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The Dunkl derivative in quantum mechanics, and some of their applications

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Dedication

To our fathers

To our dear mothers

To all my brothers and sisters

To all the family and all my friends

To all my elementary school teachers at the end of this memoir.

Fatima Zohra and Chahrazad.

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Chapitre 1

General introduction

It is well known that due to the successes provided by both the quantum theory and general relativity, each one separately, they were unable to integrate the laws of nature into a single mathematical model. For this reason, several physical models have appeared which are based on the generalization of Heisenberg's principle. Among them, are a minimal observable length, maximally observable momentum and length, etc. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. During the last years, there has been a rapid development in special functions with reflection symmetries and the harmonic analysis related with root systems. These special functions of several variables are investigated in quantum mechanics under a new deformation algebra it is called the Dunkl operator. It was first proposed by Wigner in 1950 in order to the quantization of bosons. This procedure makes the commutator of position and momentum dependent on some constants called Wigner parameters, and also parity operators [11].

In the past years, there are some physicists who have taken care of these Dunkl derivations in the quantum theory area. For example, the author Vincent X has provided the Dunkl oscillator in the plane in Ref. [12]. Furthermore,the author A. Lapointe, L. Vinet have studied the problem of the dunkl oscillator in Ref. [13]and the Dunkl oscillator in three dimensions [14] , also the coulomb potential and the Klein–Gordon Oscillator have treated in Ref. [15]. Morevere the problem of the free-particle spherical waves and the Pseudo-Harmonic Oscillator and the Mie-type Potential has obtained an exact solution of the Schrodinger equation with Dunkl derivative in Ref. [16].also the DKP-Dunkl oscillator equation is presented in Ref. [17].The author Charles Dunkl introduced a new operator by combining the usual differential and parity operators with Wigner parameter. the Wigner-Dunkl-Newton mechanics with time-reversal symmetry [18] and explicit solution to the N-body Calogero problem, the relativistic Dunkl oscillator in (2 + 1) dimensions [19], wherease in Ref. [20] has presented the thermal properties of relativistic Dunkl oscillator .

In the present analysis, we exerted much effort to understand the Dunkl derivation and its applications to quantum systems. Using the wave equation method, we can establish the exact solutions of the quantum particles in a relativistic and non-relativistic case under the generalized Heisenberg algebra of the Dunkl operator. The limit cases give the expected results of ordinary quantum mechanics.

This work is divided into the following chapters : The second chapter gives a brief overview of the Dunkl operator as well as its uses in quantum theory. Whereas in the third chapter, we give the exact solution of the non-relativistic case for a particle in box and Harmonic oscillator potential in arbitrary dimensions $D = 1$, 2, and 3. In two and three dimensions the Harmonic oscillator is calculated in polar and spherical coordinates respectively. In the fourth chapter, which is the principal part of this work, we examine the Dunkl-Klein-Gordon oscillator within Cartesian coordinates, then in spherical coordinates. Also in chapter 5, we study the effect of Dunkl-Dirac oscillator in one dimension. In the last chapter, we present a summary of the main results and our general conclusions.

Chapitre 2

Mathematical tools for Dunkl derivation

2.1 Important definitions of the Dunkl derivation

It is well known in Refs. [11] the Dunkl derivative instead of the normal derivative is defined as :

$$\hat{D}_x = \partial_x + \frac{\mu}{x}\left(1 - \hat{R}\right). \tag{2.1}$$

where μ is the Wigner parameter and $\hat{R}$ is called a reflection operator obeying

$$\hat{R}f(x) = f(-x). \tag{2.2}$$

This effect leads to two cases if $f(x)$ is an even function, here the eigenvalue is one (i.e., $\hat{R}f_{even}(x) = f_{even}(x)$). While if $f(x)$ is an odd function, here the eigenvalue of $\hat{R}$ is minus one (i.e., $\hat{R}f_{odd}(x) = -f_{odd}(x)$).

The role of the Dunkl derivative in quantum mechanics lies in the generalization of the momentum operator

$$\hat{p} = \frac{\hbar}{\imath}\hat{D}_x, \tag{2.3}$$

and the position operator remains the same $\hat{x} = x$. This quantum mechanics is called the Wigner-Dunkl quantum mechanics. To clarify further, we can extract the Wigner-Dunkl-Heisenberg commutator relation as follow :

$$[\hat{x}, \hat{p}] = -\imath\left[\hat{x}, \hat{D}_x\right]. \tag{2.4}$$

For even function we have $\hat{R}f_{even}(x) = f_{even}(x)$, Eq. (2.4) becomes as :

$$[\hat{x},\hat{p}]\, f_{even} = -\imath\, [\hat{x}, D_x]\, f_{even} \tag{2.5}$$
$$= -\imath\, (\hat{x}D_x - D_x\hat{x})\, f_{even} \tag{2.6}$$
$$= -\imath \left(\hat{x}\left(\partial_x + \frac{\mu}{x}\left(1-\hat{R}\right)\right) - \left(\partial_x + \frac{\mu}{x}\left(1-\hat{R}\right)\right)\hat{x}\right) f_{even} \tag{2.7}$$
$$= \imath f_{even} + (-\imath)\left(x\frac{\mu}{x} - \frac{\mu}{x}x\right) f_{even} + \imath\left(x\frac{\mu}{x} - \frac{\mu}{x}\hat{R}x\right) f_{even} \tag{2.8}$$
$$= \imath f_{even} + \imath\left(\mu\hat{R} + \frac{\mu}{x}x\right) f_{even} \tag{2.9}$$
$$= \left[\hat{x}, -\imath\frac{\partial}{\partial x} - \imath\frac{\mu}{x}\left(1-\hat{R}\right)\right] f_{even} \tag{2.10}$$
$$= \left[\hat{x}, -\imath\frac{\partial}{\partial x}\right] f + \left[\hat{x}, -\imath\frac{\mu}{x}\left(1-\hat{R}\right)\right] f_{even} \tag{2.11}$$

then we find

$$[\hat{x},\hat{p}] = \imath\left(1+2\mu\hat{R}\right) \tag{2.12}$$

For odd function we find the same result.

In three dimensions, the Dunkl derivative is defined as

$$\hat{D}_j = \frac{\partial}{\partial x_j} + \frac{\mu_j}{x_j}\left(1-\hat{R}_j\right). \tag{2.13}$$

The Wigner parameter υ_j is made up of positive real values. The reflection operators $\hat{R}_j$ are those that meet the following action :

$$\hat{R}_j f(x_j) = f(-x_j)\,; \hat{R}_i\hat{R}_j = \hat{R}_j\hat{R}_i; \hat{R}_{jj}x_i = -\delta_{ij}x_i\hat{R}_j; \tag{2.14}$$
$$\hat{R}_j\frac{\partial}{\partial x_i} = -\delta_{ij}\frac{\partial}{\partial x_i}\hat{R}_j.\ \text{(nosummation)}. \tag{2.15}$$

As a result, the Dunkl operators obey the algebra below :

$$p_j\hat{D}_j = -\hat{D}_j p_j, \quad \left[\hat{D}_i, \hat{D}_j\right] = 0; \tag{2.16}$$
$$\left[\hat{x}, \hat{D}_j\right] = \delta_{ij}\left(1+2\mu_{ij}\hat{R}_{ij}\right). \quad \text{(nosummation)}. \tag{2.17}$$

When using the Dunkl derivative instead of the ordinary derivative. The scalar product is :

$$\langle f \mid g\rangle = \int g^*(\|\mathbf{x}\|)\, f(\|\mathbf{x}\|)\, |x|^{2\mu}\, d^3\mathbf{x}\ . \tag{2.18}$$

With respect to the state $|\psi\rangle$, the anticipated value of an operator$O$ can be defined as

$$\left\langle \Psi \mid \hat{O}\Psi\right\rangle = \int \Psi^*(x)\,\hat{O}\Psi(x)\,|x|^{2\mu}\,dx\ . \tag{2.19}$$

2.2 The properties of the Dunkl derivative

1- The Dunkl derivative in general dimensions is linear because :

$$\hat{D}_j \left(af \left(x_j\right) + bg \left(x_j\right)\right) = a\hat{D}_j f \left(x_j\right) + b\hat{D}_j g \left(x_j\right). \tag{2.20}$$

The Leibniz rule applies to the Dunkl derivative for any function $f \left(x_j\right)$, $g \left(x_j\right)$

$$\begin{aligned} \hat{D}_j \left(f \left(x_j\right) g \left(x_j\right)\right) &= \left(\hat{D}_j f \left(x_j\right)\right) g \left(x_j\right) + f \left(x_j\right) \hat{D}_j g \left(x_j\right) \\ &- \frac{\mu_j}{x_j} \left[\left(1 - \hat{R}_j\right) f \left(x_j\right)\right] \left[\left(1 - \hat{R}_j\right) g \left(x_j\right)\right]. \end{aligned} \tag{2.21}$$

When the function$f \left(x_j\right)$ or $g \left(x_j\right)$ is even, the Leibniz relation takes the form

$$\hat{D}_j \left(f \left(x_j\right) g \left(x_j\right)\right) = \left(\hat{D}_j f \left(x_j\right) g \left(x_j\right) + f \left(x_j\right) \hat{D}_j g \left(x_j\right)\right), \tag{2.22}$$

while in one dimension the Dunkl derivative satisfies the following Leibniz rule

$$\begin{aligned} \hat{D}_x \left(f \left(x\right) g \left(x\right)\right) &= \left(\hat{D}_x f \left(x\right)\right) g \left(x\right) + f \left(x\right) \hat{D}_x g \left(x\right) \\ &- \frac{\mu}{x} \left[\left(1 - \hat{R}\right) f \left(x\right)\right] \left[\left(1 - \hat{R}\right) g \left(x\right)\right], \end{aligned} \tag{2.23}$$

also, if we have$f(x)$ or $g(x)$ is even, the ordinary Leibniz rule becomes :

$$\hat{D}_x \left(f \left(x\right) g \left(x\right)\right) = \left(\hat{D}_x f \left(x\right) g \left(x\right) + f \left(x\right) D_x g \left(x\right)\right). \tag{2.24}$$

2- Acting the Dunkl derivative on the monomial gives : In follow we calculate the Dunkl derivation on function x^n, we have :

$$\hat{D}_x x^n = \partial_x x^n + \frac{\mu}{x} x^n - \frac{\mu}{x} \hat{R} x^n. \tag{2.25}$$

After some simplified we write

$$\begin{aligned} \hat{D}_x x^n &= n x^{n-1} + \mu x^{n-1} - \frac{\mu}{x} \left(-1\right)^n x^{n-1} = \left[n + \mu + \mu \left(-1\right)^n\right] x^{n-1} \\ &= \left[n + \mu \left(1 - \left(-1\right)^n\right)\right] x^{n-1} = \left[n + \mu \left(1 - \hat{R}\right)\right] x^{n-1}. \end{aligned} \tag{2.26}$$

Or other form we write

$$\hat{D}_x x^n = \left[n\right]_\mu x^{n-1}, \tag{2.27}$$

whereμ $-$deformed number is defined by

$$\left[n\right]_\mu = n + \mu \left(1 - \left(-1\right)^n\right) \ . \tag{2.28}$$

The first $\mu-$deformed number is $([0]_\mu = 0)$, the second is $\left([1]_\mu = 1 + 2\mu\right)$ and we can conclude in general deformed case as :

$$[2k]_\mu = 2k, [2k+1]_\mu = 2k + 1 + 2\mu, (k = 0, 1, 2, ...) \ . \tag{2.29}$$

Here we know that all μ-deformed number is non-negative if $\mu > -1/2$ is imposed. The Dunkl derivative behaves like ∂_x when acted on the even function, while it behaves like $\partial_x + \frac{2\mu}{x}$ when acted on the odd function.

