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Some properties of the cohomology of finite p-groups

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Dedication

I dedicate this work to my dear parents my Mother
and my Father "May Allah have mercy on him" for their patience, love, support
and encouragement.
To my sisters ***Chahrazad,Randa,Yasmine,Mariem*** *and* ***Nour*** *and my only brother*
Mohammed Deyaa El ddine*, and my friends in life*
Nor do I forget my friends in the second Master Algebra
and Geometry, year 2017/ 2018.
Without forgetting all the teachers whether primary,
middle, secondary or higher education
To anyone who encouraged or helped me
during my studies

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Contents

Introduction

IN order to know descriptions of groups we need necessary passing by automorphisms ,we denote them :$Aut(G)$, where G is a group .inner automorphisms form a subgroup such that $Aut(G)/Inn(G)$ is isomorphic to $Out(G)$ (exterior automorphisms).
Why the automorphism $\Phi \in Aut(G)$is important ?

Because if a set $S \leq G$ is invariant under Φ, i.e $\Phi.S \leq S$,then Φ is a symmetry. This shows to us : that symmetries are leading behind the tree of automorphisms.

IN 1911 ,Burnside has conjectured :Does there exist any finite group such that G has a non inner class presering automorphisms ?
After two years Burnside found the answer as follows he constructed a group G of order p^6 ,p is an odd prime such that $Out(G) \neq 1$.

The birth of Tate cohomology introduced by john Tate in (1952) ; give a hard tool to research about non inner automorphisms .
The idea of Tate is combining between homology and cohomology in one Sequence.

A longstanding conjecture asserts that: any finite non-abelian p-group possesses at least one non-inner automorphism of order p .
The first one who gave an answer to this question is W . Gaschütz using Tate cohomology and he enjoyed this domain with good ideas: if $\hat{H}^n(G, A) = 0$ for some integer n, then A is cohomologically trivial.

K.Gruenberg says in **[19]** (one of the most ingenious application of cohomology to a purely group the oretical problem is the recent solution by Gaschütz of the question whether ever finite p-group has outer automorphism of order p); When G is a regular p- group, P–Schmid prove the theorem as follows: let G be a regular p- group and N a non-trivial normal subgroup of G. if $Q = G/N$ is not cyclic $\hat{H}^n(Q, Z(N)) \neq 0$ for all n; Thus there exist a non-inner p- automorphism .

Many others searchers have studied some kinds of p-groups: Regular p-groups, coclass, powerful p-groups, semi abelian p-groups, potent p-groups. The semi abelian p-groups are introduce by XU, and the aim of this work is to giving details and reproduce proofs of theorems belongs to this kind of p-groups

- The First chapter is an overview of the structure of p-groups.
- The Scened chapter includes other types of p -groups that are related to the p-groups; which is the following:
  - p-group semi-abelian and strongly semi-abelian.
  - Regular p-groups.
  - p-central p-groups.
  - Powerful p-groups.

  and we talked about power structure of p-groups
- In the Third chapter, which is the most important part of our work, which in turn discusses Schmid's conjecture by using cohomological technique, and some results about them. where we initially explained the cohomology of finite p-groups and As we have also mentioned for Gaschütz and Uchida theorem on the triviality of the cohomology of finite p-groups.
- In the last chapter we devoted the study on the existence of non-inner automorphism of p-power order in the finite p-groups and we find there is a relation between cohomologically trivial modules and the existence of non inner automorphism. We also discussed a new type of finite p-groups identified by potent p-group; we also asked whether these potent p-groups were verifying schmid's conjecture

Notations and Terminology

Let G be a group. p is a prime number,
$\mathrm{Aut(G)}$ automorphism group of G. Let $\sigma \in \mathrm{Aut(G)}$ is said inner if it is the form:

$$\sigma(x) = g^{-1}xg = x^g \quad \text{for some} \quad g \in G;$$

σ is called inner automorphism of G.
$\mathrm{Inn(G)}$ the inner automorphism group of G.
$\mathrm{Out(G)} = \mathrm{Aut(G)}/\mathrm{Inn(G)}$ is called the outer automorphism group of G.
for N subgroup of G, $\mathrm{Aut_N(G)}$ denotes the automorphism group σ of G which verify:

$$x^{-1}\sigma(\mathrm{x}) \in N \quad \text{for all} \quad x \in G;$$

$\mathrm{Aut_Z(G)}$ means $\mathrm{Aut_{Z(G)}}$ (central automorphism group of G).

for x and $y \in G$, we define the commutator $[x, y]$=$x^{-1}y^{-1}xy$;

Let X, Y two non-empty parts of G, $[X, Y]$ is a subgroup generated by all commutators

$$[x, y] \quad , x \in G \quad y \in G.$$

$[X,_n Y]$ is defined by $\quad [X,_1 Y] = [X, Y]$
$[X,_n Y] = [[X,_{n-1} Y], Y]$

$\mathrm{C_G(X)}$ the part centralizer $X \subseteq G$ that is to say the set of elements that commutes with all elements of X.

Z(G) =$\mathrm{C_G(G)}$ is the center of G.

$\mathrm{Z_i(G)}$ the terms of ascending central sequence of G.

$\gamma_\mathrm{i}(\mathrm{G})$ the terms of descendant central sequence of G.

$\lambda_\mathrm{i}(\mathrm{G})$ the terms p-central descendant sequence of G.

$\mathrm{G^i}$ the terms of derived sequence of G.

$\Phi(\mathrm{G})$ Frattini subgroup G.

$\mathrm{G^n}$ the subgroup generated by the elements of the form $x^n, x \in G$

$\Omega_\mathrm{i}(\mathrm{G})$ the subgroup generated by the G element of order dividing p^i.

$\Omega_{\{1\}}(\mathrm{p})$ the set of all elements of order at most p in a powerful p-groups.

$\mathrm{d(G)}$ =minimum generator number of G.

CHAPTER 1

THE p-GROUPS

1.1 p-groups structure

Definition 1.1.1 *We say that G group is a p-group if his order is a power of p, if G of order $p^n m$ with m prime to p, we say that a subgroup H of G is p-Sylow of G if H has order p^n.*

Remark 1.1.1

1. *Let S a subgroup of G, S is p-Sylow of G if and only if S is a p-group and $|G : S|$ is prime with p.*
2. *Any conjugate of p-Sylow of G is p -Sylow of G.*

Definition 1.1.2 *A central sequence is a normal subgroup chain*

$$1 = H_0 \leq H_1 \ \leq ... \leq H_r = G$$

such as:

$$H_i/H_{i-1} \leq Z(G/H_{i-1}) \quad \textit{for all } 1 \leq i \leq r$$

we can show that a finite group is nilpotent if and only if it has a central sequence .In fact, it is traditionally the definition of a nilpotent group (perhaps infinite).

Definition 1.1.3 *Let G be a group defined by :*
(i) $G^1 = G'$, the derived subgroup , and $G^r = [G^{(r-1)}, G^{(r-1)}]$.

(ii)$Z_0(G) = 1$, $Z_1 = Z(G)$, and $Z_r(G)/Z_{r-1}(G) = Z(G/Z_{r-1}(G))$, the upper central sequence.

(iii) $\gamma_1(G) = G$, $\gamma_2(G) = G'$, and $\gamma_r(G) = [\gamma_{r-1}(G), G]$ the lower central sequence.

Lemma 1.1.1 *Suppose that :*

$$1 = H_0 \leq H_i \leq ... \leq H_r = G.$$

is a central sequence for G. then $Z_i(G) \geq H_i$ and $\gamma_i(G) \leq H_{r-i-1}$ for all i; This lemma implies that if c the smallest integrates such that $Z_c(G) = G$. Then $\gamma_{c+1}(G) = 1$, and $\gamma_c(G) \neq 1$, and any central sequence has a length of at least c. This integer c is called the nilpotency class of a nilpotent group.

Proposition 1.1.1 *Let G be a group of order p^n then G is nilpotent, and if c denotes its class, then $0 \leq c \leq n-1$, $c = 0$ if and, only if G is trivial, and $c = 1$ if and only if G is abelian.*

Definition 1.1.4 *Let G be a finite group of order p^n, if c denotes its class,then the Coclass of G is the quantity $n - c$.*

Definition 1.1.5 *Let G be a finite abelian group, so G is called elementary abelian if any non-identity element has order p.*

Lemma 1.1.2 *Let G be a group non-abelian of order p, then $Z(G)$ is of order p, and $G/Z(G)$ is elementary abelian.*

1.2 Commutators

Definition 1.2.1 *Let x and y two elements of G, then the subgroup commutator or derived subgroup is defined by:*

$$G^{'} = [G,G] =< [x,y] : x,y \in G >; \text{ if } H \text{ and } K \text{ are two subgroups of } G, \text{ then } [H,K] =<[h,k] : h ,k \in G.>$$

Lemma 1.2.1 *Let G be a p-group, and H every subgroup normal of G.*

(i) *the quotient $G/G^{'}$ is abelian.*

(ii) *if G/H also abelian then $G^{'} \leq H$*

Proof.

