in G.

The last result follows immediately from the definition of the bracket in $L(G)$. Let us mention some consequences of the last proposition for $L(G)$ the Lie algebra associated to the lower series of G. First, since $\gamma_n(G)$ is generated by all the commutators of weight n, it follows that every element of γ_n/γ_{n+1} is a linear combination of elements of the form $c(\overline{x}_1, \cdots, \overline{x}_n)$, where $\overline{x}_1 \in \gamma_2(G)$ and c is a commutator of weight n, this proves the following.

Fact 01. *The Lie algebra*

$$L(G) = \bigoplus_{n\geqslant 1} \gamma_n/\gamma_{n+1}$$

is generated as an algebra by the elements of $G/\gamma_2(G)$.

The lower series of Lie algebra L is defined inductively by :

$$\gamma_1(L) = L \text{ and } \gamma_{n+1}(L) = [\gamma_n(L), L], \text{ for } n \geqslant 1$$

.

The second immediate consequence of the proposition is :

Fact 02. For the Lie algebra $L(G) = \oplus_{n\geqslant 1}\gamma_n(G)/\gamma_{n+1}$ *we have*

$$\gamma_k(L(G)) = \bigoplus_{n\geqslant k}\gamma_n(G)/\gamma_{n+1}.$$

Recall also that the group G is said to be nilpotent of class c if $\gamma_{c+1}(G) = 1$ and c is the smallest non-negative integer with this property. The same definition applies when we replace G by a Lie algebra.

Corollary 2.10 *A group G is nilpotent of class n if and only if the Lie algebra $L(G) = \bigoplus_{n\geqslant 1}\gamma_n(G)/\gamma_{n+1}$ is nilpotent of the same class.*

2.3 Dimension subgroups

Definitions

Throughout this section K denotes a commutative unital ring and G is a group.

Definition 2.11 *The algebra $K[G]$ of G over K is the set of all finite (formal) sums $\sum_{g\in G} x_g g$, with $x_g \in K$, for all $g \in G$ (so $x_g = 0$, for almost all the g's). That set is endowed with an addition and a multiplication as follows : for two elements $x = \sum_{g\in G} x_g g$ and $y = \sum_{g\in G} y_g g$ of $K[G]$*

$$x + y = \sum_{g\in G}(x_g + y_g)g$$

and

$$xy = \sum_{g,h\in G}(x_g y_h)gh.$$

It follows that $K[G]$ has a structure of an associative K-algebra, where the action of K on $K[G]$ is defined by

$$\lambda\left(\sum_{g\in G} x_g g\right) = \sum_{g\in G}(\lambda x_g)g.$$

For example consider $G = \{e, a, a^2\} \simeq \mathbb{Z}_3$, and $K = \mathbb{Z}$. Then an element of $\mathbb{Z}[G]$ has the form $n \cdot e + m \cdot a + l \cdot a^2$ for some $n, m, l \in \mathbb{Z}$. If we take $x = 2e + a$, $y = 3a - a^2$ then

$$x + y = (2e + a) + (3a - a^2) = 2e + 4a + a^2$$

$$xy = (2e + a)(3a - a^2) = -e + 6a + a^2$$

For another example, consider the Klein group $G = \{e, a, b, ab\} \simeq \mathbb{Z}_2 \rtimes \mathbb{Z}_2$; so the table of multiplication of G can be written as

	e	a	b	ab
e	e	a	b	ab
a	a	e	ab	b
b	b	ab	e	a
ab	ab	b	a	e

An element of $\mathbb{Z}[G]$ has the shape $n \cdot e + m \cdot a + l \cdot b + t \cdot ab$ for some $n, m, l, t \in \mathbb{Z}$; if we take $x = a - 100b, y = a + ab, z = e + b \in \mathbb{Z}[G]$ then

$$x + y = (a - 100b) + (a + ab) = 2a - 100b + ab$$

$$x + z = (a - 100b) + (e + b) = e + a - 99b$$

$$y + z = (a + ab) + (e + b) = e + a + ab$$

$$xy = (a - 100b)(a + ab) = e - 100a + b - 100ab$$

$$xz = (a - 100b)(e + b) = -100e + a - 200b$$

$$yz = (a + ab)(e + b) = 2a + 2ab$$

If G contains a torsion elements, that is an element different from 1 of finite order, then $\mathbb{Z}[G]$ contains zero divisors, Indeed if $g \neq 1 \in G$ satifies $g^n - 1$ for some $n \in \mathbb{N}$ then

$$(g - 1)(1 + g + \cdots + g^{n-1}) = g^n - 1 = 0.$$

A noteworthy is that a longstanding conjecture claims that *the group algebra* $\mathbb{Z}[G]$ *has no zero divisors if* G *is torsion free* (see for instance [20]).

