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THESE

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Contribution à l'analyse asymptotique des couches limites en homogénéisation périodique

Presented By:

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DEDICATION

I am dedicating this thesis to sixth beloved people who have meant and continue to mean so much to me. Although they are no longer of this world, their memories continue to regulate my life. First and foremost, to my best friend, who accompanied me on all my travels seeking knowledge from Boumerdes to Constantine to Skikda without forgetting Tlemcen where he carried the burden of traveling despite the distance, despite the hardship and despite his old age, to who his precious tears flowed in sorrow for me, to who was my first support after Allah to complete this thesis, my dear father **Boubaker**.

To who whose love for me knew no bounds and, who taught me the value of hard work,

to who accompanied me with whose prayers, to who carried my burden, my dear mother.

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NOTATIONS

➤∇ : denotes the full gradient operator.

➤ div: denotes the full divergence operator.

➤ ∇_x : denotes the gradient in the slow variable.

➤ div_x: denotes the divergence in the slow variable.

➤ ∇_y: denotes the gradient in the fast variable.

➤ div_y: denotes the divergence in the fast variable.

➤ $curl_x$: denotes the rotation vector in the slow variable in two dimensions, such that

$$curl_x = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}.$$

➤ $curl_y$: denotes the rotation vector in the fast variable in two dimensions, such that

$$curl_y = \begin{pmatrix} -\frac{\partial}{\partial y_2} \\ \frac{\partial}{\partial y_1} \end{pmatrix}.$$

➤ $<>$: denotes the mean operator which is defined by $< . >= \dfrac{1}{|Y|} \int_Y .dy$, where $|Y|$ is the measure of Y.

➤ $L^2_\sharp(Y)$: denotes the subspace of functions in $L^2_{loc}(\mathbb{R}^n)$, which are Y-periodic.

➤ $H^1_\sharp(Y)$: denotes the subspace of functions in $H^1_{loc}(\mathbb{R}^n$, which are Y-periodic.

➤ $M_s^{n\times n}$: denotes the set of $n \times n$ symmetric matrices.

$$\mathcal{M}_s(\alpha, \beta, \Omega) = \left\{ A \in L^\infty\big(\Omega; M_s^{n\times n}\big); \alpha|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2 \ for\ any\ \xi \in \mathbb{R}^n \right\}$$

where α is the uniform coercivity constant and β is L^∞- bound, with α, β are positive, such that $0 < \alpha \leq \beta$.

➤ $M^{n\times n}$: denotes the set of all possibly non-symmetric where,

$$\mathcal{M}(\alpha,\beta,\Omega) = \Big\{A \in L^\infty\big(\Omega; M^{n\times n}\big); \alpha|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2 \ for\ any\ \xi \in \mathbb{R}^n\Big\}$$

.

➤ $\mathcal{M}_s(\alpha,\beta,\Omega)$: denotes the set of $n \times n$ symmetric matrices,

$$\mathcal{M}_s(\alpha,\beta,\Omega) = \Big\{A(x) \in L^\infty\big(\Omega; M_s^{n\times n}\big) such\ that \alpha|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2 \ for\ any\ \xi \in \mathbf{R}^n\Big\}$$

where α is the uniform coercivity constant and β is L^∞- bound, with α, β are positive, such that $0 < \alpha \leq \beta$. .

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$$\mathcal{M}(\alpha,\beta,\Omega) = \Big\{A(x) \in L^\infty\big(\Omega; M^{n\times n}\big) such\ that \alpha|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2 \ for\ any\ \xi \in \mathbf{R}^n\Big\}$$

.

➤$D'(\Omega) = [C_0^\infty(\Omega)]'$.

➤ $C_\sharp^\infty(Y)$: denotes the space of infinitely differentiable functions in $\mathbb{R}^n$ which are periodic of period Y.

➤ $C_\sharp(Y)$: denotes the Banach space of continuous and Y-periodic functions. Eventually, denotes the space of infinitely smooth and compactly supported functions in Ω with values in the space $C_\sharp^\infty(Y)$.

➤ $D(\Omega; C_\sharp^\infty(Y))$: denotes the space of infinitely differentiable functions with compact support in Ω with respect to the first argument and taking values in $C_\sharp^\infty(Y)$ with respect to the second argument.

➤ $\overset{c}{\hookrightarrow}$: compactly embedded .

➤δ_{ij}: represents the symbol of Kronecker .

➤ n_j the outer normal.

INTRODUCTION

This work aims to the asymptotic study of the boundary layers in periodic homogenization of some elliptical problem.

This thesis contained two parts:

∗ The first part puts forward the improvement of the estimates obtained on the boundary layer correctors in the classical problem of homogenization in divergence form with Dirichlet boundary conditions and provides the third-order error estimates with and without the boundary layer correctors.

∗ The second part sheds new light on the homogenization of a three-dimensional piezoelectric heterogeneous structure and presents a new approach to the homogenized problem of periodic, heterogeneous and non-isotropic piezoelectric plate when the thickness and the period of this plate tending to zero simultaneously, where in both studies we have used the energy method of Tartar.

It is divided into three chapters structured as follows:

Chapter I: Homogenization and boundary layers

In the first chapter, we present a brief history of the homogenization and boundary layers and we give a general panorama of homogenization method techniques with some illustrations, we ended this chapter by a quick overview of the boundary layers in elasticity and thin elastic plates.

Chapter II: Error estimates

In the second chapter, we pose the classical problem of homogenization and we present the error estimates that have been done upon this problem. The main achievements in this chapter, including contributions to the field can be summarized as follows:

1. Error estimates of the second and third orders with and without boundary layers terms.

2. Third-order error estimates with and without the third-order boundary layer corrector in two-dimension, using the mixed-method.

Chapter III: Homogenization of a piezoelectric structure by the energy method

In this chapter, we are interested in the homogenization of a piezoelectric structure by the energy method in two cases, we started by the case of three-dimensional piezoelectric structure, then we applied the same steps for the case of periodic, heterogeneous and non-isotropic piezoelectric plate. The contributions of this chapter are presented as follows:

1. Establishing the convergence theorem using the energy method of Tartar for the case of three-dimensional piezoelectric structure.

2. Outlines the limit of the piezoelectric problem for the case of the periodic and heterogeneous and non-isotropic plate when the thickness and the period of this plate are comparable.

CHAPTER 1

HOMOGENIZATION AND BOUNDARY LAYERS

1.1 The concept of homogenization

Definition 1.1.1. *Homogenization method is a mathematical theory of averaging, which allows the calculation of composite* ***effective properties*** *knowing the topology of the composite unit cell and* ***the replacement*** *of the composite medium by an* ***"equivalent"*** *homogeneous medium to solve the global problem.*

Among it's advantage in relation to other methods that it needs only the information about the unit cell and this last can have any complex shape. Note that the homogenization method used to study:

1) Differential operators with rapidly oscillating coefficients.

2) Boundary value problems with rapidly changing boundary conditions.

3) Equations in perforated domains.

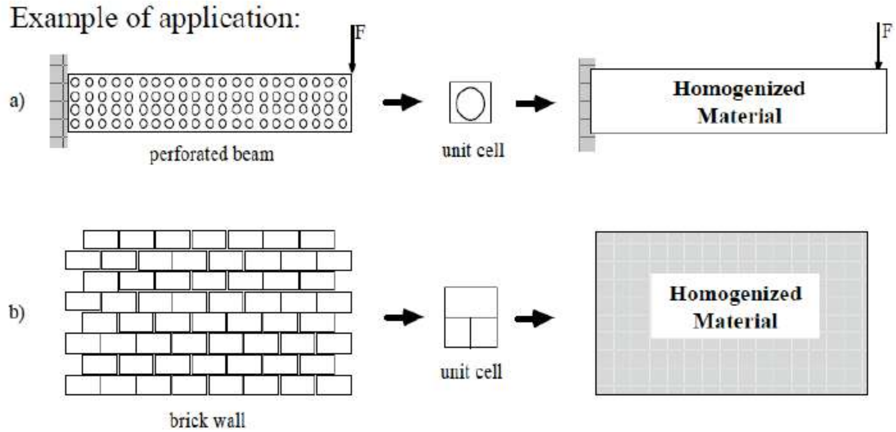

Figure 1.1: Illustration of the homogenization of perforated beam and break wall

1.2 Brief history

In this section we give a short (possibly incomplete) historical development of homogenization methods. The problem of replacement of a heterogeneous material by an equivalent homogeneous one dates back to at least the 19th century. This was raised in works by Poisson [92], Maxwell [76] and Rayleigh [95]. In 1881, Maxwell [76] studied the effective conductivity of media with small concentrations of randomly arranged inclusions, and Rayleigh [95] studied the same problem with periodically distributed inclusions in 1892.

In 1906, Einstein [47] investigated the effective viscosity of suspensions with hard spherical particles in incompressible viscous fluids. A good survey of results on this question until 1926 can be found in [69]. Striking contributions were made in the 1930s. Voight [112] calculated effective parameters of polycrystal, such as, the stiffness tensor, by averaging the appropriate values over volume and orientation, while Reuss [96] used averaging of the component of the reverse tensor (compliance) for the same problem.

Later on, Hill [[58], [59]] and Il'ushina [63] rigorously proved that Voight and Reuss methods give the upper bound and the lower bound, respectively, of those effective parameters. For results in the direction of the so-called Reuss-Voight inequality (Hill' fork), such as the Hashin-Shtrikman bounds, we refer to [[64], Chapter 6] and references therein.

It should be noted that iterated homogenization type problems were considered for the first time by Bruggeman in 1935 [23] The first asymptotically exact scheme for calculating effective parameters of laminated media was proposed in 1946 by Lifshits and Rozentsveig [[70],[71]].

In 1964, Marchenko and Khruslov [74] introduced a general approach based on asymptotic tools which could handle numerous physical problems, including for example (for the first time), boundary value problems with fine-grained boundary [[74],[75]].

From the early 1970s, further development of the mathematical study of phenomena in heterogeneous media is done by averaging differential equations with rapidly oscillating coefficients, and the first results (according to e.g., [[7], [13]] are in [[11],[14], [15], [19], [20],[41], [86],[98]].
The name homogenization was first introduced in 1974 by Babuska [12].

1.3 Homogenization techniques

Several homogenization methods were developed in the 1970s, and homogenization became a subject in Mathematics. The methods introduced include:

1.3.1 Parametrized Measures (Young Measures)

Young measures were developed by L.C Young [113]. They were initially used for treating problems of calculus of variations, until L. Tartar [102] developed it as a tool for the analysis of non-linear partial differential equations. Young measures can be used to compute the weak limit of any function of weakly converging fields. Additional information on Young measures can be found in [16], [89], [54], just to cite a few.

Definition 1.3.1. *Let K be a bounded open set in $\mathbb{R}^n$ and let $u : \Omega \mapsto \mathbb{R}^n$: be a measurable function such that $u \in K$ a.e. We define a measure μ on $\Omega \times \mathbb{R}^n$ by*

$$\langle \mu, \phi(x, \lambda) \rangle = \int_{\Omega} \phi(x, u(x)) dx.$$

for all continuous function ϕ with compact support contained in $\Omega \times \mathbb{R}^n$. μ is known as the Radon measure or the generalized measure associated to u.

Proposition 1.3.1. *The Radon measure μ has the following properties.*

1. $\mu \geq 0$ *i.e.* $\langle \mu, \phi \rangle \geq 0$ *if* $\phi \geq 0$.

2. $supp\mu \subset \overline{graph\ u}$ *i. e. if* $\phi = 0$ *on graph* u *then* $\langle \mu, \phi \rangle$=0.

3. *If* $\phi(x, \lambda) = \Psi(x)$ *then*

$$\langle \mu, \phi \rangle = \int_{\Omega} \Psi(x) dx.$$

Theorem 1.

Let K be a bounded set in $\mathbb{R}^m$ and Ω, a bounded open set in $\mathbb{R}^n$. Let $u_j : \Omega \mapsto \mathbb{R}^m$ be a sequence such that $u_j \in K$ a.e..

Then

there exists a subsequence $\{u_{jk}\}$ and a family of probability measures $\{\nu_x\}_{x \in \Omega}$ (i.e., $\nu_x \geq 0$, $\nu_x(\mathbb{R}^n) = 1$)

with *supp* $\nu_x \subset \bar{K}$ such that for F a continuous function on $\mathbb{R}^n$,

$$F(u_{jk}) \overset{*}{\rightharpoonup} \bar{f} \text{ weakly}^* \text{ in } L^{\infty}(\Omega), \text{ as } k \to \infty$$

where

$$\bar{f}(x) = \langle \nu_x, F(\lambda) \rangle = \int_{\mathbb{R}^n m} \nu_x(\lambda) F(\lambda) d\lambda \ a.e..$$

The family $\{\nu_x\}_{x \in \Omega}$ is called the Young measure associated to the subsequence $\{u_{jk}\}$.

1.3.2 Method of Asymptotic Expansion

The most traditional method in homogenization theory is the so-called method of asymptotic expansions which dates way back to the 1960s. it is widely used in mechanics and physics. It was originally introduced for mechanical problems by engineers till mathematicians began to use it in the study of problems with periodic coefficients. The method of two-scale asymptotic expansions applied to the following well-posed problem in $H_0^1(\Omega)$

$$(P_\varepsilon) \begin{cases} -div A_\varepsilon \nabla u_\varepsilon = f & in\ \Omega, \\ u_\varepsilon = 0 & on\ \ \partial\Omega, \end{cases} \tag{1.3.1}$$

postulate that the solution u_ε of (P_ε) admits the ansatz

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \varepsilon^3 u_3(x, \frac{x}{\varepsilon}) + \varepsilon^4 u_4(x, \frac{x}{\varepsilon}) + ... \tag{1.3.2}$$

where each function $u_i(x,y)$ is Y-periodic with respect to $y = \frac{x}{\varepsilon}$. When this expansion is substituted into problem (1.3.1), the terms with equal powers of ε are equated and a series of problems are obtained. Solving these problems leads to the homogenized problem and the homogenized solution. This method is systematically formalized to handle homogenization of boundary value problems with periodic rapidly oscillating coefficients by Bensoussan, Lions and Papanicolaou [18], see also Keller and Larsen [[67],[68]], and Sanchez- Palencia [99]. More details on the Asymptotic Expansions method will be stated in chapter 2.

1.3.3 G-convergence

The G-convergence is a notion of convergence associated with sequences of symmetric, second-order, elliptic operators. It was introduced in the late sixties by Spagnolo [101]. The G means Green because this type of convergence corresponds roughly to the convergence of the associated Green functions. The main result of the G-convergence is a compactness theorem in the homogenization theory which states that, for any bounded and uniformly coercive sequence of coefficients of a symmetric, second-order, elliptic equation, there exist a subsequence and a G-limit (i.e., homogenized coefficients) such that, for any source term, the corresponding subsequence of solutions converges to the solution of the homogenized equation. In physical terms, it means that the physical properties of a heterogeneous medium (such as its permeability, conductivity, or elastic moduli) can be well approximated by the properties of a homogeneous or homogenized medium if the size of the heterogeneities is small compared to the overall size of the medium. For simplicity, we introduce the notion of G-convergence for a specific example of operators, namely, a scalar diffusion process with a Dirichlet boundary condition i.e. problem (1.3.1), but all the results hold for a large class of second-order, elliptic operators and boundary conditions. Let Ω be a bounded open set in $\mathbb{R}^n$. We consider a sequence A_ε of diffusion tensors in $\mathcal{M}_s(\alpha, \beta, \Omega)$, indexed by a sequence of positive numbers ε going to 0. Here, ε is not associated with any specific length scale or statistical property of the diffusion process. In other words, no special assumptions (like periodicity or stationarity) are placed on the sequence A_ε. The G-convergence of operators associated with the sequence A_ε is defined below as the convergence of the corresponding solutions u_ε.

Definition 1.3.2. *The sequence of tensors A_ε is said to G-converge to a limit A^*, as ε goes to 0, if,* *for any right-hand side $f \in L^2(\Omega)$ in (1.3.1), the sequence of solutions u_ε converges weakly in $H_0^1(\Omega)$ to a limit u_0 which is the unique solution of the homogenized equation associated with A^*.*

$$\begin{cases} -divA^*\nabla u_0 = f & in\ \Omega, \\ u_0 = 0 & on\ \partial\Omega. \end{cases} \tag{1.3.3}$$

This definition makes sense because of the compactness of the set $\mathcal{M}_s(\alpha, \beta, \Omega)$ with respect to the G-convergence, as stated in the next theorem.

Theorem 2.

For any sequence A_ε in $\mathcal{M}_s(\alpha, \beta, \Omega)$, there exist a subsequence (still denoted by ε) and a homogenized limit A^*, belonging to $\mathcal{M}_s(\alpha, \beta, \Omega)$, such that A_ε G-converges to A^*.

The original proof of Theorem 2 (due to Spagnolo [101]) was based on the convergence of the Green functions associated with (1.3.1). Another proof uses the Γ-convergence of De Giorgi. A simpler proof was found by Tartar in the framework of the H-convergence which is a generalization of the G-convergence to the case of non-symmetric operators. The interested reader is referred to the next subsection on H -convergence for a discussion of such a proof.

Remark 1.3.1. *If a sequence A_ε converges strongly in $L^\infty(\Omega)^{n^2}$ to a limit A, then its G-limit A^* coincides with A. In general the G-limit A^* of a sequence A_ε has nothing to do with its weak-* $L^\infty(\Omega)$-limit. For example, a straightforward computation in one dimension shows that the G-limit of a sequence A_ε is given as the inverse of the weak-*$L^\infty(\Omega)$ -limit of A_ε^{-1} (the so-called harmonic limit). This last result holds true only in one dimension, and no explicit formula is available in higher dimensions.*

The G-convergence enjoys a few useful properties as enumerated in the following proposition.

Proposition 1.3.2. *1. If a sequence A_ε G-converges, its G-limit is unique.*

2. *Let A_ε and B_ε be two sequences which G-converge to A^* and B^*, respectively. Let $\omega \in \Omega$ be a subset strictly included in Ω, such that $A_\varepsilon = B_\varepsilon$ in ω. Then, $A^* = B^*$ in ω (this property is called the locality of G-convergence).*

3. *The G-limit of a sequence A_ε is independent of the source term f and of the boundary condition on $\partial\Omega$.*

4. *Let A_ε be a sequence which G-converges to A^*. Then, the associated density of energy $A_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon$ also converges to the homogenized density of energy $A^* \nabla u_0 . \nabla u_0$ in the sense of distributions in Ω.*

1.3.4 Γ-convergence

The Γ-convergence is an abstract notion of functional convergence which has been introduced by De Giorgi ([[42]] and [[43]]). It is not restricted to homogenization, and it has many applications in the calculus of variations, such as singular perturbation problems. A detailed presentation of Γ-convergence and several applications may be found in the books [[24]] and [[39]]. We first give the abstract definition of Γ-convergence and the fundamental theorem of Γ-convergence which motivates this definition.

Definition 1.3.3. *Let X be a metric space endowed with a distance d. Let ε be a sequence of positive indexes which goes to zero. Let F_ε be a sequence of functions defined on X with values in $\mathcal{R}$ The sequence F_ε is said to Γ-converge to a limit function F_0 if, for any point $x \in X$,*

1. *All sequences x_ε converging to x satisfy $F_0(x) \leq \liminf_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon)$, and*
2. *There exists at least one sequence x_ε converging to x, such that*

$$F_0(x) = \lim_{\varepsilon \to 0} F_\varepsilon(x_\varepsilon)$$

Definition 1.3.4. *A sequence F_ε is said to be d-equicoercive on X if there exists a compact set K (independent of ε) such that*

$$\inf_{x \in X} F_\varepsilon(x) = \inf_{x \in K} F_\varepsilon(x).$$

The definition of Γ-convergence makes sense because of the following fundamental theorem which yields the convergence of the minimum values and of the minimizers for an equicoercive Γ-converging sequence.

Theorem 3.

Let F_ε be a d-equicoercive sequence on X which Γ-converges to a limit F_0. Then,

1. the minima of F_ε converge to that of F_0, i.e.,

$$\min_{x\in X} F_0(x) = \lim_{\varepsilon\to 0}\left(\inf_{x\in X} F_\varepsilon(x)\right), \text{ and}$$

2. the minimizers of F_ε converge to those of F_0, i.e., if x_ε converges to x and $\lim_{\varepsilon\to 0} F_\varepsilon(x_\varepsilon) = \lim_{\varepsilon\to 0}(\inf_{x\in X} F_\varepsilon(x)), then,$ x *is a minimizer of* F_0.

Theorem 4.

Assume that the metric space X (with the distance d) is separable (i.e., contains a dense countable subset). Let F_ε be a sequence of functions defined on X. Then, there exist a subsequence $F_{\varepsilon'}$ and a Γ-limit F_0 such that $F_{\varepsilon'}$ Γ-converges to Fo.

A proof of the above theorems may be found in [39]. Their main interest is to show that the notion of Γ-convergence is, roughly speaking, equivalent to the convergence of minimizers. Note, however, that they do not give any method, in practice, for computing the Γ-limit F_0.

Remark 1.3.2. *The relevance of* Γ*-convergence to homogenization is the following.*

Consider, for example, the problem (1.3.1) of linear diffusion process in a periodic domain Ω with period ε. Assume that the tensor of diffusion is $A(\frac{x}{\varepsilon})$, where $A(y)$ is a symmetric, coercive, and bounded matrix which is Y-periodic. It is well-known that, when the matrix A is symmetric, the P. D. E. (**??**) is equivalent to the following variational problem: find $u^\varepsilon \in H_0^1(\Omega)$ which achieves the minimal value of

$$\min_{u^\varepsilon\in H_0^1(\Omega)}\left(\frac{1}{2}\int_\Omega A\left(\frac{x}{\varepsilon}\right)\nabla u^\varepsilon.\nabla u^\varepsilon - \int_\Omega f u^\varepsilon\right). \tag{1.3.4}$$

Therefore, the Γ-convergence of the functionals subject to minimization in (1.3.4) is equivalent to the homogenization of the P. D. E. (1.3.1). The advantage of the Γ-convergence is that it is not restricted to linear equations (or equivalently quadratic functionals).

1.3.5 H -convergence

The H -convergence is a generalization of the G-convergence to the case of non-symmetric problems. More than that, it provides a new constructive proof (the so-called energy method) of the main compactness theorem, which is both simpler and more general than the previous proofs. The H-convergence (H stands for "homogenization") was introduced by Murat and Tartar [79], [80] and [106] in the mid-seventies. As for the G-convergence, we simply introduce the notion of H -convergence for a scalar diffusion process with a Dirichlet boundary condition, although all the results hold for any second-order, elliptic operators and boundary conditions.

Let Ω be a bounded open set in $\mathbb{R}^n$, we consider the same problem (1.3.1)but at this once the sequence A_ε of diffusion tensors is in $\mathcal{M}(\alpha, \beta, \Omega)$. Once again, ε is not associated with any specific length scale or statistical property of the diffusion process. We emphasize that the tensors A_ε are *not necessarily symmetric*. This corresponds to a possible drift in the diffusion process.

The H-convergence of the sequence A_ε differs from the previous G-convergence in the sense that it requires more than the mere convergence of the sequence of solutions u_ε

Definition 1.3.5. *The sequence of tensors A_ε is said to H-converge to a limit A^*, as ε goes to 0, if, for any right-hand side $f \in H^{-1}(\Omega)$ in (1.3.1), the sequence of solutions u_ε converges weakly in $H_0^1(\Omega)$ to a limit u_0, and the sequence of fluxes $A_\varepsilon \nabla u_\varepsilon$ converges weakly in $L^2(\Omega)^N$ to $A^* \nabla u_0$, where u_0 is the unique solution of the homogenized equation associated with A^*.*

$$\begin{cases} -divA^* \nabla u_0 = f & in\ \Omega, \\ u_0 = 0 & on\ \partial\Omega. \end{cases} \tag{1.3.5}$$

This definition makes sense because of the following compactness result.

Theorem 5.

For any sequence A_ε in $\mathcal{M}(\alpha, \beta, \Omega)$, there exist a subsequence (still denoted by ε) and a homogenized limit A^*, belonging to $\mathcal{M}(\alpha, \frac{\beta^2}{\alpha}, \Omega)$, such that A_ε H-converges to A^*.

Remark 1.3.3. *Notice that the set $\mathcal{M}(\alpha, \beta, \Omega)$ is not stable with respect to the H-convergence (as is the case for the G-convergence), because the $L^\infty(\Omega)$-bound of the H -limit can be increased by a factor of $\frac{\beta}{\alpha} > 1$. This is a specific effect of the non-symmetry of a sequence A_ε. In physical* *terms, it means that microscopic convective phenomena can yield macroscopic diffusive effects. The proof of Theorem 5 is constructive and based on the so-called energy method described in section 1.3.7. Beyond its theoretical interest for proving the above compactness result, the energy method of Tartar is of paramount importance in practical applications because it gives a convenient recipe for homogenizing any second-order, elliptic system. A detailed proof of Theorem 5 may be found in [79]. Like G-convergence, H-convergence satisfies the same properties as stated in Proposition 1.3.2, namely, uniqueness of the H-limit, locality, independence of the H-limit with respect to the boundary condition, and convergence of the energy density. To conclude this subsection, we give a simple example which demonstrates the necessity of requiring the convergence of the fluxes* $A_\varepsilon \nabla u_\varepsilon$ *on top of that of the solutions* u_ε *to have a coherent definition of H -convergence. Let B be a constant skew-symmetric matrix, i.e., such that its entries satisfy*

$$b_{ij} = -b_{ji} \quad for\ all\ 1 \leq i, j \leq n.$$

Then, for any real-valued function u,

$$div(B\nabla u) = \textstyle\sum_{1 \leq i,j \leq n} b_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0.$$

Therefore, if u is a solution of the homogenized equation (1.3.5), it is also a solution of the following equation

$$\begin{cases} -div\Big((A^* + B)\nabla u\Big) = f & in\ \Omega, \\ u = 0 & on\ \partial\Omega. \end{cases} \tag{1.3.6}$$

Assume for a moment that the definition of H-convergence is the same as that of G-convergence (i.e., only the convergence of solutions is required). Then, if A^* *is a H -limit of a sequence* A_ε*, so is* $A^* + B$ *for any constant, skew-symmetric matrix* B*, which contradicts the uniqueness of the H-limit (a highly desirable feature of any type of convergence). Therefore, in the non-symmetric case, the definition of H-convergence must include an additional condition compared to that of G-convergence. This is precisely the role of the convergence of fluxes* $A_\varepsilon \nabla u_\varepsilon$.

1.3.6 Iterated Homogenization

In the two-scale convergence method, we considered homogenization problems in periodic media where only two different length scales were considered, namely, the macroscopic (of the order of the domain size) and the microscopic (of the order of the heterogeneities period), which have a ratio denoted by ε. Of course, in the real world, porous media are far from being periodic and usually exhibit many different length scales of heterogeneities. The very crude modeling of subsection 1.3.9 can be further improved by considering not only a single scale of heterogeneities but several periodic scales of heterogeneities (up to a countable infinite number of scales). This type of homogenization problem is called reiterated homogenization (following a terminology of [18]) because, under a mild assumption on the separation of scales, it amounts to successively homogenizing the smallest scale while keeping the larger ones fixed. Here, we shall simply state the main result of this process of reiterated homogenization on a model problem to explain the main ideas without dwelling too much on technicalities. Our model problem is a diffusion equation in a multiply periodic domain Ω (a bounded open set in $\mathbb{R}^n$). We assume that there are n scales of heterogeneities $\varepsilon_1, \varepsilon_2, ..., \varepsilon_n$ which depend on a single positive parameter ε which tends to zero. The key assumption is that all scales go to zero as ε does, while remaining ordered, ε_1 being the largest and ε_n the smallest, i.e.,

$$\lim_{\varepsilon \to 0} \varepsilon_i(\varepsilon) = 0, \quad for\ 1 \leq i \leq n, \tag{1.3.7}$$

and

$$\lim_{\varepsilon \to 0} \frac{\varepsilon_i(\varepsilon)}{\varepsilon_{i-1}(\varepsilon)} = 0, \quad for\ 2 \leq i \leq n. \tag{1.3.8}$$

For simplicity, the rescaled unit cell Y_i at each scale is assumed to be the same, equal to the unit cube $Y = (0.1)^n$. The tensor of diffusion in Ω is given by an $n \times n$ matrix $A(x, \frac{x}{\varepsilon_1}, ..., \frac{x}{\varepsilon_n})$, not necessarily symmetric, where $A(x, y_1, ..., y_n)$ is a continuous function of all variables $x \in \Omega$ and $y_i \in Y_i$ which is Y_i-periodic in y_i. Furthermore, this matrix A satisfies the usual coerciveness and boundedness assumptions: there exist two positive constants α and β, satisfying $0 \leq \alpha \leq \beta$, such that, for any constant vector $\xi \in \mathbb{R}^n$ and at any point $(x, y_1, ..., y_n)$,

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{n} A_{i,j}(x, y_1, ..., y_n)\xi_i\xi_j \leq \beta|\xi|^2,$$

where $A_{i,j}$ denotes the entries of the matrix A.

Denoting the source term by $f \in L^2(\Omega)$ and enforcing a Dirichlet boundary condition, our model problem of diffusion in a multiply periodic medium reads

$$\begin{cases} -div\Big(A(x, \dfrac{x}{\varepsilon_1}, ..., \dfrac{x}{\varepsilon_n})\nabla u_\varepsilon\Big) = f \quad in\ \Omega, \\ \qquad\qquad u_\varepsilon = 0 \quad on\ \ \partial\Omega. \end{cases} \tag{1.3.9}$$

By applying the Lax-Milgram lemma, equation (1.3.9) admits a unique solution u_ε in $H_0^1(\Omega)$. Moreover, u_ε satisfies the following a priori estimate:

$$\|u_\varepsilon\|_{H_0^1(\Omega)} \leq C\|f\|_{L^2(\Omega)}, \tag{1.3.10}$$

where C is a positive constant which does not depend on ε. It implies that the sequence u_ε is bounded in the Sobolev space $H_0^1(\Omega)$. To compute the homogenized diffusion tensor we need the following notations. Let $A_n(y_0, y_1, ..., y_n)$ be the original tensor $A(x, y_1, ..., y_n)$ (for convenience, the macroscopic variable x is denoted by y_0). For $0 \leq i \leq n-1$, a tensor $A_i(y_0, y_1, ..., y_i)$ is defined as the homogenized tensor of $A_{i+1}(y_0, y_1, ..., y_i, \frac{x}{\varepsilon})$ where all the larger scales $y_0, y_1, ..., y_i$ are kept fixed. We also denote the last homogenized tensor $A_0(y_0)$ by $A^*(x)$, for which there is no more micro-scale. In other words, the rule for computing the final homogenized tensor $A^*(x)$, is to separately and sequentially homogenize the different scales from the smallest to the largest. More precisely, at each scale $1 \leq i \leq n$, we introduce the solutions $w_p^i(y_0, y_1, ..., y_i)$ with $1 \leq p \leq n$, defined, at each point $(y_0, y_1, ..., y_{i-1})$, as the unique solutions in $H_\sharp^1(Y_i)/\mathbb{R}$ of the local problems:

$$\begin{cases} \quad - div_{y_i}\big(A_i(y_0, y_1, ..., y_i)(\vec{e}_p + \nabla_{y_i}\chi_p^i(y_0, y_1, ..., y_i)\big) = 0 \;\; in\ Y_i, \\ y_i \longrightarrow \chi_p^i(y_0, y_1, ..., y_i) \qquad\qquad\qquad\qquad Y_i - periodic, \end{cases} \tag{1.3.11}$$

with $(\vec{e}_p)_{1\leq p\leq n}$, the canonical basis of $\mathbb{R}^n$. Then, the sequence $A_i(y_0, y_1, ..., y_i)$ is defined by its entries,

$$A_i^{pq}(y_0, y_1, ..., y_i) = \int_Y A_{i+1}(y_0, y_1, ..., y_i, y_{i+1})(\vec{e}_p + \nabla_{y_{i+1}}\chi_p^{i+1}) \quad .(\vec{e}_q + \nabla_{y_{i+1}}\chi_q^{i+1})dy_{i+1}. \tag{1.3.12}$$

Formulas (1.3.11) and (1.3.12) are usually used for computing the homogenized coefficients of a single-scale periodic medium. Finally, the main result of this reiterated homogenization process is the following theorem.

Theorem 6.

The sequence u_ε of solutions of equation(1.3.9) converges weakly in $H_0^1(\Omega)$ to u. the unique solution of the homogenized problem,

$$\begin{cases} -div_x\Big(A^*\nabla u\Big) = f \qquad in\ \Omega, \\ \qquad\qquad\qquad u = 0 \qquad on\ \partial\Omega, \end{cases} \tag{1.3.13}$$

where the homogenized diffusion tensor is given by the last term A_0 of the sequence defined by (1.3.12).

Theorem 6 was first proven in [18] when the scales are successive powers of ε, i.e., $\varepsilon_i = \varepsilon^i$ (this assumption favors the use of multiple-scale asymptotic expansions). A general proof of Theorem 6 (including the case of an infinite number of scales) is given in [6], where a notion of multiple-scale convergence is introduced . Reiterated homogenization has been used in [10] for rigorously justifying the so-called differential effective scheme for computing effective coefficients in a heterogeneous medium with an infinite number of length scales. The differential effective scheme is a well-known method for estimating mechanical properties of composite materials (see, e.g., [84]). Loosely speaking, it amounts to computing homogenized coefficients as the solution of an ordinary differential equation. This differential effective scheme could also be applied to evaluate diffusion constants in porous media, but, to our knowledge, it has never been done so far.

1.3.7 The Energy Method

1.3.7.1 Setting of a Model Problem

A very elegant and efficient method for homogenizing partial differential equations has been devised by Tartar [106] and[79], which has later been called the energy method although it has nothing to do with any kind of energy. It is sometimes more appropriately called the oscillating test function method, but it is most commonly referred to as the energy method, and we shall stick to this name. The energy method is a very general method in homogenization. It does not require any geometric assumptions about the behavior of the p.d.e. coefficients: neither periodicity nor statistical properties like stationarity or ergodicity. Actually, it encompasses all other approaches in the framework of H

-convergence. As was already mentioned in the previous section, the energy method is a constructive proof for the compactness theorem of H -convergence. However, to expose the energy method in its full generality may hide the key ideas of the method in a lot of technicalities. Therefore, for clarity, we prefer to present the energy method on a model problem of periodic homogenization. Nevertheless, we reemphasize that the energy method works also for non-periodic homogenization, as the reader can be convinced by referring to [79] and [35]. We consider a model problem of diffusion in a periodic medium, a usual example in all textbooks on homogenization, but, of course, the energy method covers many other problems with slight changes.

In order to get the hang of the energy method we consider the same model problem of diffusion (1.3.1) in a periodic medium, of course the energy method covers many other problems with slight changes.

1.3.7.2 Proof of the main convergence result

In this section we give a rigorous proof of Theorem 7, following a general method due to Tartar ([106], [105]). This method relies on the construction of a class of oscillating test functions obtained by periodizing the solution of a problem set in the reference cell.

Theorem 7.

Let $f \in H^{-1}(\Omega)$ and u_ε be the solution of (1.3.1). Then,

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 \quad \textit{weakly in } H_0^1(\Omega), \\ A_\varepsilon \nabla u_\varepsilon \rightharpoonup A^* \nabla u_0 \quad \textit{weakly in } (L^2(\Omega))^n, \end{cases} \tag{1.3.14}$$

where $u_0(x)$ is the unique solution in $H_0^1(\Omega)$ of the homogenized problem

$$\begin{cases} -divA^* \nabla u_0(x) = f \quad in\ \Omega, \\ u_0 = 0 \quad on\ \ \partial\Omega, \end{cases} \tag{1.3.15}$$

In (1.3.15), the homogenized diffusion tensor $A^* = (a^*_{ij})_{1 \leq i,j \leq n}$ is constant, elliptic and given by

$$A^*_{ij} = \int_Y a_{ik}(y)\big(\delta_{kj} + \nabla_{y_k} \chi^j(y)\big) \tag{1.3.16}$$

where $\chi^j(y)$ are defined, as the unique solutions in $H^1_\sharp(Y)/\mathbb{R}$ of the so-called *cell problems*

$$\begin{cases} -\, div_y(A(\vec{e}_j + \nabla_y \chi^j(y))) = 0 \ \ inY, \\ y \longrightarrow \chi^j(y) \qquad\qquad Y - periodic, \end{cases} \tag{1.3.17}$$

with $(\vec{e}_j)_{1 \leq j \leq n}$, the canonical basis of $\mathbb{R}^n$.

Before start proving the above theorem we need to recall the weak limits theorem of rapidly oscillating periodic functions.

Theorem 8.

Let $1 \leq p \leq +\infty$ and f be a Y-periodic function in $L^p(Y)$. Set

$$f_\varepsilon(x) = f\big(\frac{x}{\varepsilon}\big) \quad a.e.\ on\ \mathbb{R}^n.$$

Then, if $p < \infty$, as $\varepsilon \to 0$

$$f_\varepsilon \rightharpoonup M_Y(f) = \frac{1}{|Y|}\int_Y f(y)dy \ weakly\ in\ L^p(\Omega),$$

for any bounded open subset Ω of $\mathbb{R}^n$.

If $p = \infty$, one has

$$f_\varepsilon \rightharpoonup M_Y(f) = \frac{1}{|Y|}\int_Y f(y)dy \ weakly^*\ in\ L^\infty(\Omega).$$

Proof. (proof of theorem 7) The proof will be divided into 3 steps.

Step 1: **Existence and uniqueness**

We start the demonstration by proving the existence and uniqueness of of the solution u_ε, the variational formulation of (1.3.1) is given by

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon \nabla v_\varepsilon = \int_\Omega fv \quad \forall v \in H_0^1(\Omega). \tag{1.3.18}$$

The existence and uniqueness of of the solution u_ε follow immediately by Lax-Milgram theorem.

Step 2: **A priori estimate**

From (1.3.10), we have that $u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_\varepsilon \to u_0$ strongly in $L^2(\Omega)$. This implies that ∇u_ε is bounded in $(L^2(\Omega))^n$, which further implies that up to a subsequence, $\nabla u_\varepsilon \rightharpoonup \nabla u_0$ weakly in $(L^2(\Omega))^n$, so

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 & weakly\ in\ H_0^1(\Omega), \\ u_\varepsilon \rightharpoonup u_0 & stongly\ in\ L^2(\Omega), \end{cases} \tag{1.3.19}$$

Introduce now

$$\xi^\varepsilon = (\xi_1^\varepsilon, \xi_2^\varepsilon, .., \xi_n^\varepsilon) = \left(\sum_{j=1}^n a_{1j}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}, a_{2j}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}, .., a_{nj}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j} \right) = A_\varepsilon \nabla u_\varepsilon. \tag{1.3.20}$$

From (1.3.18), it is easily seen that ξ^ε, satisfies

$$\int_\Omega \xi^\varepsilon \nabla v_\varepsilon = \int_\Omega fv \quad \forall v \in H_0^1(\Omega). \tag{1.3.21}$$

It is self-evident that using the ellipticity of the matrix A_ε and the a priori estimate (1.3.10), yields

$$\|\xi^\varepsilon\|_{(L^2(\Omega))^n}. \tag{1.3.22}$$

Hence, we can extract a subsequence still denoted by ξ^ε such that

$$\xi^\varepsilon \rightharpoonup \xi^* \quad \textit{weakly in } (L^2(\Omega))^n.$$

Passing to the limit in (1.3.21), leads to

$$\begin{aligned}\lim_{\varepsilon \longrightarrow 0} \int_\Omega \xi^\varepsilon \nabla v dx = \int_\Omega \xi^* \nabla v dx = \int f v dx, \forall v \in H_0^1(\Omega)\\ \Rightarrow - \int_\Omega div(\xi^*) v dx = \int f v dx, \forall v \in H_0^1(\Omega)\\ \Rightarrow -div \xi^* = f \quad in\ \Omega.\end{aligned} \tag{1.3.23}$$

Now we are left with the task of determining the equation verified by ξ^*, and this is what involves the following step.

Step 3: The limit problem (the homogenized problem)

Showing now that

$$\xi^* = A^* \nabla u_0.$$

Set

$$w_\varepsilon^j = \varepsilon w^j\left(\frac{x}{\varepsilon}\right) = -\varepsilon \hat{\chi}^j + e_j x, \quad j = 1, ..., n,$$

it is obvious that

$$\begin{cases} w_\varepsilon^j \rightharpoonup e_j x & \textit{weakly in } H^1(\Omega), \\ w_\varepsilon^j \rightarrow e_j x & \textit{strongly in } L^2(\Omega), \\ \nabla w_\varepsilon^j \rightharpoonup e_j & \textit{weakly in } (L^2(\Omega))^n. \end{cases} \tag{1.3.24}$$

where $\hat{\chi}$ are not the solutions of the cell problems, defined in (1.3.17), but that of the dual cell problems

$$\begin{cases} -div\left(A_\varepsilon^t\left(\nabla_y \hat{\chi}^j + \vec{e}_j\right)\right) = 0 & in\ Y, \\ y \rightarrow \hat{\chi}^j(y) & Y - periodic. \end{cases} \tag{1.3.25}$$

Set now

$$\eta_\varepsilon^j(\eta_1^\varepsilon,\eta_2^\varepsilon,..,\eta_n^\varepsilon) = \left(\sum_{j=1}^n a_{j1}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}, a_{j2}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}, .., a_{jn}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_j}\right) = A_\varepsilon^t \nabla w_\varepsilon^j. \tag{1.3.26}$$

Observe that

$$\begin{aligned}\eta_\varepsilon^j &= A_\varepsilon^t(\frac{x}{\varepsilon})\nabla w_\varepsilon^j(\frac{x}{\varepsilon})\\ &= \left(A_\varepsilon^t \nabla w_\varepsilon^j\right)(\frac{x}{\varepsilon})\\ &= \left(A_\varepsilon^t \nabla \hat{\chi}^j + A^t \vec{e}_j\right)(\frac{x}{\varepsilon}).\end{aligned}$$

Since $A_\varepsilon^t \nabla \hat{\chi}^j\left(\frac{x}{\varepsilon}\right)$ and $A_\varepsilon^t\left(\frac{x}{\varepsilon}\right)$ are periodic functions, Hence, applying Theorem 8 one derives the convergence

$$\eta_\varepsilon^j \rightharpoonup \langle A_\varepsilon^t \nabla w_\varepsilon^j\rangle = \langle A_\varepsilon^t \nabla \hat{\chi}^j + A^t \vec{e}_j\rangle = \left(A^*\right)^t \vec{e}_j \quad \textit{weakly in } (L^2(\Omega))^n. \tag{1.3.27}$$

We can show easily (see for instance ....) that η_ε^j verifies

$$\int_\Omega \eta_\varepsilon^j \nabla v = 0 \quad \forall v \in H_0^1(\Omega). \tag{1.3.28}$$

Let $\varphi \in \mathcal{D}(\Omega)$ and choose φw_ε^j as test function in (1.3.21) and φu_ε as test function in (1.3.28). We have respectively,

$$\begin{cases} \int_\Omega \xi^\varepsilon \nabla w_\varepsilon^j \varphi + \int_\Omega \xi^\varepsilon \nabla \varphi w_\varepsilon^j = \int_\Omega f \varphi w_\varepsilon^j, \quad \forall \varphi \in \mathcal{D}(\Omega),\\ \int_\Omega \eta_\varepsilon^j \nabla u_\varepsilon \varphi + \int_\Omega \eta_\varepsilon^j \nabla \varphi u_\varepsilon = 0, \quad \forall \varphi \in \mathcal{D}(\Omega). \end{cases} \tag{1.3.29}$$

See that from the definitions (1.3.20) and (1.3.26), one has

$$\xi^\varepsilon \nabla w_\varepsilon^j = A^\varepsilon \nabla u_\varepsilon \nabla w_\varepsilon^j = A^t \nabla w_\varepsilon^j \nabla u_\varepsilon = \eta_\varepsilon^j \nabla u_\varepsilon.$$

Therefore by subtraction, the first integrals in the expressions above cancel and we obtain

$$\int_\Omega \xi^\varepsilon \nabla \varphi w_\varepsilon^j - \int_\Omega \eta_\varepsilon^j \nabla \varphi u_\varepsilon = \int_\Omega f \varphi w_\varepsilon^j, \quad \forall \varphi \in \mathcal{D}(\Omega)$$

Making use of the convergences (1.3.19), (1.3.7.2), (1.3.27) and (1.3.24), one can now pass to the limit in this identity and get

$$\int_\Omega \xi^* \vec{e}_j x \nabla \varphi - \int_\Omega (A^*)^t \vec{e}_j u_0 \nabla \varphi = \int_\Omega f \vec{e}_j x \varphi, \quad \forall \varphi \in \mathcal{D}(\Omega). \tag{1.3.30}$$

Choosing $e_j x\varphi$ as test function in the last equation of (1.3.23)

$$\int_\Omega \xi^* \nabla\varphi x_j dx + \int_\Omega \xi^* \varphi \vec{e}_j dx = \int_\Omega f x_j \varphi dx \quad \forall\varphi \ \in \mathcal{D}(\Omega) \tag{1.3.31}$$

□

Substituting (1.3.31) in (1.3.30) gives

$$\int_\Omega \xi^* \vec{e}_j \varphi dx = -\int_\Omega \left(A^*\right)^t \vec{e}_j \nabla\varphi u_0 dx \tag{1.3.32}$$

we derivate the left-hand side integral of (1.3.32) in the sense of distribution with taking into account the fact that $\left(A^*\right)^t$ is constant, we get

$$\int_\Omega \xi_j^* \varphi dx = \int_\Omega \nabla\left(\left(A^*\right)^t u_0\right)_j \varphi dx, \quad \forall\varphi \in \mathcal{D}(\Omega). \tag{1.3.33}$$

Hence

$$\xi_j^* = \left(\left(A^*\right)^t \nabla u_0\right)_j.$$

This ends the proof of Theorem 7.