Suppose that x and y are two elements of G, then $[x,y] = g$; for some g of $G^{'}$, but from the definition of $[x,y]$ we have this $x^{-1}y^{-1}xy$ = g ; when xy=yx g, and so on $1 \in G^{'}$, xy and yx are in the same subset(coset) of $G^{'}$,so

$$(xG^{'})(yG^{'}) = (xy)G^{'} = (yx)G^{'} = (yG^{'})(xG^{'}) \text{ proving the assertion (i).}$$

If G/H is abelian, then for any element x and y of G, we have yx and xy are in the same coset of H, when $x\,y = y\,xh$;for some $h \in H$, likewise proved (i).

we have $x^{-1}y^{-1}xy = h \in H$, and so since all the generators of $G^{'}$ and this is in H, we have $G^{'} \leq H$. ■

Lemma 1.2.2 *Let G be a group and x, y, z elements of G*

(i) $[x,y,z]$=$[x,z]^y[y,z]$=$[x,z][x,z,y][y,z]$.

(ii) $[x,yz] = [x,z][x,y]^z$=$[x,z][x,y][x,y,z]$

(iii) $[x,y]$=$[y,x]^{-1}$

(iv) (Hall-Witt's identity) $[x,y^{-1},z]^y[y,z^{-1},x]^z[z,x^{-1},y]^x = 1.$

Theorem 1.2.1 ***(Three subgroups lemma)*** *Let G a group and X,Y,Z three subgroups and let N normal subgroup, of G, if $[X,Y,Z]$ and $[Y,Z,X]$ are both contained in N, so it contains $[Z,X,Y]$.*

Proof. Let $x \in X, y \in Y$ and $z \in Z$ as $[X,Y,Z]$ and $[Y,Z,X]$ in N, then $[x,y^{-1},z]^y$ and $[y,z^{-1},x]^z$ are two elements of N (since N is normal), so $([x,y^{-1},z]^y[y,z^{-1},x]^z)^{-1}$ = $[z,x^{-1},y]^x \in N$ by Hall-Witt identity since N normal, we conjugate by x^{-1} to get $[z,x^{-1},y] \in N$, and let's $x' = x^{-1}$. We have $[z,x',y] \in N$, for all $z \in Z$,
$x' \in X$ and $y \in Y$; therefore $[z,x,y] \leq N$ ■

Proposition 1.2.1 *Let G and $\gamma_n(G)$ the descendant central sequence of G, then*

$$[\gamma_n(G),\gamma_m(G)] \subseteq \gamma_{n+m}(G)$$

Proof. (by induction), for $n = 1$. $[\gamma_1(G),\gamma_m(G)] = [G,\gamma_m(G)]$ =$\gamma_{m+1}(G)\forall m \geq 1$ Suppose the proposition is true for all $n-1$, $\forall m \geq 1$

$[\gamma_{n-1}(G),\gamma_m(G)] \subseteq \gamma_{n+m-1}(G)$ such that $\gamma_n(G) = [\gamma_{n-1}(G),G]$ we put $N = \gamma_{n+m}$

1/ $[\gamma_{n-1}(G),G,\gamma_m(G)]$.

2/ $[G,\gamma_{n-1},\gamma_m(G)]$=$[\gamma_{m+1}(G),\gamma_{n-1}(G)] \subseteq \gamma_{n+1}(G) = N$.

3/ $[\gamma_m(G),\gamma_{n-1}(G),G]$

= $[\gamma_{n-1}(G),\gamma_m(G),G]$ =$[\gamma_{n+m-1}(G),G]$$\subseteq \gamma_{n+1}(G) = N$

by the lemma of three subgroup we have

$$(1) = [G,\gamma_{n-1},\gamma_m(G)] \subseteq N = \gamma_{n+m}(G); \qquad so[\gamma_n(G),\gamma_m(G)] \subseteq \gamma_{n+m}(G)$$

■

1.3 Frattini subgroup

Definition 1.3.1 *Let G be a finite group, the Frattini subgroup is the intersection of all maximal subgroups of G, it is denoted by $\Phi(G)$.*

Definition 1.3.2 *Let G a group, and $x \in G$, then x is called non-generator if for any group G generated by x and the set X, then $G =< X >$.*

Proposition 1.3.1 *Let G a group, so $\Phi(G)$ is the set of all non-generator of G, so if $G = H\Phi(G)$ for some H then $H = G$.*

Proof. Assume that x is non-generator of G, and let M a maximal subgroup of G. Then $< M, x >\geq M$,and like M is a maximal subgroup of$G$, $< M, x >= M$ so $< M, x >= G$ if $< M, x >= G$ then since x is non-generator $< M >= G$ contradiction; $< M, x >= M$; i.e ,$x \subseteq M$ this is true for all maximals subgroups and for $x \in \Phi(G)$

Conversely, suppose that x is an element of $\Phi(G)$,let G generated by some set X ,and x, so $G =< X, x >$
Denotes by N the group $< X >$.if $N \neq G$ it is contained in a maximal subgroup, say M, but $x \in M$, since x is an element of $\Phi(G) \leq M$ and as $< X, x >\leq M \leq G$ contradiction, this $< X >= G$ for any set X when $< X, x >= G$; i.e x is non-generator.

Finally we put $G = H\Phi(G) =< H, \Phi(G) >$ so $G =< H >= H$ ■

Theorem 1.3.1 *Let G be a finite group, and suppose that $N \trianglerighteq G$. contained $\Phi(G)$, if $N/\Phi(G)$ is nilpotent then N is nilpotent.*

Proof.
A finite group is nilpotent if and only if its p-Sylows subgroups are normals, we will show that the p- Sylow of N are normal, which proves that N is nilpotent ,either P a p-Sylow of N then $P\,\Phi(G)/\Phi(G)$ is a p-Sylow of $G/\Phi(G)$, since if $\mid P \mid = p^d$ and $\mid \Phi(G) \mid = p^e a$ such that $p \nmid a$, then $\mid P\Phi(G)/\Phi(G) \mid = p^{d-1}$, where the powerful of p divides $N/\Phi(G)$ .so $N/\Phi(G)$ is nilpotent, it is also $P\Phi(G)/\Phi(G)$ is a p-Sylow of $N/\Phi(G)$ is normal ,if the p-Sylow of G is normal, it is also characterized; this $P\Phi(G)/\Phi(G)$ characterized by $N/\Phi(G)$ and $P\Phi(G)$ characterized $N \trianglerighteq G$;
we observe that $P\Phi(G) \trianglerighteq G$.
Now we can use **The argument of Frattini** :
P is a p-Sylow of N ,so is a p- Sylow of $P\Phi(G)$ and therefore $G = N_G(P)P\Phi(G)$, and as $P \leqslant N_G(P)$ idem $N_G(P)P = N_G(P)$; from where $G = N_G(P)\Phi(G)$.
This proposition proves that G=$N_G(P)$idem $P \trianglerighteq G$ which implies that $P \trianglerighteq N$ ■

Corollary 1.3.1 *The Frattini subgroup of a finite group is nilpotent.*

Proof.
We take N=$\Phi(G)$ in Theorem 1.3.1 ■

Proposition 1.3.2 *Let G be a finite p-group, then $G/\Phi(G)$ is elementary abelian, and if H is a normal subgroup of G such as G/H is elementary abelian, then $\Phi(G) \leq H$*

Proof. We notice first that any maximum subgroup of p-group is normal and of index p, this means M is a maximal subgroup of G, then G/M is cyclic of order p, idem $G' \leq M$ for any maximum subgroup M, by consequently $G' \leq \Phi(G)$ and $G/\Phi(G)$ is abelian, and as G/M is of order p we know that $(Mx)^p = M\ \forall x \in G$;i.e , $x^p \in \Phi(G)$ and as $\Phi(G)x \in G/\Phi(G)$ Then $\Phi(G)x$ is of order p so it is elementary abelian.
Now we suppose that G/H is elementary abelian of order p^n then G/H is generated by n coset Hx_i of G/H; each coset is of order p, and like $G/H \cong < Hx_1 > \times ... \times < Hx_n >$. ■

Proposition 1.3.3 *Let G^p denotes the group generated by the set $\{ g^p : g \in G\}$; i.e, the smallest group that continents all the elements of order p. Then $\Phi(G) = G'G^p$.*

Proof. Since $\Phi(G)$ contains all x^p, we prove in proposition 1.3.2 $G^p \leq \Phi(G)$ thus $G^{'} \leq \Phi(G)$ since $G/\Phi(G)$ is abelian; $G^{'}G^p \leq \Phi(G)$. to prove the inverse denoted $G/G^{'}G^p$ is elementary abelian and $G^{'} \leq G^{'}G^p$ and also $x^p \in G^p \leq G^{'}G^p$ for all $x \in G$, idem any element of $G/G^{'}G^p$ is of order 1 or p. which implies that $G/G^{'}G^p$ is elementary abelian and therefore $G^{'}G^p \leq \Phi(G)$ ■

Remark 1.3.1 $G/\Phi(G)$ *can be considered as a vector space on* $\mathbb{Z}_{\mathbb{P}}$*,and the dimension of* $G/\Phi(G)$ *coincide with the generators minimal number of G,which is noted by* $\mathrm{d(G)}$.

