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Dedicated to my:

- mother

- father

-All my brothers

-All friends:(ex):Fayçal alili

- all familly

- Our colleagues at department of mathematique University Kasdi Merbah of Ouargla

I didecated this work.

C.Abderrahman

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Contents

Index of notation

$\mathcal{P}(\Omega)$	The power set of the set Ω
$\|A\|$	The cardinality of the set A
$A - B$	The set difference between the sets A and B
$\amalg$	Disjoint union,
$\mathbb{N}$	The positive integers,
$\mathbb{N}_0$	The positive integers and 0,
$\mathbb{Z}$	The integers,
$\mathbb{Q}$	The rational numbers,
$\mathbb{R}$	The real numbers,
Γ	The Grigorchuk group,
$F_n, \mathscr{F}_n$	The free group on n generators, for $n \in \mathbb{N} \cup \{\infty\}$, (free group of rank n,)
$F_2, \mathscr{F}_2$	free group of rank 2,
χ_A	Characteristic functin of A
$\langle S \rangle_G$	Subgroup of G generated dy S
$(S \cup S^{-1})^*$	Set of words over $S \cup S^{-1}$
S_n	symmetric group over $\{1, \ldots, n\}$
S_X	symmetric group over X
$l^\infty(, \mathbb{R})$	space of bounded real valued functions,
$\mathbb{Z}/n\mathbb{Z}, \mid \mathbb{Z}_n$	group of integers modulo n

Introduction

We wish to discuss the relationship between the notions of growth and amenability of finitely generated groups.For this aim,we will follow several steps .So this thesis is divided into three chapters.

The reader is not assumed to be familiar with basic group theory and, so the first chapter contains basic and essential definitions, properties and results on group theory. For more [1] and[2] are well-known to this purpose.

In Chapter 2, we shall develop the theory of growth of finitely generated groups. By defining a metric on the group. The growth function then counts the (finite) number of elements in the ball of radius n. After defining a certain equivalence relation between such growth functions, we can consider the equivalence classes under this relation, the growth type of a finit ely generated group, which turns out to be independent of the generating subset. We give examples of classes of groups having polynomial and exponential growth, respectively, and we shall refer to this by saying that the group has regular growth. Note that this term is not standard . If a group has neither polynomial nor exponential growth, we say that it has intermediate growth.

In Chapter 3, we first discuss the notion of amenability which was originally introduced in order to explain the Banach-Tarski paradox. We shall use the definition given by John von Neumann in 1929. A discrete group is amenable if there exists an invariant finitely additive probability measure on the power set of the group. The class of amenable groups, denoted by AG, has nice permanence properties and it coincides with the non-paradoxical groups, and the Følner conditions.

After that we will pass to our principal aim, by relating the two notions of amenability (and its subclass, elementary amenable groups denoted EA) and growth (and its types).

1 CHAPTER ONE : Groups and free groups

1.1 1.Basic facts

Definition 1.1.1 : (The group)

A group is a non-empty set G on which there is a binary operation $(a, b) \mapsto ab$ such that

- if a and b belong to G then ab is also in G(closure),
- $a(bc) = (ab)c$ for all a, b, c in $G(associativity)$
- there is an element $1 \in G$ such that $a1 = 1a = a$ for all $a \in G$ (identity),
- if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = 1$ (inverse)

One can easily check that this implies the unicity of the identity and of the inverse.

A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a, b \in G$ Remark. There are two standard notations for the binary group operation: either the additive notation, that is $(a, b) \mapsto a + b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a, b) \mapsto ab$ for which the identity is denoted by 1.

Examples 1.1.1 :

1) $\mathbb{Z}$ with the addition and 0 as identity is an abelian group.
2) Z with the multiplication is not a group since there are elements which are not invertible in $\mathbb{Z}$
3) The set of $n \times n$ invertible matrices with real coefficients is a group for the matrix product and identity the matrix $\mathbf{I}_n$. It is denoted by $GL_n(\mathbb{R})$ and called the general linear group. It is not abelian for $n \geq 2$

Definition 1.1.2:(The order of a group)

The order of a group G, denoted by $|G|$, is the cardinality of G, that is the number of elements in G.

We have only seen infinite groups so far. Let us look at some examples of finite groups.

Examples 1.1.2:

1• The trivial group $G = \{0\}$ may not be the most exciting group to look at, but still it is the only group of order 1.

2• The group $G = \{0, 1, 2, \ldots, n-1\}$ of integers modulo n is a group of order n. It is sometimes denoted by $\mathbb{Z}_n$

Definition 1.1.3:(the subgroup)

A subgroup H of a group G is a non-empty subset of G that forms a group under the binary operation of G.

Examples 1.1.3:

1• If we consider the group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ of integers modulo $4, H = \{0, 2\}$ is a subgroup of G.

2• The set of $n \times n$ matrices with real coefficients and determinant of 1 is a subgroup of $GL_n(\mathbb{R})$, denoted by $SL_n(\mathbb{R})$ and called the special linear group.

Proposition 1.1.1

Let G be a group. Let H be a non-empty subset of G. The following are equivalent:

1• H is a subgroup of G

2• $(a) x, y \in H$ implies $xy \in H$ for all x, y (b) $x \in H$ implies $x^{-1} \in H$

3• $x, y \in H$ implies $xy^{-1} \in H$ for all x, y

Definition 1.1.4 : (the group homomorphism)

Given two groups G and H, a group homomorphism is a map $f : G \to H$ such that

$$f(xy) = f(x)f(y) \text{ for all } x, y \in G$$

Note that this definition immediately implies that the identity 1_G of G is mapped to the identity 1_H of H. The same is true for the inverse, that is $f\left(x^{-1}\right) = f(x)^{-1}$

Example 1.1.4 :

The map exp: $(\mathbb{R}, +) \to (\mathbb{R}^*, \cdot), x \mapsto \exp(x)$ is a group homomorphism.

Definition 1.1.5: (the isomorphic group)

Two groups G and H are isomorphic if there is a group homomorphism $f : G \to H$ which is also a bijection.

Roughly speaking, isomorphic groups are "essentially the same".

Example 1.1.5 :

If we consider again the group $G = \mathbb{Z}_4 = \{0, 1, 2, 3\}$ of integers modulo 4 with subgroup $H = \{0, 2\}$, we have that H is isomorphic to $\mathbb{Z}_2$, the group of integers modulo 2.

A crucial definition is the definition of the order of a group element.

Definition 1.1.6: (The order of an element)

The order of an element $a \in G$ is the least positive integer n such that $a^n = 1$. If no such integer exists, the order of a is infinite. We denote it by $|a|$

Definition 1.1.7: (the cyclic group)

A group G is cyclic if it is generated by a single element, which we denote by $G = \langle a \rangle$. We may denote by C_n a cyclic group of n elements.

Example 1.1.6:

A finite cyclic group generated by a is necessarily abelian, and can be written (multiplicatively)

$$\left\{1, a, a^2, \ldots, a^{n-1}\right\} \text{ with } a^n = 1$$

or (additively)

$$\{0, a, 2a, \ldots, (n-1)a\} \text{ with } na = 0$$

A finite cyclic group with n elements is isomorphic to the additive group $\mathbb{Z}_n$ of integers modulo n

Example 1.1.7

An n th root of unity is a complex number z which satisfies the equation $z^n = 1$ for some positive integer n. Let $\zeta_n = e^{2i\pi/n}$ be an n th root of unity. All the n th roots of unity form a group under multiplication. It is a cyclic group, generated by ζ_n, which is called a primitive root of unity. The term "primitive" exactly refers to being a generator of the cyclic group, namely, an n th root of unity is primitive when there is no positive integer k smaller than n such that $\zeta_n^k = 1$

Example 1.1.8:

Consider the group $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, the group $\mathbb{Z}_6^*$ of invertible elements in $\mathbb{Z}_6$ is $\mathbb{Z}_6^* = \{1, 5\}$

Definition 1.1.8: (the right and the left coset)

Let H be a subgroup of a group G. If $g \in G$, the right coset of H generated by g is

$$Hg = \{hg, h \in H\}$$

and similarly the left coset of H generated by g is

$$gH = \{gh, h \in H\}$$

In additive notation, we get $H + g$ (which usually implies that we deal with a commutative group where we do not need to distinguish left and right cosets).

Example 1.1.9:

If we consider the group $Z_4 = \{0, 1, 2, 3\}$ and its subgroup $H = \{0, 2\}$ which is isomorphic to Z_2, the cosets of H in G are

$$0 + H = H, 1 + H = \{1, 3\}, 2 + H = H, 3 + H = \{1, 3\}$$

Clearly $0 + H = 2 + H$ and $1 + H = 3 + H$

Definition 1.1.9: (The index of a subgroup)

The index of a subgroup H in G is the number of right (left) cosets. It is a positive number or ∞ and is denoted by $[G : H]$

If we think of a group G as being partitioned by cosets of a subgroup H then the index of H tells how many times we have to translate H to cover the whole group.

Example 1.1.10:

The index $[\mathbb{R} : \mathbb{Z}]$ is infinite, since there are infinitely many cosets of $\mathbb{Z}$ in $\mathbb{R}$.

Theorem 1.1.1:(Lagrange's Theorem)

If H is a subgroup of G, then $|G| = |H|[G : H]$. In particular, if G is finite then $|H|$ divides $|G|$ and $[G : H] = |G|/|H|$

Example 1.1.11:

Consider $G = \mathbb{Z}, H = 3Z$, then $[G : H] = 3$

Given a group G and a subgroup H, we have seen how to define the cosets of H and thanks to Lagrange's Theorem, we already know that the number of cosets $[G : H]$ is related to the order of H and G by $|G| = |H|[G : H]$. A priori, the set of cosets of H has no structure. We are now interested in a criterion on H to give the set of its cosets a structure of group. In what follows, we may write $H \leq G$ for H is a subgroup of G.

Definition 1.1.10: (normal subgroup)

Let G be a group and $H \leq G$. We say that H is a normal subgroup of G, or that H is normal in G, if we have

$$cHc^{-1} = H, \text{ forall } c \in G$$

We denote it $H \leq G$, or $H \leftrightarrow G$ when we want to emphasize that H is a proper subgroup of G.