Remark 1.3.4. *As a final comment, let us reemphasize that the energy method is not restricted to the periodic case and works without any assumption about the behavior of the sequence of the diffusion tensor. The energy method is also valid for some nonlinear problems involving convex minimization (see Subsection 1.3.4 and references therein), and monotone operators (corresponding to non-symmetric problems).*

For more details on the energy method see.............

1.3.8 The Compensated Compactness method

This was introduced by L. Tartar [102] and F. Murat [81] in the 1970s, for the of vector-valued (systems of nonlinear PDEs). First, they proved that under certain conditions on the derivatives of weakly converging sequences, the product of two of such sequences converge to the product of their limits in the sense of distributions. This result is known as the **Div-curl lemma** which it is applicable to non-periodic problems and nonlinear homogenization problems. In the study of elliptic problems in divergence forms, this lemma comes in handy. However, one can not apply it to any quadratic product because it requires some specific conditions on the derivatives of the weakly converging quantities. See [102], [81], [50] for more details on Compensated Compactness, a prototype of the result is given below.

Definition 1.3.6. *Given a vector* $\omega \in (L^2(\Omega))^n$. *The matrix* $(curl\omega)_{ij}$ *is defined by:*

$$(curl\ \omega)_{ij} = \frac{\partial \omega_i}{\partial x_j} - \frac{\partial \omega_j}{\partial x_i} \qquad for\ i, j = 1, ..., n.$$

Lemma 1.3.1. *(Div-Curl Lemma) Let* $P_\varepsilon, P_0, V_\varepsilon,$ *and* V_0 *be vector fields in* $L^2(\Omega)$ *such that*

$$P_\varepsilon \rightharpoonup P_0, V_\varepsilon \rightharpoonup V_0\ in\ L^2(\Omega)\ as\ \varepsilon \to 0. \tag{1.3.34}$$

If in addition

$$divP_\varepsilon \rightharpoonup divP_0\ in\ H^{-1}(\Omega),\ and\ curl\ V_\varepsilon = 0, \tag{1.3.35}$$

then

$$P_\varepsilon V_\varepsilon \rightharpoonup P_0 V_0\ in\ \mathcal{D}'(\Omega), \tag{1.3.36}$$

Recall that the convergence as distributions (in $\mathcal{D}'(\Omega)$*) in (1.3.36) means that*

$$\forall \phi \in C_0^\infty(\Omega), \int_\Omega P_\varepsilon V_\varepsilon \phi\ dx \longrightarrow \int_\Omega P_0 V_0 \phi\ dx$$

Remark 1.3.5. *The name compensated compactness comes from the fact that the additional properties (1.3.35) compensate for the lack of strong convergence of the factors in the product which in general is needed for passing to weak limits in the product.*

Proof. See [21]. □

There are different variants of the div-curl lemma that can be applied to various problems, the relation between the **div-curl lemma** and the homogenization can be viewed in the proof of the following theorem for the classical problem (1.3.1)

Theorem 9.

Let u_ε be the weak solution of problem (1.3.1) with $f \in L^2(\Omega)$ and $A_\varepsilon \in \mathcal{M}_s(\alpha, \beta, \Omega)$ is Y-periodic. Then

1. $u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$,
2. $A_\varepsilon \nabla u_\varepsilon \rightharpoonup A^* \nabla u_0$ weakly in $(L^2(\Omega))^n$

Furthermore, $u_0 \in H_0^1(\Omega)$ is the weak solution to the homogenized problem:

$$\begin{cases} -div\Big(A^* \nabla u_0\Big) = f \quad in\ \Omega, \\ u_0 = 0 \quad on \ \ \partial\Omega, \end{cases} \tag{1.3.37}$$

and the coefficients of the homogenized matrix A^* are given by

$$a_{ij}^* = \int_Y [a_{ij}(y) - a_{ik}\frac{\partial \chi^j}{\partial y_k}(y)]dy,$$

where $\chi^j(y)$ are the weak solutions in $H_\sharp^1(Y)$ to the cell problem

$$\begin{cases} -div\Big(A(y)\chi^j(y)\Big) = -\dfrac{\partial a_{ij}}{\partial y_i}(y) \qquad in\ Y; \\ \displaystyle\int_Y \chi^j(y)dy = 0. \end{cases} \tag{1.3.38}$$

Proof. From (1.3.10), we have that $u_\varepsilon \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$ and $u_\varepsilon \to u_0$ strongly in $L^2(\Omega)$. This implies that ∇u_ε is bounded in $(L^2(\Omega))^n$, which further implies that up to a subsequence, $\nabla u_\varepsilon \rightharpoonup \nabla u_0$ weakly in $(L^2(\Omega))^n$. If A_ε converges strongly to A^* then we can pass to the limit. But dealing with composite materials, one cannot have a strong convergence of the matrix A_ε.

From the membership of A_ε to $\mathcal{M}_s(\alpha, \beta, \Omega)$ one has weakly* convergence of A_ε to A^* in $L^\infty(\Omega)^{n\times n}$, which implies weak convergence in $L^2(\Omega)^{n\times n}$, to A^*.

That leaves us to finding the limit of the product of two weakly convergent sequences $A_\varepsilon \nabla u_\varepsilon$. As mentioned earlier, this is not straightforward and generally, the product of two weakly convergences does not converge to the product of their limit, hence we employ the **div-curl** Lemma.

Recall the weak formulation of (1.3.1)

$$\int_\Omega A(\frac{x}{\varepsilon})\nabla u_\varepsilon.\nabla\phi = \int_\Omega f\phi \quad \forall\varphi\in H_0^1(\Omega). \tag{1.3.39}$$

To resolve the difficulty of the limit of the product of two weakly convergent sequences $A_\varepsilon\nabla u_\varepsilon$, one can choose special test functions $\phi = \phi_\varepsilon \in H_0^1(\Omega)$ which depend on ε in such a way that we can apply the Div-Curl Lemma. Given an arbitrary test function $\phi \in C_0^\infty(Y)$ (a dense subset of $H_0^1(\Omega)$), we construct a special set of oscillating test functions, ϕ_ε such that the following conditions hold:

(H_1) $\phi_\varepsilon \rightharpoonup \phi$ weakly in $L^2(\Omega)$;

(H_2) $div\Big(A(\frac{x}{\varepsilon})\nabla\phi_\varepsilon\Big) \to div\Big(A^*\nabla\phi\Big)$ strongly in $H^{-1}(\Omega)$;

(H_3) $A(\frac{x}{\varepsilon})\nabla\phi_\varepsilon \rightharpoonup A^*\nabla\phi$ weakly in $L^2(\Omega)$;

Step 1: Passing to the limit in (1.3.39). under the Assumption that there exists a family of test functions satisfying properties (H1)(H3).

Set

$$P_\varepsilon := A(\tfrac{x}{\varepsilon})\nabla\phi_\varepsilon,$$

$$P_0 = A^*\nabla\phi.$$

Note that (H3) implies that $P_\varepsilon \rightharpoonup P_0$ weakly in $L^2(\Omega)$, and that (H2) implies that $divP_\varepsilon \to divP_0$ strongly in $H^{-1}(\Omega)$. Set

$$V_\varepsilon := \nabla u_\varepsilon$$

observe that

$curlV_\varepsilon = curl\nabla u_\varepsilon = 0$. All of the hypotheses of the Div-Curl Lemma hold and thus we can pass to the limit in the product of weakly convergent sequences in the left-hand side of (1.3.39) after taking ϕ_ε as a test function

$$\int_\Omega A(\frac{x}{\varepsilon})\nabla u_\varepsilon\nabla\phi_\varepsilon \longrightarrow \int_\Omega A^*\nabla u_0\nabla\phi. \tag{1.3.40}$$

For the right-hand side of (1.3.39), use (H1) to pass to the limit

$$\int_\Omega f\phi_\varepsilon dx \longrightarrow \int_\Omega f\phi dx.$$

Thus,

$$\int_\Omega A^* \nabla u_0 \nabla \phi dx = \int_\Omega f \phi dx, \quad \forall \phi \in C_0^\infty(Y). \tag{1.3.41}$$

This holds for all test functions $\phi \in C_0^\infty(\Omega)$, and by density of $C_0^\infty(\Omega)$ in $H_0^1(\Omega)$, (1.3.41) holds for every $\phi \in H_0^1(\Omega)$. Thus, Theorem9 (homogenization limit) is proved provided that we prove existence of functions ϕ_ε with properties (H1)-(H3).

Step 2: Construction of oscillating test functions ϕ_ε.

Given $\phi \in C_0^\infty(\Omega)$, set

$$\phi_\varepsilon := \phi(x) + \varepsilon \sum_{j=1}^n \frac{\partial \phi}{\partial x_j} \chi^j\left(\frac{x}{\varepsilon}\right), \tag{1.3.42}$$

where χ^j are the solutions to the cell problem (1.3.38). Condition (H1) follows immediately from the form of (1.3.42). Indeed, for all $\psi \in L^2(\Omega)$, the Cauchy-Schwarz inequality yields

$$\lim_{\varepsilon \longrightarrow 0} \left(\varepsilon \sum_{j=1}^n \int_\Omega \frac{\partial \phi}{\partial x_j} \chi^j\left(\frac{x}{\varepsilon}\right) \psi \right) \le \lim_{\varepsilon \longrightarrow 0} \left(\varepsilon \sum_{j=1}^n \|\phi\|_{C^1(\Omega)} \|\chi^j\left(\frac{x}{\varepsilon}\right)\|_{L^2(\Omega)} \|\psi\|_{L^2(\Omega)} \right) \longrightarrow 0, \tag{1.3.43}$$

since $\chi^j \in H_\sharp^1(Y)$.

To prove (H3) observe that

$$A\left(\frac{x}{\varepsilon}\right) \nabla \phi_\varepsilon = A\left(\frac{x}{\varepsilon}\right) \left[\nabla_x \phi(x) + \sum_{j=1}^n \frac{\partial \phi}{\partial x_j} \nabla_y \chi^j\left(\frac{x}{\varepsilon}\right) \right] + \varepsilon \left[A\left(\frac{x}{\varepsilon}\right) \nabla \frac{\partial \phi}{\partial x_j} \chi^j\left(\frac{x}{\varepsilon}\right) \right] \tag{1.3.44}$$

where the L^2 norm of the last term is of order ε. Take the weak L^2 limit in the right-hand side of (1.3.44) using the Averaging Lemma (we assume that $|Y| = 1$ for simplicity):

$$A\left(\frac{x}{\varepsilon}\right) \nabla \phi_\varepsilon \rightharpoonup \int_Y A(y) \big[I + \nabla_y \chi(y) \big] dy . \nabla \phi(x). \tag{1.3.45}$$

Note that the first term in the right-hand side of (1.3.44) depends not only the fast variable $y = \frac{x}{\varepsilon}$ but also on the slow variable x. However one still can apply the Averaging Lemma using the fact that each term has the form of product of a smooth function depending on x only and periodic function depending on $\frac{x}{\varepsilon}$. Indeed, for example, considering the first term in (1.3.44) we have by Averaging Lemma

$$\int_\Omega A\left(\frac{x}{\varepsilon}\right) \nabla_x \phi(x) \psi(x) dx \longrightarrow \int_\Omega \left[\int_Y A(y) dy \right] \nabla_x \phi(x) \psi(x) dx, \tag{1.3.46}$$

for arbitrary function $\psi \in L^2(\Omega)$, therefore $A\left(\frac{x}{\varepsilon}\right)\nabla\phi_\varepsilon \rightharpoonup A^*\nabla\phi(x)$. Thus we have proved (H3).

It remains to prove the key property (H2). To this end we compute

$$div\left[A\left(\frac{x}{\varepsilon}\right)\nabla\phi_\varepsilon\right] = \frac{1}{\varepsilon}\sum_{i,j,k}\frac{\partial\phi}{\partial x_j}(x)\frac{\partial}{\partial y_i}\left[A_{ik}(y)\left(\delta_{kj}+\frac{\partial\chi^j}{\partial y_k}\right)\right]$$
$$+A\left(\frac{x}{\varepsilon}\right)\Delta\phi(x)+A\left(\frac{x}{\varepsilon}\right)\sum_{i,j}\frac{\partial^2\phi}{\partial x_j\partial x_i}\frac{\partial\chi^j}{\partial y_i}\left(\frac{x}{\varepsilon}\right) \tag{1.3.47}$$
$$+\varepsilon div\left[A\left(\frac{x}{\varepsilon}\right)\sum_j\left(\nabla\frac{\partial\phi}{\partial x_j}\right)(x)\chi^j\right] := I_{-1}^{(\varepsilon)}+I_0^{(\varepsilon)}+I_1^{(\varepsilon)}.$$

The first term $I_{-1}^{(\varepsilon)}$ actually zero since functions χ^j are solutions of the cell problem. The second term $I_0^{(\varepsilon)}$ converges weakly in L^2 to

$$\begin{aligned} I_0^{(\varepsilon)} &\rightharpoonup \int_Y\left[A(y)\Delta\phi(x)+A(y)\sum\frac{\partial^2\phi}{\partial x_j\partial x_i}\frac{\partial\chi^j}{\partial x_i}(y)\right]dy \\ &=\sum_j\frac{\partial}{\partial x_j}\int_Y\left[A(y)\frac{\partial\phi}{\partial x_j}+A(y)\sum_i\frac{\partial\phi}{\partial x_i}\frac{\partial\chi^j}{\partial x_i}\right] \\ &=\sum_i\sum_j\frac{\partial}{\partial x_j}\frac{\partial\phi}{\partial x_j}e_i\int_Y A(y)\left[e_j+\nabla_y\chi^j\right]dy \\ &= div\left[A^*\nabla\phi\right]. \end{aligned} \tag{1.3.48}$$

As above, (1.3.48) can be proved by applying the Averaging Lemma. Thus $I_0^{(\varepsilon)}$ converges to $div\left[A^*\nabla\phi\right]$ strongly in $H^{-1}(\Omega)$. Indeed, weak convergence in $L^2(\Omega)$ implies boundedness in $L^2(\Omega)$ which in turn by the compactness of the embedding $L^2(\Omega)\subset H^{-1}(\Omega)$ implies strong convergence in $H^{-1}(\Omega)$. Finally, $I_1^{(\varepsilon)}$ converges to 0 strongly in $H^{-1}(\Omega)$. Indeed, $I_1^{(\varepsilon)}$ has the form $I_1^{(\varepsilon)} = \varepsilon div F_\varepsilon$ with $F_\varepsilon := A\left(\frac{x}{\varepsilon}\right)\sum\left(\nabla\frac{\partial\phi}{\partial x_j}\right)(x)\chi^j(\frac{x}{\varepsilon})\Big)$.

Recall that for any vector-valued function $u\in L^2(\Omega)$ one can define $div\ u \in H^{-1}(\Omega)$ by the formula

$$\big(div\ u,\phi\big)=\int_\Omega u.\nabla\phi\ dx,\quad \forall\phi\in H_0^1(\Omega). \tag{1.3.49}$$

Therefore

$$\Big|\langle div F_\varepsilon,\varphi\rangle_{H^{-1},H_0^1(\Omega)}\Big| = \Big|\langle F_\varepsilon,\nabla\varphi\rangle_{L^2,L^2}\Big| \le \|F_\varepsilon\|_{L^2}\|\varphi\|_{H_0^1}\le C\|\varphi\|_{H_0^1},$$

i.e., $\|I_1^{(\varepsilon)}\|_{H^{-1}} = O(\varepsilon)$.Thus conditions (H1), (H2), and (H3) are satisfied, and the proof of Theorem 9 (the homogenization limit) is complete. □

1.3.9 Two-Scale Convergence

1.3.9.1 A Brief Presentation

Contrary to the previous homogenization methods, the two-scale convergence method is devoted only to periodic homogenization problems. It is, therefore, a less general method than the Γ, G, and H-convergence, but, because it is dedicated to periodic homogenization, it is also more efficient and simple in this context. Two-scale convergence was introduced by Nguetseng [83] and Allaire [4]. It has been further generalized to the stochastic setting of homogenization in [22], thus, considerably extending its scope. The next sub-subsection is concerned with the main theoretical results which are at the root of this method, whereas the last sub-subsection contains a detailed application of the method on a simple model problem. Before going into the details of the two-scale convergence method, let us explain its main idea and the reasons for its success. In periodic homogenization problems, it is well-known that the homogenized problem can be heuristically obtained by using the two-scale asymptotic expansions as described in many textbooks (see, e.g., [13], [18], [99]). Denoting the size of the periodic heterogeneities (a small number which goes to zero in this asymptotic process) by ε and the sequence (indexed by ε) of solutions of the considered partial differential equation with periodically oscillating coefficients by u_ε, a two-scale asymptotic expansion is an ansatz of the form,

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \varepsilon^3 u_3(x, \frac{x}{\varepsilon}) + \varepsilon^4 u_4(x, \frac{x}{\varepsilon}) + ... \tag{1.3.50}$$

where each function $u_i(x, y)$ in this series depends on two variables, x the macroscopic (or slow) variable and y the microscopic (or fast) variable, and is Y-periodic in y (Y is the unit period). Inserting the ansatz (1.3.50) in the equation satisfied by u_ε and identifying powers of ε leads to a cascade of equations for each term $u_i(x, y)$. In general, averaging with respect to y yields the homogenized equation for u_0. Unfortunately, mathematically, this method of two-scale asymptotic expansions is only formal because, a priori, there is no reason for the ansatz (1.3.50) to hold true. Thus, another step is required to rigorously justify the homogenization result obtained heuristically with this two-scale asymptotic expansion (see, for example, the energy method). Despite its frequent success in homogenizing many different types of equations, this method is not entirely satisfactory because it involves two steps, formal derivation and rigorous justification of the homogenized problem, which have little in common and are partly redundant. Consequently, there is room for a more efficient method which will combine these two steps in a single, simpler one. This is exactly the purpose of *the two-scale convergence* method which is based on a new type of convergence (see Definition 1.3.7). Roughly speaking, *the two-scale convergence* is a rigorous justification of the first term of the ansatz (1.3.50) for any bounded sequence u_ε, in the sense that it asserts the existence of a two-scale limit $u_0(x, y)$, such that u_ε, tested again any periodically oscillating test function, converges to $u_0(x, y)$:

$$\int_\Omega u_\varepsilon(x)\varphi(x, \frac{x}{\varepsilon})dx \stackrel{2-s}{\longrightarrow} \int_\Omega \int_Y u_0(x, y)\varphi(x, y)dxdy \tag{1.3.51}$$

Two-scale convergence is an improvement over the usual weak convergence because equation (1.3.51) measures the periodic oscillations of the sequence u_ε. The two-scale convergence method is based on this result: multiplying the equation satisfied by u_ε with an oscillating test function $\varphi(x, \frac{x}{\varepsilon})$ and passing to the two-scale limit automatically yields the homogenized problem.

1.3.9.2 Statement of the Principal Results

Let us begin this subsection with a few notations. Ω is an open set of $\mathbb{R}^n$ (not necessarily bounded), and $Y = (0, 1)^n$ is the unit cube.

Definition 1.3.7. *A sequence of functions u_ε in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0(x, y)$ belonging to $L^2(\Omega \times Y)$ if, for any function $\varphi(x, y)$ in $D(\Omega; C^\infty_\sharp(Y))$, it satisfies*

$$\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\varphi(x, \frac{x}{\varepsilon})dx \longrightarrow \int_\Omega \int_Y u_0(x, y)\varphi(x, y)dxdy \tag{1.3.52}$$

This notion of "two-scale convergence" makes sense because of the next compactness theorem.

Theorem 10.

From each bounded sequence u_ε in $L^2(\Omega)$, one can extract a subsequence, and there exists a limit $u_0(x, y) \in L^2(\Omega \times Y)$ such that this subsequence two-scale converges to u_0.

Proof. See [4]. □

We give now a few examples of two-scale convergences.

1. Any sequence u_ε which converges strongly in $L^2(\Omega)$ to a limit u, two-scale converges to the same limit u.

2. For any smooth function $u_0(x,y)$, Y-periodic in y, the associated sequence $u_\varepsilon(x) = u_0(x, \dfrac{x}{\varepsilon})$ two-scale converges to $u_0(x,y)$.

3. For the same smooth and Y-periodic function $u_0(x,y)$, the sequence defined by $v_\varepsilon = u_0(x, \dfrac{x}{\varepsilon^2})$ has the same two-scale limit and weak-L^2 limit, namely, $\int_Y u_0(x,y)dy$ (this is a consequence of the difference of orders in the speed of oscillations for v_ε and the test functions $\varphi(x, \dfrac{x}{\varepsilon})$. Clearly, the two-scale limit captures only the oscillations which are in resonance with those of the test functions $\varphi(x, \dfrac{x}{\varepsilon})$.

4. Any sequence u_ε which admits an asymptotic expansion of the type

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \varepsilon^3 u_3(x, \frac{x}{\varepsilon}) + \varepsilon^4 u_4(x, \frac{x}{\varepsilon}) + ...,$$

where the functions $u_i(x,y)$ are smooth and Y-periodic in y, two-scale converges to the first term of the expansion, namely, $u_0(x,y)$.

Lemma 1.3.2. *(**Generalized Averaging Lemma**) Assume that $f(x,y)$ is Y-periodic in y and $f \in C(\Omega; C_{per}(Y))$, then,*

$$\lim_{\varepsilon \to 0} \int_\Omega f(x, \frac{x}{\varepsilon}) g(x) dx = \int_\Omega \left(\int_Y f(x,y) \right) g(x) dx, \quad \forall g \in L^2(\Omega).$$

Remark 1.3.6. *Let us summarize the relations between weak-L^2, strong-L^2, and two-scale convergences:*

- *Strong-L^2 convergence implies two-scale convergence.*

- *Two-scale convergence implies weak-L^2 convergence*

So, if the strong-L^2 limit exists, then the two-scale limit also exists and the limits agree. In contrast, if the two-scale limit exists, then a weak-L^2 limit also exists but these limits may be different. Namely, the weak-L^2 limit can be obtained by averaging the two-scale limit in the y variable over its period, as the following example shows.

Example 1.3.1. *Let $u_\varepsilon = sin\big(\frac{x}{\varepsilon}\big)$, $x \in [0, 2\pi]$. Since $Y = [0, 2\pi]$ is a periodic cell, and u_ε is bounded,then we can apply the generalized Averaging Lemma to deduce*

$$\int_0^{2\pi} sin\big(\frac{x}{\varepsilon}\big)\phi(x)\Phi\big(\frac{x}{\varepsilon}\big) \longrightarrow \int_\Omega \int_Y sin(y)\phi(x)\Phi(y).$$

By definition 1.3.7 of two-scale convergence we deduce $sin(\frac{x}{\varepsilon}) \overset{2-sc}{\rightharpoonup} sin(y)$*. However, considering the weak limit, we can apply the (regular) Averaging Lemma to see that*

$$sin(\frac{x}{\varepsilon}) \rightharpoonup \int_Y sin(y) = 0 \text{ weakly in } L^2(\Omega).$$

Conversely, can we have a weakly convergent sequence that does not two-scale converge? The following example answers this question.

Example 1.3.2. *Let* $u_n = (-1)^n sin(nx),\ \varepsilon = \frac{1}{n}$*. In the weak sense, we know that* u_n *converges to*

$$\int_0^{2\pi} (-1)^n sin\left(\frac{x}{1/n}\right)dx = \int_Y sin(y) = 0.$$

If $n = 2k,\ k \in N$*, then by the generalized Averaging Lemma, we have that* $u_{2k} \overset{2-sc}{\rightharpoonup} sin(y)$ *However, when* $n = 2k+1$*, then we have* $u_{2k+1} \overset{2-sc}{\rightharpoonup} -sin(y)$*. Therefore, a two-scale limit for* u_n *does not exist.*

Theorem 11.

Let u_ε be a sequence of functions in $L^2(\Omega)$ which two-scale converges to a limit $u_0(x, y) \in L^2(\Omega \times Y)$.

1. Then, u_ε converges weakly in $L^2(\Omega)$ to $u = \int_Y u_0(x, y)dy$, and

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|^2_{L^2(\Omega)} \geq \|u_0\|^2_{L^2(\Omega \times Y)} \geq \|u\|^2_{L^2(\Omega)}. \quad (1.3.53)$$

2. Assume,further, that $u_0(x, y)$ is smooth and that

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|^2_{L^2(\Omega)} = \|u_0\|^2_{L^2(\Omega \times Y)}. \quad (1.3.54)$$

Then,

$$\|u_\varepsilon - u_0(x, \frac{x}{\varepsilon})\|^2_{L^2(\Omega)} \to 0. \quad (1.3.55)$$

Proof. See [5]. □

Remark 1.3.7. *The smoothness assumption on* u_0 *in the second part of Theorem 11 is needed only to ensure the measurability of* $u_0(x, \frac{x}{\varepsilon})$ *(which otherwise is not guaranteed for a function of* $L^2(\Omega \times Y)$*. One can further check that any function in* $L^2(\Omega \times Y)$ *is attained as a two-scale limit (see Lemma*

1.13 in [4]), which implies that two-scale limits have no extra regularity. So far we have considered only bounded sequences in $L^2(\Omega)$. The next theorem investigates the case of a bounded sequence in $H^1(\Omega)$.

Theorem 12.

Let u_ε be a bounded sequence in $H^1(\Omega)$. Then, up to a subsequence, $u_\varepsilon$two-scale converges to a limit $u \in H^1(\Omega)$, and ∇u_ε two-scale converges to

$$\nabla_x u(x) + \nabla_y u_1(x, y),$$

where the function $u_1(x, y)$ belongs to $L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$.

Proof. See [4]. □

Remark 1.3.8. *There are many generalizations of Theorem 12 which gives the precise form of the two-scale limit of a sequence of functions for which some extra estimates on part of their derivatives are available. To obtain as much information as possible on the two-scale limit is a key point in applying the two-scale convergence method, as described in the next subsection. For completeness, we give an examples below of such generalizations of Theorem 12, the proofs of which may be found in [4].*

Theorem 13.

1. Let u_ε be a bounded sequence in $L^2(\Omega)$, such that $\varepsilon \nabla u_\varepsilon$ is also bounded in $L^2(\Omega)^n$. Then, there exists a two-scale limit $u_0(x, y) \in L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$ such that, up to a subsequence, u_ε two-scale converges to $u_0(x, y)$, and $\varepsilon \nabla u_\varepsilon$ to $\nabla_y u_0(x, y)$.

2. Let u_ε be a bounded sequence of vector-valued functions in $L^2(\Omega)^n$, such that its divergence $div u_\varepsilon$ is also bounded in $L^2(\Omega)$. Then, there exists a two-scale limit $u_0(x, y) \in L^2(\Omega \times Y)^n$ which is divergence-free with respect to y, i.e., $div_y u_0 = 0$, has a divergence with respect to x, $div_x u_0$, in $L^2(\Omega \times Y)$, and such that, up to a subsequence, u_ε two-scale converges to $u_0(x, y)$ and $div u_\varepsilon$ to $div_x u_0(x, y)$.

1.3.9.3 Application to a Model Problem

This sub-subsection shows how the notion of two-scale convergence can be used for homogenizing partial differential equations with periodically oscillating coefficients. Our purpose is to give a tutorial on the two-scale convergence method, Therefore, the usual model problem of diffusion in a periodic medium is reconsidered. Of course, the principles of the two-scale convergence method are valid in many other cases with only slight changes, including nonlinear (monotone or convex) problems. We now describe the so-called two-scale convergence method for homogenizing problem (1.3.1), where the tensor of diffusion $A_\varepsilon \in L^\infty(Y)^{n\times n}$, is not necessarily symmetric.

In a **first step**, we deduce the precise form of the two-scale limit of the sequence u_ε from the a priori estimate (1.3.10). By application of Theorem 12, there exist two functions, $u \in H_0^1(\Omega)$ and $u_1(x,y) \in L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$, such that, up to a subsequence, u_ε two-scale converges to u, and ∇u_ε two-scale converges to $\nabla_x u(x) + \nabla_y u_1(x,y)$. In view of these limits, u_ε is expected to behave as $u(x) + \varepsilon u_1(x,y)$.

Then, in a **second step**, we multiply equation (1.3.1) by a test function similar to the limit of u_ε, namely, $\varphi(x) + \varepsilon\varphi_1(x, \frac{x}{\varepsilon})$, where $\varphi(x) \in D(\Omega)$ and $\varphi_1(x, \frac{x}{\varepsilon}) \in D(\Omega; C^\infty_\sharp(Y))$. This yields

$$\int_\Omega A(\frac{x}{\varepsilon})\nabla u_\varepsilon.\left(\nabla\varphi(x) + \nabla_y\varphi_1(x,\frac{x}{\varepsilon}) + \varepsilon\nabla_x\varphi_1(x,\frac{x}{\varepsilon})\right)dx = \int_\Omega f\left(\varphi(x) + \varepsilon\varphi_1(x,\frac{x}{\varepsilon})\right)dx. \quad (1.3.56)$$

Regarding $A^t(\frac{x}{\varepsilon})\left(\nabla\varphi(x) + \nabla_y\varphi_1(x, \frac{x}{\varepsilon})\right)$ as a test function for the two-scale convergence (see remark 9.4 in [32]), we pass to the two-scale limit in (1.3.56) for the sequence ∇u_ε. Although this test function is not necessarily very smooth, as required by Definition 1.3.7. Thus, the two-scale limit of equation (1.3.56) is given by

$$\int_\Omega\int_Y A(y)\left(\nabla_x u(x) + \nabla_y u_1(x,y)\right).\left(\nabla\varphi(x) + \nabla_y\varphi_1(x,y)\right)dxdy = \int_\Omega f(x)\varphi(x). \quad (1.3.57)$$

In a **third step**, we read off a variational formulation for $(u, u_1,)$ in (1.3.57). Note that (1.3.57) holds true for any (φ, φ_1) in the Hilbert space $H_0^1(\Omega) \times L^2(\Omega; H^1_\sharp(Y)/\mathcal{R})$ by density of smooth functions in this space. Endowing it with the norm

$$\sqrt{\left(\|\nabla u(x)\|^2_{L^2(\Omega)} + \|\nabla_y u_1(x,y)\|^2_{L^2(\Omega\times Y)}\right)}.$$

The assumptions of the Lax-Milgram lemma are easily checked for the variational formulation (1.3.57). The main point is the coercivity of the bilinear form defined by the left-hand side of (1.3.57). The coercivity of A yields

$$\begin{aligned}
&\int_\Omega\int_Y A(y)\Big(\nabla_x u(x)+\nabla_y u_1(x,y)\Big).\Big(\nabla u(x)+\nabla_y u_1(x,y)\Big)dxdy\\
&\geq \alpha\int_\Omega\int_Y |\nabla u(x)+\nabla_y u_1(x,y)|^2 dxdy\\
&= \alpha\int_\Omega |\nabla u(x)|^2 dx+\alpha\int_\Omega\int_Y |\nabla_y u_1(x,y)|^2 dxdy
\end{aligned} \tag{1.3.58}$$

By applying the Lax-Milgram lemma, we conclude that there exists a unique solution (u, u_1) of the variational formulation (1.3.57) in $H_0^1(\Omega)\times L^2(\Omega; H_\sharp^1(Y)/\mathcal{R})$. Consequently, the entire sequences u_ε and ∇u_ε converge to u and $\nabla_x u(x)+\nabla_y u_1(x,y)$ respectively. An easy integration by parts shows that (1.3.57) is a variational formulation associated with the following system of equations, the so-called "two-scale homogenized problem":

$$\begin{cases}
-div_y\Big(A(y)\Big(\nabla_x u(x)+\nabla_y u_1(x,y)\Big)\Big) = 0 \ in\ \Omega\times Y,\\
-div_x\Big(\int_Y A(y)\Big(\nabla_x u(x)+\nabla_y u_1(x,y)\Big)\Big) = f(x)\ in\ \Omega\\
y\to u_1(x,y) \quad Y\ periodic\\
u=0\ on\ \partial\Omega.
\end{cases} \tag{1.3.59}$$

At this point, the homogenization process could be considered achieved because the entire sequence of solutions u_ε converges to the solution of a well-posed limit problem, namely, the two-scale homogenized problem (1.3.59). However, it is usually preferable, from a physical or numerical point of view, to eliminate the microscopic variable y (one does not want to solve the small scale structure). In other words, we want to extract and decouple the usual homogenized and local (or cell) equations from the two-scale homogenized problem. Thus, in a **fourth (and optional) step**, the y variable and the u_1 unknown are eliminated from (1.3.59). It is an easy algebraic exercise to prove that u_1 can be computed in terms of the gradient of u through the relationship

$$u_1(x,y)=\sum_{i=1}^n \frac{\partial u}{\partial x_i}(x)\chi^j(x,y), \tag{1.3.60}$$

where $\chi^j(y)$ are defined, at each point $x\in\Omega$, as the unique solutions in $H_\sharp^1/\mathbb{R}$. of the cell problems (see chapter 2) with $(\vec{e}_j)_{1\leq j\leq n}$ the canonical basis of $\mathbb{R}^n$. Then, plugging formula (1.3.60) into (1.3.59) yields the usual homogenized problem for u:

$$\begin{cases} -divA^*\nabla u_0 = f \quad in\ \Omega, \\ \qquad u_0 = 0 \quad on\ \ \partial\Omega, \end{cases} \tag{1.3.61}$$

where the homogenized diffusion tensor is given by its entries

$$\begin{cases} -div_y\Big(A(y)\Big(\vec{e}_j + \nabla_y \chi^j(y)\Big)\Big) = \ 0\ in\ \ Y, \\ \qquad y \to \chi^j(y) \quad\ Y\ periodic \end{cases} \tag{1.3.62}$$

$$A^*_{i,j} = \int_Y A(y)(\vec{e}_j + \nabla_y \chi^j(y)).(\vec{e}_j + \nabla_y \chi^j(y))dy. \tag{1.3.63}$$

Of course, all the above formulas coincide with those usually obtained by using asymptotic expansions. Due to the simple form of our model problem, the two equations of (1.3.59) can be decoupled in a microscopic and a macroscopic equation, (1.3.62) and (1.3.61) respectively, but we emphasize that it is not always possible. Sometimes, it leads to very complicated forms of the homogenized equation, including integro-differential operators. Thus, the homogenized equation does not always belong to a class for which an existence and uniqueness theory is easily available, contrary to the two-scale homogenized system, which, in most cases, is of the same type as the original problem, but with double the number of variables (x and y) and unknowns ($u\ and\ u_1$).

1.3.10 H-Measures

1.3.10.1 Brief presentation

The notion of H-measure has been introduced by Gérard [52] and Tartar [104]. It is a default measure which quantifies, in the phase space (i.e. the physical space times the Fourier space of propagation directions), the lack of compactness of weakly converging sequences in $L^2(\mathbb{R}^n)$. . In other words, it indicates where in the physical space, and at which frequency in the Fourier space, are the obstructions to strong convergence. As recognized by Tartar [104], this abstract tool has many important applications in the mathematical theory of composite materials. We briefly recall the necessary results on H-measures and refer to [52], [104] for complete proofs. Note that H-Measures only apply to sequences of functions that converge weakly to zero.

Theorem 14.

(Existence of H-Measures) There exists a subsequence (still denoted by ε) and a family of complex-valued Radon measures $(\mu_{ij}(x,\xi))_{1\leq ij\leq p}$ on $\mathbb{R}^n \times S_{n-1}$ such that for every $\phi_1, \phi_2 \in C_0(\mathbb{R}^n)$ and $\Psi(\xi) \in C(S_{n-1})$, , it satisfies

$$\int_\Omega \int_{S_{n-1}} \phi_1(x)\bar{\phi}_2(x)\psi\left(\frac{\xi}{|\xi|}\right)\mu_{ij}(dx,d\xi) = \lim_{\varepsilon\longrightarrow 0}\int_{\mathbb{R}^n} \mathcal{F}(\phi_1\ u_\varepsilon^i)(\xi)\overline{\mathcal{F}(\phi_2\ u_\varepsilon^j)}(\xi)\psi\left(\frac{\xi}{|\xi|}\right)d\xi.$$

The matrix of measures $\mu = (\mu_{ij})$ is called the H-measure of the subsequence u_ε. It takes its values in the set of hermitian and non-negative matrices

$$\mu = \bar{\mu}_{ij}, \sum_{i,j=1}^{p} \lambda_i\bar{\lambda}_j\mu_{ij} \geq 0, \quad \forall \lambda \in \mathbb{C}^p.$$

Let us explain the notations of Theorem 14:S_{n-1} is the unit sphere in $\mathbb{R}^n$, $C(S_{n-1})$ is the space of continuous complex-valued functions on S_{n-1}, $C_0(S_{n-1})$ that of continuous complex-valued functions decreasing to 0 at infinity in $\mathbb{R}^n$, and $\bar{z}$ denotes the complex conjugate of the complex number z. Finally,$\mathcal{F}$ is the Fourier transform operator defined in $L^2(\mathbb{R}^n)$ by

$$(\mathcal{F}\phi))(\xi) = \int_{\mathbb{R}^n} \phi(x)e^{-2i\pi(x.\xi)}dx.$$

In Theorem 14, the role of the test functions ϕ_1 and ϕ_2 is to localize in space, while that of ψ is to localize in the directions of oscillations.

Remark 1.3.9. *Theorem 14 furnishes a representation formula for the limit of quadratic objects of the sequence u_ε. When we take $\psi = 1$, we recover the usual default measure in the physical space, i.e. $\int_{S_{n-1}} \mu_{ij}(.,d\xi)$ is just the weak * limit measure of the sequence $u_\varepsilon^i\bar{u}_\varepsilon^j$, which is bounded in $L^1(\mathbb{R}^n)$. Therefore, the H-measure gives a more precise representation of the compactness default, taking into account oscillation directions.*

Theorem14 can be easily generalized to more general quadratic forms of u_ε in the context of pseudo-differential operators (see section 18.1 in [60]). Let us recall that a standard pseudo-differential operator q is defined through its symbol $\big(q_{ij}(x,\xi)\big)_{1\leq i,j\leq p}$ in $C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$(qu)_i(x) = \sum_{j=1}^{p} \mathcal{F}^{-1}\Big(q_{ij}(x,\xi)\mathcal{F}u_j(.)\Big)$$

for any smooth and compactly supported function u. In the sequel, we shall only use so-called poly-homogeneous pseudo-differential operators of order 0, i.e. whose (principal) symbol $\big(q_{ij}(x,\xi)\big)_{1\le i,j\le p}$ is homogeneous of degree 0 in ξ and with compact support in x. Recall also that a poly-homogeneous pseudo-differential operators of order 0 is a bounded operator in $L^2(\mathbb{R}^n)^p$.

Theorem 15.

(Localization of H-measures) Suppose u_ε is a sequence converging weakly to zero in $L^2(\mathbb{R}^n;\mathbb{R}^p)$ and define an H-measure μ If u_ε is such that

$$\sum_{j=1}^{p}\sum_{k=1}^{p}\frac{\partial}{\partial x_k}\Big(A_{jk}(x)u_{j\varepsilon}\Big)\longrightarrow 0 \ \textit{strongly} \ \in H^{-1}_{loc}(\Omega),$$

then

$$\sum_{j=1}^{p}\sum_{k=1}^{p}A_{jk}(x)\xi_k\mu^{jm}=0 \ =0 \qquad in \ \Omega\times S^{n-1} \quad \forall m$$

where A_{jk} are continuous in Ω.

1.3.11 The periodic unfolding method

Periodic unfolding was introduced in 2002 by D. Cioranescu, A. Damlamian, and G. Griso, the unfolding method is particularly well adapted for perforated domains. For an extensive presentation and some applications of the unfolding method in periodic homogenization, we refer to e.g.,[36] and [34]. Loosely speaking, the main ingredient of the unfolding method in periodic homogenization is the unfolding operator.

1.3.11.1 The unfolding operator τ_ε

In $\mathbb{R}^n$, let Ω be an open set in $\mathbb{R}^n$ set Y a reference cell (ex. $]0,1[^n$). More generally Y can be replaced by an n-dimensional parallelepiped

$$Y=\big\{\lambda_1 b_1+....+\lambda_n b_n:0\le\lambda_i\le 1, i=1,...,n\big\},$$

where $b_1,...,b_n\in\mathbb{R}^n$ is an n-tuple of independent vectors. $[z]_Y$ denotes the unique integer combination $\sum_{j=1}^{n}k_jb_j$ such that $z-[z]_Y$ belongs to Y, and set $\{z\}_Y=z-[z]_Y$. The decomposition $z=[z]_Y-\{z\}_Y$

is the usual decomposition into the integer and fractional parts. Then, for each $x \in \mathbb{R}^n$ and $\varepsilon > 0$, we have

$$x = \varepsilon\left(\left[\frac{x}{\varepsilon}\right]_Y + \left\{\frac{x}{\varepsilon}\right\}_Y\right).$$

Definition 1.3.8. *define $\tau_\varepsilon(w)(x,y) \in L^p(\Omega \times Y)$ for $w \in L^p(\Omega), (p \in [1,\infty])$ by*

$$\tau_\varepsilon(w)(x,y) = \begin{cases} w\left(\varepsilon\left[\frac{x}{\varepsilon}\right]_Y + \varepsilon y\right) & a.e. \ for \ (x,y) \in \Omega \times Y, \\ 0 & a.e. \ else. \end{cases} \tag{1.3.64}$$

for any $x \in \Omega$, $\tau_\varepsilon(w)(x, \left\{\frac{x}{\varepsilon}\right\}_Y) = w(x)$ and $\tau_\varepsilon(wv) = \tau_\varepsilon(w)\tau_\varepsilon(v)$,
$\forall v, w \in L^2(\Omega)$.

The advantage of using this operator in the homogenization of different partial differential equations is that it allows to transform any series of strongly oscillating periodic functions of the form $\{f(\frac{x}{\varepsilon})\}$ into a constant sequence $\{f(y)\}$. This simplifies the demonstration of the homogenization result since there is no need to use special techniques to circumvent the difficulty due to the products of weak convergences.

Proposition 1.3.3. *(proprieties of τ_ε) One has the following integration formula:*

$$\int_\Omega w dx = \frac{1}{|Y|}\int_{\Omega \times Y} \tau_\varepsilon(w) dx dy \quad \forall w \in L^1(\Omega).$$

For $\{w_\varepsilon\} \subset L^p(\Omega)$, if $\tau_\varepsilon(w_\varepsilon) \rightharpoonup \hat{w}$ in $L^p(\Omega \times Y)$, then $w_\varepsilon \rightharpoonup w$ in $L^p(\Omega)$ where $w = \frac{1}{|Y|}\int_Y \hat{w} dy$

Proposition 1.3.4. *(relation with two-scale convergence) Let $\{w_\varepsilon\} \subset L^p(\Omega)$, $p \in (1,\infty)$, be a bounded sequence. The following are equivalent:*

(i) $\{\tau_\varepsilon(w_\varepsilon)\}_\varepsilon$ converges weakly to w in $L^p(\Omega \times Y)$,

(ii) $\{w_\varepsilon\}_\varepsilon$ two-scale converges to w.

Periodic unfolding appears to be equivalent to two-scale convergence. However, it is both simpler and more efficient.

Proposition 1.3.5. *(τ_ε and gradients) For every $w \in W^{1,p}(\Omega)$ one has*

$$\nabla_y(\tau_\varepsilon(w)) = \varepsilon(\tau_\varepsilon(\nabla_x w)).$$

If $\{w_\varepsilon\} \subset W^{1,p}(\Omega)$, *is a bounded sequence in* $L^p(\Omega)$ *such that*

$$\tau_\varepsilon(w_\varepsilon) \rightharpoonup \hat{w} \text{ in } L^p(\Omega \times Y) \text{ with } \varepsilon\|\nabla w_\varepsilon\|_{L^p(\Omega)} \leq C,$$

then

$$\varepsilon(\tau_\varepsilon(\nabla_x w_\varepsilon)) \rightharpoonup \nabla_y \hat{w} \text{ in } L^p(\Omega \times Y).$$

Furthermore, the limit function $\hat{w}$ *is* Y *-periodic, namely belongs to* $L^p(\Omega; W^{1,p}_{per}(Y))$.

1.3.11.2 Periodic unfolding and homogenization

One considers the limit behavior as ε to 0^+ of the solutions of the $\varepsilon-$problem:

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon \nabla v = \int_\Omega f v \quad \forall v \in H^1_0(\Omega), \tag{1.3.65}$$

where, for each ε, A_ε is assumed measurable and bounded in $L^\infty(\Omega)$ One also assumes uniform ellipticity

$$\alpha|\xi|^2 \leq A_\varepsilon \xi.\xi \leq \beta|\xi|^2 \qquad \text{a.e. } x \in \Omega,$$

with strictly positive constants α and β. Traditionally, A_ε is derived as $A(x, \frac{x}{\varepsilon})$ from a $A(x, y)$ which is assumed Y-periodic as a function of its second variable. With $f \in H^{-1}(\Omega)$, $\{u_\varepsilon\}$ is bounded in $H^1_0(\Omega)$ so that there is a subsequence(still denoted ε,) and some u_0 with $u_\varepsilon \rightharpoonup u_0$ in $H^1_0(\Omega)$.

Theorem 16.

(standard periodic homogenization) Suppose that A_ε and f satisfy the above hypotheses. Suppose furthermore that

$$B_\varepsilon(x,y) \doteq \tau_\varepsilon(A_\varepsilon)(x,y) \to B(x,y) \quad a.e.\Omega \times Y. \tag{1.3.66}$$

Then there exists $\hat{u} \in L^2(\Omega; H^1_{per}(Y))$ such that

$$\begin{aligned} \tau_\varepsilon(u_\varepsilon) &\rightharpoonup u_0 \quad in\ L^2(\Omega; H^1(Y)), \\ \tau_\varepsilon(\nabla u_\varepsilon) &\rightharpoonup \nabla_x u_0 + \nabla_y \hat{u} \quad in\ L^2(\Omega \times Y). \end{aligned} \tag{1.3.67}$$

The pair $(u_0, \hat{u})$ is the unique solution of the problem: $\forall \Psi \in H^1_0(\Omega), \forall \Phi \in L^2(\Omega; H^1_{per}(Y))$,

$$\frac{1}{|Y|}\int_{\Omega\times Y} B(x,y)\Big(\nabla_x u_0 + \nabla_y \hat{u}\Big)\Big(\nabla_x \Psi(x) + \nabla_y \Phi(x,y)\Big) = \int_Y f\Psi. \tag{1.3.68}$$

Remark 1.3.10. *1. Problem (1.3.68) is of standard variational form on* $\mathcal{H} = H^1_0(\Omega) \times L^2(\Omega; H^1_{per}(Y)/\mathcal{R})$.