CHAPTER 2

OTHERS KINDS OF p-GROUPS

2.1 Powerful structure of p-groups

Let G be a finite p-group, assume that

$$\text{expG} = p^e \quad \text{and} \quad e \geq 0 \quad ; \forall 0 \leq n \leq e.$$

we define $\Pi_n : G \to G$ (the n th power mapping of G)

$$\bigwedge_n(G) \text{ the kernel of } \Pi_n : G \to G$$

$$\bigvee_n(G) \text{ the image of } \Pi_n : G \to G$$

$\Omega_n(G) =< \bigwedge_n(G) >$ $\qquad$ $\mho_n(G) =< \bigvee_n(G) >$

We get the following two characteristic subgroup series .

(1) 1= $\Omega_0(G) < \Omega_1(G) \leq ... \leq \Omega_e(G) = G$, the upper power series of G.

(2) $G = \mho_0(G) > \mho_1(G) > ... > \mho_e(G) = 1$,the lower series of G.

2.1.1 The main properties of the power structure

(i) $\mho_n(G) = \bigvee_n(G)$ for all n.

(ii) $\Omega_n(G) = \bigwedge_n(G)$ for all n.

(iii) $\Pi_n : g\,\Omega_n(G) \to g^{p^n}$ is defined a bijection from $G/\Omega_n(G)$ onto $\mho_n(G)$.

(iii') $|G/\Omega_n(G)| = |\mho_n(G)|$; for all n.

We call G group with regular power structure any p-group that satisfies:

1- the exponent of $\Omega_n(G)$ is at most p^n;

2- G^{p^n} coincide with the element of the form $g^{p^n}, g \in G$;

3- $\mid G : G^{p^n} \mid = \mid \Omega_n(G) \mid$;

Definition 2.1.1 *We say that G has a regular power structure if* $\mho_n(G) = \bigvee_n(G)$ *and*

$$\Pi_n : x\ \Omega_n(G) \to \mho_n(G)$$

is a bijection.

2.2 p-group semi-abelian and strongly semi-abelian

The semi-abelian p-group are introduced and studied by Ming You XU (see**[36]**)

Definition 2.2.1 *Let G be a group we say that G is strongly semi-abelian if for all* $x, y \in G$*, we have*

$$(xy^{-1})^{p^n} = 1 \Leftrightarrow x^{p^n} = y^{p^n} \quad \textit{for any positive integer } n.$$

for $n = 1$*, G is said semi-abelian .*

Deduce that if G is strongly p*-group, then the element having an order dividing* p^n *form a subgroup which coincide with* $\Omega_n(G)$*, it is a result the application* $G/\Omega_n(G) \to G^{p^n}$ *that applies* $x\Omega_n(G)$ *to* x^{p^n} *is a bijection.This property* $|G : G^{p^n}| \leq |\Omega_n(G)|$ *for any positive integer* n*.*

Lemma 2.2.1 *Let G be a strongly semi-abelian* p*-group. Then*

$$[x^{p^n}, y] = 1 \Leftrightarrow [x, y]^{p^n} = 1 \textit{ for any } x,\ y \in G \textit{ and } n \in \mathbb{N}.$$

Proof. Suppose that $[x, y]^{p^n} = 1$, we have

$x^{p^n} = (x^{p^n})^y$=$(x[x, y])^{p^n}$ as G is strongly semi-abelian then $[x, y]^{p^n}$=$(x^{-1}x[x, y])^{p^n} = 1$.

inversely, suppose that $[x, y]^{p^n}$ so $(x^{-1}x[x, y])^{p^n} = 1$ and as

$x^{p^n} = (x[x, y])^{p^n} = (x^y)^{p^n} = (x^{p^n})^y$. This shows that

$$[x^p, y] = 1 \ \blacksquare$$

2.3 Regular p-groups

The regular p-group theory is due to P.Hall see(**[22]**).

Definition 2.3.1 *A p-group G is said regular if for all $x, y \in G$, there exists $c \in \gamma_2(< x, y >)^p$, such that $(xy)^p = x^p y^p c$*

Proposition 2.3.1 *Every regular p-group is strongly semi-abelian.*

Proof. See[**[37]**]
the reciprocal of the proposition is not necessary true. ■

Theorem 2.3.1 *Let G be a p-group, then G is regular if and only if, any section of G is semi-abelian.*

Proposition 2.3.2 *Let G a p-group; we have:*

- All abelian p-groups are regular.

-All p-group of class strictly less then p or if the exponent of G is equal to p then G is regular.

-If G doesn't have any normal subgroup of order p^{n-1}, and of exponent p then G is regular.

-If $\gamma_{p-1}(G)$ is cyclic then G is regular.

-If $\mid G : G^p \mid < p^p$, then G is regular.

-If $\mid \gamma_2(G) : \gamma_2(G)^p \mid < p^{p-1}$, then G is regular.

-All regular 2-group are abelian.

- If G is regular then exp($\Omega_n(G)$) $\leq p^n$.

*-The element of the form x^{p^n} form a subgroup and $\mid G : G^{p^n} \mid = \mid \Omega_n(G) \mid$;
for all $n \in \mathbb{N}^*$.*

*-(A.Mann**[31]**) for $p > 2$ if any subgroup of G can be generated by $(p-1)/2$, then G is regular.*

2.4 p-central groups

Definition 2.4.1 *A group G is say p-central if the center of G contains all the element of order p.*
(i.e) $G^p \leq Z(G)$ *;*
G is p-abelian if $(xy)^p = x^p y^p \quad \forall x, y \in G$

Proposition 2.4.1 *G is p-abelian $\Leftrightarrow$ G is regular and $(G^{'})^p = 1$.*

Corollary 2.4.1 *A p-group p-central ,$p \geq 3$ is not necessarily a regular power structure*

Proof. Otherwise, all the p-groups G_n has a regular power structure if the elements of the form x^p in a group G form a subgroup, so it holds true for all quotient of G, it follows that for any p-group G, G^p coincide with the set of element of the form x^p, $x \in G$ (contradiction) ∎

Proposition 2.4.2 *(Hall-Petrescu Formulla)*
Let x and y two elements of a group G,and n a positive integer. Then

$$(xy)^n = x^n y^n c_2^{\binom{2}{n}} c_3^{\binom{3}{n}} \ldots c_n^{\binom{n}{n}} \quad \text{where} \quad c_i \in \gamma_i(G), \text{for each index } i.$$

Proof. view [[**6**],Appendix 1] ∎

Definition 2.4.2 *A group G is p^i-central of height k if all the elements of G which has order dividing p^i are contained in $Z_k(G)$.*

Lemma 2.4.1 *G is a p-group if G is p-central of height $p-1$, then*

$$(xy^{-1})^p = 1 \Rightarrow x^p = y^p \text{ for all } x, y \in G.$$

Theorem 2.4.1 *Let G a p-group with $p > 2$ satisfies $\Omega_1(\gamma_{p-1}(G)) \leq Z(G)$ .Then G is strongly semi-abelian.*

Proof. [36] ∎

Theorem 2.4.2 *Let G a group. If G is p-central of height $p-2$ or p^2-central of height $p-1$; then G is strongly semi-abelian.*

Proposition 2.4.3 *Let G a finite p^k -central group of coclass r then there exists a function $f = f(k, p, r)$ such that the order of G is delimited by p^f*

2.5 powerful p-groups

Definition 2.5.1 *Let G be a group is called powerful p-group if* $\mho_1(G) \supseteq G'$.

Let N subgroup of G ,$N \trianglelefteq G$; N is said powerfully embedded in G if $\mho_1(N) \supseteq [N, G]$.

Theorem 2.5.1 *Let N, M are powerfully embedded in G , then $[N, G]$,$\mho_1(N)$,MN and $[M, N]$ are powerfully embedded in G.*

Proof.

1/ $[N, G]$ is powerfully embedded in G ; we have

$$[N, G, G, G] = 1 \Rightarrow [N, G] \in Z_2(G) \Rightarrow \mho_1([N, G]) = [\mho_1(N), G];$$

so $\mho_1([N, G])$= $[\mho_1(N), G]$$\supseteq$ $[N, G, G]$.

