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presented by:

Khadir Djamila Mayssoune

Supervisor:

M. Badidja Salim

Theme

Fibonacci Numbers and Polynomials

Represented 19/06/2022 Limb from jury:

Mr.M.T.Benmoussa	MCB, Kasdi Merbah - Ouargla	President
Mr.Y.Guerboussa	MCA, Kasdi Merbah - Ouargla	Examiner
Mr.B.Mittou	MAA, Kasdi Merbah - Ouargla	Examiner
Mr.S.Badidja	MCA, Kasdi Merbah - Ouargla	Supervisor

Dedicate

I dedicate my efforts and the harvest of my accademic years to the candle of my path and the light of my life (my parents), and to the dearest person in my life, to the one who was the reason for continuing my studies, to the one who gave me strength and determination, to my sister and my second soul *Ms. Hadda Khadir.

To my brothers and sisters, may Allah protect them, to my classmates and to my dear colleague *Hand Khawla.

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Abstract
In this thesis, we are interested in the study of Fibonacci numbers and polynomials, where
we will deal with the concept of each and some important proprties such as Binet's formula
and the generation function,then we make a generalization for the two (generalized Fibonacci
squence and generalized Fibonacci polynomial).We end our study with some applications.
Key words: Fibonacci numbers, Fibonacci polynomials, Binet's formula, generation function, generalized Fibonaccisequance, generalized Fibonaccipolynomials, Q-matrix.

Résumé
Dans ce mémoir,nous nous intédessons à l'étude des nombres et des polynomes de Fibonacci,
ou nous allons traité du concept de chacun et de certaines propriétés importantes telles
que la formule de Bienet et la fonction de génération puis nous faisons une généralisons pour
les deux(suite de Fibonacci généralisée et polynome de Fibonacci généralisée).Nous terminons
notre étude par quelques application.
Mots clé:nombres des Fibonacci, les polynomes de Fibonacci, formule de Bienet, fonction de génération, généralisons de Fibonacci, généralisons polynomes de Fibonacci, Q-matrice.

ملخص
في هذه المذكرة نهتم بدراسة أرقام وكثيرات حدود فيبوناتشي، حيث تناولنا مفهوم كل منهما وبعض الخصائص الهامة مثل صيغة بينيه ودالة التوليد وتعميم لكليهما (تسلسل فيبوناتشي المعمم ومتعدد حدود فيبوناتشي المعمم) وختمنا هذه الدراسة ببعض التطبيقات.
الكلمات المفتاحية: أرقام وكثيرات حدود فيبوناتشي، صيغة بينيه، دالة التوليد، تعميم أرقام فيبوناتشي، تعميم متعدد حدود فيبوناتشي، Qمصفوفة.

Contents

Symbols

In this paragraph, we state a list of notations and conventions which will not be explicity defined in this thesis:

1. E Category of commutative fillds of characteristic zero.
2. E_0 Se quence with values in A.
3. A[[x]] Formal series with coefficients in A.
4. $Jl_0(A)$ The algebra of formal series defined.
5. $r(k)$ Set of sequence of E_0 satisfying linear recurrences with constant coefficients.
6. $s.l.c$ Linear recurrent sequence with constant coefficients.
7. $r.l.p$ Linear recurrent sequence with polynomial coefficients. .
8. h Length of the sequence a.
9. F_n Term of the Fibonacci sequence .
10. α Positive root of the quadratic equation $x^2 - x - 1$.
11. β Negative root of the quadratic equation $x^2 - x - 1$.
12. $g(x)$ Geneating function.
13. GFS Generalized Fibonacci sequence .
14. G_n General term of generalized Fibonacci sequence .
15. PMI principle of mathematical induction.
16. $f_n(x)$ General term of Fibonacci polynomial.
17. R_1 Positive root of the quadratic equation $x^2 = xt + 1$.
18. R_2 Negative root of the quadratic equation $x^2 = xt + 1$.

Introduction

Leonardo Fibonacci $(1175 - 1250)$ was one of the most prominent mathematicians of the Middle Ages in Europe.His real name Leonardo Guliclm,he was known where the name Fibonacci,which means the son of Bonacci (Filis Bonaccio).The name that was attached to him after his death.

He was educated in the Algerian city of Bejaia,where he mastered the numbering system and the techniques of Indo-Arabic arithmetic there,then was sent to Egypt,Syria,Greece,France ,Constantinople and others to study the different arithmetic systems used at the time.

Among his most prominent works :
His famous book(Liber Abaci)which he published in 1202 through which he contributed to the dissemination of Arabic numerals in Europe.

The book (Libre Quardratorum) in the year 1225 or what is known as the Book of Squares.

The book (Parctice Geometion) in the year 1228,in which he presented a solution to many mathematical problems.Among his works that were attributed to him is the Fibonacci sequence or Fibonacci numbers,a sequence in which the term equals the sum of the previous two terms.Therefore,the first terms of the sequence are as follows $0; 1; 1; 2; 3; 5; 8; 13$....,but some schools omitted the 0term and replaced it with 1twice.In both cases,each term remains the sum of the two preceding terms.Therefore,we know the sequence F_n of the Fibonacci number with the following recurrence relation: $F_n = F_{n-1} + F_{n-2}$ considering $F_1 = F_2 = 1$.

The ratio $\frac{\alpha^n - \beta^n}{\alpha - \beta}$ or what is known as Binet's formula expresses the term of the sequence of degree n and it is clear from it that the ratio between two successive terms of the sequence when n goes to infinity devolves to1.618which is what is called the golden ratio.

In the first part we give some reminders about algebra r(K)and then introduce a new definition of r(K) and j(K)and the study the set of j(K).

In the second part of this study,we present the basic theories of the sequence such as Binet's formula and Generalized Fibonacci sequence, giving some interesting properties.

In the last part,we drop what was discussed earlier on the Fibonacci polynomial $F_n(x)$.

Leonardo Fibonacci,the ltalin mathematician,first introduced the Fibonacci sequence to the wast in the 13thcenturt.Thefibonacci number series contains unique mathematical properties and relationships that can be found today in fields as diverse as biology,physics,chemistry,electrical engineering,neurophysiology,art,music and more.Fields of life.
We will show simple examples of fibonacci numbers in nature.

Fibonnaci numbers in nature:

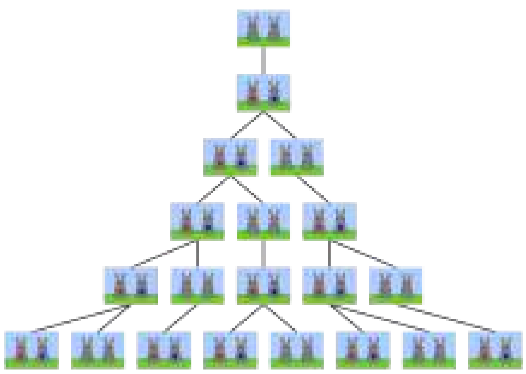
Fibonacci and the rabbit proplem:[8]

Thefollowing table shows one of the most famous primary problems (the rabbit problem).

Number of pairs	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug
Adults	0	1	1	2	3	5	8	13
Babies	1	0	1	1	2	3	5	8
Total	1	1	2	3	5	8	13	21

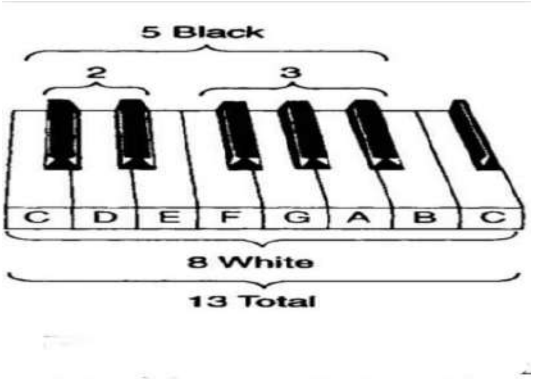
Suppose there are two newborn rabbits,one male and the other female.Let's find the numberof rabbits produced in a year if :

1. each pair takes one month to become mature.
2. each pairproduces a mixed pair every month,from the second month on.
3. no rabbits die during the course of the year.
   Suppose ,for convenience,that the original pair of rabbits was born on january 1.They take a month to become mature,so there is still only one pair on february1.On march 1,they are two months old and produce a new pair,a total of two pairs.Continuing like this,there will be three pairs on april 1, five pairs on may 1,and so on(as shown in the table above).