The square of the Dunkl derivative in one dimension is defined as

$$\hat{D}_x^2 f = \left(\partial_x + \frac{\mu}{x}\left(1 - \hat{R}\right)\right)\left(\partial_x + \frac{\mu}{x}\left(1 - \hat{R}\right)\right) f \tag{2.30}$$

$$= \partial_x^2 f + \partial\left(\frac{\mu}{x}\left(1 - \hat{R}\right) f\right) + \frac{\mu}{x}\left(1 - \hat{R}\right)\partial_x f + \frac{\mu}{x}\left(1 - \hat{R}\right) f \tag{2.31}$$

$$= \partial_x^2 f + \left(-\frac{\mu}{x^2}\left(1 - \hat{R}\right) f\right) + \frac{\mu}{x}\left(1 - \hat{R}\right)\partial_x f + \frac{\mu}{x}\left(1 - \hat{R}\right)\partial_x f \tag{2.32}$$

$$= \partial_x^2 f + \left(-\frac{\mu}{x^2}\left(1 - \hat{R}\right) f\right) + 2\frac{\mu}{x}\left(1 - \hat{R}\right)\partial_x f. \tag{2.33}$$

Where we have

$$\frac{\mu}{x}\left(1 - \hat{R}\right) f = 0. \tag{2.34}$$

Then we obtain

$$\hat{D}_x^2 = \partial_x^2 + 2\frac{\mu}{x}\left(1 - \hat{R}\right)\partial_x + -\frac{\mu}{x^2}\left(1 - \hat{R}\right). \tag{2.35}$$

The square of the Dunkl derivative in general dimension is defined as follows :

$$\hat{D}_i^2 = \frac{\partial^2}{\partial x_i^2} + 2\frac{\mu_i}{x_i}\left(1 - \hat{R}_i\right)\frac{\partial}{\partial x_i} - \frac{\mu_i}{x_i^2}\left(1 - \hat{R}_i\right). \tag{2.36}$$

It is well known the Dunkl derivative dose not obey chain rule because :

$$\hat{D}_x f\left(u\left(x\right)\right) = \left(\partial_x - \frac{\mu}{x}\left(1 - \hat{R}\right) f\left(u\left(x\right)\right)\right), \tag{2.37}$$

we introduce the derivation of $u\left(x\right)$

$$= \frac{du}{dx}\frac{d}{du} f\left(u\right) + \frac{\mu}{x}\left(1 - \hat{R}\right) f\left(u\left(x\right)\right). \tag{2.38}$$

$$= \frac{du}{dx}\frac{d}{du} f\left(u\right) + \frac{\mu}{x} f\left(u\left(x\right)\right) - \frac{\mu}{x} p f\left(u\left(x\right)\right) \tag{2.39}$$

$$= \frac{du}{dx}\frac{d}{du} f\left(u\right) + \frac{\mu}{x} f\left(u\left(x\right)\right) - \frac{\mu}{x} f\left(u\left(-x\right)\right). \tag{2.40}$$

Consequently we write

$$\hat{D}_x f(u(x)) = \frac{du}{dx}\frac{d}{du} f(u) + \frac{\mu}{x}\left(f(u(x)) - f(u(-x))\right). \tag{2.41}$$

But, for a even function $u(x)$ we have the series rule

$$\hat{D}_x f(u(x)) = \frac{d}{du} f(u) + \hat{D}_x u(x) \ . \tag{2.42}$$

2.3 The important relations in quantum mechanics

1- The time-dependent Schrodinger equation reads

$$\imath \frac{\partial}{\partial t}\Psi(x,t) = \hat{H}(x,p)\Psi(x,t) = \left(\frac{\hat{p}^2}{2m} + V(x)\right)\Psi(x,t). \tag{2.43}$$

Using the generalized momentum operator defined in Eq. (2.35) into the time-dependent Schrodinger equation (2.43), we will get the next

$$\imath \frac{\partial}{\partial t}\Psi(x,t) = \hat{H}(x,p)\Psi(x,t) \tag{2.44}$$

$$= \left(-\frac{\partial_x^2 + 2\frac{\mu}{x}\left(1-\hat{R}\right)\partial_x - \frac{\mu}{x^2}\left(1-\hat{R}\right)}{2m} + V(x)\right)\Psi(x,t). \tag{2.45}$$

Furthermore, we can find the exact solutions of this equation for the particle in a box potential and Harmonic oscillator in one dimension. Thus we must define the important relations of the Dunkl operator in quantum mechanics.

2- The inner product :

In the Hilbert space related to one-dimensional, the inner product of Wigner-Dunkl quantum mechanics is given by

$$\langle f \mid g\rangle = \int g^*(x) f(x) |x|^{2\mu} dx \ , \tag{2.46}$$

where $|x|^{2\mu}$ is a weight function.

The expectation value of a physical operator $\hat{O}$ with respect to the state $\psi(x,t)$ is defined by

$$\left\langle \hat{O} \right\rangle = \langle \Psi \mid O\Psi\rangle = \int \Psi^*(x,t) O\Psi(x,t) |x|^{2\mu} dx \ , \tag{2.47}$$

and$O$ is a Hermitian operator if it obeys

$$\langle O\Psi \mid \Psi\rangle = \langle \Psi \mid O\Psi\rangle \ . \tag{2.48}$$

The properties of the weight function :

1- For an arbitrary even function $f(x)$ we have

$$\int_{-\infty}^{+\infty} dx\,|x|^{2\mu}\,\hat{D}_x f = 0\ . \tag{2.49}$$

2- For the odd function $f(x)$ obeying $\lim_{x\to\infty} x^{2\mu} f\,(x) = 0$ and $\mu \succ 0$ we have

$$\int_{-\infty}^{+\infty} dx\,|x|^{2\mu}\,\hat{D}_x f = 0\ , \tag{2.50}$$

we have

$$\langle \psi \mid \psi \rangle = \int \psi^*\,(x)\,\psi\,(x)\,|x|^{2\mu}\,dx. \tag{2.51}$$

In order to obtain the projection relation, we must use the Eq. (2.51) and as we know $\psi\,(x) = \langle \psi \mid x \rangle$ and $\psi^*\,(x) = \langle x \mid \psi \rangle$, we find

$$\langle \psi \mid \psi \rangle = \int |x|^{2\mu}\,\langle \psi \mid x \rangle\,\langle x \mid \psi \rangle\,dx, \tag{2.52}$$

then we shorthand we have

$$\int |x|^{2\mu} \mid x \rangle \langle x \mid dx = 1. \tag{2.53}$$

To understand these relationships well, we will apply them to a set of physical examples in several dimensions and we will treat them in the relativistic and non-relativistic cases.

Chapitre 3

Applications of the Dunkl operator in non-relativistic case

3.1 Introduction

The particle in a box and harmonic oscillator potentials are one of those few problems that are important to all branches of physics. They provide useful models for a variety of vibrational phenomena that are encountered, for instance, in classical mechanics, electrodynamics, statistical mechanics, solid-state, atomic, nuclear, and particle physics. In quantum mechanics, it serves as an invaluable tool to illustrate the basic concepts and the formalism. For this, we will examine these examples in the framework of the generalized Heisenberg algebra of the Dunkl operator.

3.2 The Dunkl operator in one dimension

In this section, we will seek to find the exact solutions to the problem of a quantum particle in non-relativistic case that is subject to a harmonic oscillating potential and a particle confined to a box in one dimension with the presence of Dunkl operator.

3.2.1 Particle in a box :

Consider a particle of mass m moving in the following potential

$$V(x) = \begin{cases} 0 & (-L < x < L) \\ \infty & \text{elsewhere} \end{cases} . \tag{3.1}$$

The time-independent Schrodinger equation is expressed as follows :

$$-\frac{1}{2m}\hat{D}_x^2\Psi = E\Psi. \tag{3.2}$$

Alternatively, we can write

$$-\frac{1}{2m}\left[\partial^2 + \frac{2\mu}{x}\partial - \frac{\mu}{x^2}\left(1-\hat{R}\right)\right]\Psi = E\Psi. \tag{3.3}$$

As we know the $\hat{R}-$reflection operator forces us to divide the wave equation.(3.3) into two parts :

For the case of even parity is achieved by :

$$-\frac{1}{2m}\left[\partial^2 + \frac{2\mu}{x}\partial\right]\Psi_+ = E_+\Psi_+. \tag{3.4}$$

Using the series method, so the corresponding wave function of Eq. (3.4) writes as :

$$\Psi_+^\lambda = \sum_{n=0}^{\infty} a_n^+ x^{2n}\left|x\right|^\lambda . \tag{3.5}$$

In order to replace the form series of wave function Ψ_+^λ in Eq. (3.4), one must make the first and second derivation on Ψ_+^λ. Indeed we have,

$$\partial\Psi_+ = \sum_{n=0}^{\infty} a_n^+ \frac{(2n+\lambda)}{x} x^{2n}\left|x\right|^\lambda ,$$

and

$$\partial^2\Psi_+ = \sum_{n=0}^{\infty} a_n^+ \left(2n+1\right)\left(2n+\lambda-1\right) x^{2n}\left|x\right|^\lambda . \tag{3.6}$$

After substituting these derivations into an equation.(3.4) we get

$$-\frac{1}{2m}\left[\sum_{n=0}^{\infty} a_n^+ \frac{\left(2n+\lambda\right)\left(2n+\lambda-1\right)}{x^2} x^{2n}\left|x\right|^\lambda + \sum_{n=0}^{\infty} a_n^+ \left(2n+\lambda\right)\frac{2\mu}{x^2}\left|x\right|^\lambda\right]$$
$$= E_+\sum_{n=0}^{\infty} a_n^+ x^{2n}\left|x\right|^\lambda \tag{3.7}$$

Then we will expand the series as follow, we obtain

$$
\begin{aligned}
&-\frac{1}{2m}a_0^+\lambda\left(\lambda-1\right)x^{-2}\left|x\right|^{\lambda}+a_1^+\left(2+\lambda\right)\left(2+\lambda-1\right)\left|x\right|^{\lambda}\\
&+a_2^+\left(4+\lambda\right)\left(3+\lambda\right)x^2\left|x\right|^{\lambda}+....+a_0^+\lambda 2\mu x^{-2}\left|x\right|^{\lambda}\\
&+a_1^+\left(2+\lambda\right)2\mu\left|x\right|^{\lambda}+a_2^+\left(4+\lambda\right)2\lambda x^2\left|x\right|^{\lambda}+....\\
&=E_+\left(a_0^+\left|x\right|^{\lambda}+a_1^+x^2\left|x\right|^{\lambda}+....\right).
\end{aligned}
\tag{3.8}
$$

We find after simplification :

$$
\begin{aligned}
&-\frac{1}{2m}[a_0^+\lambda\left(\lambda-1\right)x^{-2}+a_1^+\left(2+\lambda\right)\left(2+\lambda-1\right)\\
&+a_2^+\left(4+\lambda\right)\left(3+\lambda\right)x^2+....+a_0^+\lambda 2\mu x^{-2}\\
&+a_1^+\left(2+\lambda\right)2\mu+a_2^+\left(4+\lambda\right)2\lambda x^2+....]
\end{aligned}
\tag{3.9}
$$

$$
=E_+\left(a_0^++a_1^+x^2+....\right). \tag{3.10}
$$

All the terms in order (x^{-2}) are,

$$
-\frac{1}{2m}\left[a_0^+\lambda\left(\lambda-1\right)x^{-2}+a_0^+\lambda 2\mu x^{-2}\right]=0. \tag{3.11}
$$