2/ $\mho_1(N)$ is powerfully embeded in G ; $[\mho_1(N), G, G] = 1$ and $\mho_1(\mho_1(N)) \supseteq \mho_1([N, G])$=$[\mho_1(N), G]$.

■

Remark 2.5.1 *If G is powerful p-group we have*

$$G_i, G^{(i)}, \mho_i(G), \Phi(G) \quad \textit{are powerfully embedded.}$$

CHAPTER 3

COHOMOLOGY OF FINITE p-GROUPS

Let G be a group, We can form the group ring $Z[G]$ over G; by definition it is the set of formal finite sums $\sum a_i g_i$, where $a_i \in \mathbb{Z}, g_i \in G$, and multiplication is defined in the obvious manner. We shall call an abelian group A a G-module if it is left $Z[G]$-module, This means of course that there exists a homomorphism $G \to \mathrm{Aut(A)}$. We can also make A into a right $Z[G]$-module simply writing $ag := g^{-1}a$ for all $a \in A, g \in G$. This is important for tensor products. An example of G-module is any abelian group with trivial action by G, for instance we shall in the future denote by $\mathbb{Z}$ the integers with trivial G-action. Finally, if A and B are G-modules, then a G-homomorphism between them is a map $\phi : A \to B$ which is a $Z[G]$-homomorphism, The set of G-homomorphism between A and B is denoted by $\mathrm{Hom_G(A, B)}$, it is left exact functor of A and B, covariant in B and contravariant in A. As usual its derived functors are denoted by $\mathrm{Ext^i}$.
Let A be a G-module, then we define the Cohomology groups as:

$$H^i(G, A) := \mathrm{Ext^i}_{Z[G]}(\mathbb{Z}, A).$$

Then $H^i(G, -)$ are covariant functor from the category of G-modules to the category of abelian groups. Now we have clearly $H^0(G, A) = \mathrm{Hom}(\mathbb{Z}, A)$ by basic properties of Ext over any Ring, also a $\mathbb{Z}$-homomorphism if and only if its image $a \in A$ is fixed by G, i.e $\quad ga = a$ for all $g \in G$.
Denote the set of such a by A^G. So we see that $\mathrm{Hom}(\mathbb{Z}, A) = A^G$; in particular $A \to A^G$ is a left functors, then another way of stating our definition is that $H^i(G, -)$ are the derived functors of the functor $A \to A^G$.
Recall that A is a right G-module; if we have an action $A \times G \to A$ that satisfies the following conditions:

- $a(gg^{'}) = (ag)g^{'}$;
- $a1 = a$;
- $(a + b)g = ag + bg$;

for all $g, g^{'} \in G$; and all $a, b \in A$.

3.1 Cohomology of groups :H^0 and H^1

Let G a finite group acts on abelian group M.
the action of $\sigma \in G$ on $m \in M$ will be noted $m \to m^{\sigma}$.

Definition 3.1.1 *M is also a G-module on the right if the action of G on M satisfies :*

$$m^1 = m \quad (m + m^{'})^{\sigma} = m^{\sigma} + m^{'\sigma} \quad (m^{\sigma})^{\tau} = m^{\sigma\tau}$$

Definition 3.1.2 *If M and N are two G-modules, a G-homomorphism is a homomorphism $\Phi : M \to N$ of abelian group which commutes with the action of G that is means*

$$\Phi(M^{\sigma}) = (\Phi(M))^{\sigma}, \forall m \in M, \forall \sigma \in G;$$

Given a G-module M, we can look at the largest submodule of M for which G operates trivially.

Definition 3.1.3 *The 0^{th} cohomological group of the G-module M that a note M^G or $H^0(G, M)$ is defined by*

$$M^G = H^0(G, M) = \{m \in M : m^{\sigma} = m, \forall \sigma \in G\};$$

It is the submodule of M of all these G-invariants.

Proposition 3.1.1 *Consider the exact sequence of G-modules*

$$0 \to P \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$$

(i.e Φ and Ψ are G-homomorphisms, with Φ injective, Ψ surjective $Im\Phi = \ker \Psi$)
We deduce the exact sequence of G-modules

$$0 \to H^0(G, P) \xrightarrow{\tilde{\phi}} H^0(G, M) \xrightarrow{\tilde{\psi}} H^0(G, N)$$

Proof. Firstly if $m \in H^0(G, P)$, the restriction of Φ from $H^0(G, P)$ is well defined

$$\forall \sigma \in G; m^{\sigma} = m \quad \text{from where:}$$

$$\Phi(m) = \Phi(m^{\sigma}) = (\Phi(m))^{\sigma} \to \Phi(m) \in H^0(G, M)$$

Likewise the restriction of Ψ from $H^0(G, M)$ is afraid image in $H^0(G, N)$ $\tilde{\Phi}$ is obviously injective because Φ is injective and

$$m \in \ker \tilde{\Psi} \to m \in \ker \psi \to m \in Im\phi \to m \in Im\tilde{\Psi}$$

Now , Ψ is not always surjective for the measure of this surjectivity, we define ■

Definition 3.1.4 *Let M a G-module*

a) The group of 1-cochains of G on M is by definition the set of application of G in M

$$C^1(G,M) = \{\text{maps } \xi : G \rightarrow \mathbf{M}\}$$

b) The group of 1-cocycle of G in M is defined by

$$Z^1(G,M) = \{\xi \in C^1(G,M) : \xi_{\sigma\tau} = \xi_\sigma^\tau + \xi_\tau \quad \forall \sigma, \tau \in G\}.$$

c) The group of 1-cobords from G in M is defined by

$$B^1(G,M) = \{\xi \in C^1(G,M) / \exists m \in M : \xi_\sigma = m^\sigma - m \quad \forall \sigma \in G\}.$$

Proposition 3.1.2 $B^1(G,M) \subset Z^1(G,M)$;
The 1^{st} cohomology group of G-module M is the quotient

$$H^1(G,M) = \frac{Z^1(G,M)}{B^1(G,M)}$$

Proof.
Let's show that $B^1 \subset Z^1$
Let $\xi \in B^1$, then $\exists\, m \in M \quad \forall \sigma \quad \xi_\sigma = m^\sigma - m$; proved that

$$\xi \in Z^1 \quad i.e \quad \xi_{\sigma\tau} = \xi_\sigma^\tau + \xi_\tau$$

$\xi_{\sigma\tau} = m^{\sigma\tau} - m = (m^\sigma)^\tau - m^\tau + m^\tau - m = (m^\sigma - m)^\tau + m^\tau - m = \xi_\sigma^\tau + \xi_\tau$ so $\xi \in Z^1$

$H^1(G,M)$ is also defined as the group of 1-cocycles modulo the equivalence relation of the form

$$\xi_1 \sim \xi_2 \Leftrightarrow \xi_1 - \xi_2 \in B^1(G,M)$$

that is to say $\xi_1 - \xi_2$ of the form $\sigma \rightarrow m^\sigma - m$ for a $m \in M$ ■

Example: If G operates trivially on M then
$H^0(G,M) = M$ because $\forall \sigma \in G \quad m^\sigma = m \quad \forall m \in M$

$Z^1(M) = \{\xi : G \rightarrow M \backslash \xi_{\sigma\tau} = \xi_\sigma^\tau + \xi_\tau = \xi_\sigma + \xi_\tau\}$ i.e $Z^1(M) = Hom(G,M)$
$B^1(M) = \{\xi \backslash\ \exists m\ \xi_\sigma = m^\sigma - m = 0 \quad \forall \sigma \in G\}$
that is to say $B^1(M) = 0$ and $H^1(G,M)$ is equal $Hom(G,M)$

Proposition 3.1.3 *Let $\Phi : M \rightarrow N$ a homomorphism of G-module if $\xi \in Z^1(G,M)$,*
$\Phi(\xi) \in Z^1(G,N)$

$$\begin{tikzcd} G \arrow[r, "\xi"] \arrow[dr, "\phi \circ \xi"'] & M \arrow[d, "\phi"] \\ & N \end{tikzcd}$$

Then $\xi \in Z^1(G,M)$, $\phi \circ \xi \in Z^1(G,N)$

Proof.