Definition 2.12 *The augmentation map $\varepsilon : K[G] \longrightarrow K$ is defined by*

$$\varepsilon\left(\sum_{g\in G} x_g g\right) = \sum_{g\in G} x_g.$$

Note that $\varepsilon : K[G] \longrightarrow K$ is K-algebra homomorphism that is

$$\varepsilon(x+y) = \varepsilon(x) + \varepsilon(y)$$

$$\varepsilon(xy) = \varepsilon(x)\varepsilon(y)$$

$$\varepsilon(\lambda x) = \lambda\varepsilon(x)$$

It follows that $\ker\varepsilon$ is an ideal of $K[G]$ that is usually denoted by I_G and called *the augmentation ideal* of G (with respect to K). We have

$$I_G = \left\{\sum_{g\in G} x_g g \in K[G] \;\mid\; \sum_{g\in G} x_g = 0\right\}$$

Proposition 2.13 *The augmentation ideal I_G is a free k-submodule on the set $\{g-1 \mid g\in G^*\}$, where $G^* = G - \{1\}$.*

Proof. Let $x = \sum_{g\in G} x_g g \in I_G$, so $\sum_{g\in G} x_g = 0$; if follows that

$$x = x - \sum_{g\in G} x_g.1 = \sum_{g\in G} x_g(g-1).$$

So $\{g-1 \mid g\in G\}$ is a generating set for I_G

Now, let (x_g) be a family in K such that $\sum_{g\neq 1} x_g(g-1) = 0$. Therefore

$$\sum_{g\neq 1} x_g(g-1) = \left(\sum_{g\neq 1} x_g\right).1 + \sum_{g\in G} x_g g.$$

Thus $x_g = 0$, for all $g \neq 1$. So $\{g-1 \mid g\neq 1\}$ is linearly independent. ■

Let A be a ring, and I, J two ideals of A; the product IJ is by definition the ideal generated by the all products xy, where $x\in I$ and $y\in J$. Thus IJ is the set of all finite linear combinations $\sum a_k b_k$, with $a_k\in I$ and $b_k\in J$.

Note that the ideal IJ is contained in the ideal $I\cap J$. Indeed, if $x\in IJ$, then x can be written as $x = \sum_{i=1}^{n} a_k b_k$ with $a_k \in I$ and $b_k\in J$. For each k, we have $a_k b_k \in I\cap J$, therefore $x\in I\cap J$, hence $IJ \subset I\cap J$ as desired.

For example, for $A = \mathbb{Z}$ and all integers $a, b \geq 2$, if we set $I = a\mathbb{Z}$, and $J = b\mathbb{Z}$; then $a\mathbb{Z} \cap b\mathbb{Z} = m\mathbb{Z}$ where m is the *ppcm* of a and b, $a\mathbb{Z} \cdot b\mathbb{Z} = ab\mathbb{Z} \subset m\mathbb{Z}$.

Now, for each $n \in \mathbb{N}^*$, and for each ideal I in some ring A, one can form the ideal $I^n = \underbrace{I_G \cdot I_G \cdots I_G}_{n \quad times}$. Note that (I_G^n) is a decreasing sequence of ideals of A.

Definition 2.14 *For each natural number $n \geqslant 1$, the nth dimension subgroup $D_n(G)$ is defined to be*

$$D_n(G) = G \cap (1 + I_G^n)$$

Since the definition of the augmentation ideal I_G depends on the base ring K, we have actually a dimension subgroup series for each choice of K. Any property that we prove for the subgroups $D_n(G)$ without specifying the base ring K, is in principle true for all the choices of K.

Proposition 2.15 *For every group G, the dimension subgroup series $(D_n(G))_n$ forms a filtration of G.*

Proof. First, note that the $D_n(G)$'s form a decreasing sequence of subsets of G since the powers $(I_G^n)_n$ form a decreasing sequence of ideals. Let's show that each $D_n(G)$ is a subgroup of G. Obviously, $1 = 1+0 \in 1+I_G$, so $D_n(G)$ is not empty. Now, if $x, y \in D_n(G)$, then by the identity

$$xy^{-1} - 1 = ((x-1) - (y-1))y^{-1}$$

the member on the left lies in I_G^n, hence $xy^{-1} \in D_n(G)$.

Now, let $x \in D_n(G)$ and $y \in D_m(G)$. Then by the identity

$$[x, y] - 1 = x^{-1}y^{-1}((x-1)(y-1) - (y-1)(x-1))$$

Obviuously, $(x-1)(y-1)$ and $(y-1)(x-1)$ lies I_G^{n+m} from which it follows that $[x, y] \in D_{n+m}(G)$, so $[D_n(G), D_m(G)] \subseteq D_{n+m}(G)$. This completes the proof. ■

Integral dimension subgroups

The integral dimension subgroups are the dimension subgroups taken with respect to the base ring $\mathbb{Z}$. As we have seen, the integral dimension subgroups $D_n(G)$ of a group G form a filtration for G, so in particular $\gamma_n(G) = D_n(G)$, for all $n \geq 1$.

Proposition 2.16 *Let $\mathbb{Z}[G]$ be the integral group ring of a group G then $D_1(G) = G'$ and $G/G' \simeq I_G/I_G^2$.*

Proof.

$$\begin{aligned} \varepsilon : & G \to I_G/I_G^2 \\ & 1 \mapsto \overline{1-x}. \end{aligned}$$

A proved that ε is a isomorphism, i,e homomorphism bijection $\varepsilon(xy) = \overline{1-xy}$, we have

$$\begin{aligned} 1 - xy &= 1 - x + x - xy = (1-x) + x(1-y) \\ &= (1-x) + x(1-y) - (1-y) + (1-y) \\ &= (1-x) + (1-y) - (1-y)(1-x). \end{aligned}$$

So $\varepsilon(xy) = \overline{1-xy} = (\overline{1-x}) + (\overline{1-y}) - \overline{(1-y)(1-x)}$

Notations that $G/\ker\varepsilon \simeq Im\varepsilon \subseteq I_G/I_G^2$ abLien, so $G' \subseteq \ker\varepsilon$ then

$$\begin{aligned} \overline{\varepsilon} : & G/G' \to I_G/I_G^2 \\ & \overline{x} \quad \mapsto \overline{1-x}. \end{aligned}$$

