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Szemerédi's regularity lemma

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Dedication

to my:

- mother

- father

- brother: Med elarbi, Med elhadi, Mohammed

- sonsister: Abdelhaq, Omar, Abderraouf, Hassan, Islam

-All friends:(ex):Seddik+Hamdane(chawi)

+Naimi+Nouri+Yaqub+Ilyas+Fatah

+Khaled+Fares+Med chikh

- all familly

- Our colleagues at department of mathematique University Kasdi Merbah of Ouargla

I didecated this work.

Seyf.Bougoffa

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Introduction

Szemerédi's regularity lemma is a masterpiece in graph theory; it asserts roughly that the vertices of every finite graph can be partitioned into pieces X_i, such that for almost all pairs (X_i, X_j) of these pieces, the edges between X_i and X_j have a definite uniform pattern. While this result has many important consequences in graph theory, it proved also very useful in other areas such as number theory, additive combinatorics, etc. Actually, E. Szemerédi proved a weak version of his lemma in 1975 when dealing with a conjecture of Erdös and Turán that states that every set of natural numbers of positive upper density, contains arithmetic progressions of arbitrary large lengths. The lemma in the current form has been established in 1978. The earlier applications motivated enlargements of the regularity lemma to other structures as hypergraphs, or giving it other flexible forms. A noteworthy is that B. Green and T. Tao proved their famous result on arithmetic progressions of prime numbers via such meditations on the regularity lemma.

This work aims to explain the foregoing lemma and its proof, and to show that such a seemingly non productive result has highly non trivial consequences. The thesis is organized as follows:

In the first chapter, after introducing the notion of graphs and some basic fact on them, and after giving the basic ingredients for formulating the regularity lemma, we give a full proof of it.

The second chapter treats some applications of the lemma. We begin by explaining the mentioned conjecture of Erdös and Turán. We prove an important consequence of the regularity lemma: the triangle removal lemma, and we apply the latter to prove Roth's theorem which confirms the conjecture in a particular case. This conjecture became *Szemerédi's theorem*, in 1975; while a full proof of the latter will not be given here, we discuss its first proof, and other proofs as Furstenberg's proof via ergodic theory, Gowers' proof via hypergraphs, and finally the effective bounds for the parameters in these theorems.

Chapter 1

Szemerédi's regularity lemma

1.1 Generalities on graphs

Definition 1 *A graph G is a pair (V, E), where V is a set and E is a family of subsets of V all containing two elements.*

Let $G = (V, E)$ be a graph. The elements of V will be called the vertices of G; we may denote V by $V(G)$ if we want to keep track of G. The elements of E will be termed the edges of G (we may write also $E = E(G)$).

If $x, y \in E(G)$, then it is more convenient to write $xy \in E$; we say simply that xy is an edge of G. Note that xy and yx represent the same edge.

Let $x, y \in V(G)$; we say that x and y are adjacent if xy in an edge of G. The set of the vertices of G adjacent to x will be denoted $\delta(x)$, and called the neighborhood of x; hence

$$\delta(x) = \{y \in V(G) \,|\, xy \in E(G)\}.$$

The cardinality of $\delta(x)$ will be called the degree of x and will be denoted $d(x)$; so $d(x) = |\delta(x)|$.

The graph G is finite if $V(G)$ is finite. It follows in this case that G has at most $\binom{|V|}{2}$ edges. Hence, if G is finite, then $d(x)$ is finite for every vertex x of G. More generally, if $d(x)$ is finite for every vertex x of G, we say that G is a locally finite graph.

Henceforth, all the graphs that will be considered are supposed to be finite. Every such a graph can be represented in the plan by associating a point (or a labeled point) to each vertex of the graph, and two such points are joined by an arc if the corresponding vertices are adjacent.

Example 1 *:*

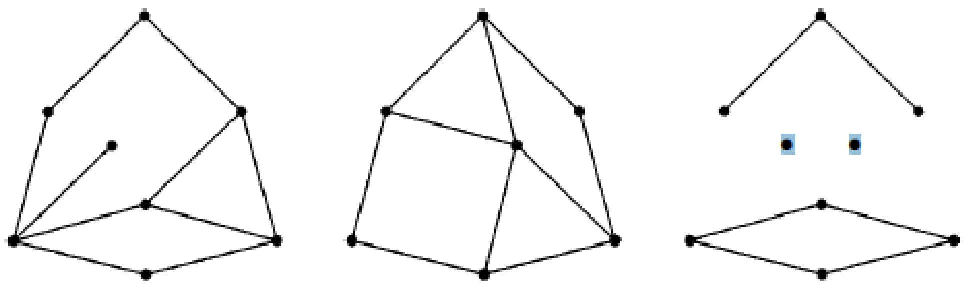


Figure 1.1:

Below are two basic nice results that hold for all graphs.

Proposition 2 *For every graph $G = (V, E)$, we have*

$$\sum_{x \in V} d(x) = 2|E|$$

Proof. Let $S = \{(x, y) \mid xy \in E\}$. On the one hand, each edge $xy \in E$ induces exactly two elements (x, y) and (y, x) in S, hence $|S| = 2|E|$. On the other hand, S can be partitioned as

$$S = \coprod_{x \in V} \{x\} \times \delta(x),$$

so

$$|S| = \sum_{x \in V} |\{x\} \times \delta(x)| = \sum_{x \in V} d(x);$$

the result follows. ■

Proposition 3 *For every graph G with $|V(G)| \geq 2$, there exist two distinct vertices with the same degree.*

Proof. We proceed by induction on n the number of vertices in G. Assume $n = 2$, and put $V(G) = \{x, y\}$. If x and y are adjacent, then $d(x) = 1 = d(y)$; otherwise, $d(x) = 0 = d(y)$; so, the claim is true in this case. Now, assume the result holds for every graph with $< n$ vertices. Let $V(G) = \{x_1, \ldots, x_n\}$, and assume for a contradiction that $d(x_1) < \ldots < d(x_n)$. If $d(x_1) = 0$, then the graph G' obtained from G by removing x_1 has the same edges as G; it follows by induction that there exist $2 \leq i < j$ such that $d(x_i) = d(x_j)$, a contradiction. On the other hand, if $d(x_1) \geq 1$, , then $d(x_2) \geq 2, \ldots$ and so $d(x_n) \geq n$; but x_n could have at most $n - 1$ neighbors, a contradiction. This shows as claimed that for some $i \neq j$, $d(x_i) = d(x_j)$. ■

Definition 4 *Let $G = (V, E)$ and $G' = (V', E')$ be two graphs. We say that G' is a subgraph of G if $V' \subseteq V$ and $E' \subseteq E$. If $V = V'$, then G' is called a spanning graph of G.*

Thus, a subgraph G' of G is an induced subgraph if $G' = G[V(G')]$.

Example 2 *:*

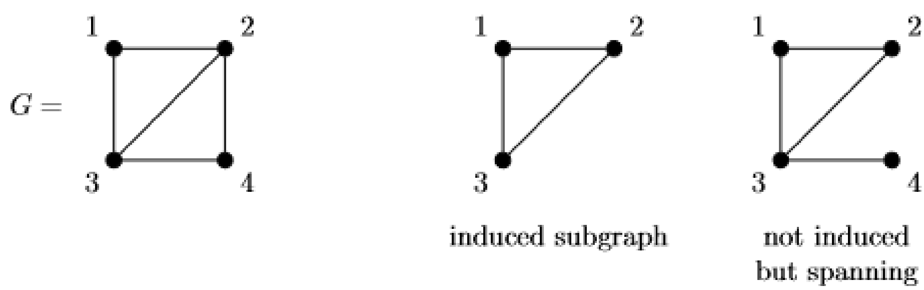
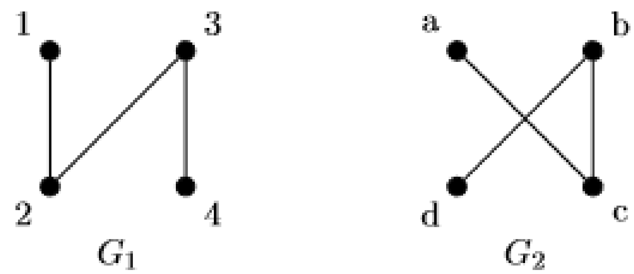


Figure 1.2:

Definition 5 *Let $G = (V, E)$ and $G' = (V', E')$ be graphs. A morphism from G to G' is a map $f : V \longrightarrow V'$ that satisfies the following property for all $x, y \in V$:*

$$xy \in E \Rightarrow f(x)f(y) \in E'.$$

We can define therefore, in the obvious way, the category of graphs and their morphisms. Note that morphism $f : G \longrightarrow G'$ in this category is *an* *isomorphism* if and only if f is bijective. The isomorphisms from a graph G onto itself form a group (under the composition of maps) which we denote $\mathrm{Aut}(G)$.