Indeed $\xi \in Z^1(G,M)$, then $\quad \forall \sigma,\tau \in G \quad \xi_{\sigma\tau} = \xi_\sigma^\tau + \xi_\tau \quad$ from where

$$\forall \sigma,\tau \in G \quad (\phi\circ\xi)_{\sigma\tau} = \phi(\xi_{\sigma\tau}) = \phi(\xi_\sigma)^\tau + \phi(\xi_\tau) = (\phi\xi)_\sigma^\tau + (\phi\xi)_\tau$$
$$\xi \in B^1(G,N) \quad \exists\, m \quad \xi_\sigma = m^\sigma - m \quad \forall\sigma \quad \to \quad \forall\sigma \quad \phi(\xi_\sigma) = \phi(\xi)_\sigma = \phi(m)^\sigma - \phi(m)$$
$$\phi(\xi)_\sigma = \phi(\xi_\sigma) = \phi(m^\sigma - m) = (\phi(m))^\sigma - \phi(m)\ ; \qquad \exists\phi(m)$$

ϕ thus induces an application $\phi : H^1(G,M) \to H^1(G,N)$, and we have the proposition. ■

Proposition 3.1.4 *Let $0 \to P \xrightarrow{\phi} M \xrightarrow{\psi} N \to 0$ an exact sequence of G-modules, then there is long exact sequence*

$$0 \to P^G \to M^G \to N^G \xrightarrow{\sigma} H^1(G,P) \to H^1(G,M) \to H^1(G,N)$$

σ is defined as either.
Let $n \in N^G = H^0(G,N)$ and $m \in M$ such that $\psi(m) = n$.
Let's define the cochaine $\xi \in C^1(G,M)$ by $\xi_\sigma = m^\sigma - m$.
Actually $\xi \in Z^1(G,P)$ and since ϕ injective and, $m^\sigma - m \in \ker\psi$
$\sigma(n)$ is the class of 1-cocycle $\xi in H^1(G,P)$. Now let H be a subgroup of G.
Obviously $\xi \in C^1(G,M)$ and by restriction $\xi \in C^1(G,N)$ and of obvious manner

$$\xi \in Z^1(G,M) \to \xi \in Z^1(H,M);$$
$$\xi \in B^1(G,M) \to \xi \in B^1(H,M);$$

and we get this homomorphism restriction

$$Res : H^1(G,M) \to H^1(H,M).$$

Suppose that $H \lhd G$ the submodule M^H of the elements fixed by H has a natural structure of G/H-module:

$$\begin{array}{ll} G/H \times M^H & \to M^H \\ (\bar\sigma, m) & \to m^\sigma \end{array};$$

Firstly $m^\sigma \in M^H$, Indeed if $\tau \in H$ $(m^{\sigma\tau}) = m^\sigma$ because $H \lhd G$;

The action is well defined :

$$\bar\sigma = \bar\tau \to \sigma\tau^{-1} \in H \to m^{\sigma\tau^{-1}} = m \to m^\sigma = m^\tau.$$

Now if $\xi : G/H \to M^H$ is a cochaine of G/H to M^H, then when composing with $s : G \to G/H$ and $i : M^H \to M$. We obtain a G-$\tau \circ M$ cochaine : $i\circ\xi\circ s : G \to M$ again note ξ.
Now if $\xi \in Z^1(G/H, M^H)$, then $i\circ\xi\circ s \in Z^1(G,M)$; and if $\xi \in B^1(G/H,M^H)$ then $i\circ\xi\circ s \in B^1(G,M)$

$$G \xrightarrow{s} G/H \xrightarrow{\xi} M^H \xrightarrow{i} M;$$

We get an inflation homomorphism

$$Inf : H^1(G/M, M^H) \to H^1(G,M)$$

3.2 ∂-Functors and ∂*-Functors

Let C be an abelian category, $C^{'}$ an additive category, and a and b be two integers (which can be equal to $\pm\infty$) such that $a+1<b$. A covariant ∂-functor from C to $C^{'}$ in degrees $a<i<b$, is a system $T=(T^i)$ of additive covariant functors from C to $C^{'}$, $(a<i<b)$, in addition to giving, for any i such that $a<i<b-1$ and for any exact sequence

$0\to A^{'}\to A\to A^{''}\to 0$, a morphism

$$\partial : T^i(A^{''})\to T^{i+1}(A^{'})$$

(the "boundary" or "connecting" homomorphism). The following axioms are assumed to be satisfied:

(i) If we have a second exact sequence $0\to B^{'}\to B\to B^{''}\to 0$ and a homomorphism from the first exact sequence to the second, the corresponding diagram commutes.

$$\begin{array}{ccc} T^i(A^{''}) & \xrightarrow{\partial} & T^{i+1}(A^{'}) \\ \downarrow & & \downarrow \\ & & \\ T^i(B^{''}) & \xrightarrow{\partial} & T^{i+1}(B^{'}) \end{array}$$

(ii) For any exact sequence $0\to A^{'}\to A\to A^{''}\to 0$, the associated sequence of morphisms

$$\ldots\to T^i(A^{'})\to T^i(A)\to T^i(A^{''})\to T^{i+1}(A)\to\ldots \qquad (*)$$

3.3 Universal ∂-functors

is a complex, i.e. the composite of two consecutive morphisms in this sequence is 0.
There is an analogous definition for a covariant ∂^*-functor, the only difference being that the ∂^* operator decreases the degree by one unit instead of increasing it. There are analogous definitions for contravariant ∂-functors and ∂^*-functors. The T^i are then contravariant additive functors and the boundary operators go from $T^i(A^{'})\to T^{i+1}(A^{''})$ or $T^i(A^{'})\to T^{i-1}(A^{''})$. If we change the sign of the i in T^i, or if we replace $C^{'}$ by its dual, the ∂-functors become ∂^*-functors. Thus, one can always stick to the study of covariant ∂-functors. Note that if $a=-\infty; b=+\infty$, a ∂-functor is a connected sequence of functors.
Given two ∂-functors T and $T^{'}$ defined in the same degrees, we call a morphism (or natural transformation) from $T\to T^{'}$ a system $f=(f^i)$ of natural transformations $f^i: T^i\to T^{'i}$ subject to the natural condition of commutativity with ∂:
for any exact sequence $0\to A^{'}\to A\to A^{''}\to 0$, the diagram

$$\begin{array}{ccc} T^i(A^{''}) & \xrightarrow{\partial} & T^{i+1}(A^{'}) \\ \downarrow & & \downarrow \\ T^{i'}(A^{''}) & \xrightarrow{\partial} & T^{i+1'}(A^{'}) \end{array}$$

commutes.
Morphisms of ∂-functors add and compose in the obvious way. Assume that $C^{'}$ is also an abelian category. A ∂-functor is exact if for any exact sequence $0 \to A^{'} \to A \to A^{''} \to 0$ in C, the corresponding sequence $(*)$ is exact. We say that a cohomological functor (respectively homological functor) is an exact ∂-functor (respectively exact ∂^*-functor) defined for all degrees.

Let $T = (T^i)$ for $0 \leq i \leq a$ be a covariant ∂-functor from C to $C^{'}$, where $a > 0$. T is called a universal ∂-functor if for any ∂-functor $T^{'} = (T^{'i})$ defined in the same degrees, any natural transformation $f^0 : T^0 \to T^{'0}$ extends to a unique ∂-functor $f : T \to T^{'}$ which reduces to f^0 in degree $0.^n$
We use the same definition for contravariant ∂-functors. In the case of ∂^*-functors we have to consider morphisms from $T^{'} \to T$ rather than $T \to T^{'}$. By definition, given a covariant functor F from C to $C^{'}$, and an integer $a > 0$, there can exist, up to unique isomorphism, at most one universal ∂-functor defined in degrees $0 \leq i \leq a$ and reducing to F in degree 0.

3.4 Tate's cohomology

In this section, we assume that G is finite group and A is a G-module. We define the trace map

$$\tau = \tau_G : A \to A$$

by setting

$$\tau(a) = \prod_{x \in G} a^x, \qquad \text{for all } a \in A.$$

Let us denote by A^G the submodule of the elements of A fixed by G, that is

$$A^G = \{a \in A \,|\, a^g = a \text{ for all } g \in G\}.$$

We denote by $[A, G]$ the submodule formed by the elements of the form $a^{-1}a^g$, where $a \in A$ and $g \in G$.

In the usual setting, we have it is well to known that

$H^0(G, A) = A^G$ and $\quad H_0(G, A) = A/[A, G]$. We shall modify these groups to obtain the Tate cohomology groups $\hat{H}^n(G, A)$, $n \in \mathbb{Z}$.

Lemma 3.4.1 *Under the above assumptions, we have*

- $A^\tau \subseteq A^G$*, where* A^τ *denotes the image of* A *by the trace map* τ*;*
- $[A, G] \subseteq \ker \tau$.

Proof. Assume $a \in A^\tau$, so there exists $b \in A$ such that $a = b^\tau = \prod_{x \in G} b^x$. Let $g \in G$; we have

$$a^g = (\prod_{x \in G} b^x)^g = \prod_{x \in G} b^{xg};$$

as the map $x \to xg$ is a permutation of G, we have

$$\prod_{x \in G} b^{xg} = \prod_{x \in G} b^x,$$

so $a^g = a$; this proves the first inclusion.