2. The only situations for which (1.3.66) is known to hold, are sums of the following four cases where B always equals A: $A(x,y) = A(y)$, $A(x,y) = A_1(x)A_2(y)$, $A \in L^1(Y; C(\Omega))$, $A \in L^1(\Omega; C(Y))$.

Some advantages of the method are:

- More cases can be treated.
- One can put together several kind of holes with deferent boundary condition (impossible using test functions).
- Some assumptions on correctors can be weaker.
- Nice for some linear problems.

1.4 An overview of boundary layers

The boundary-layer theory began with Ludwig Prandtl's paper on the motion of a fluid with very small viscosity, which was presented at the Third International Congress of Mathematicians in August, 1904, at Heidelberg and published in the Proceedings of the Congress in the following year. This paper marked an epoch in the history of fluid mechanics, opening the way for understanding the motion of real fluids.

L. Prandtl [90] showed that for a sufficiently high Reynolds number, the flow over a solid body can be divided into an outer region of inviscid flow unaffected by viscosity (the main-stream) and a region close to the surface of the body where viscosity is important (the boundary layer). He derived a system of equations for the first approximation of the velocity in the boundary layer (the boundary layer equations). On the interface between the boundary layer and the main-stream, the two flows are properly matched.

Asymptotic modeling and homogenization problems in connection with the boundary layer theory have been considered for 50 years. Averaging techniques have been used for modeling boundary layer of fluid on a porous surface having a micro-inhomogeneous structure, see [[72],[73]].

The boundary layer concept used in fluid mechanics was actually extended to all similar singular problems. Singularly perturbed partial differential equations can yield solutions with zones of rapid variation. These zones are called layers and often appear at the boundary of the domain (then are called boundary layers) and also at the interior of the domain, then are called interior layers.

The construction of an approximate solution to a partial differential equation consists in three main steps: identifying the location of layers (boundary or internal), deriving asymptotic approximations to the solution in the different zones, deriving a uniformly valid solution over the whole domain. The (slowly varying) solutions for the regular distinguished limits are called outer solutions, while the solutions obtained for the layers (singular distinguished limits) are called inner solutions.

Among the methods used for solving singularly perturbed partial differential equations, let us mention the method developed by Vishik and Lyusternik [111], called the VishikLyusternik method or the method of boundary layer functions. This method is based on the construction of an asymptotic expansion of the solution. This asymptotic expansion consists of a so-called regular series and a boundary layer series.

The notion of boundary layer is also widely used in the homogenization theory, for elliptic boundary-value problems with periodically oscillating coefficients, with small period ε, to improve the macroscopic approximations given by the homogenization procedure in the neighborhood of the boundary of the domain, one needs to introduce boundary layer correctors. Such correctors can be defined by using boundary layer functions (called sometimes boundary layers). In homogenization theory, boundary layers are solutions to problems defined on the boundary layer cell. The correctors are constructed via the boundary layers by an appropriate scaling with ε, their energies are negligibly small outside a neighborhood of the boundary. See for instance [[53], [94]] and the references therein.

1.5 Singularly Perturbed Differential Equations

Differential equations are often used as mathematical models describing processes in physics, chemistry, and biology. In the investigation of a number of applied problems, an important role is played by differential equations that contain small parameters at the highest derivatives. Such equations are called singularly perturbed differential equations. These equations describe various processes that are characterized by boundary and/or interior layers. Consider the following simple example:

Example 1.5.1. *(See [65]) Consider the following Differential equation*

$$\varepsilon\frac{du}{dt} = -u + t, \quad 0 \leq t \leq 1, \quad u(0) = 1. \tag{1.5.1}$$

where ε is a small positive parameter: $0 < \varepsilon \ll 1$. The solution of this problem is

$$u(t) = (1+\varepsilon)exp\{-1/t\} + t - \varepsilon.$$

The graph of $u_\varepsilon(t)$ for small $\varepsilon > 0$ is presented in Fig 1.2. *Note two characteristic features of this problem:*

1. *In the subinterval $[\delta, 1]$ (where δ is a small number) the solution $u_\varepsilon(t)$ is close to $\bar{u}_0(t) = t$, that is, to the solution of the equation that we obtain from (1.5.1) for $\varepsilon = 0$. We will call such equation the reduced equation. Thus, the solution $\bar{u}_0(t) = t$ of the reduced equation gives an approximation for the solution $u_\varepsilon(t)$ of Problem (1.5.1) in the subinterval $[\delta, 1]$ for small $\varepsilon > 0$.*

2. *in the subinterval $[0, \delta]$ the solution $u_\varepsilon(t)$ changes rapidly from the initial value $u_\varepsilon(0) = 1$ to values close to $\bar{u}_0(t)$. In this subinterval, $\bar{u}_0(t)$ does not approximate $u_\varepsilon(t)$. The subinterval*

*$[0, \delta]$ is called a **boundary layer**. A generalization of this example is Tikhonov's system (z and y are vector functions) [97]*

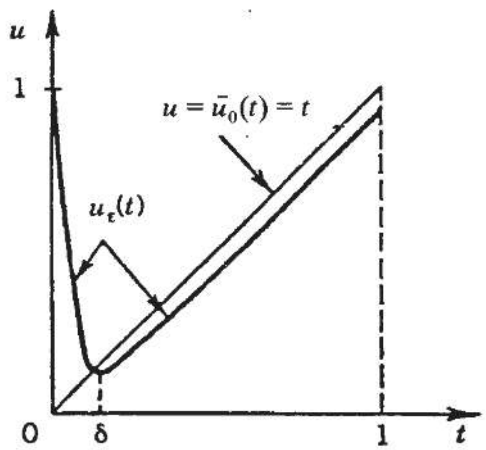

Figure 1.2: The exact solution $u_\varepsilon(t)$ and the ' solution $\bar{u}_0(t)$ of the reduced

1.5.1 The Regular and boundary Layer Parts of the Asymptotic Expansion

Consider in a bounded domain Ω (with $\partial\Omega$ itself is) the following well-posed problem in $H_0^1)(\Omega)$, with Dirichlet boundary-data

$$-\,divA_\varepsilon \nabla u_\varepsilon = f \quad in\ \Omega, \tag{1.5.2}$$

$$u_\varepsilon = 0 \quad on\ \ \partial\Omega. \tag{1.5.3}$$

We seek an asymptotic expansion of the solution of (1.5.2) in the form

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \varepsilon^3 u_3(x, \frac{x}{\varepsilon}) + \varepsilon^4 u_4(x, \frac{x}{\varepsilon}) + ... \tag{1.5.4}$$

Which is called **the regular part** of the asymptotic expansion, but this last does not generally satisfy the boundary condition (1.5.3), which requires adding a boundary layers terms $u_i^{bl,\varepsilon}(x)$ that are called **boundary Layer Part**. In the terminology of the paper of Vishik and Lyusternik [111], the regular terms of the asymptotics introduce a discrepancy into the boundary condition. The purpose of the boundary layer functions is to compensate for this discrepancy. Note that the boundary layer functions together with the regular terms must satisfy the boundary condition (1.5.3). More details on the boundary layers part will be found in Chapter 2 .

1.5.2 Corner boundary layers

The construction of an asymptotic solution in the previous subsection was carried out under an essential assumption: The boundary $\partial\Omega$ of the domain Ω is assumed to be a smooth curve. The normal to the curve exists at each point and the boundary layer functions were constructed from the solutions of ordinary differential equations with derivatives taken along these normals. In the case when the boundary of the domain is no longer smooth, but contains corner points, the structure of the asymptotic solutions becomes more complicated in vicinities of these points. The boundary layer functions constructed in the previous subsection are not sufficient to describe the asymptotic behavior of the solution near the corners, moreover it introduce additional discrepancies in the boundary conditions on the corners. Hence, again we need to introduce a new type of boundary layer functions, **corner boundary functions**, in the vicinities of the corner points, such that we seek an asymptotic expansion of the solution of (1.5.2) in the form

$$u_\varepsilon(x) = \varepsilon^i u_i(x, y) + \varepsilon^i u_i^{bl,\varepsilon}(x) + \varepsilon^i u_i^{cb,\varepsilon}(x).$$

For more examples on the subject see [65]

1.6 Boundary layers in elasticity

We consider in R^3 a bounded domain Ω made of elastic composite materials, with smooth boundary $\partial\Omega$, Moreover, we assume that its mechanical properties are periodic with a small period Y, described with the aid of a small parameter ε. The body is subjected to forces of density f, is fixed for example on a portion $\mathit{\Gamma}^1$ of its boundary and we assume that the remainder $\mathit{\Gamma}^2$ of its boundary is free. Let us set $x = (x_1, x_2, x_3)$ a point of Ω and $y = (y_1, y_2, y_3) \in Y$. One sets,the equilibrium problem defined by :

$$\begin{cases} -\dfrac{\partial \sigma_{ij}^\varepsilon(x)}{\partial x_j} = f \quad in\ \Omega, \\ \sigma_{ij}^\varepsilon(x) = C_{ijkl}(\dfrac{x}{\varepsilon})(e_{ij}(u^\varepsilon(x))) \quad in\ \Omega, \\ u^\varepsilon(x) = 0 \quad on\ \Gamma^1, \\ \sigma_{ij}^\varepsilon.n_j^\varepsilon = 0 \quad on\ \Gamma^2. \end{cases} \tag{1.6.1}$$

Where $\sigma_{ij}^{\varepsilon} = \sigma_{ji}^{\varepsilon}$ is the Cauchy tensor with $C_{ijkl} \in L^{\infty}(Y)$ are periodic and elliptic and symmetric coefficients, $u^{\varepsilon}(x)$ the displacement, $e_{ij}(u^{\varepsilon}(x))$ the strain tensor :
$e_{ij}(u^{\varepsilon}(x)) = \frac{1}{2}(\frac{\partial u_i^{\varepsilon}}{\partial x_j} + \frac{\partial u_j^{\varepsilon}}{\partial x_i})$, and n is the outside unit normal of Ω. The solution $u^{\varepsilon}(x)$ is searched under the form of an asymptotic expansion

$$u^{\varepsilon}(x) = u^0(x,y) + \varepsilon u^1(x,y) + \varepsilon^2 u^2(x,y) + ... \tag{1.6.2}$$

As a sequence, we get the corresponding expansions for strain and stress

$$\begin{aligned} e_{ij}^{\varepsilon}(x) &= e_{ij}^0(x,y) + \varepsilon e_{ij}^1(x,y) + ... \\ \sigma_{ij}^{\varepsilon} &= \sigma_{ij}^0(x,y) + \varepsilon \sigma_{ij}^1(x,y) + ..., \end{aligned} \tag{1.6.3}$$

where

$$\begin{aligned} e_{ij}^0 &= e_{ij,x}(u^0) + e_{ij,y}(u^1), \\ \sigma_{ij}^0 &= a_{ijkl} e_{kl}^0(u^0), \end{aligned} \tag{1.6.4}$$

such that

$$e_{ij,z}(w) = \frac{1}{2}\left(\frac{\partial w_i}{\partial z_j} + \frac{\partial w_j}{\partial z_i}\right), \quad (z = x, y, ...)$$

Then

$$u^1 = e_{kr,x}(u^0(x))\chi^{kr}(y) + c, \tag{1.6.5}$$

where $\chi^{kr}(y)$ are Y-periodic solutions of the local problems

$$-\frac{\partial}{\partial y_j}\{a_{ijkl}(\delta_{mk}\delta_{lr} + e_{kl}(\chi^{kr}(y)))\} = 0 \tag{1.6.6}$$

and the homogenized coefficients are

$$a_{ijmr}^h = \int_Y \left\{ a_{ijkl}\Big(\delta_{mk}\delta_{lr} + e_{kl}\big(\chi^{kr}(y)\big)\Big)\right\}. \tag{1.6.7}$$

Then u_0 is the solution of the homogenized equation and the boundary condition

$$\begin{cases} \dfrac{\partial \sigma_{ij}^h}{\partial x_j} = f_i; \quad \sigma_{ij}^h = \displaystyle\int_Y \sigma_{ij}^0 = a_{ijmr}^h e_{mr,x}(u^0(x)), \quad in\ \Omega, \\ u_0 = 0 \quad on\ \Gamma^1, \\ \langle \sigma_{ij}^h \rangle n_j = 0 \quad on\ \Gamma^2, \end{cases} \tag{1.6.8}$$

where $\langle\sigma_{ij}^h\rangle$ are the average of σ_{ij}^h. We note that $\langle\sigma_{ij}^h\rangle$ are an approximation of σ_{ij}^h on Γ^2 and that $\sigma_{ij}^h n_j \neq 0$ on Γ^2, which is a source of boundary layer phenomena. We intend to describe the influence of the periodic structure by the microscopic variable y (resp. the macroscopic variable x) in (1.6.2). To this end, we search for an expansion (1.6.2) with functions u^i Y-periodic with respect to the variable y and smooth with respect to x. Indeed, each $u^i(x,y)$ is defined on $\Omega \times Y$.
It is evident that this locally periodic expansion is fit to describe the solution in regions of Ω far from its boundary, or from regions where the local effects are not Y-periodic. But in practice, we need a more precise analysis of the local stress field, at the microscopic scale of the heterogeneities, specially near the boundaries, note that the asymptotic expansion technique allows to obtain an approximation of the micro-stresses within the material by a localization method. But in this way, the micro-stresses do not satisfy the boundary conditions of Neumann, in addition they are supposed periodic as the structure and this hypothesis must be discussed near a boundary. Consequently, the approximation obtained by the classical homogenization theory, is not very satisfactory in the neighborhood of a Neumann boundary. As a result, near the boundary $\partial\Omega$ of the body we must consider boundary layers where the solution is searched under the form (1.6.2) but now x runs in $\partial\Omega$ and y in the strip S Fig.1.3 and u^i is searched to be S-periodic instead of Y-periodic (the periodicity is parallel to the free boundary). Note, in Fig. 1.3 for instance, that S is a semi-infinite strip formed by Y-periods (plus perhaps "parts" of periods at the intersection with $\partial\Omega$).

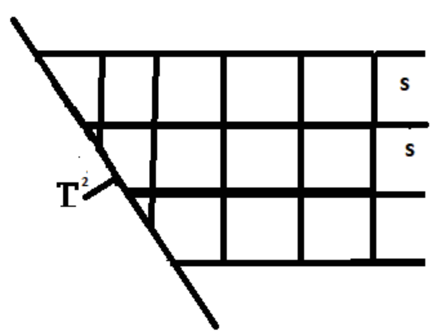

Figure 1.3: The strip s

In this case, the solution in the boundary layer region takes the form : (the superscript bl is for "boundary layer") :

$$u^\varepsilon(x) = u^{0,bl}(x,y) + \varepsilon u^{1,bl}(x,y) + ... \tag{1.6.9}$$

such that each $u^{i,bl}$ is defined for $x \in \Gamma^2$ and S-periodic in y, It is clear that two such expansions "must agree", i.e. the boundary layer contains a transition region between the genuine boundary layer and the "outer" region (outer to the boundary layer). Note that the solution u^ε equals to the sum of the two expansions, in order to guarantee that the boundary conditions are verified.

Remark 1.6.1. 1. *In homogenization theory, boundary layers are solutions to problems defined on a semi-infinite strip* $[0,1]^{n-1}\times]0,\infty[$, *whose energies decrease exponentially with respect to the second variable.*

2. *The construction of the boundary layers in general domains is up to now an open question. The only cases where results have been obtained are when the domain is a half space. Recently, Allaire and Amar studied boundary layers in rectangular domains which are either fixed or have an oscillating boundary.*

1.7 Boundary layers in thin plates

In this section we present the steps to construct a valid asymptotic expansion with boundary layers terms for the displacement $u^\eta(x)$ in thin plates (see [40] for more details).

Consider a thin plate $\Omega^\eta = \omega \times (-\eta, \eta)$, where the mean surface ω is an open subset in $\mathbb{R}^2$ and the thickness η is a small parameter designed to tend to zero. We suppose that the boundary $\partial\Omega$ is divided into horizontal boundaries $\omega \times \{\pm\eta\}$ and lateral boundary $\Gamma^\eta = \partial\omega \times (-\eta, \eta)$. There are three types of plates, such that the kind of each plate is referred to the boundary conditions imposed on the lateral boundary, i.e:

$$\begin{cases} u^\eta = 0 \; on \; \Gamma^\eta \implies hard\ clamped\ plate, \\ u^\eta.n = 0 \; and \; u_3^\eta = 0 \; on \; \Gamma^\eta \implies soft\ clamped\ plate, \\ u^\eta \times n = 0 \; and \; u_3^\eta = 0 \; on \; \Gamma^\eta \implies simply\ supported\ plate, \end{cases} \tag{1.7.1}$$

with n is the inner unit normal to Γ^η.

1.7.1 Outer and inner ansatz

In the case of thin plates, before postulating the outer and inner ansatz, it is needful to make a scaling to the domain Ω^η, the displacement u^η and the forces if they are exist, namely, we transform the studied problem into a problem posed over a fixed domain, which does not depend on η.

The outer ansatz is the same of what we were mentioned in last sections, but here we use the scaled displacement $u(\eta)$ instead of u^η i. e.

$$u(\eta)(x) = u^0(x) + \eta u^1(x) + \eta^2 u^2(x) + .. + \eta^k u^k(x) + .. \tag{1.7.2}$$

where the $u^k$are independent of η, corrected by a boundary layer expansion the inner Ansatz. These inner and outer expansions are familiar notions in the theory of matching asymptotics [62], where the idea is somewhat different: it consists of trying to describe the asymptotics either in primitive variables, or in boundary layer variables in different zones and to match both in an intermediate zone. Here we search for a combined expansion which is valid everywhere. More precisely, we find that the ingredients of a correct Ansatz are the following.

1. **Kirchhoff-Love displacements** u^k_{KL}: It is well known that the limit of $u(\varepsilon)$ is a Kirchhoff-Love displacement, namely:

$$u^k_{KL,\alpha} = \zeta^k_\alpha(x_\alpha) - x_3\partial_\alpha\zeta^k_3(x_\alpha), \qquad \zeta^k_{KL,3} = \zeta^k_3(x_\alpha). \tag{1.7.3}$$

   Indeed we find that such a displacement appears at each level of the asymptotic.

2. **Displacements with mean values zero in each vertical fiber** :

$$\int_{-1}^{1} u^k(x_\alpha, x_3)dx_3 = 0 \; \forall \; x_\alpha \in \omega, \tag{1.7.4}$$

   which are determined by the solution of a Neumann problem on the interval [-1, 1]. Added to the previous Kirchhoff- Love displacements (1.7.3), they constitute the outer expansion part of the Ansatz (1.7.2).

3. **Boundary layer terms**

$$\omega^k = \omega^k(\eta^{-1}r, s, x_3) \; with \begin{cases} r \; the \; distance \; to \; \partial\omega, \\ s \; the \; arc \; length \; in \; \partial\omega. \end{cases} \tag{1.7.5}$$

They compensate for discrepancies in imposed lateral boundary conditions and describe phenomena rapidly" varying and decreasing near Γ, their introduction allows for a complete resolution. They constitute the inner expansion part of the Ansatz. For every k, $\omega^k(t, s, x_3)$ is exponentially decreasing as $t \longrightarrow +\infty$.

With χ denoting a cut-off function equal to 1 in a neighborhood of $\partial\omega$, we consider the localized function $\chi(r)\omega^k(\eta^{-1}r, s, x_3)$.

Collecting all these features, we get the following expansion

$$u(\eta) = u^0_{KL} + \eta u^1_{KL} + \eta\chi(r)\omega^1_\alpha(\eta^{-1}r, s, x_3, 0) + \sum_{k\geq 2} \eta^k(u^k_{KL} + v^k + \chi(r)\omega^k(\eta^{-1}r, s, x_3)). \quad (1.7.6)$$

CHAPTER 2

ERROR ESTIMATES

This chapter is dedicated to the study of error estimates in the periodic homogenization of elliptic equation in divergence form with Dirichlet boundary conditions. We remind that the homogenization theory consists in substituting a non-homogeneous material for an homogeneous material with equivalent mechanic properties. Among several basic techniques in homogenization theory we are concerned in this chapter with the two-scale asymptotic expansions method (see Subsection 1.3.2, Chapter 1), through which the solution u_ε of our problem can be written as the ansatz

$$u_\varepsilon = \sum_{i=0}^{\infty} \varepsilon^i u_i(x, \frac{x}{\varepsilon}), \tag{2.0.1}$$

where ε is a small parameter ($0 < \varepsilon \leq 1$) which represents the size of the basic period $Y = (0,1)^2$, the leading term u_0 denotes the homogenized solution and u_i for $i \in \mathbb{N}^*$ are called *correctors* which are periodic with respect to the second variable. This method is very simple and powerful, but unfortunately is formal since the ansatz (2.0.1) is fit to describe the solution in regions of Ω far from its boundary and this is the most drawback of this expansion. Thus, the two-scale asymptotic expansion method is used only to guess the form of the homogenized problem. As a consequence, near the boundary, one must consider boundary layers terms, such that matching both (2.0.1) and boundary layers terms ansatz gives an asymptotic expansion for the solution u_ε which is correct everywhere.

An important point to bear in mind is that the phenomenon of boundary layer appears in PDE either due to the boundary conditions or the geometry of the domain. We note that boundary layers are often more important for improving the rate of convergence than the usual periodic correctors. For instance, taking into account the boundary layers in our problem, we obtain in the first approximation an estimate of order ε for the remainder term, whereas without the boundary layers we can only get an estimate of order $\varepsilon^{\frac{1}{2}}$. To the best of our knowledge, the only situation where there is no boundary layer is the case of periodic boundary conditions. The purpose of our study is to find the error estimates of the third-order with or without boundary layer terms in the periodic homogenization of elliptic equations in divergence form with Dirichlet boundary conditions. Thus, the originality of the present study lies in the improvement of the homogenization approximation by taking into account the third-order corrector. To our knowledge, the third-order corrector was not studied in homogenization theory.

2.1 Setting of the problem

We start by recalling the basic notions of the asymptotic homogenization method for periodic structures (see [18, 32]). Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary.
Let $A(y)$ be a square symmetric matrix with entries $a_{ij}(y)$ which are Y-periodic functions belonging to $L^\infty(Y)$. We assume that there exist two constants $0 < \lambda < \Lambda < +\infty$ such that, for a.e. $y \in Y$,

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2, \ \forall \xi \in R^n.$$

Let $A_\varepsilon(x) = A(\frac{x}{\varepsilon})$ be a periodically oscillating matrix of coefficients where ε is a small positive parameter ($0 < \varepsilon \leq 1$). For a given function $f \in L^2(\Omega)$ we consider the following well-posed problem in $H_0^1(\Omega)$

$$(P_\varepsilon)\begin{cases} -divA_\varepsilon\nabla u_\varepsilon = f \quad in\ \Omega, \\ u_\varepsilon = 0 \quad on\ \ \partial\Omega. \end{cases} \tag{2.1.1}$$

We postulate the following ansatz for the solution $u_\varepsilon(x)$

$$u_\varepsilon(x) = u_0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \varepsilon^3 u_3(x, \frac{x}{\varepsilon}) + \varepsilon^4 u_4(x, \frac{x}{\varepsilon}) + ... \tag{2.1.2}$$

where each function $u_i(x, y)$ is Y-periodic with respect to $y = \frac{x}{\varepsilon}$.
Suppose that a function $\Psi^\varepsilon(x) = \Psi^\varepsilon(x, y)$ depends on both the slow and the fast coordinates. We

make use of the chain rule of differentiation we obtain the following relations:

$$\begin{cases} \dfrac{\partial \Psi^\varepsilon(x,y)}{\partial x} = \dfrac{\partial \Psi(x,y)}{\partial x} + \dfrac{1}{\varepsilon}\dfrac{\partial \Psi(x,y)}{dy};\ y = \dfrac{x}{\varepsilon}, \\ div\Psi^\varepsilon(x) = div_x\Psi(x,y) + \dfrac{1}{\varepsilon} div_y\Psi(x,y), \\ \nabla\Psi^\varepsilon = \nabla_x\Psi(x,y) + \dfrac{1}{\varepsilon}\nabla_y\Psi(x,y). \end{cases} \tag{2.1.3}$$

Plugging the asymptotic expansion (2.1.2) in (2.1.1) taking into account (2.1.3) and identifying different powers of ε yields a cascade of equations. Defining an operator L_ε by $L_\varepsilon\varphi = -divA_\varepsilon\nabla\varphi$, one can write $L_\varepsilon = \varepsilon^{-2}L_0 + \varepsilon^{-1}L_1 + L_2$, where

$$L_0 = -\frac{\partial}{\partial y_i}(a_{ij}(y)\frac{\partial}{\partial y_j})$$
$$L_1 = -\frac{\partial}{\partial y_i}(a_{ij}(y)\frac{\partial}{\partial x_j}) - \frac{\partial}{\partial x_i}(a_{ij}(y)\frac{\partial}{\partial y_j})$$
$$L_2 = -\frac{\partial}{\partial x_i}(a_{ij}(y)\frac{\partial}{\partial x_j}).$$

So the first equation in (2.1.1) is identical to the following system

$$\begin{aligned} L_0u_0 &= 0 \\ L_0u_1 + L_1u_0 &= 0 \\ L_0u_2 + L_1u_1 + L_2u_0 &= f \\ L_0u_3 + L_1u_2 + L_2u_1 &= 0 \\ L_0u_4 + L_1u_3 + L_2u_2 &= 0 \end{aligned} \tag{2.1.4}$$

....

By application of the Fredholm alternative for periodic elliptic PDEs to (2.1.4), we deduce that each equation in (2.1.4) has a unique solution $u_i(x,y)$ (up to a constant $\tilde{u}_i$ that depends on x only). The first equation in (2.1.4) leads us to deduce that $u_0(x,y) \equiv u_0(x)$ is independent of y. The second equation gives u_1 in terms of u_0

$$u_1(x,y) = -\chi^j(y)\frac{\partial u_0}{\partial x_j}(x) + \tilde{u_1}(x), \tag{2.1.5}$$

where $\chi^j(y)$ are the unique solutions in $H^1_\sharp(Y)$ of the first cell problem

$$\begin{cases} L_0\chi^j(y) = -\dfrac{\partial a_{ij}}{\partial y_i}(y) \qquad in\ Y; \\ \displaystyle\int_Y \chi^j(y)dy = 0. \end{cases} \tag{2.1.6}$$

The third equation in (2.1.4) gives u_2

$$u_2(x,y) = \chi^{ij}(y)\frac{\partial^2 u_0}{\partial x_i \partial x_j} - \chi^j(y)\frac{\partial \tilde{u_1}}{\partial x_j}(x) + \tilde{u_2}(x) \tag{2.1.7}$$

where $\chi^{ij} \in H^1_\sharp(Y)$ are the unique solutions of the second cell problem

$$\begin{cases} L_0\chi^{ij} = b_{ij} - \int_Y b_{ij}(y)dy \qquad in\ Y; \\ \int_Y \chi^{ij}(y)dy = 0, \end{cases} \tag{2.1.8}$$

with $b_{ij} = a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k} - \frac{\partial}{\partial y_k}(a_{ik}(y)\chi^j)$.
The fourth equation in (2.1.4) gives u_3

$$u_3(x,y) = \chi^{ijk}(y)\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} + \chi^{ij}(y)\frac{\partial^2 \tilde{u_1}(x)}{\partial x_i \partial x_j} - \chi^j(y)\frac{\partial \tilde{u_2}}{\partial x_j}(x) + \tilde{u_3}(x) \tag{2.1.9}$$

where $\chi^{ijk} \in H^1_\sharp(Y)$ are the unique solutions of the third cell problem

$$\begin{cases} L_0\chi^{ijk} = c_{ijk} - \int_Y c_{ijk}(y)dy \qquad in\ Y; \\ \int_Y \chi^{ijk}(y)dy = 0, \end{cases} \tag{2.1.10}$$

with $c_{ijk} = -a_{ij}\chi^k + \frac{\partial}{\partial y_m}(a_{im}\chi^{jk}) + a_{im}\frac{\partial \chi^{jk}}{\partial y_m}$.
The fifth equation in (2.1.4) gives u_4

$$\begin{aligned} u_4(x,y) = {} & \chi^{ijmp}(y)\frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_m \partial x_p} + \chi^{ijk}(y)\frac{\partial^3 \tilde{u_1}(x)}{\partial x_i \partial x_j \partial x_k} + \chi^{ij}(y)\frac{\partial^2 \tilde{u_2}(x)}{\partial x_i \partial x_j} - \chi^j(y)\frac{\partial \tilde{u_3}}{\partial x_j}(x) \\ & + \tilde{u_4}(x) \end{aligned} \tag{2.1.11}$$

where $\chi^{ijmp} \in H^1_\sharp(Y)$ are the unique solutions of the fourth cell problem

$$\begin{cases} L_0\chi^{ijmp} = d_{ijmp} - \int_Y d_{ijmp}(y)dy \qquad in\ Y; \\ \int_Y \chi^{ijmp}(y)dy = 0, \end{cases} \tag{2.1.12}$$

with $d_{ijmp} = a_{ij}\chi^{mp} + \frac{\partial}{\partial y_k}(a_{ik}\chi^{jmp}) + a_{ik}\frac{\partial \chi^{jmp}}{\partial y_k}$.
The homogenized problem of (P^ε) is obtained by averaging the third equation in (2.1.4). It is given by

$$(P_H)\begin{cases} -divA^*\nabla u_0 = f \quad in\ \Omega, \\ u_0 = 0 \quad on\ \ \partial\Omega, \end{cases} \tag{2.1.13}$$

where the coefficients of the homogenized matrix A^* are given by

$$a^*_{ij} = \int_Y [a_{ij}(y) - a_{ik}\frac{\partial \chi^j}{\partial y_k}(y)]dy \tag{2.1.14}$$

such that (a^*_{ij}) is bounded, symmetric and uniformly elliptic. The problem (P_H) is well-posed in $H^1_0(\Omega)$.

The functions $\tilde{u_1}, \tilde{u_2}, \tilde{u_3}$ *and* $\tilde{u_4}$ are non-oscillating functions which represent the average of u_1, u_2, u_3 *and* u_4 respectively and are solutions in Ω of the equations

$$-div[A^*\nabla\tilde{u_1}(x)] =< c_{ijk} > \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}, \tag{2.1.15}$$

$$-div[A^*\nabla\tilde{u_2}(x)] =< d_{ijkl} > \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l} + < c_{ijk} > \frac{\partial^3 \tilde{u}_1}{\partial x_i \partial x_j \partial x_k}, \tag{2.1.16}$$

$$\begin{aligned} -div[A^*\nabla\tilde{u_3}(x)] =& < e_{ijklm} > \frac{\partial^5 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l \partial x_m} + < d_{ijkl} > \frac{\partial^4 \tilde{u}_1}{\partial x_i \partial x_j \partial x_k \partial x_l} \\ & + < c_{ijk} > \frac{\partial^3 \tilde{u}_2}{\partial x_i \partial x_j \partial x_k}, \end{aligned} \tag{2.1.17}$$

where

$$e_{ijklm} = a_{ij}\chi^{klm} + \frac{\partial}{\partial y_r}(a_{ir}\chi^{jklm}) + a_{ir}\frac{\partial}{\partial y_r}(\chi^{jklm}),$$

and

$$\begin{aligned} -div[A^*\nabla\tilde{u_4}(x)] =& < h_{ijklmn} > \frac{\partial^6 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l \partial x_m \partial x_n} + < e_{ijklm} > \frac{\partial^5 \tilde{u}_1}{\partial x_i \partial x_j \partial x_k \partial x_l \partial x_m} \\ & + < d_{ijkl} > \frac{\partial^4 \tilde{u}_2}{\partial x_i \partial x_j \partial x_k \partial x_l} + < c_{ijk} > \frac{\partial^3 \tilde{u}_3}{\partial x_i \partial x_j \partial x_k}, \end{aligned} \tag{2.1.18}$$

where

$$h_{ijklmn} = a_{ij}\chi^{klmn} + \frac{\partial}{\partial y_r}(a_{ir}\chi^{jklmn}) + a_{ir}\frac{\partial}{\partial y_r}(\chi^{jklmn}),$$

such that $\chi^{jklmn} \in H^1_\sharp(Y)$ are the unique solutions of the fifth cell problem

$$\begin{cases} L_0\chi^{jklmn}(y) = e_{ijklm} - < e_{ijklm} > \\ \int_Y \chi^{jklmn}(y)dy = 0. \end{cases} \tag{2.1.19}$$

Remark 2.1.1. *The functions $\tilde{u}_1$, $\tilde{u}_2$, $\tilde{u}_3$, and $\tilde{u}_4$ are not uniquely defined since the equations (2.1.15), (2.1.16), (2.1.17), and (2.1.18) haven't any boundary conditions, and it is very difficult to determine them. However, there is a special geometric case allows us to find out the boundary conditions for only $\tilde{u_1}$(see, for instance [2]).*

It is technically complicated to keep track of boundary conditions when seeking u_ε in the form (2.1.2), especially near the boundary, so we expect u_ε to behave like

$$u_\varepsilon(x) = u_0(x) + \varepsilon[u_1(x,y) + u_1^{bl,\varepsilon}(x)] + \varepsilon^2[u_2(x,y) + u_2^{bl,\varepsilon}(x)] + ... \quad (2.1.20)$$

where each boundary layer term $u_i^{bl,\varepsilon}$ satisfies

$$\begin{cases} -divA_\varepsilon \nabla u_i^{bl,\varepsilon} = 0 & in\ \Omega \\ u_i^{bl,\varepsilon} = -u_i(x, \frac{x}{\varepsilon}) & on\ \ \partial\Omega \end{cases} \quad (2.1.21)$$

Remark 2.1.2. *i) Since* $\partial\Omega$ *is Lipschitz continuous and* $u_i^{bl,\varepsilon} \in H^1(\Omega)$*, so (2.1.21) has a unique solution.*

ii) The advantage of the new ansatz (2.1.20) is that each term $u_i + u_i^{bl,\varepsilon}$ *satisfies a homogeneous Dirichlet boundary condition.*

iii) Both the coefficients and the Dirichlet boundary data in (2.1.21) are periodic and rapidly oscillating.

iv)The case where the boundary data in (2.1.21) is not oscillating and belongs to $L^p(\partial\Omega)$, $1 < p < \infty$, *was studied by Avellaneda and Lin [9].*

v) The asymptotic analysis of (2.1.21) turns out to be more difficult than that of (P_ε) *since* $u_i^{bl,\varepsilon}$ *is not uniformly bounded in the usual energy space* $H^1(\Omega)$*. More precisely we have*

$$\left\|u_i^{bl,\varepsilon}\right\|_{H^1(\Omega)} = O(\frac{1}{\sqrt{\varepsilon}}), \quad \left\|u_i^{bl,\varepsilon}\right\|_{L^2(\Omega)} = O(1), \quad \left\|u_i^{bl,\varepsilon}\right\|_{H^1(\omega)} = O(1) \text{ for all } \omega \subset\subset \Omega. \ \ (see[17])$$

The asymptotic analysis of (2.1.21) is a very difficult problem that has been addressed only for very special domain, namely with boundaries that are hyperplanes (see [94]) and the references therein). A major progress was made in the pioneering work of Gérard-Varet and Masmoudi [53] for solutions to elliptic systems of divergence type, under the assumption that Ω is a smooth, bounded and uniformly convex domain[1] of $\mathbb{R}^n$ ($n \geq 2$).They proved that, as $\varepsilon \to 0$, the unique solution $u_i^{bl,\varepsilon}$ of (2.1.21) converges strongly in $L^2(\Omega)$ to some function $u_i^{bl,*}$, which is solution of

$$\begin{cases} -divA^*\nabla u_i^{bl,*}(x) = 0 \text{ in } \Omega, \\ u_i^{bl,*}(x) = -\overline{u}_i(x) \text{ on } \partial\Omega, \end{cases}$$

[1]A convex set $\mathcal{C}$ is said to be **uniformly convex** if there exists a function $\delta(r)$ positive for $r > 0$, and zero only for $r = 0$, such that $x, y \in \mathcal{C}$ and $\left\|z - \frac{x+y}{z}\right\| \leq \delta(\|x-y\|)$ imply $z \in \mathcal{C}$.

where $A^* = (a^*_{ij})$ is defined in (2.1.14), and $\overline{u}_i$ is the homogenized Dirichlet boundary data that depends non trivially on u_i, A and Ω. More recently, Armstrong & al [8] have improved the results of Gérard-Varet and Masmoudi [53] to the case $L^p(\Omega)$ where $2 \leq p < \infty$.

2.2 An overview of some error estimates

In this section we present a brief overview of some known results on error estimates in periodic homogenization for the problem (P_ε). Let's start with the error estimate between u_ε and u_0 the unique solutions of (P_ε) and (P_H) respectively. For f smooth ($f \in C^k(\overline{\Omega})$), using the maximum principle, Bensoussan & al. [18] obtained the estimate

$$\|u_\varepsilon - u_0\|_{L^\infty(\Omega)} \leq C\varepsilon, \tag{2.2.1}$$

and for $\chi^j \in L^\infty(Y)$, Jikov et all. [64] obtained the estimate

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C\varepsilon. \tag{2.2.2}$$

The error estimate with a first-order corrector in the periodic homogenization for the problem (P_ε) was given under additional regularity assumptions on u_0 or on the cell functions χ^j. Under the assumption that $\chi^j \in W^{1,\infty}(Y)$, Bensoussan et all [18] obtained the estimate

$$\|u_\varepsilon - u_0 - \varepsilon u_1\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}. \tag{2.2.3}$$

The same estimate (2.2.3) is obtained by Jikov & al. [64], under the assumptions that $u_0 \in C^2(\overline{\Omega})$ and $\nabla_y \chi^j \in L^\infty(Y)$, and by Allaire and Amar [2] under the assumption that $u_0 \in W^{2,\infty}(\Omega)$.

The estimate (2.2.3) has a general character since it holds for a wide range of boundary value problems, and not only for the Dirichlet problem.

Without any regularity assumptions on χ^j and under the hypothesis that $u_0 \in H^2(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ regularity, Griso [56] using the periodic unfolding method introduced in [31] and further developed in [33, 34], proved the estimate

$$\|u_\varepsilon - u_0 - \varepsilon u_1\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}\,\|u_0\|_{H^2(\Omega)}, \tag{2.2.4}$$

where $u_1(x,\frac{x}{\varepsilon})=\chi^j(\frac{x}{\varepsilon})Q_\varepsilon(\frac{\partial u_0}{\partial x_j}(x))$, $x\in\widetilde{\Omega}_\varepsilon=\cup_{\xi\in Z^n}\{\varepsilon\xi+\varepsilon Y \text{ with } (\varepsilon\xi+\varepsilon Y)\cap\Omega\neq\emptyset\}$,
$Q_\varepsilon(\phi)(x)=\sum\limits_{i_1,\ldots,i_n}M_Y^\varepsilon(\phi)\,(\varepsilon\xi+\varepsilon i)\,\overline{x}_{1,\xi}^{i_1}\cdot\ldots\cdot\overline{x}_{n,\xi}^{i_n}$, $\xi=\left[\dfrac{x}{\varepsilon}\right]$ for $\phi\in L^2(\Omega)$, $i=(i_1,...,i_n)\in\{0,1\}^n$,
$M_Y^\varepsilon(\phi)(x)=\frac{1}{\varepsilon^n}\int_{\varepsilon\xi+\varepsilon Y}\phi(y)dy$ and

$$\overline{x}_{k,\xi}^{i_k}=\begin{cases}\frac{x_k-\varepsilon\xi}{\varepsilon} & \text{if } \ i_k=1\\ 1-\frac{x_k-\varepsilon\xi}{\varepsilon} & \text{if } \ i_k=0\end{cases}\qquad x\in(\varepsilon\xi+\varepsilon Y)\,.$$

For any open set $\omega\subset\subset\Omega$ compactly embedded in Ω, under the assumption that $u_0\in W^{3,\infty}(\Omega)$, Allaire and Amar (Theorem 2.3, [2]) obtained the interior estimate

$$\|u_\varepsilon-u_0-\varepsilon u_1\|_{H^1(\omega)}\leq C\varepsilon, \tag{2.2.5}$$

where C depends on ω.

Under the assumptions that Ω is a bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ regularity and $f\in L^2(\Omega)$, Griso [57] proved the same estimate above

$$\|u_\varepsilon-u_0-\varepsilon u_1\|_{H^1(\omega)}\leq C\varepsilon\,\|f\|_{L^2(\Omega)}\,, \tag{2.2.6}$$

where C_6 depends on n, A^*, ω and $\partial\Omega$, $u_1(x,\frac{x}{\varepsilon})=\chi^j(\frac{x}{\varepsilon})Q_\varepsilon(\frac{\partial u_0}{\partial x_j}(x))$.

Cioranescu & al. [34] proved the estimates (2.2.4) and (2.2.6) with

$$u_1(x,\frac{x}{\varepsilon})=\chi^j(\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_j}(x),\ \text{without } Q_\varepsilon.$$

Using the first-order boundary layer corrector defined in (2.1.21), under the assumptions that $A_\varepsilon\in C^\infty(\mathbb{R}^n)^{n\times n}$, $Y-$periodic, $u_0\in H^2(\Omega)$, where $\Omega\subset\mathbb{R}^2$ is a bounded domain with Lipschitz boundary satisfying a uniform exterior sphere condition[2], Moskow and Vogelius [78] obtained the estimate

$$\left\|u_\varepsilon-u_0-\varepsilon u_1-\varepsilon u_1^{bl,\varepsilon}\right\|_{H_0^1(\Omega)}\leq C\varepsilon. \tag{2.2.7}$$

Under the assumption that $u_0\in W^{2,\infty}(\Omega)$, Allaire and Amar [2] obtained the same estimate above.

[2] A domain $\Omega\subseteq\mathbb{R}^n$ satisfies an **exterior sphere condition** at $\xi\in\partial\Omega$ if there exists $y\in\mathbb{R}^n$ and $\rho>0$ such that $\overline{B_\varrho(y)}\cap\bar{\Omega}=\{\xi\}$.

In the general case of non-smooth periodic coefficients, where Ω is a bounded convex polyhedron or a bounded convex domain and $u_0 \in H^2(\Omega)$, inspired by Griso's idea, Onofrei and Vernescu [87] proved the estimate

$$\left\| u_\varepsilon - u_0 - \varepsilon\chi^j(\frac{x}{\varepsilon})Q_\varepsilon(\frac{\partial u_0}{\partial x_j}(x)) - \varepsilon\beta_\varepsilon \right\|_{H_0^1(\Omega)} \leq C\varepsilon \left\| u_0 \right\|_{H^2(\Omega)}, \tag{2.2.8}$$

where β_ε is the solution to (2.1.21) with $u_1(x, \frac{x}{\varepsilon}) = \chi^j(\frac{x}{\varepsilon})\frac{\partial u_0}{\partial x_j}(x)$.

For more results on first-order estimates, we also quote the references [94, 109, 91].

Taking into account the second-order corrector, under the assumptions that $f \in C^\infty(\overline{\Omega})$, $\widetilde{u}_1 = \widetilde{u}_2 = 0$ and χ^j, χ^{ij} in $W^{1,\infty}(Y)$, Cioranescu and Donato [32] obtained the estimate

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 \right\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon}. \tag{2.2.9}$$

Using the first-order boundary layer corrector, under the assumptions that Ω is a cubic domain and $u_0 \in W^{2,\infty}(\Omega)$, where u_1 is defined by (2.1.5) and $\widetilde{u}_1$ satisfies (2.1.15), Allaire and Amar [2] obtained the estimate

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 \right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{3}{2}}. \tag{2.2.10}$$

This result shows that with the help of the second-order corrector, and the first-order boundary layer corrector, one can essentially improve the order of the estimates (2.2.9) and (2.2.7) respectively. We note that the result (2.2.10) is obtained provided that $\widetilde{u}_1$ satisfies (2.1.15) otherwise, the estimate is wrong. For the case of a convex bounded domain Ω with smooth enough boundary, and under the assumptions that $u_0 \in H^3(\Omega)$, $\widetilde{u}_1 = \widetilde{u}_2 = 0$ and χ^j, χ^{ij} in $W^{1,p}(Y)$ for some $p > n$, Onofrei and Vernescu [88] proved the estimate

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 \right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{3}{2}} \left\| u_0 \right\|_{H^3(\Omega)}. \tag{2.2.11}$$

The following section sets out the principal results, such that it presents the error estimates of the third-order for the problem (2.1.1) with and without boundary layers terms. Note that these last, are stated under the assumptions that $A_\varepsilon \in \left(L^\infty(Y)\right)^{n\times n}$ (i.e. the coefficients a_{ij} are not smooth), and either the homogenized solution u_0 is smooth or the solutions of the cell problems are smooth.

Remark 2.2.1. *This section represents our published article see [107].*

2.3 Third-order corrections in periodic homogenization for elliptic problem

In this section we need more regularity for u_0 the solution of (P_H) which requires more regularity on the data, and we suppose that the functions $\widetilde{u}_i = \langle u_i \rangle \equiv 0$, $i = 1, 2, 3, 4$. Since we will not try to compute the minimal regularity required for Ω and f, we simply assume in the sequel that Ω is a bounded domain with $\partial\Omega \in C^\infty$ and $f \in C^\infty(\overline{\Omega})$ which implies, according to the regularity theory (see Evans [49]), that $u_0 \in C^\infty(\overline{\Omega})$. Using the density of $C^\infty(\overline{\Omega})$ in $W^{m,p}(\Omega)$ for all $m \in \mathbb{N}^*$ and $1 \leq p < \infty$, we have $u_0 \in W^{m,p}(\Omega)$.