To give the following equation

$$
\lambda\left(\lambda-1\right)+\lambda 2\mu=0. \tag{3.12}
$$

Consequently, in the same method we find for the other terms in order of $x^2, x^4, ...$ and x^{2k},

$$
\begin{aligned}
&-\frac{1}{2m}[a_1^+\left(2+\lambda\right)\left(2+\lambda-1\right)+a_2^+\left(4+\lambda\right)\left(3+\lambda\right)x^2\\
&+....+a_1^+\left(2+\lambda\right)2\mu+a_2^+\left(4+\lambda\right)2\lambda x^2+....\\
&=E_+\left(a_0^++a_1^+x^2+....\right).
\end{aligned}
\tag{3.13}
$$

This equation gives

$$
a_1^+=\frac{-2mE_+}{\left(2+\lambda\right)\left[\lambda+1+2\mu\right]}a_0^+, \tag{3.14}
$$

and

$$
a_2^+=\frac{-2mE_+}{\left(4+\lambda\right)\left(3+\lambda+2\mu\right)}a_1^+. \tag{3.15}
$$

The recurrence relation is discovered by

$$
a_{n+1}^+=\frac{-2mE_+}{\left(n+2+\lambda\right)\left(n+1+\lambda+2\mu\right)}a_n^+. \tag{3.16}
$$

As we know the even wave function is $\Psi_{+}^{\lambda} = \sum_{n=0}^{\infty} a_n^{+} x^{2n} |x|^{\lambda}$. In this stage we replace a_n^{+} by function of a_{n-1}^{+} with using relation (3.16), we find :

$$\Psi_{+}^{\lambda} = \sum_{n=0}^{\infty} \frac{-2mE_{+}}{(n+1+\lambda)(n+\lambda+2\mu)} a_{n-1}^{+} x^{2n} |x|^{\lambda}. \tag{3.17}$$

As a result, we obtain the recurrence relation with the parameter a_0^{+},

$$\Psi_{+}^{\lambda} = \sum_{n=0}^{\infty} \frac{-2mE_{+}}{(n+1+\lambda)(n+\lambda+2\mu)} \frac{-2mE_{+}}{(n+\lambda)(n-1+\lambda+2\mu)} \times ... \frac{-2mE_{+}}{(n+\lambda)(n-1+\lambda+2\mu)} a_0^{+} x^{2n} |x|^{\lambda}. \tag{3.18}$$

Or we can write

$$\Psi_{+}^{\lambda} = a_0^{+} \sum_{n=0}^{\infty} \frac{(-2mE_{+})^{n}}{(n+2+\lambda)(n+1+\lambda+2\mu)} x^{2n} |x|^{\lambda}. \tag{3.19}$$

We use a constant value of $a_0^{+} = 1$.

1- For the case of $\lambda = 0$, we have

$$\Psi_{+}^{\lambda=0} = \sum_{n=0}^{\infty} \frac{(-2mE_{+})^{n}}{(n+2)(n+1+2\mu)} x^{2n}. \tag{3.20}$$

This function is the polynomial hypergeometric

$$\Psi_{+}^{\lambda=0} = {}_0F_1\left(;\frac{1}{2}+\mu; \frac{-mE_{+}x^2}{2}\right). \tag{3.21}$$

2- For the case $\lambda = 1-2\mu$, we have

$$\Psi_{+}^{\lambda=1-2\mu} = \sum_{n=0}^{\infty} \frac{(-2mE_{+})^{n}}{(n+3-2\mu)(n+2)} x^{2n} |x|^{1-2\mu}. \tag{3.22}$$

the above function is

$$\Psi_{+}^{\lambda=1-2\mu} = |x|_0^{1-2\mu} F_1\left(;\frac{3}{2}-\mu; \frac{-mE_{+}x^2}{2}\right). \tag{3.23}$$

When $\mu = 0$ and $\lambda = 0$, the even parity solution is

$$\Psi_{+}^{\lambda=0} = {}_0F_1\left(;\frac{1}{2}; \frac{-mE_{+}x^2}{2}\right). \tag{3.24}$$

we pose $z = \frac{mE_{+}x^2}{2}$

$$\Psi_{+}^{\lambda=0} = {}_0F_1\left(;\frac{1}{2};-z\right) = \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n)} z^{2n}, \tag{3.25}$$

this sum series is $\cos(z)$ function, therefore we obtain,

$$
{}_0F_1\left(;\frac{1}{2};\frac{-mE_+x^2}{2}\right) = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)}\left(2\sqrt{\frac{mE_+x^2}{2}}\right)^{2n}
$$

$$
= \cos\left(2\sqrt{\frac{mE_+x^2}{2}}\right) = \cos\left(\sqrt{2mE}x\right). \tag{3.26}
$$

As a result we find,

$$
\Psi_+^{\lambda=0} = \cos\left(2\sqrt{2mE_+}x\right). \tag{3.27}
$$

The solution $\Psi_+^{\lambda=1-2\mu}$ is unacceptable since it leads to $\frac{|x|}{\sqrt{2mE_+}}\sin\sqrt{2mE_+}x$ in the limit $\mu \longrightarrow 0$. The Eq. (3.21) can be expressed as a Bessel function :

$$
J_n = \frac{\left(\frac{1}{2}x\right)^n}{n!}{}_0F_1\left(n+1;\frac{-1}{4}x^2\right). \tag{3.28}
$$

Thus, we have

$$
{}_0F_1\left(\mu+\frac{1}{2};\frac{-1}{4}\left(\sqrt{2mE_+}x\right)^2\right) = \frac{J_{\mu-\frac{1}{2}}\left(\sqrt{2mE_+}x\right)\left(\mu-\frac{1}{2}\right)!}{\left(\frac{1}{2}\sqrt{2mE_+}x\right)^{\mu+\frac{1}{2}}}. \tag{3.29}
$$

So we can write $\Psi_+^{\lambda=0}$

$$
\Psi_+^{\lambda=0} = N_+x^{\frac{1}{2}-\mu}J_{\mu-\frac{1}{2}}\left(\sqrt{2mE_+}x\right). \tag{3.30}
$$

where $N_+ = \frac{\left(\mu-\frac{1}{2}\right)!}{\left(\frac{1}{2}\sqrt{2mE_+}\right)^{\mu-\frac{1}{2}}}$ is constant normalization of $\Psi_+^{\lambda=0}$–function. As we know, the boundary condition $\Psi_+(\pm L) = 0$ determines the energy level for our system. This results,

$$
\begin{cases} \Psi_+(L) = L^{\frac{1}{2}-\mu}N_+J_{\mu-\frac{1}{2}}\left(\sqrt{2mE_+}L\right) = 0 \\ \Psi_+(-L) = -L^{\frac{1}{2}-\mu}N_+J_{\mu-\frac{1}{2}}\left(-\sqrt{2mE_+}L\right) = 0 \end{cases}, \tag{3.31}
$$

Which gives

$$
\begin{cases} J_{\mu-\frac{1}{2}}\left(\sqrt{2mE_+}L\right) = 0 \\ J_{\mu-\frac{1}{2}}\left(-\sqrt{2mE_+}L\right) = 0 \end{cases}. \tag{3.32}
$$

These equivalence by $\left(\sum_{n=0}^{\infty}\frac{\left(\sqrt{2mE_+}L\right)^n}{n!} = 0\right)$, indeed we have

$$
1+\sqrt{2mE_+}L+\frac{\left(\sqrt{2mE_+}L\right)^2}{2!}+\frac{\left(\sqrt{2mE_+}L\right)^3}{3!}+........ = 0, \tag{3.33}
$$

where

$$
\alpha_{\mu-\frac{1}{2},n} = \frac{-1}{1+\frac{\left(\sqrt{2mE_+}L\right)}{2!}+\frac{\left(\sqrt{2mE_+}L\right)^2}{3!}+......}. \tag{3.34}
$$

Thus we get the level energy for particle in one box

$$E_n^+ = \frac{1}{2mL^2}\alpha^2_{\mu-\frac{1}{2},n}, \qquad (n = 1, 2, 3, ...), \tag{3.35}$$

where $\alpha_{\mu-\frac{1}{2},n}$, is the number of zeros in $J_{\mu-\frac{1}{2}}(x)$.

The solution for odd parity Ψ_- is :

$$\frac{-1}{2m}\left[\partial^2 + \frac{2\mu}{x}\partial - \frac{2\mu}{x^2}\right]\Psi = E_-\Psi_-. \tag{3.36}$$

The corresponding wave function of Eq. (3.36) is :

$$\Psi_-^\lambda = \sum_{n=0}^{\infty} a_n^- x^{2n+1}|x|^\lambda. \tag{3.37}$$

In order to replace this series in Eq. (3.36), one must make the first and second derivation on Ψ_+^λ. Indeed we have.

$$\partial\Psi_- = \sum_{n=0}^{\infty} a_n^-(2n+1+\lambda)x^{2n}|x|^\lambda. \tag{3.38}$$

and

$$\partial^2\Psi_- = \sum_{n=0}^{\infty} a_n^-(2n+1+\lambda)(2n+\lambda)\frac{x^{2n}}{x}|x|^\lambda. \tag{3.39}$$

After substituting these derivations into an equation (3.36) we get

$$\frac{-1}{2m}\left[\begin{array}{c} \sum_{n=0}^{\infty} a_n^-(2n+1+\lambda)(2n+\lambda)\frac{x^{2n}}{x}|x|^\lambda + \\ \frac{2\mu}{x}\sum_{n=0}^{\infty} a_n^-(2n+1+\lambda)x^{2n}|x|^\lambda - \frac{2\mu}{x^2}\sum_{n=0}^{\infty} a_n^- x^{2n+1}|x|^\lambda \end{array}\right] = E_-\sum a_n^- x^{2n+1}|x|^\lambda. \tag{3.40}$$

Then we will expand this series as follow

$$\frac{-1}{2m}\left[\begin{array}{c} a_0^-(1+\lambda)\lambda + a_1^-(3+\lambda)(2+\lambda)x + a_2^-(5+\lambda)(4+\lambda)x^3 \\ +.... + a_0^-(1+\lambda)2\mu + a_1^-(3+\lambda)2\mu x + \end{array}\right] \tag{3.41}$$

After simplification Eq. (3.41) becomes as :

$$\frac{-1}{2m}\left[a_0^-(1+\lambda)\lambda + a_0^-(1+\lambda)2\mu - a_0^-\frac{2\mu}{x}\right] = 0. \tag{3.42}$$

In order (x^0), we find

$$(1+\lambda)\lambda + (1+\lambda)2\mu - 2\mu = 0. \tag{3.43}$$

As a result, we write

$$\lambda\left(1+\lambda+2\mu\right)=0. \tag{3.44}$$

Consequently, for the other terms in order of $x^3, x^5, ...$ and x^{2k+1}, we write :

$$a_1^- = \frac{-2mE_+}{\left(3+\lambda\right)\left(2+\lambda+2\mu\right)} a_0^- \tag{3.45}$$

and

$$a_2^- = \frac{-2mE_+}{\left(5+\lambda\right)\left(4+\lambda+2\mu\right)} a_1^-. \tag{3.46}$$

The recurrence relation deduced from

$$a_{n+1}^- = \frac{-2mE_-}{\left(2n+2+\lambda\right)\left(2n+3+\lambda+2\mu\right)} a_n^-. \tag{3.47}$$

In this stage we replace a_n^+ by function of a_{n-1}^+ with using relation (3.47), we find :

$$\Psi_-^\lambda = \sum_{n=0}^{\infty} \frac{-2mE_-}{\left(2n+\lambda\right)\left(2n+1+\lambda+2\mu\right)} a_{n-1}^- x^{2n} \left|x\right|^\lambda. \tag{3.48}$$

As a result, we obtain the recurrence relation with the term a_0^+,

$$\Psi_-^\lambda = \sum_{n=0}^{\infty} \frac{-2mE_+}{\left(2n+\lambda\right)\left(2n+1+\lambda+2\mu\right)} \frac{-2mE_+}{\left(2n-2+\lambda\right)\left(2n-1+\lambda+2\mu\right)} a_{n-2}^- x^{2n} \left|x\right|^\lambda. \tag{3.49}$$

Likewise for the rest, we obtain

$$\Psi_-^\lambda = \sum_{n=0}^{\infty} \frac{-2mE_-}{\left(2n+\lambda\right)\left(2n+1+\lambda+2\mu\right)} \frac{-2mE_-}{\left(2n-2+\lambda\right)\left(2n-1+\lambda+2\mu\right)} \times .... \frac{-2mE_-}{\left(2n+2+\lambda\right)\left(2n-3+\lambda+2\mu\right)} a_0^- x^{2n} \left|x\right|^\lambda, \tag{3.50}$$

so

$$\Psi_-^\lambda = \sum_{n=0}^{\infty} \frac{-2mE_-}{\left(2n+2+\lambda\right)\left(2n-3+\lambda+2\mu\right)} x^{2n} \left|x\right|^\lambda. \tag{3.51}$$

We use a constant value of $a_0^- = 1$.