Let $1-[x,y] = \overline{1-x^{-1}y^{-1}xy} = (\overline{1-x^{-1}y^{-1}}) + (\overline{1-xy})$, we defined a nother map denoted τ such that

$$\begin{aligned} \tau : & I_G \to G/G' \\ & a \quad \mapsto \prod_{x \in G} x^{-ax} \end{aligned}$$

If we take $a = \sum_{x\in G} n_x x$ and $b = \sum_{x\in G} m_x x \in I_G$ impliqe that $a+b = \sum_{x\in G}(n_x+m_x)x$,so

$$\begin{aligned} \tau(a+b) &= \overline{\prod_{x\in G} x^{-(n_x+m_x)x}} = \overline{\prod_{x\in G} x^{-n_x x} x^{-m_x x}} \\ &= \overline{\prod_{x\in G} x^{-n_x x}} . \overline{\prod_{x\in G} x^{-m_x x}} \\ &= \tau(a)\tau(b) \end{aligned}$$

Note that I_G (resp, I_G^2) is generated by the elements of the form $(1-x)$ (resp, $(1-x)(1-y)$) for all $x, y \in G$ in all that ablian group (resp,additive group), we have

$$\tau((1-x)(1-y)) = \tau(1-x-y+xy) = \overline{1xy(xy)^{-1}} = \overline{xyy^{-1}x^{-1}} = \overline{1}$$

So $\ker \tau$ contains the generators of I_G^2 from where $I_G^2 \subseteq \ker \tau$, we have

$$\begin{aligned} \overline{\tau} :& I_G/I_G^2 \to G/G' \\ & \overline{a} \mapsto \varepsilon(x). \end{aligned}$$

Note that

$$\overline{\tau} \circ \overline{\varepsilon} : G/G' \to G/G'$$

for $\overline{x} \in G/G'$:

$$\overline{\tau} \circ \overline{\varepsilon} = \overline{\tau}(\overline{1-x}) = \tau(1-x) = \overline{1^{-1}.x} = \overline{x} = 1_{G/G'}(\overline{x})$$

On the other hand

$$\overline{\varepsilon} \circ \overline{\tau} : I_G/I_G^2 \to I_G/I_G^2$$

for $\overline{a} \in I_G/I_G^2$:

$$\begin{aligned} \overline{\varepsilon} \circ \overline{\tau}(\overline{a}) = \overline{\varepsilon}(\overline{\tau}(\overline{a})) &= \overline{\varepsilon}(\overline{\prod_{x \in G} x^{-n_x x}}) = \overline{\varepsilon}(\prod_{x \in G} \overline{x}^{-n_x x}) \\ &= \sum_{x \in G} \overline{\varepsilon}(\overline{x}^{-n_x x}) = \sum_{x \in G} -n_x \overline{\varepsilon}(\overline{x}) \\ &= \sum_{x \in G} -n_x(\overline{x-1}) = \sum_{x \in G} -n_x \overline{x} - \sum_{x \in G} -n_x.1 \\ &= \overline{\sum_{x \in G} -n_x x} = \overline{a} \\ &= 1_{I_G/I_G^2}(\overline{a}) \end{aligned}$$

Hence, ε is an isomorphism ■

Corollary 2.17 *For every group G, we have $\gamma_2(G) = D_2(G)$.*

It is known moreover that $\gamma_3(G) = D_3(G)$. The problem of whether $\gamma_n(G) = D_n(G)$ is know as *the dimension subgroup problem*. Though this problem has a positive solution

for many classes of groups (Free groups, Groups with torsion free lower central factors, ect), E.Rips (1972), exhibited a finite 2-group for which $\gamma_4(G) \neq D_4(G)$; the Rips group has the following presentation :
Consider the group $G =< a_0, a_1, a_2, a_3, b_1, b_2, b_3, c | a_0, a_1, a_2, a_3, b_1, b_2, b_3, c \in G >$ and defining relations.

$$b_1^{64} = b_2^{16} = b_3^4 = c^{256} = 1$$

$$[b_2, b_1] = [b_3, b_1] = [b_3, b_2] = [c, b_1] = [c, b_2] = [c, b_3] = 1$$

$$a_0^{64} = b_1^{32}, a_1^{64} = b_2^{-4} b_3^{-2}, a_1^{62} = b_1^4 b_3^{-1}, a_3^4 = b_1^2 b_2,$$

$$[a_1, a_0] = b_1 c^2, [a_2, a_0] = b_2 c^8, [a_3, a_0] = b_3 c^{32}$$

$$[a_2, a_1] = c, [a_3, a_1] = c^2, [a_3, a_2] = c^4,$$

$$[b_1, a_1] = c^4, [b_2, a_2] = c^{16}, [b_3, a_3] = c^{64},$$

$$[b_i, a_j] = 1 \text{ if } i \neq j, [c, a_i] = 1 \text{ for } i = 0, 1, 2, 3.$$

Then $\gamma_4(G) = 1$ while the element

$$[a_1, a_2]^{128} [a_1, a_3]^{64} [a_2, a_3]^{32} = c^{128}$$

is a non-identity element in $D_4(G)$.

G. Higman reduced the Dimension subgroup problem to the class of finite p-groups, and by a result of Passi we know that $\gamma_4(G) = D_4(G)$, for G a finite p-group with p odd. So it was natural to check the problem $\gamma_4(G) = D_4(G)$, when G is a 2-group, According to the litterature there were at least three published papers all claiming to prove the DSP, so Rips counter-example is really a landmark in the subject.