Example 3 *For the two graphs G_1 and G_2 below, the function $f : G_1 \to G_2$ given by $f(1) = a$, $f(2) = c$, $f(3) = b$, and $f(4) = d$ is an isomorphism.*

For two graphs H and G, we say that G contains a copy of H if there exists an injective morphism of graphs $H \hookrightarrow G$; this means that G contains a subgraph that is isomorphic to H. We say that G is H-free if it contains no copy of H. The latter definitions arise in formulating the Graph Removal Lemma, which is a very important consequence of the regularity lemma (cf. Remark 18).

Let G be a graph. A *walk* in G is a sequence of vertices $x_0, x_1, ..., x_l$ such that $x_i x_{i+1} \in E$ for all $i \in \{0, 1, ..., l-1\}$. The number l is called the length of that walk.

Note that an edge $x_i x_{i+1}$ in a walk could appear several times; so the same vertex could appear more than once.

A walk in which all the edges are distinct is called a *trail*; it is important to note that a trail may have repeated vertices.

A trail $x_0, x_1, ..., x_l$ such that all the vertices are distinct except for x_0, x_l will be called a *path* in G; such a path is closed if $x_0 = x_l$. A closed path will be termed also a *cycle*. A cycle is a *triangle* if it has length 3, a *quadrilateral* if it has length 4, etc.

Example 4 *:*

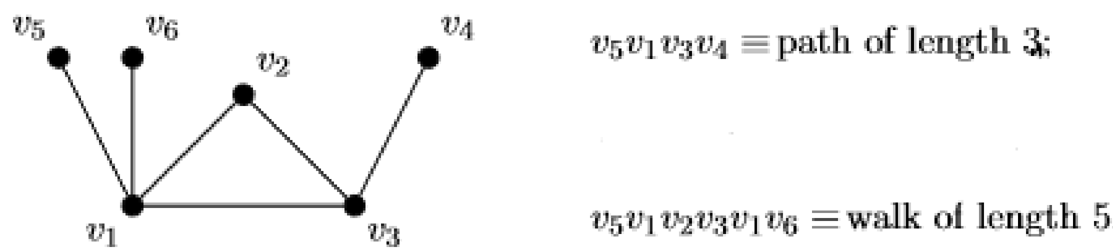


Figure 1.3:

1.2 The regularity lemma and its proof

Statement of the result

Let $\varepsilon > 0$ be a real number, and X be a finite set. Conceptually, it is convenient to say that a subset $X' \subseteq X$ is ε-large (or merely large) if $|X'| \geq \varepsilon|X|$; we say that X' is small otherwise.

Let $G = (V, E)$ be a graph. For every pair (X, Y) of disjoint subsets of V, we define $E(X, Y)$ to be the set of edges between X and Y, so

$$E(X,Y) = \{e \in E \mid \exists x \in X \text{ and } \exists y \in Y \text{ with } e = xy\}.$$

Clearly, $E(X,Y) = E(Y,X)$, and $E(X,Y)$ reaches its maximal possible size if every vertex in X is adjacent to every vertex in Y; in this case $|E(X,Y)| = |X||Y|$. Also, $|E(X,Y)| = 0$ if and only if X is isolated from Y.

Definition 6 *Let (X, Y) be a pair of disjoint sets of vertices of G. The (edge) density of (X, Y) in G is the number*

$$d(X,Y) = \frac{|E(X,Y)|}{|X||Y|}.$$

Since the largest possible value of $|E(X,Y)|$ is $|X||Y|$, it follows that

$$d(X,Y) \leq 1.$$

Definition 7 *A pair* (X, Y) *of disjoint sets of vertices of* G *is* ε-regular *if*

$$|d(X^{'}, Y^{'}) - d(X, Y)| \leq \varepsilon,$$

for all large subsets $X^{'} \subseteq X$ *and* $Y^{'} \subseteq Y$.

The above means that if $X^{'} \subseteq X$ and $Y^{'} \subseteq Y$ satisfy $|X^{'}| \geq \varepsilon|X|$ and $|Y^{'}| \geq \varepsilon|Y|$, then

$$|d(X^{'}, Y^{'}) - d(X, Y)| \leq \varepsilon;$$

(the edge density of $(X^{'}, Y^{'})$ is very close to that of (X, Y)).

Remark 8 *Some authors use the term* ε*-uniform instead for* ε*-regular (cf. for instance [1]).*

Let $\mathcal{P} = \{X_0, ..., X_k\}$ be a partition of $V = V(G)$; we say that $\mathcal{P}$ is ε-regular with exceptional set X_0, if the following conditions hold:

(a) X_0 is small in V (that is $|X_0| < \epsilon|V|$);

(b) the parts X_i have the same cardinality for $i \in \{1, ..., k\}$;

(c) the pairs (X_i, X_j) with $i < j$, are ε-regular for all but at most $\varepsilon\binom{k}{2}$ of them.

If $\mathcal{P}$ satisfy (a) and (b), but not necessarily (c), we say that $\mathcal{P}$ is an equipartition of V with exceptional set X_0.

Remark 9 *The number of pairs* (X_i, X_j) *with* $1 \leq i < j \leq k$ *is equal to* $\binom{k}{2} = \frac{1}{2}k(k-1)$*. If we denote by* U *the set of those pairs* (X_i, X_j) *that are* ε*-regular, then the condition (c) amounts to saying that*

$$\binom{k}{2} - |U| \leq \varepsilon\binom{k}{2}$$

or equivalently,

$$|U|\binom{k}{2}^{-1} \geq 1 - \varepsilon.$$

The above means that an arbitrary pair (X_i, X_j) *is* ε*-regular with probability at least* $1-\varepsilon$*. Usually,* $\varepsilon > 0$ *is interpreted as a very small quantity, so* $1-\varepsilon$ *is very close to* 1*; therefore, the previous statement means, informally, that almost all the pairs* (X_i, X_j) *is* ε*-regular.*

Now, we can state the main result.

Theorem 10 (Szemerédi's Regularity Lemma.) *Let m be a positive integer, and $\varepsilon > 0$. Then, there exists an integer M (depending on m and ϵ) such that every graph G with at least M vertices has an ε-regular partition $X_0, ..., X_k$ (with exceptional set X_0) with $m \leq k \leq M$.*

Another equivalent form of the regularity lemma is the following.

Theorem 11 *Let m be a positive integer, and $\varepsilon > 0$. Then, there exists an integer M (depending on m and ϵ) such that for every graph G with at least M vertices, there exists a partitions $X_1, \ldots X_k$ of $V(G)$ that satisfies $|X_1| \leq \ldots \leq |X_k| \leq |X_1| + 1$, and all the pairs (X_i, X_j), $1 \leq i < j \leq k$, are ε-regular except for at most εk^2 of them.*

A weaker version of that theorem was introduced in [15] to prove a conjecture of Erdös and Turán on arithmetic progressions in some sets of numbers. The result in full generality was proved by Szemerédi in [16]. Applications of this result will be discussed in the last chapter.

The Proof

We follow closely the proof presented in [2, §12.4].

Let $G = (V, E)$ be graph, and let $n = |V|$. For every pair (X, Y) of disjoint subsets of V, we define the index of regularity $\varrho(X, Y)$ by

$$\varrho(X, Y) = |X||Y|d(X, Y)^2.$$

We can extend this definition to every (finite) collection $\mathcal{P} = \{X_i\}_{i \in I}$ of disjoint subsets of V by setting

$$\varrho(\mathcal{P}) = \sum_{\substack{X, Y \in \mathcal{P} \\ X \neq Y}} \varrho(X, Y).$$

(We shall denote such a sum simply by $\sum \varrho(X, Y)$.) Note that we can define such a number in the particular case where $\mathcal{P}$ is a partition of V.