The condition for a subgroup to be normal can be stated in many slightly different ways.

Lemma 1.1.1

Let $H \leq G$. The following are equivalent:

1• $cHc^{-1} \subseteq H$ for all $c \in G$

2• $cHc^{-1} = H$ for all $c \in G$, that is $cH = Hc$ for all $c \in G$

3• Every left coset of H in G is also a right coset (and vice-versa, every right coset of H in G is also a left coset).

Example 1.1.12:

Let $GL_n(\mathbb{R})$ be the group of $n \times n$ real invertible matrices, and let $SL_n(\mathbb{R})$ be the subgroup formed by matrices whose determinant is 1. Let us see that $SL_n(\mathbb{R}) < GL_n(\mathbb{R})$ For that, we have to check that $ABA^{-1} \in SL_n(\mathbb{R})$ for all $B \in SL_n(\mathbb{R})$ and $A \in GL_n(\mathbb{R})$. This is clearly true since

$$\det\left(ABA^{-1}\right) = \det(B) = 1$$

Proposition 1.1.2:

If H is normal in G, then the cosets of H form a group.

Definition 1.1.11: (the quotient group)

The group of cosets of a normal subgroup N of G is called the quotient group of G by N. It is denoted by G/N

Recall that if $f : G \to H$ is a group homomorphism, the kernel of f is defined by

$$\operatorname{Ker}(f) = \{a \in G, f(a) = 1\}$$

It is eary to see that $\operatorname{Ker}(f)$ is a normal subgroup of G, since

$$f\left(aba^{-1}\right) = f(a)f(b)f(a)^{-1} = f(a)f(a)^{-1} = 1$$

for all $b \in \operatorname{Ker}(f)$ and all $a \in G$ The converse is more intercsting.

Proposition 1.1.2:

Let G be a group. Every normal subgroup of G is the kernel of a homomorphism. Proof. Suppose that $N \leq G$ and consider the map

$$\pi : G \to G/N, a \mapsto aN$$

To prove the result. we have to show that π is a group homomorphism whose kernel is N. First note that π is indeed a map from group to group since G/N is a group by assuming that N is normal. Then we have that

$$\pi(ab) = abN = (aN)(bN) = \pi(a)\pi(b)$$

where the second equality comes from the group structure of G/N. Finally

$$\operatorname{Ker}(\pi) = \{a \in G | \pi(a) = N\} = \{a \in G | aN = N\} = N$$

Definition 1.1.12: (the natural or canonical map or projection)

Let $N \leq G$. The group homomorphism

$$\pi : G \to G/N, a \mapsto aN$$

is called the natural or canonical map or projection.

Recall for further usage that for f a group homomorphism, we have the following characterization of injectivity: a homomorphism f is injective if and only if its kernel is trivial (that is, contains only the identity element). Indeed, if f is injective, then $\mathrm{Ker}(f) = \{a, f(a) = 1\} = \{1\}$ since $f(1) = 1$. Conversely, if $\mathrm{Ker}(f) = \{1\}$ and we assume that $f(a) = f(b)$, then

$$f\left(ab^{-1}\right) = f(a)f(b)^{-1} = f(a)f(a)^{-1} = 1$$

and $ab^{-1} = 1$ implying that $a = b$ and thus f is injective.

We have:
monomorphism : injective homomorphism
epimorphism : surjective homomorphism
isomorphism : bijective homomorphism
endomorphism : homomorphism of a group to itself
automorphism: isomorphism of a group with itself
assume that we have a group G which contains a normal subgroan N another group H, and $f : G \to H$ a group homomorphism. Let π be the canonical projection (see Definition 1.12) from G to the quoticnt group G/N

$$\begin{tikzcd} G \arrow[r, "f"] \arrow[d, "\pi"'] & H \\ G/N \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

We would like to find a homomorphism $\bar{f} : G/N \to H$ that makes the diagram commute, namely

$$f(\boldsymbol{a}) = \bar{f}(\pi(a))$$

for all $a \in G$

Theorem 1.1.2(Factorisation Theorem)

Any homomorphism f whose kernel K contains N can be factored through G/N. In other worda, there is a unique homomorphism $\bar{f} : G/N \to H$ such that $\bar{f} \circ \pi = f$. Furthermore

1. $\bar{f}$ is an epimorphism if and only if f is.
2. $\bar{f}$ is a monomorphism if and only if $K = N$
3. $\bar{f}$ is an isomorphism if and only if f is an epimorphism and $K = N$

Let us start with two groups H and K, and let $G = H \times K$ be the cartesian product of H and K, that is

$$G = \{(h, k), h \in H, k \in K\}$$

We define a binary operation on this set by doing componentwise multiplication (or addition if the binary operations of H and K are denoted additively) on G :

$$(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2) \in H \times K$$

Clearly G is closed under multiplication, its operation is associative (since both operations on H and K are), it has an identity element given by $1_G = (1_M, 1_K)$ and the inverse of (h, k) is (h^{-1}, k^{-1}). In summary, G is a group.

Definition 1.1.13 (the external direct product)

Let H, K be two groups. The group $G = H \times K$ with binary operation defined componentwise as described above is callad the external direct product of H and K

Examples 1.1.13

1• Let $\mathbb{Z}_2$ be the group of integers modulo 2 . We can build a direct product of $\mathbb{Z}_2$ with itself, namely $\mathbb{Z}_2 \times \mathbb{Z}_2$ with additive law componentwise. This is actually the Klein group, also written $C_2 \times C_2$ This group is not isomorphic to $\mathbb{Z}_4$!

2• Let $\mathbb{Z}_2$ be the group of integers modulo 2, and $\mathbb{Z}_3$ be the group of integers modulo 3. We can build a direct product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$, namely $\mathbb{Z}_2 \times \mathbb{Z}_3$ with additive law componentwise. This group is actually isomorphic to $\mathbb{Z}_6$!

3• The group $(\mathrm{R}, +) \times (\mathrm{R}, +)$ with componentwise addition is a direct product.

Definition 1.1.14: (the internal direct product)

If a group G coutains normnl subgroups H and K such that $G = HK$ and $H \cap K = \{1_G\}$, we say that G is the internal direct product of H and K

Examples 1.1.14:

1• Consider the Klein group $\mathbb{Z}_2 \times \mathbb{Z}_2$, it contains the two subgroups $H = \{(h, 0), h \in \mathbb{Z}_2\}$ and
$K = \{(0, k), k \in \mathbb{Z}_2\}$. We have that both H and K are normal, because the Klein group is commutative. We also have that $H \cap K = \{(0, 0)\}$, so the Klein group is indeed an internal direct product. On the other hand, $\mathbb{Z}_4$ only contains as subgroup $H = \{0, 2\}$, so it is not an internal direct product!

2• Consider the group $\mathbb{Z}_2 \times \mathbb{Z}_3$, it contains the two subgroups $H = \{(h, 0), h \in \mathbb{Z}_2\}$ and $K = \{(0, k), k \in \mathbb{Z}_3\}$. We have that both H and K are normal, because the group is commutative. We also have that $H \cap K = \{(0, 0)\}$ so this group is indeed an internal direct product. Also $\mathbb{Z}_6$ contains the two subgroups $H = \{0, 3\} \simeq \mathbb{Z}_2$ and $K = \{0, 2, 4\} \simeq \mathbb{Z}_3$. We have that both H and K are normal, because the group is commutative. We also have that $H \cap K = \{0\}$, so this group is indeed an internal direct product, namely the internal product of $\mathbb{Z}_2$ and $\mathbb{Z}_3$. This is in fact showing that $\mathbb{Z}_6 \simeq \mathbb{Z}_2 \times \mathbb{Z}_3$

Theorem 1.1.3 :(1st Isomorphism Theorem)

If $f : G \to H$ is a homomorphism with kernel K, then the image of f is isomorphic to G/K

$$\mathrm{Im}(f) \simeq G/\operatorname{Ker}(f)$$

Theorem 1.1.4 (2nd Isomorphism Theorem)

If H and N are subgroups of G, with N normal in G, then

$$H/(H \cap N) \simeq HN/N$$

Theorem 1.1.5(3rd Isomorphism Theorem)

If N and H are normal subgroups of G, with N contained in H, then

$$G/H \simeq (G/N)/(H/N)$$

Definition 1.1.15: (The group action)

The group G action on the set X if for all $g \in G$, there is a map

$$G \times X \to X, (g, x) \mapsto g \cdot x$$

such that

1. $h \cdot (g \cdot x) = (hg) \cdot x$ for all $g, h \in G$, for all $x \in X$
2. $1 \cdot x = x$ for all $x \in X$

The first condition says that we have two laws, the group law between elements of the group, and the action of the group on the set, which are compatible.

Examples 1.1.15

Let us consider two examples where a group G acts on itself.

1• Every group acts on itself by left multiplication. This is called the regular action.