Modular dimension subgroups

For a group G, the p-modular dimension subgroups $D_n(G)$, are the dimension subgroups, taken with respect to the finite filed $\mathbb{F}_p$, In the sequel $L_p(G)$ will denote the Lie algebra defined by the p-modular dimension subgroup, so

$$L_p(G) = \bigoplus_{n \geqslant 1} D_n / D_{n+1}.$$

The subgroups $D_n(G)$ are also known as the Jenning-Zassenhauss-Lazard subgroups of G.

Theorem 2.18 *For every group G, and each positive integer n, we have*

$$D_n(G) = \prod_{i.p^j \geq n} \gamma_i(G)^{p^j}.$$

The last formula was discovered by M. Lazard. The proof, though elementary, uses extensive commutator calculus. The reader is refered to [5, Chapter 10] for a detailed proof.

Another interesting formula for the $D_n(G)$ is the following :

$$D_n = D^p_{\lceil \frac{n}{p} \rceil} \cdot \prod_{i+j=n} [D_i, D_j]$$

From Lazard's formula, one deduces that $D_n(G)$ is generated by the commutators $[x_1, \cdots, x_k]$, $k \geq n$ and all the powers $[x_1, \cdots, x_i]^{p^j}$ with $i, p^j \geq n$. As a consequance, we have $D_n(G)/D_{n+1}(G)$ is an elementary abelian p-group, so $L_p(G)$ can be viewed as a Lie algebra over the field $\mathbb{F}_p$.

2.4 Identities and infinitesimal identities

Identities in group theory

Let $w(x_1, \cdots, x_n)$ be an element of F_n and let G be a group, we can define a map from G^n into G by sending each $(g_1, \cdots, g_n) \in G^n$ to the element $w(g_1, \cdots, g_n) \in G$ obtained by replacing each indeterminate x_i by g_i. We call the later map the evaluation of the word w on G, and $w(g_1, \cdots, g_n)$ the value of the word w on $(g_1, \cdots, g_n)$.

For example, we have

1. For $w_1 = x_1^{-1}x_2^{-1}x_1x_2 \in F_2$, the associated evaluation map is given by

$$\begin{array}{ccc} G^2 & \to & G \\ (g_1, g_2) & \mapsto & [g_1, g_2] \end{array}$$

2. Consider, $w_2 \in F_3$ defined by

$$w_2 = x_2^{-1}x_1^{-1}x_2x_1x_3^{-1}x_1^{-1}x_2^{-1}x_1x_2x_3.$$

More concisely, $w_2 = [x_1, x_2, x_3]$; hence the associated evaluation is

$$\begin{array}{ccc} G^3 & \to & G \\ (g_1, g_2, g_3) & \mapsto & [g_1, g_2, g_3] \end{array}$$

3. For $w_3 = x^n \in F_1$, we have

$$\begin{array}{ccc} G & \to & G \\ g & \mapsto & g^n \end{array}$$

Definition 2.19 *We say that w is an identity for G if the map on G induced by w is trivial ; that is to say $w(g_1, \cdots, g_n) = 1\ \forall g_1, \cdots, g_n \in G$.*

Examlpe

Define inductively a family of words in F the free group on $X = \{x_1, x_2, \cdots, x_n\}$ by $c_1(x_1) = x_1$, and $c_{n+1}(x_1, \cdots, x_{n+1}) = [c_n(x_1 \cdots, x_n), x_{n+1}]$. For instance, $c_2(x_1, x_2) = [x_1, x_2]$, $c_3(x_1, x_2, x_3) = [[x_1, x_2], x_3]$, ...
One can prove that a group G satisfies the identity $c_n(x_1, \cdots, x_n) = 1$ if and only if G is nilpotent of class at most n.

P.I Lie algebras

Let $f = f(x_1, \cdots, x_n)$ be a Lie polynomial, that is f is an element of the free Lie algebra L_X on $X = \{x_1, \cdots, x_n\}$.
For a Lie algebra L, and an n-tuple $(a_1, \cdots, a_n)$ of elements of L, the map $x_i \mapsto a_i$ for $i = \overline{1, n}$ extends to a Lie algebra homomorphism

$$\phi : L_x \to L$$

hence $f(x_1, \cdots, x_n)$ has an image in L by ϕ which we denote by $f(a_1, \cdots, a_n)$ and which we the value of the polynomial f on $(a_1, \cdots, a_n)$. In that manner, we define actually a map $f : L^n \to L$ which assigns to each n-tuple $(a_1, \cdots, a_n) \in L^n$, the element $f(a_1, \cdots, a_n)$.