As we have noticed previously, $0 \leqslant d(X,Y) \leqslant 1$, so $d(X,Y)^2 \leqslant 1$ for every pair (X,Y) in $\mathcal{P}$. It follows that

$$\varrho(\mathcal{P}) = \sum |X||Y|d(X,Y)^2 \leqslant \sum_{\substack{X,Y\in\mathcal{P}\\X\neq Y}} |X||Y|. \tag{1.1}$$

Lemme 1 *For any collection $\mathcal{P}$ of disjoint subsets of V, we have*

$$\varrho(\mathcal{P}) < \frac{n^2}{2}.$$

.

Proof. By the inequality (1.1), wa have only to show that

$$\sum_{\substack{X,Y\in\mathcal{P}\\X\neq Y}} |X||Y| \leqslant \frac{n^2}{2}.$$

We have $\bigcup_{X\in\mathcal{P}} X \subseteq V$, so $\sum_{X\in\mathcal{P}} |X| \leq n$. Now,

$$n^2 \geq (\sum_{X\in\mathcal{P}} |X|)(\sum_{Y\in\mathcal{P}} |Y|) = \sum_{X\in\mathcal{P}} |X|^2 + 2\sum |X||Y|.$$

But, $\sum_{X\in\mathcal{P}} |X|^2 > 0$ (we are assuming that $\mathcal{P} \neq \varnothing$), the result follows immediately.

■

Let $\mathcal{P}$ and $\mathcal{Q}$ be two collections of disjoint subsets of V. We say that $\mathcal{Q}$ is a refinement of $\mathcal{P}$, and we write $\mathcal{Q} \preccurlyeq \mathcal{P}$, if every part in $\mathcal{P}$ is a reunion of parts in $\mathcal{Q}$.

Every refinement of $\mathcal{P}$ can be obtained by replacing each part in $\mathcal{P}$ by an appropriate partition. Hence, the basic step in constructing refinements of $\mathcal{P}$ is to take $X \in \mathcal{P}$ and replace it by X_1, X_2, where $X = X_1 \cup X_2$ and $X_1 \cup X_2 = \emptyset$. This observation is useful in proving the next result.

Proposition 12 *Let $\mathcal{P}$ and $\mathcal{Q}$ be two collections of disjoint subsets of V. If $\mathcal{Q} \preccurlyeq \mathcal{P}$, then $\varrho(\mathcal{Q}) \geq \varrho(\mathcal{P})$.*

The above means that the regularity index does not decrease when taking refinements.

Proof. By the observation preceding the proposition, we have only to prove the claim when $\mathcal{Q}$ is obtained from $\mathcal{P}$ by replacing a part $X \in \mathcal{P}$ by X_1, X_2, where $X = X_1 \amalg X_1$ and X_1, X_2 are proper subsets of X. We have

$$\varrho(\mathcal{P}) = \sum_{\substack{Y \in \mathcal{P}, \\ Y \neq X}} \varrho(X_1, Y) + \sum_{\substack{Y \in \mathcal{P}, \\ Y \neq X}} \varrho(X_2, Y) + \varrho(X_1, X_2) + \sum_{\substack{Y, Z \in \mathcal{P} \setminus \{X\}, \\ Y \neq Z}} \varrho(Y, Z);$$

but,

$$\sum_{\substack{Y, Z \in \mathcal{P} \setminus \{X\}, \\ Y \neq Z}} \varrho(Y, Z) + \sum_{\substack{Y \in \mathcal{P}, \\ Y \neq X}} \varrho(Y, X) = \varrho(\mathcal{P});$$

therefore,

$$\varrho(\mathcal{Q}) - \varrho(\mathcal{P}) = \varrho(X_1, X_2) + \sum_{\substack{Y \in \mathcal{P}, \\ Y \neq X}} \varrho(X_1, Y) + \varrho(X_1, Y) - \varrho(X, Y).$$

By putting $A(Y) = \varrho(X_1, Y) + \varrho(X_2, Y) + \varrho(X, Y)$, we have only to show that each $A(Y)$ is ≥ 0.

Write $x = |X|$, $y = |Y|$, $x_i = |X_i|$, $d_i = d(X_i, Y)$ for $i \in 1, 2$, and $d = d(x, y)$. Note that $x = x_1 + x_2$.

Obviously, we have

$$|E(X, Y)| = |E(X_1, y)| + |E(X_2, y)|,$$

so

$$xyd = x_1 y d_1 + x_2 y d_2,$$

which simplifies to

$$d = \frac{x_1}{x} d_1 + \frac{x_2}{x} d_2. \tag{1.2}$$

We may assume that $d_1 \geq d_2$ (otherwise, we replace d_1 by d_2), so $d - d_2 =$

$\frac{x_1}{x}(d_1 - d_2)$; hence $d \geq d_2$. On the other hand,

$$\begin{aligned} A &= x_1yd_1^2 + x_2yd_1^2 - xyd_1^2 \\ &= yx(\frac{x_1}{x}d_1^2 + (1 - \frac{x_1}{x})d_2^2 - d^2) \\ &= yx(\frac{x_1}{x}(d_1^2 - d_2^2) + (d_2^2 - d^2)) \\ &= yx(\frac{x_1}{x}(d_1 - d_2)(d_1 + d_2) - (d^2 - d_2^2)) \\ &= yx(d - d_2)(d_1 + d_2) - (d - d_2)(d + d_2) \\ &= yx(d - d_2)(d_1 + d_2) - (d + d_2) \end{aligned}$$

so $A \geq 0$.

■

Assume (X, Y) is a pair of disjoint sets of vertices of G which are not ε-regular. Hence, we can pick a large subset X_1 of X, and a large subset Y_1 of Y such that

$$|d(X, Y) - d(X_1, Y_1)| > \varepsilon.$$

If we set $X_2 = X - X_1$ and $Y_2 = Y - Y_1$, then $\mathcal{P} = \{X_1, X_2, Y_1, Y_2\}$ is formed by pairwise disjoint subsets of V; so the index $\rho(\mathcal{P})$ is well defined.

The next lemma is the key to prove the main result.

Lemme 2 *Under the above assumptions, we have*

$$\rho(\mathcal{P}) - \rho(X, Y) \geq (\frac{\varepsilon^4}{1 - \varepsilon^2})|X||Y|.$$

Proof. Set $x = |X|$, $Y = |Y|$, $d = d(X, Y)$, and $x_i = |X_i|$, $y_i = |Y_i|$, $d_{ij} = d(X_i, Y_j)$ for $i, j \in \{1, 2\}$. Our assumptions can be rewritten as $x_1 \geq \varepsilon x$, $y_1 \geq \varepsilon y$, and $|d_1 - d| > \varepsilon$. Therefore,

$$x_1 \geq \varepsilon x, \quad y_1 \geq \varepsilon y, \quad \text{and } (d_{11} - d)^2 > \varepsilon^2. \tag{1.3}$$

Also, as

$$E(X, Y) = \coprod_{i,j} E(X_i, Y_j),$$

we have $xyd = \sum_{i,j=1,2} x_iy_jd_{ij}$. Hence,

$$xyd - x_1y_1d_{11} = x_1y_2d_{12} + x_2y_1d_{21} + x_2y_2d_{22} \tag{1.4}$$

Now, let $a_1 = \sqrt{x_1 y_2}$, $a_2 = \sqrt{x_2 y_1}$, $a_3 = \sqrt{x_2 y_2}$,
and
$b_1 = \sqrt{x_1 y_2} d_{12}$, $b_2 = \sqrt{x_2 y_1} d_{21}$, $b_3 = \sqrt{x_2 y_2} d_{22}$.