For the second, let $a \in [A, G]$, so there exist $b \in A$ and $g \in G$ such that $a = b^{-1}b^g$. Now,

$$\begin{aligned} a^\tau = (b^\tau)^{-1}(b^g)^\tau &= (\prod_{x \in G} b^x)^{-1}(\prod_{x \in G} b^x)^g \\ &= (\prod_{x \in G} b^x)^{-1}(\prod_{x \in G} b^{xg}) \\ &= (\prod_{x \in G} b^x)^{-1}(\prod_{x \in G} b^x) = 1 \end{aligned}$$

so $a \in \ker \tau$;this proves the second inclusion. ■

The latter result implies that we have a homomorphism of G-modules: $\tau^* = \tau_G^* : A/[A, G] \to A^G$, where $\tau^*(\bar{a}) = \tau(a)$, for all $\bar{a} \in A/[A, G]$. Therefore, we have natural homomorphisms

$$\tau_G^* : H_0(G, A) \to H^0(G, A).$$

Now, let us define

$$\hat{H}^0(G, A) = A^G/A^\tau;$$

$$\hat{H}^{-1}(G, A) = \ker \tau/[A, G];$$

$$\hat{H}^n(G, A) = H^n(G, A), \qquad \text{for } n \geq 1;$$

and

$$\hat{H}^n(G, A) = H_{-n-1}(G, A), \qquad \text{for } n \leq -2.$$

Therefore, we have a family of groups $(\hat{H}^n(G, A))_{n \in \mathbb{Z}}$; we call these the Tate cohomology groups of G with coefficients in A.

Let us show that $(\hat{H}^n(G, A))_{n \in \mathbb{Z}}$ satisfies the usual cohomological properties.

Proposition 3.4.1 *Every exact sequence of G-modules*

$$0 \to A \to B \to C \to 0$$

induces a long exact sequence of groups

$$\ldots \to \hat{H}^n(G, B) \to \hat{H}^n(G, C) \xrightarrow{\delta} \hat{H}^{n+1}(G, A) \to \hat{H}^{n+1}(G, B) \to \ldots$$

Moreover, the above construction is natural in the sense that for any morphism of exact sequences of G-modules

$$\begin{array}{ccccccccc} 0 & \to & A & \to & B & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & A' & \to & B' & \to & C' & \to & 0 \end{array}$$

induces a commutative diagram

$$\begin{array}{ccccccccccc} \dots & \to & \hat{H}^n(G,B) & \to & \hat{H}^n(G,C) & \to & \hat{H}^{n+1}(G,A) & \to & \hat{H}^{n+1}(G,B) & \to & \dots \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \to & \hat{H}^n(G,B') & \to & \hat{H}^n(G,C') & \to & \hat{H}^{n+1}(G,A') & \to & \hat{H}^{n+1}(G,B') & \to & \dots \end{array}.$$

Proof. For $n \geq 1$ and $n \leq -2$; This suite corresponds to the suites exact lengths of usual homology and cohomology. So just establish if for $n = -1$ or 0,it's not hard to see the next diagram is commutative:

$$\begin{array}{ccccccccc} H_1(G,C) & \to & H_0(G,A) & \to & H_0(G,B) & \to & H_0(G,C) & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \to & H^0(G,A) & \to & H^0(G,B) & \to & H^0(G,C) & \to & H^1(G,A) \end{array}$$

Knowing that $\hat{H}^{-1}(G,A) = \ker \tau_A^*$ and $\hat{H}^0(G,A) = coker\ \tau_A^*$.
This lemma of snake (see[**[23]**]) implies that there is a homomorphism (connecting) natural.
$\hat{H}^{-1}(G,C) \xrightarrow{\delta} H^0(G,A)$, such that the sequence

$$\dots \to \hat{H}^{-2}(G,C) \to \hat{H}^{-1}(G,A) \to \hat{H}^{-1}(G,B) \to \hat{H}^{-1}(G,C) \xrightarrow{\delta} \hat{H}^0(G,A) \to \hat{H}^0(G,B) \to \hat{H}^0(G,C) \to \hat{H}^1$$

is exact ■

Remark 3.4.1 *The above means that $\hat{H}^n(G,-)$ is a cohomological functor in the category of G-modules; in fact it is a universal cohomological functor.These later it is operas for first time in A.Grothendieck paper " sur quelque points d'algebre homologique"*
*We write the definition as it is in **[18]**.*

3.5 Cohomology trivial

We say that a G-module A is said to be cohomologically trivial if $\hat{H}^k(S,A) = 0$ for all $S \leq G$ and for all integers k.

The following important result was proved independently by Gaschütz**[14]** and Uchida **[35]**. According to Uchida this result was conjectured by Tannaka.

Theorem 3.5.1 *Let G be a finite p-group and A be a G-module that is also a finite p-group. If there exists $n \in \mathbb{Z}$ such that $\hat{H}^n(G,A) = 0$, then A is a cohomologically trivial module over G.*

For a proof we refer the reader to Gruenberg "Cohomology Topics in Group theory"**[19]**.

3.5.1 Cohomological Property of Regular p-groups

Suppose that G is a group and N is a normal subgroup of G such that G/N is finite. Then $Z(N)$ can be viewed as a G/N-module by considering the action of G/N on $Z(N)$ by conjugation i.e,

$$z^{gN} := z^g = g^{-1}zg \quad \text{for all} \quad g \in G \quad \text{and} \quad z \in Z(N);$$

Theorem 3.5.2 *Let G be a regular p-group and N a non-trivial normal subgroup of G such that $Q = G/N$ is not cyclic, then $\hat{H}^n(Q, Z(N)) \neq 0$ for all n.*

Proof. see(**[33]**) ■

Proposition 3.5.1 *Let A and G are finite p-groups, $A \neq 1$ is a cohomologically trivial G-module . Then for every subgroup K of G, the centralizer $C_G(A_K) = K$.*

Proof. Assume that $K < C_G(A_K)$ and let xK an element of order p on the center of p-group $C_G(A_K/K)$.
Then K a maximal subgroup in $H =< x, K >$, by Gaschütz and Uchida theorem we have $\hat{H}^0(H, A) = 0$. Hence $\hat{H}^0(H/k, A_k) = 0$;
(by inflation-restriction proposition 3.1.4) and as H/K acts trivially over A_K, we have

$$(A_K)_{H/K} = A_k \quad \text{and} \quad \tau_{H/K}(x) = x^p, \forall x \in A_K.$$

So $(A_K)^{\tau_{H/K}} < A_K$ thus implies $\hat{H}^0(H/K, A_K) \neq 0$ a contradiction. ■

Corollary 3.5.1 *Let G be a finite p-group, N a normal subgroup of G.*
Assume that $\hat{H}^n(G/N, Z(N)) = 0$ for some integer n. Then, for all subgroups H of G containing N, we have

$$C_G(Z(H)) = H.$$

Proof. We know that $A = Z(N)$ is cohomologically trivial as G/N-module. In view of proposition 3.5.1 it is suffices te check that $Z(H) \leq A$ for all subgroups H of G containing N.But $Z(H)$ centralizes N and

$$C_G(N) \leq C_G(A) = N.$$

■

Proposition 3.5.2 *Suppose G is a regular p-group and N a maximal subgroup of G. Let $Q = G/N$ be of order p^n. Then the trace map τ_Q of $A = Z(N)$ is just $a \to a^{p^n}$.*

Proof. Assume first that N is a maximal subgroup of G.
Let $Q =< Nx >$ and $H =< A, x >$. Then the commutator group $H' = [A, x]$. Since x^p centralizes A and H is regular, We have

$$[a, x]^p = [a, x^p] = 1 \quad \text{for all} \quad a \in A;$$

Consequently H' is of exponent p, and so $(xa)^p = x^p a^p$ for all $a \in A$, by regularity .On the other hands

$$(xa)^p = x^p a^{x \ldots x^{p-1}} = x^p a^\tau.$$

Where $\tau = \tau_Q$.Consequently $a^\tau = a^p$, as desired. ■

A Conjecture of P.Schmid
Let G be a finite p-groups, $A = Z(\Phi(G))$ and $Q = G/C_G(\Phi(G))$.Then the cohomology of A over Q is not trivial.

3.5.2 Cohomology non-trivial of p-groups semi-abelian

The conjecture of Schmid holds for all semi-abelian p-groups in fact, more is true.

Lemma 3.5.1 *Let G a p-group semi-abelian and N a normal subgroup such as G/N non-cyclic or non quaternion.*
We put $A = Z(N)$ and let S/N be a subgroup of exponent p in G/N, we have then $A^p \leqslant A_{S/N}$, and also $C_{S/N}(A^p) = S/N$.