Definition 2.20 *Let L be a Lie algebra, we say that L satisfies a polynomial identity (or L is a P.I Lie algebra) if there exists a Lie polynomial $f(x_1, \cdots, x_n)$ such that $f(a_1, \cdots, a_n) = 0$, for all $a_1, \cdots, a_n \in L$.*

Example

1) An abelian Lie algebra is a P.I Lie algebra since it satisfies the identity $[x, y] = 0$.
2) Define a family of Lie polynomials on x_1, x_2, x_n by $f_1(x_1) = x_1$ and

$$f_{n+1}(x_1, x_2, x_{n+1}) = [f_n(x_1, \cdots, x_n), x_{n+1}]$$

   A Lie algebra L satisfies the identity $f_n(x_1, \cdots, x_n) = 0$ if and only if it is nilpotent of class at most n.
3) The Engel polynomials $(e_n)_n$ are defined inductively by $e_1(x, y) = [x, y]$, and

$$e_{n+1}(x, y) = [e_n(x, y), y]$$

   A Lie algebra which satisfies the identity e_n is said to be a (left) n-Engel Lie algebras.
4) The linearized Engel polynomials E_n are defined by

$$E_n(x, x_1, \cdots, x_n) = \sum_{\delta} f(x, x_{\delta(1)}, \cdots, x_{\delta(n)})$$

   where δ runs over all the permatations on n elements, and $f_n(x_1, \cdots, x_n)$ is the polynomial defined in example 2 above. For instance the linearized Engel's polynomial E_2 is given by

$$\begin{aligned} E_2(x, x_1, x_2) &= f(x, x_1, x_2) + f(x, x_2, x_1) \\ &= [[x, x_1], x_2] + [[x, x_2], x_1]. \end{aligned}$$

For a Lie algebra L, and an element $x \in L$, define the map $ad_x : L \to L$ by $ad_x(y) = [x, y]$, for $y \in L$.
Note that

$$ad_x^n(y) = [x, \underbrace{y, \cdots, y}_{n\ times}].$$

Note that L satisfies the Engel identity E_n, or equivalently L is an n-Engel Lie algabra if and only if $ad_x^n = 0$, for all $x \in L$.
By a result of G.Higman, the associated Lie algebra $L_p(G)$ of a group G of orders p^k satisfies the linearized Engel identity E_{p^k-1} (e.g[11]).

Definition 2.21 *A Lie algebra L is said to be ad-nilpotent if ad_x is a nilpotent map, for all $x \in L$ that is for every $x \in L$, $\exists n \in \mathbb{N}$ such that $ad_x^n = 0$.*

For instance, Every n-Engel Lie algebra is ad-nilpotent. Similarly, every nilpotent Lie algebra is ad-nilpotent. whether the converce holds is known as the Kurosh Problem for Lie algabras.
It is well known that every finite dimensional Lie algebra over a field which is ad-nilpotent is nilpotent. More generally, if our Lie algebra is defined over a field of of characteristic 0, then the ad-nilpotency implies the nilpotency of the Lie algebra ; this is one of the deep results of Zelmanove on the subset. The last two results can be summarized by saying that the Kurosh Problem has positive solution for Lie algebras of finite dimension over some field, and for Lie algebras defined over fileds of characteristic 0.

Remark 2.22 *The ad-nilpotent Lie algebras are also known under the name of Engel Lie algebras.*

The result of Zelmanov on Lie algebras related to the RBP may be viewed as a solution of the Kurosh problem in the class of P.I Lie algebra. This result will be discussed in the next section.

Passing from identitis to infinitesimal identities

Recall that for every group G and for every prime p, we can associated the Lie algebra $L_p(G)$ defined by the dimension subgroup series with respect to the field $\mathbb{F}_p$.

Definition 2.23 *Let G be a group. we say that G satisfies an infinitesimal identity or for short G is infinitesimally P.I if the Lie algebra $L_p(G)$ satisfies a polynomial identity (for some prime p).*

For example, every group G of exponent p^k, then G is infinitesimally P.I, since $L_p(G)$ satisfies the linearized Engel identity E_{p^k-1}(G.Higman).
Before stating the main result of this section let us introduce the notion of coset identities. Let G be a group, and H be a subgroup of finite index in G. we say that satisfies a coset identity (with respect to H) if there exists a word $w(x_1, \cdots, x_n)$ and elemnets $g_1, \cdots, g_n \in G$ such that $w(g_1h_1, g_2h_2, \cdots, g_nh_n) = 1$, for all $h_1, \cdots, h_n \in G$.

Theorem 2.24 *(Wilson-Zelmanove) .*
A group which satisfies a coset identity is infinitesimally P.I (for every prime p).

The proof is quite technical, and we refer the reader to ([33]) for details.

Chapitre 3

Zelmanov's theorem on Lie algebras

3.1 The main result

The first result of Zelmanov which led to the solution of Restricted Burnside Problem can be stated as follows.

Theorem 3.1 *Let L be a Lie algebra over a field of characteristic $p > 0$ generanted by elements $a_1, \ldots, a_k$ and assume that there exist two positive integers n and m such that :*

(1) L statisfies the linearized Engel's identity E_n.

(2) For every commutator ρ on the generators $x_1, x_2, \ldots, x_n$ We have $ad(\rho)^m = 0$.

Then L is nilpotent.

This result was established in 1989 in the two papers [30, 31]. Later in 1993 Zelmanov established the following more general version (see [32])

Theorem 3.2 *Let L be a Lie algebra over a filed of characteristic $p > 0$ that is generated by $a_1, \ldots, a_k$ and assume that :*

(1) L statisfies a polynomial identity.

(2) Every commutator in $a_1, \ldots, a_k$ ad-nilpotent.

Then L is nilpotent.

3.1.1 On the proof of theorem 3.2

1) A Lie algebera L is said to be locally nilpotent if the Lie subalgebra generated by every finite subset $X \subseteq L$, is nilpotent. Condition (1) guarantees that there is a maximal locally nilpotent ideal $Loc(L)$ such that the quotient $L/Loc(L)$ contains no non-trivial locally nilpotent ideal. If the Lie algebra L is locally nilpotent, then the one considered in [Theorem 3.2] is nilpotent since it is finitely generated. Hence, we may assume that L is not locally nilpotent, and by replacing L by $L/Loc(L)$ we may assume that L contain no locally nilpotent ideal $\neq 0$.