The Cauchy-schwarz inequality yields

$$(\sum_{i=1}^{3} a_i^2)(\sum_{i=1}^{3} b_i^2) \geq (\sum_{i=1}^{3} a_i b_i)^2. \tag{1.5}$$

By (1.4), we have

$$(\sum_{i=1}^{3} a_i b_i)^2 = (xyd - x_1 y_1 d_{11})^2;$$

and by observing that $x = x_1 + x_2$ and $y = y_1 + y_2$, we have

$$xy - x_1 y_1 = x_1 y_2 + x_2 y_1 + x_2 y_2 = \sum_{i=1}^{3} a_i^2.$$

It follows from (1.5) that

$$x_1 y_2 d_{12}^2 + x_2 y_1 d_{21}^2 + x_2 y_2 d_{22}^2 \geq \frac{(xyd - x_1 y_1 d_{11})^2}{xy - x_1 y_1} \tag{1.6}$$

As

$$\rho(\mathcal{P}) = \sum_{i,j} x_i y_i d_{ij}^2 + x_1 x_2 d(X_1, X_2) + y_1 y_2 d(Y_1, Y_2),$$

we have

$$\rho(\mathcal{P}) \geq x_1 y_1 d_{11}^2 + \frac{(xyd - x_1 y_1 d_{11})^2}{xy - x_1 y_1}.$$

By subtracting $\rho(X, Y) = xyd^2$ from both sides, a straightforward simplification yields

$$\rho(\mathcal{P}) - \rho(X, Y) \geq \frac{xy}{xy - x_1 y_1} x_1 y_1 (d_{11} - d)^2.$$

By (1.3), the right hand side above is $\geq \frac{\varepsilon^4}{1 - \varepsilon^2} xy$; the result follows.
■

Let us prove the following before embarking on the proof of the regularity lemma.

Proposition 13 *Let $\mathcal{P} = \{X_0, ..., X_k\}$ be an equipartition of V with exceptional set X_0. If $\mathcal{P}$ is not ε-regular and $|X_0| \leq (\varepsilon - 2^{-k})n$, then there is an equipartition $Q = \{Y_0, ..., Y_l\}$ of V such that*

$$|Y_0| \leq |X_0| + \frac{n}{2^k},$$

$$k \leq l \leq 4^k k,$$

and

$$\rho(Q) - \rho(\mathcal{P}) \geq (\frac{1-\varepsilon}{1+\varepsilon})\varepsilon^5 n^2.$$

Proof. By Lemma 2, each pair (X_i, X_j) in $\mathcal{P}$ which is not ε-regular can be refined into four parts $\mathcal{P}_{ij} = \{X_{i1}, X_{i2,} X_{j1}, X_{j2}\}$ so that

$$\rho(\mathcal{P}_{ij}) - \rho(X_i, X_j) \geq (\frac{\varepsilon^4}{1-\varepsilon^2})|X_i||X_j|. \tag{1.7}$$

Since $\mathcal{P}$ is an equipartition, we have $|X_1| = ... = |X_k|$; so

$$n - |X_i|k = |X_0| < \varepsilon n;$$

(because X_0 is small). Hence, $|X_i| > \frac{n}{k}(1-\varepsilon)$; by combining this with (1.7), one obtains

$$\rho(\mathcal{P}_{ij}) - \rho(X_i, X_j) \geq \frac{\varepsilon^4}{1-\varepsilon^2}(1-\varepsilon)^2\frac{n^2}{k^2} = (\frac{1-\varepsilon}{1+\varepsilon})\varepsilon^4\frac{n^2}{k^2}.$$

For $i \geq 1$ fixed, and $j \neq i$ such that (X_i, X_j) is not $\varepsilon-$regular, one has a partition of X_i into two parts induced by $\mathcal{P}_{ij}$. Let us denote by $\mathcal{P}_i$ the coarsest [1] refinement of these partitions of X_i. Note that $|\mathcal{P}_i| \leq 2^{k-1}$ (every refinement yields at most two more parts). Let

$$\mathcal{P}' = (\bigcup_{x \in X_0}\{x\}) \cup (\bigcup_{i=1,k}\mathcal{P}_i);$$

so $\mathcal{P}'$ is a partition of V, and $\mathcal{P}' \leq \mathcal{P}$. As $|\mathcal{P}_i| \leq 2^{k-1}$, we have $|\mathcal{P}'| \leq k2^{k-1} + |X_0|$, so $|\mathcal{P}'| \leq k2^k$; moreover, as there are more than εk^2 irregular pairs, we have.

$$\rho(\mathcal{P}') - \rho(\mathcal{P}) \geq \varepsilon k^2(\frac{1-\varepsilon}{1+\varepsilon})\varepsilon^4\frac{n^2}{k^2} = (\frac{1-\varepsilon}{1+\varepsilon})\varepsilon^5 n^2$$

Now, we shall construct the desired partition $\mathcal{Q}$. Let $\{Y_1, ..., Y_l\}$ be a maximal collection of disjoint subset of V such that

[1]This means that $\mathcal{P}_i$ is a refinement of all the $\mathcal{P}_\rangle|$ and if $Q \leq \mathcal{P}_\rangle|$ for all j then $Q \leq \mathcal{P}_\rangle$.

(1) $|Y_1| = ... = |Y_l|$;

(2) $|Y_i| \leq \binom{\frac{n}{k}}{4k}$;

(3) each Y_i is contained in some part in $\mathcal{P}'$.

Let $Y_0 = V \backslash (\bigcup_{i=1,l} Y_i)$ and $\mathcal{Q} = \{Y_0, ..., Yl\}$. Therefore, $\mathcal{Q}$ is an equipartition of V with exceptional set Y_0; moreover

$$|Y_0 \cap C| \leq \frac{n}{k4^k}, \quad \text{for all } C \in \mathcal{P}'. \quad \textbf{Why?}$$

If follows that

$$|Y_0| < |X_0| + k2^k(\frac{n}{k4^k}) = |X_0| + \frac{n}{2^k} \leq \varepsilon n;$$

so $|Y_0|$ is small in V. Moreover, since $\mathcal{Q}$ is a refinement of $\mathcal{P}$, $k \leq l$; and since every $|Y_i|$ is contained in at most one part in $\mathcal{P}'$, $l \leq k4^k$, so $k \leq l \leq k4^k$. Finally, $\rho(Q) \geq \rho(\mathcal{P}')$ since the refinement of Q obtained by partitioning Y_0 into singletons is a refinement of $\mathcal{P}'$. Thus,

$$\rho(Q) - \rho(\mathcal{P}) = \rho(Q) - \rho(\mathcal{P}') + \rho(\mathcal{P}') - \rho(\mathcal{P}) \geq (\frac{1-\varepsilon}{1+\varepsilon})\varepsilon^5 n^2;$$

the result follows. ■

Finally, let us start with an equipartition $\mathcal{P}_0 = \{X_0, ..., X_k\}$ of V with exceptional set X_0. If $\mathcal{P}_0$ is not ε-regular, then by the previous proposition, we can find an equipartition $\mathcal{P}_1 \leq \mathcal{P}_0$ such that

$$\rho(\mathcal{P}_1) - \rho(\mathcal{P}_0) \geq (\frac{1-\varepsilon}{1+\varepsilon})\varepsilon^5 n^2$$

By induction, if each $\mathcal{P}_r \leq \mathcal{P}_0$ is not ε-regular then

$$\rho(\mathcal{P}_{r+1}) - \rho(\mathcal{P}_r) \geq (\frac{1-\varepsilon}{1+\varepsilon})\varepsilon^5 n^2.$$

So

$$\rho(\mathcal{P}_{r+1}) \geq r(\frac{1-\varepsilon}{1+\varepsilon})\varepsilon^5 n^2 + \rho(\mathcal{P}_0).$$

But, for r large enough, the right hand side would be greater than any given number; this contradicts Lemma 1. This proves that for some r, $\mathcal{P}_r$ is an ε-regular partition of V; the result follows.

Chapter 2

Applications and recent developments

The regularity lemma has many interesting applications in graph theory; although, we will not focus on this aspect here, and we refer the reader to the survey [13]. Rather, we shall focus on number theoretic applications of the lemma, which, historically, were the main reason for introducing it.

