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Asymptotic Expansion for some Anisotropic Singular Perturbation Problems

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Jury

President	Mabrouk Meflah,	MC(A),	Kasdi Merbah University -Ouargla,
Supervisor	Azouz Salima,	MC(B),	Ecole Normale Supérieure -Ouargla,
Examiners	Chacha Djamal Ahmed,	Prof.,	Kasdi Merbah University -Ouargla,
	Badidja Salim,	MC(B),	Kasdi Merbah University -Ouargla.

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بسم الله الرحمن الرحيم

Dedication

I dedicate this work

To my dear, respectful and wonderful parents, they have supported me throughout my life, with their patience, love and advice.

To my sisters and my family members.

To everyone who helped and supported me in my research.

To my friends and all the students and professors of the Mathematics Department of Kasdi Merbah University.

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Contents

Introduction

Singular perturbation theory concerns the study of problems featuring a parameter for which the solutions of the problem at a limiting value of the parameter are different in character from the limit of the solutions of the general problem; namely, the limit is singular. In contrast, for regular perturbation problems, the solutions of the general problem converge to the solutions of the limit-problem as the parameter approaches the limit-value. Though classically having their origins in the study of differential equations, singular perturbation problems occur in a broad array of contexts.

A great deal of the early motivation in this area arose from studies of physical problems (O'Malley 1991, Cronin and O'Malley 1999). Notable examples are Poincare's work on time-scales for periodic phenomena in celestial mechanics and Prandtl's work on fluid flow (Van Dyke 1975), Van Der Pol's work on electric circuits and oscillatory dynamics studies of biological systems and chemical reaction kinetics by Segel and others and Each of these areas yield problems whose solutions have features that vary on disparate length- or time-scales.

Particularly, in the theory of anisotropic singular perturbation boundary value problems, the solution u_ε does not converge, in the H^1 -norm on the whole domain, towards some u_0, since the perturbation is only taken in some directions. Thus we construct correctors which are simple functions that ensure the affinity of u_ε towards u_0 in H^1 -space.

In this work, we will consider the same problem as in [5] but with Dirichlet boundary conditions, then we construct a composite asymptotic approximations to get the convergence results on the whole domain.

The dissertation is organized as follow

In the first chapter, We provide the necessary basic tools for our study.

In the second chapter, we deduce the outer asymptotic expansion and some convergence results far away from the boundary layers from [2] for our problem.

Then, in the last chapter, we give the definition of the formal correctors and a justification of the proposed composite asymptotic expansion.

Chapter 1

Basic Tools

We recall in this chapter some basic notions and properties as an introduction to linear elliptic problems (see [1, 8, 9]).

1.1 L^q–spaces and Distributions

Definition 1 *Let $q \geq 1$ be a real number, Ω an open subset of $\mathbb{R}^{\mathrm{n}}$, $\mathrm{n} \geq 1$*

$$L^q(\Omega) = \left\{ v : \Omega \to \mathbb{R},\ v \text{ is measurable and } \int_\Omega |v(x)|^q \,\mathrm{d}x < \infty \right\},$$

and

$$L^\infty(\Omega) = \{ v : \Omega \to \mathbb{R},\ v \text{ is measurable} \mid \ \exists \kappa \text{ such that} |v(x)| \leq \kappa, \quad a.e.\ x \in \Omega \}.$$

Equipped with the norm

$$|v|_{q,\Omega} = \left\{ \int_\Omega |v(x)|^q \,\mathrm{d}x \right\}^{\frac{1}{q}},$$

and

$$|v|_{\infty,\Omega} = \inf \{ \kappa \text{ such that } |v(x)| \leq \kappa, \quad a.e.\ x \in \Omega \}$$

respectively, where $L^q(\Omega)$ and $L^\infty(\Omega)$) are Banach spaces.

In the following we recall some elementary properties of L^q–spaces (in particular, the case where $q = 2$).

Theorem 2 (Tonelli) *Let $u(X_1, X_2) : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a measurable function satisfying*

1. $\displaystyle\int_{\Omega_2} u(X_1, X_2)\,\mathrm{d}X_2 < \infty \qquad a.e.\ X_1 \in \Omega_1,$

2. $\displaystyle\int_{\Omega_1} \mathrm{d}X_1 \int_{\Omega_2} u(X_1, X_2)\,\mathrm{d}X_2 < \infty.$

Then $u \in L^1(\Omega_1 \times \Omega_2)$.

Theorem 3 (Fubini) *Assume that* $u \in L^1(\Omega_1 \times \Omega_2)$*. Then for a.e.* $x \in \Omega_1$, $u(X_1, X_2) \in L^1(\Omega_2)$ *and*

$$\int_{\Omega_2} u(X_1, X_2)\,\mathrm{d}X_2 \in L^1(\Omega_1).$$

Similarly, for a.e. $X_2 \in \Omega_2$,

$$u(X_1, X_2) \in L^1(\Omega_1) \quad \text{and} \quad \int_{\Omega_1} u(X_1, X_2)\,\mathrm{d}X_1 \in L^1(\Omega_2).$$

Moreover, one has

$$\int_{\Omega_1} \mathrm{d}X_1 \int_{\Omega_2} u(X_1, X_2)\,\mathrm{d}X_2 = \int_{\Omega_2} \mathrm{d}X_2 \int_{\Omega_1} u(X_1, X_2)\,\mathrm{d}X_1 = \iint_{\Omega_1 \times \Omega_2} u(X_1, X_2)\,\mathrm{d}X_2\,\mathrm{d}X_1.$$

We denote by $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω where the support of a function ρ is defined as

$$\mathrm{supp}\rho = \text{the closure of the set } \{x \in \Omega \mid \rho(x) \neq 0\}.$$

Definition 4 *Let* $\Omega \subset \mathbb{R}^n$ *be open and let* $1 \leq q \leq \infty$*. We say that a real value function* u *belongs to* $L^q_{loc}(\Omega)$ *if* $u\chi \in L^q(\Omega)$ *for every compact set* K *contained in* Ω *where* χ *is the characteristic function of* K.

Note that if $u \in L^q_{loc}(\Omega)$ then $u \in L^1_{loc}(\Omega)$. Then we have

Lemme 1 *Let* $u \in L^1_{loc}(\Omega)$ *such that*

$$\int_{\Omega} u\varphi\,\mathrm{d}x = 0, \qquad \forall \varphi \in \mathcal{D}(\Omega).$$

Then $u = 0$ *a.e. on* Ω.

If $\mathcal{X}$ is a normed linear space, then the collection of bounded linear functional on $\mathcal{X}$ is called its dual space and is denoted by $\mathcal{X}'$. In particular, for any $1 \leq q < \infty$ the dual of $L^q(\Omega)$ can be identified with $L^q(\Omega)$ where q' is the conjugate number of q which is defined by

$$\frac{1}{q} + \frac{1}{q'} = 1,$$

i.e., $q' = \frac{q}{q-1}$ with the convention that $q' = +\infty$ when $q = 1$ (in particular if $q = 2$ we also have $q' = 2$).

Theorem 5 (Holder's inequality) *Assume that $u \in L^q(\Omega)$ and $v \in L^{q'}(\Omega)$ with $1 \leq q \leq \infty$. Then $uv \in L^1(\Omega)$ and*

$$\int_\Omega |uv| \, \mathrm{d}x \leq |u|_{q,\Omega} \, |v|_{q',\Omega} \, .$$

For $q = 2$, $q' = 2$, we have the Cauchy–Schwarz inequality in $L^2(\Omega)$. We also recall the classical form of Young inequality

$$\kappa_1 \kappa_2 \leq \frac{1}{q}\kappa_1^q + \frac{1}{q'}\kappa_2^{q'} \quad \forall \kappa_1, \kappa_2 \geq 0$$

with $1 < q < \infty$. More general Young inequality can be written as

$$\kappa_1 \kappa_2 \leq \frac{\varepsilon}{q}\kappa_1^q + \frac{1}{\sqrt[q-1]{\varepsilon} q'}\kappa_2^{q'} \quad \forall \kappa_1, \kappa_2 \geq 0$$

where ε is positive constant. Note that the last inequality can be derived from the previous one by replacing κ_1 and κ_2 by $\sqrt[q]{\varepsilon}\kappa_1$ and $\kappa_2 / \sqrt[q]{\varepsilon}$ respectively.

In the following we would like to introduce the notion of distributions.

Definition 6 *A distribution T on Ω is a continuous linear functional on $\mathcal{D}(\Omega)$. We will denote by*

$$\langle T, \varphi \rangle := T(\varphi) \quad \forall \varphi \in \mathcal{D}(\Omega),$$

and by $\mathcal{D}'(\Omega)$ the space of all distributions on Ω.

In what follows, we use the abbreviation D^α, for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_\mathrm{n}) \in \mathbb{N}^\mathrm{n}$, the partial derivative given by

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_\mathrm{n}}^{\alpha_\mathrm{n}}}$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_\mathrm{n}$. In the next we will clarify the preceding definition.

Definition 7 *Let $T \in \mathcal{D}'(\Omega)$ for some open set $\Omega \subset \mathbb{R}^{\mathrm{n}}$.*

1. *If α is any multi-index, then we define $D^\alpha T$, the α–weak partial derivative, by*

$$\langle D^\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, D^\alpha \varphi \rangle .$$

2. *Given $\psi \in \mathcal{C}^\infty(\Omega)$, we define ψT by*

$$\langle \psi T, \varphi \rangle = \langle T, \psi \varphi \rangle .$$

Remark 8 *One of the key feature of distributions is the fact that we can differentiate them - in some sense - as much as we wish. Indeed, if u is a function that is k–times differentiable in Ω, $k \geq |\alpha|$, then the distribution $D^\alpha u$ coincides with the function $D^\alpha u$, the partial derivative of u in the usual sense.*

We define now the convergence in $\mathcal{D}'(\Omega)$.

Definition 9 *Let $(T_n)_{n\in\mathbb{N}}$ be a sequence of distributions on Ω. We say that when $n \to \infty$,*

$$T_n \to T \quad \text{in } \mathcal{D}'(\Omega)$$

iff

$$\langle T_n, \varphi \rangle \to \langle T, \varphi \rangle , \quad \forall \varphi \in \mathcal{D}(\Omega) .$$

Now we recall the notion of *weak convergence.* A sequence (u_n), $n \in \mathbb{N}$, in a normed space, $\mathcal{X}$ is called *weakly convergent* to an element $u \in \mathcal{X}$, if

$$f(u_n) \to f(u) \quad \text{as } n \to +\infty, \quad \forall f \in \mathcal{X}'.$$

In this case u is called a *weak limit* of the sequence and we write

$$u_n \rightharpoonup u \quad \text{as } n \to +\infty.$$

Note that

$$u_n \to v \quad \text{in } \mathcal{X} \Rightarrow u_n \rightharpoonup v \quad \text{in } \mathcal{X}, \quad \text{as } n \to +\infty.$$

The converse is not true in general. However, a weakly convergent sequence is bounded. Then we have

Proposition 10 *Let* $u_n, u \in L^q(\Omega)$. *If we suppose that when* $n \to +\infty$,

$$u_n \to u \quad in\ L^q(\Omega), \quad (respectively\ u_n \rightharpoonup u\ in\ L^q(\Omega)\ ) \quad 1 \leq q < \infty,$$

then we have

$$u_n \to u \quad in\ \mathcal{D}'(\Omega).$$

Proposition 11 *The operator* D^α, $\alpha \in \mathbb{N}^{\mathrm{n}}$ *is continuous on* $\mathcal{D}'(\Omega)$, *i.e.*

$$T_n \to T\ in\ \mathcal{D}'(\Omega) \Rightarrow D^\alpha T_n \to D^\alpha T\ in\ \mathcal{D}'(\Omega) \quad as\ n \to \infty.$$

We conclude this section by the following results often used to prove the weak convergence of a whole sequence.

Theorem 12 (Weak compactness of balls) *If* (u_n) *is a bounded sequence in a Hilbert space* H, *there exists a subsequence* (u_{n_k}) *of* (u_n) *and* $u \in H$ *such that*

$$u_{n_k} \rightharpoonup u \quad as\ n_k \to \infty.$$

Proposition 13 *Let* $\mathcal{X}$ *be a reflexive Banach space and* (u_n) *a bounded sequence in* $\mathcal{X}$. *We assume that there exists* $u \in \mathcal{X}$ *such that every weakly convergent subsequence of* (u_n) *has a limit equal to* u*; then the whole sequence* (u_n) *weakly converges to* u.