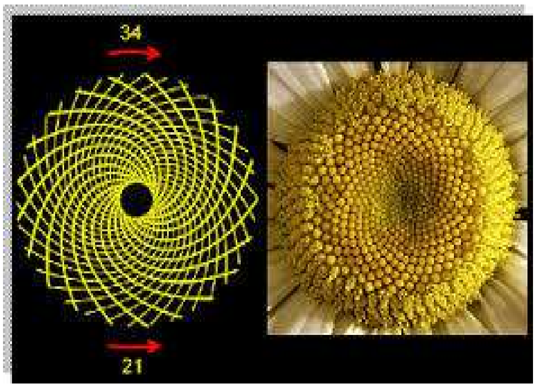
Fibonacci and music:[8]

The piano keyboard provides a great visual illustration of the link between Fibonacci numbers and music,where the octave is divided into 5 black and 8 white keys,with a total of 13 keys, the 5 black keys from two groupe,on of two keys and the other of 3 keys.
pianos are often tuned to a standard of 440 cycles per second,where(440=8.55).



Fibonacci and sunflowers:[8]

Mature sunflowers display Fibonacci numbers in a unique and remarkabie way.The seeds of the flower are tightly packed in two distinct spirals,emanating from the center of the head to the outer edge.One goes clockwise and the other counterclockwise.Studies have shown that although there are exceptions,the number of spirals, by and large,is adjacent Fibonacci numbers;usually they are 34 and 55.Hoggatt reports alarge sunflower with 89 spirals in the clockwise direction and 55 in the opposite direction,and a giganticflower with 144 spirals clockwise and 89 counterclockwise.



Chapter 1

Preliminaries

In this first chapter,we collect some basic tools and findings that will be used later.This part is only intended to provide the reader with the correct follow-up of this work,and therefore,the reader can dispence with reading this chapter,and return only to it when necessary.

1.1 Reminders

1.1.1 Hadamard algebra

Definition 1.1.1. [1] *Let A be a commutative ring with unith we denote by E_0 the set of A^N of all sequences in A; the latter is an A-algeber undes the asual operatiaus:*

$$\begin{cases} (a_n) + (b_n) = (a_n + b_n) \\ \alpha(a_n) = \alpha(a_n) \\ (a_n)(b_n) = (a_n b_n) \end{cases} \tag{1.1}$$

for $(a_n), (b_n) \in E_0$ and $\alpha \in A$.
For everg sequence $a \in E_0$,we associate a formal soures series
$f_a(x) = \sum a_n x^n$; So $f_a \in A[[x]]$
If we define on $A[[x]]$ the Hadamard product by $\sum a_n x^n \sum b_n x^n = \sum a_n b_n x$.
f:$E_0 \to A[[x]]$; define clearlg au isomorphism the A-module $A[[x]]$ and of $A-$algebra with the Hadamard product.

1.2 The linear recurrent sequence with constant coefficients

Definition 1.2.1. *Let $a \in E$,a is saist a linear recurrent sequence with constant coefficients*

$$\begin{cases} h \in N \\ P_0, ..., P_h \in k \\ P_h \neq 0 \\ P_0 \neq 0 \end{cases} \tag{1.2}$$

Then: $P_h a(n+h) + P_{h-1} a(n+h-1) + ... + P_0 a(n) = 0 \quad \forall n \in N.$

Theorem 1.2.1. *(Generation series)*
The generation series for a in K,is given: $f_a(x) = \sum_{n\geq 0} a(n)x^n$, *where a is sequence of* E.

If $f_a(x) = \frac{A(x)}{B(x)}$. *So* $a \in r(K)$.

Note 1.2.1. $r(k) = \{a \in E; a(n)\ check\ s.l.c\}$.

1.2.1 Linear operator T

Definition 1.2.2. [1] *Consider the linear operator* $T \in L_k(E)$ *defined by,* $(T_a)(a) = a(n+1)$

property 1.2.1. *T is an endomorphism of K-algebra, we have* $(T_a^n)(n) = a(n+h)$ *; where n and h in* N.
E *is* $k[T]$*module on the left.*

1.2.2 The algebra $r(k)$

Definition 1.2.3. $r(k) = \{a \in E;\ Ann(a) \neq 0\}$.

property 1.2.2. $r(k)$ *is a* $k-$*algebra.*

Theorem 1.2.2. *Let a be a sequence in* $r(k)$ *of length h and d a positive integer.*
Then for all $r \in \{0, 1, ...d-1\}$
The sequence $b(k) = a(kd+r)$ *is in* $r(k)$ *is greater than* h.

Proof. Suppose that $a \in r(k)$ in the $a(n)$ are written in the forme of :$a(n) = \sum_{i=1}^{n} p_i(n)\alpha_1^n$ and $d_p = m_i - 1;\forall n \geq 0$ and for:

$$\begin{cases} n = kd + r \\ r = 0, 1, ..., d-1 \end{cases}$$

We will have the $b(k)$ in the form:

$$\begin{cases} b(k) = \sum_{i=1}^{s} q_i(k)\beta_i^r \\ d_{p_i}^0 \leq m_i - 1 \\ q_i(k) = p_i(kd+r)\alpha_i^r \\ \beta_i = \alpha_i^r \end{cases}$$

Definition 1.2.4. a_n *is a a linear recurrent sequence with polynomials cofficients if*

$$\begin{cases} h \in N \\ q_0, ..., q_h \in P \end{cases} \tag{1.3}$$

such that, $\sum_{i=0}^{h} q_{h-i}(n)a_{(n+h-i)} = q_h(n)a_{(n+h)} + q_{h-1}a_{(n+h-1)} + ... + q_0(n)a(n)$

By sequence $b \in r(k)$ of length h. □

1.3 The linear recurrent sequence with polynomial coefficients

Theorem 1.3.1. *(Generation series) be a sequence in E and $f_a(x) = \sum_0 a(n)x^n$ its associated generation series in $k[[x]]$. Then a belongs to $j(K)$ if only if f_a is the solution of a homogeneons lineiar differential equation with polynomial coefficients.*

Note 1.3.1. $j(k) = \{a \in E; a(n)\ check\, r.l.p\}$.

1.3.1 New definition of $j(k)$

Definition 1.3.1. (*The linear operator q*) *Consider the linear operator $q \in L_k(E)$ defined by: $(qa)(n) = na(n)$.*

property 1.3.1. *we have,*

1. $T_q = (q+1)T$
2. *Let $A = K[q,T]$;$\forall f(q,T) \in A$. We have:*

$$\begin{aligned} f(q,T) &= A_0(q) + A_1(q)T + ... + A_h(q)T^h; A_1 \in k[x] \\ &= B_0(q) + TB_1(q) + ... + T^h B_n(q); B_1 \in K[x]. \\ &= A'_0(T) + A'_1(T)q + ... + A'_r(T)q^r; A'_1 \in k[T]. \\ &= B'_0(T) + qB'_1(T) + ... + q^r B'_r(T); B'_1 \in k[T]. \end{aligned}$$

3. *A is a non commutative ring.*
4. *E is a A-module on the left.*

Definition 1.3.2. (*The algebra $K_1[T]$*) *Denote by $K_1(q)$ the field of fraction of the integral domain $K[q]$ hence evry $\lambda \in k_1$ can be written as $\lambda = \frac{A(q)}{B(q)}$, where A, B in $K[q]$.*
For $a \in E, (\lambda a)(n) = \lambda a(n) = \frac{A(n)}{B(n)} a(n)$ is not defined for all n bst the set of these singular n is finite. Hence k_1 acts on E by, $(\lambda, a) \longrightarrow \lambda.a$ where $(\lambda a)(n) = \lambda(n)a(n)$, for $n \geq n_0$. (n_0 is large enough to avoid nugularities)

property 1.3.2. *The ring $K_1[T]$ is a domain, and left euclidian.*

1.3.2 The algebra $j(k)$

Definition 1.3.3. $j(k) = \{a \in E; Ann(a) \neq 0\}$.

property 1.3.3. *$j(k)$ is a K-algebra and $r(k)$ is a sub-algebra of $j(k)$.*

Theorem 1.3.2. *Let a be a sequence in j(k)of length h and d a positive integer.*
Then for all $r \in 0, 1, ..., d-1$, the sequence $b(k) = a(kd + r)$ is in $j(k)$ of is greater than equal to h.