2• Every group acts on itself by conjugation. Let us write this action as

$$g \cdot x = gxg^{-1}$$

Let us check the action is actually well defined. First, we have that

$$h \cdot (g \cdot x) = h \cdot \left(gxg^{-1}\right) = hgxg^{-1}h^{-1} = (hg)xg^{-1}h^{-1} = (hg) \cdot x$$

As for the identity, we get

$$x = 1x1^{-1} = x$$

Definition 1.1.16:(The kernel of an action)

The kernel of an action $G \times X \to X, (g, x) \mapsto g \cdot x$ is given by

$$\text{Ker} = \{g \in G, g \cdot x = x \text{ for all } x\}$$

This is the set of elements of G that fix everything in X. When the group G acts on itself, that is $X = G$ and the action is the conjugation, we have

$$\text{Ker} = \left\{g \in G, gxg^{-1} = x \text{ for all } x\right\} = \{g \in G, gx = xg \text{ for all } x\}$$

This is called the center of G, denoted by $Z(G)$

Definition 1.1.17: (The orbit)

Suppose that a group G acts on a set X. The orbit $B(x)$ of x under the action of G is defined by

$$B(x) = \{g \cdot x, g \in G\}$$

This means that we fix an element $x \in X$, and then we let g act on x when g runs through all the elements of G. By the definition of an action, $g \cdot x$ belongs to X, so the orbit gives a subset of X It is important to notice that orbits partition X

Definition 1.1.18: (Transitive action)

Suppose that a group G acts on a set X. We say that the action is transitive, or that G acts transitively on X if there is only one orbit, namely, for all $x, y \in X$, there exists $g \in G$ such that $g \cdot x = y$

Definition 1.1.19:(The stabilizer of an element)

The stabilizer of an element $x \in X$ under the action of G is defined by

$$\mathrm{Stab}(x) = \{g \in G, g \cdot x = x\}$$

Given x, the stabilizer $\mathrm{Stab}(x)$ is the set of elements of G that leave x fixed. One may check that this is a subgroup of G. We have to check that if $g, h \in \mathrm{Stab}(x)$, then $gh^{-1} \in \mathrm{Stab}(x)$. Now

$$\left(gh^{-1}\right) \cdot x = g \cdot \left(h^{-1} \cdot x\right)$$

by definition of action. since $h \in \mathrm{Stab}(x)$, we have $h \cdot x = x$ or equivalently $x = h^{-1} \cdot x$, so that

$$g \cdot \left(h^{-1} \cdot x\right) = g \cdot x = x$$

which shows that $\mathrm{Stab}(x)$ is a subgroup of G

Examples 1.1.16:

1• The regular action (see the previous example) is transitive, and

for all $x \in X = G$, we have $\mathrm{Stab}(x) = \{1\}$, since x is invertible and we can multiply $g \cdot x = x$ by x^{-1}.

2• Let us consider the action by conjugation, which is again an action of G on itself ($X = G$) : $g \cdot x = gxg^{-1}$. The action has no reason to be transitive in general, and for all $x \in X = G$, the orbit of x is given by

$$B(x) = \left\{gxg^{-1}, g \in G\right\}$$

This is called the conjugacy class of x. Let us now consider the stabilizer of an element $x \in X$:

$$\mathrm{Stab}(x) = \left\{g \in G, gxg^{-1} = x\right\} = \{g \in G, gx = xg\}$$

which is the centralizer of x, that we denote by $C_G(x)$

Definition 1.1.20: (The commutator)

Let G be a group and $x, y \in G$. The commutator of x and y is the element

$$[x, y] = x^{-1}y^{-1}xy$$

Definition 1.1.21: (The derived subgroup)

Let G be a group. The derived subgroup (or commutator subgroup) G' of G is the subgroup generated by all commutators of elements from G :

$$G' = \langle [x, y] | x, y \in G \rangle$$

Example 1.1.17:

In an abelian group G $[x, y] = x^{-1}y^{-1}xy = 1$ for all x and y , so $G' = \mathbf{1}$. (The condition $G' = \mathbf{1}$ is equivalent to G being abelian.)

Definition 1.1.22:(the chain of subgroups)

The derived series $\left(G^{(i)}\right)$ (for $i \geqslant 0$) is the chain of subgroups of the group G defined by

$$G^{(0)} = G$$

and

$$G^{(i+1)} = \left(G^{(i)}\right)' \quad \text{for } i \geqslant 0$$

So $G^{(1)} = G', G^{(2)} = (G')' = G''$, etc. We then have a chain of subgroups

$$G = G^{(0)} \geqslant G^{(1)} \geqslant G^{(2)} \geqslant \cdots$$

Definition 1.1.23: (Solvable group)

A group G is soluble (solvable in the U.S.) if $G^{(d)} = 1$ for some d. The least such d is the derived length of G

$$G = G^{(0)} > G^{(1)} > G^{(2)} > \cdots > G^{(d)} = \mathbf{1}$$

Example1.1.18:

Any abelian group is soluble with derived length 1.

Definition 1.1.24: (The commutator subgroup)

Let A and B be subgroups of a group G. Define the commutator subgroup $[A, B]$ by

$$[A, B] = \langle [a, b] | a \in A, b \in B \rangle$$

the subgroup generated by all commutators $[a, b]$ with $a \in A$ and $b \in B$
In this notation, the derived series is given recursively by $G^{(i+1)} = \left[G^{(i)}, G^{(i)}\right]$ for all i

Definition 1.1.25: (the chain of subgroups)

The lower central series $(\gamma_i(G))$ (for $i \geqslant 1$) is the chain of subgroups of the group G defined by

$$\gamma_1(G) = G$$

and

$$\gamma_{i+1}(G) = [\gamma_i(G), G] \quad \text{for } i \geqslant 1$$

Definition 1.1.26: (the nilpotency class)

A group G is *nilpotent* if $\gamma_{c+1}(G) = 1$ for some c. The least such c is the nilpotency class of G
It is easy to see that $G^{(i)} \leqslant \gamma_{i+1}(G)$ for all i (by induction on i). Thus if G is nilpotent, then G is soluble. Note also that $\gamma_2(G) = G'$

Definition 1.1.27: (Exact sequence)

Let $F, G, H, I, \dots$ be groups, and let $f, g, h, \dots$ be group homomorphisms. Consider the following sequence:

$$\dots\dots \quad F \xrightarrow{f} G \xrightarrow{g} H \xrightarrow{h} I \quad \dots\dots$$

. We say that this sequence is exact in one point (say G) if $\operatorname{Im}(f) = \operatorname{Ker}(g)$. A sequence is exact if it is exact in all points. A short exact sequence of groups is of the form

$$1 \xrightarrow{i} F \xrightarrow{f} G \xrightarrow{g} H \xrightarrow{j} 1$$

where i is the inclusion and j is the constant map 1

Proposition 1.1.3:

Let

$$1 \xrightarrow{i} F \xrightarrow{f} G \xrightarrow{g} H \xrightarrow{j} 1$$

be a short exact sequence of groups. Then $\operatorname{Im}(f)$ is normal in G and we have a group isomorphism

$$G/\operatorname{Im}(f) \simeq H$$

or equivalently

$$G/\operatorname{Ker}(g) \simeq H$$

Proof. since the sequence is exact, we have that $\operatorname{Im}(f) = \operatorname{Ker}(g)$ thus $\operatorname{Im}(f)$ is a normal subgroup of G. By the first Isomorphism Theorem, we have that

$$G/\operatorname{Ker}(g) \simeq \operatorname{Im}(g) = H$$

since $\operatorname{Im}(g) = \operatorname{Ker}(j) = H$

Definition 1.1.28: (Generating set)

- Let G be a group and let $S \subset G$ be a subset. The subgroup generated by S in G is the smallest subgroup (with respect to inclusion) of G that contains S; the subgroup generated by S in G is denoted by $\langle S \rangle_G$ The set S generates G if $\langle S \rangle_G = G$

- A group is finitely generated if it contains a finite subset that generates the group in question.

Remark 1.1.1:(Explicit description of generated subgroups)

Let G be a group and let $S \subset G$. Then the subgroup generated by S in G always exists and can be described as follows:

$$\langle S \rangle_G = \bigcap \{H | H \subset G \text{ is a subgroup with } S \subset H\}$$
$$= \{s_1^{\varepsilon_1} \cdot \dots \cdot s_n^{\varepsilon_n} | n \in \mathbb{N}, s_1, \dots, s_n \in S, \varepsilon_1, \dots, \varepsilon_n \in \{-1, +1\}\}$$

Example 1.1.19: (Generating sets).

- If G is a group, then G is a generating set of G
- The trivial group is generated by the empty set.
- The set $\{1\}$ generates the additive group $\mathbb{Z}$; moreover, also, e.g., $\{2,3\}$ is a generating set for $\mathbb{Z}$
- Let X be a set. Then the symmetric group S_X is finitely generated if and only if X is finite.

1.2 Free groups

Definition 1.2.1: (Free groups, universal property).

Let S be a set. A group F is freely generated by S if F has the following universal property: For any group G and any map $\varphi : S \to G$ there is a unique group homomorphism $\bar{\varphi} : F \to G$ extending φ

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook] & G \\ F \arrow[ur, dashed, "\bar{\varphi}"'] & \end{tikzcd}$$

A group is free if it contains a free generating set.

Example 1.2.1: (Free groups)

- The additive group $\mathbb{Z}$ is freely generated by $\{1\}$. The additive group $\mathbb{Z}$ is not freely generated by $\{2,3\}$; in particular, not every generating set of a group contains a free generating set.
- The trivial group is freely generated by the empty set.
- Not every group is free; for example, the additive groups $\mathbb{Z}/2\mathbb{Z}$ and $\mathbb{Z}^2$ are not free

The term "universal property" obliges us to prove that objects having this universal property are unique in an appropriate sense; moreover, we will see below (the text Theorem) that for every set there indeed exists a group freely generated by the given set.

Proposition 1.2.1:(Free groups, uniqueness).

Let S be a set. Then, up to canonical isomorphism, there is at most one group freely generated by S.

Proof Proposition 1.2.1 :

Let F and F' be two groups freely generated by S. We denote the inclusion of S into F and F' by φ and φ' respectively.

1. Because F is freely generated by S, the existence part of the universal property of free generation guarantees the existence of a group homomorphism $\bar{\varphi}' : F \to F'$ such that $\bar{\varphi}' \circ \varphi = \varphi'$. Analogously, there is a group homomorphism $\bar{\varphi} : F' \to F$ satisfying $\bar{\varphi} \circ \varphi' = \varphi$

$$\begin{tikzcd} S \arrow[r, "\varphi'"] \arrow[d, hook, "\varphi"'] & F' \\ F \arrow[ur, dashed, "\overline{\varphi}'"'] & \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\varphi'"'] & F \\ F' \arrow[ur, dashed, "\overline{\varphi}"'] & \end{tikzcd}$$

2. We now show that $\bar{\varphi} \circ \bar{\varphi}' = \mathrm{id}_F$ and $\bar{\varphi}' \circ \bar{\varphi} = \mathrm{id}_{F'}$ and hence that φ and φ' are isomorphisms: The composition $\bar{\varphi} \circ \bar{\varphi}' : F \to F$ is a group homomorphism making the diagram

$$\begin{tikzcd} S \arrow[r, "\varphi"] \arrow[d, hook, "\varphi"'] & F \\ F \arrow[ur, dashed, "\overline{\varphi}\circ\overline{\varphi}'"'] & \end{tikzcd}$$

commutative. Moreover, also id_F is a group homomorphism fitting into this diagram. Because F is freely generated by S, the uniqueness part of the universal property thus tells us that these two homomorphisms have to coincide.