2) **Sandwich elements and Sandwich Lie algebras**

   The notion of Sandwich element in Lie algebra was introduced by KostriKin (1959) :

   Definition 3.3 *An element a of a Lie algebra L is a sandwich element if $[L, a, a] = 0$, and $[L, a, L, a] = 0$. That is to say $[x, a, a] = 0$, and $[x, a, y, a] = 0$, for all $x, a \in L$*

   The following theorem is due to Kostrikin and Zelmanov (1989).

   Theorem 3.4 *(Sandwich Lie algebra).*

   If L is a Lie algebra that is generated by sandwich elements $a_1, \cdots, a_k$ then L is nilpotent.

3) We can replace the base field by its algebraic closure F, and so by replacing L by $L \otimes F$, we may assume that the base field is infinite. We manage to find a polynomial $f(x_1, \ldots, x_n)$ which is not an identity for L, such that $f(a_1, \ldots, a_n)$ is a Sandwich element of L for all $a_1, \ldots, a_k \in L$. If so then we have to show that the subspace generated by all the $f(a_1, \ldots, a_n)'s$ is an ideal. By the theorem on Sandwich algebra, the later ideal is locally nilpotent, which contradicts our assumption that L contains no locally nilpotent ideal $\neq 0$ which gives a contradiction. Hence all is reduced to coustructing such a sandwich valued polynomial f.

4) It may happen that our Lie algebra L containes no sandwich valued polynomial (even if its is a PI algebra). To follows the idea in (3) one has to change the Lie algebra L and even the notion of polynomials. This can be done as fallous :
   We consider the associative F-algebra E on the generators $e_1, \cdots, e_n, \cdots$ subset to the relations $e_i^2 = 0$ and $e_i e_j = e_j e_i$ for all $i, j \geq 1$.
   It follows easily that the set of elements $e_I = e_{i_1} e_{i_2} \ldots e_{i_n}, I = i_1, i_2, \cdots, i_1$ and $i_1 < i_2 < \cdots < i_1$, from a basis for E. (Exactly like the alternating algebras).
   Now, we replace L by
   $$\widehat{L} = L \otimes_F E$$
   and the prodect in $\widehat{L}$ is defined as usual by
   $$(l \otimes a).(l' \otimes a') = [l, l'] \otimes aa'.$$
   Note that every element of $\widehat{L}$ can be uniquely written as a finite sum
   $$\sum_I l_I \otimes e_I = \sum_I a_I.$$
5) Now, we let V_L be the ideal of $\widehat{L}$ consisting of all the elements $a = \sum_I a_I$, with $a_I = 0$ if $i \notin I$. It follows that $V_i^2 = 0$ and $\widehat{L} = \bigoplus_{i \geq 1} V_i$.
   Consider a finite subset A of $\widehat{L}$ such that :
   i) $|A \cap V_i| \leq 1$ for all i.
   ii) $[a, b] = 0$, for all $a, b \in A$
   Define a linear maps on $\widehat{L}$, for each $k \geq 1$ by setting
   $$U_k(A) = \sum_k ad_{a_1}.ad_{a_2}.\ldots ad_{a_k}$$
   . Where the sum runs over all the subsets of A having k elements . we have for instance $U_1(A) = \sum_{a \in A} ad_a$ For $K = 0$ we set $U_0(A) = Id$
6) Now, we define for the Lie algebra L, a set of words in the alphabets, $x_i, U_j, (,), [,]$ and ad_i, $i \geq 1$ and $j \geq 1$. These words will be called the U-words (relative to L). If only the letters $x_1, \ldots, x_n$ occur in such a word w, we write $w = w(x_1, \ldots, x_n)$.

And if the variables x_i are replaced by elements a_i of $\widehat{L}$ then the value $w(a_1, \dots, a_n)$ is a linear map on $\widehat{L}$. The U-words on L are defind by the fallowing axioms :

i) If ρ is a commutator in $x_1, \dots, x_n$ then the word $w = ad(\rho)$ is a U-word. The value of this word on $a_1, \dots, a_n$ is the map $ad(\rho(a_1, \dots, a_n))$.

ii) If $w(x_1, \dots, x_r)$ is a U-word then $ad(x_i w)$ is a U-word. The value of that word on $a_1, \dots, a_r, a_i$ is the linear map $ad(a_i w(a_1, \dots, a_n))$.

iii) If v and w are U-words, then vw is also a U-woord, and its associated value is the product of the linear maps associated to v and w.

iv) If $w = w(x_1, \dots, x_r)$ is a U-word such that for all $a, b, a_1, \dots a_r \in \widehat{L}$ we have

$$[aw(a_1, \cdots, a_r), bw(a_1, \dots, a_r)] = 0$$

and x does not occun in w ; then $w' = U_k(xw)$ is a U-word. If we replace $x_1, \dots, x_r$ and x respectively by $a_1, \dots, a_r$ and $a = \sum_I a_I$; then the set :

$$A = a_I w(a_1, \dots, a_r)$$

satisfies the condition of paragraph (5).

The value of w' on $a_1, \dots, a_r$ and a is just the map $U_k(A)$.