Theorem 1.3.3. *If $a \in j(k)$, then T_a and $q_a \in j(k)$.*

Proof. The proof only requires the definitions of the linear operators T and q. □

Lemma 1.3.1. *Let $f_a \in j(k)$the generating series associated with the sequence a,and we consider the function $\delta_{d,r}$ defined by:* $\delta_{d,r} = \sum_k x^{kd+r}$

$\delta_{d,r}(x) = x^r \sum_{k\geq 0}((x^d))^k = \dfrac{x^r}{1-x^d} \in R(k).$

As a result $\delta_{d,r} \in j(k)$. Let's put: $f_a(x) = \sum_{n\geq 0} a(n)x^n$,then the product de Hadamard of f_a by $\delta_{d,r}$ gives:

$$f_a(x)\delta_{d,r}(x) = \sum_{k\geq 0} a(kd+r)x^{kd+r} \in j(k)$$

Therefore $b \in j(k)$.

1.4 The units of $j(k)$

1.4.1 Hypergeometric sequences

A sequence belongin to $j(k)$ is said hypergeometric if:

$$\begin{cases} d_{p_0} = 1 \\ F_a = T - \lambda \\ a(n+1) = \lambda_0 a(n) \\ \lambda_0 = \frac{U(n)}{V(n)} \end{cases} \tag{1.4}$$

If U and V not vanish in N. We have $a(n) = \frac{U_{(n-1)}U_{(n-2)}......U_0}{V_{(n-1)}V_{(n-2)}.....V_0}a_0$

property 1.4.1. *The sequence a is said to be regular if $a(n) = 0$. $\forall n \in N$.*

Example 1.4.1. *The sequence $a(n) = n!$ of $j(k)$ is a hypergeometric.*

Theorem 1.4.1. *Let G_0 be the set of regular hypoergeometric sequence in k.Then G_0 is a sub-group of the unit group j^* of j.*

Proof. Let a and b be two sequences of G.We then have :

$$\begin{cases} a(n) \neq 0 \\ b(n) \neq 0 \\ \forall n \in N \end{cases}$$

Moreover:

$$\begin{cases} T_a = \lambda a \\ T_b = \mu b \\ where \lambda, \mu \in k(q) \end{cases}$$

Thus: as $T_{ab} = T_a T_b$ it comes while

$$\begin{aligned} T_a T_b &= \lambda a.\mu b. \\ &= (\lambda\mu)ab. \end{aligned}$$

Theere fore a,b belongs to G_0: $ab \in G_0$.
On the other hand we have : $a = T_{\frac{1}{a}} = \frac{1}{\lambda} = \frac{1}{a}$, so the inverse $\frac{1}{a}$ of a is G_0 □

Definition 1.4.1. *Let a belong to j,a is said a hypergeometric sequence nested if there esciste* $m \in N$*,such that the sequence* $a(r + tm)$ *is in* G*. For all* $r \in 0, 1, ..., m - 1$

Corollary 1.4.1. *Let* G *be the set of nested hypergeometric sequences then* G *is a sub-group of* $j^*(k)$*.*

Lemma 1.4.1. *The unit group of* $j(k)$ *is* $j^*(k) = G$*.*

Chapter 2

Fibonacci sequence

In this chapter we deal with the concept of the Fibonacci sequence,which is a sequence defined by a recurrence relation,and we also deal with its most important properties such as Binet's formula and generating function.We also generalize the Fibonacci sequence to the generalized Fibonacci sequence by giving two qualitative intial conditions.

2.1 Definition of Fibonacci sequence

Definition 2.1.1. *The sequence of Fibonacci numbers* (F_n),*is defined by the following recurrence relation:* $\forall n \geq 3, F_n = F_{n-1} + F_{n-2}$ *where* $F_1 = F_2 = 1$.
The following numbers represent :1,1,2,3,5,8,13,...is the Fibonacci sequence.

property 2.1.1. [8]

1. $\sum_{1}^{n} F_i = F_{n+2} - 1.$

2. $\sum_{1}^{n} F_{2i-1} = F_{2n}.$

3. $\sum_{1}^{n} F_{2i} = F_{2n+1} - 1.$

4. $\sum_{1}^{n} F_i^2 = F_n F_{n+1}.$

Proof. 1. Using the Fibonacci recurrence relation we have:
$F_1 = F_3 - F_2$
$F_2 = F_4 - F_3$
$F_3 = F_5 - F_4$
$F_{n-1} = F_{n+1} - F_n$
$F_n = F_{n+2} - F_{n+1}$
Adding these equations,we get: $\sum_{1}^{n} F_i = F_{n+2} - F_2 = F_{n+2} - 1.$

2. Using the Fibonacci recurrence relation we have:
   $F_1 = F_2 - F_0$
   $F_3 = F_4 - F_2$
   $F_5 = F_6 - F_4$
   $F_{2n-3} = F_{2n-2} - F_{2n-4}$
   $F_{2n-1} = F_{2n} - F_{2n-2}$
   Adding these equations,we get: $\sum_{1}^{n} F_{2i-1} = F_{2n} - F_0 = F_{2n}$.

3. By the Fibonacci recurrence relation:

$$\sum_{1}^{n} F_{2i} = \sum_{1}^{2n} F_i - \sum_{1}^{n} F_{2i-1} = (F_{2n+2} - 1) - F_{2n} = (F_{2n+2} - F_{2n}) - 1 = F_{2n+1} - 1$$

   .

4. Assume it is true for an arbitrary positive integer k; $\sum_{1}^{k} F_i^2 = F_k F_{k+1}$, Then $\sum_{1}^{k+1} F_i^2 = \sum_{1}^{k} F_i^2 + F_{k+1}^2$. So $\sum_{1}^{k+1} F_i^2 = F_k F_{k+1} + F_{k+1}^2 = F_{k+1}(F_k + F_{k+1}) = F_{k+1} F_{k+2}$.
   So the statement is true when $n = k + 1$, thus it is trus for every positive integer n.

□

2.2 Binet's formula[8]

In 1843,french mathematician Jacques Philippe Marie Binet;introduced a formula for Fibonacci numbers using the generation function.This formula is named Binet's formula in his honor,as this formula is used to find the n term of the Fibonacci sequence.

Theorem 2.2.1. *Binet's formula is given as follows :* $F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}$.
where $\alpha = \frac{1+\sqrt{5}}{2}$ *and* $\beta = \frac{1-\sqrt{5}}{2}$.

Proof. (Since we have proven this formula using a generation function,we will give a detailed explanation of what a generation function is later).
Our way of proving Binet's formula will be to find the coefficients of the Taylor series that correspond directly to the Fibonnaci numbers.
By definition we have :

$$\begin{cases} F_n = F_{n-1} + F_{n-2}. \\ F_1 = F_2 = 1. \end{cases} \tag{2.1}$$

The general coefficients of the Taylor series are given:
$F(x) = F_0 + F_1 x + F_2 x^2 + \ldots$
$-xF(x) = F_0 x - F_1 x^2 - F_2 x^3 - \ldots$
$-x^2 F(x) = F_0 x^2 - F_1 x^3 - \ldots$
Combining these equations,we get, $F(x) - xF(x) - x^2 F(x) = F_0 + (F_1 - F_0)x = F_0 = 1$
thus: $F(x) = \frac{1}{1-x-x^2} = \frac{-1}{x^2+x-1}$.
Using the quadratic equation : $x^2 + x - 1$ we find the roots $x_1 = \frac{-1+\sqrt{5}}{2}$; $x_2 = \frac{-1-\sqrt{5}}{2}$.