These isomorphisms are canonical in the following sense: They induce the identity map on S, and they are (by the uniqueness part of the universal property) the only isomorphisms between F and F' extending the identity on S

Theorem 1.2.1 : (Free groups, construction).

Let S be a set. Then there exists a group freely generated by S. (By the previous proposition, this group is unique up to isomorphism.

Proof Theorem 1.2.1 :

The idea is to construct a group consisting of "words" made up of elements of S and their "inverses" using only the obvious cancellation rules for elements of S and their "inverses." More precisely, we consider the alphabet

$$A := S \cup \bar{S}$$

where $\bar{S} := \{\bar{s} | s \in S\}$; i.e., $\bar{S}$ contains an element for every element in S, and $\bar{s}$ will play the rôle of the inverse of s in the group that we will construct.

- As first step, we define A^* to be the set of all (finite) sequences (" words") over the alphabet A; this includes in particular the empty word ε. On A^* we define a composition $A^* \times A^* \to A^*$ by concate nation of words. This composition is associative and ε is the neutral element.

- As second step we define
$$F(S) := A^* / \sim$$
where $\sim$ is the equivalence relation generated by
$$\begin{aligned} &\forall_{x,y \in A^*} \quad \forall_{s \in S} \quad xs\bar{s}y \sim xy \\ &\forall_{x,y \in A^*} \quad \forall_{s \in S} \quad x\bar{s}sy \sim xy \end{aligned}$$

i.e., $\sim$ is the smallest equivalence relation in $A^* \times A^*$ (with respect to inclusion) satisfying the above conditions. We denote the equivalence classes with respect to the equivalence relation $\sim$ by $[\cdot]$

It is not difficult to check that concatenation induces a well-defined composition $\cdot : F(S) \times F(S) \to F(S)$ via

$$[x] \cdot [y] = [xy]$$

for all $x, y \in A^*$ The set $F(S)$ together with the composition "." given by concatenation is a group: Clearly, $[\varepsilon]$ is a neutral element for this composition, and associativity of the composition is inherited from the associativity of the composition in A^*. For the existence of inverses we proceed as follows: Inductively (over the length of sequences), we define a map $I : A^* \to A^*$ by $I(\varepsilon) := \varepsilon$ and

$$I(sx) := I(x)\bar{s}$$
$$I(\bar{s}x) := I(x)s$$

for all $x \in A^*$ and all $s \in S$. An induction shows that $I(I(x)) = x$ and

$$[I(x)] \cdot [x] = [I(x)x] = [\varepsilon]$$

for all $x \in A^*$ (in the last step we use the definition of $\sim$). This shows that inverses exist in $F(S)$

The group $F(S)$ is freely generated by S : Let $i : S \to F(S)$ be the map given by sending a letter in $S \subset A^*$ to its equivalence class in $F(S)$; by construction, $F(S)$ is generated by the subset $i(S) \subset F(S)$. As we do not know yet that i is injective, we take a little detour and first show that $F(S)$ has the following property, similar to the universal property of groups freely generated by S : For every group G and every map $\varphi : S \to G$ there is a unique group homomorphism $\bar{\varphi} : F(S) \to G$ such that $\bar{\varphi} \circ i = \varphi$ Given φ, we construct a map

$$\varphi^* : A^* \to G$$

inductively by

$$\varepsilon \longmapsto e$$
$$sx \longmapsto \varphi(s) \cdot \varphi^*(x)$$
$$\bar{s}x \longmapsto (\varphi(s))^{-1} \cdot \varphi^*(x)$$

for all $s \in S$ and all $x \in A^*$. It is easy to see that this definition of φ^* is compatible with the equivalence relation $\sim$ on A^* (because it is compatible with the given generating set of $\sim$) and that $\varphi^*(xy) = \varphi^*(x) \cdot \varphi^*(y)$ for all $x, y \in A^*$; thus, φ^* induces a well-defined map

$$\begin{aligned} \bar{\varphi} : F(S) &\longrightarrow G \\ [x] &\longmapsto [\varphi^*(x)] \end{aligned}$$

which is a group homomorphism. By construction $\bar{\varphi} \circ i = \varphi$. Moreover, because $i(S)$ generates $F(S)$ there cannot be another such group homomorphism.

In order to show that $F(S)$ is freely generated by S, it remains to prove that i is injective (and then we ic
Let $s_1, s_2 \in S$. We consider the map $\varphi : S \to \mathbb{Z}$ given by $\varphi(s_1) := 1$ and $\varphi(s_2) := -1$. Then the corresponding homomorphism $\bar{\varphi} : F(S) \to G$ satisfies

$$\bar{\varphi}(i(s_1)) = \varphi(s_1) = 1 \neq -1 = \varphi(s_2) = \bar{\varphi}(i(s_2))$$

in particular, $i(s_1) \neq i(s_2)$. Hence, i is injective.

Definition 1.2.2 : (Free group F_n).

Let $n \in \mathbb{N}$ and let $S = \{x_1, \ldots, x_n\}$ where $x_1, \ldots, x_n$ are n distinct elements. Then we write F_n for "the" group freely generated by S, and call F_n the free group of rank n

Proposition 1.2.2 :

If F_1 is free on X_1 and F_2 is free on X_2 and if $|X_1| = |X_2|$, then

$$F_1 \simeq F_2$$

Proof Proposition 1.2.2 :

See [1] .

Corollary 1.2.1 :

Every group is a quotient of a free group.

Proof Corollary 1.2.1 ;

See [1] or [2] .

Theorem 1.2.2 : (NIELSEN-SCHREIER)

Subgroups of free groups are free.

Proof Theorem 1.2.2 :

see [2] .

2 CHAPTER TOW :Group growth

2.1 The word metric

A group G is finitely generated if there exists a finite subset $S \subseteq G$ such that any group element can be written as a product of elements in S. The generating subset S is said to symmetric if $S=S^{-1}$. If G is finitely generated by a subset $S \subseteq G$, then $S \cup S^{-1}$ is a symmetric and finite generating subset of G.

Definition 2.1.1 : (S-word-length).

Let G be a finitely generated group and let $S \subseteq G$ be a finite symmetric generating subset. For each fixed $g \in G$, we define the S-word-length as

$$\ell_S^G(g)=\min \left\{n \geqslant 0 \mid g=s_1 \cdots s_n, s_1, \ldots, s_n \in S\right\} \tag{2.1}$$

When there is no fear of ambiguity we shall omit the superscript.

Note that $\ell_S^G(g)=0$ if and only if $g=e_G$. In the following, let $S \subseteq G$ be a finite symmetric generating subset of a group G.

Proposition 2.1.1 :

For all $g, h \in G$, we have

$$\ell_S(g)=\ell_S\left(g^{-1}\right) \tag{2.2}$$

$$\ell_S(gh) \leqslant \ell_S(g)+\ell_S(h) \tag{2.3}$$

Proof Proposition 2.1.1 :

Take $g, h \in G$. If any of these is the ident ity element the statement is trivial, so suppose $g, h \neq e_G$. There exist natural numbers n and m such that $g=s_1 \cdots s_n$ and $h=t_1 \cdots t_m$, where $s_1, \ldots, s_n$ and $t_1, \ldots, t_m$ are elements in S. since $g^{-1}=s_n^{-1} \cdots s_1^{-1}$, we have $\ell_S\left(g^{-1}\right) \leqslant n=\ell_S(g)$. The converse follows analogously. Also, $gh=s_1 \cdots s_n t_1 \cdots t_m$, hence $\ell_S(gh) \leqslant n+m=\ell_S(g)+\ell_S(h)$

This construction induces a metric on the group G.

proposition 2.1.2 :

(the word metric) . the map $G \times G \rightarrow \mathbb{N}_0$ given by

$$d_S(g, h)=\ell_S\left(g^{-1} h\right), \quad g, h \in G \tag{2.4}$$

defines a metric on the group G.

Proof proposition 2.1.2 :

Let $g, h, k \in G$. Note that $d_S(g,h) = \ell_S(g^{-1}h) = 0$ if and only if $g^{-1}h = e_G$, i.e., $g = h$ Symmetry follows from **(2)**; namely,

$$d_S(g,h) = \ell_S\left(g^{-1}h\right) = \ell_S\left(h^{-1}g\right) = d_S(h,g) \tag{2.5}$$

By **(3)** we also have the triangle inequality:

$$d_S(g,h) = \ell_S\left(g^{-1}h\right) = \ell_S\left(g^{-1}kk^{-1}h\right) \leqslant \ell_S\left(g^{-1}k\right) + \ell_S\left(k^{-1}h\right) = d_S(g,k) + d_S(k,h) \tag{2.6}$$

This proves the claim.

The ball of radius $n \geqslant 0$ in G centered at $g_0 \in G$ is denoted by $BS\left(g_0, n\right) = \{g \in G \mid ds\left(g, g_0\right) \leqslant n\}$. If $g_0 = e_G$ we simply write $B_S^G(n) = \{g \in G \mid \ell_S(g) \leqslant n\}$. We shall omit the superscript when there is no ambiguity. Next, we show that the word metric is invariant under left multiplication.

Proposition 2.1.3 :

The action of G on itseif is isometric with respect to the word metric, i.e.,

$$d_S\left(g'g, g'h\right) = d_S(g,h) \tag{2.7}$$

for all $g', g, h \in G$

Proof Proposition 2.1.3 :

The following simple calculation,

$$d_S\left(g'g, g'h\right) = \ell_S\left(g^{-1}\left(g'\right)^{-1}g'h\right) = \ell_S\left(g^{-1}h\right) = d_S(g,h) \tag{2.8}$$

gives the result.