Proof. Let be $x \in S$ and $a \in A$. We have $x^p \in N$, hence

$$x^p = (x^p)^a = (x^a)^p = (x[x,a])^p;$$

As G is semi-abelian, we have $[x,a]^p = 1$.It follows that $(a^{-1}a[a,x])^p = [a,x]^p = 1$, and again since G is semi-abelian

$$a^p = (a[a,x])^p = (a^x)^p = (a^p)^x.$$

This shows that A^p is centralized by every element of S/N. ■

Lemma 3.5.2 *Let G a p-group semi-abelian and N a normal subgroup such as G/N neither cyclic nor a generalized quaternion group. Then $\hat{H}^n(G/N, Z(N)) \neq 0$, for all integers n.*

Proof. Assume for a contradiction that $\hat{H}^n(G/N, A) = 0$ for some integer n, where A denotes $Z(N)$. As G/N is not cyclic and different from the group Q_{2^m}, there is in G/N a subgroup S/N of exponent p and order at least p.It follows from theorem 3.5.1 that $\hat{H}^n(S/N, A) = 0$, so A is a cohomologically trivial S/N-module. Let $K/N \leq S/N$ be a subgroup of order p 3.5.1implies that $\hat{H}^0(K/N, A) = 0$. We have $\hat{H}^0(K/N, A) = A_{K/N}/A^\tau = 0$, where A^τ is the image of A under the trace homomorphism $\tau : A \to A$ induced by K/N. As K/N is cyclic of order p, our trace map is given by

$$a^\tau = aa^x...a^{x^{p-1}} \quad \text{for} \quad a \in A \quad \text{and any fixed} \quad x \in K - N,$$

from which it follows that

$$a^\tau = (ax^{-1})^p x^p.$$

Now a G is semi-abelian, $a \in \ker \tau$, if and only if $a^p = 1$; that is, $\ker \tau = \Omega_1(A)$. This implies that $\mid A^\tau \mid = \mid A^p \mid$. As $A_{K/N} = A^\tau$, and $A^p \leq A_{K/N}$ by proposition 3.5.2 implies that $S/N = K/N$, a contradiction. ■

CHAPTER 4

NON-INNER AUTOMORPHISM OF p-GROUPS

Let G be a group. An automorphism $\sigma : G \to G$ is termed inner if there exists $g \in G$ such that

$$\sigma(x) = g^{-1}xg, \text{ for all } x \in G.$$

In the above case, we say that σ is the inner automorphism induced by g; we may denote σ by c_g. The inner automorphisms of G form a normal subgroup $\mathrm{Inn}(G)$ of $\mathrm{Aut}(G)$. Note that the map $c : G \to \mathrm{Inn}(G)$ that maps every $g \in G$ to c_g is an epimorphism, whose kernel is the center $Z(G)$; it follows $G/Z(G) \cong \mathrm{Inn}(G)$, and that two elements $g, g' \in G$ induce the same inner automorphism if and only if $gZ(G) = g'Z(G)$.

4.1 p -groups with non-inner automorphism

A result of W.Gaschütz says that if G is a finite non-abelian p-group, then G has an automorphism of p-power order which is not inner.
It is an open problem whether every non-abelian p-group G has an automorphism of order p. The latter question has positive answer whenever G satisfies one the following condition:

1. G is nilpotent of class 2 or 3;
2. G is a regular p-group;
3. $G/Z(G)$ is a powerful p-group;
4. G satisfies: $C_G(Z(\Phi(G))) \neq \Phi(G)$;
5. If the commutator subgroup of G is cyclic;
6. If G of Coclass 2;

In most of the above cited results on the conjecture, it is proved that G has often a noninner automorphism of order p leaving the center $Z(G)$ or Frattini subgroup $\Phi(G)$ of G elementwise fixed.

4.2 Cohomologically trivial modules and non-inner p-automorphisms

There is a relation between non-triviality of Tate cohomology $\hat{H}^n(G/N, Z(N))$ and the existence of non-inner automorphisms of p-power order in $\mathrm{Aut(G)}$.

Proposition 4.2.1 *Suppose that N is a normal subgroup of a group G, and we have $C_G(N) = Z(N)$. Then there is a natural isomorphism*

$$\Phi : Z^1(G/N, Z(N)) \to C_{Aut(G)}(N, G/N) \quad \textit{given by}$$

$$g^{\varphi(f)} = g(gN)^f, \textit{for all} \quad g \in G, f \in Z^1(G/N, Z(N)).$$

The image of $B^1(G/N, Z(N))$under Φ is the group of inner automorphism of G induced by $Z(N)$. Here $C_{Aut(G)}(N, G/N)$ denotes all automorphism α of G such that
$x^\alpha = x$ for all $x \in N$ and $g^{-1}g^\alpha \in N$ for all $g \in G$.

Theorem 4.2.1 *Assume that N is a normal subgroup of G since that $C_G(N) = Z(N)$ and $\hat{H}^1(G/N, Z(N)) \neq 0$. Then $C_{Aut(G)}(N, G/N)$ is not contained in $Inn(G)$.*

Proof. view**[3]** ■

4.2.1 Semi-abelian p-groups and non-inner automorphism

In this section we need the definition of the Crossed homomorphism; let us again write
A 1-Cocycle is a map $\delta : G \to A$ such that $\delta\xi = 0$, thus $Z^1(G, A)$ consists of all δ satisfying

$$\delta(xy) = \delta(x)^y \delta(y) \quad \text{for all} x, y \in G$$

Thus function δ called a derivation or a crossed homomorphism.
If $\delta_a(x) = a^{-1}a^x$ for some fixed $a \in A$. Then δ is said to be inner or a principal crossed homomorphism, writing $\mathrm{Der(G, A)}$ for the group of derivations and $\mathrm{Ider(G, A)}$ for the subgroup of inner derivation, and we have

$$H^1(G, A) \cong \mathrm{Der(G, A)/Ider(G, A)}$$

Each element δ of $\mathrm{Der(G, A)}$, determines an endomorphism Φ_δ of G, given by

$$\Phi_\delta(x) = x\delta(x), x \in G.$$

This map Φ defines a bijection between $\mathrm{Der(G, A)}$ and $\mathrm{End}_A(G)$, where

$$\mathrm{End_A(G)} = \{\Theta \in \mathrm{End(G)}/x^{-1}\Theta(x) \in A, \quad \text{for all} \quad x \in G\}.$$

If we consider only the set $\mathrm{Der(G/C_G(A), A)}$ of derivations, that are trivial on $C_G(A)$, then the map Φ induces an isomorphism between $\mathrm{Der(G/C_G(A), A)}$ and the group $\tilde{C}(A)$ of the automorphisms of G acting trivially on $C_G(A)$ and $G = A$.

It is straightforward to see that this isomorphism maps the inner derivations, $\mathrm{Ider}(\mathrm{G}/\mathrm{C_G}(\mathrm{A}), \mathrm{A})$, into a group of inner automorphisms lying in $\tilde{C}(A)$, though an inner automorphism lying in $\tilde{C}(A)$ need not necessarily be induced by an inner derivation. This case can be avoided by assuming that $C_G(C_G(A)) = A$; indeed, if $\Phi_\delta(x) = x^g$ for some $g \in G$ and all $x \in G$, then $g \in C_G(C_G(A))$, so g lies in A and δ is the inner derivation induced by g^{-1}.

Proposition 4.2.2 *There is an isomorphism between* $\mathrm{Der}(\mathrm{G}/\mathrm{C_G}(\mathrm{A}), \mathrm{A})$ *and* $\tilde{C}(A)$*, which maps* $\mathrm{Ider}(\mathrm{G}/\mathrm{C_G}(\mathrm{A}), \mathrm{A}$ *exactly to the inner automorphisms lying in* $\tilde{C}(A)$*.*

Theorem 4.2.2 *Let G be a semi-abelian fini p-group . Then G has a noninner automorphism of order p.*

Proof. Assume for a contradiction that every automorphism of G of order p is inner. Let $A = Z(\Phi(G))$. By Proposition , we have $C_G(A) = \Phi(G)$, and so $C_G(C_G(A)) = A$. If we prove that $Der(G/C_G(A), A) = \mathrm{Der}(\mathrm{G}/\Phi(\mathrm{G}), \mathrm{Z}(\Phi(\mathrm{G}))$ has exponent p, then our first assumption together with Proposition 4.2.2 implies that

$$\hat{H}^1(G/\Phi(G); Z(\Phi(G))) = 0;$$

which contradicts Lemma 3.5.1. So we only need to prove, for any derivation
$\delta \in \mathrm{Der}(\mathrm{G}, \mathrm{Z}(\Phi(\mathrm{G}))$ which is trivial on $\Phi(G)$, that $\delta(x)^p = 1$; for all $x \in G$. Indeed,

$$\delta(x^p) = \delta(x)\delta(x)^x ... \delta(x)^{x^{p-1}} = (\delta(x)x^{-1})^p x^p;$$

As δ is trivial on $\Phi(G)$, we have $\delta(x^p) = (\delta(x)x^{-1})^p x^p = 1$, and since G is semi-abelian it follows that $\delta(x)^p = 1$. ■

4.3 Potent p-groups

Let G be a finite p-group. We say that G is *potent* if $\gamma_{p-1}(G) \subseteq G^p$ (for $p = 2$, we require $\gamma_2(G) \subseteq G^4$). This class of groups was introduced by A. Jaikin Zapiarain and G. Sanchez in **[21]**; their aim was to show that several properties of the powerful p-groups can be extended to the potent p-groups. Below we list some of the main properties of these groups.
Note that for $p = 2$ and $p = 3$ to be potent is the same as powerful. In general, any powerful p-group is also potent.