Then we use the partial fractions method:

$$\begin{aligned} F(x) &= \frac{-1}{x^2+x-1}. \\ &= \frac{-1}{(x-\frac{-1+\sqrt{5}}{2})(x-\frac{-1-\sqrt{5}}{2})}. \\ &= \frac{\frac{-1}{\sqrt{5}}}{x+\frac{1-\sqrt{5}}{2}} + \frac{\frac{1}{\sqrt{5}}}{x+\frac{1+\sqrt{5}}{2}}. \end{aligned}$$

We will now use more aggregate methods to complete this proof:
$\binom{-1}{k} = \frac{(-1)(-2)...(-1-k+1)}{k!} = (-1)^k$.

$$\begin{aligned} F(x) &= \frac{\frac{-1}{\sqrt{5}}}{x+\frac{1-\sqrt{5}}{2}} + \frac{\frac{1}{\sqrt{5}}}{x+\frac{1+\sqrt{5}}{2}} \\ &= \frac{-1}{\sqrt{5}}\sum_0^\infty \binom{-1}{k} x^k \left(\frac{1-\sqrt{5}}{2}\right)^{-1-k} + \frac{1}{\sqrt{5}}\sum_0^\infty \binom{-1}{k} x^k \left(\frac{1+\sqrt{5}}{2}\right)^{-1-k} \\ &= \frac{1}{\sqrt{5}}\sum_0^\infty \left[(-1)^{k+1}\left(\frac{1-\sqrt{5}}{2}\right)^{-1-k} + (-1)^k\left(\frac{1+\sqrt{5}}{2}\right)^{-1-k}\right] x^k. \\ &= \frac{1}{\sqrt{5}}\sum_0^\infty \left[\left(\frac{-2}{1-\sqrt{5}}\right)^{k+1} - \left(\frac{-2}{1+\sqrt{5}}\right)^{k+1}\right] x^k. \end{aligned}$$

We can simplify this equation to the initial form:

$$F_n = \frac{(\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n}{\sqrt{5}} = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{2.2}$$

Thus,we have proven Binet's formula using the method of generation.
For example,if we want to know the 16 term of the Fibonacci sequence without prior knowledge of the terms 14,15.

$$F_{16} = \frac{\alpha^{16} - \beta^{16}}{\alpha - \beta} = \frac{(\frac{1+\sqrt{5}}{2})^{16} - (\frac{1-\sqrt{5}}{2})^{16}}{\sqrt{5}} = 987.$$

□

Note 2.2.1. $|\beta^n| \simeq (0.618)^n < \frac{1}{2}$ *if* $n > 2$.

Rule 2.2.1. *To get* F_n *compute* $\frac{1}{\sqrt{5}}\alpha^n$ *and round* $F_n = \frac{1}{\sqrt{5}}\alpha^n$.

Rule 2.2.2. *To get* F_{n+1} *multiply* F_n *by* α *and round* $F_{n+1} = \alpha F_n$.

Proof. We have $F_n = \frac{1}{\sqrt{5}}\alpha^n$, so $F_{n+1} = \frac{1}{\sqrt{5}}\alpha^{n+1} = \alpha(\frac{1}{\sqrt{5}}\alpha^n) = \alpha F_n$ □

Example 2.2.1. *1.* $F_8 = \frac{1}{\sqrt{5}}\alpha^8 = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^8 \simeq 21.0095$. *So* $F_8 = 21$.

2. $F_9 = \alpha F_8 = 21(\frac{1+\sqrt{5}}{2}) \simeq 33.97$ *.So* $F_9 = 34$.

property 2.2.1. $F_{n+1}^2 + F_n^2 = F_{2n+1}$
$F_{n+1}^2 - F_{n-1}^2 = F_{2n}$

Proof. We have once aside:

$$\begin{aligned} F_{n+1}^2 + F_n^2 &= \left(\frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}\right)^2 + \left(\frac{\alpha^n - \beta^n}{\alpha - \beta}\right)^2 \\ &= \frac{\alpha^{2n+2} + \beta^{2n+2} + \alpha^{2n} + \beta^{2n} - 2(\alpha\beta)^{n+1} - 2(\alpha\beta)^n}{(\alpha - \beta)^2} \\ on \;\; the \;\; other \;\; hand : F_{2n+1} &= \frac{(\alpha^{2n+1} - \beta^{2n+1})(\alpha - \beta)}{(\alpha - \beta)^2} \\ &= \frac{\alpha^{2n+2} + \beta^{2n+2} - \alpha\beta^{2n+1} - \beta\alpha^{2n+1}}{(\alpha - \beta)^2} \\ &= \frac{\alpha^{2n+2} + \beta^{2n+2} - (\alpha\beta)\beta^{2n} - (\beta\alpha)\alpha^{2n}}{(\alpha - \beta)^2} \\ &= \frac{\alpha^{2n+2} + \beta^{2n+2} + \beta^{2n} + \alpha^{2n}}{(\alpha - \beta)^2} \end{aligned}$$

In the same way we prove ($F_{n+1}^2 - F_{n-1}^2 = F_{2n}$). □

Corollary 2.2.1. *We can extend the definition of Fibonacci numbers to negative numbers using the orevious theorem by following the following method.We have:*
$F_{-n} = \frac{\alpha^{-n} - \beta^{-n}}{\alpha - \beta} = \frac{\left(\frac{1}{\alpha}\right)^n - \left(\frac{1}{\beta}\right)^n}{\alpha - \beta} = \frac{(-\beta)^n - (-\alpha)^n}{\alpha - \beta} = (-1)^{n+1} \frac{\alpha^n - \beta^n}{\alpha - \beta} = (-1)^{n+1} F_n$.

Example 2.2.2. $F_{-1} = (-1)^{1+1} F_1 = 1$ $F_{-2} = -1$ $F_{-3} = 2; F_{-4} = -3$.
Thus the Fibonacci series of negative numbers is as follows: $1; -1; 2; -3; 5; -8; 13; -21...$

Theorem 2.2.2. [8] *(Binomial writing of Fibonacci numbers)*
$F_n = \frac{1}{2^{n-1}} \sum_{0}^{[\frac{n-1}{2}]} \binom{n}{2i+1} 5^i$.

Proof. we have

$$\begin{cases} 2\alpha = 1 + \sqrt{5}. \\ 2\beta = 1 - \sqrt{5}. \end{cases}$$

So:

$$\begin{cases} (2\alpha)^n = 2 \sum_{0}^{n} \binom{n}{i} (\sqrt{5})^i. \\ (2\beta)^n = 2 \sum_{0}^{n} \binom{n}{i} (-1)^i (\sqrt{5})^i. \end{cases} \tag{2.3}$$

Therfor;

$$\begin{aligned}(2\alpha)^n-(2\beta)^n &= 2\sum_{0}^{n}\binom{n}{i}(\sqrt{5})^i.\\ &= 2\sum_{0}^{[\frac{n-1}{2}]}(\sqrt{5})^{2i+1}\binom{n}{2i+1}.\\ &= 2\sqrt{5}\sum_{0}^{[\frac{n-1}{2}]}5^i\binom{n}{2i+1}.\end{aligned}$$

So this yields : $2^n(\alpha^n-\beta^n)=\sqrt{5}\sum_{0}^{[\frac{n-1}{2}]}\binom{n}{2i+1}5^i$, therfor : $\frac{\alpha^n-\beta^n}{\sqrt{5}}=\frac{1}{2^{n-1}}\sum_{0}^{[\frac{n-1}{2}]}\binom{n}{2i+1}5^i=F_n$.

□

2.3 Geneating function

In 1718,the french mathematician Abraham De Moivre. Invented generating functions in order to solve the Fibonacci recurrence relation.
Geneating functions provide a powerful tool for solving linear homogeneous recurrence relations with constant coefficients.As we saw earlier.
More generally,we make the following definition.

Definition 2.3.1. [8] *Let $a_0, a_1, a_2, ...$ be a sequence of real numbers.then the functiong $g(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n + ...$ is called the generating function for the sequence a_n.*

For example :$g(x) = 1 + 2x + 3x^2 + ... + (n + 1)x^n + ...$ is the generating functin for the sequence of positive integers.

Note 2.3.1. *We can also define generating functions for the finite sequence $a_0, a_1, ..., a_n$by letting $a_i = 0$ for $i > n$;thus $g(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ is the generating function for the finite sequence $a_0 + a_1x + ... + a_n$.*

Lemma 2.3.1. [8]*(Multiplication of generating functions) Let $f(x) = \sum_{0}^{\infty} a_nx^n$ and $g(x) = \sum_{0}^{\infty} b_nx^n$ be two generating functions.Then $f(x)g(x) = \sum_{0}^{\infty}(\sum_{0}^{n} a_ib_{n-i})x^n$.*
For exemple

$$\begin{aligned}\frac{1}{(1-x)^2} &= \frac{1}{1-x}.\frac{1}{1-x}\\ &= (\sum_{0}^{\infty}x^i)(\sum_{0}^{\infty}x^i)\\ &= \sum_{0}^{\infty}(\sum_{0}^{n}1.1)x^n\\ &= \sum_{0}^{\infty}(n+1)x^n\\ &= 1+2x+3x^2+...+(n+1)x^n+...\end{aligned}$$

Theorem 2.3.1. [8] *Applying the generating function to the Fibonacci series, we fid:*
$g(x) = F_0 + F_1x + F_2x^2 + ... + F_nx^n + ... = \sum_0^\infty F_nx^n$
is called the generating function of the Fibonacci recurrence relation.