We also have

Theorem 14 *Let* u_n, v_n *be two sequences in* H *such that* $u_n \to u$ *and* $v_n \rightharpoonup v$ *when* n *tends to* $+\infty$. *Then we have*

$$\langle u_n, v_n \rangle \to \langle u, v \rangle \quad in\ \mathbb{R}, \quad as\ n \to +\infty.$$

1.2 Lax–Milgram Theorem

Lax–Milgram theorem is a key tool for solving elliptic partial differential equations. Instead of a scalar product one can consider more generally a continuous bilinear form. Suppose that $(H, \langle ., . \rangle_H)$ is a Hilbert space, $\mathcal{V} \subseteq H$ is a linear subspace of H, and $\langle ., . \rangle_{\mathcal{V}}$ is an inner product on $\mathcal{V}$ that turns $\mathcal{V}$ into a Hilbert space.

Moreover, we suppose that the embedding $\mathcal{V} \hookrightarrow H$ is continuous, i.e. there is a positive constant κ such that

$$|v|_H \leq \kappa \, |v|_{\mathcal{V}} \quad \forall v \in \mathcal{V}. \tag{1.1}$$

Definition 15 *A bilinear form $a : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ is said to be*

1. ***continuous** if there is a positive constant Λ such that*

$$|a\,(u, v)| \leq \Lambda \, |u|_{\mathcal{V}} \, |v|_{\mathcal{V}} \quad \forall u, v \in \mathcal{V},$$

2. ***coercive** if there is a positive constant λ such that*

$$a\,(v, v) \geq \lambda \, |v|_{\mathcal{V}}^2 \quad \forall v \in \mathcal{V}.$$

Then we have the following theorem.

Theorem 16 (Lax–Milgram) *Assume that $a(.,.)$ is a continuous coercive bilinear form on $\mathcal{V}$. Then, given any $f \in \mathcal{V}'$, there exists a unique element $u \in \mathcal{V}$ such that*

$$a(u, v) = \langle f, v \rangle_{\mathcal{V}} \quad \forall v \in \mathcal{V}.$$

1.3 Essential Features of the Sobolev Spaces

The Sobolev spaces are very convenient tools to study the partial differential equations. In this section we concentrate ourselves to the most simple ones. Then, we firstly define $W^{1,q}(\Omega)$ as follows

$$W^{1,q}\,(\Omega) = \{v \in L^q\,(\Omega) \mid \partial_{x_i} v \in L^q\,(\Omega)\,, \quad i = 1, ..., \mathrm{n}\}\,,$$

where $\partial_{x_i} v$ denotes the derivative of v in the weak sense. Equipped with the norm $|.|_{q,\Omega}$ is a Banach space and it is reflexive for $1 < q < \infty$.

In the next, we focus on the properties of Hilbert spaces. For $q = 2$, Ω be an open subset of $\mathbb{R}^{\mathrm{n}}$, $\mathrm{n} \geq 1$. We denote by $H^1\,(\Omega)$ the subset of $L^2\,(\Omega)$ defined by $H^1\,(\Omega) := W^{1,q}\,(\Omega)$ which is a Hilbert space equipped with the scalar product

$$(u, v)_{1,\Omega} = \int_\Omega (\nabla u \cdot \nabla v + uv) \; \mathrm{d}x,$$

where "∇u" denotes the Gradient vector $(\partial_{x_1}u, \partial_{x_2}u, \ldots, \partial_{x_\mathrm{n}}u)^T$. However, here we can also state

Theorem 17 *Assume that Ω is a bounded Lipschitz domain of $\mathbb{R}^\mathrm{n}$, then the canonical embedding from $H^1(\Omega)$ into $L^2(\Omega)$ is compact.*

We use of the above theorem in partial differential equations as follows. From any bounded sequence in $H^1(\Omega)$ we can extract a subsequence converges weakly in $H^1(\Omega)$, strongly in $L^2(\Omega)$, and almost everywhere in Ω.

Now, let us denote by Γ_0 a subset of $\Gamma = \partial\Omega$ the boundary of Ω, such that

$$|\Gamma_0| > 0,$$

where $|\Gamma_0|$ denotes the measure of Γ_0. Define $\mathcal{C}_0^1\left(\bar{\Omega}, \Gamma_0\right)$ as the set of continuously differentiable functions on $\bar{\Omega}$ (Γ is a Lipschitz boundary). An important subspace of $H^1(\Omega)$ is $\mathcal{V}$ the space defined as

$$\mathcal{V} = \text{the closure of } \mathcal{C}_0^1\left(\bar{\Omega}, \Gamma_0\right) \text{ in } H^1(\Omega).$$

The mapping

$$|u|_{\mathcal{V}} = \left\{\int_\Omega |\nabla u(x)|^2 \,\mathrm{d}x\right\}^{\frac{1}{2}} = |\nabla u|_{2,\Omega} \tag{1.2}$$

defines a norm on $\mathcal{V}$ which is equivalent the H^1–norm when Ω is bounded. This is an immediate consequence of the following lemma.

Lemme 2 (Poincaré's inequality on $\mathcal{V}$) *Let Ω be a bounded open subset of $\mathbb{R}^\mathrm{n}$. There exists a positive constant κ such that*

$$|v|_{2,\Omega} \leq \kappa\,|v|_{\mathcal{V}} = \kappa\,|\nabla v|_{2,\Omega} \quad \forall v \in \mathcal{V}.$$

In particular when $\Gamma_0 = \Gamma = \partial\Omega$ we set $H_0^1(\Omega) := H_0^1(\Omega, \Gamma)$ which is the closure of $\mathcal{D}(\Omega)$ in $H^1(\Omega)$. We denote by $H^{-1}(\Omega)$ the dual space of $H_0^1(\Omega)$.

An other most important properties of the trace are the following

Theorem 18 (Green formula) *For all $u, v \in H^1(\Omega)$, we have*

$$\int_{\Omega} v\,(x)\,\partial_{x_i} u\,(x)\ \mathrm{d}x = -\int_{\Omega} \partial_{x_i} v\,(x)\,u\,(x)\ \mathrm{d}x + \int_{\partial\Omega} \gamma_0\,(v)\,\gamma_0\,(u)\,\nu_i\,\mathrm{d}\sigma$$

where $\nu = (\nu_1, ..., \nu_{\mathrm{n}})$ the outward unit normal to $\Gamma = \partial\Omega$ and γ_0 (respectively $\mathrm{d}\sigma$) denotes the operator trace (respectively the superficial measure on $\Gamma = \partial\Omega$).

Chapter 2

Outer Asymptotic Expansion

In this chapter we construct an outer asymptotic development to the weak solution of anisotropic singular perturbation problems of elliptic type. Since the perturbation is only taken in one direction, the convergence of this expansion is ensured in the Sobolev Space far away from the boundary layer.

2.1 Position of the Problem

In order to describe the class of problems that we would like to address, we first introduce some basic notations and hypotheses. Let Ω be a bounded open cylinder in $\mathbb{R}^{\mathrm{n}}$ i.e.

$$\Omega = \omega_1 \times \omega_2 = (-1, 1) \times \omega_2,$$

where ω_2 is bounded lipschitz domain of $\mathbb{R}^{\mathrm{n}-1}$ ($\mathrm{n} > 1$), we denote by $x = (x_1, \dots, x_{\mathrm{n}}) = (X_1, X_2)$ the points of $\mathbb{R}^{\mathrm{n}}$ where

$$X_1 = (x_1), \qquad X_2 = (x_2, \dots, x_{\mathrm{n}}),$$

i.e., we split the coordinates into two parts. Throughout this manuscript ∂_{x_i} denotes the partial derivative in the x_i-direction. With these notation we set

$$\nabla u = (\partial_{x_1} u, \dots, \partial_{x_{\mathrm{n}}} u)^T = \begin{pmatrix} \partial_{X_1} u \\ \nabla_{X_2} u \end{pmatrix},$$

where

$$\nabla_{X_2} u = (\partial_{x_2} u, \dots, \partial_{x_{\mathrm{n}}} u)^T .$$

For a function

$$f \in L^2(\Omega), \tag{2.1}$$

we ensure the existence and the uniqueness, (from Lax-Milgram theorem), of a weak solution u_ε to

$$\begin{cases} -\varepsilon^2 \partial_{X_1}^2 u_\varepsilon - \nabla_{X_2} \cdot (\nabla_{X_2} u_\varepsilon) = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

in the following sense

$$\begin{cases} u_\varepsilon \in H_0^1(\Omega), \\ \varepsilon^2 \int_\Omega \partial_{X_1} u_\varepsilon \partial_{X_1} v \,\mathrm{d}x \quad + \int_\Omega \nabla_{X_2} u_\varepsilon \cdot \nabla_{X_2} v \,\mathrm{d}x = \int_\Omega f v \,\mathrm{d}x, \quad \forall v \in H_0^1(\Omega). \end{cases} \tag{2.3}$$

2.2 Formal Asymptotic Expansion

As it is shown in (see [4]) the limit u_0 of u_ε is the unique solution, for a.e. $X_1 \in \omega_1$, to the following lower dimension problem

$$\begin{cases} -\nabla_{X_2} \cdot (\nabla_{X_2} u_0(X_1, \cdot)) = f(X_1, \cdot) & \text{in } \omega_2, \\ u_0(X_1, \cdot) = 0 & \text{on } \partial\omega_2, \end{cases} \tag{2.4}$$

in the following sense, for a.e. $X_1 \in \omega_1$

$$\begin{cases} u_0(X_1, \cdot) \in H_0^1(\omega_2), \\ \int_{\omega_2} \nabla_{X_2} u_0(X_1, X_2) \cdot \nabla_{X_2} v \,\mathrm{d}X_2 = \int_{\omega_2} f(X_1, X_2) v \,\mathrm{d}X_2, \qquad \forall v \in H_0^1(\omega_2). \end{cases} \tag{2.5}$$

The existence and the uniqueness of u_0 is followed from the Lax-Milgram theorem, since for a.e. $X_1 \in \omega_1 = (-1, 1)$, $f(X_1, \cdot) \in L^2(\omega_2)$. The convergence holds out on the whole domain Ω, but with respect to topologies weaker than that of the space of existence of u_ε; $H^1(\Omega)$. We mean the following functional space

$$\mathcal{V}(\Omega) = \left\{ u \in L^2(\Omega) \mid \nabla_{X_2} u \in L^p(\Omega) \right\},$$

equipped with the norm

$$|u|_{\mathcal{V}(\Omega)}^2 = \int_\Omega \left(|u(x)|^2 + |\nabla_{X_2} u(x)|^2 \right) \,\mathrm{d}x.$$

The improvements related to the convergence $u_\varepsilon \to u_0$ investigated on one hand the topology type by considering the standard Sobolev space on domains located far away from the boundary layer $\{-1,1\} \times \omega_2$ and on the other hand the rate of convergence. Unfortunately, these improvements are limited by the nature of the problem. It can go until an exponential rate of convergence if u_0 be independent of x_1, however, a rate of convergence as

$$u_\varepsilon - u_0 = o\left(\varepsilon\right) \quad \text{in } L^2\left(\omega_1' \times \omega_2\right), \qquad \omega_1' \subset\subset \omega_1$$

where $\omega_1' = (-a, a) \subset\subset (-1, 1)$, $0 < a < 1$, can not take place in general case.

Now, in order to reduce the approximation error we can propose an asymptotic development of u_ε, i.e. it should be expressed as a power series of ε in the form

$$u_\varepsilon = u_0 + \varepsilon u_1 + \cdots . \tag{2.6}$$

Consequently, this allows to chose U_ε as a polynomial in ε, i.e.

$$U_\varepsilon^N = u_0 + \varepsilon u_1 + \cdots + \varepsilon^N u_N.$$

Formally, if we substitute the asymptotic expansion of (2.6) into (2.2) and expand the left-hand side in powers of ε, we then deduce, after equating coefficients of equal powers of ε, that the coefficient u_N are solutions of the following system of boundary value problems, defined on the section ω_2 for a.e. $X_1 \in \omega_1 = (-1, 1)$,

$$\begin{cases} -\nabla_{X_2} \cdot (\nabla_{X_2} u_0(X_1, \cdot)) = f(X_1, \cdot) & \text{in } \omega_2, \\ u_0(X_1, \cdot) \in H_0^1(\omega_2), \end{cases} \tag{2.7}$$

$$\begin{cases} -\nabla_{X_2} \cdot (\nabla_{X_2} u_1(X_1, \cdot)) = 0, & \text{in } \omega_2, \\ u_1(X_1, \cdot) \in H_0^1(\omega_2). \end{cases} \tag{2.8}$$

and for $N \geq 2$,

$$\begin{cases} -\nabla_{X_2} \cdot (\nabla_{X_2} u_N(X_1, \cdot)) & = \partial_{X_1} \cdot (\partial_{X_1} u_{N-2}(X_1, \cdot)) \quad \text{in } \omega_2, \\ u_N(X_1, \cdot) \in H_0^1(\omega_2). \end{cases} \tag{2.9}$$

Our perturbed problem is now reduced to a sequence of the elliptic boundary value problems (2.7), (2.8) and (2.9) which can be easily solved iteratively once the solution of (2.7) has been constructed and has the necessary smoothness.

