NILPOTENT GROUPS OF AUTOMORPHISMS

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Abstract

The aim of this work is to discuss some results of Kaloujnine and P. Hall on the automorphism groups centralizing some subgroup series. The conclusion about such groups is that they are nilpotent of a class bounded in terms of the length of the series that they centralize.

Keywords: Nilpotent groups, Automorphisms, commutators.

1. Introduction

Let G be a group and $(N_i)_{1\leq i\leq n+1}$ a decreasing series of subgroups of G such that $N_1 = G$ and $N_{n+1} = \{1\}$.
Roughly speaking, a subgroup A of the automorphism group of G stabilizes the series $(N_i)_{1\leq i\leq n+1}$ if it fixes each element in quotient set N_i/N_{i+1} for all i.

Our work treats three cases. In the first case we assume that each N_i is normal in G, and the conclusion is that A is nilpotent of class not exceeding $n-1$ (this result is due to Kaloujnine). In the second case, we assume that N_i is only normal in N_{i-1} for every i; this case is settled by P. Hall who has proved that A should be nilpotent of class at most $\frac{1}{2}n(n-1)$. In the last case, no assumption on the subgroups N_i's is made; as far as we know this case has not been treated yet, and still we have no insight on what the answer might be.

2. Commutators

Let G be a group, and $x, y \in G$. The commutator $[x, y]$ is defined by

$$[x, y] = x^{-1}y^{-1}xy.$$

Note that $xy = yx[x, y]$; so x and y commute if and only if $[x, y] = 1$. We write x^y to denote $y^{-1}xy$; hence

$$[x, y] = x^{-1}x^y. \tag{2.1}$$

One sees the commutator satisfies the following identities:

$$[x, y]^{-1} = [y, x]. \tag{2.2}$$
$$[xy, z] = [x, z]^y[y, z]. \tag{2.3}$$
$$[x, yz] = [x, z][x, y]^z. \tag{2.4}$$
$$[[x, y^{-1}], z]^y[[y, z^{-1}], x]^z[[z, x^{-1}], y]^x = 1. \tag{2.5}$$

Let X and Y be two non-empty subsets of G. We define $[X, Y]$ to be the subgroup generated by all the commutators $[x, y]$, with $x \in X$ and $y \in Y$. It follows immediately from the identity (2.2) that $[X, Y] = [Y, X]$.

The identity (2.5) is known as the Hall-Witt formula; it is very similar to the Jacobi identity for Lie algebras. A very important consequence of the latter is the so called the Three Subgroups Lemma.

Lemma. *Let G be a group, X, Y, Z three subgroups of G, and $N \lhd G$. If N contains two of the subgroups $[X, Y, Z]$, $[Y, Z, X]$ and $[Z, X, Y]$, then it contains the third.*

3. Nilpotent groups

We define $(\gamma_n(G))_{n\geq 1}$ the lower central series if G by induction:

$$\gamma_1(G) = G, \quad \text{and} \quad \gamma_{n+1}(G) = [\gamma_n(G), G].$$

One sees by induction that every subgroup $\gamma_n(G)$ is normal in G, so we have a decreasing series of normal subgroups:

$$\ldots \subseteq \gamma_n(G) \subseteq \ldots \subseteq \gamma_2(G) \subseteq \gamma_1(G)$$

Lemma. *Let G be a group. We say that G nilpotent if there exists a positive integer c such that $\gamma_{c+1}(G) = \{1\}$. The smallest integer satisfying the latter property is called the class of nilpotence of G.*

For example, the trivial group is nilpotent of class 0. A group is nilpotent of class 1 if and only if it is abelian.
More generally, for any commutative unital ring k, the upper triangular matrices of type $n \times n$ with 1 on the main diagonal form a group $\mathrm{U}_n(k)$; this group is nilpotent of class $n-1$.

Below are some basic properties of nilpotent groups:

1. If G is nilpotent, and N is a normal non-trivial subgroup of G, then N intersects the center of G non-trivially, that is to say $N \cap Z(G) \neq \{1\}$.
   In particular, the center of every non-trivial nilpotent group is non-trivial.
2. Every nilpotent group satisfies the normalizer condition; in other words, if G is nilpotent and H is a proper subgroup of G, then $H < N_G(H)$; where $N_G(H) = \{g \in G\, g^{-}Hg \subseteq H\}$.
3. It follows from the latter, that every maximal subgroup of a nilpotent group G is normal. This property implies for instance that if G is finite, then every Sylow subgroup of G is normal.
4. The last property leads to a characterization of the finite nilpotent groups: A finite group is nilpotent if and only if it is a dirct product of its Sylow subgroups.

4. Stability groups

Let G be a group, and A a subgroup of the automorphism group of G. Let

$$1 = N_{n+1} \subset N_n \subseteq \ldots \subseteq N_1 = G$$

be a series of normal subgroups of G. We say that A stabilizes this series if $[N_i, A] \subseteq N_{i+1}$ for all $i \in \{1, n\}$. The latter means that for all $i \in \{1, n\}$, $x_i^{-1}\sigma(x_i) \in N_{i+1}$ for all $x_i \in N_i$ and $\sigma \in A$.

The first main result that we treat in our thesis is the following.

Theorem. (Kaloujnine 1953) *Let G be a group, and A a group of automorphisms of G stabilizing a normal series of length n in G. Then A is nilpotent of class at most $n-1$.*

5. A generalization by P. Hall

We aim here to present a sort of generalization of Kaloujnine's theorem given by P. Hall in 1958.

Hall has proved that the series $(N_i)_{1\leq i\leq n+1}$ is subnormal, that is to say every N_{i+1} is normal in N_i, then A is nilpotent of class at most$\binom{n}{2}$

6. The general case: stability groups of series satisfying no extra-condition

Here, we assume that the N_i's are just subgroups of G. We could work in the semi-direct product AG, and the only obvious conclusion that we can extract is that

$$[G, A, \ldots, A] = \{1\}$$

where A appears n times.

Naturally, we have either to show that A is nilpotent of a class bounded in terms of n; or A is nilpotent and can have arbitrary large class; or of course to exhibit a counter example. We tend to believe that a sort of the fist guess is true.

References

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