It is possible to define another metric on the finitely generated group where the distance between elements is interpreted in terms of lengths of paths. In order to do this we need the not ion of a Caule u ora ph.

2.2 Growth functions

In this section we shall consider the notion of a growth function. We shall first consider the notion of growth functions in general.

A growth function is a non-decreasing map $\gamma : \mathbb{N}_0 \to [0, \infty)$. If γ and γ' are two growth functions we say that γ' dominates γ if there exists a positive integer c such that

$$\gamma(n) \leqslant c\gamma'(cn), \quad n \geqslant 1 \tag{2.9}$$

in which case we write $\gamma \preceq \gamma'$. Note that the statement is trivially true for $n = 0$. If, in addition, $\gamma' \preceq \gamma$, then we write $\gamma \sim \gamma'$ and say that the two growth functions are equivalent.

Proposition 2.2.1 :

The relation $\sim$ is an equivalence relation.

Proof Proposition 2.2.1:

Reflexivity and symmetry are obvious properties of $\sim$ from the definition. In order to show transitivity let $\gamma_1, \gamma_2, \gamma_3 : \mathbb{N}_0 \to [0, \infty)$ be three growth functions such that $\gamma_1 \sim \gamma_2$ and $\gamma_2 \sim \gamma_3$. In particular, there exist positive integers c_1 and c_2 such that $\gamma_1(n) \leqslant c_1\gamma_2(c_1 n)$ and $\gamma_2(n) \leqslant c_2\gamma_3(c_2 n)$. Put $c = c_1c_2$ and observe that

$$\gamma_1(n) \leqslant c_1\gamma_2(c_1 n) \leqslant c_1c_2\gamma_3(c_1c_2n) = c\gamma_3(cn), \quad n \geqslant 1 \tag{2.10}$$

i.e., $\gamma_1 \preceq \gamma_3$ and the relation $\preceq$ is transitive. The converse follows analogously.

We shall denote the equivalence class under $\sim$ of a growth function γ by $[\gamma]$. Given two growth functions γ_1 and γ_2 we write $[\gamma_1] \preceq [\gamma_2]$ if γ_2 dominates γ_1. To see that this is welldefined let $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$ be growth functions such that $\gamma_1 \sim \gamma_1', \gamma_2 \sim \gamma_2'$ and $\gamma_1 \preceq \gamma_2$. In particular, we have $\gamma_1' \preceq \gamma_1, \gamma_1 \preceq \gamma_2$ and $\gamma_2 \preceq \gamma_2'$. By virtue of the transitivity of $\preceq$ we obtain $\gamma_1' \preceq \gamma_2'$. Note that $\preceq$ now defines a partial order relation on the set of equivalence classes of growth functions. Reflexivity and anti-symmetry are obvious from the definition while transitivity was proved above.

We can define a product on the set of equivalence classes of growth functions as follows. Given two growth functions γ_1 and γ_2 set $[\gamma_1] \cdot [\gamma_2] = [\gamma_1\gamma_2]$. Let us first clarify why this is well-defined: Let $\gamma_1, \gamma_2, \gamma_1', \gamma_2' : \mathbb{N}_0 \to [0, \infty)$ be growth functions such that $\gamma_1 \sim \gamma_1'$ and $\gamma_2 \sim \gamma_2'$ Pick positive integers c_1 and c_2 such that $\gamma_1(n) \leqslant c_1\gamma_1'(c_1n)$ and $\gamma_2 \leqslant c_2\gamma_2'(c_2n)$, for all $n \geqslant 1$ and set $c = c_1c_2$. The product of growth functions is itself a growth function and

$$(\gamma_1\gamma_2)(n) = \gamma_1(n)\gamma_2(n) \leqslant c_1c_2\gamma_1'(c_1c_2n)\gamma_2'(c_1c_2n) = c(\gamma_1'\gamma_2')(cn) \tag{2.11}$$

for all $n \geqslant 1$. The converse follows analogously, so the above product is well-defined. Thus $\gamma_1\gamma_2 \sim \gamma_1'\gamma_2'$

Example 2.2.1 :

(i) Let $a, b \in [0, \infty)$. For all $n \geqslant 1$, we have $n^a \leqslant n^b$ (resp. $, n^a = b^b$) if and only if $a \leqslant b$ (resp., $a = b$). Thus $n^a \preceq n^b$ (resp., $n^a \sim n^b$) if and only if $a \leqslant b$ (resp.,$a = b$)

(ii) Let the growth function $\gamma : \mathbb{N}_0 \to [0, \infty)$ be a polynomial of degree $d \geqslant 0$. Then $\gamma \sim n^d$

(iii) Let $a, b \in (1, \infty)$ such that $a \leqslant b$. Obviously, $a^n \leqslant b^n$ for all $n \geqslant 1$, so $a^n \preceq b^n$. On the other hand, put $c = \lfloor \log_a b \rfloor + 1$ where $\lfloor \cdot \rfloor$ denotes the integer part. Note that $c > 1$. For all $n \geqslant 1$

$$b^n = a^{(\log_a b)n} \leqslant a^{(\lfloor \log_a b \rfloor + 1)n} \leqslant c a^{cn} \tag{2.12}$$

which shows that $b^n \preceq a^n$, hence $a^n \sim b^n$. In particular, we have $a^n \sim \exp(n)$, for all$a \in (1, \infty)$

(iv) Let $d \geqslant 0$ be an integer. Then $n^d \preceq \exp(n)$ but $n^d \nsim \exp(n)$. Note $\lim_{n \to \infty} \frac{n^d}{\exp(n)} = 0$ so the sequence $\left(\frac{n^d}{\cos(n)}\right)_{n \in N}$ is bounded. This ensures the existence of a positive integer c such that

$$\frac{n^d}{\exp(n)} \leqslant c \tag{2.13}$$

from which it follows that $n^d \leqslant c \exp(cn)$; i.e., $n^d \preceq \exp(n)$. In order to reach a contradiction, assume that $\exp(n) \preceq n^d$. Then there exists a positive integer c such that $exp(n) \leqslant c^{d+1} n^d$, i.e.,

$$\frac{\exp(n)}{n^d} \leqslant c^{d+1} \tag{2.14}$$

for all $n \geqslant 1$. This is in contradiction with the fact that the sequence is not bounded. Thus $n^d \nsim \exp(n)$ We are now in position to turn our attention toward the growth function of finit ely generated groups.

Definition 2.2.1 : (The growth function)

Let G be a group and let $S \subseteq G$ be a finite symmetric generating subset. The growth function of G relative to S is the map $\gamma_S^G : \mathbb{N}_0 \to \mathbb{N}$ given by

$$\gamma_S^G(n) := \left|B_S^G(n)\right| = \left|\{g \in G \mid \ell_S(g) \leqslant n\}\right|, \quad n \geqslant 0 \tag{2.15}$$

When there is no fear of ambiguity twe shall omit the superscript. Observe that $\gamma s(0) = |\{e_G\}| = 1$ and $\gamma_S(n) \leqslant \gamma_S(n+1)$, for all $n \geqslant 0$; thus, γ_S^G is indeed a growth function. It follows from the construction that the map $\varphi : (S \cup \{e_G\})^n \to Bs(n)$ given by $\varphi(s_1, \ldots, s_n) = s_1 \cdots s_n$ with $s_i \in S \cup \{e_G\}$ is surjective (however, not injective). Hence γ_S sat isfies

$$\gamma s(n) \leqslant |S \cup \{e_G\}|^n, \quad n \geqslant 0 \tag{2.16}$$

In particular, γ_S^G takes only finite values.

Lemma 2.2.1 :

Let G be a finitely generated group . Let S and S' be two finite symmetric generating subsets of G and put $c = \max\{\ell_{S'}(s) \mid s \in S\}$. Then

(i) $\ell_{S'}(g) \leqslant c\ell_S(g), \quad g \in G$

(ii) $d_{S'}(g, h) \leqslant c d_S(g, h), \quad g, h \in G$

(iii) $B_S(n) \subseteq B_{S'}(cn), \quad n \geqslant 0$

Proof Lemma 2.2.1 :

(i) Fix $g \in G$ and suppose $\ell_S(g) = n$. Then there exist $s_1, \ldots, s_n \in S$ such that$g = s_1 \cdots s_n$ and

$$\ell_{S'}(g) = \ell_{S'}(s_1 \cdots s_n) \leqslant \sum_{i=1}^{n} \ell_{S'}(s_i) \leqslant cn \tag{2.17}$$

wherein the first inequality follows from **(3)**.

(ii) Fix $g, h \in G$. By (i), we have

$$d_{S'}(g,h) = \ell_{S'}\left(g^{-1}h\right) \leqslant c\ell_S\left(g^{-1}h\right) = cd_S(g,h) \tag{2.18}$$

(iii) To show the indusion, let $g \in B_S(n)$ for some $n \geqslant 0$; i.e., $\ell_S(g) \leqslant n$. By (i) this implies that $\ell_{S'}(g) \leqslant cn$, which is equivalent to $g \in B_{S'}(cn)$.

We say that a finitely generated group G has exponential (resp., subexponential) growth if $\gamma(G) \sim \exp(n)$ (resp., $\gamma(G) \nsim \exp(n)$); it has polynomial growth if there is an integer $d \geqslant 0$ such that $\gamma(G) \preceq n^d$. If the growth type is neither exponential nor polynomial we say that G has intermediate growth. Also, we say that a group has regular growth if it has polynomial or exponent ial growth. We state the result of the previous Example (iv) and formula **(16)** as a proposition.