The first result concerns the second-order error estimate with boundary layers correctors. In this case, we need the regularity $H^3(\Omega)$ for u_0.

Theorem 17.

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H) respectively, with $\Omega \subset \mathbb{R}^n$ is a strictly convex bounded domain with $\partial\Omega \in C^\infty$. Assume that $f \in C^\infty(\overline{\Omega})$ and $\chi^{ijk} \in W^{1,\infty}(Y)$. Then

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} \right\|_{H_0^1(\Omega)} \leq C\varepsilon^2 \left\| u_0 \right\|_{H^3(\Omega)}. \tag{2.3.1}$$

Definition 2.3.1. *The domain Ω is strictly convex if the open straight segment joining any two points of $\partial\Omega$ lies entirely in Ω.*

Proof. Defining $r_\varepsilon(x) = \dfrac{1}{\varepsilon^2}(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon})$,
it satisfies

$$\begin{cases} -divA_\varepsilon \nabla r_\varepsilon = \dfrac{1}{\varepsilon^2}(f + divA_\varepsilon \nabla u_0) + \dfrac{1}{\varepsilon} divA_\varepsilon \nabla u_1 + divA_\varepsilon \nabla u_2 & in \;\; \Omega \\ \qquad r_\varepsilon = 0 & on \; \partial\Omega. \end{cases} \tag{2.3.2}$$

Using the relations (2.1.3), (2.1.4) and the fact that u_0 is independent of y, we get

$$f + divA_\varepsilon \nabla u_0 = f - L_2 u_0 - \frac{1}{\varepsilon} L_1 u_0 = L_0 u_2 + L_1 u_1 - \frac{1}{\varepsilon} L_1 u_0$$

$$divA_\varepsilon \nabla u_1 = -L_2 u_1 - \frac{1}{\varepsilon} L_1 u_1 - \frac{1}{\varepsilon^2} L_0 u_1$$

$$divA_\varepsilon \nabla u_2 = -L_2 u_2 - \frac{1}{\varepsilon} L_1 u_2 - \frac{1}{\varepsilon^2} L_0 u_2.$$

So the equation (2.3.2) is reduced to

$$\begin{aligned} -divA_\varepsilon \nabla r_\varepsilon &= \frac{1}{\varepsilon^2}\left(L_0 u_2 + L_1 u_1 - \frac{1}{\varepsilon} L_1 u_0\right) + \frac{1}{\varepsilon}\left(-L_2 u_1 - \frac{1}{\varepsilon} L_1 u_1 - \frac{1}{\varepsilon^2} L_0 u_1\right) \\ &+ \left(-L_2 u_2 - \frac{1}{\varepsilon} L_1 u_2 - \frac{1}{\varepsilon^2} L_0 u_2\right) \\ &= -\frac{1}{\varepsilon^3}\left(L_1 u_0 + L_0 u_1\right) + \frac{1}{\varepsilon^2}\left(L_0 u_2 + L_1 u_1 - L_1 u_1 - L_0 u_2\right) - \frac{1}{\varepsilon}\left(L_2 u_1 + L_1 u_2\right) - L_2 u_2 \\ &= \frac{1}{\varepsilon} L_0 u_3 - L_2 u_2. \end{aligned}$$

Then the variational formulation of (2.3.2) is

$$\begin{cases} \textit{Find } r_\varepsilon \in H_0^1(\Omega) \textit{ such that} \\ \int_\Omega A_\varepsilon \nabla r_\varepsilon \nabla \phi dx = \frac{1}{\varepsilon} \int_\Omega \left(L_0 u_3\right) \phi dx - \int_\Omega \left(L_2 u_2\right) \phi dx, \quad \forall \phi \in H_0^1(\Omega). \end{cases}$$

We have for all $\phi \in H_0^1(\Omega)$ the estimate

$$\begin{aligned} \left|\int_\Omega A_\varepsilon \nabla r_\varepsilon \nabla \phi dx\right| &= \left|\frac{1}{\varepsilon}\int_\Omega (L_0 u_3)\phi dx - \int_\Omega (div_x A_\varepsilon \nabla_y u_3)\phi dx + \int_\Omega (div_x A_\varepsilon \nabla_y u_3)\phi dx - \int_\Omega (L_2 u_2)\phi dx\right| \\ &\leq \left|\frac{1}{\varepsilon}\int_\Omega (L_0 u_3)\phi dx - \int_\Omega (div_x A_\varepsilon \nabla_y u_3)\phi dx\right| + \left|\int_\Omega (div_x A_\varepsilon \nabla_y u_3)\phi dx - \int_\Omega (L_2 u_2)\phi dx\right| \\ &= \left|-\int_\Omega (divA_\varepsilon \nabla_y u_3)\phi dx\right| + \left|\int_\Omega (div_x A_\varepsilon(\nabla_x u_2 + \nabla_y u_3))\phi dx\right| \\ &= \left|\int_\Omega A_\varepsilon \nabla_y u_3 \nabla \phi dx\right| + \left|-\int_\Omega A_\varepsilon(\nabla_x u_2 + \nabla_y u_3)\nabla \phi dx\right| \\ &\leq 2\left|\int_\Omega A_\varepsilon \nabla_y u_3 \nabla \phi dx\right| + \left|\int_\Omega A_\varepsilon \nabla_x u_2 \nabla \phi dx\right|. \end{aligned}$$

Using the L^∞ boundedness of A_ε, and that $\|\nabla_y u_3\|_{L^2(\Omega)} \leq C_{13}\|u_0\|_{H^3(\Omega)}$ and $\|\nabla_x u_2\|_{L^2(\Omega)} \leq C\|u_0\|_{H^3(\Omega)}$, we get

$$\left|\int_\Omega A_\varepsilon \nabla r_\varepsilon \nabla \phi dx\right| \leq C\|u_0\|_{H^3(\Omega)} \|\phi\|_{H_0^1(\Omega)}, \ \forall \phi \in H_0^1(\Omega).$$

By taking $\phi = r_\varepsilon$ and using the ellipticity of A_ε, we obtain

$$\lambda \|r_\varepsilon\|^2_{H_0^1(\Omega)} \leq \int_\Omega A_\varepsilon \nabla r_\varepsilon \nabla r_\varepsilon dx \leq C\|u_0\|_{H^3(\Omega)} \|r_\varepsilon\|_{H_0^1(\Omega)}$$

which implies that

$$\|r_\varepsilon\|_{H_0^1(\Omega)} \leq C\|u_0\|_{H^3(\Omega)}.$$

□

The second result deals to the third-order error estimate without the boundary layer correctors. For this case, we need u_0 to be in $H^4(\Omega)$.

Theorem 18.

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H) respectively, with $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^\infty$. Assume that $f \in C^\infty(\overline{\Omega})$ and χ^j, χ^{ij} and $\chi^{ijk} \in W^{1,\infty}(Y)$. Then

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 \right\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon} \tag{2.3.3}$$

For the proof of this theorem we need to use the following tools:

Proposition 2.3.1. *Let F be in $H^{-1}(\Omega)$. Then, there exist $n+1$ functions $f_0, f_1, ..., f_n$ in $L^2(\Omega)$ such that*

$$F = f_0 + \textstyle\sum_{i=1}^{n} \dfrac{\partial f_i}{\partial x_i}$$

in the sense of distributions. Moreover

$$\|F\|^2_{H^{-1}(\Omega)} = inf \textstyle\sum_{i=0}^{n} \|f_i\|^2_{L^2(\Omega)},$$

where the infimum is taken over all the vectors $(f_0, f_1, ...f_n) \in [L^2(\Omega)]^{n+1}$. Conversely, if $(f_0, f_1, ..., f_n)$ is a vector in $[L^2(\Omega)]^{n+1}$, then $F \in H^{-1}(\Omega)$ and it satisfies

$$\|F\|^2_{H^{-1}(\Omega)} \leq \textstyle\sum_{i=0}^{n} \|f_i\|^2_{L^2(\Omega)}.$$

(See [Proposition 3.42, [32]]).

Lemma 2.3.1. *Let Ω be a bounded domain with a smooth boundary and*

$$B_\delta = \{x \in \Omega, \rho(x, \partial\Omega) < \delta\} \text{ with } \delta > 0.$$

Then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ and every $v \in H^1(\Omega)$ we have

$$\|v\|_{L^2(B_\delta)} \leq C\delta^{\frac{1}{2}} \|v\|_{H^1(\Omega)},$$

where $\rho(x, \partial\Omega)$ denotes the distance of $x \in \Omega$ from the set $\partial\Omega$, and C_{18} is a constant independent of δ and v.

Proof. (See [Chapter 1, Lemma 1.5, [85]]). □

Theorem 19.

Let $A(\frac{x}{\varepsilon})$ be an uniformly elliptic bounded matrix and $\partial\Omega$ be Lipschitz continuous. Suppose that

$f \in H^{-1}(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$ then, there exists a unique $u_\varepsilon \in H^1(\Omega)$ solution to

$$\begin{cases} -div(A(\frac{x}{\varepsilon})\nabla u_\varepsilon) = f \quad in\ \Omega \\ u_\varepsilon = g \quad on\ \ \partial\Omega \end{cases}$$

and

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)} + C\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

Proof. (See [Theorem 23.4, [48]]). □

We now give the proof of Theorem 18.

Proof. We set:

$$Z_\varepsilon = u_\varepsilon - (u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3),$$

$$u_0 = u_0(x),$$

$$u_1 = -\chi^j \frac{\partial u_0}{\partial x_j},$$

$$u_2 = \chi^{ij} \frac{\partial^2 u_0}{\partial x_i \partial x_j},$$

$$u_3 = \chi^{ijk} \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k},$$

then,

$$\begin{aligned} L_\varepsilon Z_\varepsilon &= L_\varepsilon u_\varepsilon - L_\varepsilon(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3) \\ &= L_\varepsilon u_\varepsilon - (\varepsilon^{-2} L_0 + \varepsilon^{-1} L_1 + L_2)(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3) \\ &= L_\varepsilon u_\varepsilon - \varepsilon^{-2} L_0 u_0 - \varepsilon^{-1}(L_0 u_1 + L_1 u_0) - (L_0 u_2 + L_1 u_1 + L_2 u_0) \\ &- \varepsilon(L_0 u_3 + L_2 u_1 + L_1 u_2) - \varepsilon^2(L_1 u_3 + L_2 u_2) - \varepsilon^3(L_2 u_3). \end{aligned}$$

Using the equations of (2.1.4), we get

$$L_\varepsilon Z_\varepsilon = -\varepsilon^2(L_1 u_3 + L_2 u_2) - \varepsilon^3(L_2 u_3).$$

Since

$$\frac{\partial}{\partial x_i} = \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}, \quad and \ \frac{\partial}{\partial y_i} = \varepsilon\frac{\partial}{\partial x_i},$$

a simple computation shows that:

$$L_1 u_3 = -a_{lm}(\frac{x}{\varepsilon})\frac{\partial \chi^{ijk}}{\partial y_m}\frac{\partial^4 u_0}{\partial x_l \partial x_i \partial x_j \partial x_k} - \varepsilon\frac{\partial}{\partial x_l}(a_{lm}(\frac{x}{\varepsilon})\chi^{ijk}(\frac{x}{\varepsilon})\frac{\partial^4 u_0}{\partial x_m \partial x_i \partial x_j \partial x_k})$$
$$- \varepsilon L_2 u_3,$$
$$L_2 u_2 = -a_{lm}(\frac{x}{\varepsilon})\chi^{ij}(\frac{x}{\varepsilon})\frac{\partial^4 u_0}{\partial x_l \partial x_m \partial x_i \partial x_j}$$

then:

$$L_\varepsilon Z_\varepsilon = -\varepsilon^2(a_{lm}(\frac{x}{\varepsilon})\frac{\partial \chi^{ijk}}{\partial y_m}\frac{\partial^4 u_0}{\partial x_l \partial x_i \partial x_j \partial x_k}) - \varepsilon^3(\frac{\partial}{\partial x_l}(a_{lm}(\frac{x}{\varepsilon})\chi^{ijk}(\frac{x}{\varepsilon})\frac{\partial^4 u_0}{\partial x_m \partial x_i \partial x_j \partial x_k}))$$
$$- \varepsilon^2 a_{lm}(\frac{x}{\varepsilon})\chi^{ij}(\frac{x}{\varepsilon})\frac{\partial^4 u_0}{\partial x_l \partial x_m \partial x_i \partial x_j}.$$

Taking into account that u_ε and u_0 vanish on the boundary $\partial\Omega$, then it follows easily that Z_ε satisfies

$$\begin{cases} L_\varepsilon Z_\varepsilon = \varepsilon^2 F^\varepsilon & in\ \Omega \\ Z_\varepsilon = \varepsilon G^\varepsilon & on\ \ \partial\Omega, \end{cases}$$

where

$$\begin{cases} F^\varepsilon = -a_{lm}(\frac{x}{\varepsilon})\frac{\partial \chi^{ijk}}{\partial y_m}\frac{\partial^4 u_0}{\partial x_l \partial x_i \partial x_j \partial x_k} - a_{lm}(\frac{x}{\varepsilon})\chi^{ij}(\frac{x}{\varepsilon})\frac{\partial^4 u_0}{\partial x_l \partial x_m \partial x_i \partial x_j} \\ \quad - \varepsilon(\frac{\partial}{\partial x_l}(a_{lm}(\frac{x}{\varepsilon})\chi^{ijk}(\frac{x}{\varepsilon})\frac{\partial^4 u_0}{\partial x_m \partial x_i \partial x_j \partial x_k})), \\ G^\varepsilon = -u_1 - \varepsilon u_2 - \varepsilon^2 u_3. \end{cases}$$

We put

$F_0 = -a_{lm}(\frac{x}{\varepsilon})\frac{\partial \chi^{ijk}}{\partial y_m}\frac{\partial^4 u_0}{\partial x_l \partial x_i \partial x_j \partial x_k} - a_{lm}(\frac{x}{\varepsilon})\chi^{ij}(\frac{x}{\varepsilon})\frac{\partial^4 u_0}{\partial x_l \partial x_m \partial x_i \partial x_j},$

$F_l = -a_{lm}(\frac{x}{\varepsilon})\chi^{ijk}(\frac{x}{\varepsilon})\frac{\partial^4 u_0}{\partial x_m \partial x_i \partial x_j \partial x_k}.$

Under the assumptions on $a_{lm}, u_0, \chi^{ij} and\ \chi^{ijk}\ we\ get$

$$\|F_0\|_{L^2(\Omega)} \leq C, \tag{2.3.4}$$

$$\|F_l\|_{L^2(\Omega)} \leq C. \tag{2.3.5}$$

Using the Proposition 2.3.1, then from (2.3.4) and (2.3.5) we obtain $F^\varepsilon \in H^{-1}(\Omega)$.

Let us now look at the function G_ε . We prove the following estimate:

$$\|G_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\varepsilon^{\frac{-1}{2}}.$$

At this point, we need to introduce the function $m_\varepsilon \in D(\Omega)$ defined as follows

$$\begin{cases} m_\varepsilon = 1 \ if \ \rho(x, \partial\Omega) \leq \varepsilon \\ m_\varepsilon = 0 \ if \ \rho(x, \partial\Omega) \geq 2\varepsilon \\ \|\nabla m_\varepsilon\|_{L^\infty(\Omega)} \leq \dfrac{C}{\varepsilon}. \end{cases}$$

For the existence of such kind of functions see [32] and the references therein.

Set

$$V_\varepsilon = m_\varepsilon G_\varepsilon$$

supp $V_\varepsilon = \overline{\{x, \rho(x, \partial\Omega) < 2\varepsilon\}}$ which will be denoted by U_ε.

Using the H^1-norm, we have

$$\|V_\varepsilon\|_{H^1(U_\varepsilon)} = \|V_\varepsilon\|_{L^2(U_\varepsilon)} + \|\nabla V_\varepsilon\|_{L^2(U_\varepsilon)}.$$

Clearly, from the definition of m_ε and the regularity properties of $u_0, \chi^j, \chi^{ij} and\ \chi^{ijk}$, one has that

$$\|V_\varepsilon\|_{L^2(U_\varepsilon)} \leq C.$$

On the other hand, we have

$$\begin{aligned} \frac{\partial V^\varepsilon}{\partial x_i}(x) \;=\; & m_\varepsilon(x) \left[\frac{1}{\varepsilon} \frac{\partial \chi^k}{\partial y_i}(\frac{x}{\varepsilon}) \frac{\partial u_0(x)}{\partial x_k} + \chi^k(\frac{x}{\varepsilon}) \frac{\partial^2 u_0(x)}{\partial x_i \partial x_k} - \frac{\partial \chi^{kl}}{\partial y_i}(\frac{x}{\varepsilon}) \frac{\partial^2 u_0(x)}{\partial x_k \partial x_l} - \right. \\ & \left. \varepsilon \chi^{kl}(\frac{x}{\varepsilon}) \frac{\partial^3 u_0(x)}{\partial x_i \partial x_k \partial x_l} - \varepsilon \frac{\partial \chi^{klm}}{\partial y_i}(\frac{x}{\varepsilon}) \frac{\partial^3 u_0(x)}{\partial x_k \partial x_l \partial x_m} - \varepsilon^2 \chi^{klm}(\frac{x}{\varepsilon}) \frac{\partial^4 u_0(x)}{\partial x_i \partial x_k \partial x_l \partial x_m} \right] \\ & + \frac{\partial m_\varepsilon}{\partial x_i} \left[\chi^k(\frac{x}{\varepsilon}) \frac{\partial u_0(x)}{\partial x_k} - \varepsilon \chi^{kl}(\frac{x}{\varepsilon}) \frac{\partial^2 u_0(x)}{\partial x_k \partial x_l} - \varepsilon^2 \chi^{klm}(\frac{x}{\varepsilon}) \frac{\partial^3 u_0(x)}{\partial x_k \partial x_l \partial x_m} \right]. \end{aligned}$$

Again, on the account of the above definition of m_ε and the regularity properties of u_0, χ^k, χ^{kl} and χ^{klm}, it is easy to check that

$$\|\nabla V^\varepsilon\|_{L^2(U_\varepsilon)} \leq \frac{1}{\varepsilon} C \|u_0\|_{H^1(U_\varepsilon)} + C,$$

and owing to Lemma 2.3.1, we derive that

$$\|u_0\|_{H^1(U_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)} .$$

Then we conclude that

$$\begin{aligned}\|V_\varepsilon\|_{H^1(U_\varepsilon)} &\leq C + \varepsilon^{-1} C_{26} \|u_0\|_{H^1(U_\varepsilon)} \\ &\leq C + \varepsilon^{-1} C(C\varepsilon^{\frac{1}{2}} \|u_0\|_{H^2(\Omega)}) \\ &\leq C\varepsilon^{\frac{-1}{2}}.\end{aligned}$$

On $\partial\Omega$, $V_\varepsilon = G_\varepsilon$, this gives that

$$\|G_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|V_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\|V_\varepsilon\|_{H^1(\Omega)} = C\|V_\varepsilon\|_{H^1(U_\varepsilon)} \leq C\varepsilon^{\frac{-1}{2}}.$$

Using the regularity results of Theorem 19, we deduce that

$$\|Z_\varepsilon\|_{H^1(\Omega)} \leq \varepsilon^2 \|F^\varepsilon\|_{H^{-1}(\Omega)} + \varepsilon \|G_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\varepsilon^{\frac{1}{2}},$$

which proves the theorem. □

The third result is about the third-order error estimate without the third boundary layer corrector. In this case, we need u_0 to be in $W^{4,\infty}(\Omega)$.

Using the Sobolev embedding result (see Adams [1]): Let $l \in \mathbb{N}$, $m \in \mathbb{N}^*$ and $1 \leq p < \infty$. If either $(m-l)p > n$ or $m - l = n$ and $p = 1$, then $W^{m,p}(\Omega) \hookrightarrow W^{l,q}(\Omega)$, for $p \leq q \leq \infty$. So we have $W^{n+4,1}(\Omega) \hookrightarrow W^{4,\infty}(\Omega)$ and like $u_0 \in C^\infty(\overline{\Omega}) \subset W^{m,p}(\Omega)$ for all $m \in \mathbb{N}^*$ and $1 \leq p < \infty$, then $u_0 \in W^{4,\infty}(\Omega)$.

Theorem 20.

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H) respectively, with $\Omega \subset \mathbb{R}^n$ is a strictly convex bounded domain with $\partial\Omega \in C^\infty$. Assume that $f \in C^\infty(\overline{\Omega})$ and χ^{ijk}, $\chi^{ijkl} \in W^{1,\infty}(Y)$. Then

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3 \right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{5}{2}}. \tag{2.3.6}$$

In order to proof this theorem we need the following Lemma :

Lemma 2.3.2. *Let ϕ_ε be a sequence of functions in $W^{1,\infty}(\Omega)$ such that*

$$\|\phi_\varepsilon\|_{L^\infty(\Omega)} \leq C \quad \textit{and} \quad \|\nabla\phi_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}.$$

Let $z_\varepsilon \in H^1(\Omega)$ be the solution of

$$\begin{cases} -div A_\varepsilon \nabla z_\varepsilon = 0 & in \quad \Omega, \\ z_\varepsilon = \phi_\varepsilon & on \ \partial\Omega. \end{cases}$$

Then it satisfies

$$\|z_\varepsilon\|_{H^1(\Omega)} \leq \frac{C}{\sqrt{\varepsilon}}.$$

Proof. For the proof we refer the reader to (Lemma 2.6, [2]) □

Proof of Theorem 20

Defining $r_\varepsilon(x) = \dfrac{1}{\varepsilon^3}(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3)$,

it satisfies

$$\begin{cases} -div A_\varepsilon \nabla r_\varepsilon = \dfrac{1}{\varepsilon^3}(f + div A_\varepsilon \nabla u_0) + \dfrac{1}{\varepsilon^2} div A_\varepsilon \nabla u_1 + \dfrac{1}{\varepsilon} div A_\varepsilon \nabla u_2 + div A_\varepsilon \nabla u_3 & in \quad \Omega, \\ r_\varepsilon = -u_3(x, \dfrac{x}{\varepsilon}) & on \quad \partial\Omega. \end{cases} \tag{2.3.7}$$

We decompose $r_\varepsilon = r_\varepsilon^1 + r_\varepsilon^2$, where r_ε^1 satisfies

$$\begin{cases} -div A_\varepsilon \nabla r_\varepsilon^1 = \dfrac{1}{\varepsilon^3}(f + div A_\varepsilon \nabla u_0) + \dfrac{1}{\varepsilon^2} div A_\varepsilon \nabla u_1 + \dfrac{1}{\varepsilon} div A_\varepsilon \nabla u_2 + div A_\varepsilon \nabla u_3 \ in \ \Omega, \\ r_\varepsilon^1 = 0 \qquad on \ \partial\Omega, \end{cases} \tag{2.3.8}$$

and r_ε^2 satisfies

$$\begin{cases} -div A_\varepsilon \nabla r_\varepsilon^2 = 0 & in \ \Omega, \\ r_\varepsilon^2 = -u_3(x, \dfrac{x}{\varepsilon}) = -\chi^{ijk}(\dfrac{x}{\varepsilon}) \dfrac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} & on \ \partial\Omega. \end{cases} \tag{4.6}$$

Using the fact that $u_3(x, \dfrac{x}{\varepsilon})$ satisfies

$$\|u_3\|_{L^\infty(\Omega)} \leq C \quad \text{and} \quad \|\nabla u_3\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon},$$

then Lemma 2.3.2 gives that $\|r_\varepsilon^2\|_{H^1(\Omega)} \leq \dfrac{C}{\sqrt{\varepsilon}}$. On the other hand, we will now estimate r_ε^1 the solution of the problem (2.3.8). Using the results obtained in the proof of Theorem 17 and the fact that

$$div A_\varepsilon \nabla u_3 = -L_2 u_3 - \frac{1}{\varepsilon} L_1 u_3 - \frac{1}{\varepsilon^2} L_0 u_3$$

we get

$$-divA_\varepsilon\nabla r_\varepsilon^1 = -L_2u_3 - \frac{1}{\varepsilon}\left(L_1u_3 + L_2u_2\right) = -L_2u_3 + \frac{1}{\varepsilon}L_0u_4.$$

The variational formulation of (2.3.8) is

$$\begin{cases} \text{Find } r_\varepsilon^1 \in H_0^1(\Omega) \text{ such that} \\ \int_\Omega A_\varepsilon\nabla r_\varepsilon^1\nabla\phi dx = \frac{1}{\varepsilon}\int_\Omega \left(L_0u_4\right)\phi dx - \int_\Omega \left(L_2u_3\right)\phi dx, \quad \forall\phi \in H_0^1(\Omega). \end{cases}$$

We have for all $\phi \in H_0^1(\Omega)$ the estimate

$$\begin{aligned} \left|\int_\Omega A_\varepsilon\nabla r_\varepsilon^1\nabla\phi dx\right| &= \left|\frac{1}{\varepsilon}\int_\Omega \left(L_0u_4\right)\phi dx - \int_\Omega \left(div_xA_\varepsilon\nabla_yu_3\right)\phi dx + \int_\Omega \left(div_xA_\varepsilon\nabla_yu_3\right)\phi dx - \int_\Omega \left(L_2u_3\right)\phi dx\right| \\ &\leq \left|\frac{1}{\varepsilon}\int_\Omega \left(L_0u_4\right)\phi dx - \int_\Omega \left(div_xA_\varepsilon\nabla_yu_4\right)\phi dx\right| + \left|\int_\Omega \left(div_xA_\varepsilon\nabla_yu_4\right)\phi dx - \int_\Omega \left(L_2u_3\right)\phi dx\right| \\ &= \left|-\int_\Omega \left(divA_\varepsilon\nabla_yu_4\right)\phi dx\right| + \left|\int_\Omega \left(div_xA_\varepsilon(\nabla_xu_3 + \nabla_yu_4)\right)\phi dx\right| \\ &= \left|\int_\Omega A_\varepsilon\nabla_yu_4\nabla\phi dx\right| + \left|-\int_\Omega A_\varepsilon(\nabla_xu_3 + \nabla_yu_4)\nabla\phi dx\right| \\ &\leq 2\left|\int_\Omega A_\varepsilon\nabla_yu_4\nabla\phi dx\right| + \left|\int_\Omega A_\varepsilon\nabla_xu_3\nabla\phi dx\right|. \end{aligned}$$

Using the L^∞ boundedness of A_ε, ∇_yu_4 and ∇_xu_3 we get

$$\left|\int_\Omega A_\varepsilon\nabla r_\varepsilon^1\nabla\phi dx\right| \leq C\left\|\phi\right\|_{H_0^1(\Omega)}, \ \forall\phi \in H_0^1(\Omega).$$

By taking $\phi = r_\varepsilon^1$ and using the ellipticity of A_ε, we obtain

$$\lambda\left\|r_\varepsilon^1\right\|_{H_0^1(\Omega)}^2 \leq \int_\Omega A_\varepsilon\nabla r_\varepsilon^1\nabla r_\varepsilon^1 dx \leq C_{41}\left\|r_\varepsilon^1\right\|_{H_0^1(\Omega)}$$

which implies that

$$\left\|r_\varepsilon^1\right\|_{H_0^1(\Omega)} \leq C.$$

Finally, we get $\varepsilon^3\|r_\varepsilon\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{5}{2}}$ which establishes the desired estimate. □

The fourth result concerns the third-order error estimate with boundary layers correctors. In this case, we need u_0 to be in $W^{4,\infty}(\Omega)$.

Theorem 21.

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H) respectively, with $\Omega \subset \mathbb{R}^n$ is a strictly convex bounded domain with $\partial\Omega \in C^\infty$. Assume that $f \in C^\infty(\overline{\Omega})$ and $\chi^{ijkl} \in W^{1,\infty}(Y)$. Then

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3 - \varepsilon^3 u_3^{bl,\varepsilon} \right\|_{H_0^1(\Omega)} \leq C\varepsilon^3. \tag{2.3.9}$$

Proof. Defining $r_\varepsilon(x) = \dfrac{1}{\varepsilon^3}(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3 - \varepsilon^3 u_3^{bl,\varepsilon})$, it satisfies

$$\begin{cases} -div A_\varepsilon \nabla r_\varepsilon = \dfrac{1}{\varepsilon^3}(f + div A_\varepsilon \nabla u_0) + \dfrac{1}{\varepsilon^2} div A_\varepsilon \nabla u_1 + \dfrac{1}{\varepsilon} div A_\varepsilon \nabla u_2 + div A_\varepsilon \nabla u_3 \ in\ \Omega, \\ r_\varepsilon = 0 \qquad on\ \partial\Omega. \end{cases} \tag{4.7}$$

This problem is the same as (2.3.8), so the solution r_ε has the same estimate of r_ε^1 the solution of (2.3.8), i.e.

$$\|r_\varepsilon\|_{H_0^1(\Omega)} = \|r_\varepsilon^1\|_{H_0^1(\Omega)} \leq C.$$

Thus,

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3 - \varepsilon^3 u_3^{bl,\varepsilon} \right\|_{H_0^1(\Omega)} = \varepsilon^3 \|r_\varepsilon\|_{H_0^1(\Omega)} \leq C\varepsilon^3.$$

Which completes the proof. □

Remark 2.3.1. *In accordance with the results obtained in Theorems 17, 18, 20, 21and the estimates (2.2.3) and (2.2.9), we infer that* the correctors have no influence *on the improvement of the order of the error in the estimates. However,* the introduction of boundary layers terms improves these estimates.

The conditions posed on the homogenized solution u_0 and on the solutions of the cell-problems χ^{ijk} *and* χ^{ijmp} in Theorems 20 and 21 in the above section, bring us to the following question : if we assume minimal regularity assumptions, can one prove differently and obtain the third-order error estimates as stated in theorems 20 and 21?

Our study succeed to answer this question in dimensions two, and this is exactly what will be shown by the following section.

2.4 Third-order corrections in periodic homogenization using mixed method

For the study carried out in this section we need the following results.

Lemma 2.4.1. *(Lemma 1.3.1 [110]) A function* $v \in L^2_\sharp(Y)^2, (v \in L^2_\sharp(Y)^3)$ *satisfies*

$$div\ v = 0 \ \ and \ \ \int_Y v = 0.$$

iff there exists a function $\phi \in H^1_\sharp(Y)^2, \left(\phi \in H^1_\sharp(Y)\right)^3\big)$, *such that*

$$v = curl\phi.$$

Lemma 2.4.2. *Let* $f \in L^2_\sharp(Y)$ *be a periodic function. There exists a solution in* $H^1_\sharp(Y)$ *(unique up to an additive constant) of*

$$\begin{cases} -div_y A(y)\nabla w(y) = f \quad in\ Y, \\ y \longrightarrow w(y) \quad Y-periodic. \end{cases} \tag{2.4.1}$$

iff $\int_Y f(y)dy = 0$ *(this is called the Fredholm alternative). Such that* $L^2_\sharp(Y)$ *and* $H^1_\sharp(Y)$ *denote the subspaces of functions in* $L^2_{loc}(\mathbb{R}^n)$ *and* $H^1_{loc}(\mathbb{R}^n)$*, respectively, which are Y-periodic.*

Proof. See [18]. □

Proposition 2.4.1. *(Proposition 3.31 [32])*

Suppose that $\partial\Omega$ *is Lipschitz continuous. Then there exists a constant* C_4 *such that*

$$\|\gamma(u)\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_4(\Omega)\|u\|_{H^1(\Omega)}, \quad \forall u \in H^1(\Omega),$$

where $\gamma(u)$ *denotes the trace of* u.

2.4.1 Position of the problem

Let us consider the same problem as in the previous section. Let $u_\varepsilon \in H^1_0(\Omega)$ denotes the solution to the following well-posed problem

$$(P_\varepsilon) \begin{cases} -div A_\varepsilon \nabla u_\varepsilon = f \quad in\ \Omega, \\ u_\varepsilon = 0 \quad on\ \ \partial\Omega, \end{cases} \tag{2.4.2}$$

where Ω be a bounded open subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary, such that Ω satisfies a uniform exterior sphere condition. Let $A(y)$ be a square symmetric matrix with entries $a_{ij}(y)(i,j=1,2)$, which are Y-periodic functions belonging to $C^\infty(Y)$ and satisfying

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2,\ \forall\xi \in R^2,\ where\ 0<\lambda<\Lambda<+\infty.$$

Let $A_\varepsilon(x) = A(\frac{x}{\varepsilon})$ be a periodically oscillating matrix of coefficients where ε is a small positive parameter ($0<\varepsilon\leq 1$). For a fixed $f\in L^2(\Omega)$, We search u_ε in the form of an asymptotic expansion i.e.

$$u_\varepsilon = u_0(x,\frac{x}{\varepsilon}) + \varepsilon u_1(x,\frac{x}{\varepsilon}) + .. + \varepsilon^i u_i(x,\frac{x}{\varepsilon}) + ...,\tag{2.4.3}$$

From the previous section, we know that

$$\begin{aligned}
u_0 &\equiv u_0(x),\\
u_1(x,y) &= -\chi^j(y)\frac{\partial u_0}{\partial x_j}(x),\\
u_2(x,y) &= \chi^{ij}(y)\frac{\partial^2 u_0}{\partial x_i \partial x_j},\\
u_3(x,y) &= \chi^{ijk}(y)\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k},\\
u_4(x,y) &= \chi^{ijmp}(y)\frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_m \partial x_p}
\end{aligned}\tag{2.4.4}$$

Remark 2.4.1. *Since $a(y)$ is $C^\infty(Y)$, then, according to the regularity theory (see Evans [49]), it follows immediately that $\chi^j, \chi^{ij}, \chi^{ijk}$ and χ^{ijmp} are $C^\infty(Y)$.*

In the sequel of this section, we assume that $f\in H^2(\Omega)$, which implies, according to the regularity theory that $u_0\in H^4(\Omega)$. It is straightforward to verify that (P_ε) can be written as

$$\begin{cases} A_\varepsilon \nabla u_\varepsilon - v_\varepsilon = 0,\\ -div v_\varepsilon = f.\end{cases}\tag{2.4.5}$$

We expected that v_ε behaves like

$$v_\varepsilon = v_0(x,\frac{x}{\varepsilon}) + \varepsilon v_1(x,\frac{x}{\varepsilon}) + .. + \varepsilon^j v_j(x,\frac{x}{\varepsilon}) + ...,\tag{2.4.6}$$

where each v_j is Y-periodic in the fast variable "$y=\frac{x}{\varepsilon}$".

Remark 2.4.2. *The benefit of finding an equivalent problem to (P_ε) is to compute v_j which are very important in the proof of our first main result.*

By taking into account that $\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$ *and* $div = div_x + \frac{1}{\varepsilon} div_y$ together with identifying the different powers of ε we get

$$(\varepsilon^{-1}) \begin{cases} a(y)\nabla_y u_0 = 0 \\ -div_y v_0 = 0, \end{cases} \tag{2.4.7}$$

$$(\varepsilon^{0}) \begin{cases} a(y)\nabla_x u_0 + a(y)\nabla_y u_1 - v_0 = 0 \\ -div_x v_0 - div_y v_1 = f, \end{cases} \tag{2.4.8}$$

$$(\varepsilon^{1}) \begin{cases} a(y)\nabla_x u_1 + a(y)\nabla_y u_2 - v_1 = 0 \\ -div_x v_1 - div_y v_2 = 0, \end{cases} \tag{2.4.9}$$

$$(\varepsilon^{2}) \begin{cases} a(y)\nabla_x u_2 + a(y)\nabla_y u_3 - v_2 = 0 \\ -div_x v_2 - div_y v_3 = 0. \end{cases} \tag{2.4.10}$$

The task now is to determine v_j. Let us start by v_0. It is clear from $(2.4.8)_1$ that

$$v_0 = a(y)\nabla_x u_0 + a(y)\nabla_y u_1.$$

Furthermore, we have

$$\begin{aligned} -div_y(v_0)_i = -div_y\left\{a_{ij} - a_{ik}\frac{\partial \chi^j}{\partial y_k}\right\}\frac{\partial u_0}{\partial x_j} &= -div_y\{a_{ij}\}\frac{\partial u_0}{\partial x_j} + div_y\left\{a_{ik}\frac{\partial \chi^j}{\partial y_k}\right\}\frac{\partial u_0}{\partial x_j} \\ &= \{-div_y a_{ij} + div_y a_{ij}\}\frac{\partial u_0}{\partial x_j} \ (from\ (2.1.6)) \\ &= 0. \end{aligned} \tag{2.4.11}$$

So that we recover $(2.4.7)_2$, hence, we can conclude that

$$(v_0) \begin{cases} (v_0)_i = \left(a_{ij} - a_{ik}\frac{\partial \chi^j}{\partial y_k}\right)\frac{\partial u_0}{\partial x_j}, \\ \langle (v_0)_i \rangle = a^H_{ij}\frac{\partial u_0}{\partial x_j} = \langle b_{ij}\rangle \frac{\partial u_0}{\partial x_j}, \\ -\langle div_x v_0\rangle = f, \\ -div_y(v_0) = 0. \end{cases} \tag{2.4.12}$$

It is obvious that under $(2.4.8)_2$ and $(2.4.9)_1$, one can have

$$(v_1)\begin{cases} v_1 = a(y)\nabla_x u_1 + a(y)\nabla_y u_2, \quad i.e.\ (v_1)_k = \left(-a_{ki}\chi^j + a_{kl}\dfrac{\partial \chi^{ij}}{\partial y_l}\right)\dfrac{\partial^2 u_0}{\partial x_i \partial x_j}, \\ \langle (v_1)_k \rangle = \langle c_{ijk}(y)\rangle \dfrac{\partial^2 u_0}{\partial x_i \partial x_j}, \\ \langle div_x v_1 \rangle = 0, \\ div_y v_1 = -div_x v_0 - f. \end{cases} \tag{2.4.13}$$

From $(2.4.9)_2$ and $(2.4.10)_1$, we obtain

$$(v_2)\begin{cases} v_2 = a(y)\nabla_x u_2 + a(y)\nabla_y u_3, \quad i.e.\ (v_2)_m = \left(a_{mk}\chi^{ij} + a_{ml}\dfrac{\partial \chi^{ijk}}{\partial y_l}\right)\dfrac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}, \\ \langle (v_2)_m \rangle = \langle d_{mijk}\rangle \dfrac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}, \\ \langle div_x v_2 \rangle = 0, \\ div_y v_2 = -div_x v_1. \end{cases} \tag{2.4.14}$$

It remains to determine v_3, the construction of v_3 will be divided into three steps :

Step 1: The construction of the function $q(x,y)$.

From $(2.4.7)_2$, we have

$$-div_y v_0 = -div_y(v_0 - A^H \nabla u_0) = 0. \tag{2.4.15}$$

According to $(2.4.12)_2$, it is simple matter to show that

$$\langle (v_0 - A^H \nabla u_0) \rangle = 0. \tag{2.4.16}$$

Combining (2.4.15) with (2.4.16), then by using Lemma 2.4.1, we deduce that there exists a function $q(x,y)$ such that:

$$v_0 - A^H \nabla u_0 = curl_y q. \tag{2.4.17}$$

Due to the fact that $v_0 - A^H \nabla u_0$ is a function of separated variables x and y, q itself is and factors into

$$q(x,y) = \psi(y)\nabla u_0. \tag{2.4.18}$$

Since a_{ij} and χ^j are Y-periodic and belonging to $C^\infty(Y)$, then the function $\psi = (\psi^\alpha(y))_{1\leq\alpha\leq 2}$, also is Y-periodic and belonging to $(C^\infty(Y))^2$. As u_0 is assumed to be in $H^4(\Omega), q(x,y)$ *is in* $H^3(\Omega)$ with respect to x. Furthermore we have

$$\begin{aligned} div_y(curl_x q(x,y)) &= -\frac{\partial^2 q(x,y)}{\partial y_1 \partial x_2} + \frac{\partial^2 q(x,y)}{\partial y_2 \partial x_1} \\ &= -div_x(curl_y q(x,y)) \\ &= -div_x v_0 - f. \end{aligned} \tag{2.4.19}$$

Remark 2.4.3. *We see at once that $q(x,y)$ is Y-periodic and depends linearly on $\nabla_x u_0$, thus one can obtain*

$$\sup_{y\in Y}|\nabla_x q(x,y)| \leq C \sum_{i,j} \left| \frac{\partial^2 u_0}{\partial x_i \partial x_j} \right| \quad a.e.\ x \in \Omega. \tag{2.4.20}$$

Step 2: The construction of $p(x,y)$ in terms of $q(x,y)$.

Taking advantage of (2.4.19) and the definition of v_1 from (2.4.13), on the one hand

$$\begin{aligned} div_y(v_1 - curl_x q(x,y) - \langle v_1 \rangle + \langle curl_x q(x,y) \rangle) &= div_y(v_1 - curl_x q(x,y)) \\ &= -div_x v_0 - f + div_x(v_0) + f \\ &= 0, \end{aligned} \tag{2.4.21}$$

and on the other hand

$$\Big\langle \big(v_1 - curl_x q(x,y) - \langle v_1 \rangle + \langle curl_x q(x,y) \rangle\big) \Big\rangle = 0. \tag{2.4.22}$$

It follows that one can apply Lemma 2.4.1, and get a function $p(x,y)$ the unique solution to

$$curl_y p(x,y) = v_1 - curl_x q(x,y) - \langle v_1 \rangle + \langle curl_x q(x,y) \rangle. \tag{2.4.23}$$

On account of the fact that $v_1 - curl_x q(x,y) - \langle v_1 \rangle + \langle curl_x q(x,y) \rangle$ is a function of separated variables x and y, $p(x,y)$ itself is and factors into

$$p(x,y) = \omega(y) \frac{\partial^2 u_0}{\partial x_i \partial x_j}. \tag{2.4.24}$$

Again, since a_{ij}, χ^j, χ^{ij} and the function $\psi(y)$ (defined in (2.4.18)) are Y-periodic and belonging to $C^\infty(Y)$, then the function $\omega = (\omega^\alpha(y))_{1\leq\alpha\leq 2}$, also is Y-periodic and belonging to $(C^\infty(Y))^2$. As u_0 is assumed to be in $H^4(\Omega)$, $p(x,y)$ is in $H^2(\Omega)$ with respect to x. Furthermore we have

$$\begin{aligned} div_y(curl_x p(x,y)) &= -div_x(curl_y p(x,y)) \\ &= -div_x v_1 + div_x curl_x q(x,y) + div_x \langle v_1 \rangle - div_x \langle curl_x q(x,y) \rangle \\ &= -div_x v_1. \end{aligned} \tag{2.4.25}$$

Remark 2.4.4. *Owing to the fact that $p(x,y)$ depends linearly on $\dfrac{\partial^2 u_0}{\partial x_i \partial x_j}$, then one able to get*

$$\sup_{y\in Y} |\nabla_x p(x,y)| \leq C \sum_{i,j,k} \left| \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \right| \quad a.e.\ x \in \Omega. \tag{2.4.26}$$

.

<u>**Step 3:**</u> The determination of $K(x,y)$ and v_3.

Under (2.4.14) and (2.4.25), makes it obvious that

$$\begin{aligned} div_y\big(v_2(x,y) - curl_x p(x,y) - \langle v_2 \rangle + \langle curl_x p(x,y) \rangle\big) &= div_y\big(v_2 - curl_x p(x,y)\big) \\ &= 0. \end{aligned} \tag{2.4.27}$$

On the other hand, we have

$$\Big\langle \big(v_2(x,y) - curl_x p(x,y) - \langle v_2 \rangle + \langle curl_x p(x,y) \rangle\big) \Big\rangle = 0. \tag{2.4.28}$$

Combining (2.4.27) with (2.4.28) and by applying Lemma 2.4.1, we could find a function $K(x,y)$ solution to

$$curl_y K(x,y) = v_2(x,y) - curl_x p(x,y) - \langle v_2 \rangle + \langle curl_x p(x,y) \rangle. \tag{2.4.29}$$

Using the fact that $(v_2(x,y) - curl_x p(x,y) - \langle v_2 \rangle + \langle curl_x p(x,y) \rangle)$ is a function of separated variables x and y, $K(x,y)$ itself is and factors into

$$K(x,y) = \Phi(y) \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}. \tag{2.4.30}$$

Since a_{ij}, χ^{ij}, χ^{ijk} and the function $\omega(y)$ (defined in (2.4.24)) are Y-periodic and belonging to $C^\infty(Y)$, then the function

$\Phi = (\Phi^\alpha(y))_{1\leq\alpha\leq 2}$, also is Y-periodic and belonging to $(C^\infty(Y))^2$. As u_0 is assumed to be in $H^4(\Omega)$, $K(x,y)$ is in $H^1(\Omega)$ with respect to x.