1- For the case of $\lambda = 0$, we have

$$\Psi_-^{\lambda=0} = \frac{-2mE_-}{2\left(3+2\mu\right)} x^3. \tag{3.52}$$

This function is the polynomial hypergeometric

$$\Psi_-^{\lambda=0} = x_0F_1\left(;\frac{3}{2}+\mu;\frac{-mE_-x^2}{2}\right). \tag{3.53}$$

2- For the case $\lambda = 1 - 2\mu$, we have,

$$\Psi_{-}^{\lambda=-1-2\mu} = \frac{-2mE_{-}}{(2-1-2\mu)(3-1-2\mu+2\mu)} x^{3} |x|^{-1-2\mu}, \tag{3.54}$$

the above function is

$$\Psi_{-}^{\lambda=-1-2\mu} = |x|^{-1-2\mu} x {}_0F_1\left(;\frac{1}{2}-\mu;\frac{-mE_{-}x^{2}}{2}\right). \tag{3.55}$$

We have the odd parity solution for$\mu = 0$

$$\Psi_{-} \longrightarrow \sin\sqrt{2mE_{-}}x. \tag{3.56}$$

The solution $\Psi_{+}^{\lambda=1-2\mu}$ is unacceptable since it leads to $\frac{|x|}{\sqrt{2mE_{+}}}\cos\sqrt{2mE_{-}}x$in the limit $\mu \longrightarrow 0$. The Eq. (3.53) can be expressed as a Bessel function :

$$J_{n} = \frac{\left(\frac{1}{2}x\right)^{n}}{n!} {}_0F_1\left(n+1;\frac{-1}{4}x^{2}\right). \tag{3.57}$$

Thus, we have

$${}_0F_1\left(\mu+\frac{3}{2};\frac{-1}{4}\left(\sqrt{2mE_{-}}x\right)^{2}\right) = \frac{J_{\mu+\frac{1}{2}}\left(\sqrt{2mE_{-}}x\right)\left(\mu+\frac{1}{2}\right)!}{\left(\frac{1}{2}\sqrt{2mE_{-}}x\right)^{\mu+\frac{1}{2}}}. \tag{3.58}$$

So we can write $\Psi_{-}^{\lambda=0}$

$$\Psi_{-} = x^{\frac{1}{2}-\mu} N_{-} J_{\mu+\frac{1}{2}}\left(\sqrt{2mE_{-}}x\right). \tag{3.59}$$

where $N_{+} = \frac{\left(\mu-\frac{1}{2}\right)!}{\left(\frac{1}{2}\sqrt{2mE_{+}}\right)^{\mu-\frac{1}{2}}}$ is a constant normalization of $\Psi_{-}^{\lambda=0}-$function. As we know, the boundary condition $\Psi_{-}(\pm L) = 0$ determines the energy level for our system. This results is

$$\begin{cases} \Psi_{-}(L) = L^{\frac{1}{2}-\mu} N_{-} J_{\mu+\frac{1}{2}}\left(\sqrt{2mE_{-}}L\right) = 0 \\ \Psi_{-}(-L) = -L^{\frac{1}{2}-\mu} N_{-} J_{\mu+\frac{1}{2}}\left(-\sqrt{2mE_{-}}L\right) = 0 \end{cases}. \tag{3.60}$$

Which gives

$$\begin{cases} J_{\mu+\frac{1}{2}}\left(\sqrt{2mE_{-}}L\right) = 0 \\ J_{\mu+\frac{1}{2}}\left(-\sqrt{2mE_{-}}L\right) = 0 \end{cases}. \tag{3.61}$$

This gives ($\sum\limits_{n=0}^{\infty} \frac{\left(\sqrt{2mE_{-}L}\right)^{n}}{n!} = 0$), indeed we have

$$1 + \sqrt{2mE_{-}}L + \frac{\left(\sqrt{2mE_{-}}L\right)^{2}}{2!} + \frac{\left(\sqrt{2mE_{-}}L\right)^{3}}{3!} + ... = 0, \tag{3.62}$$

which gives

$$\sqrt{2mE_{-}}L\left(1+\frac{\left(\sqrt{2mE_{-}}L\right)}{2!}+\frac{\left(\sqrt{2mE_{-}}L\right)^{2}}{3!}+........\right)=-1, \tag{3.63}$$

where

$$\alpha_{\mu+\frac{1}{2},n}=\frac{-1}{1+\frac{\left(\sqrt{2mE_{-}}L\right)}{2!}+\frac{\left(\sqrt{2mE_{-}}L\right)^{2}}{3!}+......}. \tag{3.64}$$

Thus, we get the level energy for particle in one box :

$$E_{n}^{-}=\frac{1}{2mL^{2}}\alpha_{\mu+\frac{1}{2},n}^{2}, \qquad (n=1,2,3,....), \tag{3.65}$$

where $\alpha_{\mu+\frac{1}{2},n}$,n is the number of zeros in $J_{\mu+\frac{1}{2}}(x)$.

3.2.2 Harmonic Oscillator potential

The Hamiltonian of a particle of mass m which oscillates with frequency ω under the influence of a one-dimensional harmonic potential is $\hat{H}=\frac{\hat{p}^{2}}{2m}+\frac{1}{2}m\omega^{2}x^{2}$. The problem is how to find the energy eigenvalues and eigenstates of this Hamiltonian in the existence of the Dunkl derivation. Two methods can study this problem, firstly by direct calculation (differential equation), and the second method, called the ladder method, does not deal with solving the Schrodinger equation, it deals instead with operator algebra involving operators known as the creation and annihilation operators.

The Schrodinger equation for the reflection symmetry problem of harmonic oscillators potential in one dimension is written as

$$\left[-\frac{1}{2m}\hat{D}_{x}^{2}+\frac{1}{2}m\omega^{2}x^{2}\right]\Psi=E\Psi. \tag{3.66}$$

If we use the new variable$\sqrt{m\omega}x=\zeta$, we get

$$\hat{D}_{\zeta}^{2}\Psi+\zeta^{2}\Psi=\epsilon\Psi. \tag{3.67}$$

where

$$\epsilon=\frac{2E}{\omega}. \tag{3.68}$$

We assume the following Anzast Ψ :

$$\Psi(\zeta)=e^{-\frac{\zeta^{2}}{2}}y(\zeta). \tag{3.69}$$

From Eq. (3.69) and Eq. (3.67) we have

$$\hat{D}_{\zeta}^{2} y-\hat{D}_{\zeta}(\zeta y)-\zeta \hat{D}_{\zeta} y+\epsilon y=0. \tag{3.70}$$

or

$$\hat{D}_{\zeta}^{2} y-2 \zeta \hat{D}_{\zeta} y+(\epsilon-1-2 \mu p) y=0. \tag{3.71}$$

Even solution

We assume the even solution by the following series

$$y=\sum_{n=0}^{\infty} a_{n} \zeta^{2 n}. \tag{3.72}$$

On the monomial, the Dunkl derivative is used

$$\hat{D}_{\zeta} \zeta^{2 n}=[2 n]_{\mu} \zeta^{2 n-1}. \tag{3.73}$$

and

$$\hat{D}_{\zeta}\left(\hat{D}_{\zeta} \zeta^{2 n}\right)=\hat{D}_{\zeta}\left([2 n]_{\mu} \zeta^{2 n-1}\right)=[2 n]_{\mu}[2 n-1]_{\mu} \zeta^{2 n-2}. \tag{3.74}$$

From Eqs. (3.73) and (3.74) we substitute them in Eq. (3.71) , we find

$$\begin{aligned} \sum_{n=0}^{\infty} a_{n}[2 n]_{\mu}[2 n-1]_{\mu} \zeta^{2 n-2}-2 \zeta \sum_{n=0}^{\infty} a_{n}[2 n]_{\mu} \zeta^{2 n-1} \\ +(\epsilon-1-2 \mu) \sum_{n=0}^{\infty} a_{n} \zeta^{2 n}=0. \end{aligned} \tag{3.75}$$

We find out after simplifying it

$$\begin{aligned} (\epsilon-1-2 \mu) a_{0} \zeta^{0}+a_{1}[2]_{\mu}[1]_{\mu} \zeta^{0}-2 a_{1}[2]_{\mu} \zeta^{2}+(\epsilon-1-2 \mu) a_{1} \zeta^{2} \\ +a_{2}[4]_{\mu}[3]_{\mu} \zeta^{2}-2 a_{2}[4]_{\mu} \zeta^{4}+(\epsilon-1-2 \mu) a_{2} \zeta^{4}=0. \end{aligned} \tag{3.76}$$

The recurrence relation given by

$$\begin{aligned} a_{1} & =\frac{2[0]_{\mu}+1+2 \mu-\epsilon_{+}}{(2)(1+2 \mu)} a_{0} \\ & =\frac{2[0]_{\mu}+1+2 \mu-\epsilon_{+}}{[2]_{\mu}[1]_{\mu}} a_{0}. \end{aligned} \tag{3.77}$$

and

$$\begin{aligned} a_{2} & =\frac{2[2]_{\mu}+1+2 \mu-\epsilon_{+}}{(4)(3+2 \mu)} a_{1} \\ & =\frac{2[2]_{\mu}+1+2 \mu-\epsilon_{+}}{[4]_{\mu}[3]_{\mu}} a_{1}. \end{aligned} \tag{3.78}$$

also

$$a_3 = \frac{2\,[4]_\mu + 1 + 2\mu - \epsilon_+}{(6)\,(5+2\mu)} a_2 = \frac{2\,[4]_\mu + 1 + 2\mu - \epsilon_+}{[6]_\mu\,[5]_\mu} a_2 = \frac{2\,[4]_\mu + 1 + 2\mu - \epsilon_+}{[2*2+2]_\mu\,[2*2+1]_\mu} a_2. \tag{3.79}$$