Proof. Since the orders of $F_{n-1}$and F_{n-2} are 1 and 2 less than the order of F_n,respectively,find $xg(x)$ and $x^2g(x)$:
$xg(x) = F_1x^2 + F_2x^3 + F_3x^4 + ... + F_{n-1}x^n + ...$
$x^2g(x) = F_1x^3 + F_2x^4 + F_3x^5 + ... + F_{n-2}x^n + ...$
$g(x) - xg(x) - x^2g(x) = x$
we know that: $F_1 = F_2$ and $F_n = F_{n-1} + F_{n-2}$ thus; $(1 - x - x^2)g(x) = x$
$g(x) = \frac{x}{1-x-x^2}$

$$\begin{aligned} g(x) = \frac{x}{1-x-x^2} &= \frac{1}{\sqrt{5}}\left[\frac{1}{1-\alpha x} - \frac{1}{1-\beta x}\right] \\ &= \sqrt{5}g(x) = \frac{1}{1-\alpha x} - \frac{1}{1-\beta x} \\ &= \sum_0^{+\infty} \alpha^n x^n - \sum_0^{+\infty} \beta^n x^n \\ &= \sum_0^{+\infty} (\alpha^n - \beta^n)x^n \end{aligned}$$

So $g(x) = \sum_0^{+\infty} \frac{(\alpha^n - \beta^n)}{\sqrt{5}} x^n = \sum_0^{\infty} F_nx^n$ □

2.4 Generalized Fibonacci sequance

Definition 2.4.1. [6] *A generalized Fibonacci sequence* (G_n) *is any sequence that observes-the same recurrence equqtion as the Fibonacci numbers but has different initial conditions for integers a, b enpress as* $G_n = G_{n-1} + G_{n-2}$ *where* $G_1 = a$ *and* $G_2 = b$*.*

Corollary 2.4.1. *Using the defenition we get: a; b; a+b; a+2b; 2a+3b... this is known as the generalized Fibonacci sequence*

Theorem 2.4.1. *Let* $G_n = aF_{n-2} + bF_{n-1}\,(n \geq 3)$ *the nth term of the GFS.*

Proof. (by PMI). Since $G_3 = a + b = aF_1 + bF_2$, the statemient is true when $n = 3$
Let k be an arbitrary integer $n \geq 3$. Assume the given statement is true for all integers i. Where $3 \leq i \leq k : G_i = aF_{i-2} + bF_{i-1}.Then : G_{k+1} = G_k + G_{k-1}$

$$\begin{aligned} &= (aF_{k-2} + bF_{k-1}) + (aF_{k-3} + bF_{k-2}) \\ &= a(F_{k-2} + F_{k-3}) + b(F_{k-1} + F_{k-2}) \\ &= aF_{k-1} + bF_k \end{aligned}$$

Thus, by the(PMI)the formula holds for every integer$n \geq 3$.

Note 2.4.1. *That this theorem is in fact true for all* $n \geq 1$.

□

Example 2.4.1. $G_9 = aF_{n-7} + bF_{n-8} = 13a + 21b.$

property 2.4.1. $\sum_{i=1}^{n} G_{k+i} = G_{n+k-2} - G_{k+2}$ *where* $3 \leq i \leq k$.

Proof.

$$\begin{aligned}
\sum_{i=1}^{n} G_{k+i} &= \sum_{i=1}^{n} (aF_{k+i-2} + bF_{k+i-1}) \\
&= \sum_{i=1}^{n} (aF_{k+i-2}) + \sum_{i=1}^{n} (bF_{k+i-1}) \\
&= a\sum_{i=1}^{n} (F_{k+i-2}) + b\sum_{i=1}^{n} F_{k+i-1}) \\
&= a(F_{n+k} + bF_{n+k+1}) - (F_k + bF_{k+1}) \\
&= G_{n+k-2} - G_{k+2}.
\end{aligned}$$

□

property 2.4.2. [6]

1. $G_{4n} + b = (G_{2n} + G_{2n-1})F_{2n-1}$.
2. $G_{4n+3} + a = (G_{2n+3} + G_{2n+1})F_{2n+1}$.
3. $G_{4n+1} - a = (G_{2n+2} + G_{2n})F_{2n}$.
4. $G_{4n+2} - b = (G_{2n+3} + G_{2n+1})F_{2n}$.

2.4.1 Bient's formula

Theorem 2.4.2. *We have* $G_n = \frac{c\alpha^n - d\beta^n}{\alpha - \beta}$ *is Binet's formula of GFS. where* $c = a + (a-b)\beta$ *and* $d = a + (a-b)\alpha$.

Proof. We have $G_n = aF_{n-2} + bF_{n-1}$; $F_n = \frac{\alpha^n - \beta^n}{\sqrt{5}}$, So

$$\begin{aligned}
G_n &= aF_{n-2} + bF_{n-1} \\
&= a\left(\frac{\alpha^{n-2} - \beta^{n-2}}{\sqrt{5}}\right) + b\left(\frac{\alpha^{n-1} - \beta^{n-1}}{\sqrt{5}}\right)
\end{aligned}$$

$$\begin{aligned}
\sqrt{5}G_n &= a(\alpha^{n-2} - \beta^{n-2}) + b(\alpha^{n-1} - \beta^{n-1}) \\
&= \alpha^n(\frac{a}{\alpha^2} + \frac{b}{\alpha}) - \beta^n(\frac{a}{\beta^2} + \frac{b}{\beta}) \\
&= \alpha^n(a\beta^2 - b\beta) - \beta^n(a\alpha^2 - b\alpha) \\
&= \alpha^n[a + (a-b)\beta] - \beta^n[a + (a-b)\alpha]
\end{aligned}$$

Thus, $G_n = \frac{\alpha^n[a+(a-b)\beta] - \beta^n[a+(a-b)\alpha]}{\alpha - \beta}$.We put $[a + (a-b)\beta] = c$ and $[a + (a-b)\alpha] = d$

So $G_n = \frac{c\alpha^n - d\beta^n}{\alpha - \beta}$ □

2.4.2 Generating function

Use generating function to solve the generalized Fibonacci number $G_n = G_{n-1} + G_{n-2}$ where $G_1 = a$ and $G_2 = b$.
let $g(x) = a + bx + (a+b)x^2 + (a+2b)x^3 + (2a+3b)x^4 + ...$
be the generating function of the generalized Fibonacci number .

2.5 Convergence of the Fibonacci series

Theorem 2.5.1. *The Fibonacci series converges to the sum:* $S = \sum_0^\infty \frac{F_i}{k^{i+1}}$ *where* k *is a positive integer.*

Proof. Suppose the series converges,then by Binet's formula :

$$\begin{aligned}
\sum_0^\infty \frac{F_i}{K^{i+1}} &= \frac{1}{\sqrt{5}k}\left[\sum_0^\infty \left(\frac{1+\sqrt{5}}{2k}\right)^i - \left(\frac{1-\sqrt{5}}{2k}\right)^i\right] \\
&= \frac{1}{\sqrt{5}k}\left[\frac{1}{1-\left(\frac{1+\sqrt{5}}{2k}\right)}\frac{1}{1-\left(\frac{1-\sqrt{5}}{2k}\right)}\right] \\
&= \frac{2}{\sqrt{5}k}\left(\frac{1}{2k-1-\sqrt{5}-}\frac{1}{2k-1+\sqrt{5}}\right) \\
&= \frac{1}{k^2-k-1}
\end{aligned}$$

it is well known that the power series $\frac{1}{1-x} = \sum_0^\infty x^i$ converges if and only if $|x| < 1$ .Consequently the Fibonacci power series converges if and only if $|\alpha| < k$ and $|\beta| < k$. that is if and anly if $k > max|\alpha|, |\beta|$
but $\alpha = |\alpha| > |\beta|$.thus the series...converges if and only if $k > \alpha$.That is if and only if $k \geq 2$;which was somewhat obvious. □

property 2.5.1. [8]

1. $\sum_0^\infty \frac{F_i}{2^{i+1}} = 1 = \frac{1}{F_1}$.