Remark 2.4.5. *It is easily seen that $K(x,y)$ is Y-periodic and depends linearly on $\dfrac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}$, so that one has the estimate*

$$\sup_{y\in Y} |\nabla_x K(x,y)| \leq C \sum_{i,j,k,l} \left| \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l} \right| \quad a.e.\ x \in \Omega. \tag{2.4.31}$$

Thus, it is convenient to take $v_3 = curl_x K(x,y)$, by a simple manipulations we can conclude that

$$(v_3)\begin{cases} v_3 = curl_x K(x,y), \\ div_x v_3 = 0, \\ div_y v_3 = -div_x v_2. \end{cases} \tag{2.4.32}$$

Making use of (2.4.31) we get

$$\sup_{y\in Y}|v_3| \le C\sum_{i,j,k,l}\left|\frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l}\right|. \tag{2.4.33}$$

2.4.2 The boundary layers terms

Under the assumption that $u_0 \in H^4(\Omega)$, so the functions u_1, u_2, u_3 defined in (2.4.4) have a traces in $H^{\frac{1}{2}}(\partial\Omega)$, consequently, and owing to Proposition 2.4.1 we can extract the following estimates:

$$\begin{aligned} \|u_1\|_{H^{\frac{1}{2}}(\partial\Omega)} &\le C\|u_0\|_{H^4(\Omega)}, \\ \|u_2\|_{H^{\frac{1}{2}}(\partial\Omega)} &\le C\|u_0\|_{H^4(\Omega)}, \\ \|u_3\|_{H^{\frac{1}{2}}(\partial\Omega)} &\le C\|u_0\|_{H^4(\Omega)}. \end{aligned} \tag{2.4.34}$$

Therefore we can introduce the boundary layers functions $u_1^{bl,\varepsilon}$, $u_2^{bl,\varepsilon}$ and $u_3^{bl,\varepsilon}$ the unique solutions to $(P_{u_1^{bl,\varepsilon}})$, $(P_{u_2^{bl,\varepsilon}})$ and $(P_{u_3^{bl,\varepsilon}})$ respectively, where

$$(P_{u_1^{bl,\varepsilon}})\begin{cases} div(a(\frac{x}{\varepsilon})\nabla u_1^{bl,\varepsilon}) = 0 \quad in\ \Omega, \\ u_1^{bl,\varepsilon} = u_1 \quad on\ \partial\Omega, \end{cases} \tag{2.4.35}$$

and

$$(P_{u_2^{bl,\varepsilon}})\begin{cases} div(a(\frac{x}{\varepsilon})\nabla u_2^{bl,\varepsilon}) = 0 \quad in\ \Omega, \\ u_2^{bl,\varepsilon} = u_2 \quad on\ \partial\Omega, \end{cases} \tag{2.4.36}$$

and

$$(P_{u_3^{bl,\varepsilon}})\begin{cases} div(a(\frac{x}{\varepsilon})\nabla u_3^{bl,\varepsilon}) = 0 \quad in\ \Omega, \\ u_3^{bl,\varepsilon} = u_3 \quad on\ \partial\Omega. \end{cases} \tag{2.4.37}$$

Remark 2.4.6. *The existence and uniqueness of $u_1^{bl,\varepsilon}$, $u_2^{bl,\varepsilon}$ and $u_3^{bl,\varepsilon}$ can be deduced immediately from Theorem 19.*

From the L^2-estimates proved in ([9]) and the formula for each $u_i(x,y)$, it follows that

$$\begin{aligned}
\|u_1^{bl,\varepsilon}\|_{L^2(\Omega)} &\leq C\|u_1(x,\frac{x}{\varepsilon})\|_{L^2(\partial\Omega)} \leq C\|u_0\|_{H^4(\Omega)}, \\
\|u_2^{bl,\varepsilon}\|_{L^2(\Omega)} &\leq C\|u_2(x,\frac{x}{\varepsilon})\|_{L^2(\partial\Omega)} \leq C\|u_0\|_{H^4(\Omega)}, \\
\|u_3^{bl,\varepsilon}\|_{L^2(\Omega)} &\leq C\|u_3(x,\frac{x}{\varepsilon})\|_{L^2(\partial\Omega)} \leq C\|u_0\|_{H^4(\Omega)}.
\end{aligned} \tag{2.4.38}$$

2.4.3 The main results

The first result concerns the third-order error estimate with the third-order boundary layer corrector. For this case we need the regularity $H^4(\Omega)$ for u_0.

Theorem 22.

Let u_ε and u_0 denote the unique solutions of (P_ε) and (P_H) respectively, suppose that $f \in H^2(\Omega)$ then

$$\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\|_{H_0^1(\Omega)} \leq C\varepsilon^3 \|u_0\|_{H^4(\Omega)}$$

Proof. The proof will be divided into three steps.

Step 1: *The definitions of ψ_ε and ξ_ε .*

Let

$$\begin{aligned}
\psi_\varepsilon &= u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3, \\
\xi_\varepsilon &= a(\frac{x}{\varepsilon})\nabla u_\varepsilon - v_0 - \varepsilon v_1 - \varepsilon^2 v_2 - \varepsilon^3 v_3,
\end{aligned}$$

such that

$$
\begin{aligned}
a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon &= a(\frac{x}{\varepsilon})\nabla u_\varepsilon - a(\frac{x}{\varepsilon})\nabla u_0 - \varepsilon a(\frac{x}{\varepsilon})\nabla u_1 - \varepsilon^2 a(\frac{x}{\varepsilon})\nabla u_2 - \varepsilon^3 a(\frac{x}{\varepsilon})\nabla u_3 \\
div\xi_\varepsilon &= div(a(\frac{x}{\varepsilon}))\nabla u_\varepsilon) - div_x v_0 - \frac{1}{\varepsilon}div_y v_0 - \varepsilon div_x v_1 - div_y v_1 - \varepsilon^2 div_x v_2 - \varepsilon div_y v_2 \\
&- \varepsilon^3 div_x v_3 - \varepsilon^2 div_y v_3 \\
&= -f(x) - div_x v_0 - \varepsilon div_x v_1 + div_x v_0 + f(x) - \varepsilon^2 div_x v_2 - \varepsilon div_y v_2 - \varepsilon^3 div_x v_3 \\
&- \varepsilon^2 div_y v_3 \\
&= -\varepsilon div_x v_1 - \varepsilon div_y v_2 \\
&= 0. \\
a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon &= a(\frac{x}{\varepsilon})\nabla u_\varepsilon - a(\frac{x}{\varepsilon})\nabla u_0 - \varepsilon a(\frac{x}{\varepsilon})\nabla u_1 - \varepsilon^2 a(\frac{x}{\varepsilon})\nabla u_2 - \varepsilon^3 a(\frac{x}{\varepsilon})\nabla u_3 - a(\frac{x}{\varepsilon})\nabla u_\varepsilon + v_0 \\
&+ \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 \\
&= -a(\frac{x}{\varepsilon})\nabla_x u_0 - \varepsilon a(\frac{x}{\varepsilon})\nabla_x u_1 - a(\frac{x}{\varepsilon})\nabla_y u_1 - \varepsilon^2 a(\frac{x}{\varepsilon})\nabla_x u_2 - \varepsilon a(\frac{x}{\varepsilon})\nabla_y u_2 - \varepsilon^3 a(\frac{x}{\varepsilon})\nabla_x u_3 \\
&- \varepsilon^2 a(\frac{x}{\varepsilon})\nabla_y u_3 + a(\frac{x}{\varepsilon})\nabla_x u_0 + a(\frac{x}{\varepsilon})\nabla_y u_1 + \varepsilon a(\frac{x}{\varepsilon})\nabla_x u_1 + \varepsilon a(\frac{x}{\varepsilon})\nabla_y u_2 \\
&+ \varepsilon^2 a(\frac{x}{\varepsilon})\nabla_y u_3 + \varepsilon^2 a(\frac{x}{\varepsilon})\nabla_x u_2 + \varepsilon^3 v_3 \\
&= \varepsilon^3(v_3 - a(\frac{x}{\varepsilon})\nabla_x u_3).
\end{aligned}
\tag{2.4.39}
$$

<u>Step2</u>: *The estimation of* $\|a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon\|_{L^2(\Omega)}$.

Since χ^{ijk} are in $C^\infty(Y)$ and $u_0 \in H^4(\Omega)$ we see that

$$
\sup_{y\in Y}|\nabla_x u_3| \leq C \sum_{i,j,k,l} \left|\frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l}\right|. \tag{2.4.40}
$$

Therefore from (2.4.33) and (2.4.40) we conclude that

$$
\begin{aligned}
\|a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon\|_{L^2(\Omega)} &\leq \varepsilon^3\|v_3\|_{L^2(\Omega)} + \varepsilon^3\|a(\frac{x}{\varepsilon})\nabla_x u_3\|_{L^2(\Omega)} \\
&\leq C\varepsilon^3\|u_0\|_{H^4(\Omega)}.
\end{aligned}
\tag{2.4.41}
$$

<u>Step3</u>:*The estimation of* $\|\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\|_{H_0^1(\Omega)}$.

Let $g \in L^2(\Omega)$ and $\omega_\varepsilon \in H_0^1(\Omega)$ the solution to

$$
\begin{cases}
-div(a(\frac{x}{\varepsilon})\nabla\omega_\varepsilon) = g \quad in\ \Omega, \\
\omega_\varepsilon = 0 \quad on\ \ \partial\Omega.
\end{cases}
\tag{2.4.42}
$$

Since $\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon} \in H_0^1(\Omega)$, so by using the Green Formula the integration yields

$$\begin{aligned}\int_\Omega (\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}) g \; dx &= \int_\Omega -div(a(\frac{x}{\varepsilon})\nabla\omega_\varepsilon)(\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}) dx \\ &= \int_\Omega a(\frac{x}{\varepsilon})(\nabla\psi_\varepsilon + \varepsilon\nabla u_1^{bl,\varepsilon} + \varepsilon^2\nabla u_2^{bl,\varepsilon} + \varepsilon^3\nabla u_3^{bl,\varepsilon}).\nabla\omega_\varepsilon \; dx \\ &= \int_\Omega a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon.\nabla\omega_\varepsilon - \int_\Omega \underbrace{div(a(\frac{x}{\varepsilon})(\varepsilon\nabla u_1^{bl,\varepsilon} + \varepsilon^2\nabla u_2^{bl,\varepsilon} + \varepsilon^3\nabla u_3^{bl,\varepsilon}))}_{\to 0} \; \omega_\varepsilon dx \\ &= \int_\Omega a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon.\nabla\omega_\varepsilon dx\end{aligned} \tag{2.4.43}$$

Making use of (2.4.39) and taking advantage of the ellipticity of A_ε, we get

$$\begin{aligned}\int_\Omega a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon.\nabla\omega_\varepsilon dx &= \int_\Omega (a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon).\nabla\omega_\varepsilon + \int_\Omega \xi_\varepsilon.\nabla\omega_\varepsilon dx \\ &= \int_\Omega (a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon).\nabla\omega_\varepsilon - \int_\Omega \underbrace{div\xi_\varepsilon\omega_\varepsilon dx}_{\to 0} \\ &= \int_\Omega (a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon).\nabla\omega_\varepsilon \; dx \\ &\le \|a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon\|_{L^2(\Omega)}\|\omega_\varepsilon\|_{H_0^1(\Omega)} \\ &\le C\|a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon\|_{L^2(\Omega)}\|g\|_{H^{-1}(\Omega)}.\end{aligned} \tag{2.4.44}$$

Using the estimate obtained in (2.4.41), it follows that:

$$\Big| \int_\Omega (\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}) \; g \; dx \Big| \le C\|a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon\|_{L^2(\Omega)}\|g\|_{H^{-1}(\Omega)},$$

by dividing by $\|g\|_{H^{-1}(\Omega)}$ *and taking the supremum over all* $g \neq 0,$ *we immediately conclude that*

$$\begin{aligned}\sup \frac{| \int_\Omega (\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}) g |}{\|g\|_{H^{-1}(\Omega)}} &\le C\|a(\frac{x}{\varepsilon})\nabla\psi_\varepsilon - \xi_\varepsilon\|_{L^2(\Omega)} \\ &\le C\varepsilon^3\|u_0\|_{H^4(\Omega)}.\end{aligned} \tag{2.4.45}$$

Hence, it seems clear that

$$\|\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\|_{H_0^1(\Omega)} \le C\varepsilon^3\|u_0\|_{H^4(\Omega)}, \tag{2.4.46}$$

which establishes the formula. □

The second result is about the third-order error estimate without the third-order boundary layer corrector. Again, for this case we need the regularity $H^4(\Omega)$ for u_0.

Theorem 23.

Let u_ε and u_0 denote the unique solutions of (P_ε) and (P_H) respectively, suppose that $f \in H^2(\Omega)$, then

$$\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon}\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{5}{2}}\|u_0\|_{H^4(\Omega)}.$$

Proof. Using the result obtained in Theorem 22, we have

$$\begin{aligned}
&\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon}\|_{H^1(\Omega)} \\
&= \|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon} - \varepsilon^3 u_3^{bl,\varepsilon}\|_{H^1(\Omega)} \\
&\leq \|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\|_{H_0^1} + \varepsilon^3\|u_3^{bl,\varepsilon}\|_{H^1(\Omega)} \\
&\leq C\varepsilon^3\|u_0\|_{H^4(\Omega)} + \varepsilon^3\|u_3^{bl,\varepsilon}\|_{H^1(\Omega)}.
\end{aligned} \tag{2.4.47}$$

The task is now to estimate $\|u_3^{bl,\varepsilon}\|_{H^1(\Omega)}$. Since u_3 has a trace in $H^{\frac{1}{2}}(\partial\Omega)$, consequently, owing to Theorem 19 we can conclude that

$$\|u_3^{bl,\varepsilon}\|_{H^1(\Omega)} \leq C_{33}\|u_3\|_{H^{\frac{1}{2}}(\partial\Omega)}.$$

The proof is completed by showing that

$$\|u_3\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C_{34}\varepsilon^{\frac{-1}{2}}. \tag{2.4.48}$$

For this purpose, we define the function $\kappa_\varepsilon(x) \in D(\Omega)$, such that

$$\begin{cases} \kappa_\varepsilon = 1 \; if \;\; \rho(x,\partial\Omega) \leq \varepsilon, \\ \kappa_\varepsilon = 0 \; if \;\; \rho(x,\partial\Omega) \geq 2\varepsilon, \\ \|\nabla\kappa_\varepsilon\|_{L^\infty(\Omega)} \leq \dfrac{C}{\varepsilon}. \end{cases} \tag{2.4.49}$$

For the existence of such kind of functions see [32] and the references therein.

Let us put

$$V_\varepsilon = \kappa_\varepsilon u_3,$$

such that

$$supp\; V_\varepsilon = \{x, \rho(x,\partial\Omega) \leq 2\varepsilon\},$$

which will be denoted by U_ε.

At this stage, the only point remaining to get (2.4.48), is the estimation of $\|V_\varepsilon\|_{H^1(U_\varepsilon)}$.

Making use of H^1-norm, we get

$$\|V_\varepsilon\|_{H^1(U_\varepsilon)} = \|V_\varepsilon\|_{L^2(U_\varepsilon)} + \|\nabla V_\varepsilon\|_{L^2(U_\varepsilon)}.$$

Clearly, from the definition of κ_ε, and the assumption that $u_0 \in H^4(\Omega)$, with taking advantage of $a_{ij}(y), \chi^{ijk} \in C^\infty(Y)$, we obtain

$$\begin{aligned} \|V_\varepsilon\|_{L^2(U_\varepsilon)} &= \left\| \kappa_\varepsilon(x)\chi^{ijk}(\frac{x}{\varepsilon})\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \right\|_{L^2(U_\varepsilon)} \\ &\leq \left\| \chi^{ijk}(\frac{x}{\varepsilon}) \right\|_{L^\infty(Y)} \left\| \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \right\|_{L^2(U_\varepsilon)} \\ &\leq C \left\| \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \right\|_{L^2(U_\varepsilon)} \\ &\leq C\|u_0\|_{H^3(U_\varepsilon)}. \end{aligned} \tag{2.4.50}$$

Hence

$$\|V_\varepsilon\|_{L^2(U_\varepsilon)} \leq C\|u_0\|_{H^3(U_\varepsilon)}. \tag{2.4.51}$$

Let us now estimate the gradient of V_ε, first we have

$$\begin{aligned} \frac{\partial V_\varepsilon}{\partial x_l}(x) = \kappa_\varepsilon(x) &\left\{ \frac{1}{\varepsilon}\frac{\partial \chi^{ijk}}{\partial y_l}(\frac{x}{\varepsilon})\frac{\partial^3 u_0(x)}{\partial x_i \partial x_j \partial x_k} + \chi^{ijk}(\frac{x}{\varepsilon})\frac{\partial^4 u_0(x)}{\partial x_i \partial x_j \partial x_k \partial x_l} \right\} \\ + \frac{\partial \kappa_\varepsilon(x)}{\partial x_l} &\left\{ \chi^{ijk}(\frac{x}{\varepsilon})\frac{\partial^3 u_0(x)}{\partial x_i \partial x_j \partial x_k} \right\}. \end{aligned} \tag{2.4.52}$$

Again, from the above definition of κ_ε, and the assumption that $u_0 \in H^4(\Omega)$, with taking advantage of

$a_{ij}(y), \chi^{ijk} \in C^\infty(Y)$, one can have

$$\|\nabla V_\varepsilon\|_{L^2(U_\varepsilon)} \leq \frac{C}{\varepsilon}\left\| \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial_k} \right\|_{L^2(U_\varepsilon)} + C\left\| \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l} \right\|_{L^2(U_\varepsilon)}, \tag{2.4.53}$$

however,

$$\|\nabla V_\varepsilon\|_{L^2(U_\varepsilon)} \leq C\varepsilon^{-1}\|u_0\|_{H^3(U_\varepsilon)} + C\|u_0\|_{H^4(U_\varepsilon)}. \tag{2.4.54}$$

Furthermore, by applying Lemma 2.3.1, we derive that

$$\|u_0\|_{H^3(U_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}\|u_0\|_{H^4(\Omega)}. \tag{2.4.55}$$

Combining (2.4.51) with (2.4.54) and making use of (2.4.55), we conclude that:

$$\begin{aligned}\|V_\varepsilon\|_{H^1(U_\varepsilon)} &\le C\|u_0\|_{H^3(U_\varepsilon)} + C\varepsilon^{-1}\|u_0\|_{H^3(U_\varepsilon)} + C\|u_0\|_{H^4(U_\varepsilon)} \\ &\le C\varepsilon^{\frac{1}{2}}\|u_0\|_{H^4(\Omega)} + C\varepsilon^{-1}(C\varepsilon^{\frac{1}{2}}\|u_0\|_{H^4(\Omega)}) + C\|u_0\|_{H^4(\Omega)} \\ &\le C\varepsilon^{\frac{-1}{2}}\|u_0\|_{H^4(\Omega)}.\end{aligned} \tag{2.4.56}$$

On $\partial\Omega,\ V_\varepsilon = u_3$, so

$$\|u_3\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|V_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\|V_\varepsilon\|_{H^1(\Omega)} = C\|V_\varepsilon\|_{H^1(U_\varepsilon)} \le C\varepsilon^{\frac{-1}{2}}\|u_0\|_{H^4(\Omega)}. \tag{2.4.57}$$

Using the regularity results of Theorem 19, we deduce that

$$\|u_3^{bl,\varepsilon}\|_{H^1(\Omega)} \le C\|u_3\|_{H^{\frac{1}{2}}(\partial\Omega)} \le C\varepsilon^{\frac{-1}{2}}\|u_0\|_{H^4(\Omega)}. \tag{2.4.58}$$

Substituting (2.4.58) in (2.4.47), we get

$$\begin{aligned}\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon}\|_{H^1(\Omega)} &\le C\varepsilon^3\|u_0\|_{H^4(\Omega)} + \varepsilon^3\|u_3^{bl,\varepsilon}\|_{H^1(\Omega)} \\ &\le C\varepsilon^3\|u_0\|_{H^4(\Omega)} + C\varepsilon^{\frac{5}{2}}\|u_0\|_{H^4(\Omega)} \\ &\le C\varepsilon^{\frac{5}{2}}\|u_0\|_{H^4(\Omega)},\end{aligned}$$

which is precisely the assertion of the theorem. □

CHAPTER 3

HOMOGENIZATION OF A PIEZOELECTRIC STRUCTURE BY THE ENERGY METHOD

3.1 Case of 3D structure

The generation of electric charges in certain crystals when subjected to mechanical force was discovered in 1880 by Pierre et Jacques Curie and is nowadays known as piezoelectric effect (or direct piezoelectric effect). The inverse phenomenon, that is, the generation of mechanical stress and strain in crystals when subjected to electric fields is called inverse piezoelectric effect and was predicted in 1881 by Lippmann (see [61]). The effect is found useful in applications such as the production and detection of sound, generation of high voltages, electronic frequency generation, micro-balances, and ultra fine focusing of optical assemblies. It is also the basis of a number of scientific instrumental techniques with atomic resolution, the scanning probe microscopies such as STM, AFM, MTA, SNOM, etc., and everyday uses such as acting as the ignition source for cigarette lighters and push-start propane barbecues.

3.1.1 Notations and geometry

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain occupied by a piezoelectric material with Lipschitz boundary $\Gamma = \partial\Omega$, points of Ω are denoted by $x = (x_1, x_2, x_3)$.

We consider two decompositions of the boundary Γ,

$$\Gamma = \Gamma_0^M \cup \Gamma_1^M \ with\ \Gamma_0^M \cap \Gamma_1^M = \emptyset,\ and\ meas(\Gamma_0^M) > 0,$$
$$\Gamma = \Gamma_0^E \cup \Gamma_1^E \ with\ \Gamma_0^E \cap \Gamma_1^E = \emptyset,\ and\ meas(\Gamma_0^E) > 0. \tag{3.1.1}$$

Let $Y = [0, Y_1] \times [0, Y_2] \times [0, Y_3]$, denotes the basic period, points of Y are denoted by

$$y = (y_1, y_2, y_3) = (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}),$$

where ε denotes the size of the period.

In the sequel we consider the following three-dimensional piezoelectric model

$$\begin{cases} -\dfrac{\partial}{\partial x_j}[\sigma_{ij}^\varepsilon] = f_i \ in\ \Omega \\ \dfrac{\partial}{\partial x_i}[D_i^\varepsilon] = r \qquad in\ \Omega \\ \sigma_{ij}^\varepsilon n_i = g_i\ on \quad \Gamma_1^M \\ D_i^\varepsilon n_i = 0 \ \ on\ \ \Gamma_1^E \\ u^\varepsilon = 0 \ \ on\ \ \Gamma_0^M \\ \varphi^\varepsilon = 0 \ \ on\ \ \Gamma_0^E \\ where \\ \qquad \sigma_{ij}^\varepsilon = C_{ijkl}^\varepsilon(x) e_{kl}(u^\varepsilon) - P_{kij}^\varepsilon(x) \dfrac{\partial \varphi^\varepsilon}{\partial x_k} \\ \qquad D_i^\varepsilon = P_{ikl}^\varepsilon(x) e_{kl}(u^\varepsilon) - \epsilon_{ik}^\varepsilon(x) \dfrac{\partial \varphi^\varepsilon}{\partial x_k}. \end{cases} \tag{3.1.2}$$

Note that the unknown of the piezoelectric structure model (3.1.2) is the pair $(u^\varepsilon, \varphi^\varepsilon)$, where

Notation	Designation
f	is the density of the mechanical volume force.
g	is the the density of the mechanical surface traction.
r	is the density of the electric volume charge.
$u^\varepsilon : \Omega \to \mathbb{R}^3$	denotes the displacement vector field.
$\varphi : \Omega \to \mathbb{R}$	is the electric potential, that is a scalar field.
$\sigma^\varepsilon_{ij} : \Omega \to \mathbb{R}^9$	is the stress tensor.
$D^\varepsilon_i : \Omega \to \mathbb{R}^3$	is the electric displacement vector.
$e_{kl}(u^\varepsilon)$	is the linear strain tensor.
$C^\varepsilon_{ijkl}(x) = C_{ijkl}\left(\frac{x}{\varepsilon}\right)$	is the elastic fourth order tensor field.
$P^\varepsilon_{ijk}(x) = P_{ijk}\left(\frac{x}{\varepsilon}\right)$	is the piezoelectric third order tensor field,.
$\epsilon^\varepsilon_{ij}(x) = \epsilon_{ij}\left(\frac{x}{\varepsilon}\right)$	is the dielectric second order tensor field.

Table 3.1: Notaions and designations of the piezoelectric problem

We assume that

$f \in (L^2(\Omega))^3$, $r \in (L^2(\Omega))^3$, $g \in (L^2(\Gamma_1^M))^3$ and

the elastic tensor C_{ijkl} is symmetric, positive defined, it verifies

$$\begin{cases} C_{ijkl} = C_{klij} = C_{jikl} = C_{ijlk}, \\ C_{ijkl} \in L^\infty(\Omega), \\ \exists C > 0 : C_{ijkl}(x) X_{ij} X_{kl} \geq C X_{ij} X_{kl}, \forall x \in \Omega, for\ every\ symmetric\ 3 \times 3 real\ matrix\ X_{ij}, \end{cases} \tag{3.1.3}$$

the piezoelectric third order tensor P_{ijk} is symmetric, it verifies

$$\begin{cases} P_{ijk} = P_{ikj} \\ P_{ijk} \in L^\infty(\Omega) \\ Y - periodic, \end{cases} \tag{3.1.4}$$

the dielectric tensor ϵ_{ij} is symmetric, positive defined, it verifies

$$\begin{cases} \epsilon_{ij} = \epsilon_{ji}, \\ \epsilon_{ij} \in L^{\infty}(\Omega), \\ \exists C > 0 : \epsilon_{ij}(x) X_i X_j \geq C, \forall x \in \Omega, for\ any\ vector\ X_i \in \mathbb{R}^3, \end{cases} \tag{3.1.5}$$

3.1.2 Homogenization by the energy method of Tartar

The previous works (see for instance, Racila and Boubaker [93], Mechkour [77]) have only focused on the formal asymptotic analysis or on the two-scale convergence methods [77] to homogenize the piezoelectric problem, however, until now there is no result on the homogenization of (3.1.2) by the energy method.Indeed, the major difficulty in establishing such theorem using Tartar's method (see chapter 1, subsection 1.3.7) is the choice of the oscillating test functions and this is the most challenge, in fact, if one follows the same steps as in [[32],chapter 8] to prove the convergence theorem by the energy method, for the case of piezoelectric problem, he will fined him-self in wild tangle because, a lot of terms will not be canceled after the subtraction of the resulting equations and furthermore they not converge, that is why by proving the following theorem of convergence using Tartar's method, we believe that we have designed an innovative solution to this problem by choosing a suitable oscillating test functions.

Theorem 24.

Let $\left(u^\varepsilon, \varphi^\varepsilon\right) \in (H^1(\Omega))^2$ be the unique solutions of (3.1.2), then

$$\begin{cases} u^\varepsilon \overset{H^1(\Omega)}{\rightharpoonup} u^0, \\ \varphi^\varepsilon \overset{H^1(\Omega)}{\rightharpoonup} \varphi^0, \\ \sigma^\varepsilon_{ij} \overset{L^2(\Omega)}{\rightharpoonup} \sigma^*_{ij} = C^h_{ijkl}\dfrac{\partial u^0_k}{\partial x_l} + P^h_{kij}\dfrac{\partial \varphi^0}{\partial x_k}, \\ D^\varepsilon_i \overset{L^2(\Omega)}{\rightharpoonup} D^*_i = P^h_{ikl}\dfrac{\partial u^0_k}{\partial x_l} + \epsilon^h_{ij}\dfrac{\partial \varphi^0}{\partial x_j}, \end{cases} \tag{3.1.6}$$

where $\left(u^0, \varphi^0\right)$ are the unique solutions in $H^1(\Omega)^2$ of the homogenized problem

$$\begin{cases} C^h_{ijkl}\dfrac{\partial^2 u^0_k}{\partial x_j \partial x_l} + P^h_{kij}\dfrac{\partial^2 \varphi^0}{\partial x_j \partial x_k} = f, \\ P^h_{ikl}\dfrac{\partial^2 u^0_k}{\partial x_i \partial x_l} + \epsilon^h_{ij}\dfrac{\partial^2 \varphi^0}{\partial x_i \partial x_j} = r, \\ u^0 = 0 \quad on\ \Gamma^M_0, \\ \langle \sigma^*_{ij}\rangle n_j = g_i \quad on\ \Gamma^M_1, \\ \varphi^0 = 0 \quad on\ \Gamma^E_0, \\ \langle D^*_i\rangle n_i = 0, \quad on\ \Gamma^E_1, \end{cases} \tag{3.1.7}$$

where the homogenized coefficients $C^h_{ijkl}, P^h_{kij}, P^h_{ikl}, \epsilon^h_{ij}$,

$$\begin{aligned} C^h_{ijkl} &= \frac{1}{|Y|}\int_Y \left\{C_{ijmn}(y)e_{mn,y}(\chi^{kl}) + C_{ijkl}(y) + P_{mij}(y)\frac{\partial \psi^{kl}}{\partial y_m}\right\}dy, \\ P^h_{kij} &= \frac{1}{|Y|}\int_Y \left\{C_{ijmn}(y)e_{mn,y}(\Phi^k) + P_{mij}(y)\frac{\partial\left(R^k + y_k\right)}{\partial y_m}\right\}dy, \\ P^h_{ikl} &= \frac{1}{|Y|}\int_Y \left\{P_{imn}(y)e_{mn,y}(\chi^{kl}) + P_{ikl}(y) - \epsilon_{im}(y)\frac{\partial \psi^{kl}}{\partial y_m}\right\}dy, \\ \epsilon^h_{ij} &= \frac{1}{|Y|}\int_Y \left\{P_{jmn}(y)e_{mn,y}(\Phi^i) - \epsilon_{jm}(y)\frac{\partial\left(R^i + y_i\right)}{\partial y_m}\right\}dy. \end{aligned} \tag{3.1.8}$$

Proof. The proof will be divided into 4 steps.

Step 1: **The variational formulation** Let us define the two following spaces:

$V = \{\upsilon \mid \upsilon \in H^1(\Omega)^3, \upsilon = 0 \ on\ \Gamma^M_0\}$,

$\Psi = \{\psi \mid \psi \in H^1(\Omega), \psi = 0 \ on\ \Gamma^E_0\}$, equipped with the two norms (equivalent to the usual norm H^1)

$$\|\upsilon\|_V = \left(\sum_{i,j=1}^{3} \int_\Omega \left(\frac{\partial \upsilon_i}{\partial x_j} \right)^2 \right)^{\frac{1}{2}},$$
$$\|\psi\|_\Psi = \left(\sum_{i=1}^{3} \int_\Omega \left(\frac{\partial \psi}{\partial x_i} \right)^2 \right)^{\frac{1}{2}}, \|\upsilon, \psi\|_{\upsilon\times\psi} \; = \|\upsilon\|_V + \|\psi\|_\Psi. \tag{3.1.9}$$

Multiplying the first equation by a test function $\upsilon \in V$ and the second one by $\psi \in \Psi$, and summing the two obtained equations we get the following variational problem:

$$\int_\Omega \left[C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) + P^\varepsilon_{kij} \frac{\partial \varphi^\varepsilon}{\partial x_k} \right] e_{ij}(\upsilon) dx - \int_\Omega \left[P^\varepsilon_{jkl} e_{kl}(u^\varepsilon) - \epsilon^\varepsilon_{jk} \frac{\partial \varphi^\varepsilon}{\partial x_k} \right] \frac{\partial \psi}{\partial x_j} = \int_\Omega f \upsilon dx + \int_{\Gamma_1^M} g \upsilon d\Gamma_1^M + \int_\Omega r\psi \tag{3.1.10}$$

<u>Step 2</u>: **<u>A priori estimates</u>**

Lemma 3.1.1. *The solutions $(u^\varepsilon, \varphi^\varepsilon)$ of (3.1.2) are bounded.*

Proof. We take $\upsilon = u^\varepsilon$ and $\psi = \varphi^\varepsilon$ in (3.1.10) we get:

$$\int_\Omega C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) e_{ij}(u^\varepsilon) dx + \int_\Omega \epsilon^\varepsilon_{jk} \frac{\partial \varphi^\varepsilon}{\partial x_k} \frac{\partial \varphi^\varepsilon}{\partial x_j} dx = \int_\Omega f u^\varepsilon dx + \int_{\Gamma_1^M} g u^\varepsilon d\Gamma_1^M + \int_\Omega r \varphi^\varepsilon dx. \tag{3.1.11}$$

On the one hand,

taking advantage of the ellipticity of C^ε_{ijkl} and $\epsilon^\varepsilon_{jk}$ we obtain

$$C \left\| \frac{\partial \varphi^\varepsilon}{\partial x_k} \right\|^2_{L^2(\Omega)} + C\|e_{ij}(u^\varepsilon)\|^2_{L^2(\Omega)} \leq \int_\Omega C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) e_{ij}(u^\varepsilon) dx + \int_\Omega \epsilon^\varepsilon_{jk} \frac{\partial \varphi^\varepsilon}{\partial x_k} \frac{\partial \varphi^\varepsilon}{\partial x_j} dx \tag{3.1.12}$$

Applying *Korn inequality* together with the relation $(a+b)^2 \leq 2(a^2+b^2)$ on (3.1.12) we get:

$$C\Big(\|\varphi^\varepsilon\|_\Psi + \|u^\varepsilon\|_V \Big)^2 \leq \int_\Omega C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) e_{ij}(u^\varepsilon) dx + \int_\Omega \epsilon^\varepsilon_{jk} \frac{\partial \varphi^\varepsilon}{\partial x_k} \frac{\partial \varphi^\varepsilon}{\partial x_j} dx, \tag{3.1.13}$$

and on the other hand,

making use of Cauchy Schwarz inequality and trace theorem, we have

$$\int_\Omega C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) e_{ij}(u^\varepsilon) dx + \int_\Omega \epsilon^\varepsilon_{jk} \frac{\partial \varphi^\varepsilon}{\partial x_k} \frac{\partial \varphi^\varepsilon}{\partial x_j} dx \leq \|f\|_{L^2(\Omega)} \|u^\varepsilon\|_{L^2(\Omega)} + C\|g\|_{L^2(\Gamma_1^M)} \|u^\varepsilon\|_{H^1(\Omega)}. + \|r\|_{L^2(\Omega)} \|\varphi^\varepsilon\|_{L^2(\Omega)}. \tag{3.1.14}$$

Using the Poincaré inequality, we get

$$\int_\Omega C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) e_{ij}(u^\varepsilon) dx + \int_\Omega \epsilon^\varepsilon_{jk} \frac{\partial \varphi^\varepsilon}{\partial x_k} \frac{\partial \varphi^\varepsilon}{\partial x_j} dx, \leq C\|f\|_{L^2(\Omega)} \|u^\varepsilon\|_V + C\|g\|_{L^2(\Gamma_1^M)} \|u^\varepsilon\|_V + C\|r\|_{L^2(\Omega)} \|\varphi^\varepsilon\|_\Psi, \tag{3.1.15}$$

which implies that

$$\int_{\Omega} C^{\varepsilon}_{ijkl} e_{kl}(u^{\varepsilon}) e_{ij}(u^{\varepsilon}) dx + \int_{\Omega} \epsilon^{\varepsilon}_{jk} \frac{\partial \varphi^{\varepsilon}}{\partial x_k} \frac{\partial \varphi^{\varepsilon}}{\partial x_j} dx \leq C \|(u^{\varepsilon}, \varphi^{\varepsilon})\|_{V \times \Psi}. \tag{3.1.16}$$

Combining (3.1.13) and (3.1.16), we deduce that:

$$C\Big(\|\varphi^{\varepsilon}\|_{\Psi} + \|u^{\varepsilon}\|_{V} \Big)^2 \leq \tilde{C} \|(u^{\varepsilon}, \varphi^{\varepsilon})\|_{V \times \Psi}, \tag{3.1.17}$$

which leads to

$$\|\varphi^{\varepsilon}\|_{\Psi} + \|u^{\varepsilon}\|_{V} \leq C, \tag{3.1.18}$$

which means that

$$\begin{cases} \|u^{\varepsilon}\|_{V} \leq C, \\ \|\varphi^{\varepsilon}\|_{\Psi} \leq C. \end{cases} \tag{3.1.19}$$

So, we can extract a subsequences still denoted by $u_{\varepsilon}, \varphi_{\varepsilon}$ such that

$$\begin{cases} u^{\varepsilon} \overset{H^1(\Omega)}{\rightharpoonup} u^0, \\ \varphi^{\varepsilon} \overset{H^1(\Omega)}{\rightharpoonup} \varphi^0. \end{cases} \tag{3.1.20}$$

Using (Rellich Kondrachov theorem) $H^1(\Omega) \underset{c}{\hookrightarrow} L^2(\Omega)$, so

$$\begin{cases} u^{\varepsilon} \overset{L^2(\Omega)}{\longrightarrow} u^0, \\ \varphi^{\varepsilon} \overset{L^2(\Omega)}{\longrightarrow} \varphi^0. \end{cases} \tag{3.1.21}$$

Furthermore, from (3.1.19) we can extract a subsequence still denoted by $\frac{\partial u_{\varepsilon}}{\partial x_j}$ such that

$$\frac{\partial u^{\varepsilon}}{\partial x_j} \overset{L^2(\Omega)}{\rightharpoonup} \xi_j,$$

then, the derivate in the sense of distributions yields

$$\int_{\Omega} \frac{\partial u^{\varepsilon}}{\partial x_j} \vartheta \ dx = \int_{\Omega} -u^{\varepsilon} \frac{\partial \vartheta}{\partial x_j} \ dx \qquad \forall\ \vartheta \in D(\Omega),$$

passing to the limit in the previous equation

$$\lim_{\varepsilon \to 0} \int_{\Omega} \frac{\partial u^{\varepsilon}}{\partial x_j} \vartheta \ dx = -\lim_{\varepsilon \to 0} \int_{\Omega} u^{\varepsilon} \frac{\partial \vartheta}{\partial x_j} \ dx \quad \forall\ \vartheta \in D(\Omega),$$

gives

$$\begin{aligned}
&\int_\Omega \xi_j \vartheta \ dx = -\int_\Omega u^0 \frac{\partial \vartheta}{\partial x_j} \quad \forall \ \vartheta \in D(\Omega) \\
&\Rightarrow \int_\Omega \xi_j \vartheta \ dx = \int_\Omega \frac{\partial u^0}{\partial x_j} \vartheta \ dx \quad \forall \vartheta \in D(\Omega) \\
&\Rightarrow \int_\Omega \left(\xi_j - \frac{\partial u^0}{\partial x_j} \right) \vartheta dx = 0 \quad \forall \vartheta \in D(\Omega) \\
&\Rightarrow \xi_j = \frac{\partial u^0}{\partial x_j}. \\
&\Rightarrow \boxed{\frac{\partial u^\varepsilon}{\partial x_j} \overset{L^2(\Omega)}{\rightharpoonup} \frac{\partial u^0}{\partial x_j}.}
\end{aligned} \tag{3.1.22}$$

Again, from (3.1.19) we can extract a subsequence still denoted by $\frac{\partial \varphi^\varepsilon}{\partial x_j}$ such that

$$\frac{\partial \varphi^\varepsilon}{\partial x_j} \overset{L^2(\Omega)}{\rightharpoonup} \lambda_j,$$

then, the derivation in the sense of distributions yields

$$\int_\Omega \frac{\partial \varphi^\varepsilon}{\partial x_j} \Psi dx = \int_\Omega -\varphi^\varepsilon \frac{\partial \Psi}{\partial x_j} dx \quad \forall \ \Psi \in D(\Omega),$$

passing to the limit in the previous equation

$$\lim_{\varepsilon \to 0} \int_\Omega \frac{\partial \varphi^\varepsilon}{\partial x_j} \Psi \ dx = -\lim_{\varepsilon \to 0} \int_\Omega \varphi^\varepsilon \frac{\partial \Psi}{\partial x_j} \ dx \quad \forall \ \Psi \in D(\Omega),$$

gives

$$\begin{aligned}
&\int_\Omega \lambda_j \Psi \ dx = -\int_\Omega \varphi^0 \frac{\partial \Psi}{\partial x_j} \ dx \quad \forall \ \Psi \in D(\Omega) \\
&\Rightarrow \int_\Omega \lambda_j \Psi \ dx = \int_\Omega \frac{\partial \varphi^0}{\partial x_j} \Psi \ dx \quad \forall \ \Psi \in D(\Omega) \\
&\Rightarrow \int_\Omega \left(\lambda_j - \frac{\partial \varphi^0}{\partial x_j} \right) \Psi \ dx = 0 \quad \forall \ \Psi \in D(\Omega) \\
&\Rightarrow \lambda_j = \frac{\partial \varphi^0}{\partial x_j}. \\
&\Rightarrow \boxed{\frac{\partial \varphi^\varepsilon}{\partial x_j} \overset{L^2(\Omega)}{\rightharpoonup} \frac{\partial \varphi^0}{\partial x_j}.}
\end{aligned} \tag{3.1.23}$$

So, we conclude

$$\begin{aligned}
u^\varepsilon &\overset{H^1(\Omega)}{\rightharpoonup} u^0,\\
\varphi^\varepsilon &\overset{H^1(\Omega)}{\rightharpoonup} \varphi^0,\\
u^\varepsilon &\overset{L^2(\Omega)}{\longrightarrow} u^0,\\
\varphi^\varepsilon &\overset{L^2(\Omega)}{\longrightarrow} \varphi^0,\\
\frac{\partial u^\varepsilon}{\partial x_j} &\overset{L^2(\Omega)}{\rightharpoonup} \frac{\partial u^0}{\partial x_j},\\
\frac{\partial \varphi^\varepsilon}{\partial x_j} &\overset{L^2(\Omega)}{\rightharpoonup} \frac{\partial \varphi^0}{\partial x_j}.
\end{aligned} \tag{3.1.24}$$

□

Set

$$\begin{aligned}
\Sigma^\varepsilon_{ij} &= C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) + P^\varepsilon_{kij}\frac{\partial \varphi^\varepsilon}{\partial x_k},\\
\Lambda^\varepsilon_j &= P^\varepsilon_{jkl} e_{kl}(u^\varepsilon) - \epsilon^\varepsilon_{jk}\frac{\partial \varphi^\varepsilon}{\partial x_k}.
\end{aligned} \tag{3.1.25}$$

$$\begin{aligned}
\|\Sigma^\varepsilon_{ij}\|^2_{L^2(\Omega)} &= \int_\Omega \left|\Sigma^\varepsilon_{ij}\right|^2 \, dx\\
&= \int_\Omega \left| C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) + P^\varepsilon_{kij}\frac{\partial \varphi^\varepsilon}{\partial x_k}\right|^2 \, dx\\
&\le \int_\Omega 2\left| C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon)\right|^2 + 2\left| P^\varepsilon_{kij}\frac{\partial \varphi^\varepsilon}{\partial x_k}\right|^2 \, dx\\
&\le C\|e_{kl}(u^\varepsilon)\|^2_{L^2(\Omega)} + C\left\|\frac{\partial \varphi^\varepsilon}{\partial x_k}\right\|^2_{L^2(\Omega)}\\
&\le C\|\nabla u^\varepsilon\|^2_{L^2(\Omega)} + C\left\|\frac{\partial \varphi^\varepsilon}{\partial x_k}\right\|^2_{L^2(\Omega)}\\
&= C\|u_\varepsilon\|^2_V + C\|\varphi^\varepsilon\|^2_\Psi\\
&\le C \ (from \ (3.1.19))\\
&\Rightarrow \|\Sigma^\varepsilon_{ij}\|_{L^2(\Omega)} \le C.
\end{aligned} \tag{3.1.26}$$

Hence, we deduce that we can extract a subsequence still denoted by Σ^ε_{ij} such that $\Sigma^\varepsilon_{ij} \overset{L^2(\Omega)}{\rightharpoonup} \Sigma^*_{ij}$.

$$
\begin{aligned}
\|\Lambda_j^\varepsilon\|^2_{L^2(\Omega)} &= \int_\Omega \left|\Lambda_j^\varepsilon\right|^2\, dx \\
&= \int_\Omega \left| P^\varepsilon_{jkl} e_{kl}(u^\varepsilon) - \epsilon^\varepsilon_{jk} \frac{\partial \varphi^\varepsilon}{\partial x_k} \right|^2\, dx \\
&\le \int_\Omega 2\left| P^\varepsilon_{jkl} e_{kl}(u^\varepsilon) \right|^2 + 2\left| \epsilon^\varepsilon_{jk} \frac{\partial \varphi^\varepsilon}{\partial x_k} \right|^2\, dx \\
&\le C\|e_{kl}(u^\varepsilon)\|^2_{L^2(\Omega)} + C\left\| \frac{\partial \varphi^\varepsilon}{\partial x_k} \right\|^2_{L^2(\Omega)} \\
&\le C\|\nabla u^\varepsilon\|^2_{L^2(\Omega)} + C\left\| \frac{\partial \varphi^\varepsilon}{\partial x_k} \right\|^2_{L^2(\Omega)} \\
&= C\|u_\varepsilon\|^2_V + C\|\varphi^\varepsilon\|^2_\Psi \\
&\le C\ (from\ (3.1.19)) \\
&\Rightarrow \|\Lambda_j^\varepsilon\|_{L^2(\Omega)} \le C.
\end{aligned} \tag{3.1.27}
$$

Thus, we deduce that we can extract a subsequence still denoted by Λ_j^ε such that $\Lambda_j^\varepsilon \overset{L^2(\Omega)}{\rightharpoonup} \Lambda_j^*$.

So, we conclude

$$
\boxed{
\begin{aligned}
\Sigma^\varepsilon_{ij} &\overset{L^2(\Omega)}{\rightharpoonup} \Sigma^*_{ij}, \\
\Lambda^\varepsilon_j &\overset{L^2(\Omega)}{\rightharpoonup} \Lambda^*_j.
\end{aligned} \qquad (3.1.28)
}
$$

It is worth noting that Σ^*_{ij} satisfies

$$
-\frac{\partial \Sigma^*_{ij}}{\partial x_j} = f_i \quad in\ \Omega. \tag{3.1.29}
$$

Indeed, taking $\psi = 0$ in (3.1.10) brings us to

$$\int_\Omega \left[C^\varepsilon_{ijkl} e_{kl}(u^\varepsilon) + P^\varepsilon_{kij}\frac{\partial \varphi^\varepsilon}{\partial x_k}\right] e_{ij}(\upsilon)dx = \int_\Omega f\upsilon dx + \int_{\Gamma_1^M} g\upsilon d\Gamma_1^M$$

$$\Leftrightarrow \int_\Omega \Sigma^\varepsilon_{ij} e_{ij}(\upsilon)dx = \int_\Omega f\upsilon dx + \int_{\Gamma_1^M} g\upsilon d\Gamma_1^M, \forall \upsilon \in V.$$

Passing to the limit (taking $\upsilon \in \mathcal{D}(\Omega)$*)*

$$\begin{aligned} &\lim_{\varepsilon\to 0}\int_\Omega \Sigma^\varepsilon_{ij} e_{ij}(\upsilon)dx = \int_\Omega \Sigma^*_{ij} e_{ij}(\upsilon)dx \\ &= \int_\Omega f\upsilon dx \\ &\Rightarrow \int_\Omega -\frac{\partial \Sigma^*_{ij}}{\partial x_j}\upsilon_i dx = \int_\Omega f\upsilon dx \\ &\Rightarrow \int_\Omega \Big(-\frac{\partial \Sigma^*_{ij}}{\partial x_j} - f_i\Big)\upsilon_i = 0 \quad \forall \upsilon \in V \\ &\Rightarrow \boxed{-\frac{\partial \Sigma^*_{ij}}{\partial x_j} = f_i}. \end{aligned} \tag{3.1.30}$$

Also, Λ^*_j verifies

$$\boxed{-\frac{\partial \Lambda^*_j}{\partial x_j} = r}, \tag{3.1.31}$$

which is easy to check following the same techniques above.