The recurrence relation deduced from

$$a_{n+1} = \frac{2\,[2n]_\mu + 1 + 2\mu - \epsilon_+}{[2n+2]_\mu\,[2n+1]_\mu} a_n. \tag{3.80}$$

Series termination request, we have

$$2\,[2n]_\mu + 1 + 2\mu - \epsilon_+ = 0. \tag{3.81}$$

which gives

$$(\epsilon_+)_N = 2\,[2N]_\mu + 1 + 2\mu, \qquad N = 0, 1, 2, \ \dots\ . \tag{3.82}$$

From Eq. (3.82) and $\epsilon_+ = \frac{2E}{\omega}$, the energy level is written as

$$E_N^+ = \frac{\omega}{2}\left(2\,[2N]_\mu + 1 + 2\mu\right). \tag{3.83}$$

In order to obtain the corresponding wave function, we give a polynomial's recurrence relation, we have,

$$a_{n+1} = \frac{2\,[2n]_\mu - 2\,[2N]_\mu}{[2n+2]_\mu\,[2n+1]_\mu} a_n. \tag{3.84}$$

At this stage we define the H_N^+ 's in the function$y$ corresponding to$N$:

$$\begin{cases} H_0^+(x) = 1. \\ H_1^+(x) = 1 + \frac{2\left([0]_\mu - [2]_\mu\right)}{[2]_\mu[1]_\mu} x^2 = 1 - \frac{2}{[1]_\mu} x^2 \\ H_2^+(x) = 1 + \frac{2\left([0]_\mu - [4]_\mu\right)}{[2]_\mu[1]_\mu} x^2 + \frac{2\left([2]_\mu - [4]_\mu\right)}{[2]_\mu[3]_\mu} \frac{2[4]_\mu}{[2]_\mu} x^4 \end{cases}. \tag{3.85}$$

Therefore the final solution is

$$\Psi_N(\zeta) = e^{\frac{-\zeta^2}{2}} H_N^+, \qquad N = 0.1.2.3\ \dots\ .$$

Odd solution

We put the even solution

$$y = \sum_{n=0}^{\infty} b_n \zeta^{2n+1}. \tag{3.86}$$

We insert the Eq.(3.86) into Eq.(3.71), we have

$$\hat{D}_\zeta \zeta^{2n+1} = [2n+1]_\mu \zeta^{2n}. \tag{3.87}$$

and

$$\hat{D}_\zeta \left(\hat{D}_\zeta \zeta^{2n+1} \right) = \hat{D}_\zeta \left([2n+1]_\mu \zeta^{2n} \right) = [2n+1]_\mu [2n]_\mu \zeta^{2n-1}. \tag{3.88}$$

These lead to

$$\sum_{n=0}^{\infty} b_n [2n+1]_\mu [2n]_\mu \zeta^{2n-1} - 2\zeta \sum_{n=0}^{\infty} b_n [2n+1]_\mu \zeta^{2n} + (\epsilon - 1 + 2\mu) \sum_{n=0}^{\infty} b_n \zeta^{2n+1} = 0. \tag{3.89}$$

after simplification

$$\begin{aligned} -2b_0 [1]_\mu \zeta^1 + (\epsilon - 1 + 2\mu) b_0 \zeta^1 + b_1 [3]_\mu [2]_\mu \zeta^1 - 2b_1 [2]_\mu \zeta^3 \\ +b_2 [5]_\mu [4]_\mu \zeta^3 - 2b_2 [5]_\mu \zeta^5 + (\epsilon - 1 + 2\mu) b_2 \zeta^5 = 0. \end{aligned} \tag{3.90}$$

In accordance with the terms we receive

$$b_1 = \frac{2 [1]_\mu + 1 - 2\mu - \epsilon_-}{[3]_\mu [2]_\mu} b_0, \tag{3.91}$$

and

$$b_2 = \frac{2 [3]_\mu + 1 - 2\mu - \epsilon_-}{[5]_\mu [4]_\mu} b_1, \tag{3.92}$$

also

$$b_3 = \frac{2 [5]_\mu + 1 - 2\mu - \epsilon_-}{[7]_\mu [6]_\mu} b_2. \tag{3.93}$$

The recurrence relation deduced from

$$b_{n+1} = \frac{2 [2n+1]_\mu + 1 - 2\mu - \epsilon_-}{[2n+3]_\mu [2n+2]_\mu} b_n \tag{3.94}$$

Requiring that the series be terminated, we have

$$(\epsilon_-)_N = 2 [2N+1]_\mu + 1 - 2\mu, \qquad N = 0, 1, 2, \ \dots \ . \tag{3.95}$$

From Eq. (3.95) and $\epsilon_- = \frac{2E}{\omega}$, the energy level is written as

$$E_N^- = \frac{\omega}{2} \left(2 [2N+1]_\mu + 1 - 2\mu \right). \tag{3.96}$$

Then there's the polynomial solution, which has the following recurrence relation :

$$b_{n+1} = \frac{2\left[2n+1\right]_{\mu} + 1 - 2\mu - 2\left[2N+1\right]_{\mu} - 1 + 2\mu}{\left[2n+3\right]_{\mu}\left[2n+2\right]_{\mu}} b_n. \tag{3.97}$$

In order to obtain the corresponding wave function, we give a polynomial's recurrence relation, we have,

$$b_{n+1} = \frac{2\left[2n+1\right]_{\mu} - 2\left[2N+1\right]_{\mu}}{\left[2n+3\right]_{\mu}\left[2n+2\right]_{\mu}} b_n \tag{3.98}$$

Therefore the finale solution is

$$\Psi_N\left(\zeta\right) = e^{\frac{-\zeta^2}{2}} H_N^-, \quad N = 0, 1, 2 \ldots . \tag{3.99}$$

At this stage we define the H_N^- 's in the function y corresponding to N

$$\begin{cases} H_0^-\left(x\right) = x \\ H_1^-\left(x\right) = x + \frac{2\left(\left[1\right]_{\mu} - \left[3\right]_{\mu}\right)}{\left[3\right]_{\mu}\left[2\right]_{\mu}} x^3 = x - \frac{2\left(\left[3\right]_{\mu} - \left[1\right]_{\mu}\right)}{\left[3\right]_{\mu}\left[2\right]_{\mu}} x^3 \\ \quad = x - \frac{2\left(\left[3\right]_{\mu} - \left[1\right]_{\mu}\right)}{\left[3\right]_{\mu}!} x^3 \\ H_2^-\left(x\right) = x - \frac{2\left(\left[5\right]_{\mu} - \left[1\right]_{\mu}\right)}{\left[3\right]_{\mu}\left[2\right]_{\mu}} + \frac{2\left(\left[5\right]_{\mu} - \left[3\right]_{\mu}\right)}{\left[5\right]_{\mu}\left[4\right]_{\mu}} \frac{2\left(\left[5\right]_{\mu} - \left[1\right]_{\mu}\right)}{\left[3\right]_{\mu}\left[2\right]_{\mu}} x^5 \\ \quad = x - \frac{2\left(\left[5\right]_{\mu} - \left[1\right]_{\mu}\right)}{\left[3\right]_{\mu}!} + \frac{2^2\left(\left[5\right]_{\mu} - \left[3\right]_{\mu}\left[5\right]_{\mu} - \left[1\right]_{\mu}\right)}{\left[5\right]_{\mu}!} x^5. \end{cases} . \tag{3.100}$$

Use odd and even solutions

$$\Psi_M\left(\zeta\right) = e^{\frac{-\zeta^2}{2}} H_M^{\mu}, \quad M = 0, 1, 2 \ldots . \tag{3.101}$$

The energy level is

$$E_M = \frac{\omega}{2}\left(\left[M\right]_{\mu} + \left[M+1\right]_{\mu}\right). \tag{3.102}$$

3.2.3 Operator method for harmonic oscillator potential

Now we have presented the Hamiltonian operator for the reflection symmetry problem in harmonic oscillators in one dimension by the creation and annihilation operators a^+ and a method takes the following form :

$$\hat{a} = \sqrt{\frac{m\omega}{2}}\left(\hat{x} + \frac{1}{m\omega}\hat{D}_x\right), \quad \hat{a}^+ = \sqrt{\frac{m\omega}{2}}\left(\hat{x} - \frac{1}{m\omega}\hat{D}_x\right). \tag{3.103}$$

Which given by

$$\hat{H} = \frac{\omega}{2}\left(\hat{a}\hat{a}^+ + \hat{a}^+\hat{a}\right). \tag{3.104}$$

The commutation relation between $\hat{a}$ and $\hat{a}^+$ is developed as follows :

$$\begin{aligned}\left[\hat{a},\hat{a}^+\right] &= \left[\sqrt{\frac{m\omega}{2}}\left(\hat{x}+\frac{1}{m\omega}\hat{D}_x\right),\sqrt{\frac{m\omega}{2}}\left(\hat{x}-\frac{1}{m\omega}\hat{D}_x\right)\right] \\ &= \frac{m\omega}{2}\left([\hat{x},\hat{x}]+\left[\hat{x},\frac{-1}{m\omega}\hat{D}_x\right]+\left[\frac{1}{m\omega}\hat{D}_x,\hat{x}\right]+\left[\frac{1}{m\omega}\hat{D}_x,\frac{-1}{m\omega}\hat{D}_x\right]\right). \end{aligned} \tag{3.105}$$

As we know $[\hat{x},\hat{x}]=0$ and $\left[\hat{D}_x,\hat{D}_x\right]=0$, then we conclude that,

$$\begin{aligned}\left[\hat{a},\hat{a}^+\right] &= \frac{m\omega}{2}\left(\frac{-\imath}{m\omega}\left[\hat{x},\frac{1}{\imath}\hat{D}_x\right]+\left[\frac{\imath}{m\omega}\hat{D}_x,\hat{x}\right]\right) \\ &= \frac{m\omega}{2}\left(\frac{1}{m\omega}\right)\left[\left(1+2\mu\hat{R}\right)-\imath\left(\imath\left(1+2\mu\hat{R}\right)\right)\right]\end{aligned} \tag{3.106}$$

As a result,

$$\left[\hat{a},\hat{a}^+\right]=1+2\mu\hat{R}. \tag{3.107}$$

If we use the number operator $\hat{N}=\hat{a}^+\hat{a}$, the commutation relation $\left[\hat{N},\hat{a}^+\right]$ is developed by :

$$\begin{aligned}\left[\hat{N},\hat{a}^+\right] &= \left[\hat{a}^+\hat{a},\hat{a}^+\right]=\left[\hat{a}^+\hat{a}\hat{a}^+-\hat{a}^+\hat{a}^+\hat{a}\right] \\ &= \hat{a}^+\left[\hat{a}\hat{a}^+-\hat{a}^+\hat{a}\right]=\hat{a}^+\left[N+1-N\right] \\ &= \hat{a}^+.\end{aligned} \tag{3.108}$$

The same for the commutation $\left[\hat{N},\hat{a}\right]$ relation is given by

$$\begin{aligned}\left[\hat{N},\hat{a}\right] &= \left[\hat{a}^+\hat{a},\hat{a}\right]=\left[\hat{a}^+\hat{a}\hat{a}-\hat{a}\hat{a}^+\hat{a}\right] \\ &= \left[\hat{a}\hat{a}^+-\hat{a}^+\hat{a}\right]\hat{a}=\left[\hat{N}-\hat{N}-1\right]\hat{a} \\ &= -\hat{a}.\end{aligned} \tag{3.109}$$

Consequently we conclude that,

$$\left[\hat{N},\hat{a}^+\right]=\hat{a}^+,\ \left[\hat{N},\hat{a}\right]=-\hat{a}. \tag{3.110}$$

In order to obtain the eigenvalues for the Hamiltonian operator (3.104) we use the following relations

$$\hat{a}^+\hat{a}=\left[\hat{N}\right]_\mu,\quad \hat{a}\hat{a}^+=\left[\hat{N}+1\right]_\mu. \tag{3.111}$$

then by describing the Fock space :

$$\hat{N}\mid n\rangle=n\mid n\rangle,\quad n=0,1,2,3..... \tag{3.112}$$

As a result, the energy level is

$$E_n = \frac{\omega}{2}\left([n]_\mu + [n+1]_\mu\right), \; n = 0, 1, 2 \; ... \; . \tag{3.113}$$

The same result of Eq. (3.113) is given in previous section (see, Eq. (3.102)).