Theorem 4.3.1 *Let G be a finite potent p-group.*

1. *if $p = 2$ then*

   (a) *- the exponent of $\Omega_i(G)$ is at most 2^{i+1} and, even more*

   $$[\Omega_i(G), G]^{2i} = \Omega_i(G^2)^{2i} = 1;$$

   (b) *-the nilpotency class of $\Omega_i(G)$ is at most $[(i+3)/2]$;*

(c) - *if $N \lhd G$ and $N \leq G^2$ then N is power abelian;*

(d) - *if $N \lhd G$ and $N \leq G^4$ then N is powerful;*

2. *if $p > 2$ then*

   (a) - *the exponent of $\Omega_i(G)$ is at most p^i;*

   (b) -*the nilpotency class of $\Omega_i(G)$ is at most $(p-2)i+1$;*

   (c) - *if $N \lhd G$ then N is power abelian;*

   (d) - *if $N \lhd G$ and $N \leq G^p$ then N is powerful;*

Theorem 4.3.2 *Let G be a potent p-group. then the following proprieties hold:*

1. *if $p = 2$, then $\gamma_{k+1}(G) \leq \gamma_k(G)^4$, and if $p > 2$ then $\gamma_{p-1+k}(G) \leq (\gamma_{k+1}(G))^p$;*

2. *$\gamma_i(G)$ is potent;*

3. *$< x, [G, G] >$ is potent, for all $x \in G$;*

4. *G is p-powered;*

5. *if N is a normal subgroup of G then G/N is potent;*

Theorem 4.3.3 *Let G be a finite p-group and N, M normal subgroups of G.*

$$\textit{If} \quad N \leq M[N,G]N^p \quad \textit{then} \quad N \leq M.$$

Corollary 4.3.1 *Let G be a group and $x_1, ..., x_k$ elements of G. Then*

$$(x_1...x_k)^{p^n} \equiv x_1^{p^n}...x_k^{p^n} (\mod \gamma_2(L)^{p^n}\gamma_p(L)^{p^{n-1}}\gamma_{p^2}(L)^{p^{n-2}}...\gamma_{p^n}(L));$$

Where $L =< x_1, ..., x_k >$. In particular, we have

$$\Omega_i^{p^n} \leq \Omega_{i-k}\gamma_2(\Omega_i)^{p^n}\gamma_p(\Omega_i)^{p^{n-1}}\gamma_{p^2}(\Omega_i)^{p^{n-2}}...\gamma_{p^n}(\Omega_i);$$

Where $\Omega_l = \Omega_l(G)$ for $l \geq 1$ and $\Omega_l = 1$ for $l \leq 1$.

4.3.1 Potently embedded subgroups

We say that N is potently embedded in G if $[N, G] \leq N^4$ for $p = 2$ and $[N,_{p-2} G] \leq N^p$ for p odd. (Since when p=2 a potent 2-group is powerful).

Theorem 4.3.4 *Let N and M be potently embedded subgroups of G. Then*

$$[N^p, M] = [N, M]^p.$$

Proof. By P.Hall's formula [1.2.2] we have

$$[N^p, M] \leq [N, M]^p[N,_p N] \leq [N, M]^p[[M^p, N], N].$$

Now we apply again P.Hall's formula to $[N^p, M]$ and we get that

$$[N^p, M] \leq [N, M]^p[N^p, M, M, N].$$

Therefore, by theorem [], $[N^p, M] \leq [N, M]^p$.
In order to prove the converse, we can assume that

$$[N^p, M] = [[N, M]^p, G] = 1.$$

Note that since M is potently embedded in G, $[M,_p N] \leq [M^p, N, N]$
By the previous argument, $[M^p, N] \leq [M, N]^p$ so

$$[M,_p N] \leq [[M, N]^p, N] = 1.$$

Now, let us prove by the revers induction in k, that $[N, M, \gamma_k(N)]^p = 1$ for all $k \geq 1$.
This is clear when k is big enough. So, suppose that $[M, N, \gamma_{k+1}(N)]^p = 1$. Then, we have

$$[N, M, \gamma_k(N)]^p \leq [[N, M]^p, \gamma_k(N)][[N, M], \gamma_{k+1}(N)]^p$$

$[N, M, \gamma_k(N),_{p-2} N] = 1$.
Therefore, we have that

$$[N, M]^p \leq [N^p, M][N, M, N]^p[M,_p N] = 1.$$

■

Theorem 4.3.5 *Let G be a potent p-group and N, M potently embedded subgroups of G. Then we have that*

1. *$NM$is potently embedded;*
2. *$[N, G]$is potently embedded;*
3. *$N^p$is potently embedded;*

Proof.

1. This is obvious because
$$[NM,_{p-2} G] \leq [N,_{p-2} G][M,_{p-2} G].$$
2. we have that
$$[[N, G],_{p-2} G] \leq [N^p, G];$$
and by the previous theorem,
$$[[N, G],_{p-2} G] \leq [N, G]^p;$$
3. By the previous theorem
$$[N^p,_{p-2} G] \leq [N,_{p-2} G]^p \leq (N^p)^p;$$
Therefore N^p is potently embedded.

■

4.3.2 Normal subgroups of potent p-groups are power abelian

Proposition 4.3.1 *We call a finite p-group G power abelian if satisfies:*

- $G^{p^i} = \{g^{p^i}/g \in G\}$;
- $\Omega_i(G) = \{g \in G/o(g) \leq p^i\}$;
- $\mid G^{p^i} \mid = \mid G : \Omega_i(G) \mid$, *for all i;*

Theorem 4.3.6 *Let G be a potent p-group and $N \leq G^2$ a normal subgroup of G. Then N is power abelian.*

Proof. We have already prover that
$N^{p^i} = \{n^{p^i}/n \in N\}$, $\Omega_i(N) = \{n \in N/n^{p^i} = 1\}$, and $|N^p| = |N : \Omega_1(N)|$. So we just need to see that for any $i \geq 1 |N^{p^i}| = |N : \Omega_i(N)|$. We work by induction on i, then

$$|N^{p^{i+1}}| = |(N^p)^{p^i}| = |N^p : \Omega_i(N^p)|.$$

But $\Omega_i(N^p) = \Omega_i(N) \cap N^p$, and we also have that

$$N^p/(\Omega_i(N) \cap N^p) \cong N^p\Omega_i(N)/\Omega_i(N) = (N/\Omega_i(N))^p;$$

Hence $|N^{p^{i+1}}| = |(N/\Omega_i(N))^p|$. By applying the case $i = 1$ to the potent p-group $G/\Omega_i(N)$ and its normal subgroup $N/\Omega_i(N)$, we get that

$$|(N/\Omega_i(N))^p| = |N/\Omega_i(N) : \Omega_1(N/\Omega_i(N))| = |N/\Omega_i(N) : \Omega_{i+1}(N)/\Omega_i(N)| = |N : \Omega_{i+1}(N)|;$$

which concludes the proof. ■

Conclusion

The study of existence of Non-inner automorphisms of p-power order is an open problem that who is verify in some type of p-groups;
In our work we proved this existence by using cohomologically technique according to Tate cohomology and his property. And we also showed a nice results about P.schmid's conjuncture and about the power structure of the p-groups in the semi-abelian p-groups that introduced early by Ming You Xu .
Our aim is to find an positive answer for the class of potent p-groups. Which it is open for a longtime.

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Abstract

This work treats a conjecture of Peter Schmid on the Tate cohomology of finite p-groups. We examine some cases in which the conjecture has an affirmative answer, as well as the relevance of this conjecture to studying automorphisms of finite p-groups.

Résumé

Ce travail traite le conjecture de Peter Schmid sur la cohomologie Tate des p-groupes finis. Nous examinons certains cas dans lesquels la conjecture a une réponse affirmative, ainsi que la pertinence de cette conjecture pour étudier les automorphismes des p-groupes finis.