7) Finally, we define the notion of U-polynomials by the following process : Consider homogeneous Lie polynomials $f(t_i, y_j, z_k)$ whose variables are devided into three disjoint subsets $\{t_i\}, \{y_j\}$ and $\{z_k\}$ and two U-words $w = w(x_1, \dots, x_r)$, $w' = w'(x_1, \dots, x_r)$.

If the variables t_i, y_i, z_j are given value $a_i, b_j, c_k \in \widehat{L}$, respectively ; and $x_1, \dots, x_r$ are given the values $d_1, \dots, d_r$. The value of the U-polynomial $(f(t_i; y_j; z_k); w; w')$ is :

$$f(a_i w(d_1, \dots, d_r), b_j w'(d_1, \dots, d_r), C_k).$$

Proposition 3.5 *Let L be a Lie algebra over a field of characteristic p>0 ; and suppose that L satisfies a PI. Then there exists a U-polynomial $f = f(t_i w, y_j, w', z_k)$ which is not identically* 0 *on $\widehat{L}$ but all the values of f on $\widehat{L}$ are sandwiches.*

By the above proposition we construct a polynomial $\widehat{f}$ on $\widehat{L}$ all of its values are linear combination of a bounded number of Sandwiches.

At this point, the idea of using Sandwich algebras can be applied to obtain the main result.

3.2 CONSEQUENCES FOR RESTRICTED BURNSIDE PROBLEM

We have seen in Section 1.3 that a positive solution of the RBP will follow if we could prove the finiteness of the restricted Burnside groups $R(d, p^n)$, where p is prime

Theorem 3.6 *For all positive integers d, n and every prime p the group $R(d, p^n)$ is finite*

Before completing the proof of Theorem 3.6, we need the following result of M.lazard.

Proposition 3.7 *let G be a groups and $x \in G$ such that $x^{p^n} = 1$ for some prime power p^n. Assume that $x \in D_k(G) - D_{k+1}$ and let $\overline{x}$ be the corresponding element in $D_k(G)/D_{k+1} \subseteq L_p(G)$, Then $\overline{x}$ is ad-nilpotent of degree at most p^n, that is $ad_{\overline{x}}^{p^n} = 0$.*

For a proof we refer the reader to lazard's thes's " sur les groupes nilpotents et les anneaux de Lie"

Proof of Theorem 3.6.

Let $G = R(d, p^n)$ and assume that $a_1, a_2, \cdots, a_d$ are generators for G, also let L be the subalgebra of $L_p(G)$ generated by $\overline{a_1}, \cdots, \overline{a_d}$, where $\overline{a_i}$ denotes $a_i D_2(G)$.

Any commutator $c(\overline{a_1}, \cdots, \overline{a_n}) \in L$ correspods to the group commutator $c(a_1, \cdots, a_k)$ mod D_{k+1}. Since $c(a_1, \cdots, a_d)^{p^n} = 1$, it follows from lazard's result that $c\overline{a_1, \cdots, a_d} = c(\overline{a_1}, \cdots, \overline{a_d})$ is ad-nilpotent of degre $\leq p^n$.

We know on the other hand that $L_p(G)$ satisfies the linearized Engel identity E_{p^n-1}, so in particular L satisfies a P.I (alternatively we can use Theorem (2.24) to establish the leter). It follows that L satisfies the conditions in Zelmanov's theorem thus L is nilpotent so $\exists c > 0$ such that $\gamma_c(L) = 0$.

Now, recall that $D_k(G)$ is generated by all the elements $[x_1, \cdots, x_i]^{p^j}$ where $ip^j \geq kj$, in other words. $D_k(G)$ is generated by all the elements ρ^{p^i} where ρ is a left normed commutator of weight at least i such that $ip^j \geq k$.

Hence from $\gamma_c(L) = 0$ one deduces that every commutator in $a_1, \ldots, a_d$ lies in $D_c(G)$, and so can be written as a product of powers $\rho_i^{p^{s_i}}$, where each ϱ_i is a left normed commutator of weight $l(\varrho_{i)}$ satisfying $l(\varrho_i)p^{s_i} \geq c$.

Let $m \geq 1$, $g \in G$, and $\rho_1, \ldots, \rho_r$ be all the left normed commutators in $a_1, \ldots, a_d$ of weight $< c$ then we can write

$$g = \varrho_1^{k_1} \varrho_2^{k_2} \ldots \varrho_r^{k_r} g^{'}$$

for some $g^{'} \in D_c$ and some positive integers $k_1, \ldots, k_r$. Since G is periodic, let p^s be the largest among the orders of the ϱ_i's ; thus each $\varrho_L^{k_i}$ in the expression of G can have at most p^s values, so

$$|G/D_n(G)| \leq rp^s.$$

As $\bigcap_n D_n(G) = 1$, it follows that $|G| \leq p^{s_r}$. So G is finite. ■

3.3 Other fectures of zelmanov's theorem

Periodic compact groups

An old Burnside-like problem asks whether every compact periodic group is finite. By a result of Platonov, every periodic compact group is a profinite group.