3.3 The Dunkl operator in 2 and 3 dimensions

In this section, we examine how to solve the Schrodinger equation for spinless particles moving in two and three dimensions potentials. We carry out this study of the Box and oscillator harmonic potentials in two different dimensions in Cartesian coordinates system.

3.3.1 The Box Potential in Cartesian Coordinates

We consider the case of a spinless particle of mass m confined in rectangular box of sides L_x, L_y, which can be defined as $V(x,y) = V_x(x) + V_y(y)$,

$$V_{x_j}(x_{j=1,2}) = \begin{cases} 0, & 0 \prec x_j \prec L_{x_j} \\ \infty & \text{elsewhere,} \end{cases} \tag{3.114}$$

The wave function $\psi(x,y)$ must vanish at the walls of the box. We have seen in previous section the solutions for this potential with even case are of the form

$$\Psi_+^{\lambda=0} = N_+^x x^{\frac{1}{2}-\mu_x} J_{\mu_x-\frac{1}{2}}\left(\sqrt{2mE_+}x\right). \tag{3.115}$$

and the corresponding levels energy are

$$E_{n_x}^+ = \frac{1}{2mL^2}\alpha^2_{\mu_x-\frac{1}{2},n}, \qquad (n = 1, 2, 3, \; ...). \tag{3.116}$$

where $\alpha_{\mu_x-\frac{1}{2},n}$ is a n-th of zeros in $J_{\mu-\frac{1}{2}}(x)$, and $J_{\mu-\frac{1}{2}}(x)$ is the Bessel function. From these expressions we can conclude the corresponding eigenfunctions and their levels energy :

$$\Psi_+^{\lambda=0}(x,y) = N_+^x N_+^y x^{\frac{1}{2}-\mu_x} y^{\frac{1}{2}-\mu_y} \times J_{\mu_x-\frac{1}{2}}\left(\sqrt{2mE_+}x\right) J_{\mu_y-\frac{1}{2}}\left(\sqrt{2mE_+}y\right). \tag{3.117}$$

and

$$E_{n_x,n_y}^+ = \frac{1}{2mL_x^2}\frac{1}{2mL_y^2}\alpha^2_{\mu_x-\frac{1}{2},n_x}\alpha^2_{\mu_y-\frac{1}{2},n_y}. \tag{3.118}$$

The same remark we will do in odd case.

In the case when $D = 3$ dimensions we find the level energy for a particle in box as follow

$$E^{+}_{n_x,n_y,n_z} = \frac{1}{2mL_x^2}\frac{1}{2mL_y^2}\frac{1}{2mL_z^2}\alpha^2_{\mu_x-\frac{1}{2},n_x}\alpha^2_{\mu_y-\frac{1}{2},n_y}\alpha^2_{\mu_z-\frac{1}{2},n_z}. \tag{3.119}$$

3.3.2 Oscillator Harmonic in Cartesian coordinates

The Schrodinger equation for the reflection symmetry problem of harmonic oscillators potential in two dimensions is developed by :

$$\left[-\frac{\hbar^2}{2m}\hat{D}_1^2 - \frac{\hbar^2}{2m}\hat{D}_2^2 + \frac{m\omega^2}{2}(x_1^2 + x_2^2)\right]\psi = E\psi. \tag{3.120}$$

We use the separation method variable, the solution of Eq. (4.38), indeed we have,

$$\psi = \psi(x_1)\,\psi(x_2)\,. \tag{3.121}$$

and

$$\hat{H} = \hat{H}_1 + \hat{H}_2. \tag{3.122}$$

where

$$H_j = -\hat{D}_j^2 + \frac{m^2\omega^2}{\hbar^2}x_j^2 \quad \text{with } j = 1, 2. \tag{3.123}$$

We assume the following definition

$$\frac{2m}{\hbar^2}E = \varepsilon_1 + \varepsilon_2 \tag{3.124}$$

Consequently, the Eq. (4.38) reduces by two wave equation for each value of j,

$$\left\{\frac{\partial^2}{\partial x_j^2} + 2\frac{\mu_j}{x_j}\left(\mathbb{I} - \hat{R}_j\right)\frac{\partial}{\partial x_j} - \frac{\mu_j}{x_j^2}(\mathbb{I} - \hat{R}_j) - \frac{m^2\omega^2}{\hbar^2}x_j^2 + \frac{2m}{\hbar}E\right\}\psi(x_j) = 0 \tag{3.125}$$

In addition, since the commutator between $\hat{H}_j$ and $\hat{R}_j$ equals zero (i.e., $\left[\hat{H}_j, \hat{R}_j\right] = 0$) the eigenfunctions could be selected as they have a definite parity, $\hat{R}_j\psi(x_j) = s_j\psi(x_j)$ with $s_j = \pm 1$. Eq (4.43) becomes as

$$\left\{\frac{\partial^2}{\partial x_j^2} + 2\frac{\mu_j}{x_j}(1 - s_j)\frac{\partial}{\partial x_j} - \frac{\mu_j}{x_j^2}(1 - s_j) - \frac{m^2\omega^2}{\hbar^2}x_j^2 - \varepsilon_j\right\}\psi(x_j) = 0 \tag{3.126}$$

Then, we set the following transformations

$$\xi_j = \frac{m\omega}{\hbar}x_j^2, \quad \psi(\xi_j) = \xi_j^{\frac{1-s_j}{4}} e^{-\frac{\xi_j}{2}}\phi^{s_j}(\xi_j)\,. \tag{3.127}$$

Thus, Eq. (4.43) becomes as

$$\left\{\xi_j \frac{\partial^2}{\partial\ \xi_j^2} + \left(1 + \mu_j - \frac{s_j}{2} - \xi_j\right) \frac{\partial}{\partial \xi_j} + n_j\right\} \phi^{s_j}\left(\xi_j\right) = 0 \tag{3.128}$$

As we know the above equation (4.46) is similar to the problem which has solved in the previous chapter, and it becomes the following solution

$$\phi_n^{s_j}\left(\xi\right) = cF\left(a, b; \xi\right) = cF\left(\frac{\left(2\mu_j + 1\right)\left(1 - s_j\right)}{4} - \frac{\hbar \varepsilon_j}{4m\omega}, 1 - \frac{s_j}{2} + \mu, \xi\right). \tag{3.129}$$

The first argument is equal to as, the confluent hypergeometric function simplifies to a polynomial of degree n in y. We conclude that,

$$\frac{\left(2\mu_j + 1\right)\left(1 - s_j\right)}{4} - \frac{\hbar \varepsilon_j}{4m\omega} = -n_j, \tag{3.130}$$

$$-\left(\mu_j + \frac{1}{2}\right)\left(1 - s_j\right) + \frac{\hbar \varepsilon_j}{2m\omega} = 2n_j, \tag{3.131}$$

$$\frac{\hbar \varepsilon_j}{2m\omega} = 2n_j + \left(\mu_j + \frac{1}{2}\right)\left(1 - s_j\right), \tag{3.132}$$

where n_j is non-negative integer quantum numbers. The energy eigenvalue function, which is parity dependent, is quantized as follows

$$\varepsilon_{n_j}^{s_j} = 2\frac{\hbar}{m\omega}\left[2n_j + \left(\frac{1}{2} + \mu_j\right)\left(1 - s_j\right)\right]. \tag{3.133}$$

The generic solution of Equation .(4.46) can therefore be expressed in terms of the associated Laguerre polynomials as

$$\phi_{n_j}^{s_j}\left(\xi_j\right) = C_{s_j} L_{n_j}^{\mu_j - \frac{s_j}{2}}\left(\frac{m\omega}{\hbar} x_j^2\right). \tag{3.134}$$

where C_{s_j} is a constant normalization term that can be calculated using Equation.(4.13). We have,

$$\int x^{\alpha} e^{-x} L_{n_j}^{\mu_j - \frac{s_j}{2}}\left(x\right) L_{m_j}^{\mu_j - \frac{s_j}{2}}\left(x\right) = \delta_{nm} \frac{(n-\alpha)!}{n!}. \tag{3.135}$$

This relation is transformed by following new variable

$$\xi = \frac{m\omega}{\hbar} x^2 \Longrightarrow dx = \frac{d\xi}{2\sqrt{\frac{m\omega}{\hbar}\xi}}. \tag{3.136}$$

Therefore Eq. (4.13) is simplified by

$$\langle \psi \mid \psi \rangle = \left|c_{s_j}\right|^2 \int \frac{d\xi}{2\sqrt{\frac{m\omega}{\hbar}\xi}} \xi^{\frac{1-s_j}{2}} e^{-\xi} \left(L_{n_j}^{\mu_j - \frac{s_j}{2}}(\xi)\right)^2 \left|\sqrt{\frac{\xi}{m\omega}}\right|^{2\mu_j} \tag{3.137}$$

$$= \frac{\left|c_{s_j}\right|^2}{2\sqrt{\frac{m\omega}{\hbar}}} \int d\xi \xi^{\frac{-s_j}{2}} e^{-\xi} \left(L_{n_j}^{\mu_j - \frac{s_j}{2}}(\xi)\right)^2 \left|\sqrt{\frac{\xi}{\frac{m\omega}{\hbar}}}\right|^{2\mu_j} \tag{3.138}$$

$$= \frac{\left|c_{s_j}\right|^2}{2\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}+\mu_j}} \int \xi^{\mu_j + \frac{s_j}{2}} e^{-\xi} \left(L_{n_j}^{\mu_j - \frac{s_j}{2}}(\xi)\right)^2 d\xi. \tag{3.139}$$

This given

$$\langle \psi \mid \psi \rangle = \frac{\left|c_{s_j}\right|^2}{2\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}+\mu_j}} \frac{(n_j + \mu_j - s_j)!}{n_j!} = 1$$

$$\Longrightarrow \left|c_{s_j}\right|^2 = 2\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}+\mu_j} \frac{n_j!}{(n_j + \mu_j - s_j)!}. \tag{3.140}$$

We get the following expression for the normalization constancy after a computation

$$c_{s_j} = \sqrt{2\left(\frac{m\omega}{\hbar}\right)^{\frac{1}{2}+\mu_j} \frac{n_j!}{(n_j + \mu_j - s_j)!}}. \tag{3.141}$$

In order to obtain the level energy for our system we use the equation (4.42) . We can write,

$$E = \frac{\hbar(\varepsilon_1 + \varepsilon_2)}{2m}, \tag{3.142}$$

and ε_j are calculated in Eqs. (4.51), then we find,

$$E_{n_1,n_2}^{(s)} = 2\frac{\hbar}{m\omega}\left[\left[2n_1 + \left(\frac{1}{2} + \mu_1\right)(1 - s_1)\right]\right.$$
$$\left.\left[2n_2 + \left(\frac{1}{2} + \mu_2\right)(1 - s_2)\right]\right]. \tag{3.143}$$

where $n_{j=1,2} = 0, 1, 2, \ldots$.