Remark 19 *It is clear by using the problem (2.7)-(2.9) defined each coefficients u_N, $N \in \mathbb{N}$, that we have only the pairs coefficients of the asymptotic representation (2.6), i.e. $u_i = 0$ for all $i = 2k+1$, $k \in \mathbb{N}$.*

As it is mentioned above, the problem (2.9) will be solved iteratively. That is to say, in order to define u_N, for $N \geq 2$, as a solution of (2.9) in the following weak sense, for a.e. $X_1 \in \omega_1$,

$$\begin{cases} u_N(X_1,\cdot) \in H_0^1(\omega_2) \\ \displaystyle\int_{\omega_2} \nabla_{X_2} u_N(X_1,.) \cdot \nabla_{X_2} v \, \mathrm{d}X_2 \quad = \int_{\omega_2} \partial_{X_1} \left(\partial_{X_1} u_{N-2}(X_1,.)\right) v \, \mathrm{d}X_2 \quad \forall v \in H_0^1(\omega_2). \end{cases} \tag{2.10}$$

we need to ensure the smoothness of u_{N-2}, in the following sense

$$\partial_{X_1}^2 u_{N-2} \in L^2(\Omega). \tag{2.11}$$

Again we have to ensure the existence and the uniqueness of u_{N-2}, of course as solutions of the same problem (2.10) replacing N by $N-2$ respectively, as well as their smoothness in (2.11). So, to simplify the study, in the following we suppose that we have all the necessary smoothness regularity for the existence of the coefficients $u_N \in H^1(\Omega)$, where N be an even integer, i.e. $N = 2k, k \in \mathbb{N}$.

2.3 Asymptotic Convergence Results

Now we pass to the main outer asymptotic expansion result.

Theorem 20 (Outer Asymptotic Expansion of Higher Order) *Under the sufficient assumptions to get $u_N \in H^1(\Omega), N \in \mathbb{N}$, and for any $\omega_1' \subset\subset (-1,1)$, it holds that, when $\varepsilon \to 0$,*

$$R_N(\cdot;\varepsilon) = O\left(\varepsilon^{N+1}\right), \quad \nabla_{X_2} R_N(\cdot;\varepsilon) = O\left(\varepsilon^{N+1}\right), \quad \partial_{X_1} R_N(\cdot;\varepsilon) = O\left(\varepsilon^{N}\right) \tag{2.12}$$

in $L^2(\omega_1' \times \omega_2)$ where $R_N(\cdot;\varepsilon) = u_\varepsilon - \sum_{i=0}^{N} \varepsilon^i u_i$.

Proof. First, we notice that for $N = 0$, the same results of this theorem are shown in [4, Theorem 3]. For $N > 0$, we proceed by induction on N and suppose that the estimates (2.12) take place for any $N' < N$, and we will prove this result for N. Strating from (2.10) written for $k = 2, 4, \dots, N$ and multiplying each $k-$identity by ε^k then summing up over $k = 2, 4, \dots, N$, we obtain, for $v \in H_0^1(\Omega)$

$$\varepsilon^2 \int_\Omega \partial_{X_1} \left(\sum_{i=0}^{N-2} \varepsilon^i u_i \right) \partial_{X_1} v \, \mathrm{d}x + \int_\Omega \nabla_{X_2} \left(\sum_{i=0}^{N} \varepsilon^i u_i \right) \cdot \nabla_{X_2} v \, \mathrm{d}x = \int_\Omega f \, v \, \mathrm{d}x$$

Of course the identity (2.5) is added and we integrated over ω_1. Then subtracting the above identity from (2.3), we get

$$\varepsilon^2 \int_\Omega \partial_{X_1} R_{N-2}(x;\varepsilon) \partial_{X_1} v \, \mathrm{d}x + \int_\Omega \nabla_{X_2} R_N(x;\varepsilon) \cdot \nabla_{X_2} v \, \mathrm{d}x = 0. \tag{2.13}$$

Next, we consider a smooth function $\rho = \rho(X_1)$ supported in $\omega_1'' \subset\subset \omega_1$, $\rho = 1$ on $\omega_1' = (-a, a), 0 < a < 1$ and $0 \leq \rho \leq 1$. This allowed to take $v = R_N(\cdot;\varepsilon)\rho^2(X_1) \in H_0^1(\Omega)$ as a test function in (2.13) that leads to

$$\begin{aligned} & \varepsilon^2 \int_\Omega \rho^2 \partial_{X_1} R_{N-2}(x;\varepsilon) \partial_{X_1} R_N(x;\varepsilon) \, \mathrm{d}x + \int_\Omega \rho^2 \nabla_{X_2} R_N(x;\varepsilon) \cdot \nabla_{X_2} R_N(x;\varepsilon) \, \mathrm{d}x \\ = \; & -2\varepsilon^2 \int_\Omega \partial_{X_1} R_{N-2}(x;\varepsilon) \left(\partial_{X_1} \rho\right) \rho R_N(x;\varepsilon) \, \mathrm{d}x. \end{aligned}$$

Thus

$$\begin{aligned} & \int_\Omega \rho^2 \nabla_\varepsilon R_N(x;\varepsilon) \cdot \nabla_\varepsilon R_N(x;\varepsilon) \, \mathrm{d}x \\ = \; & -2\varepsilon^2 \int_\Omega \partial_{X_1} R_{N-2}(x;\varepsilon) \left(\partial_{X_1} \rho\right) \rho R_N(x;\varepsilon) \, \mathrm{d}x - \varepsilon^{N+2} \int_\Omega \rho^2 \partial_{X_1} u_N \partial_{X_1} R_N(x;\varepsilon) \, \mathrm{d}x. \end{aligned}$$

where $\nabla_\varepsilon = \begin{pmatrix} \varepsilon \partial_{X_1} \cdot \\ \nabla_{X_2} \cdot \end{pmatrix}$. Next applying the Poincaré and the Cauchy-Schwarz inequalities, we get

$$\begin{aligned} |\rho \nabla_\varepsilon R_N(\cdot;\varepsilon)|_{2,\Omega}^2 \; \leq \; & 2\varepsilon^2 \sqrt{C_{\omega_2}} \, |\partial_{X_1} \rho|_{\infty,\omega_1''} \, |\partial_{X_1} R_{N-2}(\cdot;\varepsilon)|_{2,\omega_1'' \times \omega_2} \, |\rho \nabla_{X_2} R_N(\cdot;\varepsilon)|_{2,\Omega} \\ & + \varepsilon^{N+1} \, |\partial_{X_1} u_N|_{2,\Omega} \, \varepsilon \, |\rho \partial_{X_1} R_N(\cdot;\varepsilon)|_{2,\Omega} \, , \end{aligned} \tag{2.14}$$

where C_{ω_2} is the Poincaré constant. Then, using Young's inequality and the convergences (2.12) on $\omega_1'' = (-b, b), 0 < a < b < 1$ for all $N' < N$, we end up with

$$\varepsilon^2 \, |\rho \partial_{X_1} R_N(\cdot;\varepsilon)|_{2,\Omega}^2 + |\rho \nabla_{X_2} R_N(\cdot;\varepsilon)|_{2,\Omega}^2 \leq C\varepsilon^{2N+2}.$$

This completes the proof. ■

The authors M.Chipot and S.Guesmia improves in [4] the results (2.13) by

$$|u_\varepsilon - u_0|_{2,\omega_1'\times\omega_2} = O(\varepsilon^2), \qquad |\nabla_{X_2}(u_\varepsilon - u_0)|_{2,\omega_1'\times\omega_2} = O(\varepsilon^2), \tag{2.15}$$

and

$$|\partial_{X_1}(u_\varepsilon - u_0)|_{2,\omega_1'\times\omega_2} = O(\varepsilon), \tag{2.16}$$

under more smoothness assumptions on the data, i.e. the function f.

Then we end up with

Theorem 21 *Let $N = 2k$, $k \in \mathbb{N}\backslash\{0\}$, we have*

$$|R_N(\cdot;\varepsilon)|_{2,\omega_1'\times\omega_2} = O(\varepsilon^{N+2}), \qquad |\nabla_{X_2} R_N(\cdot;\varepsilon)|_{2,\omega_1'\times\omega_2} = O(\varepsilon^{N+2}), \tag{2.17}$$

and

$$|\partial_{X_1} R_N(\cdot;\varepsilon)|_{2,\omega_1'\times\omega_2} = O(\varepsilon^{N+1}), \tag{2.18}$$

where $R_N(\cdot;\varepsilon) = u_\varepsilon - \sum_{i=0}^{N/2} \varepsilon^{2i} u_{2i}$.

Proof. Using (2.14), we have

$$\begin{aligned}\int_\Omega \rho^2 \nabla_\varepsilon R_N(.;\varepsilon)\cdot\nabla_\varepsilon R_N(.;\varepsilon)\,\mathrm{d}x = -2\varepsilon^2 \int_\Omega \partial_{X_1} R_{N-2}(.;\varepsilon)\,(\partial_{X_1}\rho)\,\rho R_N(.;\varepsilon)\,\mathrm{d}x \\ -\varepsilon^{N+2}\int_\Omega \rho^2 \partial_{X_1} u_N \partial_{X_1} R_N(.;\varepsilon)\,\mathrm{d}x.\end{aligned} \tag{2.19}$$

We suppose that

$$\rho^2 \partial_{X_1} u_N \in H_0^1(\omega_1). \tag{2.20}$$

Thus, by using Green's formula it follows that

$$\begin{aligned}\int_\Omega \rho^2 \nabla_\varepsilon R_N(.;\varepsilon)\cdot\nabla_\varepsilon R_N(.;\varepsilon)\,\mathrm{d}x = -2\varepsilon^2 \int_\Omega \partial_{X_1} R_{N-2}(.;\varepsilon)\,(\partial_{X_1}\rho)\,\rho R_N(.;\varepsilon)\,\mathrm{d}x \\ +\varepsilon^{N+2}\int_\Omega \partial_{X_1}\left(\rho^2 \partial_{X_1} u_N\right) R_N(.;\varepsilon)\,\mathrm{d}X_1\,\mathrm{d}X_2,\end{aligned} \tag{2.21}$$

so

$$\int_{\Omega} \rho^2 \nabla_\varepsilon R_N(x;\varepsilon) \cdot \nabla_\varepsilon R_N(x;\varepsilon)\, \mathrm{d}x$$

$$\begin{aligned} &= -2\varepsilon^2 \int_{\Omega} \partial_{X_1} R_{N-2}(x;\varepsilon)\,(\partial_{X_1}\rho)\,\rho R_N(x;\varepsilon)\,\mathrm{d}x \\ &+2\varepsilon^{N+2}\int_{\Omega} \partial_{X_1} u_N\,(\partial_{X_1}\rho)\,\rho R_N(x;\varepsilon)\,\mathrm{d}x \\ &+\varepsilon^{N+2}\int_{\Omega} \left(\partial^2_{X_1} u_N\right)\rho^2 R_N(x;\varepsilon)\,\mathrm{d}x, \end{aligned} \tag{2.22}$$

Using the Cauchy-Schwarz and Poincaré inequalities, we get

$$\begin{aligned} |\rho\nabla_\varepsilon R_N(.;\varepsilon)|^2_{2,\Omega} \quad &\le +2\varepsilon^2\sqrt{C_{\omega_2}}\,|\partial_{X_1}R_{N-2}(\cdot;\varepsilon)|_{2,\omega_1''\times\omega_2}\,|\partial_{X_1}\rho|_{\infty,\omega_1''}\,|\rho\nabla_{X_2}R_N(\cdot;\varepsilon)|_{2,\Omega} \\ &+2\varepsilon^{N+2}\sqrt{C_{\omega_2}}\,|\nabla_{X_1}u_N|_{2,\Omega}\,|\nabla_{X_1}\rho|_{\infty,\omega_1''}\,|\rho\nabla_{X_2}R_N(\cdot;\varepsilon)|_{2,\Omega} \\ &+\varepsilon^{N+2}\sqrt{C_{\omega_2}}\,\left|\partial^2_{X_1}u_N\right|_{2,\Omega}|\rho\nabla_{X_2}R_N(\cdot;\varepsilon)|_{2,\Omega} \end{aligned} \tag{2.23}$$

Using Young's inequality and (2.18) reads for $N-2$, we get

$$|\rho\nabla_\varepsilon R_N(.;\varepsilon)|_{2,\Omega} \le C\varepsilon^{N+2}. \tag{2.24}$$

This completes the proof of the theorem. ∎

Corollary 22 *We can also get*

$$|R_N(\cdot;\varepsilon)|_{H^1(\omega_1'\times\omega_2)} = O(\varepsilon^{N+2}). \tag{2.25}$$

Proof. Using the Triangular inequality, we have

$$|\partial_{X_1}R_N(\cdot;\varepsilon)|_{2,\omega_1'\times\omega_2} \le |\partial_{X_1}R_{N+2}(\cdot;\varepsilon)|_{2,\omega_1'\times\omega_2} + \varepsilon^{N+2}\,|\partial_{X_1}u_{N+2}|_{2,\omega_1'\times\omega_2}. \tag{2.26}$$

Using (2.18) in the above inequality, we obtain

$$|\partial_{X_1}R_N(\cdot;\varepsilon)|_{2,\omega_1'\times\omega_2} \le C\varepsilon^{N+2}, \tag{2.27}$$

since $u_{N+1}=0$. Then, from (2.17), we get

$$|R_N(\cdot;\varepsilon)|_{2,\omega_1'\times\omega_2} \le C\varepsilon^{N+2}.$$

This completes the proof of the corollary. ∎

Chapter 3

Composite Asymptotic Expansion

In this chapter, we will construct a composite asymptotic expansion on the whole domain.