2.1 A conjecture of Erdös and Turàn

Let k be a positive integer, and R be a ring. We call *an arithmetic progression of length k* in R every finite sequence $a_0, \ldots, a_{k-1}$ of distinct elements of R such that the difference $a_{i+1} - a_i$ remains constant; in other words, there exists $b \in R$ ($b \neq 0$) such that $a_i = a_0 + ib$ for $i \in \{0, \ldots, k-1\}$. For $R = \mathbb{Z}$ we recover the usual notion of arithmetic progressions of integers. Usually, if a result on arithmetic progressions of integers is proved, one encounters naturally the problem of extending that result to $\mathbb{Z}^n$. Note that an arithmetic progression (of length k) in $\mathbb{Z}^n$ has the form $(a_1 + ib_1, \ldots, a_n + ib_n)$, where $i \in \{0, \ldots, k-1\}$, and $a_j, b_j \in \mathbb{Z}$. In the sequel, by an arithmetic progression we mean always an arithmetic progression of integers.

In 1921, P. Baudet[1] formulated the following conjecture: *for any finite partition $P_1, \ldots, P_r$ of the set $\mathbb{N}$, at least one of the parts P_i contains arith-*

[1] Actually, Baudet formulated the conjecture for partitions having two parts, and Emil Artin extended it to the this form.

metic progressions of arbitrary large lengths (hence, an arithmetic progression of length k, for any given k).

For example, the partition of $\mathbb{N}$ induced by a subgroup $b\mathbb{Z}$, $b \geq 0$, is formed by the b parts $C_0, \ldots, C_{b-1}$, where $C_i = \{i + bj \mid j \in \mathbb{N}\}$. Clearly, the elements of each C_i form an arithmetic progression, and Baudet's conjecture holds trivially in this case; moreover, this conjecture claims that the distribution of the cosets $a + b\mathbb{Z}$, for various a, b, controls in some sense the finite partitions of $\mathbb{N}$.

van der Waerden (1927) confirmed Baudet's conjecture by proving the following (the formulation of the conjecture in the form below is due to O. Schreier).

Theorem 14 *Let k and r be positive integers. Then there exists a positive integer M depending on k and r such that : for every partition of the set $\{1, 2, \ldots, M\}$ into r parts, at least one of these parts contains an arithmetic progression of length k.*

Let us show how Baudet's conjecture follows from the latter theorem. Assume $P_1, \ldots, P_r$ is a partition of $\mathbb{N}$, k a positive integer, and let $M = M(r, k)$ be as in the above theorem. If we set $A = \{1, 2, \ldots, M\}$, then $\{A \cap P_1, \ldots, A \cap P_r\}$ is a partition of A; by the theorem, there is $i_k \in \{1, \ldots, r\}$ such that $A \cap P_{i_k}$ contains an arithmetic progression of length k. By choosing one $i_k \in \{1, \ldots, r\}$ for each value of k, one obtains a map $k \mapsto i_k$ from $\mathbb{N}^*$ to $\{1, \ldots, r\}$. Therefore, for some $i \in \{1, \ldots, r\}$, the set $\{k \mid i_k = i\}$ is infinite; this means that P_i contains arithmetic progressions of arbitrary large length.

Remark 15 *The least integer $M = M(r, k)$ in Theorem 14, is usually denoted by $W(r, k)$; these are called the van der Waerden numbers; the values of $W(r, k)$ are known for very few values of r and k, and determining them is a very open problem*

In 1936, P. Erdös and P. Turán (cf. [3]) have considered the following related problem:

Let N and k be positive integers, and assume $A \subseteq \{1, \ldots, N\}$ with the property that A contains no arithmetic progression of length k. What is the largest possible size of A? We may define $r_k(N)$ to be the size of such an A; hence, the question is about determining $r_k(N)$.

A noteworthy is that a subset $A \subseteq \mathbb{Z}$ contains a k-arithmetic progression if, and only if, $x + A = \{x + A \,|\, a \in A\}$ does, for all $x \in \mathbb{Z}$. In other words, the property of containing (or not containing) a k-arithmetic progression is invariant under translations. This allows us, for instance, to suppose always that $1 \in A$ in the above setting. Moreover, it follows at once that

$$r_k(n+m) \leq r_k(n) + r_k(m), \qquad \text{for all } n, m \in \mathbb{N}^*. \tag{2.1}$$

In fact, Erdös and Turán considered only the case $k = 3$. The problem could be better approached by a concrete example; consider for instance $N = 8$, and let us try to find $A \subseteq \{1, \ldots, 8\}$ of maximal possible size subject to having no arithmetic progression of length 3. As we have mentioned, we may suppose $1 \in A$. If $2 \in A$, then $3 \notin A$ (otherwise, $1, 2, 3$ is an arithmetic progression of length 3). Hence, the next minimal possible choice is $4 \in A$, so we may suppose that $1, 2, 4 \in A$. We couldn't have $6 \in A$, as this yields the progression $2, 4, 6$; and similarly, $7 \notin A$ (because of $1, 4, 7$); the remaining possibilities are $5 \in A$ or $8 \in A$. The case $5 \in A$ makes no problem, and if so then $8 \notin A$ because this yields $2, 5, 8$. Thus, $A = \{1, 2, 4, 5\}$ has a maximal possible size. Another possible choice is $A = \{1, 3, 4, 6\}$. Anyway, we have $r_3(8) = 4$.

Similarly, one sees that $r_3(9) = r_3(10) = 5$ (consider for instance $A = \{1, 3, 4, 6, 9\}$); $r_3(11) = r_3(12) = 6$; $A = \{1, 3, 4, 6, 10, 11, 13, 14\}$ contains no 3-arith. prog., so $r_3(13) = 7$ and $r_3(14) = 8$. It follows from (2.1) that

$$r_3(16) \leq r_3(8) + r_3(8) = 8;$$

$$r_3(18) \leq r_3(8) + r_3(10) = 9;$$

$$r_3(20) \leq r_3(10) + r_3(10) = 10;$$

$$r_3(22) \leq r_3(10) + r_3(12) = 11.$$

The above values suggest that (except for $N = 7$):

$$r_3(2N) \leq N, \qquad \text{for all } N \geq 4.$$

The latter is the first main result of Erdös and Turán in [3]. To see it, we may assume that $N \geq 8$. If the result holds for an integer some N, then by (2.1) we should have

$$r_3(2(N+4)) \leq r_3(2N) + r_3(8) \leq N + 4.$$

As the result holds for $N = 8, 9, 10, 11$, then it holds for $N = 12, 13, 14, 15$; and more generally for $N = 8+4q, 9+4q, 10+4q, 11+4q$, with $q \geq 0$. Since every integer $N \geq 8$ has one of the previous forms, the result follows.

Following similar lines, it was observed also in the same paper that for every $\varepsilon > 0$, there exists $n_0 = n_0(\varepsilon)$ such that

$$r_3(N) < (\frac{3}{8} + \varepsilon)N, \quad \text{for all } N \geq n_0.$$

Just after, Erdös and Turán said:

> "At present this is the best result for $r_3(N)$. It is probable that $r_3(N) = o(N)^2$."

The latter became the Erdös-Turán conjecture for arithmetic progressions, which was formulated thereafter to include all the functions $r_k(N)$, $k \geq 3$.

Let $\varepsilon > 0$. The fact $r_k(N) < \varepsilon N$ means that the maximal possible size of a subset $A \subseteq \{1, 2, \ldots, N\}$ containing no k-arithmetic progression is $< \varepsilon N$; in other words, if $A \subseteq \{1, 2, \ldots, N\}$ satisfies $|A| \geq \varepsilon N$, then A contains an arithmetic progression of length k.

If the Erdös-Turán conjecture holds true, then we should have $r_k(N) < \varepsilon N$ for all N large enough; so for any fixed such a N, if $A \subseteq \{1, 2, \ldots, N\}$ satisfies $|A| \geq \varepsilon N$, then A contains an arithmetic progression of length k. Conversely, assume the latter holds for some N. Then as we observed above, we should have $r_k(N) < \varepsilon N$. By (2.1), we have for every $q \geq 0$,

$$r_k(qN) < q\varepsilon N = \varepsilon qN.$$

This means that $\lim\limits_{q} \frac{r_k(qN)}{qN} = 0$; but since $(r_k(n))_n$ is sub-additive, $\lim\limits_{n} \frac{r_k(n)}{n}$ exists, so certainly $\lim\limits_{n} \frac{r_k(n)}{n} = 0$. This proves that the Erdös-Turán conjecture is equivalent to the following.