<u>Step 3</u>: **The introduction of the oscillating test functions**

Let

$$\begin{aligned} \rho_i^{\varepsilon,mn}(x) &= \varepsilon\chi_i^{mn}(\frac{x}{\varepsilon}) + \delta_{im}x_n, \\ \Theta^{\varepsilon,mn}(x) &= \varepsilon\Psi^{mn}(\frac{x}{\varepsilon}), \\ \pi_i^{\varepsilon,m}(x) &= \varepsilon\Phi_i^m(\frac{x}{\varepsilon}), \\ I^{\varepsilon,m}(x) &= \varepsilon R^m + x_m, \end{aligned} \tag{3.1.32}$$

where $\big(\chi^{mn}(y), \Psi^{mn}(y)\big)$ and $\big(\Phi^m(y), R^m(y)\big)$ are the unique solutions in $H^1_\sharp(Y)$ with zero average of the cell problems $\big(P_{\chi^{mn},\Psi^{mn}}\big)$ and $\big(P_{\Phi^m,R^m}\big)$, respectively

$$\big(P_{\chi^{mn},\Psi^{mn}}\big)\begin{cases} -\dfrac{\partial}{\partial y_j}\Big\{C_{ijkl}(y)\big(e_{kl,y}(\chi^{mn}(y)) + \tau^{kl}_{mn}\big) + P_{kij}(y)\dfrac{\partial \Psi^{mn}(y)}{\partial y_k}\Big\} = 0 \quad in\ Y, \\ -\dfrac{\partial}{\partial y_j}\Big\{P_{jkl}(y)\big(e_{kl,y}(\chi^{mn}(y)) + \tau^{kl}_{mn}\big) - \epsilon_{jk}(y)\dfrac{\partial \Psi^{mn}(y)}{\partial y_k}\Big\} = 0 \quad in\ Y, \\ \displaystyle\int_Y \chi^{mn} = 0, \int_Y \Psi^{mn} = 0 \quad \chi^{mn}, \Psi^{mn} \quad Y-periodic, \end{cases} \tag{3.1.33}$$

and

$$\left(P_{\Phi^m,R^m}\right)\begin{cases} -\dfrac{\partial}{\partial y_j}\left\{C_{ijkl}(y)e_{kl,y}(\Phi^m(y)) + P_{kij}(y)\left(\delta_{km} + \dfrac{\partial R^m(y)}{\partial y_k}\right)\right\} = 0 \quad in\ Y, \\ -\dfrac{\partial}{\partial y_j}\left\{P_{jkl}(y)e_{kl,y}(\Phi^m(y)) - \epsilon_{jk}(y)\left(\delta_{km} + \dfrac{\partial R^m(y)}{\partial y_k}\right)\right\} = 0 \quad in\ Y, \\ \displaystyle\int_Y \Phi^m = 0, \int_Y R^m = 0 \quad \Phi^m, R^m \quad Y - periodic, \end{cases} \tag{3.1.34}$$

with

$$\tau_{mn}^{kl} = \tfrac{1}{2}\left[\delta_{km}\delta_{ln} + \delta_{kn}\delta_{lm}\right] \quad 1 \leq k, l, m, n \leq 3$$

is the unit tensor of the fourth-order. Such that $\left(\rho_i^{\varepsilon,mn}(x), \Theta^{\varepsilon,mn}(x)\right)$ and $\left(\pi_i^{\varepsilon,m}(x), I^{\varepsilon,m}(x)\right)$ are the solutions of $\left(P_{\rho,\Theta}^{\varepsilon}\right)$ and $\left(P_{\pi,I}^{\varepsilon}\right)$ respectively, i.e.

$$\left(P_{\rho,\Theta}^{\varepsilon}\right)\begin{cases} -\dfrac{\partial}{\partial x_j}\left\{C_{ijkl}^{\epsilon}e_{kl}(\rho^{\varepsilon,mn}) + P_{kij}^{\varepsilon}\dfrac{\partial\Theta^{\varepsilon,mn}}{\partial x_k}\right\} = 0, \\ \dfrac{\partial}{\partial x_j}\left\{P_{jkl}^{\varepsilon}e_{kl}(\rho^{\varepsilon,mn}) - \epsilon_{jk}^{\varepsilon}\dfrac{\partial\Theta^{\varepsilon,mn}}{\partial x_k}\right\} = 0, \end{cases} \tag{3.1.35}$$

and

$$\left(P_{\pi,I}^{\varepsilon}\right)\begin{cases} -\dfrac{\partial}{x_j}\left\{C_{ijkl}^{\varepsilon}e_{kl}(\pi^{\varepsilon,m}) + P_{kij}^{\varepsilon}\dfrac{\partial I^{\varepsilon,m}}{\partial x_k}\right\} = 0, \\ \dfrac{\partial}{\partial x_j}\left\{P_{jkl}^{\varepsilon}e_{kl}(\pi^{\varepsilon,m}) - \epsilon_{jk}^{\varepsilon}\dfrac{\partial I^{\varepsilon,m}}{\partial x_k}\right\} = 0. \end{cases} \tag{3.1.36}$$

Lemma 3.1.2. *We have the following convergences ($\varepsilon \to 0$):*

$$\begin{aligned} &1)\ \rho_i^{\varepsilon,mn}(x) \overset{L^2(\Omega)}{\longrightarrow} \delta_{im}x_n, \\ &2)\ \Theta^{\varepsilon,mn}(x) \overset{L^2(\Omega)}{\longrightarrow} 0, \\ &3)\ I^{\varepsilon,m}(x) \overset{L^2(\Omega)}{\longrightarrow} x_m, \\ &4)\ \pi_i^{\varepsilon,m}(x) \overset{L^2(\Omega)}{\longrightarrow} 0. \end{aligned} \tag{3.1.37}$$

Proof. We only give the main ideas of the proof.

1. Since $\rho_i^{\varepsilon,mn}, \Theta^{\varepsilon,mn}(x), I^{\varepsilon,m}(x)$ and $\pi_i^{\varepsilon,m}(x)$ are bounded functions independently of ε in $L^2(\Omega)$, it follows that they are convergent.

2. Taking advantage of the periodicity of each functions, and making use of Theorem 8, the lemma follows.

□

Set

$$\Im^{1,\varepsilon}_{ijmn} = C^{\varepsilon}_{ijkl} e_{kl}(\rho^{\varepsilon,mn}) + P^{\varepsilon}_{kij} \frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}, \qquad S^{1,\varepsilon}_{jmn} = P^{\varepsilon}_{jkl} e_{kl}(\rho^{\varepsilon,mn}) - \epsilon^{\varepsilon}_{jk} \frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}. \tag{3.1.38}$$

We see at once that $\Im^{1,\varepsilon}_{ijmn}$ and $S^{1,\varepsilon}_{jmn}$ verify the problems $(P_{\Im^{1,\varepsilon}_{ijmn}})$ and $(P_{S^{1,\varepsilon}_{jmn}})$, respectively, i. e.

$$\begin{cases} (P_{\Im^{1,\varepsilon}_{ijmn}}): \quad -\dfrac{\partial \Im^{1,\varepsilon}_{ijmn}}{\partial x_j} = 0, \\ (P_{S^{1,\varepsilon}_{jmn}}): \quad \dfrac{\partial S^{1,\varepsilon}_{jmn}}{\partial x_j} = 0, \end{cases} \tag{3.1.39}$$

which is clear from (3.1.35). Set now

$$\Im^{2,\varepsilon}_{ijm} = C^{\varepsilon}_{ijkl} e_{kl}(\pi^{\varepsilon,m}) + P^{\varepsilon}_{kij} \frac{\partial I^{\varepsilon,m}}{\partial x_k}, \qquad S^{2,\varepsilon}_{jm} = P^{\varepsilon}_{jkl} e_{kl}(\pi^{\varepsilon,m}) - \epsilon^{\varepsilon}_{jk} \frac{\partial I^{\varepsilon,m}}{\partial x_k}. \tag{3.1.40}$$

From (3.1.36) it is a simple mater to check that $\Im^{2,\varepsilon}_{ijm}$ and $S^{2,\varepsilon}_{jm}$ verify the problems $(P_{\Im^{2,\varepsilon}_{ijm}})$ and $(P_{S^{2,\varepsilon}_{jm}})$, respectively,i.e.

$$\begin{cases} (P_{\Im^{2,\varepsilon}_{ijm}}): \quad -\dfrac{\partial \Im^{2,\varepsilon}_{ijm}}{\partial x_j} = 0, \\ (P_{S^{2,\varepsilon}_{jm}}): \quad \dfrac{\partial S^{2,\varepsilon}_{jm}}{\partial x_j} = 0. \end{cases} \tag{3.1.41}$$

Since $\Im^{1,\varepsilon}_{ijmn}, \Im^{2,\varepsilon}_{ijm}, S^{1,\varepsilon}_{jmn}$ and $S^{2,\varepsilon}_{jm}$ are Y-periodic, thus owing to Theorem 8 one has the following convergences:

$$
\begin{aligned}
\Im^{1,\varepsilon}_{ijmn} \rightharpoonup M_Y(\Im^{1,\varepsilon}_{ijmn}) &= M_Y\left(C^{\varepsilon}_{ijkl}(\frac{x}{\varepsilon})e_{kl}(\rho^{\varepsilon,mn}) + P^{\varepsilon}_{kij}(\frac{x}{\varepsilon})\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}\right) \\
&= \frac{1}{|Y|}\int_Y \left(C_{ijkl}(y)e_{kl,y}(\rho^{mn}) + P_{kij}(y)\frac{\partial \Theta^{mn}}{\partial y_k}\right)dy \\
&= \frac{1}{|Y|}\int_Y \left(C_{ijkl}(y)e_{kl,y}(\chi^{mn}) + C_{ijmn} + P_{kij}(y)\frac{\partial \Psi^{mn}}{\partial y_k}\right)dy. \\
\Im^{2,\varepsilon}_{ijm} \rightharpoonup M_Y(\Im^{2,\varepsilon}_{ijm}) &= M_Y\left(C^{\varepsilon}_{ijkl}(\frac{x}{\varepsilon})e_{kl}(\pi^{\varepsilon,m}) + P^{\varepsilon}_{kij}(\frac{x}{\varepsilon})\frac{\partial I^{\varepsilon,m}}{\partial x_k}\right) \\
&= \frac{1}{|Y|}\int_Y \left(C_{ijkl}(y)e_{kl,y}(\pi^{m}) + P_{kij}(y)\frac{\partial I^{m}}{\partial y_k}\right)dy \\
&= \frac{1}{|Y|}\int_Y \left(C_{ijkl}(y)e_{kl,y}(\Phi^{m}) + P_{kij}(y)\frac{\partial \big(R^m + y_m\big)}{\partial y_k}\right)dy. \\
S^{1,\varepsilon}_{jmn} \rightharpoonup M_y(S^{1,\varepsilon}_{jmn}) &= M_Y\left(P^{\varepsilon}_{jkl}(\frac{x}{\varepsilon})e_{kl}(\rho^{\varepsilon,mn}) - \epsilon^{\varepsilon}_{jk}(\frac{x}{\varepsilon})\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}\right) \\
&= \frac{1}{|Y|}\int_Y \left(P_{jkl}(y)e_{kl,y}(\rho^{mn}) - \epsilon_{jk}(y)\frac{\partial \Theta^{mn}}{\partial y_k}\right)dy. \\
&= \frac{1}{|Y|}\int_Y \left(P_{jkl}(y)e_{kl,y}(\chi^{mn}) + P_{jmn} - \epsilon_{jk}(y)\frac{\partial \Psi^{mn}}{\partial y_k}\right)dy. \\
S^{2,\varepsilon}_{jm} \rightharpoonup M_Y(S^{2,\varepsilon}_{jm}) &= M_Y\left(P^{\varepsilon}_{jkl}(\frac{x}{\varepsilon})e_{kl}(\pi^{\varepsilon,m}) - \epsilon^{\varepsilon}_{jk}(\frac{x}{\varepsilon})\frac{\partial I^{\varepsilon,m}}{\partial x_k}\right) \\
&= \frac{1}{|Y|}\int_Y \left(P_{jkl}(y)e_{kl,y}(\pi^{m}) - \epsilon_{jk}(y)\frac{\partial I^{m}}{\partial y_k}\right)dy \\
&= \frac{1}{|Y|}\int_Y \left(P_{jkl}(y)e_{kl,y}(\Phi^{m}) - \epsilon_{jk}(y)\frac{\partial \big(R^m + y_m\big)}{\partial y_k}\right)dy.
\end{aligned}
\tag{3.1.42}
$$

Step 4: **The homogenized coefficients**

We can write the equation (3.1.10) as

$$
\begin{aligned}
&\int_\Omega \left[C^{\varepsilon}_{ijkl}e_{kl}(u^{\varepsilon})e_{ij}(\upsilon) + P^{\varepsilon}_{kij}\left[e_{ij}(\upsilon)\frac{\partial \varphi^{\varepsilon}}{\partial x_k} - e_{ij}(u^{\varepsilon})\frac{\partial \psi}{\partial x_k}\right] + \epsilon^{\varepsilon}_{jk}\frac{\partial \varphi^{\varepsilon}}{\partial x_k}\frac{\partial \psi}{\partial x_j}\right]dx \\
&= \int_\Omega f\upsilon dx + \int_\Omega r\psi dx + \int_{\Gamma_1^M} g\upsilon d\Gamma_1^M,
\end{aligned}
\tag{3.1.43}
$$

taking in (3.1.43)

$$
\upsilon_i(x) = -w(x)\rho_i^{\varepsilon,mn}(x),
$$

where $w \in \mathcal{D}(\Omega)$,

then,

$$
e_{ij}(\upsilon) = e_{ij}(-w(x)\rho^{\varepsilon,mn}(x)) = -w(x)e_{ij}(\rho^{\varepsilon,mn}) - \frac{\partial w}{\partial x_j}\rho_i^{\varepsilon,mn}(x)
$$

and taking

$$
\psi(x) = w(x)\Theta^{\varepsilon,mn}(x)
$$

then,

$$\frac{\partial \psi}{\partial x_k} = w(x)\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k} + \frac{\partial w}{\partial x_k}\Theta^{\varepsilon,mn}(x).$$

We obtain

$$\begin{aligned}
&-\int_\Omega \left\{ C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)\left[we_{ij}(\rho^{\varepsilon,mn}) + \frac{\partial w}{\partial x_j}\rho_i^{\varepsilon,mn}(x) \right\}dx \\
&-\int_\Omega \left\{ P^\varepsilon_{kij}(x)\frac{\partial \varphi^\varepsilon}{\partial x_k}\left[we_{ij}(\rho^{\varepsilon,mn}) + \frac{\partial w}{\partial x_j}\rho_i^{\varepsilon,mn}(x) \right]\right\}dx \\
&-\int_\Omega \left\{ P^\varepsilon_{kij}(x)e_{ij}(u^\varepsilon)\left[w\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k} + \frac{\partial w}{\partial x_k}\Theta^{\varepsilon,mn}(x) \right]\right\}dx \qquad (3.1.44)\\
&+\int_\Omega \left\{ \epsilon^\varepsilon_{jk}(x)\frac{\partial \varphi^\varepsilon}{\partial x_k}\left[w\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_j} + \frac{\partial w}{\partial x_j}\Theta^{\varepsilon,mn}(x) \right]\right\}dx \\
&= -\int_\Omega f_i\rho_i^{\varepsilon,mn}w dx + \int_\Omega r\Theta^{\varepsilon,mn}w dx.
\end{aligned}$$

Now, we multiply the first equation of (3.1.39) by a test function $\upsilon \in V$ and the second one by $\psi \in \Psi$, summing the two obtained equations yields

$$\int_\Omega \left\{ C^\varepsilon_{ijkl}(x)e_{kl}(\rho^{\varepsilon,mn})e_{ij}(\upsilon) + P^\varepsilon_{kij}(x)\left[\frac{\partial \Theta^{\varepsilon,mn}(x)}{\partial x_k}e_{ij}(\upsilon) - e_{ij}(\rho^{\varepsilon,mn})\frac{\partial \psi}{\partial x_k} \right] + \epsilon^\varepsilon_{jk}(x)\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}\frac{\partial \psi}{\partial x_j} \right\} = 0, \qquad (3.1.45)$$

taking in (3.1.45)

$$\upsilon_i(x) = -w(x)u_i^\varepsilon(x),$$

where $w \in \mathcal{D}(\Omega)$,

then

$$e_{ij}(\upsilon) = e_{ij}(-wu^\varepsilon) = -w(x)e_{ij}(u^\varepsilon)(x) - \frac{\partial w}{\partial x_j}u_i^\varepsilon(x)$$

and taking

$$\psi(x) = w(x)\varphi^\varepsilon(x),$$

we get

$$\begin{aligned}
&-\int_\Omega \left\{ C^\varepsilon_{ijkl}(x)e_{kl}(\rho^{\varepsilon,mn})\left[we_{ij}(u^\varepsilon) + \frac{\partial w}{\partial x_j}u_i^\varepsilon \right]\right\}dx \\
&-\int_\Omega \left\{ P^\varepsilon_{kij}(x)\frac{\partial \Theta^{\varepsilon,mn}(x)}{\partial x_k}\left[we_{ij}(u^\varepsilon) + \frac{\partial w}{\partial x_j}u_i^\varepsilon \right]\right\}dx \\
&-\int_\Omega \left\{ P^\varepsilon_{kij}(x)e_{ij}(\rho^{\varepsilon,mn})\left[\frac{\partial w}{\partial x_k}\varphi^\varepsilon + \frac{\partial \varphi^\varepsilon}{\partial x_k}w \right]\right\}dx \qquad (3.1.46)\\
&+\int_\Omega \left\{ \epsilon^\varepsilon_{jk}(x)\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}\left[\frac{\partial w}{\partial x_j}\varphi^\varepsilon + \frac{\partial \varphi^\varepsilon}{\partial x_j}w \right]\right\}dx \\
&= 0.
\end{aligned}$$

Subtracting (3.1.46) from (3.1.44), gives

$$\begin{aligned}
&-\int_\Omega \left\{C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)\left[we_{ij}(\rho^{\varepsilon,mn})+\frac{\partial w}{\partial x_j}\rho_i^{\varepsilon,mn}\right\}dx\\
&-\int_\Omega \left\{P^\varepsilon_{kij}(x)\frac{\partial \varphi^\varepsilon}{\partial x_k}\left[we_{ij}(\rho^{\varepsilon,mn})+\frac{\partial w}{\partial x_j}\rho_i^{\varepsilon,mn}\right]\right\}dx\\
&-\int_\Omega \left\{P^\varepsilon_{kij}(x)e_{ij}(u^\varepsilon)\left[w\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}+\frac{\partial w}{\partial x_k}\Theta^{\varepsilon,mn}\right]\right\}dx\\
&+\int_\Omega \left\{\epsilon^\varepsilon_{jk}(x)\frac{\partial \varphi^\varepsilon}{\partial x_k}\left[w\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_j}+\frac{\partial w}{\partial x_j}\Theta^{\varepsilon,mn}\right]\right\}dx\\
&+\int_\Omega \left\{C^\varepsilon_{ijkl}(x)e_{kl}(\rho^{\varepsilon,mn})\left[we_{ij}(u^\varepsilon)+\frac{\partial w}{\partial x_j}u_i^\varepsilon\right]\right\}dx\\
&+\int_\Omega \left\{P^\varepsilon_{kij}(x)\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}\left[we_{ij}(u^\varepsilon)(x)+\frac{\partial w}{\partial x_j}u_i^\varepsilon\right]\right\}dx\\
&+\int_\Omega \left\{P^\varepsilon_{kij}(x)e_{ij}(\rho^{\varepsilon,mn})\left[w\frac{\partial \varphi^\varepsilon}{\partial x_k}+\frac{\partial w}{\partial x_k}\varphi^\varepsilon\right]\right\}dx\\
&-\int_\Omega \left\{\epsilon^\varepsilon_{jk}(x)\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}\left[w\frac{\partial \varphi^\varepsilon}{\partial x_j}+\frac{\partial w}{\partial x_j}\varphi^\varepsilon\right]\right\}dx\\
&=-\int_\Omega f_i\rho_i^{\varepsilon,mn}wdx+\int_\Omega rw\Theta^{\varepsilon,mn}dx,
\end{aligned} \tag{3.1.47}$$

At this level, we must shed light on the importance of using the energy method, which apears in the above substraction, this end allows one to cancel the terms where one cannot identify the limit since they contain products of only weakly convergent sequences. Moreover, as we show below, the other terms all pass to the limit and the limit expression will be found easily.

Equation (3.1.47) follows that

$$\begin{aligned}
&-\int_\Omega \left\{C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)+P^\varepsilon_{kij}(x)\frac{\partial \varphi^\varepsilon}{\partial x_k}\right\}\frac{\partial w}{\partial x_j}\rho_i^{\varepsilon,mn}dx\\
&-\int_\Omega \left\{P^\varepsilon_{jkl}(x)e_{kl}(u^\varepsilon)-\epsilon^\varepsilon_{jk}(x)\frac{\partial \varphi^\varepsilon}{\partial x_k}\right\}\frac{\partial w}{\partial x_j}\Theta^{\varepsilon,mn}dx\\
&+\int_\Omega \left\{C^\varepsilon_{ijkl}(x)e_{kl}(\rho^{\varepsilon,mn})+P^\varepsilon_{kij}(x)\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}\right\}\frac{\partial w}{\partial x_j}u_i^\varepsilon dx\\
&+\int_\Omega \left\{P^\varepsilon_{jkl}(x)e_{kl}(\rho^{\varepsilon,mn})-\epsilon^\varepsilon_{jk}(x)\frac{\partial \Theta^{\varepsilon,mn}}{\partial x_k}\right\}\frac{\partial w}{\partial x_j}\varphi^\varepsilon dx\\
&=-\int_\Omega f_i\rho_i^{\varepsilon,mn}wdx+\int_\Omega rw\Theta^{\varepsilon,mn}dx,
\end{aligned} \tag{3.1.48}$$

which leads from (3.1.25) and (3.1.38) to

$$\begin{aligned}
&-\int_\Omega \Sigma^\varepsilon_{ij}\ \rho_i^{\varepsilon,mn}\frac{\partial w}{\partial x_j}dx-\int_\Omega \Lambda^\varepsilon_j\ \Theta^{\varepsilon,mn}\frac{\partial w}{\partial x_j}dx+\int_\Omega \Im^{1,\varepsilon}_{ijmn}u_i^\varepsilon\frac{\partial w}{\partial x_j}dx+\int_\Omega S^{1,\varepsilon}_{jmn}\varphi^\varepsilon\frac{\partial w}{\partial x_j}dx.\\
&=-\int_\Omega f_i\rho_i^{\varepsilon,mn}wdx+\int_\Omega rw\Theta^{\varepsilon,mn}dx.
\end{aligned} \tag{3.1.49}$$

Letting $\varepsilon \to 0$ in the resulting integrals in (3.1.49) and taking advantage of the convergences(3.1.24), (3.1.37), (3.1.28) and (3.1.42), brings

$$
\begin{aligned}
& -\int_\Omega \Sigma^*_{ij}\big(\delta_{im}x_n\big)\frac{\partial w}{\partial x_j}dx + \int_\Omega \langle \Im^1_{ijmn}\rangle u^0_i \frac{\partial w}{\partial x_j}dx + \int_\Omega \langle S^1_{jmn}\rangle \varphi^0 \frac{\partial w}{\partial x_j}dx \\
& = -\int_\Omega f_i\big(\delta_{im}x_n\big) w dx. \\
& \Leftrightarrow + \int_\Omega \frac{\partial \Sigma^*_{ij}}{\partial x_j}\delta_{im}x_n w + \Sigma^*_{ij}\delta_{im}\frac{\partial x_n}{x_j} w + \int_\Omega \langle \Im^1_{ijmn}\rangle u^0_i \frac{\partial w}{\partial x_j}dx + \int_\Omega \langle S^1_{jmn}\rangle \varphi^0 \frac{\partial w}{\partial x_j}dx \\
& = -\int_\Omega f_i\big(\delta_{im}x_n\big) w dx
\end{aligned}
$$

This gives by using (3.1.29) *that*

$$
\int_\Omega \left(\Sigma^*_{mn} - \langle \Im^1_{ijmn}\rangle \frac{\partial u^0_i}{\partial x_j} - \langle S^1_{jmn}\rangle \frac{\partial \varphi^0_i}{\partial x_j}\right) w dx = 0, \quad \forall w \in \mathcal{D}(\Omega). \tag{3.1.50}
$$

$$
\begin{aligned}
& \Rightarrow \Sigma^*_{mn} = \langle \Im^1_{ijmn}\rangle \frac{\partial u^0_i}{\partial x_j} + \langle S^1_{jmn}\rangle \frac{\partial \varphi^0}{\partial x_j}. \\
& \Rightarrow \Sigma^*_{mn} = \left[\frac{1}{|Y|}\int_Y \left(C_{ijkl}(y)e_{kl,y}(\rho^{mn}) + P_{kij}(y)\frac{\partial \Theta^{mn}}{\partial y_k}\right)dy\right]\frac{\partial u^0_i}{\partial x_j} \\
& + \left[\frac{1}{|Y|}\int_Y \left(P_{jkl}(y)e_{kl,y}(\rho^{mn}) - \epsilon_{jk}(y)\frac{\partial \Theta^{,mn}}{\partial y_k}\right)dy\right]\frac{\partial \varphi^0}{\partial x_j}. \\
& = \left[\frac{1}{|Y|}\int_Y \left(C_{ijkl}(y)e_{kl,y}(\chi^{mn}) + C_{ijmn} + P_{kij}(y)\frac{\partial \Psi^{mn}}{\partial y_k}\right)dy\right]\frac{\partial u^0_i}{\partial x_j} \\
& + \left[\frac{1}{|Y|}\int_Y \left(P_{jkl}(y)e_{kl,y}(\chi^{mn}) + P_{jmn} - \epsilon_{jk}(y)\frac{\partial \Psi^{mn}}{\partial y_k}\right)dy\right]\frac{\partial \varphi^0}{\partial x_j}.
\end{aligned}
$$

Taking now in (3.1.43)

$$
\upsilon_i(x) = -w(x)\pi^{\varepsilon,m}_i(x),
$$

where $w \in \mathcal{D}(\Omega)$,

then,

$$
e_{ij}(\upsilon) = e_{ij}(-w\pi^{\varepsilon,m})(x) = -w(x)e_{ij}(\pi^{\varepsilon,m}) - \frac{\partial w}{\partial x_j}\pi^{\varepsilon,m}_i(x)
$$

and taking

$$
\psi(x) = w(x) I^{\varepsilon,m}(x)
$$

then,

$$
\frac{\partial \psi}{\partial x_k} = w(x)\frac{\partial I^{\varepsilon,m}}{\partial x_k} + \frac{\partial w}{\partial x_k} I^{\varepsilon,m}(x).
$$

We get

$$
\begin{aligned}
&-\int_\Omega \left\{ C^\varepsilon_{ijkl}(x)e_{kl}(u^\varepsilon)\left[we_{ij}(\pi^{\varepsilon,m}) + \frac{\partial w}{\partial x_j}\pi^{\varepsilon,m}\right]\right\}dx \\
&-\int_\Omega \left\{ P^\varepsilon_{kij}(x)\frac{\partial \varphi^\varepsilon}{\partial x_k}\left[we_{ij}(\pi^{\varepsilon,m}) + \frac{\partial w}{\partial x_j}\pi_i^{\varepsilon,m}\right]\right\}dx \\
&-\int_\Omega \left\{ P^\varepsilon_{kij}(x)e_{ij}(u^\varepsilon)\left[w\frac{\partial I^{\varepsilon,m}}{\partial x_k} + \frac{\partial w}{\partial x_k}I^{\varepsilon,m}\right]\right\}dx \\
&+\int_\Omega \left\{ \epsilon^\varepsilon_{jk}(x)\frac{\partial \varphi^\varepsilon}{\partial x_k}\left[w\frac{\partial I^{\varepsilon,m}}{\partial x_j} + \frac{\partial w}{\partial x_j}I^{\varepsilon,m}\right]\right\}dx \\
&= -\int_\Omega f_i\ w\pi_i^{\varepsilon,m}dx + \int_\Omega rwI^{\varepsilon,m}dx.
\end{aligned}
\tag{3.1.51}
$$

Now, we multiply the first equation of (3.1.41) by a test function $\upsilon \in V$ and the second one by $\psi \in \Psi$, summing the two obtained equations yields

$$
\int_\Omega \left\{ C^\varepsilon_{ijkl}e_{kl}(\pi^{\varepsilon,m})e_{ij}(\upsilon) + P^\varepsilon_{kij}(x)\left[\frac{\partial I^{\varepsilon,m}}{\partial x_k}e_{ij}(\upsilon) - e_{ij}(\pi^{\varepsilon,m})\frac{\partial \psi}{\partial x_k}\right] + \epsilon^\varepsilon_{jk}(x)\frac{\partial I^{\varepsilon,m}}{\partial x_k}\frac{\partial \psi}{\partial x_j}\right\} = 0, \tag{3.1.52}
$$

taking in (3.1.52)

$$
\upsilon_i(x) = -w(x)u_i^\varepsilon(x),
$$

where $w \in \mathcal{D}(\Omega)$,

then

$$
e_{ij}(\upsilon) = e_{ij}(-wu^\varepsilon) = -w(x)e_{ij}(u^\varepsilon)(x) - \frac{\partial w}{\partial x_j}u_i^\varepsilon(x)
$$

and taking

$$
\psi(x) = w(x)\varphi^\varepsilon(x),
$$

then

$$
\frac{\partial \psi}{\partial x_j} = w(x)\frac{\partial \varphi^\varepsilon}{\partial x_j} + \frac{\partial w}{\partial x_j}\varphi^\varepsilon
$$

.

We get

$$
\begin{aligned}
&-\int_\Omega \left\{ C^\varepsilon_{ijkl}e_{kl}(\pi^{\varepsilon,m})\left[we_{ij}(u^\varepsilon) + \frac{\partial w}{\partial x_j}u_i^\varepsilon\right]\right\}dx \\
&-\int_\Omega P^\varepsilon_{kij}(x)\frac{\partial I^{\varepsilon,m}}{\partial x_k}\left[we_{ij}(u^\varepsilon)(x) + \frac{\partial w}{\partial x_j}u_i^\varepsilon\right]\Bigg\}dx \\
&-\int_\Omega P^\varepsilon_{kij}(x)e_{ij}(\pi^{\varepsilon,m})\left[w\frac{\partial \varphi^\varepsilon}{\partial x_k} + \frac{\partial w}{\partial x_k}\varphi^\varepsilon\right]\Bigg\}dx \\
&+\int_\Omega \epsilon^\varepsilon_{jk}\frac{\partial I^{\varepsilon,m}}{\partial x_k}\left[w\frac{\partial \varphi^\varepsilon}{\partial x_j} + \frac{\partial w}{\partial x_j}\varphi^\varepsilon\right]\Bigg\}dx \\
&= 0.
\end{aligned}
\tag{3.1.53}
$$

Subtracting (3.1.53) from (3.1.51), gives

$$
\begin{aligned}
&-\int_{\Omega}\left\{C^{\varepsilon}_{ijkl}(x)e_{kl}(u^{\varepsilon})\left[we_{ij}(\pi^{\varepsilon,m})+\frac{\partial w}{\partial x_j}\pi^{\varepsilon,m}\right]\right\}dx\\
&-\int_{\Omega}\left\{P^{\varepsilon}_{kij}(x)\frac{\partial \varphi^{\varepsilon}}{\partial x_k}\left[w(x)e_{ij}(\pi^{\varepsilon,m})+\frac{\partial w}{\partial x_j}\pi_i^{\varepsilon,m}(x)\right]\right\}dx\\
&-\int_{\Omega}\left\{P^{\varepsilon}_{kij}(x)e_{ij}(u^{\varepsilon})\left[w\frac{\partial I^{\varepsilon,m}}{\partial x_k}+\frac{\partial w}{\partial x_k}I^{\varepsilon,m}\right]\right\}dx\\
&+\int_{\Omega}\left\{\epsilon^{\varepsilon}_{jk}\frac{\partial \varphi^{\varepsilon}}{\partial x_k}\left[w\frac{\partial I^{\varepsilon,m}}{\partial x_k}+\frac{\partial w}{\partial x_k}I^{\varepsilon,m}\right]\right\}dx\\
&+\int_{\Omega}\left\{C^{\varepsilon}_{ijkl}e_{kl}(\pi^{\varepsilon,m})\left[we_{ij}(u^{\varepsilon})+\frac{\partial w}{\partial x_j}u_i^{\varepsilon}\right]\right\}dx\\
&+\int_{\Omega}P^{\varepsilon}_{kij}(x)\frac{\partial I^{\varepsilon,m}}{\partial x_k}\left[we_{ij}(u^{\varepsilon})+\frac{\partial w}{\partial x_j}u_i^{\varepsilon}\right]\Bigg\}dx\\
&+\int_{\Omega}P^{\varepsilon}_{kij}(x)e_{ij}(\pi^{\varepsilon,m})\left[w\frac{\partial \varphi^{\varepsilon}}{\partial x_k}+\frac{\partial w}{\partial x_k}\varphi^{\varepsilon}\right]\Bigg\}dx\\
&-\int_{\Omega}\epsilon^{\varepsilon}_{jk}(x)\frac{\partial I^{\varepsilon,m}}{\partial x_k}\left[w\frac{\partial \varphi^{\varepsilon}}{\partial x_j}+\frac{\partial w}{\partial x_j}\varphi^{\varepsilon}\right]\Bigg\}dx\\
&=-\int_{\Omega}f_i(x)\ w\pi_i^{\varepsilon,m}dx+\int_{\Omega}rwI^{\varepsilon,m}dx.
\end{aligned}
\tag{3.1.54}
$$

Equation (3.1.54) yields

$$
\begin{aligned}
&-\int_{\Omega}\left\{C^{\varepsilon}_{ijkl}(x)e_{kl}(u^{\varepsilon})+P^{\varepsilon}_{kij}(x)\frac{\partial \varphi^{\varepsilon}}{\partial x_k}\right\}\pi_i^{\varepsilon,m}\frac{\partial w}{\partial x_j}dx\\
&-\int_{\Omega}\left\{P^{\varepsilon}_{jkl}(x)e_{kl}(u^{\varepsilon})-\epsilon^{\varepsilon}_{jk}(x)\frac{\partial \varphi^{\varepsilon}}{\partial x_k}\right\}I^{\varepsilon,m}\frac{\partial w}{\partial x_j}dx\\
&+\int_{\Omega}\left\{C^{\varepsilon}_{ijkl}(x)e_{kl}(\pi^{\varepsilon,m})+P^{\varepsilon}_{kij}(x)\frac{\partial I^{\varepsilon,m}}{\partial x_k}\right\}u_i^{\varepsilon}\frac{\partial w}{\partial x_j}dx\\
&+\int_{\Omega}\left\{P^{\varepsilon}_{jkl}(x)e_{kl}(\pi^{\varepsilon,m})-\epsilon^{\varepsilon}_{jk}\frac{\partial I^{\varepsilon,m}}{\partial x_k}\right\}\varphi^{\varepsilon}\frac{\partial w}{\partial x_j}dx\\
&=-\int_{\Omega}f_i\ w\pi_i^{\varepsilon,m}dx+\int_{\Omega}rwI^{\varepsilon,m}dx.
\end{aligned}
\tag{3.1.55}
$$

From (3.1.25) and (3.1.40) , we deduce that (3.1.55) is equivalent to

$$
\begin{aligned}
&-\int_{\Omega}\Sigma^{\varepsilon}_{ij}\ \pi_i^{\varepsilon,m}\frac{\partial w}{\partial x_j}dx-\int_{\Omega}\Lambda^{\varepsilon}_{j}\ I^{\varepsilon,m}\frac{\partial w}{\partial x_j}dx+\int_{\Omega}\Im^{2,\varepsilon}_{ijm}u_i^{\varepsilon}\frac{\partial w}{\partial x_j}dx+\int_{\Omega}S^{2,\varepsilon}_{jm}\varphi^{\varepsilon}\frac{\partial w}{\partial x_j}dx.\\
&=-\int_{\Omega}f_i\pi_i^{\varepsilon,m}wdx+\int_{\Omega}rI^{\varepsilon,m}wdx.
\end{aligned}
\tag{3.1.56}
$$

Letting $\varepsilon\to 0$ in (3.1.56) and taking advantage of the convergences (3.1.24), (3.1.37), (3.1.28) and (3.1.42),

we get

$$
\begin{aligned}
&-\int_\Omega \Lambda_j^*\ x_m \frac{\partial w}{\partial x_j}dx + \int_\Omega \langle \Im^2_{ijm}\rangle u_i^0 \frac{\partial w}{\partial x_j}dx + \int_\Omega \langle S^2_{jm}\rangle \varphi^0 \frac{\partial w}{\partial x_j}dx \\
&= \int_\Omega r x_m w dx. \\
&\Leftrightarrow \int_\Omega \frac{\partial\big(\Lambda_j^*\ x_m\big)}{\partial x_j} w - \int_\Omega \langle \Im^2_{ijm}\rangle \frac{\partial u_i^0}{\partial x_j} w dx - \int_\Omega \langle S^2_{jm}\rangle \frac{\partial \varphi^0}{\partial x_j} w dx \\
&= \int_\Omega r x_m w dx. \\
&\Rightarrow \int_\Omega \frac{\partial \Lambda_j^*}{\partial x_j} x_m w\ dx + \int_\Omega \Lambda_m^* w dx - \int_\Omega \langle \Im^2_{ijm}\rangle \frac{\partial u_i^0}{\partial x_j} w dx - \int_\Omega \langle S^2_{jm}\rangle \frac{\partial \varphi^0}{\partial x_j} w dx \\
&= \int_\Omega r x_m w dx. \\
&\Rightarrow \int_\Omega \left(\frac{\partial \Lambda_j^*}{\partial x_j} x_m - r x_m + \Lambda_m^* - \langle \Im^2_{ijm}\rangle \frac{\partial u_i^0}{\partial x_j} - \langle S^2_{jm}\rangle \frac{\partial \varphi^0}{\partial x_j}\right) w dx \\
&= 0, \quad \forall w \in \mathcal{D}(\Omega), \\
&\Rightarrow \frac{\partial \Lambda_j^*}{\partial x_j} x_m - r x_m + \Lambda_m^* - \langle \Im^2_{ijm}\rangle \frac{\partial u_i^0}{\partial x_j} - \langle S^2_{jm}\rangle \frac{\partial \varphi^0}{\partial x_j} \qquad (3.1.57) \\
&= 0. \\
&\Rightarrow \cancelto{0}{\left(\frac{\partial \Lambda_j^*}{\partial x_j} - r\right)} x_m + \Lambda_m^* - \langle \Im^2_{ijm}\rangle \frac{\partial u_i^0}{\partial x_j} - \langle S^2_{jm}\rangle \frac{\partial \varphi^0}{\partial x_j} \quad \big(from\ (3.1.31)\big) \\
&= 0, \\
&\Rightarrow \Lambda_m^* = \langle \Im^2_{ijm}\rangle \frac{\partial u_i^0}{\partial x_j} + \langle S^2_{jm}\rangle \frac{\partial \varphi^0}{\partial x_j}. \\
&\Rightarrow \Lambda_m^* = \left[\frac{1}{|Y|}\int_Y \left(C_{ijkl}(y) e_{kl,y}(\pi^m) + P_{kij}(y)\frac{\partial I^m}{\partial y_k}\right) dy\right] \frac{\partial u_i^0}{\partial x_j} \\
&+ \left[\frac{1}{|Y|}\int_Y \left(P_{jkl}(y) e_{kl,y}(\pi^m) - \epsilon_{jk}(y)\frac{\partial I^m}{\partial y_k}\right) dy\right] \frac{\partial \varphi^0}{\partial x_j}. \\
&= \left[\frac{1}{|Y|}\int_Y \left(C_{ijkl}(y) e_{kl,y}(\Phi^m) + P_{kij}(y)\frac{\partial\big(R^m + y_m\big)}{\partial y_k}\right) dy\right] \frac{\partial u_i^0}{\partial x_j} \\
&+ \left[\frac{1}{|Y|}\int_Y \left(P_{jkl}(y) e_{kl,y}(\Phi^m) - \epsilon_{jk}(y)\frac{\partial\big(R^m + y_m\big)}{\partial y_k}\right) dy\right] \frac{\partial \varphi^0}{\partial x_j}.
\end{aligned}
$$

□

3.2 Homogenization of a periodic piezoelectric heterogeneous plate

A plate is a mechanical structure a dimension of which (the thickness) is very much smaller than the others. According to the small thickness of the plate, the three-dimensional elasticity-equations may be approached by two-dimensional equations set on the middle plane of the plate. P. G. Ciarlet and P. Destuynder [26] and [27], P. G. Ciarlet and S. Kesavan [28], P. G. Ciarlet and P. Rabier [29] and P. Destuynder [44] showed, for different cases, that the displacements of slendered three dimensional body converge to the solutions of the two dimensional equations when the thickness tends to zero. Now, in the previous works upon the homogenization of elastic plates (G. Duvaut et A. M. Metellus [46]), the considered equations are the two-dimensional equations of plates, then, in order to use these results to calculate the homogenized coefficients of a periodic plate, this plate must have a thickness very much smaller than the size of the period. This hypothesis is not always satisfied, the structure of some composite plates (see e.g. [66]) shows that the period and thickness of the plate are sometimes comparable. In the sequel we are interested with the case when the thickness η and the period ε of an **periodic piezoelectric plate** are of the same order, the specific feature of such structures is that the periodicity occurs only in two directions. This section is devoted to the study of the limit behavior of $(u^{\varepsilon\eta}, \varphi^{\varepsilon\eta})$ when η and ε are tending together towards zero and we prove a convergence result with the aid of Tartar's method following the same steps as the previous section, note that such study had already done by Cioranescu &al [37] for the case of three-dimensional lattice structures and by D. Caillerie [25] for the case of thin elastic and periodic Plates, which calls us into question can one do the same study upon an periodic piezoelectric plate? the answer of this important question will be found in Theorem 25.