Proposition 2.2.2 :

Let G be a finitely generated group and let S and S' be two finite symmetric generating subsets of G. Then

(i) The two word metrics d_S and $d_{S'}$ induce the same topology,

(ii) The growth functions γ_S and $\gamma_{S'}$ are equivalent,

Proof Proposition 2.2.2 :

(i) Put $c := \max\{\ell_{S'}(s) \mid s \in S\}$ and $c' := \max\{\ell_S(s') \mid s' \in S'\}$ and take $g, h \in G$. If $c = 0$ or $c' = 0$ there is nothing to prove, so suppose $c, c' \neq 0$. By the previous Lemma (ii)we have

$$\frac{1}{c} d_s(g,h) \leqslant d_{S'}(g,h) \leqslant cd_S(g,h) \tag{2.19}$$

which shows that the two metrics induce the same topology.

(ii) An immediate consequence of the previous Lemma(iii)is

$$\gamma s(n) = |B_S(n)| \leqslant |B_{S'}(cn)| = \gamma_{S'}(cn) \leqslant c\gamma_{S'}(cn), \quad n \geqslant 1 \tag{2.20}$$

Thus $\gamma s \preceq \gamma s'$. The converse follows analogously by using $c' = \max\{\ell_S(s') \mid s' \in S'\}$

Definition 2.2.2 :(Growth type)

Let $S \subseteq G$ be a finite symmetric generating subset of a group G. The equivalence class $[\gamma_S]$ associated to the growth function of G relative to S is called the growth type of G and we write $\gamma(G)$

Note that the growth type is independent of a generating subset. By abuse of notation, we shall some times write $\gamma(G) \sim \exp(n)$ or $\gamma(G) \sim n^d$ for some integer d, if the group G has exponential or polynomial growth, respectively.

Example 2.2.2 :

(i) Consider the group $\mathbb{Z}$ and let $S=\{-1,1\}$ be the generating subset. The ball of radius n is

$$B_S^Z(n)=\left\{g\in\mathbb{Z}\mid \ell_S^Z(g)\leqslant n\right\}=\{-n,-(n-1),\ldots,-1,0,1,\ldots,n-1,n\}$$

so $\gamma_S^Z(n)=2n+1$. It follows that $\gamma(\mathbb{Z})\sim n$; in particular, $\mathbb{Z}$ has polynomial growth.

(ii) Consider the freegroup on two generators $\mathscr{F}_2$ and let $S=\{a,a^{-1},b,b^{-1}\}$ be the generating subset. For each increase element in the ball of radius $n-1$, three distinct words are included in the ball of radius n (when $n>1$). Thus,

$$\gamma_S^{\mathscr{F}_2}(n)=1+4\sum_{i=0}^{n-1}3^i=23^n-1$$

This shows that $\gamma\left(\mathscr{F}_2\right)\sim 3^n$. In particular, $\mathscr{F}_2$ has exponential growth.
The next result states that all finite groups have the same growth type.

Proposition 2.2.3 :

Let G be a finitely generated group. G is finite if and only if $\gamma(G)\sim 1$.

Proof Proposition 2.2.3 :

Let $S\subseteq G$ be a finite symmetric generating subset. Suppose G is finite. As $\gamma s(0)=1$ we have $1(n)\leqslant\gamma_S(n)$, for all $n\geqslant 0$; i.e., $1\preceq\gamma_S$. Note that $\gamma_S(n)\leqslant|G|=|G|1(|G|n)$, for all $n\geqslant 0$. That is, $\gamma_S\preceq 1$ Conversely, suppose that $\gamma(G)\sim 1$. In particular, $\gamma_S(G)\preceq 1$, so there exists a positive integer $c\geqslant 1$ such that $\gamma_S(n)\leqslant c1(cn)=c$. It follows that $|G|\leqslant c$

Proposition 2.2.4 :

if G is an infinite finitely generated group, then $n\preceq\gamma(G)$.

Proof Proposition 2.2.4 :

Choose a finite symmetric generating subset $S\subseteq G$ and consider the inclusions

$$\{e_G\}=B_S(0)\subseteq B_S(1)\subseteq\cdots B_S(n)\subseteq B_S(n+1)\subseteq\cdots \tag{2.21}$$

We widn to show that these inclusions are in fact all strict. In order to do this, we shall first show that if $Bs(n)=B_S(n+1)$ for some $n\geqslant 0$, then $Bs(m)=Bs(n)$ for all $m\geqslant n$. We proceed by induction on m. The start is trivial.

Suppose $Bs(n)=Bs(n+1)$ implies $Bs(n)=B_S(n+1)=\cdots=B_S(m-1)=B_S(m)$ for some $m>n$. We are to show that $B_S(m+1)=B_S(n)$. It is clear that $B_S(n)=B_S(m)\subseteq B_S(m+1)$. Conversely, take $g\in B_S(m+1)$; there exist clements $g'\in B_S(m)$ and $s\in S$ such that $g=g's$. By hypothesis $g'\in B_S(m-1)$, so

$$g\in B_S(m-1)S\subseteq B_S(m) \tag{2.22}$$

This shows that $B_S(m+1)\subseteq B_S(m)$, hence $B_S(m+1)=B_S(m)=B_S(n)$ Now, if it were the case that $B_S(n)=B_S(n+1)$ for some $n\geqslant 0$, then $B_S(m)=B_S(n)$ for all $m\geqslant n$ and consequently $G=Bs(n)$, which is untenable since G is assumed infinite. Thus all the inclusions must be strict and we obtain

$$n\leqslant|B_S(n)|=\gamma_S(n) \tag{2.23}$$

for all $n\geqslant 0$; hence $n\preceq\gamma s$. This shows that $n\preceq\gamma(G)$

2.3 The growth rate, growth of subgroups and quotients

Lemma 2.3.1 :

Let $(a_n)_{n\geqslant 1}$ be a sequence of positive real numbers satisfying $a_{n+m} \leqslant a_n a_m$ for all natural numbers m and n. Then

$$\lim_{n\to\infty} a_n^{1/n} = \inf_{n\geqslant 1} a_n^{1/n} \tag{2.24}$$

Proposition 2.3.1 :

Let G be finitely generated group and let $S \subseteq G$ be finite symmetric generating subset. Then the number defined by

$$\lambda_S^G = \lim_{n\to\infty} \gamma_S(n)^{1/n} \tag{2.25}$$

is finite. In particular, it follows that $\lambda_S \in [1,\infty)$. This proposition renders the notion of a growth rate well-defined.

Definition 2.3.1 :

We call the number $\lambda_S^G \in [1,\infty)$, the growth rate of G relative to S. We may onit the stperscript when there is no ambiguity.

Proposition 2.3.2 :

Let G be a finitely generated group and let $N \subseteq G$ be a normal subgroup. Then the quotient G/N is finitely generated and $\gamma(G/N) \preceq \gamma(G)$. If, in addition, N is finite, then $\gamma(G/N) = \gamma(G)$

Proof Proposition 2.3.3 :

Let $S \subseteq G$ be a finite symmetric generating subset and let $\pi : G \to G/N$ denote the canonical epimorphism. Then $S' := \pi(S) \subseteq G/N$ is a finite and symmetric subset. Let $h \in G/N$. By surjectivity there exists $g \in G$ such that $\pi(g) = h$. If we write $g = s_1 \ldots s_n$, where $s_1, \ldots, s_n \in S$, then $h = \pi(s_1 \cdots s_n) = \pi(s_1) \cdots \pi(s_n) \in S'$; thus S' generates G/N. Using surjectivity again we see that $.B_{S'}^{G/N}(n) = \pi(B_S^G(n))$, for all $n \geqslant 1$. Thus $\mid B_S^G(n)| \leqslant |B_{S'}^{G/N}(n) \mid$ or equivalently, $\gamma_S^{G/N}(n) \leqslant \gamma_S^G(n)$ for all $n \geqslant 1$; i.e., $\gamma(G/N) \preceq \gamma(G)$

Suppose, in addition, that $|N|$ is finite. since $B_{S'}^{G/N}(n) = \pi\left(B_S^G(n)\right)$ we observe that $B\}(n) \subseteq \pi^{-1}\left(B_{S'}^{G/N}(n)\right)$ and

$$\left|B_S^G(n)\right| \leqslant \left|\pi^{-1}\left(B_{S'}^{G/N}(n)\right)\right| = \left|N\|B_{S'}^{G/N}(n)\right| \leqslant |N|\left|B_{S'}^{G/N}(|N|n)\right| = |N|\gamma_{S'}^{G/N}(|N|n)$$

This shows that $\gamma_S \leq \gamma_{S'}^{G/N}$; consequently $\gamma(G/N) = \gamma(G)$

Proposition 2.3.4:

Let H be a finitely generated subgroup of a finitely generated group p. Then $\gamma(\boldsymbol{H}) \preceq \gamma(\boldsymbol{G})$ Proef. Let S_G and S_H be finite symmetric generating subset of G and H respectively and put $S = S_H \cup S_G$. Note that S is a finite symmetric generating subset of G. For all $n \geqslant 1$, we have $B_{S_H}^H(n) \subseteq B_S^G(n)$ since $S_H \subseteq S$ and therefore $\left|B_{S_N}^H(n)\right| \leqslant \left|B_S^G(n)\right|$, i.e., $\gamma_{S_N}(n) \leqslant \gamma_S(n)$, for all $n \geqslant 1$. Thus $\gamma_{S_H} \preceq \gamma_S$ and $\gamma(H) \preceq \gamma(G)$

2.4 The growth of finitely generated nilpotent groups

Theorem 2.4.1 : (Dixmier, Milnor, Wolf 1960 s)

Finitely generated nilpotent group has polynomial growth.

Theorem 2.4.2 : (Gromov 1980)

Any finitely generated almost nilpotent group has polynomial growth. The converse implication is, in fact, also true, whence a group has polynomial growth if and only if it is almost nilpotent. This is a remarkable result due to Gromov, see [9] The proof requires several steps, the proof is much too big. We refer the reader to [9] .