2. $\sum_0^\infty \frac{F_i}{3^{i+1}} = \frac{1}{5} = \frac{1}{F_5}$.

3. $\sum_0^\infty \frac{F_i}{8^{i+1}} = \frac{1}{55} = \frac{1}{F_{10}}$.

4. $\sum_0^\infty \frac{F_i}{10^{i+1}} = \frac{1}{89} = \frac{1}{F_{11}}$.

2.6 Golden ratio

The number1.618 is called "Q",which is also known as the golden ratio.The exact reciprocal of 1.618is 0.618.This number appears frequently in nature,architecture,and paintings such as the Mona Lisa.It also appears in hurricanes,tree branches,galaxies and outer space,we get it by dividing any number by the number that follows it,such as 89/144=0.618.

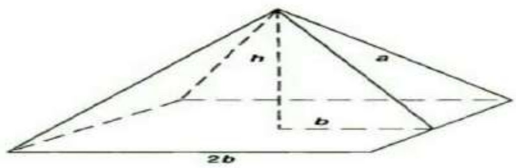
Theorem 2.6.1. *An interesting observation in the Fibonacci sequence is that the ratio* $\frac{F_{n+1}}{F_n}$ *the closer to the value 1.618,the more you take n the larger the value.Any* $\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=1.618$

Proof. From the Fibonacci recurrence relation we have:

$$\begin{aligned}\frac{F_{n+1}}{F_n} &= 1+\frac{F_{n-1}}{F_n}\\ &= 1+\frac{1}{F_n/F_{n-1}}\end{aligned}$$

As $n \longrightarrow \infty$;this yields the equation $x^2-x-1=0$;since the limit is positive we take the positive root of the quadratic equation $\alpha=\frac{1+\sqrt{5}}{2}=1.618$.Therefore:$\lim_{n\to\infty}\frac{F_{n+1}}{F_n}=\alpha=1.618$ □

The golden ratio was used in ancient times by the Pharaohs and the Greeks in the art of architecture such as building the pyramids and other buildings.
The height of the Great Pyramid is 484.4 feet,or about 5813 inches,which is the three successive Fibonacci numbers in height 5,8 and 13.



The artist Leonardo Davinci also used the golden ratio in drawing all parts of the Mona Lisa as the ratio of perfection in beauty.

Lemma 2.6.1. *We have:*

1. $\alpha_1\alpha_2 = -1$.
2. $\alpha_1^2 - \alpha_2^2 = \sqrt{5}$.
3. $\alpha_1^2 + \alpha_2^2 = 3$.

property 2.6.1. [8] *For all positive integers n, we have* $F_n = \frac{\alpha_1^{2n}+1}{\alpha_1^n\sqrt{5}}$

Proof. we have :

$$\begin{aligned} F_n &= \frac{1}{\sqrt{5}}(\alpha_1^n - \alpha_2^n) \\ &= \frac{1}{\sqrt{5}}(\alpha_1^n + \frac{1}{\alpha_1^n}) \\ &= \frac{\alpha_1^{2n}+1}{\alpha_1^n\sqrt{5}} \end{aligned}$$

□

Note 2.6.1. *When n is even,* $\frac{F_{n+1}}{F_n} > \alpha$,*and when it is odd* $\frac{F_{n+1}}{F_n} < \alpha$.
for example:
$n = 6 : \frac{F_{n+1}}{F_n} = \frac{F_7}{F_6} = \frac{13}{8} = 1.625$.
$n = 9 : \frac{F_{n+1}}{F_n} = \frac{F_{10}}{F_9} = \frac{55}{34} = 1.617$.

2.7 Fibonacci matrix

One of the most popular methods for study of the sequence defined recursively is to use matrices.Application of the so-called matrix generator or generating matrix not only allows us to derive many identities for given sequence but also provides relatively simple proofs of them.It is well known that the numbers of the Fibonacci sequence are generated by so-called Q-matrix.

2.7.1 The Q-matrix

Definition 2.7.1. [8] *The Fibonacci Q-matrix is the matrix defined by:*

$$Q = \begin{bmatrix} F_2 & F_1 \\ F_2 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Theorem 2.7.1. *Let* $n \geq 1$ *. Then*

$$Q^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$$

Proof. By (PMI).When $n = 1$

$$Q^1 = \begin{bmatrix} F_2 & F_1 \\ F_1 & F_0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = Q.$$

So the result is true. Now, assume it is true for an arbitrary positive integer k:
$Q^k = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix}$

Then $\quad Q^{k+1} = Q^k Q^1 = \begin{bmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} F_{k+1} + F_k & F_k \\ F_k + F_{k-1} & F_k \end{bmatrix} == \begin{bmatrix} F_{k+2} & F_{k+1} \\ F_{k+1} & F_k \end{bmatrix}.$

Thus the result follows by induction. □

Corollary 2.7.1. $F_{n-1}F_{n+1} - F_n^2 = (-1)^n$; $\forall(n \geq 1)$.

Proof. Since $|Q| = -1$;that $|Q^n| = (-1)^n$. We have: $|Q^n| = F_{n+1}F_{n-1} - F_n^2$.
Thus: $F_{n+1}F_{n-1} - F_n^2 = (-1)^n$. □

property 2.7.1. [8] *For positives integers n, m we have:*

1. $F_{m+n+1} = F_{m+1}F_{n+1} + F_m F_n$.
2. $F_{m+n} = F_{m+1}F_n + F_m F_{n-1}$.
3. $F_{m+n} = F_m F_{n+1} + F_{m-1}F_n$.
4. $F_{m+n-1} = F_m F_n + F_{m-1}F_{n-1}$.

2.7.2 The M- matrix

Definition 2.7.2. *The Fibonacci matrix defined by:* $M = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

Theorem 2.7.2. *Let $n \geq 1$. Then,* $M^n = \begin{bmatrix} F_{2n-1} & F_{2n} \\ F_{2n} & F_{2n+1} \end{bmatrix}$.

Note 2.7.1. *Where $n \geq 1$;* $\frac{M^n}{F_{2n-1}} = \begin{bmatrix} 1 & F_{2n}/F_{2n-1} \\ F_{2n}/F_{2n-1} & F_{2n+1}/F_{2n-1} \end{bmatrix}$

Since $\lim_{k\to\infty} \frac{F_k}{F_{k-1}} = \alpha$,*it follows that*

$\lim_{n\to\infty} \frac{M^n}{F_{F_{2}n-1}} = \begin{bmatrix} 1 & \alpha \\ \alpha & \alpha^2 \end{bmatrix} = \begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix}$

That is,the sequence $\{M^n/F_{2n-1}\}$ of Fibonacci matrices with leading entries converges to the matrix $\begin{bmatrix} 1 & \alpha \\ \alpha & 1+\alpha \end{bmatrix}$.

Likewise,the sequence $\{Q^n/F_{2n-1}\}$ converge to the matrix $\begin{bmatrix} 1+\alpha & \alpha \\ \alpha & 1 \end{bmatrix}$.

2.7.3 The P- matrix

Definition 2.7.3. [8] *The P-Fibonacci matrix is defined by* $P = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

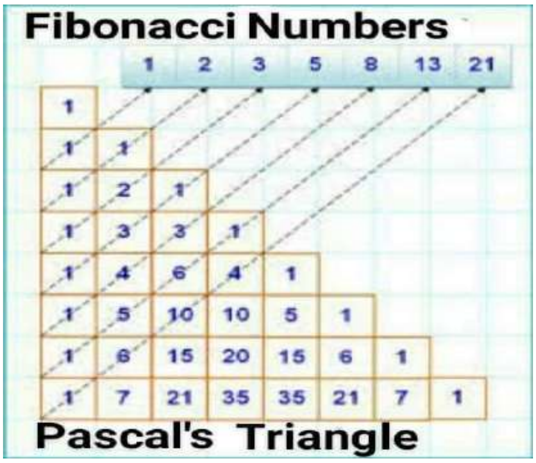
Theorem 2.7.3. *Let* $n \geq 1$. *Then* $P^n = \begin{bmatrix} F_{n-1}^2 & F_{n-1}F_n & F_n^2 \\ 2F_{n-1}F_n & F_{n+1}^2 - F_{n-1}F_n & 2F_nF_{n+1} \\ F_n^2 & F_nF_{n+1} & F_{n+1}^2 \end{bmatrix}$.