Conjecture. (Erdös-Turán) *Let k be a positive integer and let $\delta > 0$. There exists a positive integer N depending on k and δ such that every subset of $\{1, 2, \ldots, N\}$ of size at least δN contains an arithmetic progression of length k.*

[2] Recall that this means $r_3(N)/N \to 0$ as $N \to +\infty$

Remark 16 *For a subset $A \subseteq \mathbb{N}^*$, the upper density of A, denoted $\bar{d}(A)$, is defined by the formula*

$$\bar{d}(A) = \limsup \frac{|A \cap \{1, 2, \ldots, n\}|}{n}.$$

Many authors cite the Erdös-Turán conjecture under the form: Every subset $A \subseteq \mathbb{N}^*$ with positive upper density (that is $\bar{d}(A) > 0$), contains arithmetic progressions of arbitrary large lengths.

We recall that for any sequence (a_n) of real numbers, we could define a subsequence $(\bar{a}_n)_n$ by setting

$$\bar{a}_n = \sup\{a_m \,|\, m \geq n\}, \quad \textit{for every } n.$$

Clearly, the sequence $(\bar{a}_n)_n$ is decreasing, so it has a limit that coincides with $\inf \bar{a}_n$ *(we may have* $\inf \bar{a}_n = +\infty$*). By definition,*

$$\limsup a_n = \lim \bar{a}_n.$$

This conjecture was first confirmed for $k = 3$ by K. Roth in 1953. Later, Szemerédi proved it for $k = 4$ in 1969, and he proved it for all $k \geq 4$ in 1975 (cf. [15]). Roth's original proof uses analytic methods; later I. Ruzsa and Szemerédi proved a very interesting graph theoretic consequence of the regularity lemma, namely *the triangle removal lemma*, and observed that Roth's theorem follows from it in a straightforward manner. The next lines are devoted to this combinatorial proof of Roth's theorem.

Theorem 17 (Triangle removal lemma) *For every $0 < \alpha < 1$, there exists β (depending on α) such that the following property holds. If G is a graph with n vertices and at most βn^3 triangles, then G can be made triangle-free by removing at most αn^2 edges.*

Let us prove a lemma before proceeding with the proof of this theorem. Recall that a graph $G = (V, E)$ is said to be *tripartite* if V can be partitioned into three subsets A, B, C such that no edge of G has extremities in the same component.

For a vertex $x \in V$ and $Y \subseteq V$, we set $\delta_Y(x) = \delta(x) \cap Y$; hence

$$\delta_Y(x) = \{y \in V \,|\, xy \in E\}.$$

Lemme 3 *Let $\varepsilon > 0$, and let $G = (V, E)$ be a* tripartite *graph, with $V = A \cup B \cup C$. Assume that all the pairs $(A, B), (A, C), (B, C)$ are ε-regular, and that*

$$c = d(A, B) \geq 2\varepsilon, \quad b = d(A, C) \geq 2\varepsilon, \quad b = d(A, C) \geq 2\varepsilon.$$

Then the number of triangles in G is at least

$$(1 - 2\varepsilon)(a - \varepsilon)(c - \varepsilon)(d - \varepsilon)|A||B||C|.$$

In particular, there are at least $(1 - 2\varepsilon)\varepsilon^3|A||B||C|$ triangles in G.

Proof. Let

$$S = \{x \in A \,|\, |\delta_B(x)| < (c - \varepsilon)|B|\},$$

and

$$T = \{x \in A \,|\, |\delta_C(x)| < (b - \varepsilon)|C|\}.$$

We claim that S and T are ε-small in A (that is $|S| < \varepsilon|A|$, and $|T| < \varepsilon|A|$). We have only to prove the claim for S as the case of T follows similarly. If we assume that S is large, then since (A, B) is ε-regular, we should have

$$c - \varepsilon \leq d(S, B) \leq c + \varepsilon;$$

on the other hand, we have by assumption

$$d(S, B) = \frac{|E(S, B)|}{|S||B|} < \frac{|S|(c - \varepsilon)|B|}{|S||B|} = c - \varepsilon;$$

a contradiction. This proves the claim.

Now, let $A' = A \setminus (S \cup T)$; the above fact implies $|A'| > (1 - 2\varepsilon)|A|$. Fix a vertex $x \in A'$, and put $B' = \delta_B(x)$ and $C' = \delta_C(x)$. The definition of A' implies at once that $|B'| \geq (c - \varepsilon)|B|$, and $|C'| \geq (b - \varepsilon)|C|$. By our assumptions on b and c, it follows that B' is large in B, and C' is large in C; so the assumption that (B, C) is ε-regular yields

$$a - \varepsilon \leq d(B', C') \leq a + \varepsilon;$$

hence,

$$|E(B', C')| \geq (a - \varepsilon)|B'||C'| \geq (a - \varepsilon)(b - \varepsilon)(c - \varepsilon)|B||C|.$$

Observe now that every vertex bc in $E(B', C')$ yields a triangle bxc; thus the number of (A', B', C') triangles is at least

$$(1 - 2\varepsilon)(a - \varepsilon)(c - \varepsilon)(d - \varepsilon)|A||B||C|,$$

which implies the result immediately.

■

Proof of the triangle removal lemma. Let G be a graph on n vertices, $0 < \alpha < 1$ a real number; put $\varepsilon = \alpha/4$ and $m = 8/\alpha$. By the regularity lemma, there is an ε-regular partition $X_0, X_1, \dots, X_k$ of $V(G)$, with $8/\alpha \le k \le M(\varepsilon)$.

We shall construct a new graph G' by removing some edges from G according to the following rules:

- For each $i \ge 1$, remove the edges inside X_i; that is every edge whose extremities lie in X_i. As each X_i has $\frac{1}{k}(n - |X_0|) \le n/k$ vertices, we could remove at most $\binom{n/k}{2} < n^2/2k^2$ edges in each X_i; therefore, we remove in total $k\frac{n^2}{2k^2}$ edges, and so at most $\frac{\alpha}{4}n^2$ edges.
- For each irregular pair (X_i, X_j), $1 \le i < j \le n$, remove all the edges between X_i and X_j. The number of irregular pairs is at most εk^2, and

  $$|E(X_i, X_j)| \le |X_i|^2 \le n^2/k^2;$$

  the number of edges that could be removed here is at most $\varepsilon n^2 = \frac{\alpha}{4}n^2$.
- Remove the edges between all the pairs (X_i, X_j), $1 \le i < j \le n$, having density $< 2\varepsilon$. For such a pair,

  $$|E(X_i, X_j)| < 2\varepsilon|X_i|^2 \le 2\varepsilon n^2/k^2.$$

  As the number of such pairs does not exceed $\binom{k}{2} \le k^2/2$, the number of edges that could be removed is at most $\varepsilon n^2 = \frac{\alpha}{4}n^2$.
- Finally, remove all the edges with an extremity in X_0. As $|X_0| < \varepsilon n$ and every vertex in X_0 could be joined to less than n vertices, the number of such edges is $< \varepsilon n^2 = \frac{\alpha}{4}n^2$

In total, G' is obtained from G by removing at most αn^2 edges. By construction, if G' contains a triangle, then it arises from some triplet (X_i, X_j, X_l), $1 \leq i < j < l$, in which every pair is ε-regular of density $\geq \varepsilon/2$; the foregoing lemma implies that there are at least $(1-2\varepsilon)\varepsilon^3|X_i|^3$ triangles in G'. Observe that $|X_i| > \frac{\alpha}{8}n$, and let

$$\beta = \frac{1}{(32)^3}(1 - \frac{\alpha}{2})\alpha^6.$$

Either G' contains no triangles or contains more than βn^3 of them. Now, if G contains at most βn^3 triangles, then certainly G' contains no triangles; in other words, G became triangle free after removing at most αn^2 edges; the result follows.

■

Remark 18 *There is a more general analogue of the triangle removal lemma:* the graph removal lemma. *The latter can be stated as follows. For any graph H, and every $0 < \alpha < 1$, there exists β (depending on α) such that the following property holds. If G is a graph with n vertices and at most βn^h copies of H, where h denotes the number of vertices of H, then G can be made H-free by removing at most αn^2 edges.*

The graph removal lemma could be proved using the regularity lemma; a new proof that avoids the regularity lemma and gives better bounds on β has been given recently by J. Fox (cf. [5]).