3 CHAPTE THREE: Amenability and group growth

3.1 Amenability

Definition 3.1.1 :(Aminability via measure)

A group G is amenable, if it is possible to define on it
a non-trivial finite, finitely additive, translation-invariant measure.
That means that we can find a function $\mu : 2^G \to \mathbb{R} \geq 0$, which associates to each subset A of G a number $\mu(A) \geq 0$, and which satisfies:

(1) If $A, B \subseteq G$ and $A \cap B = \emptyset$, then $\mu(A \cup B) = \mu(A) + \mu(B)$

(2) If $A \subseteq G$ and $x \in G$, then $\mu(Ax) = \mu(A)$

(3) $\mu(G) > 0$

The difference between this notion and the more customary notions of measure, such as Lebesgue or Haar measures, is that, first, we require μ to be defined for all subsets of G, and, on the other hand, we require additivity only for finite unions, not countable ones. From here on, whenever we say "measure", we usually mean one which satisfies properties $(1)-(3)$ above.
Obviously multiplying μ by any positive constant yields another measure with the same properties, hence we will always assume that $\mu(G) = 1$. We required our measure to be invariant under right translations, but we can find also a left-invariant one, by defining $\nu(A) = \mu\left(A^{-1}\right)$

(Aminabilty via means) viewed as a generalised averaging operation for bounded functions. If X is a set, then $\ell^\infty(X, \mathbb{R})$ denotes the set of all bounded functions of type $X \to \mathbb{R}$. Pointwise addition and scalar multiplication turn $\ell^\infty(X, \mathbb{R})$ into a real vector space. If G is a group, then every left G -action on X induces a left G -action on $\ell^\infty(X, \mathbb{R})$ via

$$\begin{aligned} G \times \ell^\infty(X, \mathbb{R}) &\longrightarrow \ell^\infty(X, \mathbb{R}) \\ (g, f) &\longmapsto \left(x \mapsto f\left(g^{-1} \cdot x\right)\right) \end{aligned}$$

A group G is amenable if there exists a G -invariant mean on $\ell^\infty(G, \mathbb{R})$, i.e., an $\mathbb{R}$ -linear map $m : \ell^\infty(G, \mathbb{R}) \longrightarrow \mathbb{R}$ with the following properties:

- Normalisation. We have $m(1) = 1$
- Positivity. We have $m(f) \geq 0$ for all $f \in \ell^\infty(G, \mathbb{R})$ that satisfy $f \geq 0$ pointwise.
- Left-invariance. For all $g \in G$ and all $f \in \ell^\infty(G, \mathbb{R})$ we have

$$m(g \cdot f) = m(f)$$

with respect to the left G -action on $\ell^\infty(G, \mathbb{R})$ induced from the left translation action of G on G

Example 3.1.1 : (Amenability of finite groups)

Finite groups are amenable: If G is a finite group, then

$$\begin{aligned} \ell^\infty(G, \mathbb{R}) &\to \mathbb{R} \\ f &\to \frac{1}{|G|} \cdot \sum_{g \in G} f(g) \end{aligned}$$

is a G -invariant mean on $\ell^{\infty}(G, \mathbb{R})$. Or by measure $\mu(A)=|A|/|G|$ is a measure, and clearly it is the only possible one.

(Non-amenability of free groups). The free group F_2 of rank 2 is not amenable.

Let $F_2=\langle x, y\rangle$ be a free group of rank 2, with free generators x and y. Assume that μ is a (left-invariant) measure on F_2. Then $\mu\left(F_2-\{1\}\right)=1$, and $F_2-\{1\}=A \cup B \cup C \cup D$, where A, B, C, D are the sets of reduced words starting by x, x^{-1}, y, y^{-1}, respectively. Then $A-\{x\}=xA \cup xC \cup xD$, a disjoint union. since $\mu(A)=\mu(xA)$, it follows that $\mu(C)=\mu(D)=0$. A similar argument shows that $\mu(A)=\mu(B)=0$ and thus $\mu\left(F_2\right)=0$, a contradiction. Thus F_2 is not amenable.

(Amenability of Abelian groups). Every Abelian group is amenable.

The proof relies on the Markov-Kakutani fixed point theorem from functional analysis [7]

On any space with a measure we can develop the notion of the integral defined by that measure. This is particularly simple in the case of an amenable group G, because all subsets of G are measurable, and thus all functions are measurable. Let $f(x)$ be a bounded real function on G and let $a \leq f(x)<b$ (for all $x \in G$). Divide the interval $[a, b]$ in some way to subintervals, i.e. choose points $a=a_0<a_1<\cdots<a_k=b$, let Δ denote this subdivision, define $A_i=\left\{x \in G \mid a_{i-1} \leq f(x)<a_i\right\}$ The sets A_i constitute a partition of G. Put $S_{\Delta}=\sum_{i=1}^k \mu\left(A_i\right) a_i$. Then $S_{\Delta} \geq a \sum_{i=1}^k \mu\left(A_i\right) \geq a$. Similarly, if we put $s_{\Delta}=\sum_{i=1}^k \mu\left(A_i\right) a_{i-1}$, then $s_{\Delta} \leq b$. If E is a refinement of Δ, then $S_E \leq S_{\Delta}$ and $s_E \geq s_{\Delta}$. Any two divisions have a common refinement, therefore for any two divisions we have $s_E \leq S_{\Delta}$. Also $0 \leq S_{\Delta}-s_{\Delta} \leq \max \left(a_i-a_{i-1}\right)$. It follows that the infimum of the numbers S_{Δ}, taken over all subdivisions, equals the supremum of s_{Δ}. This common value is defined to be the integral of f relative to μ, denoted as usual by $\int f d\mu$

It is clear that the integral is additive and right-invariant, in the sense that given any bounded function and an element $y \in G$, if we define $f_y(x)=f(xy)$, then $\int f_y d\mu=\int f d\mu$ As a first application of integration we have:

Proposition 3.1.1 :

If G is amenable, there exists on it a measure that is both right- and left-invariant.

Proof

Let μ be a measure on G, and define a new one by $\nu(A)=\int \mu(xA) d\mu$. The verification that this is a two-sided invariant measure is immediate.

Remark The class of amenable groups is denoted by AG.

Definition 3.1.2 :

Let $\mathcal{P}$ be any property of groups. A group G is locally $\mathcal{P}$ if each finitely generated subgroup of G has property $\mathcal{P}$.

Theorem 3.1.1 :(Inheritance properties of amenable groups).

(i) Subgroups and factor groups of amenable groups are amenable.

(ii) An extension of an amenable group by an amenable group is amenable.

(iii) A locally amenable group is amenable.

Proof

(i) Let G be amenable and $H \leq G$ and $N \diamond G$. A subset A of G/N is a collection $x_\alpha N$ of cosets of N, and we put $\mu(A) = \mu\left(\bigcup x_\alpha N\right)$. Next, let R be a set of representatives of the left cosets of H in G, and for $A \subseteq H$ put $\mu(A) = \mu(RA)$. This defines measures on G/N and H

(ii) Let ν be a measure on N, and let σ be a left-invariant measure on G/N. For a coset $Nx \in G/N$ define $f(Nx) = \nu(N \cap Ax)$. This does not depend on the choice of the representative x, because if $y \in N$, then $\nu(N \cap Axy) = \nu(N \cap Ax)y = \nu(N \cap Ax)$, and thus f is a function on G/N For $A \subseteq G$, put $\mu(A) = \int f d\sigma$. This is an additive function. Replacing A by $Az(z \in G)$ we change $f(Nx)$ to $f(Nzx) = f(zNx)$, implying $\mu(Az) = \mu(A)$ by the left invariance of the integral with respect to ν

(iii) We first consider all functions from 2^G to the closed interval [0,1] This can be seen as the cartesian power $[0,1]^{2^G}$, and with the product topology it is a compact space. The set of all finitely additive invariant measures μ, satisfying $\mu(G) = 1$, is defined by various equalities between values of μ, and therefore it is a closed set (possibly empty). Let H be a finitely generated subgroup of G, let μ be a measure on H, and extend it to G by setting $\mu(A) = \mu(A \cap H)$. The resulting measure on G is finitely additive and gives G measure 1, but it is invariant only with respect to multiplication by elements of H. The set $\mathcal{M}_H$ of all measures on G which are finitely additive and H -invariant is also closed, and we have just seen that it is not empty. Taking several finitely generated subgroups $H_1, \ldots, H_r$, and putting $K = \langle H_1, \ldots, H_r \rangle$, the sets M_{H_i} intersect in M_K, and therefore the intersection is not empty. since the sets M_H are closed in a compact space, the intersection of all of them is not empty, and any function in that intersection is an invariant measure on G

Definition 3.1.3 : (Elementary amenable).

The class of elementary amenable groups is the smallest class of groups that contains the finite and the abelian groups and is stable under the four operations of taking (1) subgroups, (II) quotients, (III) extensions and (IV) direct unions. The class is denoted by EG or $\mathcal{E}$.

In the previous, we saw that the class of amenable groups is stable under the four operations (I)-(IV). Since finite and abelian groups are amenable it is clear that EG is contained in AG

Lemma 3.1.1 :

Let G be a group and let $N \subseteq G$ be a normal subgroup. If both N and G/N are amenable, then G is amenable.

Proposition 3.1.2 :

All solvable groups are amenable.

Proof.

Let G be a solvable group. We shall proceed by induction on the solvability degree i. If $i = 0$, then $G = \{e_G\}$ and there is nothing to prove. Suppose G has solvability degree $i + 1$ The derived subgroup $D(G)$ is solvable of degree i hence amenable by hypothesis. Also, the quotient $G/D(G)$ is abelian, hence amenable, we conclude that G is amenable.

Every nilpotent group is solvable so nilpotent groups are amenable.

3.2 Further caracteristions of amenabiliy

The notion of amenability revolves around the (almost) invariance. We have seen the definition via invariant means. In the following, we will study equivalent characterisations of amenability and their use cases, focusing on geometric properties:

- almost invariant subsets (Følner sequences),
- paradoxical decompositions and the Banach-Tarski paradox,

Another geometric characterisation of amenability is based on:

Definition 3.2.1 :

Let the group G be generated by the set S, and let $A \subseteq G$. The boundary ∂A of A is the set of elements at distance 1 from A, i.e. the elements x such that $x \notin A$, but there exists an element $y \in A$ and a generator $s \in S$ such that $x = ys$ or $x = ys^{-1}$

Definition 3.2.2 :

A finitely generated group G satisfies the Følner condition, , if $\inf |\partial X|/|X| = 0$, the infimum being taken over all finite subsets X of G. Equivalently, there exists in G a Følner sequence , i.e. a sequence F_n of finite subsets of G such that $\lim_{n\to\infty} \frac{|\partial F_n|}{|F_n|} = 0$

Definition 3.2.3 :

Let S be any set. A filter on S is a family $\mathcal{F}$ of subsets of S with the following properties:

(i) $\phi \notin \mathcal{F}$(ϕ is the empty set).