Proof. By (PMI)

□

2.8 Pascal's triangle

Pascal's triangle is one of the most important methodological methods that enable us to calculate the Fibonacci numbers by adding the numbers along the eastern diagonals.



The sums are $1; 1; 2; 3; 5; 8; ...$ and they seen to be the Fibonacci numbers.

Theorem 2.8.1. *For every positive integer;we have:* $F_{n+1} = \sum_{i=0}^{[n/2]} \binom{n-i}{i} \quad n \geq 0$
(The following theorem shows the relationship between Pascal's triangle and Fibonacci numbers)

Proof. By PMI
When $n = 0$, $\sum_{i=0}^{0} \binom{0-i}{i} = \binom{0}{0} = 1 = F_1$, so the statement is true when $n = 0$.
Now assume it is true for all positive integers $\leq k$,where k is an arbitrary positive integer:
$F_{k+1} = \sum_{i=0}^{[k/2]} \binom{k-i}{i}$
By Pascal's identity

$$\sum_{i=0}^{[(k+2)/2]} \binom{k+2-i}{i} = \sum_{i=0}^{[(k+2)/2]} \binom{k+1-i}{i-1} + \sum_{i=0}^{[(k+2)/2]} \binom{k+1-i}{i}$$

Suppose k is even.Then

$$\begin{aligned}
\sum_{i=0}^{[(k+2)/2]} \binom{k+2-i}{i} &= \sum_{j=0}^{[k/2]} \binom{k-j}{j} + \sum_{i=0}^{[k/2]} \binom{k+1-i}{i} + \binom{k/2}{k/2+1} \\
&= \sum_{j=0}^{[(k+1)/2]} \binom{k-j}{j} + \sum_{i=0}^{[(k+1)/2]} \binom{k+1-i}{i} + 0 \\
&= F_k + F_{k+1} \\
&= F_{k+2}.
\end{aligned}$$

It can be shown similarly that,when k is odd,

$$\sum_{i=0}^{[(k+2]/2} \binom{k+2-i}{i} = F_{k+2}.$$

Thus,by the strong verision of PMI,the formula holds for all integers $n \geq 0$. □

Example 2.8.1. *We have:*

1. $F_6 = \sum_0^2 \binom{5-i}{i} = \binom{5}{0} + \binom{4}{1} + \binom{3}{2} = 1 + 4 + 3 = 8.$
2. $F_7 = \sum_0^3 \binom{6-i}{i} = \binom{6}{0} + \binom{5}{1} + \binom{4}{2} + \binom{3}{3} = 1 + 5 + 6 + 1 = 13.$

Chapter 3

Fibonacci polynomials

The famous Fibonacci polynomial $f_n(x)$,was studied beginning in 1883 by the Belgian mathematician Eugene C.Catalan,and later the German mathematician Ernst Jacobthal(1882-1965).In this chapter we will introduce the concept of Fibonacci polynomials as well as the basic theorems.

3.1 Difinition of Fibonacci polynom

Definition 3.1.1. [8] *The sequence of Fibonacci polynomials* $f_n(x)$ *, is defined by the relation recurrence following:* $f_n(x) = xf_{n-1}(x) + f_{n-2}(x)$*,where* $f_1(x) = 1; f_2(x) = x$*.*
The first terms of the Fibonacci polynomials:
$f_1(x) = 1,$
$f_2(x) = x,$
$f_3(x) = x^2 + 1,$
$f_4(x) = x^3 + 2x,$
$f_5(x) = x^4 + 3x^2 + 1.$

Corollary 3.1.1. *That the Fibonacci polynomials can be constructed using the binomials exoansions of* $(x + 1)^n$*,* *where* $n \geq 0$*, As shows in the following table*

n	*Expansion of* $(x+1)^n$
0	1
1	$x + 1$
2	$x^2 + 2x + 1$
3	$x^3 + 3x^2 + 3x + 1$
4	$x^4 + 4x^3 + 6x^2 + 4x + 1$
5	$x^5 + 5x^4 + 10x^3 + 10x^2 + 5x + 1$

We note that the sum of the elements along the diagonal beginning at row n is $f_{n+1}(x)$ *for instance the sum of the elements along the diagonal beginning at row 3 is*$f_4(x)$*.*

Theorem 3.1.1. $x\sum_{1}^{n} f_i(x) = f_{n+1}(x) + f_n(x) - 1$

Proof. Using the recurrence relation $\sum_{1}^{n} f_{i+1}(x) = x\sum_{i=1}^{n} f_i(x) + \sum_{i=1}^{n} f_{i-1}(x)$

that is $f_n(x) + f_{n+1}(x) = x\sum_{i=1}^{n} f_i(x) + f_0(x) + f_1(x)$. We have $f_0(x) = 0, f_1(x) = 1$.

So $x\sum_{i=1}^{n} f_i(x) = f_{n+1}(x) + f_n(x) - 1$. □

Example 3.1.1. *We take value $n = 3$.*
$x\sum_{i=1}^{3} f_i = x(f_1 + f_2 + f_3) = x(1 + x + x^2 + 1) = x^3 + x^2 + 2x$
$f_{i+1}(x) + f_i(x) - 1 = f_4(x) + f_3(x) - 1 = (x^3 + 2x) + (x^2 + 1) - 1 = x^3 + x^2 + 2x$

Corollary 3.1.2. $\sum_{i=1}^{n} F_i = F_{n+2}(x) - 1$.
This corollary follows from the previous theorem,since $f_i = F_i$

3.2 Bient's formula

Theorem 3.2.1. [8] *The characteristic equation of Fibonacci polynomials is:*
$t^2 - xt - 1 = 0.$, *where* $\alpha(x) = \frac{x+\sqrt{x^2+4}}{2}$ *and* $\beta(x) = \frac{x-\sqrt{x^2+4}}{2}$,*be the solutions of the quadratic equation.*

Hence Binet's formula for the Fibonacci polynomials :$f_n(x) = \dfrac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}$.

Note 3.2.1. $\alpha(1) = \alpha$ *and* $\beta(1) = \beta$
$\alpha(2) = 1 + \sqrt{2}$ *and* $\beta(2) = 1 - \sqrt{2}$
are the characteristic rootts of the pell recurrence relation $X^2 - 2X - 1 = 0$.

3.3 Generating function

Theorem 3.3.1. [8] *The generating function for $f_n(x)$is given as follows:*
$g(t) = \sum_{n=0}^{\infty} f_n(x)t^n$

Proof. We have $g(t) = \sum_{0}^{\infty} f_n(x)t^n$. Thus $xtg(t) = \sum_{0}^{\infty} xf_n(x)t^{n+1}$. and $t^2g(t) = \sum_{0}^{\infty} f_n(x)t^{n+2}$.
By subtracting the equations,we find :
$g(t) - xtg(t) - t^2g(t) = f_0(x) - tf_1(x) - xtf_0(x)$.
$g(t)(1 - xt - t^2) = t$.
So $g(t) = \frac{t}{1-xt-t^2} = \sum_{0}^{\infty} f_n(x)t^n$. □

Theorem 3.3.2. $f_{m+n+1}(x) = f_{m+1}(x)f_{n+1}(x) + f_m(x)f_n(x)$.

Proof. We have

$$\frac{y}{1 - xy - y^2} = \sum_{0}^{\infty} f_n(x)y^n.$$

Therefore

$$\frac{yf_m(x)}{1-xy-y^2} = \sum_0^\infty f_m(x)f_n(x)y^n.$$

Then

$$\frac{f_{m+1}(x)}{1-xy-y^2} = \sum_0^\infty f_{m+1}(x)f_{n+1}y^n.$$

$$\frac{f_{m+1(x)}+f_m(x)y}{1-xy-y^2} = \sum_0^\infty f_{m+n+1}(x)y^n.$$

□

This theorem illustrates an alternate method for constructing new members of the family of Fibonacci polynomials.