In three dimensions we can find the same result in Ref.[18].

3.4 The Dunkl operator in polar coordinates

Using polar and spherical coordinates, we describe the motion of a particle moving in polar and spherically symmetric potentials. After presenting a general treatment of method separation in existence the reflection operator, we will conclude this section by calculating the energy levels of the isotropic harmonic oscillator.

Harmonic oscillator in two dimensions

The Cartesian coordinates (x, y) of a vector $\mathbf{x}$ are related to its spherical polar coordinates (ρ, φ) by

$$x_1 = \rho\cos\varphi, \qquad y = \rho\sin\varphi. \tag{3.144}$$

From Eq. (3.120) and Eq. (3.144) the Hamiltonian $\hat{H}$ operator can be written as

$$\hat{H} = A_\rho + \frac{1}{\rho^2}B_\varphi. \tag{3.145}$$

where $\hat{A}_\rho$ and $\hat{B}_\varphi$ are given by respectively

$$\hat{A}_\rho = -\frac{1}{2}\left[\partial_\rho^2 + \frac{1}{\rho}\partial_\varphi\right] - \frac{1}{\rho}\left(\mu_x + \mu_y\right)\partial_\rho + \frac{1}{2}\rho^2. \tag{3.146}$$

$$\hat{B}_\varphi = -\frac{1}{2}\partial_\varphi^2 + \left(\mu_x\tan\varphi + \mu_y\cot\varphi\right)\partial_\varphi + \mu_x\frac{\left(I - \hat{R}_x\right)}{2\cos^2\varphi} + \mu_y\frac{\left(I - \hat{R}_y\right)}{2\sin^2\varphi}. \tag{3.147}$$

with $m = \omega = \hbar = 1$.

In polar coordinates, the reflection operator act on the wave function gives to the following transformations :

$$\hat{R}_x f\left(\rho, \varphi\right) = f\left(\rho, \pi - \varphi\right), \tag{3.148}$$

$$\hat{R}_y f\left(\rho, \varphi\right) = f\left(\rho, -\varphi\right),. \tag{3.149}$$

After this step we can use the separation method with $\Psi\left(\rho, \varphi\right) = P\left(\rho\right)\Phi\left(\varphi\right)$. This gives

$$\left(\hat{A}_\rho + \frac{1}{\rho^2}\hat{B}_\varphi\right)P\left(\rho\right)\Phi\left(\varphi\right) - \varepsilon P\left(\rho\right)\Phi\left(\varphi\right) = 0. \tag{3.150}$$

Or in other form

$$\Phi\left(\varphi\right)\rho^2\hat{A}_\rho P\left(\rho\right) - \Phi\left(\varphi\right)P\left(\rho\right)\varepsilon\rho^2 + P\left(\rho\right)\hat{B}_\varphi\Phi\left(\varphi\right) = 0. \tag{3.151}$$

After this we devise Eq. (3.151) by $P\left(\rho\right)\Phi\left(\varphi\right)$, we find

$$\frac{1}{P\left(\rho\right)}\rho^2\hat{A}_\rho P\left(\rho\right) - \rho^2\varepsilon + \Phi\left(\varphi\right)\hat{B}_\varphi\Phi\left(\varphi\right) = 0. \tag{3.152}$$

Consequently, Eq. (3.152) leads to the following two equations

$$\hat{A}_\rho P\left(\rho\right) - \varepsilon P\left(\rho\right) = -\frac{m^2}{2\rho^2}P\left(\rho\right). \tag{3.153}$$

and

$$\hat{B}_\varphi \Phi(\varphi) = \frac{m^2}{2}\Phi(\varphi). \tag{3.154}$$

where $\frac{m^2}{2}$ is the separation parameter. Also, we can write the Eqs. (3.153) and (3.154) as follow :

$$\hat{A}_\rho P(\rho) - \varepsilon P(\rho) + \frac{m^2}{2\rho^2}P(\rho) = 0. \tag{3.155}$$

and

$$\hat{B}_\varphi \Phi(\varphi) - \frac{m^2}{2}\Phi(\varphi) = 0. \tag{3.156}$$

We start by examining the angular equation 3.156 it has the explicit form

$$\Phi'' - 2(\mu_x \tan\varphi - \mu_y \cot\varphi)\Phi' - \mu_x \frac{\left(I - \hat{R}_x\right)}{\cos^2\varphi} - \mu_y \frac{\left(I - \hat{R}_y\right)}{\sin^2\varphi} + m^2\Phi = 0. \tag{3.157}$$

The reflection operator commutes with the Hamiltonian $\left[\hat{H}, \hat{R}_x\right] = \left[\hat{H}, \hat{R}_y\right] = 0$, we shall label the eigenstates by the eigenvalues $s_x, s_y = \pm 1$ of the reflection operators $\hat{R}_x$ and $\hat{R}_y$.

At this stage when we use $s_x, s_y = +1$. The Eq. (3.157) written by

$$\left[\frac{d^2}{dx^2} - 2\left(\mu_x \frac{\sin\varphi}{\cos\varphi} - \mu_y \frac{\cos\varphi}{\sin\varphi}\right)\frac{d}{dx} - m^2\right]\Phi^{++} = 0. \tag{3.158}$$

After this we use a new variable $x = -\cos 2\varphi$, we will find :

$$\frac{dx}{d\varphi} = 2\sin 2\varphi. \tag{3.159}$$

$$\frac{d^2}{d\varphi^2} = \frac{d}{d\varphi}\left(2\sin 2\varphi \frac{d}{dx}\right) = 4\cos 2\varphi \frac{d}{dx} + 4\sin^2 2\varphi \frac{d^2}{dx^2}. \tag{3.160}$$

as we know $x = -\cos 2\varphi$, we can obtain

$$\cos 2\varphi = \cos^2\varphi - \sin^2\varphi$$
$$\Rightarrow \cos^2\varphi = \frac{1-x}{2}. \tag{3.161}$$

and

$$1 - \sin^2\varphi = \frac{1-x}{2} \Rightarrow \sin^2\varphi = \frac{1+x}{2}. \tag{3.162}$$

As a result

$$\cot\varphi = \sqrt{\frac{1-x}{1+x}}, \quad \tan\varphi = \sqrt{\frac{1+x}{1-x}}. \tag{3.163}$$

We substitute in Eq. (3.158)

$$\left[4\cos 2\varphi \frac{d}{dx} + 4\sin^2 2\varphi \frac{d^2}{dx^2} - 2(\mu_x \tan\varphi - \mu_y \cot\varphi)\, 2\sin 2\varphi \frac{d}{dx} + m^2\right]\Phi^{++} = 0, \tag{3.164}$$

or

$$\left[4\cos 2\varphi \frac{d}{dx} + 4\sin^2 2\varphi \frac{d^2}{dx^2} - 8\left(\mu_x \sin^2\varphi - \mu_y \cos^2\varphi\right)\frac{d}{dx} + m^2\right]\Phi^{++} = 0, \tag{3.165}$$

which gives

$$\left[-4x\frac{d}{dx} + 4\left(1-x^2\right)\frac{d^2}{dx^2} - 8\left(\mu_x\left(\frac{1+x}{2}\right) - \mu_y\left(\frac{1-x}{2}\right)\right)\frac{d}{dx} + m^2\right]\Phi^{++} = 0. \tag{3.166}$$

After simplification we obtain :

$$\left[\left(1-x^2\right)\frac{d^2}{dx^2} + \left(\mu_y - \mu_x - \left(1+\mu_y+\mu_x\right)x\right)\frac{d}{dx} + \frac{m^2}{4}\right]\Phi^{++} = 0. \tag{3.167}$$

In accordance with the general form of Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$ equation is defined as follows,

$$\left[\left(1-z^2\right)\frac{d^2}{dz^2} + \left(\beta - \alpha - \left(\alpha+\beta+2\right)z\right)\frac{d}{dz} + n(n+\alpha+\beta+1)\right]P_n^{(\alpha,\beta)}(z) = 0. \tag{3.168}$$

we find

$$\alpha + \beta + 2 \equiv 1 + \mu_y + \mu_x. \tag{3.169}$$

$$\beta - \alpha \equiv \mu_y - \mu_x. \tag{3.170}$$

$$n(n+\alpha+\beta+1) \equiv \frac{m^2}{4} \tag{3.171}$$

these lead as

$$2\beta = 2\mu_y - 1 \Rightarrow \beta = \mu_y - \frac{1}{2}. \tag{3.172}$$

$$2\alpha = 2\mu_x - 1 \Rightarrow \alpha = \mu_x - \frac{1}{2}. \tag{3.173}$$

$$m^2 = 4n(n + \mu_y + \mu_x) \tag{3.174}$$

At this stage we can write $\Phi^{++} = c_n P_n^{\left(\mu_x - \frac{1}{2}, \mu_y - \frac{1}{2}\right)}(x)$ this solution corresponds to the eigenvalue $m^2 = 4n(n + \mu_x + \mu_y)$ with $n \in N$ and c_n is a normalization constant, we can obtain it by the following relation :

$$\int_0^{2\pi} |c_n|^2 P_n^{(\mu_x - 1/2, \mu_y - 1/2)} P_m^{(\mu_x - 1/2, \mu_y - 1/2)} \left|\cos^2(\varphi)\right|^{\mu_x} \left|\sin^2(\varphi)\right|^{\mu_y} d\varphi = 1. \tag{3.175}$$

or

$$\frac{1}{2}\int_0^{2\pi} |c_n|^2 P_n^{(\mu_x - 1/2, \mu_y - 1/2)} P_m^{(\mu_x - 1/2, \mu_y - 1/2)} |1-x|^{\mu_x} |1+x|^{\mu_y} \frac{dx}{(1-x)^{1/2}(1+x)^{1/2}} = 1. \tag{3.176}$$

when $\varphi \to 0 \Rightarrow x \to -1$ and $\varphi \to \pi/2 \Rightarrow x \to 1$, then we obtain

$$\frac{2}{2^{\mu_x+\mu_y}} \int_{-1}^{1} |c_n|^2 P_n^{(\mu_x-1/2,\mu_y-1/2)} P_m^{(\mu_x-1/2,\mu_y-1/2)} |1-x|^{\mu_x} |1+x|^{\mu_y} \frac{dx}{(1-x)^{1/2}(1+x)^{1/2}} = 1. \tag{3.177}$$

we use the following standard relation (see, Ref.[**?**])

$$\int_{-1}^{1} (1-x)^{\alpha} (1+x)^{\beta} P_n^{(\alpha,\beta)} P_m^{(\alpha,\beta)} dx = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)\, n!} \delta_{n,m}. \tag{3.178}$$

So Eq. (3.177) writes as

$$2|c_n|^2 \frac{1}{2n+\mu_x+\mu_y} \frac{\Gamma(n+\mu_x+1/2)\Gamma(n+\mu_y+1/2)}{\Gamma(n+\mu_x+\mu_y)\, n!} = 1. \tag{3.179}$$

which gives

$$c_n = \sqrt{\frac{(2n+\mu_x+\mu_y)\Gamma(n+\mu_x+\mu_y)\, n!}{2\Gamma\left(n+\mu_x+\frac{1}{2}\right)\Gamma\left(n+\mu_y+\frac{1}{2}\right)}}. \tag{3.180}$$