(ii) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$

(iii) $A \in \mathcal{F}, A \subseteq B \Rightarrow B \in \mathcal{F}$

A maximal filter is called an ultrafilter. We now fix a non-principal ultrafilter $\mathcal{F}$ on $\mathbb{N}$, the set of natural numbers. Let T be any topological space, and let $\{x_n\}$ be a sequence in T For each $x \in T$, and each neighbourhood U of x, write $O(x, U) = \{n \in \mathbb{N} \mid x_n \in U\}$

Definition 3.2.4 :

We say x is the $\mathcal{F}$ -limit of $\{x_n\}$, if for each U the subset $O(x, U)$ belongs to $\mathcal{F}$. We write $x = \mathcal{F} \lim x_n$

Thus x is the $\mathcal{F}$ -limit of some sequence, if each neighbourhood of x contains almost all members of that sequence.

Corollary 3.2.1 :

Any bounded sequence of real numbers $\mathcal{F}$-converges, and its $\mathcal{F}$ -limit is unique. **Corollary 3.2.2 :** Let x_n and y_n be two bounded real sequences, and c a real number.

(a) $\mathcal{F} \lim (x_n + y_n) = \mathcal{F} \lim x_n + \mathcal{F} \lim y_n$

(b) $\mathcal{F} \lim (cx_n) = c\mathcal{F} \lim x_n$

(c) If $x_n \leq y_n$ for all n, then $\mathcal{F}\lim x_n \leq \mathcal{F}\lim y_n$

Theorem 3.2.1 :

A group satisfying the Følner condition is amenable.

Proof Theorem 3.2.1 :

Let X_n be a Følner sequence in G, fix a nonprincipal ultrafilter $\mathcal{F}$, and for each subset A of G define $A_n = A \cap X_n$ $\mu_n(A) = |A_n| / |X_n|$, and $\mu(A) = \mathcal{F}\lim \mu_n(A)$. The additivity of μ is clear. Let $s \in S$ be one of the generators of G. Then $|As \cap X_n| = |A \cap X_n s^{-1}|$ and $A \cap X_n s^{-1} \subseteq A_n \cup \partial X_n$, and thus $|As \cap X_n| / |X_n| \leq \mu_n(A) + |\partial X_n| / |X_n|$, yielding, upon passage to the limit, $\mu(As) \leq \mu(A)$ In the same way we obtain $\mu(A) = \mu\left(As \cdot s^{-1}\right) \leq \mu(As)$, and thus $\mu(As) = \mu(A)$

The converse of the theorem is also true; thus a group is amenable iff it satisfies the Følner conditions.

Definition 3.2.5 : (Paradoxical group).

A group G is paradoxical if it admits a paradoxical decomposition. A paradoxical decomposition of G is a pair $\left((A_g)_{g\in K}, (B_h)_{h\in L}\right)$ where $K, L \subset G$ are finite and $(A_g)_{g\in K}, (B_h)_{h\in L}$ are families of subsets of G with the property that

$$G = \left(\bigcup_{g\in K} A_g\right) \cup \left(\bigcup_{h\in L} B_h\right), \quad G = \bigcup_{g\in K} g \cdot A_g, \quad G = \bigcup_{h\in L} h \cdot B_h$$

are disjoint unions.

Example 3.2.1 :(Non-Abelian free groups are paradoxical).

Free groups of rank at least 2 are paradoxical.

Proof Example 3.2.1 :

We use the description of free groups in terms of reduced words. In order to keep notation simple, we consider the case of rank 2 (higher ranks basically work in the same way). Let F be a free group of rank 2, freely generated by $\{a, b\}$. We then define the following subsets of F.

1. Let A^+ be the set of all reduced words starting with a positive power of a
2. Let A^- be the set of all reduced words starting with a negative power of a
3. Let B^+ be the set containing the neutral element, all powers of b as well as all reduced words starting with a positive power of b.
4. Let B^- be the set of all reduced words starting with a negative power of b, excluding the powers of b

Then
$F_2 = A^+ \cup A^-\ B^-$, $F_2 = A^- \cup a^{-1} \cdot A^+$ $F_2 = B^- \cup b^{-1} \cdot B^+$ are disjoint unions. So $\left((A_e, A_a - 1)_{\{e,a-1\}}\right\}, (B_e, B_b - 1)\{e, b-1\}\Big)$ is a paradoxical decomposition of F, where $A_e := A^-, A_{a^{-1}} := A^+, B_e := B^-$ and $B_{h-1} := B^+$

Theorem 3.2.2 : (Tarski's theorem).

Let G be a group. Then G is paradoxical if and only if G is not amenable.

Proof Theorem 3.2.2 :

Let G be paradoxical and let $\left((A_g)_{g\in K}, (B_h)_{h\in L}\right)$ be a paradoxical decomposition of G. Assume for a contradiction that G is amenable, and let m be an invariant mean for G. Because the corresponding unions all are disjoint and m is left-invariant. Denoting characteristic functions of subsets with $\chi \ldots$, we obtain

$$1 = m(\chi_G) = \sum_{g\in K} m(\chi_g \cdot A_g) = \sum_{g\in K} m\left(g \cdot \chi_{A_g}\right) = \sum_{g\in K} m\left(\chi_{A_g}\right)$$

$$1 = \sum_{h\in L} m(\chi_{B_h})$$

and hence

$$1 = m(\chi_G) = \sum_{g\in K} m\left(\chi_{A_g}\right) + \sum_{h\in L} m(\chi_{B_h}) = 1 + 1 = 2$$

which is a contradiction. Therefore, G is not amenable

By the previous results:

Theorem 3.2.3 : [Tarski & Følner]:

Let G be a finitely generated group. The following conditions on G are equivalent:

(a) G is amenable

(b) G satisfying the Følner condition

(c) G is not paradoxical.

3.3 The relationship between growth and amenability

Proposition 3.3.1 :

A group of subexponential growth is amenable.

Proof 3.3.1 :

Let G has subexponential growth, and let $B_S(n) = \{g \in G \mid \ell_S(g) \leqslant n\}$. Then $|B_S(n)| = \gamma_S(n), \partial B_S(n)$ is the set of elements of length $n+1$, and $\lim_{n\to\infty} \inf \frac{\gamma_S(n+1)}{\gamma_S(n)} \leqslant \lim_{n\to\infty} \gamma_S(n)^{1/n} = \lambda_S^G = 1$. But $\gamma_S(n) \leqslant \gamma_S(n+1)$, and thus $\lim_{n\to\infty} \inf \frac{\gamma S(n+1)}{\gamma_S(n)} = 1$. But $\gamma_S(n+1) = |B_S(n)| + |\partial B_S(n)|$, and thus $\lim_{n\to\infty} \inf \frac{|\partial B_S(n)|}{|B_S(n)|} = 0$ and there exists a subsequence of $\{B_S(n)\}$ which is a Følner sequence.

Theorem 3.3.1 : (Pansu)

Let G be a finitely generated group of polynomial growth. Then for any finite generating set S, the sequence of balls $\{Bs(n)\}_{n=1}^{\infty}$ is a Følner sequence.

Proof Theorem 3.3.1 :

Idea is that $|B_S(n)| \sim n^d$ so

$$\lim_{n\to\infty} \frac{|\partial B_S(n)|}{|B_S(n)|} = \lim_{n\to\infty} \frac{|B_S(n+1)\backslash B_S(n)|}{|B_S(n)|} = \lim_{n\to\infty} \frac{\left|(n+1)^d - n^d\right|}{|n^d|} = 0$$

Corollary 3.3.1 ;

Groups of polynomial growth are amenable.

Theorem 3.3.2 :

A finitely generated elementary amenable group either has exponential growth or is nilpotent-by-finite. As this requires more advanced group theory, we shall omit the proof. It can be found in [16] .

Corollary 3.3.2 :

All finitely generated elementary amenable groups have regular growth.

Corollary 3.3.3 :

The groups of intermediate growth are amenable but not elementary amenable.

This is clear from the last two results. The Grigorchuk group Γ([5]) was the first example of an amenable group that was not elementarily amenable.

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Abstract

In this note, we represent the notions of growth and amenability of groups and discuss the relationship between them.

Key words: group, subgroup , free group, finitely generated group, word length, word metric, growth function, group growth, exponential, polynomial growth ,subexponential growth ,regular and intermediate growth ,growth rate, amenability , measure, mean, amenable group , paradoxical group, folner condition, Tariski theorem, Grigorchuk group.

Résumé

Dans ce mémoire nous représentons les notions de la croissance des groups et la moyennabilité des groupes et discutons la relation entre eux.

Mots clés : groupe, sous-groupe, groupe libre, groupe finiment engendré, longueur de mot métrique des mots, fonction de croissance, croissance exponentielle, polynomiale, ou sous exponentielle, croissance régulière et intermédiaire, moyennablité, mesure, moyenne, groupe moyennable, groupe paradoxal, condition de folner, théorème de Tarisk, groupe de Grigorchuk.

ملخص

في هذه المذكرة قدمنا مفهومي التزايد و الطواعية وناقشنا العلاقة بينهما .

كلمات مفتاحيه : زمرة، زمرة جزئية، زمرة حرة، زمرة منتهية التوليد، طول الكلمة، دالة تزايد، تزايد أسي، كثير حدودي، لا أسي، منتظم، لا منتظم، نسبة تزايد، طواعية، قياس،، متوسط، زمرة قابلة للطواعية ،شرط فولنر، نظرية تاريسكي، زمرة غريغورتشوك .