Example 3.3.1. *Take $m = 2$ and $n = 3$. Then*
$f_{m+1}(x)f_{n+1}(x) + f_m(x)f_n(x) = f_3(x)f_4(x) + f_2(x)f_3$.

$$\begin{aligned} &= (x^2+1)(x^3+2x) + x(x^2+1) \\ &= x^5 + 4x^3 + 3x \\ &= f_6(x) = f_{m+n+1}(x) \end{aligned}$$

3.4 Generalized Fibonacci polynomials

3.4.1 Definition

The generalized Fibonacci polynomials defined by:

$$f_{n+1}(x) = \begin{cases} S \quad , if \quad n = 0 \\ Sx \quad , if \quad n = 1 \\ xf_n(x) + f_{n-1}(x) \quad , if \quad n \geq 2. \end{cases}$$

Note 3.4.1. *If $S = 1$, then we obtained classical Fibonacci polynomial sequence.*

3.4.2 Binets formula

The nth generalized Fibonacci polynomials is given by:
$f_n(x) = S\frac{R_1^n - R_2^n}{R_1 - R_2}$. Where $R_1 = \frac{x+\sqrt{x^2+4}}{2}$ and $R_2 = \frac{x-\sqrt{x^2+4}}{2}$.

Proof. Use the principle of mathematical induction on n, it is clear the result is true for $n = 0$ and $n = 1$.
By hypotheses;Assume that is true for i such that $0 \leq i \leq r+1$.Then $f_i(x) = S\frac{R_1^i - R_2^i}{R_1 - R_2}$
it follows from definition of generalized Fibonacci polynomials and from equation(3.2.1)

$$f_{r+2(x)} = xf_{t+1}(x) + f_r(x) = S\frac{R_1^{r+2} - R_2^{r+2}}{R_1 - R_2}.$$

thus,the fromula is true for any positive integer n. So $f_n(x) = S\frac{R_1^n - R_2^n}{R_1 - R_2}$. □

Corollary 3.4.1. *Binets formula allows us to express the generalized Fibonqcci polynomial in function of roots* R_1 *and* R_2 *of the following characteristic equation. Associated to the recurrence relation,* $x^2 = xt + 1$.

property 3.4.1. 1. *For any integer* $n \geq 1$, $R_1^{n+2} = xR_1^{n+1} + R_1^n$; $R_2^{n+2} = xR_2^{n+1} + R_2^n$.

2. *For any integern* ≥ 1, $s(R_1^n + R_2^n) = f_{n+1}(x) + f_{n-1}(x)$.

3. *For any integer n:*

$$(x^2+4)f_n^2(x) + 4S^2(-1)^n = \begin{cases} S^2(R_1^n + R_2^n)^2 & if \quad n \quad even, \\ S^2(R_1^n - R_2^n)^2 & if \quad n \quad odd. \end{cases}$$

Proof. 1. Since R_1 and R_2 are the roots of the characteristic equation $x^2 = xt + 1$, then $R_1^1 = xR_1 + 1$ and $R_2^2 = xR_2 + 2$. Now, multiplying both sides of these equations by R_1^n and R_2^n respectively ,we obtain the desired result.

2. Taking in to account that $R_1 = \frac{-1}{R_2}$ it is obtained

$$\begin{aligned} &= s\frac{R_1^{n+1} - R_2^{n+1}}{R_1 - R_2} + s\frac{R_1^{n-1} - R_2^{n-1}}{R_1 - R_2} \\ &= \frac{s}{R_1 - R_2}\left(R_1^{n+1} + R_1^{n-1} - R_2^{n+1} - R_2^{n-1}\right) \\ &= \frac{s}{R_1 - R_2}\left\{R_1^n\left(R_1 + \frac{1}{R_1}\right) - R_2^n\left(R_2 + \frac{1}{R_2}\right)\right\} \\ &= s(R_1^n + R_2^n) \end{aligned}$$

3. From the Binet formula of generalized Fibonacci polynomials:
$f_n^2(x) = \frac{S^2}{(R_1-R_2)^2}[(R_1^{2n} - 2(R_1R_2)^n + R_2^{2n}]$

$$\begin{cases} (x^2+4)f_n^2(x) + 4S^2 = (SR_1^n + SR_2^2)^2.if \quad n \quad is \quad even \\ (x^2+4)f_n^2(x) - 4S^2 = (SR_1^n - SR_2^2)^2.if \quad n \quad is \quad odd \end{cases}$$

□

3.4.3 Generating function

Generating function of generalized Fibonqcci polynomials is: $\sum_0^{\infty} f_n(x)t^{n-1} = S(1 - xt - t^2)^{-1}$.

property 3.4.2. *If* $f_n(x)$ *are generalized Fibonacci polynomials then:*
$\sum_0^{\infty} f_n(x)\frac{t^{n-1}}{n} = Se_1^{xt}F_0(n+1, -, t^2)$

Proof. We have

$$\begin{aligned}\sum_0^\infty f_n(x)t^{n-1} &= S(1-xt-t^2)^{-1}.\\ &= s\sum_0^\infty t^n(x+t)^n\\ &= s\sum_0^\infty t^n\sum_0^n t^r x^{n-r}C_r\\ &= s\sum_0^\infty\sum_0^n \frac{t^{n+r}x^{n-r}n}{r(n-r)}\\ &= s\sum_0^\infty\sum_0^\infty \frac{t^{n+2r}x^n(n+r)}{rn}\\ &= s\sum_0^\infty \frac{(xt)^n}{n}\sum_0^\infty \frac{t^{2r}}{r}n(n+r)\end{aligned}$$

So $\sum_0^\infty f_n(x)\frac{t^{n-1}}{n} = Se_1^{xt}F_0(n+1,-,t^2).$

□

Proposition 3.4.1. *1.* $\sum_0^\infty (1+n)t^n f_n(x) = \frac{St(2-xt)}{(1-xt-t^2)}.$

2. $\sum_0^\infty f_n(x)+f_{n+1}(x)t^n = \frac{S(1+t)}{t(1-xt-t^2)}.$

Proof. Using the generating function the proof is clear. □

3.4.4 Q-matrix generator for Fibonacci polynomials

Definition 3.4.1. *The matrix defined by:*

$$Q(x) = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$$

Theorem 3.4.1. *Let* $n \geq 1$ *. Then* $Q^n(x) = \begin{bmatrix} f_{n+1}(x) & f_n(x) \\ f_n(x) & f_{n-1}(x) \end{bmatrix}$

Corollary 3.4.2. $f_{n+1}(x)f_{n-1}(x) - f_n^2(x) = (-1)^n$; $\forall n \geq 1$.

Proof. Since $|Q| = -1$;that $|Q^n| = (-1)^n$.

We have : $|Q^n| = f_{n+1}(x)f_{n-1}(x) - f_n^2(x).Thus : f_{n+1}(x)f_{n-1}(x) - f_n^2(x) = (-1)^n.$ □

Theorem 3.4.2. *Let n,m be a positive integers. We have :*

1. $f_{n+1}(x)f_{n-1}(x) - f_n^2(x) = (-1)^n$.
2. $f_{n+1}(x) = f_{n+1}(x)f_n(x) + f_n(x)f_{n-1}(x)$.

Proof. 1. Since $|Q| = -1, |Q^n| = |Q|^n = (-1)^n$. So $f_{n+1}(x)f_{n-1}(x) - f_n^2(x) = (-1)^n$.

2. We have $|Q|^{n+m} = |Q^n||Q^m|$ and

$$Q^{m+n}(x) = Q^m(x)Q^n(x) = \begin{bmatrix} f_{m+1}(x)f_{n+1}(x) + f_m(x)f_n(x) & f_{m+1}(x)f_n(x) + f_m(x)f_{n-1}(x) \\ f_m(x)f_{n+1}(x) + f_{m-1}(x)f_n(x) & f_m(x)f_n(x) + f_{m+1}(x)f_{n-1}(x) \end{bmatrix}$$

So cosequently $f_{m+n}(x) = f_{m+1}(x)f_n(x) + f_m(x)f_{n-1}(x)$.

□

Applications:

The numbers and polynomails of are used to solve many mathematical problems, including differential equations

Example 3.4.1. *We consider the following differential equation:*

$$y'(t)(1 - xt - t^2) - (x + 2t)y(t) = 1.$$

The solution to this equation is given as $y(t) = \sum_{n=0}^{+\infty} f_n(x)t^n = \dfrac{t}{1 - xt - t^2}$
where $f_n(x)$ *are the Fibonacci polynomials.*

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