3.1 Formal Correctors and Properties

For the construction of the correctors to each coefficient of the outer asymptotic expansion u_N, $N \in \mathbb{N}$, given in Chapter 2, we denote by S_l^+, S_l^- the half-cylinders

$$\begin{aligned} S_l^+ &= (l, +\infty) \times \omega_2, \\ S_l^- &= (-\infty, l) \times \omega_2, \end{aligned}$$

where $l \in \mathbb{R}$. Then, we define the following functions

$$\rho_{[1]}(x) = \begin{cases} 1 - x & \text{on } [0, 1] \\ 0 & \text{on } (1, +\infty) \end{cases}, \tag{3.1}$$

$$\rho_{[-1]}(x) = \begin{cases} 0 & \text{on } (-\infty, -1) \\ 1 + x & \text{on } [-1, 0] \end{cases}, \tag{3.2}$$

$$\breve{\rho}_{[1]}(x) = \begin{cases} 0 & \text{on } (-\infty, 0) \\ x & \text{on } [0, 1] \\ 1 & \text{on } (1, +\infty) \end{cases}, \tag{3.3}$$

$$\breve{\rho}_{[-1]}(x) = \begin{cases} 1 & \text{on } (-\infty, -1) \\ -x & \text{on } [-1, 0] \\ 0 & \text{on } (0, +\infty) \end{cases}. \tag{3.4}$$

For the first boundary layer, i.e. near $x_1 = 1$, we introduce $u_N^{[1]}$ as the solution, for $\varepsilon > 0$, $N \in \mathbb{N}$, to

$$\begin{cases} u_N^{[1]} \in H_0^1(S_0^+), \\ \displaystyle\int_{S_0^+} \nabla u_N^{[1]} \cdot \nabla v \,\mathrm{d}x = \int_{S_0^+} \nabla \left\{\rho_{[1]}(X_1) u_N(1 - X_1, X_2)\right\} \cdot \nabla v \,\mathrm{d}x \qquad \forall v \in H_0^1(S_0^+). \end{cases} \tag{3.5}$$

The existence and the uniqueness of $u_N^{[1]}$, $N \in \mathbb{N}$, follow from the Lax-Milgram theorem.

Then we set

$$w_N^{[1]}(X_1, X_2) = u_N^{[1]}(X_1, X_2) - \rho_{[1]}(X_1) u_N(1 - X_1, X_2), \qquad \text{a.e.}\, x \in S_0^+, \tag{3.6}$$

and denote by $\theta_N^{[\varepsilon,1]}$ the function defined as

$$\theta_N^{[\varepsilon,1]}(X_1, X_2) = w_N^{[1]}(\frac{1 - X_1}{\varepsilon}, X_2) \qquad \text{a.e.}\, x \in \Omega. \tag{3.7}$$

Then, for $\varepsilon > 0$ and $N \in \mathbb{N}$, we can consider $w_N^{[1]}$ as the weak solution to

$$\begin{cases} \Delta w_N^{[1]} = 0 & \text{in } S_0^+, \\ w_N^{[1]} = -u_N(1, X_2) & \text{on } \{0\} \times \omega_2, \\ w_N^{[1]} = 0 & \text{on } (0, +\infty) \times \partial\omega_2. \end{cases} \tag{3.8}$$

In other part, for the second boundary layer, i.e. near $x_1 = -1$, we introduce also $u_N^{[-1]}$ as the solution, for $\varepsilon > 0, N \in \mathbb{N}$, to

$$\begin{cases} u_N^{[-1]} \in H_0^1(S_0^-), \\ \displaystyle\int_{S_0^-} \nabla u_N^{[-1]} \cdot \nabla v \,\mathrm{d}x = \int_{S_0^-} \nabla \left\{\rho_{[-1]}(X_1) u_N(-1 - X_1, X_2)\right\} \cdot \nabla v \,\mathrm{d}x \qquad \forall v \in H_0^1(S_0^-), \end{cases} \tag{3.9}$$

The existence and the uniqueness of $u_N^{[-1]}$, $N \in \mathbb{N}$, follow from the Lax-Milgram theorem.

Then we set

$$w_N^{[-1]}(X_1, X_2) = u_N^{[-1]}(X_1, X_2) - \rho_{[-1]}(X_1) u_N(-1 - X_1, X_2) \qquad \text{a.e.}\, x \in S_0^-, \tag{3.10}$$

and denote by $\theta_N^{[\varepsilon,-1]}$ the function defined as

$$\theta_N^{[\varepsilon,-1]}(X_1, X_2) = w_N^{[-1]}(\frac{-1 - X_1}{\varepsilon}, X_2) \qquad \text{a.e.}\, x \in \Omega. \tag{3.11}$$

Then, for $\varepsilon > 0$ and $N \in \mathbb{N}$, $w_N^{[-1]}$ is the weak solution to

$$\begin{cases} \Delta w_N^{[-1]} = 0 & \text{in } S_0^-, \\ w_N^{[-1]} = -u_N(-1, X_2) & \text{on } \{0\} \times \omega_2, \\ w_N^{[-1]} = 0 & \text{on } (-\infty, 0) \times \partial\omega_2. \end{cases} \tag{3.12}$$

Notation 23 *We denote*

- *For* $v \in V_+ = \{v \in H^1(\Omega^+) \mid v = 0 \text{ on } \partial\Omega^+ \setminus \{0\} \times \omega_2\}$, $\Omega^+ = (0,1) \times \omega_2$

$$\hat{v}(X_1, X_2) = \begin{cases} v(X_1, X_2), & X_1 \geq 0, X_2 \in \omega_2, \\ v(-X_1, X_2) & X_1 < 0, X_2 \in \omega_2. \end{cases} \tag{3.13}$$

- *For* $v \in V_- = \{v \in H^1(\Omega^-) \mid v = 0 \text{ on } \partial\Omega^- \setminus \{0\} \times \omega_2\}$, $\Omega^- = (-1,0) \times \omega_2$

$$\tilde{v}(X_1, X_2) = \begin{cases} v(-X_1, X_2), & X_1 \geq 0, X_2 \in \omega_2, \\ v(X_1, X_2) & X_1 < 0, X_2 \in \omega_2. \end{cases} \tag{3.14}$$

In the following, we mention some properties of the above formal boundary layer functions (or correctors).

Lemme 3 *For* $N \in \mathbb{N}$, *the following identities hold*

i) For every $v \in V_+$

$$\int_{\Omega^+} \nabla_\varepsilon \theta_N^{[\varepsilon,1]} \cdot \nabla_\varepsilon v \, \mathrm{d}x = -\int_{\Omega^-} \left(\varepsilon^2 \partial_{X_1} \theta_N^{[\varepsilon,1]} \partial_{X_1} \hat{v} + \nabla_{X_2} \theta_N^{[\varepsilon,1]} \cdot \nabla_{X_2} \hat{v} \right) \mathrm{d}x \tag{3.15}$$

ii) For every $v \in V_-$

$$\int_{\Omega^-} \nabla_\varepsilon \theta_N^{[\varepsilon,-1]} \cdot \nabla_\varepsilon v \, \mathrm{d}x = -\int_{\Omega^+} \left(\varepsilon^2 \partial_{X_1} \theta_N^{[\varepsilon,-1]} \partial_{X_1} \tilde{v} + \nabla_{X_2} \theta_N^{[\varepsilon,-1]} \cdot \nabla_{X_2} \tilde{v} \right) \mathrm{d}x. \tag{3.16}$$

Proof.

i) For $l > 0$, we set $\Omega_l^+ = (0, l) \times \omega_2$. Then first note that for $v \in V_+$ we have $\hat{v}\,(1 - \varepsilon X_1, X_2) \in H_0^1(\Omega^+_{\frac{2}{\varepsilon}})$. Then, if we replace the test function in the weak formulation of (3.8) by $\hat{v}(1 - \varepsilon X_1, X_2)$, we have

$$\int_{\Omega^+_{\frac{2}{\varepsilon}}} \nabla w_N^{[1]}(X_1, X_2) \cdot \nabla \hat{v}(1 - \varepsilon X_1, X_2) \, \mathrm{d}x = 0, \tag{3.17}$$

whence

$$\int\limits_{\Omega^+_{\frac{1}{\varepsilon}}} \nabla w^{[1]}_N \cdot \nabla \hat{v}(1-\varepsilon X_1, X_2)\,\mathrm{d}x = - \int\limits_{\Omega^+_{\frac{2}{\varepsilon}} \backslash \Omega^+_{\frac{1}{\varepsilon}}} \nabla w^{[\varepsilon,1]}_N \cdot \nabla \hat{v}(1-\varepsilon X_1, X_2)\,\mathrm{d}x. \tag{3.18}$$

Using (3.7) in (3.18) and we set that $X_1' = 1-\varepsilon X_1$, we obtain respectively

$$\begin{aligned}&\int\limits_{\Omega^+_{\frac{1}{\varepsilon}}} \nabla w^{[1]}_N \cdot \nabla \hat{v}(1-\varepsilon X_1, X_2)\,\mathrm{d}x \\ &= \int\limits_{\Omega^+_{\frac{1}{\varepsilon}}} \partial_{X_1}\theta^{[\varepsilon,1]}_N\left(X_1', X_2\right)\partial_{X_1} v\left(X_1', X_2\right)\,\mathrm{d}x + \int\limits_{\Omega^+_{\frac{1}{\varepsilon}}} \nabla_{X_2}\theta^{[\varepsilon,1]}_N\left(X_1', X_2\right)\cdot \nabla_{X_2} v\left(X_1', X_2\right)\,\mathrm{d}x,\end{aligned} \tag{3.19}$$

and

$$- \int\limits_{\Omega^+_{\frac{2}{\varepsilon}} \backslash \Omega^+_{\frac{1}{\varepsilon}}} \nabla w^{[1]}_N \cdot \nabla \hat{v}(1-\varepsilon X_1, X_2)\,\mathrm{d}x = - \int\limits_{\Omega^+_{\frac{2}{\varepsilon}} \backslash \Omega^+_{\frac{1}{\varepsilon}}} \nabla \theta^{[\varepsilon,1]}_N(X_1', X_2)\cdot \nabla \hat{v}(X_1', X_2)\,\mathrm{d}x. \tag{3.20}$$

Making the change of variable $X_1 \to 1-\varepsilon X_1$ in the integrals above (3.19) and (3.20), we obtain respectively

$$\int\limits_{\Omega^+_{\frac{1}{\varepsilon}}} \nabla w^{[1]}_N \cdot \nabla \hat{v}(1-\varepsilon X_1, X_2)\,\mathrm{d}x = \frac{1}{\varepsilon}\int\limits_{\Omega^+} \nabla_\varepsilon \theta^{[\varepsilon,1]}_N \cdot \nabla_\varepsilon v\,\mathrm{d}x \tag{3.21}$$

and

$$- \int\limits_{\Omega^+_{\frac{2}{\varepsilon}} \backslash \Omega^+_{\frac{1}{\varepsilon}}} \nabla w^{[1]}_N \cdot \nabla \hat{v}(1-\varepsilon X_1, X_2)\,\mathrm{d}x = -\frac{1}{\varepsilon}\int\limits_{\Omega^-} \varepsilon^2 \partial_{X_1}\theta^{[\varepsilon,1]}_N \partial_{X_1}\hat{v} + \nabla_{X_2}\theta^{[\varepsilon,1]}_N \cdot \nabla_{X_2}\hat{v}\,\mathrm{d}x \tag{3.22}$$

We compensate (3.21) and (3.22) in (3.18) this completes the proof of assertion i).