3.2.1 General description of the plate

Let $\omega \subset \mathbb{R}^2$ be a bounded domain with Lipschitz continuous boundary $\partial\omega$, γ_0, γ_e subsets of $\partial\omega$ with $meas\ (\gamma_0) > 0$, The domain ω is covered periodically by cell $Y = [0, Y_1] \times [0, Y_2] \times [-1, 1]$ a point $y \in Y$ is given by $y = (y_1, y_2, y_3) = (\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, \frac{x_3}{\eta})$ where ε denotes the size of the periods. We denote $\gamma_1 := \partial\omega \setminus \gamma_0, \gamma_s := \partial\omega \setminus \gamma_e$. We consider $\Omega^{\varepsilon\eta} = \omega \times \big(-\eta, \eta\big)$ a thin plate with middle plane ω and thickness 2η and the boundary sets

$$\begin{aligned}
\Gamma^\eta_\pm &= \omega \times \{\pm\eta\},\\
\Gamma^\eta_{mD} &= \gamma_0 \times (-\eta,\eta),\\
\Gamma^\eta_1 &= \gamma_1 \times (-\eta,\eta),\\
\Gamma^\eta_{eN} &= \gamma_s \times (-\eta,\eta),\\
\Gamma^\eta_{eD} &= \Gamma^\eta_\pm \cup \big(\gamma_e \times (-\eta,\eta)\big).
\end{aligned} \tag{3.2.1}$$

Points of $\Omega^{\varepsilon\eta}$ are denoted by $x^\eta = (x^\eta_1, x^\eta_2, x^\eta_3)$ We now give the classical equations defining the mechanical and electric equilibrium state of the plate $\Omega^{\varepsilon\eta}$.

$$\begin{cases}
-\dfrac{\partial}{\partial x^\eta_j}[\sigma^{\varepsilon\eta}_{ij}] = f_i{}^\eta \; in \;\; \Omega^\eta\\
\dfrac{\partial}{\partial x^\eta_i}[D^{\varepsilon\eta}_i] = r^\eta \qquad in \; \Omega^\eta\\
\sigma^{\varepsilon\eta}_{ij} n^\eta_i = g^\eta_i \; on \quad \Gamma^\eta_+ \cup \Gamma^\eta_-\\
D^{\varepsilon\eta}_i n^\eta_i = 0 \;\; on \;\; \Gamma^\eta_{eN}\\
u^{\varepsilon\eta} = 0 \;\; on \;\; \Gamma^\eta_{mD}\\
\varphi^{\varepsilon\eta} = 0 \;\; on \;\; \Gamma^\eta_{eD}\\
where\\
\sigma^{\varepsilon\eta}_{ij} = C^{\varepsilon\eta}_{ijkl}(x^\eta)\dfrac{\partial u^{\varepsilon\eta}_k}{\partial x^\eta_l} - P^{\varepsilon\eta}_{kij}(x^\eta)\dfrac{\partial \varphi^{\varepsilon\eta}}{\partial x^\eta_k}\\
D^{\varepsilon\eta}_i = P^{\varepsilon\eta}_{ikl}(x^\eta)\dfrac{\partial u^{\varepsilon\eta}_k}{\partial x^\eta_l} - \epsilon^{\varepsilon\eta}_{ik}(x^\eta)\dfrac{\partial \varphi^{\varepsilon\eta}}{\partial x^\eta_k}.
\end{cases} \tag{3.2.2}$$

3.2.2 Change of scale

In order to study the limit when the both (ε and $\eta \longrightarrow 0$) we shall make a relation between ε and η to ensure that the two go to zero in the same time, so, we set $\eta = k\varepsilon$, where k is a positive constant. Let us define the two following spaces

$$V = \{\upsilon \; |\upsilon \in H^1(\Omega)^3, \upsilon = 0 \; on \; \Gamma^\eta_{mD}\},$$

$$\Psi = \{\psi \; |\psi \in H^1(\Omega), \psi = 0 \; on \; \Gamma^\eta_{eD}\},$$

equipped with the norms (equivalent to the usual norm H^1)

$$\|\upsilon\|_V = \Big(\sum_{i,j=1}^{3}\int_\Omega\Big(\frac{\partial\upsilon_i}{\partial x_j}\Big)^2\Big)^{\frac{1}{2}},$$

$$\|\psi\|_\Psi = \Big(\sum_{i=1}^{3}\int_\Omega\Big(\frac{\partial\psi}{\partial x_i}\Big)^2\Big)^{\frac{1}{2}}, \qquad (3.2.3)$$

$$\|\upsilon,\psi\|_{\upsilon\times\psi} = \|\upsilon\|_V + \|\psi\|_\Psi.$$

To work on a fixed domain independent of η, we make a dilatation in the x_3-direction, defined by

$$z_\alpha = x_\alpha \text{ and } z_3 = \tfrac{x_3}{k\varepsilon}.$$

After the dilatation, system (3.1.10) can be written in the following variational form

$$\begin{cases}
(k\varepsilon)\displaystyle\int_\Omega\Big\{C^\varepsilon_{i\alpha h\gamma}\frac{\partial u^\epsilon_h}{\partial z_\gamma}\frac{\partial\upsilon_i}{\partial z_\alpha} + (k\varepsilon)^{-1}\Big(C^\varepsilon_{i\alpha h3}\frac{\partial u^\epsilon_h}{\partial z_3}\frac{\partial\upsilon_i}{\partial z_\alpha}) + C^\varepsilon_{i3h\alpha}\frac{\partial u^\epsilon_h}{\partial z_\alpha}\frac{\partial\upsilon_i}{\partial z_3}\Big) + (k\varepsilon)^{-2}C^\varepsilon_{i3h3}\frac{\partial u^\epsilon_h}{\partial z_3}\frac{\partial\upsilon_i}{\partial z_3}\Big\}dz \\
+(k\varepsilon)\displaystyle\int_\Omega\Big\{P^\varepsilon_{\gamma i\alpha}\frac{\partial\varphi^\epsilon}{\partial z_\gamma}\frac{\partial\upsilon_i}{\partial z_\alpha} + (k\varepsilon)^{-1}\Big(P^\varepsilon_{3i\alpha}\frac{\partial\varphi^\epsilon}{\partial z_3}\frac{\partial\upsilon_i}{\partial z_\alpha} + P^\varepsilon_{\alpha i3}\frac{\partial\varphi^\epsilon}{\partial z_\alpha}\frac{\partial\upsilon_i}{\partial z_3}\Big) + (k\varepsilon)^{-2}P^\varepsilon_{3i3}\frac{\partial\varphi^\epsilon}{\partial z_3}\frac{\partial\upsilon_i}{\partial z_3}\Big\}dz \\
-(k\varepsilon)\displaystyle\int_\Omega\Big\{P^\varepsilon_{\gamma i\alpha}\frac{\partial u^\varepsilon_i}{\partial z_\alpha}\frac{\partial\psi}{\partial z_\gamma} + (k\varepsilon)^{-1}(P^\varepsilon_{\alpha i3}\frac{\partial u^\varepsilon_i}{\partial z_3}\frac{\partial\psi}{\partial z_\alpha} + P^\varepsilon_{3i\alpha}\frac{\partial u^\varepsilon_i}{\partial z_\alpha}\frac{\partial\psi}{\partial z_3}\Big) + (k\varepsilon)^{-2}P^\varepsilon_{3i3}\frac{\partial u^\varepsilon_i}{\partial z_3}\frac{\partial\psi}{\partial z_3}\Big\}dz \\
+(k\varepsilon)\displaystyle\int_\Omega\Big\{\epsilon^\varepsilon_{\alpha\gamma}\frac{\partial\varphi^\varepsilon}{\partial z_\gamma}\frac{\partial\psi}{\partial z_\alpha} + (k\varepsilon)^{-1}\Big(\epsilon^\varepsilon_{\alpha3}\frac{\partial\varphi^\varepsilon}{\partial z_3}\frac{\partial\psi}{\partial z_\alpha} + \epsilon^\varepsilon_{3\alpha}\frac{\partial\varphi^\varepsilon}{\partial z_\alpha}\frac{\partial\psi}{\partial z_3}\Big) + (k\varepsilon)^{-2}\epsilon^\varepsilon_{33}\frac{\partial\varphi^\varepsilon}{\partial z_3}\frac{\partial\psi}{\partial z_3}\Big\}dz \\
= (k\varepsilon)\displaystyle\int_\Omega f^\varepsilon\upsilon + \int_{\Gamma_+} g^\varepsilon_+\upsilon dz_1dz_1 + \int_{\Gamma_-} g^\varepsilon_-\upsilon dz_1dz_1 + (k\varepsilon)\int_\Omega r^\varepsilon\psi
\end{cases} \qquad (3.2.4)$$

We devise on $(k\varepsilon)$ we obtain

$$\begin{cases}
\displaystyle\int_\Omega\Big\{C^\varepsilon_{i\alpha h\gamma}\frac{\partial u^\epsilon_h}{\partial z_\gamma}\frac{\partial\upsilon_i}{\partial z_\alpha} + (k\varepsilon)^{-1}\Big(C^\varepsilon_{i\alpha h3}\frac{\partial u^\epsilon_h}{\partial z_3}\frac{\partial\upsilon_i}{\partial z_\alpha} + C^\varepsilon_{i3h\alpha}\frac{\partial u^\epsilon_h}{\partial z_\alpha}\frac{\partial\upsilon_i}{\partial z_3}\Big) + (k\varepsilon)^{-2}C^\varepsilon_{i3h3}\frac{\partial u^\epsilon_h}{\partial z_3}\frac{\partial\upsilon_i}{\partial z_3}\Big\}dz \\
+\displaystyle\int_\Omega\Big\{P^\varepsilon_{\gamma i\alpha}\frac{\partial\varphi^\epsilon}{\partial z_\gamma}\frac{\partial\upsilon_i}{\partial z_\alpha} + (k\varepsilon)^{-1}\Big(P^\varepsilon_{3i\alpha}\frac{\partial\varphi^\epsilon}{\partial z_3}\frac{\partial\upsilon_i}{\partial z_\alpha} + P^\varepsilon_{\alpha i3}\frac{\partial\varphi^\epsilon}{\partial z_\alpha}\frac{\partial\upsilon_i}{\partial z_3}\Big) + (k\varepsilon)^{-2}P^\varepsilon_{3i3}\frac{\partial\varphi^\epsilon}{\partial z_3}\frac{\partial\upsilon_i}{\partial z_3}\Big\}dz \\
-\displaystyle\int_\Omega\Big\{P^\varepsilon_{\gamma i\alpha}\frac{\partial u^\varepsilon_i}{\partial z_\alpha}\frac{\partial\psi}{\partial z_\gamma} + (k\varepsilon)^{-1}\Big(P^\varepsilon_{\alpha i3}\frac{\partial u^\varepsilon_i}{\partial z_3}\frac{\partial\psi}{\partial z_\alpha} + P^\varepsilon_{3i\alpha}\frac{\partial u^\varepsilon_i}{\partial z_\alpha}\frac{\partial\psi}{\partial z_3}\Big) + (k\varepsilon)^{-2}P^\varepsilon_{3i3}\frac{\partial u^\varepsilon_i}{\partial z_3}\frac{\partial\psi}{\partial z_3}\Big\}dz \\
+\displaystyle\int_\Omega\Big\{\epsilon^\varepsilon_{\alpha\gamma}\frac{\partial\varphi^\varepsilon}{\partial z_\gamma}\frac{\partial\psi}{\partial z_\alpha} + (k\varepsilon)^{-1}\Big(\epsilon^\varepsilon_{\alpha3}\frac{\partial\varphi^\varepsilon}{\partial z_3}\frac{\partial\psi}{\partial z_\alpha} + \epsilon^\varepsilon_{3\alpha}\frac{\partial\varphi^\varepsilon}{\partial z_\alpha}\frac{\partial\psi}{\partial z_3}\Big) + (k\varepsilon)^{-2}\epsilon^\varepsilon_{33}\frac{\partial\varphi^\varepsilon}{\partial z_3}\frac{\partial\psi}{\partial z_3}\Big\}dz \\
=\displaystyle\int_\Omega f^\varepsilon\upsilon + (k\varepsilon)^{-1}\int_{\Gamma_+} g^\varepsilon_+\upsilon dz_1dz_1 + (k\varepsilon)^{-1}\int_{\Gamma_-} g^\varepsilon_-\upsilon dz_1dz_1 + \int_\Omega r^\varepsilon\psi dz
\end{cases} \qquad (3.2.5)$$

We now let $\varepsilon \to 0$ in system (3.2.5). We have the following homogenization result.

3.2.3 Limit for $\varepsilon \to 0$: Homogenization and reduction of the dimension

Theorem 25.

Assume that

$$\begin{cases} f^\varepsilon \overset{L^2(\Omega)}{\rightharpoonup} f^*, \\ (k\varepsilon)^{-1} g^\varepsilon_\pm \overset{L^2(\omega)}{\rightharpoonup} g^*_\pm, \\ r^\varepsilon \overset{L^2(\Omega)}{\rightharpoonup} r^*. \end{cases} \tag{3.2.6}$$

Then there exist two functions $u^0 \in H^1(\omega)$ and $\varphi^0 \in H^1(\omega)$ such that when $\varepsilon \to 0$,

$$\begin{aligned} u^\varepsilon &\overset{H^1(\Omega)}{\rightharpoonup} u^0, \\ \varphi^\varepsilon &\overset{H^1(\Omega)}{\rightharpoonup} \varphi^0. \end{aligned} \tag{3.2.7}$$

And the functions $u^0 \in H^1(\omega)$ and $\varphi^0 \in H^1(\omega)$ satisfying the homogenized system

$$\begin{aligned} C^h_{i\alpha m\gamma} \frac{\partial^2 u^0_i}{\partial z_\alpha \partial z_\gamma} + P^h_{\alpha m\gamma} \frac{\partial^2 \varphi^0}{\partial z_\alpha \partial z_\gamma} &= \int_{-1}^{1} f^*_m dz_3 + g^{*,+}_m + g^{*,-}_m \quad in\ \omega, \\ P^h_{i\alpha\gamma} \frac{\partial^2 u^0_i}{\partial z_\alpha \partial z_\gamma} + \epsilon^h_{\alpha\gamma} \frac{\partial^2 \varphi^0}{\partial z_\alpha \partial z_\gamma} &= \int_{-1}^{1} r^* dz_3 \quad in\ \omega, \\ u^0 &= 0\ on\ \gamma_0, \\ \varphi^0 &= 0\ on\ \gamma_e. \end{aligned} \tag{3.2.8}$$

The homogenized coefficients are given by

$$\begin{aligned} C^h_{i\alpha m\gamma} &= \frac{1}{|Y|} \int_Y \Bigg(C_{i\alpha h\beta}(y) \frac{\partial(\chi^{m\gamma}_h + \delta_{im} y_\gamma)}{\partial y_\beta} + P_{\beta i\alpha}(y) \frac{\partial \Psi^{m\gamma}}{\partial y_\beta} \\ &\quad + k^{-1} \left\{ C_{i\alpha h3} \frac{\partial \chi^{m\gamma}_h}{\partial y_3} + P^\varepsilon_{3i\alpha} \frac{\partial \Psi^{m\gamma}}{\partial y_3} \right\} \Bigg) dy, \\ P^h_{i\alpha\gamma} &= \frac{1}{|Y|} \int_Y \Bigg(C_{i\alpha h\beta}(y) \frac{\partial \Phi^\gamma_h}{\partial y_\beta} + P_{\beta i\alpha}(y) \frac{\partial (R^\gamma + y_\gamma)}{\partial y_\beta} \\ &\quad + k^{-1} \left\{ C_{i\alpha h3} \frac{\partial \Phi^\gamma_h}{\partial y_3} + P_{3i\alpha} \frac{\partial R^\gamma}{\partial y_3} \right\} \Bigg) dy. \\ P^h_{\alpha m\gamma} &= \frac{1}{|Y|} \int_Y \Bigg(P_{\alpha h\beta}(y) \frac{\partial(\chi^{m\gamma}_h + \delta_{im} y_\gamma)}{\partial y_\beta} - \epsilon_{\alpha\beta}(y) \frac{\partial \Psi^{m\gamma}}{\partial y_\beta} \\ &\quad + k^{-1} \left\{ P_{\alpha h3} \frac{\partial \chi^{m\gamma}_h}{\partial y_3} - \epsilon^\varepsilon_{\alpha 3} \frac{\partial \Psi^{m\gamma}}{\partial y_3} \right\} \Bigg) dy, \\ \epsilon^h_{\alpha\gamma} &= \frac{1}{|Y|} \int_Y \Bigg(P_{\alpha h\beta}(y) \frac{\partial \Phi^\gamma_h}{\partial y_\beta} - \epsilon_{\alpha\beta}(y) \frac{\partial (R^\gamma + y_\gamma)}{\partial y_\beta} \\ &\quad + k^{-1} \left\{ P_{\alpha h3} \frac{\partial \Phi^\gamma_h}{\partial y_3} - \epsilon^\varepsilon_{\alpha 3} \frac{\partial R^\gamma}{\partial y_3} \right\} \Bigg) dy. \end{aligned} \tag{3.2.9}$$

Remark 3.2.1. *The functions* $\left(\chi^{m\gamma}(\frac{z_1}{\varepsilon}, \frac{z_2}{\varepsilon}, z_3), \Psi^{m\gamma}(\frac{z_1}{\varepsilon}, \frac{z_2}{\varepsilon}, z_3)\right)$ *and* $\left(\Phi^{\gamma}(\frac{z_1}{\varepsilon}, \frac{z_2}{\varepsilon}, z_3), R^{\gamma}(\frac{z_1}{\varepsilon}, \frac{z_2}{\varepsilon}, z_3)\right)$ *are defined in (3.2.42) and (3.2.43) respectively.*

Proof. **(Proof of Theorem 25)** The proof is done in several steps. In the first one, we obtain a priori estimates. In the second step, by choosing appropriate test functions, we derive a limit equation whose coefficients are identified in the last step by making use of Tartar's variational method.

3.2.4 A priori estimates

Lemma 3.2.1. *The solutions* $(u^\varepsilon, \varphi^\varepsilon)$ *of (3.2.5) are bounded.*

Proof. Taking $\upsilon = u^\varepsilon$ and $\psi = \varphi^\varepsilon$ in (3.2.5), gives

$$\begin{aligned}
&\int_\Omega C^\varepsilon_{i\alpha h\gamma} \frac{\partial u^\varepsilon_h}{\partial z_\gamma} \frac{\partial u^\varepsilon_i}{\partial z_\alpha} + 2C^\varepsilon_{i\alpha h3}\left(\frac{1}{k\varepsilon}\frac{\partial u^\varepsilon_h}{\partial z_3}\right)\frac{\partial u^\varepsilon_i}{\partial z_\alpha} + C^\varepsilon_{i3h3}\left(\frac{1}{k\varepsilon}\frac{\partial u^\varepsilon_h}{\partial z_3}\right)\left(\frac{1}{k\varepsilon}\frac{\partial u^\varepsilon_i}{\partial z_3}\right)dz \\
&+ \int_\Omega \epsilon^\varepsilon_{\alpha\gamma} \frac{\partial \varphi^\varepsilon}{\partial z_\gamma}\frac{\partial \varphi^\varepsilon}{\partial z_\alpha} + 2\epsilon^\varepsilon_{\alpha 3}\left(\frac{1}{k\varepsilon}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right)\frac{\partial \varphi^\varepsilon}{\partial z_\alpha} + \epsilon^\varepsilon_{33}\left(\frac{1}{k\varepsilon}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right)\left(\frac{1}{k\varepsilon}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right)dz \\
&= \int_\Omega f^\varepsilon u^\varepsilon + (k\varepsilon)^{-1}\int_{\Gamma_+} g^\varepsilon_+ u^\varepsilon dz_1 dz_2 + (k\varepsilon)^{-1}\int_{\Gamma_-} g^\varepsilon_- u^\varepsilon dz_1 dz_2 + \int_\Omega r^\varepsilon \varphi^\varepsilon.
\end{aligned} \tag{3.2.10}$$

Set

$$\begin{aligned}
\varrho^\varepsilon_{h\gamma} &= \frac{\partial u^\varepsilon_h}{\partial z_\gamma}, \qquad & \varrho^\varepsilon_{h3} &= \left(\frac{1}{k\varepsilon}\right)\frac{\partial u^\varepsilon_h}{\partial z_3}, \\
\theta^\varepsilon_\gamma &= \frac{\partial \varphi^\varepsilon}{\partial z_\gamma}, \qquad & \theta^\varepsilon_3 &= \left(\frac{1}{k\varepsilon}\right)\frac{\partial \varphi^\varepsilon}{\partial z_3}.
\end{aligned} \tag{3.2.11}$$

Substituting ϱ^ε_{hl}, and θ^ε_i in the right hand-side of (3.2.10), it becomes

$$\begin{aligned}
&\int_\Omega C^\varepsilon_{i\alpha h\gamma}\varrho^\varepsilon_{h\gamma}\varrho^\varepsilon_{i\alpha} + 2C^\varepsilon_{i\alpha h3}\varrho^\varepsilon_{h3}\varrho^\varepsilon_{i\alpha} + C^\varepsilon_{i3h3}\varrho^\varepsilon_{h3}\varrho^\varepsilon_{i3} dz \\
&+ \int_\Omega \epsilon^\varepsilon_{\alpha\gamma}\theta^\varepsilon_\gamma\theta^\varepsilon_\alpha + 2\epsilon^\varepsilon_{\alpha 3}\theta^\varepsilon_3\theta^\varepsilon_\alpha + \epsilon^\varepsilon_{33}\theta^\varepsilon_3\theta^\varepsilon_3 dz.
\end{aligned} \tag{3.2.12}$$

Which is equivalent to

$$\int_\Omega C^\varepsilon_{ijhl}\varrho^\varepsilon_{hl}\varrho^\varepsilon_{ij} dz + \int_\Omega \epsilon^\varepsilon_{ij}\theta^\varepsilon_i\theta^\varepsilon_j dz. \tag{3.2.13}$$

Using the ellipticity of the coefficients C^ε_{ijhl} and $\epsilon^\varepsilon_{ij}$ we get the estimate

$$\begin{aligned}
C\left(\sum_{i=1}^{3}\|\theta_i\|_{L^2(\Omega)} + \sum_{h=1}^{3}\sum_{l=1}^{3}\|\varrho_{hl}\|_{L^2(\Omega)}\right)^2 &\le C\sum_i \|\theta^\varepsilon_i\|^2_{L^2(\Omega)} + C\sum_{h,l}\|\varrho^\varepsilon_{hl}\|^2_{L^2(\Omega)} \\
&\le \int_\Omega C^\varepsilon_{ijhl}\varrho^\varepsilon_{hl}\varrho^\varepsilon_{ij} dz + \int_\Omega \epsilon^\varepsilon_{ij}\theta^\varepsilon_i\theta^\varepsilon_j dz.
\end{aligned} \tag{3.2.14}$$

The above formula leads us to

$$C\left(\|\varphi^\varepsilon\|_\Psi + \|u^\varepsilon\|_V\right)^2 \leq \int_\Omega C^\varepsilon_{i\alpha h\gamma}\frac{\partial u^\varepsilon_h}{\partial z_\gamma}\frac{\partial u^\varepsilon_i}{\partial z_\alpha} + 2C^\varepsilon_{i\alpha h3}\left(\frac{1}{k\varepsilon}\frac{\partial u^\varepsilon_h}{\partial z_3}\right)\frac{\partial u^\varepsilon_i}{\partial z_\alpha} + C^\varepsilon_{i3h3}\left(\frac{1}{k\varepsilon}\frac{\partial u^\varepsilon_h}{\partial z_3}\right)\left(\frac{1}{k\varepsilon}\frac{\partial u^\varepsilon_i}{\partial z_3}\right)dz$$
$$+\int_\Omega \epsilon^\varepsilon_{\alpha\gamma}\frac{\partial \varphi^\varepsilon}{\partial z_\gamma}\frac{\partial \varphi^\varepsilon}{\partial z_\alpha} + 2\epsilon^\varepsilon_{\alpha 3}\left(\frac{1}{k\varepsilon}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right)\frac{\partial \varphi^\varepsilon}{\partial z_\alpha} + \epsilon^\varepsilon_{33}\left(\frac{1}{k\varepsilon}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right)\left(\frac{1}{k\varepsilon}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right)dz. \tag{3.2.15}$$

On the other hand, making use of Cauchy Schwarz and Poincaré inequalities together with trace theorem applied on the left-hand of (3.2.5) and owing to assumptions (3.2.6), one can have

$$\begin{cases} \|u^\varepsilon\|_V \leq C, \\ \qquad \Leftrightarrow \begin{cases} \displaystyle\sum_{h=1}^{3}\left\|\frac{\partial u^\varepsilon_h}{\partial z_\alpha}\right\|_{L^2(\Omega)} \leq C, \\ \displaystyle\sum_{h=1}^{3}\left\|\frac{\partial u^\varepsilon_h}{\partial z_3}\right\|_{L^2(\Omega)} \leq (k\varepsilon)C, \end{cases} \\ \|\varphi^\varepsilon\|_\Psi \leq C, \\ \qquad \Leftrightarrow \begin{cases} \left\|\dfrac{\partial \varphi^\varepsilon}{\partial z_\alpha}\right\|_{L^2(\Omega)} \leq C, \\ \left\|\dfrac{\partial \varphi^\varepsilon}{\partial z_3}\right\|_{L^2(\Omega)} \leq (k\varepsilon)C. \end{cases} \end{cases} \tag{3.2.16}$$

So, we can extract a subsequences still denoted by $u^\varepsilon, \varphi^\varepsilon$ such that

$$\begin{cases} u^\varepsilon \overset{H^1(\Omega)}{\rightharpoonup} u^0 = u(z), \\ \varphi^\varepsilon \overset{H^1(\Omega)}{\rightharpoonup} \varphi^0 = \varphi(z). \end{cases} \tag{3.2.17}$$

And since $H^1(\Omega) \underset{c}{\hookrightarrow} L^2(\Omega)$, so

$$\begin{cases} u^\varepsilon \overset{L^2(\Omega)}{\longrightarrow} u^0(z), \\ \varphi^\varepsilon \overset{L^2(\Omega)}{\longrightarrow} \varphi^0(z) \end{cases} \tag{3.2.18}$$

Furthermore, from (3.2.16) we can extract a subsequence still denoted by $\frac{\partial u_\varepsilon}{\partial z_\alpha}$ such that

$$\frac{\partial u^\varepsilon}{\partial z_\alpha} \overset{L^2(\Omega)}{\rightharpoonup} \xi_\alpha,$$

then, the derivate in the sense of distributions yields

$$\int_\Omega \frac{\partial u^\varepsilon}{\partial z_\alpha}\vartheta\ dz = \int_\Omega -u^\varepsilon\frac{\partial \vartheta}{\partial z_\alpha}\ dz \qquad \forall\ \vartheta \in D(\Omega),$$

passing to the limit in the previous equation

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{\partial u^\varepsilon}{\partial z_\alpha}\vartheta\ dz = -\lim_{\varepsilon\to 0}\int_\Omega u^\varepsilon\frac{\partial \vartheta}{\partial z_\alpha}\ dz \quad \forall\ \vartheta \in D(\Omega),$$

gives

$$
\begin{aligned}
&\int_\Omega \xi_\alpha \vartheta \, dz = -\int_\Omega u^0 \frac{\partial \vartheta}{\partial z_\alpha} \quad \forall\, \vartheta \in D(\Omega) \\
&\Rightarrow \int_\Omega \xi_\alpha \vartheta \, dz = \int_\Omega \frac{\partial u_0}{\partial z_\alpha} \vartheta \, dz \quad \forall \vartheta \in D(\Omega) \\
&\Rightarrow \int_\Omega \left(\xi_\alpha - \frac{\partial u^0}{\partial z_\alpha} \right) \vartheta dz = 0 \quad \forall \vartheta \in D(\Omega) \\
&\Rightarrow \xi_\alpha = \frac{\partial u^0}{\partial z_\alpha}. \\
&\Rightarrow \boxed{\tfrac{\partial u^\varepsilon}{\partial z_\alpha} \overset{L^2(\Omega)}{\rightharpoonup} \tfrac{\partial u^0}{\partial z_\alpha}.}
\end{aligned}
\tag{3.2.19}
$$

Again, from (3.2.16) we can extract a subsequence still denoted by $\frac{\partial \varphi^\varepsilon}{\partial z_\alpha}$ such that

$$
\frac{\partial \varphi^\varepsilon}{\partial z_\alpha} \overset{L^2(\Omega)}{\rightharpoonup} T_\alpha,
$$

then, the derivation in the sense of distributions yields

$$
\int_\Omega \frac{\partial \varphi^\varepsilon}{\partial z_\alpha} \Psi dz = \int_\Omega -\varphi^\varepsilon \frac{\partial \Psi}{\partial z_\alpha} dz \quad \forall\, \Psi \in D(\Omega),
$$

passing to the limit in the previous equation

$$
\lim_{\varepsilon \to 0} \int_\Omega \frac{\partial \varphi^\varepsilon}{\partial z_\alpha} \Psi \, dz = -\lim_{\varepsilon \to 0} \int_\Omega \varphi^\varepsilon \frac{\partial \Psi}{\partial z_\alpha} \, dx \quad \forall\, \Psi \in D(\Omega),
$$

gives

$$
\begin{aligned}
&\int_\Omega T_\alpha \Psi \, dz = -\int_\Omega \varphi^0 \frac{\partial \Psi}{\partial z_\alpha} \, dz \quad \forall\, \Psi \in D(\Omega) \\
&\Rightarrow \int_\Omega T_\alpha \Psi \, dz = \int_\Omega \frac{\partial \varphi^0}{\partial z_\alpha} \Psi \, dz \quad \forall\, \Psi \in D(\Omega) \\
&\Rightarrow \int_\Omega \left(T_\alpha - \frac{\partial \varphi^0}{\partial z_\alpha} \right) \Psi \, dz = 0 \quad \forall\, \Psi \in D(\Omega) \\
&\Rightarrow T_\alpha = \frac{\partial \varphi^0}{\partial z_\alpha}. \\
&\Rightarrow \boxed{\tfrac{\partial \varphi^\varepsilon}{\partial z_\alpha} \overset{L^2(\Omega)}{\rightharpoonup} \tfrac{\partial \varphi^0}{\partial z_\alpha}.}
\end{aligned}
\tag{3.2.20}
$$

Also, from (3.2.16) it follows immediately that we can extract a subsequence still denoted by $\frac{\partial u_h^\varepsilon}{\partial z_3}$ and $\frac{\partial \varphi^\varepsilon}{\partial z_3}$ such that

$$
\begin{cases}
\dfrac{\partial u_h^\varepsilon}{\partial z_3} \overset{L^2(\Omega)}{\rightharpoonup} 0, \\
\dfrac{\partial \varphi^\varepsilon}{\partial z_3} \overset{L^2(\Omega)}{\rightharpoonup} 0.
\end{cases}
\tag{3.2.21}
$$

Indeed, (3.2.16) point out that $\frac{\partial u_h^\varepsilon}{\partial z_3}$ is bounded, hence we can extract a subsequence still denoted by $\frac{\partial u_h^\varepsilon}{\partial z_3}$ such that

$$\frac{\partial u_h^\varepsilon}{\partial z_3} \rightharpoonup \zeta,$$

in view of the proprieties of the weak convergence , one can get

$$\begin{aligned}
\|\zeta\|_{L^2(\Omega)} &\leq \lim_{\varepsilon\to 0} inf \left\| \frac{\partial u_h^\varepsilon}{\partial z_3} \right\|_{L^2(\Omega)} \\
&\leq \lim_{\varepsilon\to 0} inf (k\varepsilon) C, \\
&\leq 0, \\
&\Rightarrow \|\zeta\|_{L^2(\Omega)} = 0, \\
&\Rightarrow \boxed{\tfrac{\partial u_h^\varepsilon}{\partial z_3} \overset{L^2(\Omega)}{\rightharpoonup} 0.}
\end{aligned} \tag{3.2.22}$$

A quick glance at (3.2.19) shows that

$$\frac{\partial u_h^\varepsilon}{\partial z_3} \overset{L^2(\Omega)}{\rightharpoonup} \frac{\partial u_h^0}{\partial z_3},$$

and from (3.2.21), it is self-evident that

$$\frac{\partial u_h^0}{\partial z_3} = 0. \tag{3.2.23}$$

Which implies that

$$u^0(z_1, z_2, z_3) = u^0(z_1, z_2) \tag{3.2.24}$$

By analogy, we can find that

$$\begin{cases} From\ (3.2.21) : \dfrac{\partial \varphi^\varepsilon}{\partial z_3} \overset{L^2(\Omega)}{\rightharpoonup} 0, \\ From\ (3.2.17) : \dfrac{\partial \varphi^\varepsilon}{\partial z_3} \overset{L^2(\Omega)}{\rightharpoonup} \dfrac{\partial \varphi^0}{\partial z_3}. \end{cases} \tag{3.2.25}$$

$$\Rightarrow \begin{cases} \dfrac{\partial \varphi^0}{\partial z_3} = 0, \\ \varphi^0(z_1, z_2, z_3) = \varphi^0(z_1, z_2). \end{cases} \tag{3.2.26}$$

However, we are able to draw a number of conclusions

$$
\begin{aligned}
&u^\varepsilon \overset{H^1(\Omega)}{\rightharpoonup} u^0,\\
&u^\varepsilon \overset{L^2(\Omega)}{\longrightarrow} u^0,\\
&\frac{\partial u^\varepsilon}{\partial z_\alpha} \overset{L^2(\Omega)}{\rightharpoonup} \frac{\partial u^0}{\partial z_\alpha},\\
&\frac{\partial u^0}{\partial z_3} \overset{L^2(\Omega)}{\rightharpoonup} 0,\\
&u^0(z_1,z_2,z_3) = u^0(z_1,z_2),\\
&\varphi^\varepsilon \overset{H^1(\Omega)}{\rightharpoonup} \varphi^0,\\
&\varphi^\varepsilon \overset{L^2(\Omega)}{\longrightarrow} \varphi^0,\\
&\frac{\partial \varphi^\varepsilon}{\partial z_\alpha} \overset{L^2(\Omega)}{\rightharpoonup} \frac{\partial \varphi^0}{\partial z_\alpha},\\
&\frac{\partial \varphi^\varepsilon}{\partial z_3} \overset{L^2(\Omega)}{\rightharpoonup} 0,\\
&\varphi^0(z_1,z_2,z_3) = \varphi^0(z_1,z_2).
\end{aligned}
\tag{3.2.27}
$$

□

Set

$$
\xi_{ij}^\varepsilon = C_{ijh\gamma}^\varepsilon \frac{\partial u_h^\epsilon}{\partial z_\gamma} + P_{\gamma ij}^\varepsilon \frac{\partial \varphi^\varepsilon}{\partial z_\gamma} + (k\varepsilon)^{-1}\left\{ C_{ijh3}^\varepsilon \frac{\partial u_h^\varepsilon}{\partial z_3} + P_{3ij}^\varepsilon \frac{\partial \varphi^\varepsilon}{\partial z_3} \right\},
$$

Taking $\psi = 0$ in (3.2.5), then from the definitions of $\xi_{i\alpha}^\varepsilon$ and ξ_{i3}^ε it is apparent that

$$
\int_\Omega \xi_{i\alpha}^\varepsilon \frac{\partial \upsilon_i}{\partial z_\alpha} + (k\varepsilon)^{-1} \xi_{i3}^\varepsilon \frac{\partial \upsilon_i}{\partial z_3} = \int_\Omega f^\varepsilon \upsilon dz + (k\varepsilon)^{-1} \int_{\Gamma_\pm} g_\pm^\varepsilon \upsilon dz_1 dz_2. \tag{3.2.28}
$$

Another consequence of (3.2.16) is that

$$
\|\xi_{ij}^\varepsilon\|_{L^2(\Omega)} \leq C, \tag{3.2.29}
$$

It follows that up to a subsequence

$$
\xi_{ij}^\varepsilon \overset{L^2(\Omega)}{\rightharpoonup} \xi_{ij}^*. \tag{3.2.30}
$$

The following lemma gives the proprieties of $\xi_{i\alpha}^*$ and ξ_{i3}^* the limits of $\xi_{i\alpha}^\varepsilon$ and ξ_{i3}^ε, respectively.

Lemma 3.2.2.

$$
\begin{aligned}
&1) \quad \frac{\partial}{\partial z_\alpha}\left(\int_{-1}^1 \xi_{i\alpha}^* dz_3 \right) = \left(\int_{-1}^1 f_i^* \right) + g_{i,+}^* + g_{i,-}^* \;\; in \;\; \omega,\\
&2) \quad \xi_{i3}^* = 0 \qquad in \;\; \Omega.
\end{aligned}
\tag{3.2.31}
$$

Proof. Let $w_i \in \mathcal{D}(\Omega), i \in 1,2,3$ and set

$$\upsilon_i = (k\varepsilon)\int_0^{z_3} w_i(z_1, z_2, t)dt,$$

as a test function in (3.2.28), we obtain

$$\begin{aligned}(k\varepsilon)\int_\Omega \xi^\varepsilon_{i\alpha}\Big(\int_0^{z_3}\frac{\partial w_i(z_1,z_2,t)}{\partial z_\alpha}dt\Big)dz + \int_\Omega \xi^\varepsilon_{i3} w_i(z_1,z_2,z_3)dz \\ = (k\varepsilon)\int_\Omega f_i^\varepsilon\Big(\int_0^{z_3} w_i(z_1,z_2,t)dt\Big)dz \\ + (k\varepsilon)\int_{\Gamma_+}\big((k\varepsilon)^{-1}g_i^{+\varepsilon}\big)\Big(\int_0^{z_3} w_i(z_1,z_2,t)dt\Big)dz_1dz_2 \\ + (k\varepsilon)\int_{\Gamma_-}\big((k\varepsilon)^{-1}g_i^{-\varepsilon}\big)\Big(\int_0^{z_3} w_i(z_1,z_2,t)dt\Big)dz_1dz_2\end{aligned} \tag{3.2.32}$$

and by passing to the limit for $\varepsilon \to 0$, we get

$$\xi^*_{i3} = 0.$$

Taking now $\upsilon_i \in \mathcal{D}(\omega)$ in (3.2.28), then

$$\int_\Omega \xi^\varepsilon_{i\alpha}\frac{\partial \upsilon_i}{\partial z_\alpha}dz = \int_\Omega f_i^\varepsilon \upsilon_i dz + (k\varepsilon)^{-1}\int_\omega g_+^\varepsilon \upsilon dz_1dz_2 + (k\varepsilon)^{-1}\int_\omega g_-^\varepsilon \upsilon dz_1dz_2.$$

Integration by parts yields

$$-\int_\omega \frac{\partial}{\partial z_\alpha}\Big(\int_{-1}^1 \xi^\varepsilon_{i\alpha}dz_3\Big)\upsilon_i dz_1dz_2 = \int_\omega\Big(\int_{-1}^1 f_i^\varepsilon dz_3\Big)\upsilon_i dz_1dz_2 + (k\varepsilon)^{-1}\int_\omega\Big(g_+^\varepsilon + g_-^\varepsilon\Big)\upsilon dz_1dz_2$$

$$\forall \upsilon \in \mathcal{D}(\omega),$$

passing to the limit for $\varepsilon \to 0$ gives

$$-\frac{\partial}{\partial z_\alpha}\Big(\int_{-1}^1 \xi^*_{i\alpha}dz_3\Big) = \Big(\int_{-1}^1 f_i^*\Big) + g_+^* + g_-^* \quad in\ \omega. \tag{3.2.33}$$

□

Introduce now

$$\zeta_j^\varepsilon = P^\varepsilon_{jh\gamma}\frac{\partial u_h^\varepsilon}{\partial z_\gamma} - \epsilon^\varepsilon_{j\gamma}\frac{\partial \varphi^\varepsilon}{\partial z_\gamma} + (k\varepsilon)^{-1}\Big\{P^\varepsilon_{jh3}\frac{\partial u_h^\varepsilon}{\partial z_3} - \epsilon^\varepsilon_{j3}\frac{\partial \varphi^\varepsilon}{\partial z_3}\Big\},$$

Taking $\upsilon = 0$ in (3.2.5), then from the definitions of ζ^ε_α and ζ^ε_3 it is obvious that

$$-\int_\Omega \zeta^\varepsilon_\alpha \frac{\partial \psi}{\partial z_\alpha} - (k\varepsilon)^{-1}\zeta_3^\varepsilon\frac{\partial \psi}{\partial z_3} = \int_\Omega r^\varepsilon \psi dz, \quad \forall \psi \in \Psi. \tag{3.2.34}$$

Based on the results stated on (3.2.16), we get

$$\|\zeta_j^\varepsilon\|_{L^2(\Omega)} \leq C, \tag{3.2.35}$$

It follows that up to a subsequence

$$\zeta_j^\varepsilon \overset{L^2(\Omega)}{\rightharpoonup} \zeta_j^*. \tag{3.2.36}$$

The following lemma gives the proprieties of ζ_α^* and ζ_3^*, the limits of ζ_α^ε and ζ_3^ε, respectively.

Lemma 3.2.3.

$$\begin{aligned} &1) \quad \frac{\partial}{\partial z_\alpha}\left(\int_{-1}^{1} \zeta_\alpha^* dz_3\right) = \int_{-1}^{1} r^* dz_3, \quad in\ \omega, \\ &2) \quad \zeta_3^* = 0 \quad in\ \Omega.. \end{aligned} \tag{3.2.37}$$

Proof. Let $w \in \mathcal{D}(\Omega)$ and taking in (3.2.5)

$$\psi = (k\varepsilon)\int_0^{z_3} w(z_1, z_2, t)dt,$$

with $\upsilon = 0$, then one has

$$-(k\varepsilon)\int_\Omega \zeta_\alpha^\varepsilon\left(\int_0^{z_3} \frac{\partial w(z_1, z_2, t)}{\partial z_\alpha} dt\right)dz - \int_\Omega \zeta_3^\varepsilon w dz = (k\varepsilon)\int_\Omega r_\varepsilon\left(\int_0^{z_3} w(z_1, z_2, t)dt\right)dz. \tag{3.2.38}$$

Letting $\varepsilon \to 0$, results

$$\begin{aligned} &\int_\Omega \zeta_3^* w dz = 0, \quad \forall w \in \mathcal{D}(\Omega), \\ &\Rightarrow \boxed{\zeta_3^* = 0\ .} \end{aligned} \tag{3.2.39}$$

Taking now $\psi \in \mathcal{D}(\omega)$ in (3.2.34), then we get

$$-\int_{\Omega} \zeta_{\alpha}^{\varepsilon} \frac{\partial \psi}{\partial z_{\alpha}} dz = \int_{\Omega} r^{\varepsilon} \psi dz.$$

Integration by parts yields

$$\int_{\omega} \frac{\partial}{\partial z_{\alpha}^{\varepsilon}} \left(\int_{-1}^{1} \zeta_{\alpha} dz_3 \right) \psi dz_1 dz_2 = \int_{\omega} \left(\int_{-1}^{1} r^{\varepsilon} dz_3 \right) \psi dz_1 dz_2$$

Letting $\varepsilon \to 0,$ *leads to*

$$\begin{aligned} &\int_{\omega} \frac{\partial}{\partial z_{\alpha}} \left(\int_{-1}^{1} \zeta_{\alpha}^{*} dz_3 \right) \psi dz_1 dz_2 = \int_{\omega} \left(\int_{-1}^{1} r^{*} dz_3 \right) \psi dz_1 dz_2 \quad \forall \psi \in \mathcal{D}(\omega) \\ &\Rightarrow \frac{\partial}{\partial z_{\alpha}} \left(\int_{-1}^{1} \zeta_{\alpha}^{*} dz_3 \right) = \int_{-1}^{1} r^{*} dz_3. \end{aligned} \tag{3.2.40}$$

□

3.2.5 Application of Tartar's method

It remains to express $\xi_{i\alpha}^{*}$ and ζ_{α}^{*} in terms of u^0 and φ^0. we will apply the method of oscillating test functions due to Tartar. Let

$$\begin{aligned} \rho_h^{\varepsilon,m\gamma}(z) &= \varepsilon \chi_h^{m\gamma}(z) + \delta_{hm} z_{\gamma} \\ \Theta^{\varepsilon,m\gamma}(z) &= \varepsilon \Psi^{m\gamma}(z) \\ \pi_h^{\varepsilon,\gamma}(z) &= \varepsilon \Phi_h^{\gamma}(z) \\ I^{\varepsilon,\gamma}(z) &= = \varepsilon R^{\gamma} + z_{\gamma}, \end{aligned} \tag{3.2.41}$$

where $\left(\chi^{m\gamma}((\frac{z_1}{\varepsilon},\frac{z_2}{\varepsilon},z_3)),\Psi^{m\gamma}((\frac{z_1}{\varepsilon},\frac{z_2}{\varepsilon},z_3))\right)$ and $\left(\Phi^{\gamma}(\frac{z_1}{\varepsilon},\frac{z_2}{\varepsilon},z_3),R^{\gamma}(\frac{z_1}{\varepsilon},\frac{z_2}{\varepsilon},z_3)\right)$ the unique solutions in $H^1_\sharp(Y)$ with zero average of the cell problems $\left(P_{\chi^{m\gamma},\Psi^{m\gamma}}\right)$ and $\left(P_{\Phi^\gamma,R^\gamma}\right)$, respectively

$$\left(P_{\chi^{m\gamma},\Psi^{m\gamma}}\right)\begin{cases} -\dfrac{\partial}{\partial y_\alpha}\Big\{C_{i\alpha h\beta}(y)\dfrac{\partial}{y_\beta}\Big(\chi_h^{m\gamma}(y)+\Upsilon_h^{m\gamma}\Big)+P_{\beta i\alpha}(y)\dfrac{\partial\Psi^{m\gamma}(y)}{\partial y_\beta}\Big\}\\ -k^{-1}\dfrac{\partial}{\partial y_\alpha}\Big\{C_{i\alpha h3}(y)\dfrac{\partial\chi_h^{m\gamma}(y)}{\partial y_3}+P_{3i\alpha}(y)\dfrac{\partial\Psi^{m\gamma}(y)}{\partial y_3}\Big\}\\ -k^{-1}\dfrac{\partial}{\partial y_3}\Big\{C_{i3h\beta}(y)\dfrac{\partial}{\partial y_\beta}\Big(\chi_h^{m\gamma}(y))+\Upsilon_h^{m\gamma}\Big)+P_{\beta i3}(y)\dfrac{\partial\Psi^{m\gamma}(y)}{\partial y_\beta}\Big\}\\ -k^{-2}\dfrac{\partial}{\partial y_3}\Big\{C_{i3h3}(y)\dfrac{\partial\chi_h^{m\gamma}(y)}{\partial y_3}+P_{3i3}(y)\dfrac{\partial\Psi^{m\gamma}(y)}{\partial y_3}\Big\}=0\quad in\ Y,\\ \dfrac{\partial}{\partial y_\alpha}\Big\{P_{\alpha h\beta}(y)\dfrac{\partial}{\partial y_\beta}\Big(\chi_h^{m\gamma}(y)+\Upsilon_h^{m\gamma}\Big)-\epsilon_{\alpha\beta}(y)\dfrac{\partial\Psi^{m\gamma}(y)}{\partial y_\beta}\Big\}\\ +k^{-1}\dfrac{\partial}{\partial y_\alpha}\Big\{P_{\alpha h3}(y)\dfrac{\partial\chi_h^{m\gamma}(y)}{\partial y_3}-\epsilon_{\alpha3}(y)\dfrac{\partial\Psi^{m\gamma}(y)}{\partial y_3}\Big\}\\ +k^{-1}\dfrac{\partial}{\partial y_3}\Big\{P_{3h\beta}(y)\dfrac{\partial}{y_\beta}\Big(\chi_h^{m\gamma}(y))+\Upsilon_h^{m\gamma}\Big)-\epsilon_{3\beta}(y)\dfrac{\partial\Psi^{m\gamma}(y)}{\partial y_\beta}\Big\}\\ +k^{-2}\dfrac{\partial}{\partial y_3}\Big\{P_{3h3}(y)\dfrac{\partial\chi_h^{m\gamma}(y)}{\partial y_3}-\epsilon_{33}(y)\dfrac{\partial\Psi^{m\gamma}(y)}{\partial y_3}\Big\}=0\quad in\ Y,\\ \displaystyle\int_Y\chi^{m\gamma}=0,\int_Y\Psi^{m\gamma}=0\quad \Phi^\gamma,R^\gamma\quad y_1,y_2-periodic,\end{cases}\tag{3.2.42}$$

where

$$\Upsilon_h^{m\gamma}=\delta_{hm}y_\gamma\quad 1\le h,m\le 3.$$

And

$$\left(P_{\Phi^\gamma,R^\gamma}\right)\begin{cases} -\dfrac{\partial}{\partial y_\alpha}\Big\{C_{i\alpha h\beta}(y)\dfrac{\partial\Phi_h^\gamma(y)}{\partial y_\beta}+P_{\beta i\alpha}(y)\dfrac{\partial}{\partial y_\beta}\big(R^\gamma(y)+y_\gamma\big)\Big\}\\ -k^{-1}\dfrac{\partial}{\partial y_\alpha}\Big\{C_{i\alpha h3}(y)\dfrac{\partial\Phi_h^\gamma(y)}{\partial y_3}+P_{3i\alpha}(y)\dfrac{\partial R^\gamma(y)}{\partial y_3}\Big\}\\ -k^{-1}\dfrac{\partial}{\partial y_3}\Big\{C_{i3h\beta}(y)\dfrac{\partial\Phi_h^\gamma(y)}{\partial y_3}+P_{\beta i3}(y)\dfrac{\partial}{\partial y_\beta}\Big(R^\gamma(y)+y_\gamma\Big)\Big\}\\ -k^{-2}\dfrac{\partial}{\partial y_3}\Big\{C_{i3h3}(y)\dfrac{\partial\Phi_h^\gamma(y)}{\partial y_3}+P_{3i3}(y)\dfrac{\partial R^\gamma(y)}{\partial y_3}\Big\}=0\quad in\ Y,\\ \dfrac{\partial}{\partial y_\alpha}\Big\{P_{\alpha h\beta}(y)\dfrac{\partial\Phi_h^\gamma(y)}{\partial y_\beta}-\epsilon_{\alpha\beta}(y)\dfrac{\partial}{\partial y_\beta}\big(R^\gamma(y)+y_\gamma\big)\Big\}\\ +k^{-1}\dfrac{\partial}{\partial y_\alpha}\Big\{P_{\alpha h3}(y)\dfrac{\partial\Phi_h^\gamma(y)}{\partial y_3}-\epsilon_{\alpha3}(y)\dfrac{\partial R^\gamma(y)}{\partial y_3}\Big\}\\ +k^{-1}\dfrac{\partial}{\partial y_3}\Big\{P_{3h\beta}(y)\dfrac{\partial\Phi_h^\gamma(y)}{\partial y_\beta}-\epsilon_{3\beta}(y)\dfrac{\partial}{\partial y_\beta}\big(R^\gamma(y)+y_\gamma\big)\Big\}\\ +k^{-2}\dfrac{\partial}{\partial y_3}\Big\{P_{3h3}(y)\dfrac{\partial\Phi_h^\gamma(y)}{\partial y_3}-\epsilon_{33}(y)\dfrac{\partial R^\gamma(y)}{\partial y_3}\Big\}=0\quad in\ Y\\ \displaystyle\int_Y\Phi^\gamma=0,\int_Y R^\gamma=0\quad \Phi^\gamma,R^\gamma\quad y_1,y_2-periodic.\end{cases}\tag{3.2.43}$$