Before establishing Roth's theorem, let us prove a very nice consequence of the triangle removal lemma. For convenience, we denote $\{1, \ldots, n\}$ simply by $[n]$. Fora positive integer r, we denote by $[n]^r$ the cartesian product of r copies of $[n]$. We call a corner in $[n]^2$ every triple of points of $[n]^2$ of the form $(x, y), (x + d, y), (x, y + d)$, with $d > 0$; alternatively, a corner in $[n]^2$ is an isosceles triangle in the real plane, whose vertices have coordinates in $\{1, \ldots, n\}$.

Theorem 19 (Corners theorem) *For every $\alpha > 0$, there exists a positive integer N such that every subset $A \subseteq [N]^2$ of size at least αN^2 contains a corner.*

Proof. Let A be a subset of $[n]^2$ of density at least α (that is $|A| \geq \alpha n^2$). Define a tripartite graph G by taking $V(G) = X \cup Y \cup Z$, with $X = Y = [n]$

and $Z = [2n]$; for $x \in X$, $y \in Y$ and $z \in Z$, we join x to y if $(x, y) \in A$, and x to z if $(x, z - x) \in A$, and we join y to $z \in Z$ if $(z - y, y) \in A$.

If we have a triangle xyz in the graph G, then (x, y), $(x, y+d)$ and $(x+d, y)$ lie in A, where $d = z - x - y$. This yields a corner if and only if $z - x - y \neq 0$. If we denote by T the set of all triangles xyz in G such that $z = x + y$ (degenerate triangles), then clearly $|T| = |A| \geq \alpha n^2$; moreover, any two distinct triangles in T have no common edges, so we couldn't remove T by removing less than αn^2. It follows that G contains more than βn^3 triangles, where β is the constant determined by α in the triangle removal lemma. For n is sufficiently large, $\beta n^3 > n^2 \geq |T|$, so G contains at least one non-degenerate triangle in G, and hence A has a corner. ∎

Theorem 20 (K. Roth 1953**)** *For all $\delta > 0$, there exists a positive integer N such that every subset A of the set $\{1, 2, \ldots, N\}$ satisfying $|A| \geq \delta N$, contains an arithmetic progression of length 3.*

Proof. Consider the map $f : [n]^2 \to [3n]$ defined by $f(x, y) = x + 2y$. For every corner $(x, y), (x + d, y), (x, y + d)$ in $[n]^2$, the images $f(x, y), f(x + d, y), f(x, y + d)$ form an arithmetic progression of length three. Our claim follows now from the corners theorem.

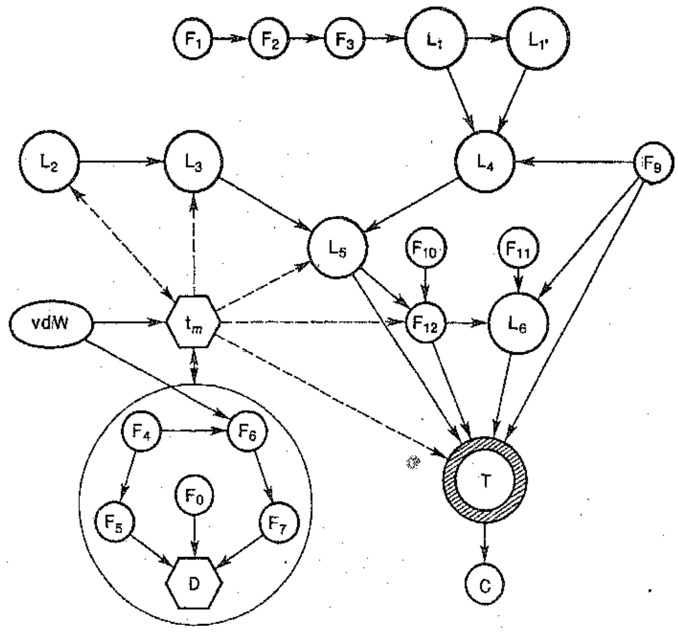
∎

We finish this section by mentioning Szemerédi's theorem; we shall not give a proof, although, several relevant remarks on it will be discussed in the next section.

Theorem 21 *Let k be a positive integer and let $\delta > 0$. There exists a positive integer N depending on k and δ such that every subset of the set $\{1, 2, \ldots, N\}$ of size at least δN contains an arithmetic progression of length k.*

2.2 Generalizations and other approach to the regularity lemma

The reason for not including a proof of Szemerédi's theorem is that it does not follows easily from the regularity lemma. In fact, Szemerédi's original proof is quite intricate; below is the scheme of the proof drawn by Szemerédi in [15].



The diagram represents an approximate flow chart for the accompanying proof of Szemerédi's theorem. The various symbols have the following meanings: F_k = Fact k, L_k = Lemma k, T = Theorem, C = Corollary, D = Definitions of B, S, P, α, β, etc., t_m = Definition of t_m, vdW = van der Waerden's theorem, F_0 = "If $f: \mathbb{R}^+ \to \mathbb{R}^+$ is subadditive then $\lim_{n\to\infty} \frac{f(n)}{n}$ exists".

Figure 2.1:

As another illustration of the situation, we quote from Terence Tao, a leading specialist in the area (see https://terrytao.wordpress.com/):

> A few days ago, Endre Szemerédi was awarded the 2012 Abel prize "for his fundamental contributions to discrete mathematics and theoretical computer science, and in recognition of the profound and lasting impact of these contributions on additive number theory and ergodic theory." ...
>
> As I was on the Abel prize committee this year, I won't comment further on the prize, but will instead focus on what is arguably Endre's most well known result, namely Szemerédi's theorem on arithmetic progressions...
>
> Szemerédi's original proof of this theorem is a remarkably in-

tricate piece of combinatorial reasoning. Most proofs of theorems in mathematics – even long and difficult ones – generally come with a reasonably compact "high-level" overview, in which the proof is (conceptually, at least) broken down into simpler pieces. There may well be technical difficulties in formulating and then proving each of the component pieces, and then in fitting the pieces together, but usually the "big picture" is reasonably clear. To give just one example, the overall strategy of Perelman's proof of the Poincaré conjecture can be briefly summarised as follows: to show that a simply connected three-dimensional manifold is homeomorphic to a sphere, place a Riemannian metric on it and perform Ricci flow, excising any singularities that arise by surgery, until the entire manifold becomes extinct. By reversing the flow and analysing the surgeries performed, obtain enough control on the topology of the original manifold to establish that it is a topological sphere.

In contrast, the pieces of Szemerédi's proof are highly interlocking, particularly with regard to all the epsilon-type parameters involved; it takes quite a bit of notational setup and foundational lemmas before the key steps of the proof can even be stated, let alone proved. Szemerédi's original paper contains a logical diagram of the proof (reproduced in Gowers' recent talk) which already gives a fair indication of this interlocking structure. (Many years ago I tried to present the proof, but I was unable to find much of a simplification, and my exposition is probably not that much clearer than the original text.) Even the use of nonstandard analysis, which is often helpful in cleaning up armies of epsilons, turns out to be a bit tricky to apply here.

The above situation suggests that other approach to Szemerédi's theorem are highly desirable. This philosophy opened the door to several fruitful ideas; some of these are mentioned briefly below.

1. The ergodic approach: The work of Hillel Furstenberg.

   Let $(X, \mathcal{B}, \mu)$ be a measurable space; this means that X is a set, $\mathcal{B}$ is a family of subsets of X which is stable under taking complements, countable unions, and $\emptyset \in \mathcal{B}$, and μ is a map on $\mathcal{B}$ with values in

$[0, +\infty]$ which satisfies

$$\mu(\emptyset) = 0, \quad \text{and } \mu(\bigcup_n A_n) = \sum_n \mu(A_n),$$

for any countable family (A_n) of pairwise disjoint subsets from $\mathcal{B}$.

The elements of $\mathcal{B}$ are usually termed the measurable subsets of X. A map $T : X \to X$ is measurable if $T^{-1}A$ is a measurable subset whenever A is.