ii) For the second identity in the lemma, we set $\Omega^-_l = (l, 0)\times\omega_2$ with $l < 0$. Then first note that for $v \in V_-$ we have $\tilde{v}(-1-\varepsilon X_1, X_2) \in H^1_0(\Omega^-_{\frac{-2}{\varepsilon}})$. Then if we replace the test function in the weak formulation of (3.12) by $\tilde{v}(-1-\varepsilon X_1, X_2)$, we get

$$\int_{\Omega^-_{\frac{-2}{\varepsilon}}} \nabla w_N^{[-1]}(X_1,X_2)\cdot\nabla\tilde{v}(-1-\varepsilon X_1,X_2)\,\mathrm{d}x = 0, \tag{3.23}$$

whence

$$\int_{\Omega^-_{\frac{-1}{\varepsilon}}} \nabla w_N^{[-1]}\cdot\nabla\tilde{v}(-1-\varepsilon X_1,X_2)\,\mathrm{d}x = -\int_{\Omega^-_{\frac{-2}{\varepsilon}}\setminus\Omega^-_{\frac{-1}{\varepsilon}}} \nabla w_N^{[-1]}\cdot\nabla\tilde{v}(-1-\varepsilon X_1,X_2)\,\mathrm{d}x. \tag{3.24}$$

Using (3.11) in (3.24) and setting that $X_1' = -1-\varepsilon X_1$, we get respectively

$$\begin{aligned}\int_{\Omega^-_{\frac{-1}{\varepsilon}}} \nabla w_N^{[-1]} &\cdot\nabla\tilde{v}(-1-\varepsilon X_1,X_2)\,\mathrm{d}x\\ &= \int_{\Omega^-_{\frac{-1}{\varepsilon}}} \partial_{X_1}\theta_N^{[\varepsilon,-1]}\left(X_1',X_2\right)\partial_{X_1}v\left(X_1',X_2\right)\,\mathrm{d}x\\ &\quad+\int_{\Omega^-_{\frac{-1}{\varepsilon}}} \nabla_{X_2}\theta_N^{[\varepsilon,-1]}\left(X_1',X_2\right)\cdot\nabla_{X_2}v\left(X_1',X_2\right)\,\mathrm{d}x,\end{aligned} \tag{3.25}$$

and

$$\int_{\Omega^-_{\frac{-2}{\varepsilon}}\setminus\Omega^-_{\frac{-1}{\varepsilon}}} \nabla w_N^{[-1]}\cdot\nabla\tilde{v}(X_1',X_2)\,\mathrm{d}x = \int_{\Omega^-_{\frac{-2}{\varepsilon}}\setminus\Omega^-_{\frac{-1}{\varepsilon}}} \nabla\theta_N^{[\varepsilon,-1]}(X_1',X_2)\cdot\nabla\tilde{v}(X_1',X_2)\,\mathrm{d}x. \tag{3.26}$$

Making the change of variable $X_1 \to -1-\varepsilon X_1$ in (3.25) and (3.26), we have respectively

$$\int_{\Omega^-_{\frac{-1}{\varepsilon}}} \nabla w_N^{[-1]}\cdot\nabla\tilde{v}(-1-\varepsilon X_1,X_2)\,\mathrm{d}x = \frac{1}{\varepsilon}\int_{\Omega^-} \nabla_\varepsilon\theta_N^{[\varepsilon,-1]}\cdot\nabla_\varepsilon v\,\mathrm{d}x, \tag{3.27}$$

and

$$\int_{\Omega^-_{\frac{-2}{\varepsilon}}\setminus\Omega^-_{\frac{-1}{\varepsilon}}} \nabla w_N^{[-1]}\cdot\nabla\tilde{v}(X_1',X_2)\,\mathrm{d}x = \frac{1}{\varepsilon}\int_{\Omega^+} \varepsilon^2\partial_{X_1}\theta_N^{[\varepsilon,-1]}\partial_{X_1}\tilde{v} + \nabla_{X_2}\theta_N^{[\varepsilon,-1]}\cdot\nabla_{X_2}\tilde{v}\,\mathrm{d}x. \tag{3.28}$$

We compensate (3.27) and (3.28) in (3.24).This completes the proof of the lemma. ■

Lemme 4 *There exist positive constants* $C, \alpha > 0$ *independent of* ε *such that for every* $N \in \mathbb{N}$*, we have*

$$\int_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \leq Ce^{\frac{-\alpha}{\varepsilon}} \int_{S^+_0} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \tag{3.29}$$

and

$$\int_{S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x \leq Ce^{\frac{-\alpha}{\varepsilon}} \int_{S^-_0} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x. \tag{3.30}$$

Proof. Without loss of generality, we assume that $\varepsilon < 1$.

i) Let $\gamma^+_\varepsilon : [0, +\infty) \longrightarrow [0, 1]$ be a continuous function such that

$$\gamma^+_\varepsilon(x) = \begin{cases} 0 & \text{on } (0, \frac{1}{\varepsilon} - 1), \\ x + 1 - \frac{1}{\varepsilon} & \text{on } \left[\frac{1}{\varepsilon} - 1, \frac{1}{\varepsilon}\right], \\ 1 & \text{on } (\frac{1}{\varepsilon}, +\infty). \end{cases}$$

Since $\gamma^+_\varepsilon(X_1) w_N^{[1]} \in H^1_0(S^+_0)$, we have

$$\int_{S^+_0} \nabla w_N^{[1]} \cdot \nabla \left(\gamma^+_\varepsilon(X_1) w_N^{[1]}\right) \mathrm{d}x = 0. \tag{3.31}$$

Thus

$$\int_{S^+_0} \nabla w_N^{[1]} \cdot \nabla w_N^{[1]} \gamma^+_\varepsilon(X_1)\, \mathrm{d}x = - \int_{S^+_{\frac{1}{\varepsilon}-1} \setminus S^+_{\frac{1}{\varepsilon}}} \partial_{X_1} w_N^{[1]} \partial_{X_1} \gamma^+_\varepsilon(X_1) w_N^{[1]}\, \mathrm{d}x$$

Then, we have

$$\int_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \leq \int_0^{1/\varepsilon} \int_{\omega_2} \left|\nabla w_N^{[1]}\right|^2 \gamma^+_\varepsilon(X_1)\, \mathrm{d}x + \int_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \leq \int_{S^+_{\frac{1}{\varepsilon}-1} \setminus S^+_{\frac{1}{\varepsilon}}} \left|\partial_{X_1} w_N^{[1]}\right| \left|w_N^{[1]}\right| \mathrm{d}x.$$

Young's inequality

$$\int_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \leq \frac{1}{2} \int_{S^+_{\frac{1}{\varepsilon}-1} \setminus S^+_{\frac{1}{\varepsilon}}} \left|\partial_{X_1} w_N^{[1]}\right|^2 \mathrm{d}x + \frac{1}{2} \int_{S^+_{\frac{1}{\varepsilon}-1} \setminus S^+_{\frac{1}{\varepsilon}}} \left|w_N^{[1]}\right|^2 \mathrm{d}x.$$

Applying the Poincaré inequality in X_2-direction, we get

$$\begin{aligned}\int\limits_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x &\leq \frac{1}{2} \int\limits_{S^+_{\frac{1}{\varepsilon}-1} \setminus S^+_{\frac{1}{\varepsilon}}} \left|\partial_{X_1} w_N^{[1]}\right|^2 \mathrm{d}x + \frac{C_{\omega_2}}{2} \int\limits_{S^+_{\frac{1}{\varepsilon}-1} \setminus S^+_{\frac{1}{\varepsilon}}} \left|\nabla_{X_2} w_N^{[1]}\right|^2 \mathrm{d}x \\ &\leq \frac{(1+C_{\omega_2})}{2} \int\limits_{S^+_{\frac{1}{\varepsilon}-1} \setminus S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \\ &= \frac{(1+C_{\omega_2})}{2} \int\limits_{S^+_{\frac{1}{\varepsilon}-1}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x - \frac{(1+C_{\omega_2})}{2} \int\limits_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x\end{aligned}$$

so that

$$\int\limits_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \leq r \int\limits_{S^+_{\frac{1}{\varepsilon}-1}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x,$$

where $r = \frac{\left(1+C_{\omega_2}\right)}{2+\left(1+C_{\omega_2}\right)} < 1$. Iterating $[\frac{1}{\varepsilon}]$ times this formula, we then obtain

$$\int\limits_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \leq r^{[\frac{1}{\varepsilon}]} \int\limits_{S^+_{\frac{1}{\varepsilon}-[\frac{1}{\varepsilon}]}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x.$$

Since $\frac{1}{\varepsilon} - 1 < [\frac{1}{\varepsilon}] \leq \frac{1}{\varepsilon}$, we deduce

$$\begin{aligned}\int\limits_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x &\leq r^{\frac{1}{\varepsilon}-1} \int\limits_{S^+_{\frac{1}{\varepsilon}-[\frac{1}{\varepsilon}]}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \leq \frac{1}{r} e^{\frac{1}{\varepsilon}\ln r} \int\limits_{S^+_0} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x \\ &\leq C e^{\frac{-\alpha}{\varepsilon}} \int\limits_{S^+_0} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x.\end{aligned}$$

where $\alpha = -\ln r$.

ii) Let $\gamma_\varepsilon^- : (-\infty, 0] \longrightarrow [0,1]$ be a continuous function such that

$$\gamma_\varepsilon^-(x) = \begin{cases} 1 & \text{on } (-\infty, \frac{-1}{\varepsilon}), \\ -x+1-\frac{1}{\varepsilon} & \text{on } \left[\frac{-1}{\varepsilon}, \frac{-1}{\varepsilon}+1\right], \\ 0 & \text{on } (\frac{-1}{\varepsilon}+1, 0). \end{cases}$$

Since $\gamma_\varepsilon^-(X_1) w_N^{[-1]} \in H_0^1(S_0^-)$, we have

$$\int\limits_{S_0^-} \nabla w_N^{[-1]} \cdot \nabla \left(\gamma_\varepsilon^-(X_1) w_N^{[-1]}\right) \mathrm{d}x = 0. \tag{3.32}$$

Thus

$$\int_{S_0^-} \nabla w_N^{[-1]} \cdot \nabla w_N^{[-1]} \gamma_\varepsilon^-(X_1)\, \mathrm{d}x = - \int_{S^-_{\frac{-1}{\varepsilon}+1} \setminus S^-_{\frac{-1}{\varepsilon}}} \partial_{X_1} w_N^{[-1]} \partial_{X_1} \gamma_\varepsilon^-(X_1) w_N^{[-1]}\, \mathrm{d}x.$$

So, we have

$$\int_{S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x \leq \int_{S^-_{\frac{-1}{\varepsilon}+1} \setminus S^-_{\frac{-1}{\varepsilon}}} \left|\partial_{X_1} w_N^{[-1]}\right| \left|w_N^{[-1]}\right| \mathrm{d}x.$$

Young's inequality

$$\int_{S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x \leq \frac{1}{2} \int_{S^-_{\frac{-1}{\varepsilon}+1} \setminus S^-_{\frac{1}{\varepsilon}}} \left|\partial_{X_1} w_N^{[-1]}\right|^2 \mathrm{d}x + \frac{1}{2} \int_{S^-_{\frac{-1}{\varepsilon}+1} \setminus S^-_{\frac{-1}{\varepsilon}}} \left|w_N^{[-1]}\right|^2 \mathrm{d}x.$$

Applying the Poincaré inequality in X_2-direction, we obtain

$$\begin{aligned}
\int_{S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x &\leq \frac{1}{2} \int_{S^-_{\frac{-1}{\varepsilon}+1} \setminus S^-_{\frac{-1}{\varepsilon}}} \left|\partial_{X_1} w_N^{[-1]}\right|^2 \mathrm{d}x + \frac{C_{\omega_2}}{2} \int_{S^-_{\frac{-1}{\varepsilon}+1} \setminus S^-_{\frac{-1}{\varepsilon}}} \left|\nabla_{X_2} w_N^{[-1]}\right|^2 \mathrm{d}x \\
&\leq \frac{(1+C_{\omega_2})}{2} \int_{S^-_{\frac{-1}{\varepsilon}+1} \setminus S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x \\
&= \frac{(1+C_{\omega_2})}{2} \int_{S^-_{\frac{-1}{\varepsilon}+1}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x - \frac{(1+C_{\omega_2})}{2} \int_{S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x,
\end{aligned}$$

and thus

$$\int_{S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x \leq r \int_{S^-_{\frac{-1}{\varepsilon}+1}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x,$$

where $r = \frac{\left(1+C_{\omega_2}\right)}{2+\left(1+C_{\omega_2}\right)}$. Iterating $[\frac{1}{\varepsilon}]$ times this formula, we then obtain

$$\int_{S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x \leq r^{[\frac{1}{\varepsilon}]} \int_{S^-_{\frac{-1}{\varepsilon}+[\frac{1}{\varepsilon}]}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x.$$

Since $\frac{1}{\varepsilon} - 1 < [\frac{1}{\varepsilon}] \leq \frac{1}{\varepsilon}$ and $r < 1$, we deduce

$$\begin{aligned}\int\limits_{S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x &\leq r^{\frac{1}{\varepsilon}-1} \int\limits_{S^-_{\frac{-1}{\varepsilon}+[\frac{1}{\varepsilon}]}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x \\ &\leq Ce^{\frac{-\alpha}{\varepsilon}} \int\limits_{S_0^-} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x.\end{aligned}$$

This completes the proof of the lemma. ■

Lemme 5 *There exist positive constants C, $\alpha > 0$ independent of ε such that for every $N \in \mathbb{N}$, we have*

$$\int\limits_{\Omega} |\nabla_\varepsilon \Phi_N(\cdot;\varepsilon)|^2 \,\mathrm{d}x \leq Ce^{\frac{-\alpha}{\varepsilon}} \left\{ \int\limits_{S_0^+} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x + \int\limits_{S_0^-} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x \right\} \tag{3.33}$$

where $\Phi_N(x;\varepsilon) = \breve{\rho}_{[-1]}(X_1)\theta_N^{[\varepsilon,1]}(X_1,X_2) + \breve{\rho}_{[1]}(X_1)\theta_N^{[\varepsilon,-1]}(X_1,X_2)$ in Ω.