Definition 3.8 *A profinite group G is a topological group that is compact such that the normal subgroups of finite index form a basis for the neighborhoods of the identity.*

A profinite group can be defined alternatively to be an inverse limite of a system of finite groups (each endowed with the disrete topology). Indeed, for a profinite group one can from the system $(G/N)_{N \lhd_\circ G}$ where $N \lhd_\circ G$ means that N is an open subgroup of G, (each $N \lhd_\circ G$ has finite index in G since G is compact), and define the transition morphisms

$$G/N \to G/M$$

$$xN \to xM$$

whenever $N \subseteq M$ and $N, M \triangleleft_\circ G$. For this system we have a natural morphism

$$G \to \varprojlim G/N$$

$$x \to (xM)_{N \triangleleft_\circ G}$$

which is a homemorphism. Conversely, if $(G_i, \phi_{ij})_{i>j}$ is an inverse system of finite groups. Then $\prod_{i \in I} G_i$ is a compact group by Tychonov theorem, since

$$\varprojlim G_i = \{(x_i) \in \prod_{i \in I} G_i | \phi_{ij}(x_i) = x_j, \text{ for } i > j\}$$

is a closed subgroup of $\prod_{i \in I} G$, then $\varprojlim G_i$ is compact. Moreover, the subgroups $\prod_{i \in I} X_i$, where $X_i = G_i$ except for a finite number of indices form a basis for the neighbors of 1, so their traces on $\varprojlim G_i$ form also a basis for the neighbors of 1. This shows that $\varprojlim G_i$ is a profinite group.

By the classification of finite simple groups (CFSG), Wilson reduced the problem on one about pro-p-groups.

Definition 3.9 *A pro-p-group is a compact topological group in which the normal subgroups of p-power index from a basis for the neighbors of 1.*

Obviously, Every pro-p-group is profinite.

Theorem 3.10 *(Zelmanov* 1993*)*
Every finitely generated periodic pro-p-group is finite.

Note here that a topological group G is said to be finitely generated if it contains a finite subset X such that $G = \overline{\langle X \rangle}$, That is the subgroup $\langle X \rangle$ is dense in G.
The proof of this Theorem 3.10 follows the lines of that of the theorem in the previous section :

1) First using Lazard's result, one shows that every commutator in the generators of $L_p(G)$ is ad-nilpotent.
2) To show that $L_p(G)$ is a PI algebra we can use Theorem (2.24) of Section (2.4) to this end, we need a formalution of the Baire categoty theorem.

Proposition 3.11 *Let G be a profinite group, and $(C_n)_n$ be a family of closed subsets in G such that $\sqcup_n C_n$ contains a non-empty open subset, then some C_n contains a non-empty open subset.*

For a periodic pro-p-group G, we set for each $n \geq 1$, $C_n = \{x \in G | x^{p^n} = 1\}$. Since the mapppings $x \mapsto x^{p^n}$ are continuous, each subset C_n is closed, and obviously $G = \sqcup_{n \geq 1} C_n$. Thus by the Baire category theorem $\exists n \geq 1$ such that C_n contains a non-empty open subset of G. Since G is profinite, that means that C_n contains some coset gH, where H is an open subgroup of G, so $gH \subseteq C_n$, and it follows that $(gh)^{p^n} = 1$ for all $h \in H$. Thus G satisfies a coset identity with respect to H and $w = x^{p^n}$ the word in $L_p(G)$ satisfies PI by Theorem 2.24 G is infinitisimally P.I, or equivalenty $L_p(G)$ satisfies a P.I [Theorem (2.24)].

3) At this stage, we can dednce that $L_p(G)$ is nilpotent, as we did in the last theorem, we many dednce that $|G/\overline{D_n(G)}|$ is bouded by a c_n termes of p and the nilpotency class of G, Since the later holds for all n, it follows that G has a bouded order.

Remark 3.12 *1) The Restricted Burnside Problem can be formulated in the class of pro-p-groups by saying that for every finitely generated pro-p-group G, the subgroup $\overline{G^{p^k}}$ is open for all $k \geq 1$, or equivalently that every finitely generated pro-p-group G such that $x^{p^k} = 1, \forall x \in G$ is finite. Every such a group is periodic and so finite.*

(2) Similar ideas lead to the fact that every profinite Engel group, which is finitely generated is nilpotent (see [33]).

Conclusion

Though the RBP has a positive solution, it is of great importance to estimate the order of magnitude of the groups $R(d, p^k)$. Computational methods led to some very particular results in this direction, concerning for instance $d = 2$ and $p^k = 5$. Adian and Repin Showed that the nilpotency class of $R(d, p)$ must grow at least exponentially with p. An interresting conjucture in this direction claims that :

There is a constant C such that every d-generated finite group of exponent p is nilpotent of class at most dC^p.

If this conjecture is true, then $R(d, p)$ should have order at most p^{dc^p}.

Recent work shows that still Zelmanov's ideas can be used to attack other related problems. We wish to follow these ideas to establish significant results in this direction in our Ph.d project.

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Abstract :

This work is a survey of the restricted Burnside problem. We have provided an overview of the most important solution to this problem, which Zelmanov reached in 1989, as well as related findinge and topics.

keywords : Lie algebra, dimension subgroup, filtration.

Résumé :

Ce travail est une étude de détermination de la portée du problème Burnside restreint. Nous avons fourni un aperçu de la solution la plus importante à ce problème et a atteint Zelmanov en 1989, en plus des résultats et des sujets liés à eux.

Mot clés : algèbre de Lie, sous-groupe de dimension, filtration.

ملخص:

هذه المذكرة عبارة عن دراسة استقصائية لمشكلة برنسايد المقيدة. و قد قدمنا لمحة عن أهم حل لهذه المشكلة و الذي توصل إليه زيلمانوف سنة 1989, بالإضافة إلى نتائج و مواضيع متعلقة بها.

كلمات مفتاحية: جبرلي, زمرة جزئية بعدية, الترشيح.