Let $T : X \to X$ be a bijection such that T and T^{-1} are both measurable, and $\mu(TA) = \mu(A)$ for every measurable subset A (such a T is also called a measure-preserving map).

The following result is known as Furstenberg multiple recurrence theorem (cf. [6]).

Theorem 22 *Under the above notation; for any positive integer k, and for any $E \in \mathcal{B}$ such that $\mu(E) > 0$, there exists an integer $n > 0$ such that*

$$\mu(E \cap T^n E \cap \cdots \cap T^{n(k-1)} E) > 0.$$

This theorem is equivalent to Szemeredi's theorem. Just to highlight this relevance; define Z to be the set of all maps from $\mathbb{Z}$ to $\{0, 1\}$. We can endow $Z = \{0, 1\}^{\mathbb{Z}}$ with the product topology, and by Tychonoff's theorem Z is a compact topological space. Note that we can identify the set of all parts of $\mathbb{Z}$ to Z by identifying each $A \subseteq \mathbb{Z}$ to its chracteristic function. Define a map $T : Z \to Z$ by setting $(Tf)(n) = f(n+1)$, for all $f \in Z$. In other words, T maps each $A \subseteq \mathbb{Z}$ to $TA = A + 1$ where $A + 1 = \{x + 1 \,|\, x \in A\}$.

When having a subset $A \subseteq \mathbb{Z}$ of positive upper density, we define a topological space X to be the topological closure of the set $\{T^n A \,|\, n \in \mathbb{Z}\}$ (so X is compact). Then, we define a measure on X that preserves T (and so a σ-algebra $\mathcal{B}$) by an appropriate limit process:

First, for each $N > 0$, we define a measure μ_N on X by setting

$$\mu_N = \frac{1}{2N+1} \sum_{n=-N}^{N} \delta_{T^n A},$$

where δ_b is the usual Dirac measure: for every $M \subseteq X$, $\delta_b(M) = 1$ if $b \in M$, and $\delta_b(M) = 0$ otherwise.

For $E = \{B \in X \mid 0 \in B\}$, we have $\delta_{T^n A}(E) = 1$ if and only if $0 \in A+n$, or equivalently $-n \in A$; it follows that

$$\mu_N(E) = \frac{|A| \cap [-N, N]}{2N+1},$$

, so $\mu_N(E)$ is the density of A in $[-N, N] = \{-N, \dots 0, \dots, N\}$. Since A has positive upper density, we could find a sequence N_n of positive integers N_n tends to a positive integer as $n \to \infty$. Informally, we define μ as the limit of μ_{N_n} as $n \to \infty$. It follows that $\mu(E)$ is the upper density of A, so $\mu(E) > 0$. Moreover,

$$T\mu_N - \mu_N = \frac{1}{2N+1}(\delta_{T^{N+1}A} + \delta_{T^{-N}A}),$$

and the latter has total mass (the measure of X) $\frac{2}{2N+1}$, which tends to 0 with n, hence $T\mu = \mu$, and T is a measure preserving map. Now, the foregoing theorem implies that

$$\mu(E \cap T^n E \cap \cdots \cap T^{n(k-1)} E) > 0,$$

for every positive integer k. This is interpreted as A contains k-arithmetic progression, for all k.

The ideas of Furstenberg led to a multidimensional version of Szemerédi's theorem. For a subset $A \subseteq \mathbb{Z}^r$, we can consider the finite sets $A \cap [-n, n]^r$ for every positive integer n; if a_n denotes the cardinality of the latter set, we define the density of A in the box $[-n, n]^r$ to be $a_n/(2n+1)^r$. The quantity $\limsup a_n/(2n+1)^r$ is the upper density of A.

Theorem 23 *Every subset $A \subseteq \mathbb{Z}^r$ of positive upper density contains arithmetic progressions (in the ring $\mathbb{Z}^r$) of arbitrary large length.*

The above theorem could be proved via the following generalization of Theorem 22 (cf. [7]).

Theorem 24 *Let $(X, \mathcal{B}, \mu)$ be a measurable space such that $\mu(X) = 1$ (so X is a probabilistic space), and let $T_1, \dots, T_k : X \to X$ be measure preserving isomorphisms such that $T_iT_j = T_jT_i$ for all i, j. Then f for any $E \in \mathcal{B}$ with $\mu(E) > 0$, there exists an integer $n > 0$ such that*

$$\mu(T_1^n E \cap T_2^n E \cap \cdots \cap T_k^n E) > 0.$$

Let us finally mention another application of this ergodic approach due to Bergelson and Leibman (1996); this result might be thought as a polynomial version of Szemerédi's theorem.

Theorem 25 *Let $P_1, \dots, P_k : \mathbb{Z} \to \mathbb{Z}^r$ be polynomial maps such that $P_i(0) = 0$ for all of them. Then every subset $A \subseteq \mathbb{Z}^r$ of positive upper density contains a sequence of the form $w + P_1(n), \dots, w + P_k(n)$, for some $w \in \mathbb{Z}^r$, and some $n > 0$.*

2. The regularity lemma for hypergraphs.

   A hypergraph $H = (V, E)$ is a set V together with a family E of subsets of V. Such a hypergraph is said to be n-uniform if all the elements of E have cardinality n. So, for instance, the 2-uniform hypergraphs are exactly the usual graphs.

   T. Gowers claimed that there is a regularity lemma for uniform hypergraphs; such a generalization was important in relevance to Szemerédi's theorem. The idea is that we may proceed as for Roth's theorem: a higher regularity lemma would implies an appropriate removal lemma, appropriate enough to imply Szemerédi's theorem in a clear way. Gowers succeeded in obtaining such a generalization in [9]; moreover, his proof yields better effective bounds. The statement of the hypergraph regularity lemma is somewhat sophisticated; we refer the interested reader to Gowers, *loc. cit.*

Time does not allowed us to discuss the probabilistic approach by T. Tao [17]; the latter gives much clarification of the work of Gowers mentioned above. Another interesting problem is related to finding effective bounds on the function $\varepsilon \mapsto M(m, \varepsilon)$ in the regularity lemma (cf. [4]). The same problem arises in relevance to the removal lemma (cf. [5]), and Szemerédi's theorem.

Abstract: Szemerédi´s Regularity Lemma is a result in graph theory. The lemma states that for every large enough graph, the set of nodes can be dvided into subsets of about the same size so that the edges between different subsets behave almost randomly. In 1975, Szemerédi introduced a weak version of this lemma, restricted to socalled bipartite graphs, in order to prove his famous theorem about arithmetic progressions. In 1978 he proved the full lemma. A graph consists of nodes and edges. The edges are connections between the nodes, and between two nodes there might or might not be an edge.

Résumé: Régularité de Szemerédi Le lemme est un résultat de la théorie des graphes. Le lemme indique que pour chaque graphe suffisamment grand, l'ensemble des noeuds peut être divisé en sous-ensembles d'environ la même taille, de sorte que les arêtes entre les différents sous-ensembles se comportent presque de manière aléatoire. En 1975, Szemerédi a introduit une version faible de ce lemme, limitée aux graphes bipartites, afin de prouver son fameux théorème sur les progressions arithmétiques. En 1978, il a prouvé le lemme complet. Un graphique est constitué de nœuds et d'arêtes. Les arêtes sont des connexions entre les noeuds et entre deux noeuds, il peut y avoir ou non une arête.

الملخص: Szemerédi´s Regularity Lemma هو نتيجة في نظرية الرسم البياني. ينص مصطلح lemma على أنه بالنسبة لكل رسم بياني كبير ، يمكن تحويل مجموعة العقد إلى مجموعات فرعية بنفس الحجم تقريباً بحيث تتصرف الحواف بين مجموعات فرعية مختلفة بشكل عشوائي تقريبًا. في عام 1975 ، قدم Szemerédi نسخة ضعيفة من هذه اللفظة ، مقتصرة على الرسوم البيانية ثنائية الطبقة الاجتماعية ، من أجل إثبات نظريته الشهيرة حول التقدم الحسابي. في عام 1978 أثبت أنه lemma الكامل. يتكون الرسم البياني من العقد والحواف. الحواف عبارة عن وصلات بين العقد ، وقد يكون أو لا يكون هناك حافة بين عقدتين.

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