Such that $\left(\rho_h^{\varepsilon,m\gamma}(x), \Theta^{\varepsilon,m\gamma}(x)\right)$ and $\left(\pi_h^{\varepsilon,\gamma}(x), I^{\varepsilon,\gamma}(x)\right)$ are the solutions of $\left(P^\varepsilon_{\rho,\Theta}\right)$ and $\left(P^\varepsilon_{\pi,I}\right)$ respectively, i.e.

$$\left(P^\varepsilon_{\rho,\Theta}\right)\begin{cases} -\dfrac{\partial}{\partial z_j}\left\{C^\varepsilon_{ijhl}\dfrac{\partial \rho_h^{\varepsilon,m\gamma}}{\partial z_l} + P^\varepsilon_{hij}\dfrac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_h}\right\} = 0, \\ \dfrac{\partial}{\partial z_j}\left\{P^\varepsilon_{jhl}\dfrac{\partial \rho_h^{\varepsilon,m\gamma}}{\partial z_l} - \epsilon^\varepsilon_{jh}\dfrac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_h}\right\} = 0. \end{cases} \tag{3.2.44}$$

And

$$\left(P^\varepsilon_{\pi,I}\right)\begin{cases} -\dfrac{\partial}{\partial z_j}\left\{C^\varepsilon_{ijhl}\dfrac{\partial \pi_h^{\varepsilon,\gamma}}{\partial z_l} + P^\varepsilon_{hij}\dfrac{\partial I^{\varepsilon,\gamma}}{\partial z_h}\right\} = 0, \\ \dfrac{\partial}{\partial z_j}\left\{P^\varepsilon_{jhl}\dfrac{\partial \pi_h^{\varepsilon,\gamma}}{z_l} - \epsilon^\varepsilon_{jh}\dfrac{\partial I^{\varepsilon,\gamma}}{\partial z_h}\right\} = 0. \end{cases} \tag{3.2.45}$$

Lemma 3.2.4. *We have the following convergences:*

$$\begin{aligned} &1)\ \rho_h^{\varepsilon,m\gamma}(z) \overset{L^2(\Omega)}{\longrightarrow} \delta_{hm} z_\gamma, \\ &2)\ \Theta^{\varepsilon,m\gamma}(z) \overset{L^2(\Omega)}{\longrightarrow} 0, \\ &3)\ I^{\varepsilon,\gamma}(z) \overset{L^2(\Omega)}{\longrightarrow} z_\gamma, \\ &4)\ \pi_h^{\varepsilon,\gamma}(z) \overset{L^2(\Omega)}{\longrightarrow} 0. \end{aligned} \tag{3.2.46}$$

Proof. See the proof of Lemma 3.1.2. □

Set:

$$\begin{aligned} \Im^{1,\varepsilon}_{ijm\gamma} &= C^\varepsilon_{ijh\beta}\frac{\partial \rho_h^{\varepsilon,m\gamma}}{\partial z_\beta} + P^\varepsilon_{\beta ij}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_\beta} + (k\varepsilon)^{-1}\left\{C^\varepsilon_{ijh3}\frac{\partial \rho_h^{\varepsilon,m\gamma}}{\partial z_3} + P^\varepsilon_{3i3}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_3}\right\}. \\ S^{1,\varepsilon}_{jm\gamma} &= P^\varepsilon_{jh\beta}\frac{\partial \rho_h^{\varepsilon,m\gamma}}{z_\beta} - \epsilon^\varepsilon_{j\beta}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_\beta} + (k\varepsilon)^{-1}\left\{P^\varepsilon_{jh3}\frac{\partial \rho_h^{\varepsilon,m\gamma}}{z_3} - \epsilon^\varepsilon_{j3}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_3}\right\}. \end{aligned} \tag{3.2.47}$$

From (3.2.44) we see at once that $\Im^{1,\varepsilon}_{ijm\gamma}$ and $S^{1,\varepsilon}_{jm\gamma}$ verify the problems $(P_{\Im^{1,\varepsilon}_{ijm\gamma}})$ and $(P_{S^{1,\varepsilon}_{jm\gamma}})$, respectively, i. e.

$$\begin{cases} (P_{\Im^{1,\varepsilon}_{ijm\gamma}}): \quad -\dfrac{\partial \Im^{1,\varepsilon}_{i\alpha m\gamma}}{\partial z_\alpha} - (k\varepsilon)^{-1}\dfrac{\partial \Im^{1,\varepsilon}_{i3m\gamma}}{\partial z_3} = 0, \\ (P_{S^{1,\varepsilon}_{jm\gamma}}): \quad \dfrac{\partial S^{1,\varepsilon}_{\alpha m\gamma}}{\partial z_\alpha} + (k\varepsilon)^{-1}\dfrac{\partial S^{1,\varepsilon}_{3m\gamma}}{\partial z_3} = 0. \end{cases} \tag{3.2.48}$$

Multiplying the first equation of (3.2.48) by a test function $\upsilon \in V$ and the second one by $\psi \in \Psi$, we get the following variational problem

$$\begin{cases} \displaystyle\int_\Omega \Im^{1,\varepsilon}_{i\alpha m\gamma}\frac{\partial \upsilon_i}{\partial z_\alpha} + (k\varepsilon)^{-1}\int_\Omega \Im^{1,\varepsilon}_{i3m\gamma}\frac{\partial \upsilon_i}{\partial z_3} = 0, \quad \forall \upsilon \in V, \\ \displaystyle -\int_\Omega S^{1,\varepsilon}_{\alpha m\gamma}\frac{\partial \psi}{\partial z_\alpha} - (k\varepsilon)^{-1}\int_\Omega S^{1,\varepsilon}_{3m\gamma}\frac{\partial \psi}{\partial z_3} = 0, \quad \forall \psi \in \Psi. \end{cases} \tag{3.2.49}$$

Set now

$$\Im^{2,\varepsilon}_{ij\gamma} = C^{\varepsilon}_{ijh\beta}\frac{\partial \pi^{\varepsilon,\gamma}_h}{\partial z_\beta} + P^{\varepsilon}_{\beta ij}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_\beta} + (k\varepsilon)^{-1}\Big\{C^{\varepsilon}_{ijh3}\frac{\partial \pi^{\varepsilon,\gamma}_h}{\partial z_3} + P^{\varepsilon}_{\beta ij}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_3}\Big\},$$
$$S^{2,\varepsilon}_{j\gamma} = P^{\varepsilon}_{jh\beta}\frac{\partial \pi^{\varepsilon,\gamma}_h}{\partial z_\beta} - \epsilon^{\varepsilon}_{j\beta}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_\beta} + (k\varepsilon)^{-1}\Big\{P^{\varepsilon}_{jh3}\frac{\partial \pi^{\varepsilon,\gamma}_h}{\partial z_3} - \epsilon^{\varepsilon}_{j3}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_3}\Big\}. \tag{3.2.50}$$

From (3.2.45) it is a simple mater to check that $\Im^{2,\varepsilon}_{ijm\gamma}$ and $S^{2,\varepsilon}_{jm\gamma}$ verify the problems $(P_{\Im^{2,\varepsilon}_{ij\gamma}})$ and $(P_{S^{2,\varepsilon}_{j\gamma}})$, respectively, i.e.

$$\begin{cases} (P_{\Im^{2,\varepsilon}_{ij\gamma}}): & -\dfrac{\partial \Im^{2,\varepsilon}_{i\alpha\gamma}}{\partial z_\alpha} - (k\varepsilon)^{-1}\dfrac{\partial \Im^{2,\varepsilon}_{i3\gamma}}{\partial z_3} = 0, \\ (P_{S^{2,\varepsilon}_{j\gamma}}): & \dfrac{\partial S^{2,\varepsilon}_{j\gamma}}{\partial x_j} + (k\varepsilon)^{-1}\dfrac{\partial S^{2,\varepsilon}_{3\gamma}}{\partial z_3} = 0. \end{cases} \tag{3.2.51}$$

Multiplying the first equation of (3.2.51) by a test function $\upsilon \in V$ and the second one by $\psi \in \Psi$, we get the following variational problem

$$\begin{cases} \displaystyle\int_\Omega \Im^{2,\varepsilon}_{i\alpha\gamma}\frac{\partial \upsilon_i}{\partial z_\alpha} + (k\varepsilon)^{-1}\int_\Omega \Im^{2,\varepsilon}_{i3\gamma}\frac{\partial \upsilon_i}{\partial z_3} = 0, & \forall \upsilon \in V, \\ \displaystyle -\int_\Omega S^{2,\varepsilon}_{\alpha\gamma}\frac{\partial \psi}{\partial z_\alpha} - (k\varepsilon)^{-1}\int_\Omega S^{2,\varepsilon}_{3\gamma}\frac{\partial \psi}{\partial z_3} = 0, & \forall \psi \in \Psi. \end{cases} \tag{3.2.52}$$

Since $\Im^{1,\varepsilon}_{ijm\gamma}, \Im^{2,\varepsilon}_{ij\gamma}, S^{1,\varepsilon}_{jm\gamma}$ and $S^{2,\varepsilon}_{j\gamma}$ are Y-periodic, thus owing to Theorem 8 one has the following convergences

$$\begin{aligned}
\Im^{1,\varepsilon}_{ijm\gamma} \rightharpoonup \frac{1}{|Y|}\int_Y \Big(& C_{ijh\beta}(y)\frac{\partial(\chi^{m\gamma}_h + \delta_{hm}y_\gamma)}{\partial y_\beta} + P_{\beta ij}(y)\frac{\partial \Psi^{m\gamma}}{\partial y_\beta} \\
& + k^{-1}\Big\{C_{ijh3}\frac{\partial \chi^{m\gamma}_h}{\partial y_3} + P^{\varepsilon}_{3ij}\frac{\partial \Psi^{m\gamma}}{\partial z_3}\Big\}\Big)dy, \\
\Im^{2,\varepsilon}_{ij\gamma} \rightharpoonup \frac{1}{|Y|}\int_Y \Big(& C_{ijh\beta}(y)\frac{\partial \Phi^{\gamma}_h}{\partial y_\beta} + P_{\beta ij}(y)\frac{\partial(R^\gamma + y_\gamma)}{\partial y_\beta} \\
& + k^{-1}\Big\{C_{ijh3}\frac{\partial \Phi^{\gamma}_h}{\partial y_3} + P_{3ij}\frac{\partial R^\gamma}{\partial z_3}\Big\}\Big)dy. \\
S^{1,\varepsilon}_{jm\gamma} \rightharpoonup \frac{1}{|Y|}\int_Y \Big(& P_{jh\beta}(y)\frac{\partial(\chi^{m\gamma}_h + \delta_{hm}y_\gamma)}{\partial y_\beta} - \epsilon_{j\beta}(y)\frac{\partial \Psi^{m\gamma}}{\partial y_\beta} \\
& + k^{-1}\Big\{P_{jh3}\frac{\partial \chi^{m\gamma}_h}{\partial y_3} - \epsilon^{\varepsilon}_{j3}\frac{\partial \Psi^{m\gamma}}{\partial z_3}\Big\}\Big)dy, \\
S^{2,\varepsilon}_{j\gamma} \rightharpoonup \frac{1}{|Y|}\int_Y \Big(& P_{jh\beta}(y)\frac{\partial \Phi^{\gamma}_h}{\partial y_\beta} - \epsilon_{j\beta}(y)\frac{\partial(R^\gamma + y_\gamma)}{\partial y_\beta} \\
& + k^{-1}\Big\{P_{jh3}\frac{\partial \Phi^{\gamma}_h}{\partial y_3} - \epsilon^{\varepsilon}_{j3}\frac{\partial R^\gamma}{\partial z_3}\Big\}\Big)dy.
\end{aligned} \tag{3.2.53}$$

The variational equation (3.2.5) is equivalent to

$$\int_\Omega \xi^\varepsilon_{i\alpha}\frac{\partial \upsilon_i}{\partial z_\alpha} + (k\varepsilon)^{-1}\xi^\varepsilon_{i3}\frac{\partial \upsilon_i}{\partial z_3} - \int_\Omega \zeta^\varepsilon_\alpha \frac{\partial \psi}{\partial z_\alpha} - (k\varepsilon)^{-1}\zeta^\varepsilon_3\frac{\partial \psi}{\partial z_3} = \int_\Omega f^\varepsilon \upsilon dz + (k\varepsilon)^{-1}\int_{\Gamma_+\cup\Gamma_-}\left(g^\varepsilon_+ + g^\varepsilon_-\right)\upsilon dz_1 dz_2 + \int_\Omega r^\varepsilon \psi dz. \tag{3.2.54}$$

Taking in (3.2.54)

$$\upsilon_i = -w\rho_i^{\varepsilon,m\gamma},$$

where $w \in \mathcal{D}(\omega)$,

then,

$$\frac{\partial \upsilon_i}{\partial z_\alpha} = -w\frac{\partial \rho_i^{\varepsilon,m\gamma}}{\partial z_\alpha} - \frac{\partial w}{\partial z_\alpha}\rho_i^{\varepsilon,m\gamma},$$
$$\frac{\partial \upsilon_i}{\partial z_3} = -w\frac{\partial \rho_i^{\varepsilon,m\gamma}}{\partial z_3}.$$

and taking

$$\psi = w\Theta^{\varepsilon,m\gamma},$$

then,

$$\frac{\partial \psi}{\partial z_\alpha} = w\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\Theta^{\varepsilon,m\gamma},$$
$$\frac{\partial \psi}{\partial z_3} = w\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_3},$$

We obtain

$$\begin{aligned}
&-\int_\Omega \left\{C^\varepsilon_{i\alpha h\beta}\frac{\partial u^\epsilon_h}{\partial z_\beta} + P^\varepsilon_{\beta i\alpha}\frac{\partial \varphi^\varepsilon}{\partial z_\beta}\right\}\left[w\frac{\partial \rho_i^{\varepsilon,m\gamma}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\rho_i^{\varepsilon,m\gamma}\right]dz \\
&-(k\varepsilon)^{-1}\int_\Omega \left\{C^\varepsilon_{i\alpha h3}\frac{\partial u^\varepsilon_h}{\partial z_3} + P^\varepsilon_{3i\alpha}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right\}\left[w\frac{\partial \rho_i^{\varepsilon,m\gamma}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\rho_i^{\varepsilon,m\gamma}\right]dz \\
&-(k\varepsilon)^{-1}\int_\Omega \left\{C^\varepsilon_{i3h\beta}\frac{\partial u^\epsilon_h}{\partial z_\beta} + P^\varepsilon_{\beta i3}\frac{\partial \varphi^\varepsilon}{\partial z_\beta}\right\}\left(w\frac{\partial \rho_i^{\varepsilon,m\gamma}}{\partial z_3}\right)dz \\
&-(k\varepsilon)^{-2}\int_\Omega \left\{C^\varepsilon_{i3h3}\frac{\partial u^\varepsilon_h}{\partial z_3} + P^\varepsilon_{3i3}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right\}\left(w\frac{\partial \rho_i^{\varepsilon,m\gamma}}{\partial z_3}\right)dz \\
&-\int_\Omega \left\{P^\varepsilon_{\alpha h\beta}\frac{\partial u^\varepsilon_h}{\partial z_\beta} - \epsilon^\varepsilon_{\alpha\beta}\frac{\partial \varphi^\varepsilon}{\partial z_\beta}\right\}\left[w\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\Theta^{\varepsilon,m\gamma}\right]dz \\
&-(k\varepsilon)^{-1}\int_\Omega \left\{P^\varepsilon_{\alpha h3}\frac{\partial u^\varepsilon_h}{\partial z_3} - \epsilon^\varepsilon_{\alpha 3}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right\}\left[w\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\Theta^{\varepsilon,m\gamma}\right]dz \\
&-(k\varepsilon)^{-1}\int_\Omega \left\{P^\varepsilon_{3h\beta}\frac{\partial u^\varepsilon_h}{\partial z_\beta} - \epsilon^\varepsilon_{3\beta}\frac{\partial \varphi^\varepsilon}{\partial z_\beta}\right\}\left(w\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_3}\right)dz \\
&-(k\varepsilon)^{-2}\int_\Omega \left\{P^\varepsilon_{3h3}\frac{\partial u^\varepsilon_h}{\partial z_3} - \epsilon^\varepsilon_{33}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right\}\left(w\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_3}\right)dz \\
&= -\int_\Omega f^\varepsilon_i \rho_i^{\varepsilon,m\gamma} w dz - (k\varepsilon)^{-1}\int_\omega g^{\varepsilon,+}_{+,i}\rho_i^{\varepsilon,m\gamma}w - (k\varepsilon)^{-1}\int_\omega g^\varepsilon_{-,i}\rho_i^{\varepsilon,m\gamma}w + \int_\Omega r^\varepsilon w\Theta^{\varepsilon,m\gamma}dz.
\end{aligned} \tag{3.2.55}$$

Taking in the first equation of (3.2.49)

$$\upsilon_i = -w u_i^{\varepsilon}(z),$$

where $w \in \mathcal{D}(\omega)$.

then,

$$\begin{aligned}\frac{\partial \upsilon_i}{\partial z_\alpha} &= -w\frac{\partial u_i^{\varepsilon}}{\partial z_\alpha} - \frac{\partial w}{\partial z_\alpha}u_i^{\varepsilon},\\ \frac{\partial \upsilon_i}{\partial z_3} &= -w\frac{\partial u_i^{\varepsilon}}{\partial z_3}.\end{aligned}$$

and taking in the second equation of (3.2.49)

$$\psi = w\varphi^{\varepsilon},$$

then,

$$\begin{aligned}\frac{\partial \psi}{\partial z_\alpha} &= w\frac{\partial \varphi^{\varepsilon}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\varphi^{\varepsilon},\\ \frac{\partial \psi}{\partial z_3} &= w\frac{\partial \varphi^{\varepsilon}}{\partial z_3}.\end{aligned}$$

Summing the two obtained equations yields

$$\begin{aligned}
&-\int_\Omega \left\{C^{\varepsilon}_{i\alpha h\beta}\frac{\partial \rho_h^{\epsilon,m\gamma}}{\partial z_\beta} + P^{\varepsilon}_{\beta i\alpha}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_\beta}\right\}\left[w\frac{\partial u_i^{\varepsilon}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}u_i^{\varepsilon}\right]dz\\
&-(k\varepsilon)^{-1}\int_\Omega \left\{C^{\varepsilon}_{i\alpha h3}\frac{\partial \rho_h^{\varepsilon,m\gamma}}{\partial z_3} + P^{\varepsilon}_{3i\alpha}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_3}\right\}\left[w\frac{\partial u_i^{\varepsilon}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}u_i^{\varepsilon}\right]dz\\
-(k\varepsilon)^{-1}&\int_\Omega \left\{C^{\varepsilon}_{i3h\beta}\frac{\partial \rho_h^{\epsilon,m\gamma}}{\partial z_\beta} + P^{\varepsilon}_{\beta i3}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_\beta}\right\}\left(w\frac{\partial u_i^{\varepsilon}}{\partial z_3}\right)dz\\
-(k\varepsilon)^{-2}&\int_\Omega \left\{C^{\varepsilon}_{i3h3}\frac{\partial \rho_h^{\epsilon,m\gamma}}{\partial z_3} + P^{\varepsilon}_{3i3}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_3}\right\}\left(w\frac{\partial u_i^{\varepsilon}}{\partial z_3}\right)dz\\
&-\int_\Omega \left\{P^{\varepsilon}_{\alpha h\beta}\frac{\partial \rho_h^{\epsilon,m\gamma}}{\partial z_\beta} - \epsilon^{\varepsilon}_{\alpha\beta}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_\beta}\right\}\left[w\frac{\partial \varphi^{\varepsilon}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\varphi^{\varepsilon}\right]dz\\
&-(k\varepsilon)^{-1}\int_\Omega \left\{P^{\varepsilon}_{\alpha h3}\frac{\partial \rho_h^{\epsilon,m\gamma}}{\partial z_3} - \epsilon^{\varepsilon}_{\alpha 3}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_3}\right\}\left[w\frac{\partial \varphi^{\varepsilon}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\varphi^{\varepsilon}\right]dz\\
&-(k\varepsilon)^{-1}\int_\Omega \left\{P^{\varepsilon}_{3h\beta}\frac{\partial \rho_h^{\epsilon,m\gamma}}{\partial z_\beta} - \epsilon^{\varepsilon}_{3\beta}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_\beta}\right\}\left(w\frac{\partial \varphi^{\varepsilon}}{\partial z_3}\right)dz\\
&-(k\varepsilon)^{-2}\int_\Omega \left\{P^{\varepsilon}_{3h3}\frac{\partial \rho_h^{\epsilon,m\gamma}}{\partial z_3} - \epsilon^{\varepsilon}_{33}\frac{\partial \Theta^{\varepsilon,m\gamma}}{\partial z_3}\right\}\left(w\frac{\partial \varphi^{\varepsilon}}{\partial z_3}\right)dz\\
&= 0.
\end{aligned} \tag{3.2.56}$$

Subtracting (3.2.56) from (3.2.55), gives

$$\begin{aligned}
&-\int_\Omega \xi^{\varepsilon}_{i\alpha}\rho_h^{\epsilon,m\gamma}\frac{\partial w}{\partial z_\alpha}dz - \int_\Omega \zeta^{\varepsilon}_{\alpha}\frac{\partial w}{\partial z_\alpha}\Theta^{\varepsilon,m\gamma}dz + \int_\Omega \Im^{\varepsilon,1}_{i\alpha m\gamma}u_i^{\varepsilon}\frac{\partial w}{\partial z_\alpha}dz + \int_\Omega S^{1,\varepsilon}_{\alpha m\gamma}\varphi^{\varepsilon}\frac{\partial w}{\partial z_\alpha}dz\\
&= -\int_\Omega f_i^{\varepsilon}\rho_i^{\varepsilon,m\gamma}w dz - (k\varepsilon)^{-1}\int_\omega g^{\varepsilon}_{+,i}\rho_i^{\varepsilon,m\gamma}w dz_1dz_2 - (k\varepsilon)^{-1}\int_\omega g^{\varepsilon}_{-,i}\rho_i^{\varepsilon,m\gamma}w dz_1dz_2 + \int_\Omega r^{\varepsilon}w\Theta^{\varepsilon,m\gamma}dz.
\end{aligned} \tag{3.2.57}$$

Making use of the convergences (3.2.27), (3.2.30), (3.2.36), (3.2.46), (3.2.53) and the assumptions (3.2.6) together with letting $\varepsilon \to 0$, brings

$$\begin{aligned} &-\int_\omega \left(\int_{-1}^1 \xi^*_{m\alpha} dz_3\right) z_\gamma \frac{\partial w}{\partial z_\alpha} dz_1 dz_2 + \int_\omega \langle \Im^1_{i\alpha m\gamma}\rangle u_i^0 \frac{\partial w}{\partial z_\alpha} dz_1 dz_2 + \int_\omega \langle S^1_{\alpha m\gamma}\rangle \varphi^0 \frac{\partial w}{\partial z_\alpha} dz_1 dz_2 \\ &= \int_\omega \left(\int_{-1}^1 f^*_m dz_3\right) z_\gamma\, w\, dz_1 dz_2 - \int_\omega g^*_{+,m} z_\gamma\, w\, dz_1 dz_2 - \int_\omega g^*_{-,m} z_\gamma\, w\, dz_1 dz_2. \end{aligned} \tag{3.2.58}$$

Integration by parts, yields

$$\begin{aligned} &\int_\omega \frac{\partial}{\partial z_\alpha}\left(\int_{-1}^1 \xi^*_{m\alpha} dz_3\right) z_\gamma w dz_1 dz_2 + \int_\omega \int_{-1}^1 \xi^*_{m\gamma} w dz_1 dz_2 - \int_\omega \langle \Im^1_{i\alpha m\gamma}\rangle \frac{\partial u_i^0}{\partial z_\alpha} w\, dz_1 dz_2 \\ &- \int_\omega \langle S^1_{\alpha m\gamma}\rangle \frac{\partial \varphi^0}{\partial z_\alpha}\, w\, dz_1 dz_2 \\ &= \int_\omega \left(\int_{-1}^1 f^*_m dz_3\right) z_\gamma\, w\, dz_1 dz_2 - \int_\omega g^*_{+,m} z_\gamma\, w\, dz_1 dz_2 - \int_\omega g^*_{-,m} z_\gamma\, w\, dz_1 dz_2. \end{aligned} \tag{3.2.59}$$

Owing to Lemma 3.2.2, we get

$$\begin{aligned} &\int_\omega \int_{-1}^1 \xi^*_{m\gamma} w dz_1 dz_2 - \int_\omega \langle \Im^1_{i\alpha m\gamma}\rangle \frac{\partial u_i^0}{\partial z_\alpha} w\, dz_1 dz_2 - \int_\omega \langle S^1_{\alpha m\gamma}\rangle \frac{\partial \varphi^0}{\partial z_\alpha}\, w\, dz_1 dz_2 \\ &= 0, \forall \omega \in \mathcal{D}(\omega), \\ &\Rightarrow \int_{-1}^1 \xi^*_{m\gamma} dz_1 dz_2 = \langle \Im^1_{i\alpha m\gamma}\rangle \frac{\partial u_i^0}{\partial z_\alpha} + \langle S^1_{\alpha m\gamma}\rangle \frac{\partial \varphi^0}{\partial z_\alpha}. \end{aligned} \tag{3.2.60}$$

Again, using derivative on z_γ and Lemma 3.2.2, leads to

$$\langle \Im^1_{i\alpha m\gamma}\rangle \frac{\partial^2 u_i^0}{\partial z_\alpha \partial z_\gamma} + \langle S^1_{\alpha m\gamma}\rangle \frac{\partial^2 \varphi^0}{\partial z_\alpha \partial z_\gamma} = \int_{-1}^1 f^*_m dz_3 + g^*_{+,m} + g^*_{-,m} \quad in\ \omega. \tag{3.2.61}$$

Which is equivalent to

$$\begin{aligned} &\left(\frac{1}{|Y|}\int_Y C_{i\alpha h\beta}(y) \frac{\partial(\chi_h^{m\gamma} + \delta_{im} y_\gamma)}{\partial y_\beta} + P_{\beta i\alpha}(y) \frac{\partial \Psi^{m\gamma}}{\partial y_\beta} + k^{-1}\left\{C_{i\alpha h3} \frac{\partial \chi_h^{m\gamma}}{\partial y_3} + P^\varepsilon_{3i\alpha} \frac{\partial \Psi^{m\gamma}}{\partial z_3}\right\} dy_1 dy_2\right) \frac{\partial^2 u_i^0}{\partial z_\alpha \partial z_\gamma} \\ &+ \left(\frac{1}{|Y|}\int_Y P_{\alpha h\beta}(y) \frac{\partial(\chi_h^{m\gamma} + \delta_{im} y_\gamma)}{\partial y_\beta} - \epsilon_{\alpha\beta}(y) \frac{\partial \Psi^{m\gamma}}{\partial y_\beta} + k^{-1}\left\{P_{\alpha h3} \frac{\partial \chi_h^{m\gamma}}{\partial y_3} - \epsilon^\varepsilon_{\alpha 3} \frac{\partial \Psi^{m\gamma}}{\partial z_3}\right\} dy_1 dy_2\right) \frac{\partial^2 \varphi^0}{\partial z_\alpha \partial z_\gamma} \\ &= \int_{-1}^1 f^*_m dz_3 + g^{*,+}_m + g^{*,-}_m \quad in\ \omega. \end{aligned} \tag{3.2.62}$$

Taking in (3.2.54)

$$\upsilon_i = -w \pi_i^{\varepsilon,\gamma},$$

then,

$$\begin{aligned} \frac{\partial \upsilon_i}{\partial z_\alpha} &= -w \frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_\alpha} - \frac{\partial w}{\partial z_\alpha} \pi_i^{\varepsilon,\gamma}, \\ \frac{\partial \upsilon_i}{\partial z_3} &= -w \frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_3}. \end{aligned}$$

and taking

$$\psi = wI^{\varepsilon,\gamma}$$

then,

$$\frac{\partial \psi}{\partial z_\alpha} = w\frac{\partial I^{\varepsilon,\gamma}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha} I^{\varepsilon,\gamma},$$
$$\frac{\partial \psi}{\partial z_3} = w\frac{\partial I^{\varepsilon,\gamma}}{\partial z_3}.$$

We obtain

$$\begin{aligned}
&- \int_\Omega \left\{ C^\varepsilon_{i\alpha h\beta}\frac{\partial u^\epsilon_h}{\partial z_\beta} + P^\varepsilon_{\beta i\alpha}\frac{\partial \varphi^\varepsilon}{\partial z_\beta}\right\}\left[w\frac{\partial \pi^{\varepsilon,\gamma}_i}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\pi^{\varepsilon,\gamma}_i\right]dz \\
&- (k\varepsilon)^{-1}\int_\Omega \left\{ C^\varepsilon_{i\alpha h3}\frac{\partial u^\varepsilon_h}{\partial z_3} + P^\varepsilon_{3i\alpha}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right\}\left[w\frac{\partial \pi^{\varepsilon,\gamma}_i}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\pi^{\varepsilon,\gamma}_i\right]dz \\
&-(k\varepsilon)^{-1}\int_\Omega \left\{ C^\varepsilon_{i3h\beta}\frac{\partial u^\epsilon_h}{\partial z_\beta} + P^\varepsilon_{\beta i3}\frac{\partial \varphi^\varepsilon}{\partial z_\beta}\right\}\left(w\frac{\partial \pi^{\varepsilon,\gamma}_i}{\partial z_3}\right)dz \\
&-(k\varepsilon)^{-2}\int_\Omega \left\{ C^\varepsilon_{i3h3}\frac{\partial u^\varepsilon_h}{\partial z_3} + P^\varepsilon_{3i3}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right\}\left(w\frac{\partial \pi^{\varepsilon,\gamma}_i}{\partial z_3}\right)dz \\
&- \int_\Omega \left\{ P^\varepsilon_{\alpha h\beta}\frac{\partial u^\varepsilon_h}{\partial z_\beta} - \epsilon^\varepsilon_{\alpha\beta}\frac{\partial \varphi^\varepsilon}{\partial z_\beta}\right\}\left[w\frac{\partial I^{\varepsilon,\gamma}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}I^{\varepsilon,\gamma}\right]dz \\
&- (k\varepsilon)^{-1}\int_\Omega \left\{ P^\varepsilon_{\alpha h3}\frac{\partial u^\varepsilon_h}{\partial z_3} - \epsilon^\varepsilon_{\alpha 3}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right\}\left[w\frac{\partial I^{\varepsilon,\gamma}}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}I^{\varepsilon,\gamma}\right]dz \\
&- (k\varepsilon)^{-1}\int_\Omega \left\{ P^\varepsilon_{3h\beta}\frac{\partial u^\varepsilon_h}{\partial z_\beta} - \epsilon^\varepsilon_{3\beta}\frac{\partial \varphi^\varepsilon}{\partial z_\beta}\right\}\left(w\frac{\partial I^{\varepsilon,\gamma}}{\partial z_3}\right)dz \\
&- (k\varepsilon)^{-2}\int_\Omega \left\{ P^\varepsilon_{3h3}\frac{\partial u^\varepsilon_h}{\partial z_3} - \epsilon^\varepsilon_{33}\frac{\partial \varphi^\varepsilon}{\partial z_3}\right\}\left(w\frac{\partial I^{\varepsilon,\gamma}}{\partial z_3}\right)dz \\
&= -\int_\Omega f^\varepsilon_i\pi^{\varepsilon,\gamma}_i w dz - (k\varepsilon)^{-1}\int_\omega g^\varepsilon_{+,i}\pi^{\varepsilon,\gamma}_i w\ dz_1dz_2 - (k\varepsilon)^{-1}\int_\omega g^\varepsilon_{-,i}\pi^{\varepsilon,\gamma}_i w dz_1dz_2 + \int_\Omega r^\varepsilon I^{\varepsilon,\gamma}w dz.
\end{aligned} \tag{3.2.63}$$

Taking in the first equation of (3.2.52)

$$\upsilon_i = -wu^\varepsilon_i(z),$$

then

$$\frac{\partial \upsilon_i}{\partial z_\alpha} = -w\frac{\partial u^\varepsilon_i}{\partial z_\alpha} - \frac{\partial w}{\partial z_\alpha}u^\varepsilon_i,$$
$$\frac{\partial \upsilon_i}{\partial z_3} = -w\frac{\partial u^\varepsilon_i}{\partial z_3}.$$

and taking in the second equation of (3.2.49)

$$\psi = w\varphi^\varepsilon,$$

then,

$$\frac{\partial \psi}{\partial z_\alpha} = w\frac{\partial \varphi^\varepsilon}{\partial z_\alpha} + \frac{\partial w}{\partial z_\alpha}\varphi^\varepsilon,$$
$$\frac{\partial \psi}{\partial z_3} = w\frac{\partial \varphi^\varepsilon}{\partial z_3}.$$

Summing the two obtained equations yields

$$
\begin{aligned}
&-\int_\Omega \left\{C^\varepsilon_{i\alpha h\beta}\frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_\beta}+P^\varepsilon_{\beta i\alpha}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_\beta}\right\}\left[w\frac{\partial u_i^\varepsilon}{\partial z_\alpha}+\frac{\partial w}{\partial z_\alpha}u_i^\varepsilon\right]dz\\
&-(k\varepsilon)^{-1}\int_\Omega \left\{C^\varepsilon_{i\alpha h3}\frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_3}+P^\varepsilon_{3i\alpha}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_3}\right\}\left[w\frac{\partial u_i^\varepsilon}{\partial z_\alpha}+\frac{\partial w}{\partial z_\alpha}u_i^\varepsilon\right]dz\\
&-(k\varepsilon)^{-1}\int_\Omega \left\{C^\varepsilon_{i3h\beta}\frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_\beta}+P^\varepsilon_{\beta i3}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_\beta}\right\}\left(w\frac{\partial u_i^\varepsilon}{\partial z_3}\right)dz\\
&-(k\varepsilon)^{-2}\int_\Omega \left\{C^\varepsilon_{i3h3}\frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_3}+P^\varepsilon_{3i3}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_3}\right\}\left(w\frac{\partial u_i^\varepsilon}{\partial z_3}\right)dz\\
&-\int_\Omega \left\{P^\varepsilon_{\alpha h\beta}\frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_\beta}-\epsilon^\varepsilon_{\alpha\beta}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_\beta}\right\}\left[w\frac{\partial \varphi^\varepsilon}{\partial z_\alpha}+\frac{\partial w}{\partial z_\alpha}\varphi^\varepsilon\right]dz \qquad (3.2.64)\\
&-(k\varepsilon)^{-1}\int_\Omega \left\{P^\varepsilon_{\alpha h3}\frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_3}-\epsilon^\varepsilon_{\alpha 3}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_3}\right\}\left[w\frac{\partial \varphi^\varepsilon}{\partial z_\alpha}+\frac{\partial w}{\partial z_\alpha}\varphi^\varepsilon\right]dz\\
&-(k\varepsilon)^{-1}\int_\Omega \left\{P^\varepsilon_{3h\beta}\frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_\beta}-\epsilon^\varepsilon_{3\beta}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_\beta}\right\}\left(w\frac{\partial \varphi^\varepsilon}{\partial z_3}\right)dz\\
&-(k\varepsilon)^{-2}\int_\Omega \left\{P^\varepsilon_{3h3}\frac{\partial \pi_i^{\varepsilon,\gamma}}{\partial z_3}-\epsilon^\varepsilon_{33}\frac{\partial I^{\varepsilon,\gamma}}{\partial z_3}\right\}\left(w\frac{\partial \varphi^\varepsilon}{\partial z_3}\right)dz\\
&=0.
\end{aligned}
$$

Subtracting (3.2.64) from (3.2.63), gives

$$
\begin{aligned}
&-\int_\Omega \xi^\varepsilon_{i\alpha}\pi_i^{\varepsilon,\gamma}\frac{\partial w}{\partial z_\alpha}dz-\int_\Omega \zeta^\varepsilon_\alpha\frac{\partial w}{\partial z_\alpha}I^{\varepsilon,\gamma}dz+\int_\Omega \Im^{\varepsilon,2}_{i\alpha\gamma}u_i^\varepsilon\frac{\partial w}{\partial z_\alpha}dz+\int_\Omega S^{2,\varepsilon}_{\alpha\gamma}\varphi^\varepsilon\frac{\partial w}{\partial z_\alpha}dz\\
&=-\int_\Omega f_i^\varepsilon\pi_i^{\varepsilon,\gamma}wdz-(k\varepsilon)^{-1}\int_\omega g^\varepsilon_{+,i}\pi_i^{\varepsilon,\gamma}wdz_1dz_2-(k\varepsilon)^{-1}\int_\omega g^\varepsilon_{-,i}\pi_i^{\varepsilon,\gamma}wdz_1dz_2+\int_\Omega r^\varepsilon wI^{\varepsilon,\gamma}dz.
\end{aligned}
\qquad (3.2.65)
$$

Making use of the convergences (3.2.27), (3.2.30), (3.2.36), (3.2.46), (3.2.53) and the assumptions (3.2.6) together with letting $\varepsilon\to 0$, brings

$$
\begin{aligned}
&-\int_\omega\left(\int_{-1}^1\zeta^*_\alpha dz_3\right)z_\gamma\ \frac{\partial w}{\partial z_\alpha}dz_1dz_2+\int_\omega\langle\Im^2_{i\alpha\gamma}\rangle u_i^0\ \frac{\partial w}{\partial z_\alpha}dz_1dz_2+\int_\omega\langle S^2_{\alpha\gamma}\rangle\varphi^0\ \frac{\partial w}{\partial z_\alpha}\ dz_1dz_2\\
&=\int_\omega\left(\int_{-1}^1 r^*dz_3\right)z_\gamma\ w\ dz_1dz_2.
\end{aligned}
\qquad (3.2.66)
$$

Integration by parts, yields

$$
\begin{aligned}
&\int_\omega\frac{\partial}{\partial z_\alpha}\left(\int_{-1}^1\zeta^*_\alpha dz_3\right)z_\gamma wdz_1dz_2\ +\int_\omega\int_{-1}^1\zeta^*_\gamma wdz_1dz_2-\int_\omega\langle\Im^2_{i\alpha\gamma}\rangle\ \frac{\partial u_i^0}{\partial z_\alpha}w\ dz_1dz_2-\int_\omega\langle S^2_{\alpha\gamma}\rangle\frac{\partial\varphi^0}{\partial z_\alpha}\ w\ dz_1dz_2\\
&=\int_\omega\left(\int_{-1}^1 r^*dz_3\right)z_\gamma\ w\ dz_1dz_2.
\end{aligned}
\qquad (3.2.67)
$$

Owing to Lemma 3.2.3, we get

$$\int_\omega \int_{-1}^{1} \zeta_\gamma^* w dz_1 dz_2 - \int_\omega \langle \Im^2_{i\alpha\gamma} \rangle \; \frac{\partial u_i^0}{\partial z_\alpha} w \; dz_1 dz_2 - \int_\omega \langle S^2_{\alpha\gamma} \rangle \frac{\partial \varphi^0}{\partial z_\alpha} \; w \; dz_1 dz_2$$
$$= 0, \forall \omega \in \mathcal{D}(\omega), \tag{3.2.68}$$
$$\Rightarrow \int_{-1}^{1} \zeta_\gamma^* w dz_1 dz_2 = \langle \Im^2_{i\alpha\gamma} \rangle \; \frac{\partial u_i^0}{\partial z_\alpha} + \langle S^2_{\alpha\gamma} \rangle \frac{\partial \varphi^0}{\partial z_\alpha}.$$

Again, using derivative on z_γ and Lemma 3.2.3, leads to

$$\langle \Im^2_{i\alpha\gamma} \rangle \frac{\partial^2 u_i^0}{\partial z_\alpha \partial z_\gamma} + \langle S^2_{\alpha\gamma} \rangle \frac{\partial^2 \varphi^0}{\partial z_\alpha \partial z_\gamma} = \int_{-1}^{1} r^* dz_3 \quad in \; \omega. \tag{3.2.69}$$

Which is equivalent to

$$\left(\frac{1}{|Y|} \int_Y \left(C_{i\alpha h\beta}(y) \frac{\partial \Phi_h^\gamma}{\partial y_\beta} + P_{\beta i\alpha}(y) \frac{\partial (R^\gamma + y_\gamma)}{\partial y_\beta} + k^{-1} \left\{ C_{i\alpha h3} \frac{\partial \Phi_h^\gamma}{\partial y_3} + P_{3i\alpha} \frac{\partial R^\gamma}{\partial z_3} \right\} \right) dy_1 dy_2 \right) \frac{\partial^2 u_i^0}{\partial z_\alpha \partial z_\gamma}$$
$$+ \left(\frac{1}{|Y|} \int_Y \left(P_{\alpha h\beta}(y) \frac{\partial \Phi_h^\gamma}{\partial y_\beta} - \epsilon_{\alpha\beta}(y) \frac{\partial (R^\gamma + y_\gamma)}{\partial y_\beta} + k^{-1} \left\{ P_{\alpha h3} \frac{\partial \Phi_h^\gamma}{\partial y_3} - \epsilon^\varepsilon_{\alpha 3} \frac{\partial R^\gamma}{\partial z_3} \right\} dy_1 dy_2 \right) \frac{\partial^2 \varphi^0}{\partial z_\alpha \partial z_\gamma}$$
$$= \int_{-1}^{1} r^* dz_3 \; in \; \omega. \tag{3.2.70}$$

□

CONCLUSION

In the first part of the present work, we have outlined the error estimates of the third order with and without the third-order boundary layer corrector for the classical problem of homogenization, in a bounded domain of $\mathbb{R}^n$, as a first step, the comparison of our obtained results and the previous findings has led us to conclude that the correctors do not influence in the improvement of the error estimate order, however, the boundary layer correctors do.

As a second step, we tried to answer the question: if we assume minimal regularity assumptions, can one obtain the third-order error estimates as stated in Theorems 20 and 21? effectively, we have succeeded to carry out the error estimates of the third-order with and without the third-order boundary layer corrector under minimal regularity assumptions on the solution of the homogenized problem u_0 in two-dimension, using the mixed method.

In the second part, we have started by describing the homogenized problem and the convergence of the solution by using the energy method of Tartar, for a 3D-piezoelectric structure as a first step, in the second step, we have done our study on periodic, heterogeneous and non-isotropic plate, and we have approached the three-dimensional piezoelectric equation by two-dimensional one set on the middle of the plate, the key which allowed as to do such passage, is to take the thickness η and the period ε of the same order, then, tending one of them to zero, give the desired limit. Note that our tow-dimensional piezoelectric equation is very different from that obtained in the literature (in the piezoelectric and the elasticity two-dimensional equations, see for instance [77] and the references therein for the piezoelectric case, and [30] for elasticity equation problem), where the two-dimensional equation is always divided into a two-dimensional membrane and flexural equations and the displacement converges to the Kirchhoff-Love displacement field, hence our results given a new model of two-dimensional equation of piezoelectric problem, which led us to deduce that to approach a three-dimensional piezoelectric plate by a two-dimensional one, is not necessary to obtain a Kirchhoff-Love displacement nor membrane and flexural equations. Notice that in this study we have also used the energy method of Tartar.

As future work, we hope to extend the error estimates obtained in two dimensions into the hole $\mathbb{R}^n$, for the first part, for the second part, we will attempt to applicate the results obtained in the first part on a piezoelectric plate model. Finlay, we hope that our researches will serve as a base for future studies either on the boundary layers or on the homogenization of the piezoelectric plate.

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