Proof. From the definition of $\Phi_N(\cdot;\varepsilon)$, we have

$$\int\limits_{\Omega} |\nabla_\varepsilon \Phi_N(x;\varepsilon)|^2 \,\mathrm{d}x = \int\limits_{\Omega^-} \left|\nabla_\varepsilon \left\{\tilde{\rho}_{[-1]}(X_1)\theta_N^{[\varepsilon,1]}(x)\right\}\right|^2 \mathrm{d}x + \int\limits_{\Omega^+} \left|\nabla_\varepsilon \left\{\tilde{\rho}_{[1]}(X_1)\theta_N^{[\varepsilon,-1]}(x)\right\}\right|^2 \mathrm{d}x. \tag{3.34}$$

For the first term of the right-hand side of (3.34), we obtain

$$\begin{aligned} T_1 &= \int\limits_{\Omega^-} \left|\nabla_\varepsilon \left\{\tilde{\rho}_{[-1]}(X_1)\theta_N^{[\varepsilon,1]}(X_1,X_2)\right\}\right|^2 \mathrm{d}x \\ &= \int\limits_{\Omega^-} \left(\varepsilon^2 \left[\partial_{X_1}\left\{\tilde{\rho}_{[-1]}(X_1)\theta_N^{[\varepsilon,1]}(X_1,X_2)\right\}\right]^2 + \left|\tilde{\rho}_{[-1]}(X_1)\nabla_{X_2}\theta_N^{[\varepsilon,1]}(X_1,X_2)\right|^2\right) \mathrm{d}x \\ &= \int\limits_{\Omega^-} \left(\varepsilon^2 \left[\partial_{X_1}\tilde{\rho}_{[-1]}(X_1)\theta_N^{[\varepsilon,1]} + \tilde{\rho}_{[-1]}(X_1)\partial_{X_1}\theta_N^{[\varepsilon,1]}\right]^2 + \left|\tilde{\rho}_{[-1]}(X_1)\nabla_{X_2}\theta_N^{[\varepsilon,1]}\right|^2\right) \mathrm{d}x. \end{aligned}$$

So

$$\begin{aligned} T_1 \leq\ & 2\varepsilon^2 \int\limits_{\Omega^-} \left|\partial_{X_1}\tilde{\rho}_{[-1]}(X_1)\right|^2 \left|\theta_N^{[\varepsilon,1]}\right|^2 \mathrm{d}x + 2\varepsilon^2 \int\limits_{\Omega^-} \left|\tilde{\rho}_{[-1]}(X_1)\right|^2 \left|\partial_{X_1}\theta_N^{[\varepsilon,1]}\right|^2 \mathrm{d}x \\ & + \int\limits_{\Omega^-} \left|\tilde{\rho}_{[-1]}(X_1)\right|^2 \left|\nabla_{X_2}\theta_N^{[\varepsilon,1]}\right|^2 \mathrm{d}x. \end{aligned}$$

Using the Poincaré inequality in the above inequality, we obtain

$$\begin{aligned} T_1 &\leq 2C_{\omega_2} \left|\nabla_{X_2}\theta_N^{[\varepsilon,1]}(X_1,X_2)\right|^2_{2,\Omega^-} + 2\varepsilon^2 \left|\partial_{X_1}\theta_N^{[\varepsilon,1]}(X_1,X_2)\right|^2_{2,\Omega^-} + \left|\nabla_{X_2}\theta_N^{[\varepsilon,1]}(X_1,X_2)\right|^2_{2,\Omega^-} \\ &\leq 2\varepsilon^2 \left|\partial_{X_1}\theta_N^{[\varepsilon,1]}\right|^2_{2,\Omega^-} + (2C_{\omega_2}+1)\left|\nabla_{X_2}\theta_N^{[\varepsilon,1]}\right|^2_{2,\Omega^-} \\ &\leq 2(C_{\omega_2}+1)\int_{\Omega^-}\left(\varepsilon^2\left|\partial_{X_1}\theta_N^{[\varepsilon,1]}\right|^2 + \left|\nabla_{X_2}\theta_N^{[\varepsilon,1]}\right|^2\right)\mathrm{d}x. \end{aligned}$$

Making the change of variable $X_1 \to \frac{1-X_1}{\varepsilon}$, we get

$$T_1 \leq 2(C_{\omega_2}+1)\varepsilon \int_{S^+_{\frac{1}{\varepsilon}}} \left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x.$$

Using (3.29), we obtain

$$\int_{\Omega^-}\left|\nabla_\varepsilon\left\{\tilde{\rho}_{[-1]}(X_1)\theta_N^{[\varepsilon,1]}(X_1,X_2)\right\}\right|^2 \mathrm{d}x \leq Ce^{\frac{-\alpha}{\varepsilon}}\int_{S_0^+}\left|\nabla w_N^{[1]}\right|^2 \mathrm{d}x. \tag{3.35}$$

Now, for the second term of the right-hand side of (3.34), we get

$$\begin{aligned} T_2 &= \int_{\Omega^+}\left|\nabla_\varepsilon\left\{\tilde{\rho}_{[1]}(X_1)\theta_N^{[\varepsilon,-1]}(X_1,X_2)\right\}\right|^2 \mathrm{d}x \\ &= \int_{\Omega^+}\left(\varepsilon^2\left[\partial_{X_1}\left\{\tilde{\rho}_{[1]}(X_1)\theta_N^{[\varepsilon,-1]}(X_1,X_2)\right\}\right]^2 + \left|\tilde{\rho}_{[1]}(X_1)\nabla_{X_2}\theta_N^{[\varepsilon,-1]}(X_1,X_2)\right|^2\right)\mathrm{d}x \\ &= \int_{\Omega^+}\left(\varepsilon^2\left[\partial_{X_1}\tilde{\rho}_{[1]}(X_1)\theta_N^{[\varepsilon,-1]} + \tilde{\rho}_{[1]}(X_1)\partial_{X_1}\theta_N^{[\varepsilon,-1]}\right]^2 + \left|\tilde{\rho}_{[1]}(X_1)\nabla_{X_2}\theta_N^{[\varepsilon,-1]}\right|^2\right)\mathrm{d}x, \end{aligned}$$

So

$$\begin{aligned} T_2 \leq\ & 2\varepsilon^2\int_{\Omega^+}\left|\partial_{X_1}\tilde{\rho}_{[1]}(X_1)\right|^2\left|\theta_N^{[\varepsilon,-1]}\right|^2 \mathrm{d}x + 2\varepsilon^2\int_{\Omega^+}\left|\tilde{\rho}_{[1]}(X_1)\right|^2\left|\partial_{X_1}\theta_N^{[\varepsilon,-1]}\right|^2 \mathrm{d}x \\ &+ \int_{\Omega^+}\left|\tilde{\rho}_{[1]}(X_1)\right|^2\left|\nabla_{X_2}\theta_N^{[\varepsilon,-1]}\right|^2 \mathrm{d}x. \end{aligned}$$

Using the Poincaré inequality,

$$\begin{aligned} T_2 &\leq 2C_{\omega_2}\left|\nabla_{X_2}\theta_N^{[\varepsilon,-1]}(X_1,X_2)\right|^2_{2,\Omega^+} + 2\varepsilon^2\left|\partial_{X_1}\theta_N^{[\varepsilon,-1]}(X_1,X_2)\right|^2_{2,\Omega^+} + \left|\nabla_{X_2}\theta_N^{[\varepsilon,-1]}(X_1,X_2)\right|^2_{2,\Omega^+} \\ &\leq 2\varepsilon^2\left|\partial_{X_1}\theta_N^{[\varepsilon,-1]}(X_1,X_2)\right|^2_{2,\Omega^+} + (2C_{\omega_2}+1)\left|\nabla_{X_2}\theta_N^{[\varepsilon,-1]}(X_1,X_2)\right|^2_{2,\Omega^+} \\ &\leq 2(C_{\omega_2}+1)\int_{\Omega^+}\left(\varepsilon^2\left|\partial_{X_1}\theta_N^{[\varepsilon,-1]}\right|^2 + \left|\nabla_{X_2}\theta_N^{[\varepsilon,-1]}\right|^2\right)\mathrm{d}x. \end{aligned}$$

Making the change of variable $X_1 \to \frac{-1-X_1}{\varepsilon}$, we obtain

$$T_2 \leq 2(C_{\omega_2}+1)\varepsilon \int\limits_{S^-_{\frac{-1}{\varepsilon}}} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x.$$

Using (3.30), we get

$$\int\limits_{\Omega^+} \left|\nabla_\varepsilon \left\{\tilde{\rho}_{[1]}(X_1)\theta_N^{[\varepsilon,-1]}(X_1,X_2)\right\}\right|^2 \mathrm{d}x \leq Ce^{\frac{-\alpha}{\varepsilon}} \int\limits_{S_0^-} \left|\nabla w_N^{[-1]}\right|^2 \mathrm{d}x. \tag{3.36}$$

We compensate (3.35) and (3.36) in (3.34). This completes the proof of the lemma. ■

Remark 24 *There exists, for every $N \in \mathbb{N}$, a positive constant $C>0$ independent of ε such that*

$$\int\limits_{S_0^+} \left|\nabla_{X_2} w_N^{[1]}\right|^2 \mathrm{d}x \leq C, \tag{3.37}$$

$$\int\limits_{S_0^-} \left|\nabla_{X_2} w_N^{[-1]}\right|^2 \mathrm{d}x \leq C. \tag{3.38}$$

Indeed, it is clear by (3.8) (resp. (3.12)).

Remark 25 *From what we done above, we can easily show that*

$$\left(u_N + \theta_N^{[\varepsilon,1]} + \theta_N^{[\varepsilon,-1]} - \Phi_N\right) \in H_0^1(\Omega), \qquad N \in \mathbb{N}.$$

Now, we can show the theorem below

Theorem 26 *Let u_ε and u_N, $N \geq 0$, be the solutions to (2.3) and (2.10). Then, there exist two constants $C, \alpha > 0$ independent of ε, such that*

$$\begin{aligned}\frac{3}{4}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|^2_{2,\Omega} \leq\ & Ce^{\frac{-\alpha}{\varepsilon}} \sum_{i=0}^{N} \left\{ \int\limits_{S_0^+} \left|\nabla w_i^{[1]}\right|^2 \mathrm{d}x + \int\limits_{S_0^-} \left|\nabla w_i^{[-1]}\right|^2 \mathrm{d}x \right\} \\ & -\varepsilon^{N+2} \int\limits_{\Omega} \partial_{X_1} u_N \partial_{X_1} RW_N(x;\varepsilon)\,\mathrm{d}x,\end{aligned} \tag{3.39}$$

where $RW_N(\cdot;\varepsilon) = R_N(\cdot;\varepsilon) - \sum_{i=0}^{N/2} \varepsilon^{2i}\left[\theta_{2i}^{[\varepsilon,1]} + \theta_{2i}^{[\varepsilon,-1]} - \Phi_{2i}(\cdot;\varepsilon)\right] \in H_0^1(\Omega)$.

