Some properties of the cohomology of finite p-groups

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Abstract

This work treats a conjecture of Peter Schmid on the Tate cohomology of finite p-groups. We examine some cases in which the conjecture has an affirmative answer, as well as the relevance of this conjecture to studying automorphisms of finite p-groups.
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1. Introduction

As we mentioned in the abstract, we study in this work a conjecture of Schmid on the cohomology of finite p-groups; this conjecture is made more precise in Section 3. Before, we recall some basic fact on the Tate cohomology groups and we discuss a theorem of Gaschutz and Uchida on the triviality of the cohomology of finite p-groupss; this theorem is of great importance in studying the mentioned conjecture. Finally, we give some results that affirm the conjecture in some particular cases. A noteworthy, is that exactly 10 sporadic counterexamples (which are 2-groups of order 256) to the conjecture are known, and it is still widely open.

2. The Tate cohomology groups

Let G be a finite group, and A a right G-module; the latter means that we have an action $A \times G \to A$ that satisfies the following conditions:

1. $a(gg') = (ag)g'$;
2. $a1 = a$;
3. $(a+b)g = ag + bg$;

for all $g, g' \in G$, and all $a, b \in A$.

Roughly speaking, the Tate cohomology groups of G with coefficients in A is a family $\hat{H}^n(G,A)$ of abelian groups indexed by $n \in \mathbb{Z}$. Some properties of these groups are given below:

1. Every short exact sequence of G-modules $0 \to A \to B \to C \to 0$ gives raise to a long exact sequence of abelian groups
$$\ldots \to \hat{H}^n(G,B) \to \hat{H}^n(G,C) \xrightarrow{\delta} \hat{H}^{n+1}(G,A)$$
$$\to \hat{H}^{n+1}(G,B) \to \ldots$$
2. We define the norm map $\tau : A \to A$ by $\tau(a) = \sum_{x\in G} ax$. The submodule of fixed point A^G is
$$A^G = \{a \in A \mid ax = a \text{ for all } x \in G\};$$
and we define $[A,G]$ to be the submodule of A formed by the elements $ax - a$, where $a \in A$ and $x \in G$. One sees immediately that A^τ the image of A by τ is contained in A^G; and $[A,G] \subseteq \ker\tau$. We have by definition
$$\hat{H}^0(G,A) = A^G/A^\tau$$
and
$$\hat{H}^{-1}(G,A) = \ker\tau/[A,G].$$
3. $\hat{H}^n(G,A)$ coincides with the usual cohomology groups $H^n(G,A)$ for $n \geq 1$.
4. $\hat{H}^n(G,A)$ coincides with the usual homology groups $H_{-n-1}(G,A)$ for $n \leq -2$.

3. Modules of trivial cohomology and a conjecture of Schmid

We say that A is a *cohomologically trivial* module over G if $\hat{H}^n(S,A) = 0$ for all $S \leq G$ and all integers n.

If G and A are finite p-groups, we have the following nice result proved independently by W. Gaschütz and K. Ushida:

Theorem 3.1 *Let G be a finite p-group, and A be a G-module which is also a finite p-group. If $\hat{H}^n(G,A) = 0$ for some integer n, then A is cohomologically trivial.*

Let N be a normal subgroup of a finite group G. Then, $Z(N)$ the center of N can be viewed as a Q-module, where $Q = G/C_G(N)$; the action here is induced by the conjugation in G. Therefore, we can consider the Tate cohomology groups $\hat{H}^n(Q, Z(N)) = 0$.

We shall be mainly interested in the case where G is a finite p-group, and $N = \Phi(G)$ the Frattini subgroup of G. Recall that in general $\Phi(G)$ is defined to be the intersection of the maximal subgroups of G, and in the particular case where G is a finite p-group, $\Phi(G) = G^p[G,G]$.

A conjecture of Schmid. *Let G be a finite p-group, $A = Z(\Phi(G))$ and $Q = G/C_G(\Phi(G))$. Then, the cohomology of A over Q is not trivial.*

4. Some results on Schmid's conjecture

Let G be a finite p-group. We say that G is *strongly semi- p-abelian*, if the following property holds in G :

$$(xy^{-1})^{p^n} = 1 \Leftrightarrow x^{p^n} = y^{p^n} \text{ for any postive integer } n.$$

For brevity, we shall use the term *semi-abelian* for such a group.

The objective of our work is to prove the following theorem.

Theorem 4.1 *Schmid's conjecture holds for all semi-abelian p-groups.*

In fact, more is true.

Theorem 4.2 *Let G be a semi-abelian p-group, and $1 < N \lhd G$ such that G/N is neither cyclic nor a generalized quaternion group. Then $\hat{H}^n(G/N, Z(N)) \neq 0$, for all integers n.*

An important subclass of the semi-abelian p-groups is the class of the so called regular p-groups. The last theorem generalizes a result of Schmid that confirms his conjecture for regular p-groups [8].

The latter results are of some importance in studying the automorphisms of finite p-groups; details can be found in our thesis (in progress).

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