Proof. Taking $v = RW_N(\cdot;\varepsilon) \in H_0^1(\Omega)$ in (2.13), we get

$$\varepsilon^2 \int\limits_{\Omega} \partial_{X_1} R_{N-2}(x;\varepsilon) \partial_{X_1} RW_N(x;\varepsilon)\,\mathrm{d}x + \int\limits_{\Omega} \nabla_{X_2} R_N(x;\varepsilon) \cdot \nabla_{X_2} RW_N(x;\varepsilon)\,\mathrm{d}x = 0.$$

Then

$$\begin{aligned}
\int\limits_{\Omega} \nabla_\varepsilon RW_N(x;\varepsilon) \cdot \nabla_\varepsilon RW_N(x;\varepsilon)\,\mathrm{d}x &= -\varepsilon^{N+2} \int\limits_{\Omega} \partial_{X_1} u_N \partial_{X_1} RW_N(x;\varepsilon)\,\mathrm{d}x \\
&\quad - \sum_{i=0}^{N} \varepsilon^i \int\limits_{\Omega} \nabla_\varepsilon \theta_i^{[\varepsilon,1]} \cdot \nabla_\varepsilon RW_N(x;\varepsilon)\,\mathrm{d}x \\
&\quad - \sum_{i=0}^{N} \varepsilon^i \int\limits_{\Omega} \nabla_\varepsilon \theta_i^{[\varepsilon,-1]} \cdot \nabla_\varepsilon RW_N(x;\varepsilon)\,\mathrm{d}x \\
&\quad + \sum_{i=0}^{N} \varepsilon^i \int\limits_{\Omega} \nabla_\varepsilon \Phi_i(x;\varepsilon) \cdot \nabla_\varepsilon RW_N(x;\varepsilon)\,\mathrm{d}x,
\end{aligned}$$

whence

$$\begin{aligned}
|\nabla_\varepsilon RW_N(\cdot;\varepsilon)|_{2,\Omega}^2 &= -\varepsilon^{N+2} \int\limits_{\Omega} \partial_{X_1} u_N \partial_{X_1} RW_N(.;\varepsilon)\,\mathrm{d}x \\
&\quad - \sum_{i=0}^{N} \varepsilon^i \left\{ \int\limits_{\Omega^-} \nabla_\varepsilon \theta_i^{[\varepsilon,1]} \cdot \nabla_\varepsilon RW_N(.;\varepsilon)\,\mathrm{d}x + \int\limits_{\Omega^+} \nabla_\varepsilon \theta_i^{[\varepsilon,1]} \cdot \nabla_\varepsilon RW_N(.;\varepsilon)\,\mathrm{d}x \right\} \\
&\quad - \sum_{i=0}^{N} \varepsilon^i \left\{ \int\limits_{\Omega^-} \nabla_\varepsilon \theta_i^{[\varepsilon,-1]} \cdot \nabla_\varepsilon RW_N(.;\varepsilon)\,\mathrm{d}x + \int\limits_{\Omega^+} \nabla_\varepsilon \theta_i^{[\varepsilon,-1]} \cdot \nabla_\varepsilon RW_N(.;\varepsilon)\,\mathrm{d}x \right\} \\
&\quad + \sum_{i=0}^{N} \varepsilon^i \int\limits_{\Omega} \nabla_\varepsilon \Phi_i(.;\varepsilon) \cdot \nabla_\varepsilon RW_N(.;\varepsilon)\,\mathrm{d}x.
\end{aligned}$$

Using (3.15) and (3.16) -since $RW_N(\cdot;\varepsilon) \in V_+$, $RW_N(\cdot;\varepsilon) \in V_-$, we get

$$
\begin{aligned}
|\nabla_\varepsilon RW_N(\cdot;\varepsilon)|^2_{2,\Omega} \;=\; & -\varepsilon^{N+2}\int\limits_{\Omega} \partial_{X_1} u_N \partial_{X_1} RW_N(.;\varepsilon)\,\mathrm{d}x \\
& -\sum_{i=0}^{N}\varepsilon^i \int\limits_{\Omega^-} \nabla_\varepsilon \theta_i^{[\varepsilon,1]}\cdot\nabla_\varepsilon RW_N(.;\varepsilon)\,\mathrm{d}x + \sum_{i=0}^{N}\varepsilon^i \int\limits_{\Omega^-} \nabla_\varepsilon \theta_i^{[\varepsilon,1]}\cdot\nabla_\varepsilon R\widehat{W_N(.;}\varepsilon)\,\mathrm{d}x \\
& +\sum_{i=0}^{N}\varepsilon^i \int\limits_{\Omega^+} \nabla_\varepsilon \theta_i^{[\varepsilon,-1]}\cdot\nabla R\widetilde{W_N(.;}\varepsilon)\,\mathrm{d}x - \sum_{i=0}^{N}\varepsilon^i \int\limits_{\Omega^+} \nabla_\varepsilon \theta_i^{[\varepsilon,-1]}\cdot\nabla_\varepsilon RW_N(.;\varepsilon)\,\mathrm{d}x \\
& +\sum_{i=0}^{N}\varepsilon^i \int\limits_{\Omega} \nabla_\varepsilon \Phi_i(x;\varepsilon)\cdot\nabla_\varepsilon RW_N(;\varepsilon)\,\mathrm{d}x,
\end{aligned}
$$

Using the Cauchy-Schwarz inequality for the three last lines, we obtain

$$
\begin{aligned}
|\nabla_\varepsilon RW_N(\cdot;\varepsilon)|^2_{2,\Omega} \;\le\; & -\varepsilon^{N+2}\int\limits_{\Omega} \partial_{X_1} u_N \partial_{X_1} RW_N(x;\varepsilon)\,\mathrm{d}x \\
& +\sum_{i=0}^{N}\varepsilon^i \left|\nabla_\varepsilon \theta_i^{[\varepsilon,1]}\right|_{2,\Omega^-} |\nabla_\varepsilon RW_N(\cdot;\varepsilon)|_{2,\Omega^-} \\
& +\sum_{i=0}^{N}\varepsilon^i \left|\nabla_\varepsilon \theta_i^{[\varepsilon,1]}\right|_{2,\Omega^-} \left|\nabla_\varepsilon R\widehat{W_N(\cdot;}\varepsilon)\right|_{2,\Omega^-} \\
& +\sum_{i=0}^{N}\varepsilon^i \left|\nabla_\varepsilon \theta_i^{[\varepsilon,-1]}\right|_{2,\Omega^+} \left|\nabla_\varepsilon R\widetilde{W_N(\cdot;}\varepsilon)\right|_{2,\Omega^+} \\
& +\sum_{i=0}^{N}\varepsilon^i \left|\nabla_\varepsilon \theta_i^{[\varepsilon,-1]}\right|_{2,\Omega^+} |\nabla_\varepsilon RW_N(\cdot;\varepsilon)|_{2,\Omega^+} \\
& +\sum_{i=0}^{N}\varepsilon^i |\nabla_\varepsilon \Phi_i(\cdot;\varepsilon)|_{2,\Omega} |\nabla_\varepsilon RW_N(\cdot;\varepsilon)|_{2,\Omega}\,.
\end{aligned}
$$

Using Young's inequality, we have

$$
\begin{aligned}
|\nabla_\varepsilon RW_N(\cdot;\varepsilon)|_{2,\Omega}^2 \leq\ & -\varepsilon^{N+2}\int\limits_\Omega \partial_{X_1}u_N\partial_{X_1}RW_N(x;\varepsilon)\,\mathrm{d}x \\
& + \sum_{i=0}^{N}\left(\mu\left|\nabla_\varepsilon\theta_i^{[\varepsilon,1]}\right|_{2,\Omega^-}^2 + \frac{1}{4\mu}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|_{2,\Omega^-}^2\right) \\
& + \sum_{i=0}^{N}\left(\mu\left|\nabla_\varepsilon\theta_i^{[\varepsilon,1]}\right|_{2,\Omega^-}^2 + \frac{1}{4\mu}\left|\nabla_\varepsilon \widehat{RW_N(\cdot;\varepsilon)}\right|_{2,\Omega^-}^2\right) \\
& + \sum_{i=0}^{N}\left(\mu\left|\nabla_\varepsilon\theta_i^{[\varepsilon,-1]}\right|_{2,\Omega^+}^2 + \frac{1}{4\mu}\left|\nabla_\varepsilon \widetilde{RW_N(\cdot;\varepsilon)}\right|_{2,\Omega^+}^2\right) \\
& + \sum_{i=0}^{N}\left(\mu\left|\nabla_\varepsilon\theta_i^{[\varepsilon,-1]}\right|_{2,\Omega^+}^2 + \frac{1}{4\mu}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|_{2,\Omega^+}^2\right) \\
& + \sum_{i=0}^{N}\left(\mu\left|\nabla_\varepsilon\Phi_i(\cdot;\varepsilon)\right|_{2,\Omega}^2 + \frac{1}{4\mu}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|_{2,\Omega}^2\right).
\end{aligned}
$$

Choosing $\mu = 3(N+1)$, we get

$$
\begin{aligned}
\frac{3}{4}|\nabla_\varepsilon RW_N(\cdot;\varepsilon)|_{2,\Omega}^2 \leq\ & -\varepsilon^{N+2}\int\limits_\Omega \partial_{X_1}u_N\partial_{X_1}RW_N(x;\varepsilon)\,\mathrm{d}x \\
& + \sum_{i=0}^{N} 3(N+1)\left|\nabla_\varepsilon\Phi_i(\cdot;\varepsilon)\right|_{2,\Omega}^2 \\
& + \sum_{i=0}^{N} 6(N+1)\left|\nabla_\varepsilon\theta_i^{[\varepsilon,1]}\right|_{2,\Omega^-}^2 + \sum_{i=0}^{N} 6(N+1)\left|\nabla_\varepsilon\theta_i^{[\varepsilon,-1]}\right|_{2,\Omega^+}^2 .
\end{aligned}
$$

Using (3.33) and making the change of variable $X_1 \to \frac{1-X_1}{\varepsilon}$ (resp. $X_1 \to \frac{-1-X_1}{\varepsilon}$) in the last line, we obtain

$$
\begin{aligned}
\frac{3}{4}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|^2_{2,\Omega} \;\le\; & -\varepsilon^{N+2}\int\limits_{\Omega}\partial_{X_1}u_N\partial_{X_1}RW_N(x;\varepsilon)\,\mathrm{d}x \\
& +\sum_{i=0}^{N}3(N+1)Ce^{\frac{-\alpha}{\varepsilon}}\left\{\int\limits_{S^+_{\frac{1}{\varepsilon}}}\left|\nabla w_i^{[1]}\right|^2\mathrm{d}x+\int\limits_{S^-_{\frac{-1}{\varepsilon}}}\left|\nabla w_i^{[-1]}\right|^2\mathrm{d}x\right\} \\
& +\sum_{i=0}^{N}6(N+1)\int\limits_{S^+_{\frac{1}{\varepsilon}}}\left|\nabla w_i^{[1]}\right|^2\mathrm{d}x \\
& +\sum_{i=0}^{N}6(N+1)\int\limits_{S^-_{\frac{-1}{\varepsilon}}}\left|\nabla w_i^{[-1]}\right|^2\mathrm{d}x.
\end{aligned}
$$

Using (3.29) and (3.30), we get

$$
\begin{aligned}
\frac{3}{4}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|^2_{2,\Omega} \;\le\; & -\varepsilon^{N+2}\int\limits_{\Omega}\partial_{X_1}u_N\partial_{X_1}RW_N(x;\varepsilon)\,\mathrm{d}x \\
& +\sum_{i=0}^{N}3(N+1)Ce^{\frac{-\alpha}{\varepsilon}}\left\{\int\limits_{S^+_0}\left|\nabla w_i^{[1]}\right|^2\mathrm{d}x+\int\limits_{S^-_0}\left|\nabla w_i^{[-1]}\right|^2\mathrm{d}x\right\} \\
& +\sum_{i=0}^{N}6(N+1))Ce^{\frac{-\alpha}{\varepsilon}}\int\limits_{S^+_0}\left|\nabla w_i^{[1]}\right|^2\mathrm{d}x \\
& +\sum_{i=0}^{N}6(N+1)Ce^{\frac{-\alpha}{\varepsilon}}\int\limits_{S^-_0}\left|\nabla w_i^{[-1]}\right|^2\mathrm{d}x,
\end{aligned}
$$

so that

$$
\begin{aligned}
\frac{3}{4}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|^2_{2,\Omega} \;\le\; & Ce^{\frac{-\alpha}{\varepsilon}}\sum_{i=0}^{N}\left\{\int\limits_{S^+_0}\left|\nabla w_i^{[1]}\right|^2\mathrm{d}x+\int\limits_{S^-_0}\left|\nabla w_i^{[-1]}\right|^2\mathrm{d}x\right\} \\
& -\varepsilon^{N+2}\int\limits_{\Omega}\partial_{X_1}u_N\partial_{X_1}RW_N(x;\varepsilon)\,\mathrm{d}x.
\end{aligned}
$$

This completes the proof of the theorem. ■

Now we turn to our composite asymptotic expansion where we start by the first composite asymptotic approximation on the whole domain Ω in the following section.

3.2 First Composite Asymptotic Approximation

As the first convergence results, we have

Theorem 27 *Under the sufficient assumptions on the data, we have the following error estimate:*

$$\left| u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} - \theta_0^{[\varepsilon,-1]} \right|_{2,\Omega} = O(\varepsilon), \quad \left| \nabla_{X_2} \left(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} - \theta_0^{[\varepsilon,-1]} \right) \right|_{2,\Omega} = O(\varepsilon), \tag{3.40}$$

and

$$\left| \partial_{X_1} \left(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} - \theta_0^{[\varepsilon,-1]} \right) \right|_{2,\Omega} = O(1). \tag{3.41}$$

Proof. Using (3.39) reads for $N = 0$, we get

$$\frac{3}{4} \left| \nabla_\varepsilon RW_0(\cdot;\varepsilon) \right|_{2,\Omega}^2 \leq Ce^{\frac{-\alpha}{\varepsilon}} \left\{ \int_{S_0^+} \left| \nabla w_0^{[1]} \right|^2 \mathrm{d}x + \int_{S_0^-} \left| \nabla w_0^{[-1]} \right|^2 \mathrm{d}x \right\} - \varepsilon^2 \int_\Omega \partial_{X_1} u_0 \partial_{X_1} RW_0(x;\varepsilon)\, \mathrm{d}x.$$

Using (3.37) and (3.38), we obtain

$$\frac{3}{4} \left| \nabla_\varepsilon RW_0(\cdot;\varepsilon) \right|_{2,\Omega}^2 \leq Ce^{\frac{-\alpha}{\varepsilon}} - \varepsilon^2 \int_\Omega \partial_{X_1} u_0 \partial_{X_1} RW_0(x;\varepsilon)\, \mathrm{d}x,$$

since $RW_0(\cdot;\varepsilon) \in H_0^1(\Omega)$. Using the Cauchy-Schwarz and Poincaré inequalities, we get

$$\frac{3}{4} \left| \nabla_\varepsilon RW_0(\cdot;\varepsilon) \right|_{2,\Omega}^2 \leq Ce^{\frac{-\alpha}{\varepsilon}} + \varepsilon^2 \left| \partial_{X_1} u_0 \right|_{2,\Omega} \left| \partial_{X_1} RW_0(\cdot;\varepsilon) \right|_{2,\Omega}.$$

Young's inequality

$$\frac{1}{2} \left| \nabla_\varepsilon RW_0(\cdot;\varepsilon) \right|_{2,\Omega}^2 \leq Ce^{\frac{-\alpha}{\varepsilon}} + \varepsilon^2 \left| \partial_{X_1} u_0 \right|_{2,\Omega}^2,$$

so that

$$\left| \nabla_\varepsilon RW_0(\cdot;\varepsilon) \right|_{2,\Omega}^2 \leq Ce^{\frac{-\alpha}{\varepsilon}} + C\varepsilon^2.$$

Using the Triangular inequality, (3.33), (3.37) and (3.38), we obtain

$$\left| u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} - \theta_0^{[\varepsilon,-1]} \right|_{2,\Omega}, \quad \left| \nabla_{X_2} \left(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} - \theta_0^{[\varepsilon,-1]} \right) \right|_{2,\Omega} \leq C\varepsilon,$$

and

$$\left| \partial_{X_1} \left(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} - \theta_0^{[\varepsilon,-1]} \right) \right|_{2,\Omega} \leq C.$$

This completes the proof. ∎

We can improve the rate of convergence as follow

Theorem 28 *The solution u_0 to (2.5) is a strong limit of the sequence $u_\varepsilon - \left[\theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]}\right]$ in $H^1(\Omega)$ and the following error estimate is valid:*

$$\left|u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]}\right|_{2,\Omega} = o\left(\varepsilon\right), \quad \left|\nabla_{X_2}\left(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]}\right)\right|_{2,\Omega} = o\left(\varepsilon\right), \tag{3.42}$$

and

$$\left|\partial_{X_1}\left(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]}\right)\right|_{2,\Omega} = o\left(1\right), \tag{3.43}$$

Proof. Firstly, the estimates (3.40) and (3.41) are valuable. Then, we can extract a weakly convergent subsequence of $\partial_{X_1}(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]} + \Phi_0(\cdot;\varepsilon))$ in $L^2(\Omega)$, since the boundedness of $\left|\partial_{X_1}(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]} + \Phi_0(\cdot;\varepsilon))\right|_{2,\Omega}$, and according to (3.40), it follows that the whole sequence converges weakly to zero, i.e.

$$\partial_{X_1}(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]} + \Phi_0(\cdot;\varepsilon)) \rightharpoonup 0, \qquad \text{in } L^2(\Omega). \tag{3.44}$$

Going back to (3.39),

$$\begin{aligned}\frac{3}{4}\left|\nabla_\varepsilon RW_0(\cdot;\varepsilon)\right|^2_{2,\Omega} \leq \ & Ce^{\frac{-\alpha}{\varepsilon}}\left\{\int\limits_{S_0^+}\left|\nabla w_0^{[1]}\right|^2 \mathrm{d}x + \int\limits_{S_0^-}\left|\nabla w_0^{[-1]}\right|^2 \mathrm{d}x\right\} \\ & -\varepsilon^2\int\limits_{\Omega}\partial_{X_1}u_0\partial_{X_1}RW_0(x;\varepsilon)\,\mathrm{d}x.\end{aligned} \tag{3.45}$$

Applying the weak convergence (3.44), we get

$$\left|\partial_{X_1}RW_0(\cdot;\varepsilon)\right|_{2,\Omega} = o\left(1\right), \qquad \left|\nabla_{X_2}RW_0(\cdot;\varepsilon)\right|_{2,\Omega} = o\left(\varepsilon\right). \tag{3.46}$$

Using the Poincaré inequality in the X_2-direction, with the help of the estimates (3.46) we complete the proof of the theorem. ■

We can also improve the rate of convergence again if we assume more smoothness assumption on the data as in the following theorem.

Theorem 29 *We have, as $\varepsilon \to 0$*

$$\left|u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]}\right|_{2,\Omega} = O\left(\varepsilon^2\right), \quad \left|\nabla_{X_2}\left(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]}\right)\right|_{2,\Omega} = O\left(\varepsilon^2\right), \tag{3.47}$$

and

$$\left|\partial_{X_1}\left(u_\varepsilon - u_0 - \theta_0^{[\varepsilon,1]} + \theta_0^{[\varepsilon,-1]}\right)\right|_{2,\Omega} = O\left(\varepsilon\right), \tag{3.48}$$

Proof. Integrating by part the last integral of (3.45), we get

$$\frac{3}{4}\left|\nabla_\varepsilon RW_0(\cdot;\varepsilon)\right|_{2,\Omega}^2 \leq Ce^{\frac{-\alpha}{\varepsilon}} + \varepsilon^2 \int\limits_\Omega \partial_{X_1}\left(\partial_{X_1} u_0\right) RW_0(x;\varepsilon)\,\mathrm{d}x,$$

since $RW_0(\cdot;\varepsilon) \in H_0^1(\Omega)$. Then

$$\frac{3}{4}\left|\nabla_\varepsilon RW_0(\cdot;\varepsilon)\right|_{2,\Omega}^2 \leq Ce^{\frac{-\alpha}{\varepsilon}} + \varepsilon^2 \int\limits_\Omega \partial_{X_1}^2 u_0 RW_0(x;\varepsilon)\,\mathrm{d}x.$$

Using the Cauchy-Schawrz, Poincaré and Young's inequalities, we obtain

$$\frac{1}{2}\left|\nabla_\varepsilon RW_0(\cdot;\varepsilon)\right|_{2,\Omega}^2 \leq Ce^{\frac{-\alpha}{\varepsilon}} + 2\varepsilon^2\left|\partial_{X_1}^2 u_0\right|_{2,\Omega}^2.$$

This completes the proof. ■

In the following we will show the convergence results for the higher order composite asymptotic expansion.

3.3 Composite Asymptotic Expansion of Higher Order

We have the following convergence results

Theorem 30 *Under the sufficient assumption on the data, the solution* u_0 *to (2.5) is a strong limit of the sequence* $u_\varepsilon - \sum\limits_{i=1}^N \varepsilon^i u_i - \sum\limits_{i=0}^N \varepsilon^i\left[\theta_i^{[\varepsilon,1]} + \theta_i^{[\varepsilon,-1]}\right]$ *in* $H^1(\Omega)$ *and the following error estimate is valid:*

$$\left|u_\varepsilon - \sum_{i=0}^N \varepsilon^i\left[u_i + \theta_i^{[\varepsilon,1]} + \theta_i^{[\varepsilon,-1]}\right]\right|_{2,\Omega}, \quad \left|\nabla_{X_2}\left(u_\varepsilon - \sum_{i=0}^N \varepsilon^i\left[u_i + \theta_i^{[\varepsilon,1]} + \theta_i^{[\varepsilon,-1]}\right]\right)\right|_{2,\Omega} = O\left(\varepsilon^{N+1}\right), \tag{3.49}$$

and

$$\left|\partial_{X_1}\left(u_\varepsilon - \sum_{i=0}^N \varepsilon^i\left[u_i + \theta_i^{[\varepsilon,1]} + \theta_i^{[\varepsilon,-1]}\right]\right)\right|_{2,\Omega} = O\left(\varepsilon^N\right), \tag{3.50}$$

where u_{2i}*,* $i = 1, 2, \ldots, N/2$*, is the solution to (2.10).*

Proof. We have

$$\begin{aligned}\frac{3}{4}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|^2_{2,\Omega} &\le Ce^{\frac{-\alpha}{\varepsilon}}\sum_{i=0}^{N}\left\{\int_{S_0^+}\left|\nabla w_i^{[1]}\right|^2\,\mathrm{d}x+\int_{S_0^-}\left|\nabla w_i^{[-1]}\right|^2\,\mathrm{d}x\right\}\\ &\quad -\varepsilon^{N+2}\int_{\Omega}\partial_{X_1}u_N\partial_{X_1}RW_N(x;\varepsilon)\,\mathrm{d}x,\end{aligned}$$

since $RW_N(\cdot;\varepsilon)\in H_0^1(\Omega)$. Using (3.37), (3.38), Cauchy-Schwarz and Poincaré inequalities, we get

$$\frac{3}{4}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|^2_{2,\Omega}\le Ce^{\frac{-\alpha}{\varepsilon}}+\varepsilon^{N+1}\left|\partial_{X_1}u_N\right|_{2,\Omega}\varepsilon\left|\partial_{X_1}RW_N(\cdot;\varepsilon)\right|_{2,\Omega}.$$

Young's inequality

$$\frac{1}{2}\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|^2_{2,\Omega}\le Ce^{\frac{-\alpha}{\varepsilon}}+\varepsilon^{2N+2}\left|\partial_{X_1}u_N\right|^2_{2,\Omega},$$

so that

$$\left|\nabla_\varepsilon RW_N(\cdot;\varepsilon)\right|^2_{2,\Omega}\le Ce^{\frac{-\alpha}{\varepsilon}}+C\varepsilon^{2N+2}.$$

Using the Triangular inequality, (3.33), (3.37) and (3.38), we obtain

$$\left|u_\varepsilon-\sum_{i=0}^{N}\varepsilon^i\left[u_i+\theta_i^{[\varepsilon,1]}+\theta_i^{[\varepsilon,-1]}\right]\right|_{2,\Omega},\quad\left|\nabla_{X_2}\left(u_\varepsilon-\sum_{i=0}^{N}\varepsilon^i\left[u_i+\theta_i^{[\varepsilon,1]}+\theta_i^{[\varepsilon,-1]}\right]\right)\right|_{2,\Omega}\le C\varepsilon^{N+1},$$

and

$$\left|\partial_{X_1}\left(u_\varepsilon-\sum_{i=0}^{N}\varepsilon^i\left[u_i+\theta_i^{[\varepsilon,1]}+\theta_i^{[\varepsilon,-1]}\right]\right)\right|_{2,\Omega}\le C\varepsilon^N.$$

since $e^{\frac{-\alpha}{\varepsilon}}=O(\varepsilon^{2N+2})$. This completes the proof. ■

Conclusion

From this study, we see that;
we can improve the convergence results by the higher order composite asymptotic expansion in some particular case, for instance where we have more smoothness data.

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الملخص.
في نظرية مسائل القيم الحديّة ذات الاضطرابات المتباينة الغير منتظمة، الحل u_ε لا يتقارب ، بالنسبة لنظيم الفضاء H^1 على كامل الميدان، نحو u_0. في هذا العمل نشكل تقريبات مقاربة مركبة للحل الضعيف لمسألة خطية اهليجية مرفقة بشروط ديريكلي للحصول على التّقارب على كامل الميدان.
الكلمات المفتاحية .
متباين، الاضطرابات المتباينة، اهليجي، مسائل خطية، النشور المقاربة المنتظمة و المركبة، معدل التّقارب، المصححات.

Abstract. In the theory of anisotropic singular perturbation boundary value problems, the solution u_ε does not converge, in the H^1 -norm on the whole domain, towards some u_0. In this work, we construct a composite asymptotic approximations for the weak solution of a linear elliptic problem with Dirichlet bouandary conditions to get the asymptotic convergence on the whole domain.
KeyWords. Anisotropic, singular perturbations, elliptic, linear problems, regular and composite asymptotic expansions, rate of convergence, correctors.