Thus, if $s_x = s_y = +1$, the equation (3.157) has the solution

$$\Phi^{++} = \sqrt{\frac{(2n+\mu_x+\mu_y)\Gamma(n+\mu_x+\mu_y)\, n!}{2\Gamma\left(n+\mu_x+\frac{1}{2}\right)\Gamma\left(n+\mu_y+\frac{1}{2}\right)}} P_n^{\left(\mu_x+\frac{1}{2};\mu_y+\frac{1}{2}\right)}(x). \tag{3.181}$$

In the same method we can calculate the general solution of the cases : $|s_x, s_y\rangle = |-1,-1\rangle$, $|1,-1\rangle$, $|-1,1\rangle$ as follow

$$\Phi'' - 2(\mu_x \tan\varphi - \mu_y \cot\varphi)\Phi' - \mu_x \frac{(1-s_x)}{\cos^2\varphi} - \mu_y \frac{(1-s_y)}{\sin^2\varphi} + m^2\Phi = 0. \tag{3.182}$$

we take the following Anzast

$$\Phi = \sin^k\varphi \cos^p\varphi \tilde{\Phi}(\varphi) \tag{3.183}$$

The Eq. (3.182) transforms by

$$\begin{aligned} &\tilde{\Phi}'' - 2(\mu_x \tan\varphi - \mu_y \cot\varphi)(-p\tan(\varphi) + k\cot(\varphi))\tilde{\Phi} \\ &- (2(\mu_x(1-s_x)+p)\tan\varphi - (2\mu_y(1-s_y)-k)\cot\varphi)\tilde{\Phi}' \\ &+ (-p\tan(\varphi) + k\cot(\varphi))^2 \tilde{\Phi} \\ &- \frac{(\mu_x(1-s_x)+p)}{\cos^2\varphi}\tilde{\Phi} - \frac{(\mu_y(1-s_y)+k)}{\sin^2\varphi}\tilde{\Phi} + m^2\tilde{\Phi} = 0. \end{aligned} \tag{3.184}$$

With some simplifications

$$\begin{gathered}\tilde{\Phi}''(\varphi)+2\left[(k+\mu_y)\cot\varphi-(p+\mu_x)\tan\varphi\right]\tilde{\Phi}'(\varphi)\\+\left[-(k+p)^2-2(k+p)(\mu_x+\mu_y)+m^2\right]\tilde{\Phi}(\varphi)\\+\left[\frac{k(k-1)-2\mu_y\left(\frac{(1-s_y)}{2}-k\right)}{\sin^2\varphi}+\frac{p(p-1)-2\mu_x\left(\frac{(1-s_x)}{2}-p\right)}{\cos^2\varphi}\right]\tilde{\Phi}(\varphi)=0.\end{gathered} \tag{3.185}$$

we put $x=-\cos 2\varphi$ we find

$$\begin{gathered}\left(1-x^2\right)\tilde{\Phi}''(x)+\\+\left[-(1+(k+\mu_y)+(p+\mu_x))x+((k+\mu_y)-(p+\mu_x))\right]\tilde{\Phi}'(x)\\+\frac{1}{4}\left[-(k+p)^2+m^2-2(k+p)(\mu_x+\mu_y)\right]\tilde{\Phi}(x)\\+\left[\frac{1}{2}\left(\frac{k(k-1)-2\mu_y\left(\frac{(1-s_y)}{2}-k\right)}{1+x}+\frac{p(p-1)-2\mu_x\left(\frac{(1-s_x)}{2}-p\right)}{1-x}\right)\right]\tilde{\Phi}(x)=0.\end{gathered} \tag{3.186}$$

The values of k and p can be determined them if the above equation is the Jacobi polynomial differential equation $P_n^{(\alpha,\beta)}(x)$. Which leads to the following equations

$$k(k-1)-2\mu_y\left(\frac{1-s_y}{2}-k\right)=0. \tag{3.187}$$

$$p(p-1)-2\mu_x\left(\frac{1-s_x}{2}-p\right)=0. \tag{3.188}$$

Their solution are given by the following equations

$$k=\frac{1-s_y}{2},\quad p=\frac{1-s_x}{2}. \tag{3.189}$$

While the other parameters are

$$\alpha+\beta+2\equiv(1+(k+\mu_y)+(p+\mu_x)). \tag{3.190}$$

$$\beta-\alpha\equiv((k+\mu_y)-(p+\mu_x)). \tag{3.191}$$

$$n(n+\alpha+\beta+1)\equiv\left[-(k+p)^2-2(k+p)(\mu_x+\mu_y)+m^2\right]. \tag{3.192}$$

we obtain

$$\beta\equiv k+\mu_y-\frac{1}{2}. \tag{3.193}$$

$$\alpha\equiv p+\mu_x-\frac{1}{2}. \tag{3.194}$$

The normalization in general case is written as

$$c_{n,s_x,s_y} = \sqrt{\frac{(2n+\mu_x+\mu_y)\,\Gamma\left(n+\mu_x+\mu_y+\frac{k}{2}+\frac{p}{2}\right)\left(n-\frac{k}{2}-\frac{p}{2}\right)!}{2\Gamma\left(n+\mu_x+\frac{k}{2}-\frac{p}{2}+\frac{1}{2}\right)\Gamma\left(n+\mu_y-\frac{k}{2}+\frac{p}{2}+\frac{1}{2}\right)}}. \tag{3.195}$$

we find

$$\Phi^{s_x s_y} = \sqrt{\frac{(2n+\mu_x+\mu_y)\,\Gamma\left(n+\mu_x+\mu_y+\frac{k}{2}+\frac{p}{2}\right)\left(n-\frac{k}{2}-\frac{p}{2}\right)!}{2\Gamma\left(n+\mu_x+\frac{k}{2}-\frac{p}{2}+\frac{1}{2}\right)\Gamma\left(n+\mu_y-\frac{k}{2}+\frac{p}{2}+\frac{1}{2}\right)}}\sin^k\varphi\cos^p\varphi P_{n-\frac{k}{2}-\frac{p}{2}}^{\left(p+\mu_x-\frac{1}{2};k+\mu_y-\frac{1}{2}\right)}(x)\,. \tag{3.196}$$

In particularly cases, we have, when $s_x = s_y = -1$, the solution reads

$$\Phi^{--} = \sqrt{\frac{(2n+\mu_x+\mu_y)\,\Gamma(n+\mu_x+\mu_y+1)\,(n-1)!}{2\Gamma\left(n+\mu_x+\frac{1}{2}\right)\Gamma\left(n+\mu_y+\frac{1}{2}\right)}}\sin\varphi\cos\varphi P_{n-1}^{\left(\mu_x+\frac{1}{2};\mu_y+\frac{1}{2}\right)}(x)\,. \tag{3.197}$$

It is understood that $P_{-1}^{(\alpha,\beta)}(x) = 0$ and hence that$\Phi^{--} = 0$. Also, when $s_x = +1$and $s_y = -1$, the solutions reads :

$$\Phi^{+-} = \sqrt{\frac{(2n+\mu_x+\mu_y)\,\Gamma\left(n+\mu_x+\mu_y+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)!}{2\Gamma(n+\mu_x)\,\Gamma(n+\mu_y+1)}}\sin\varphi P_{n-\frac{1}{2}}^{\left(\mu_x-\frac{1}{2};\mu_y+\frac{1}{2}\right)}(x)\,. \tag{3.198}$$

Lastly, when $s_x = -1$ and $s_y = 1$, the solution to the angular equation has the expression :

$$\Phi^{-+} = \sqrt{\frac{(2n+\mu_x+\mu_y)\,\Gamma\left(n+\mu_x+\mu_y+\frac{1}{2}\right)\left(n-\frac{1}{2}\right)!}{2\Gamma\left(n+\mu_x+\frac{1}{2}\right)\Gamma(n+\mu_y)}}\sin\varphi P_{n-\frac{1}{2}}^{\left(\mu_x+\frac{1}{2};\mu_y-\frac{1}{2}\right)}(x)\,. \tag{3.199}$$

After this stage we can find the radial solution of Eq. (3.155) as

$$\left[\partial_\rho^2 + \frac{1}{\rho}(1+2\mu_x+2\mu_y)\,\partial_\rho + \left(2\varepsilon - \rho^2 - \frac{m^2}{\rho^2}\right)\right]P(\rho) = 0. \tag{3.200}$$

This equation has for solutions

$$P(\rho) = c_{n,k}e^{-\rho^2/2}\rho^{2n}L_k^{(2n+\mu_x+\mu_y)}\left(\rho^2\right). \tag{3.201}$$

Using the orthogonality relation of the Laguerre polynomials, one finds that the radial wave-function normalization obeys

$$\int_0^\infty P_k(\rho)\,P_{k'}(\rho)\,\rho^{1+2\mu_x+2\mu_y}d\rho = \delta_{k,k'}. \tag{3.202}$$

we obtain

$$c_{n,k} = \sqrt{\frac{2k!}{\Gamma(k+2n+\mu_x+\mu_y+1)}}, \tag{3.203}$$

with the energy eigenvalues

$$\varepsilon = 2k + 2n + \mu_x + \mu_y + 1. \tag{3.204}$$

Harmonic oscillator in three dimensions

In spherical coordinates, the Cartesian coordinates (x_1, x_2, x_3) of a vector $\mathbf{x}$ are related to its spherical polar coordinates (r, θ, φ) by

$$x_1 = r\sin\theta\cos\varphi, \quad x_2 = r\sin\theta\sin\varphi, \quad x_3 = r\cos\theta. \tag{3.205}$$

As we know from the previous transformations, we have

$$\begin{aligned}\frac{\partial}{\partial x_1} &= \frac{\partial}{\partial r}\frac{\partial}{\partial x} + \frac{\partial}{\partial \theta}\frac{\partial \theta}{\partial x} + \frac{\partial}{\partial \varphi}\frac{\partial \varphi}{\partial x} \\ &= \sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi} \qquad (3.206)\\ \frac{\partial}{\partial x_2} &= \sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}. \qquad (3.207)\\ \frac{\partial}{\partial x_3} &= \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}. \qquad (3.208)\end{aligned}$$

also we can show that the Laplacian operator reduces to

$$\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} = \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]. \tag{3.209}$$

Substituting Eqs. (3.206)-(3.208) into Eq. (3.209) we find

$$\begin{aligned}&\left\{\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\right]\right.\\ &+2\frac{\mu_1}{x_1}\left(\mathbb{I} - \hat{R}_x\right)\left[\sin\theta\cos\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\cos\varphi\frac{\partial}{\partial\theta} - \frac{\sin\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]\\ &+2\frac{\mu_2}{x_2}\left(\mathbb{I} - \hat{R}_y\right)\left[\sin\theta\sin\varphi\frac{\partial}{\partial r} + \frac{1}{r}\cos\varphi\sin\varphi\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}\right]\\ &+2\frac{\mu_3}{x_3}\left(\mathbb{I} - \hat{R}_z\right)\left[\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right] - \frac{\mu_1}{x_1^2}(\mathbb{I} - \hat{R}_x)\\ &\left. - \frac{\mu_2}{x_2^2}(\mathbb{I} - \hat{R}_y) - \frac{\mu_3}{x_3^2}(\mathbb{I} - \hat{R}_z) - \frac{m^2\omega^2}{\hbar^2}r^2 + \frac{2m}{\hbar}E\right\}\psi(r,\theta,\varphi) = 0. \qquad (3.210)\end{aligned}$$

After simplification, Eq. (4.63) reduces to

$$\left[\hat{J}_r + \frac{\hat{J}_\varphi}{r^2\sin^2\theta} + \frac{\hat{J}_\theta}{r^2} + \frac{2m}{\hbar}E\right]\Psi = 0 \tag{3.211}$$

where the operators $\hat{J}_r$, J_φ and J_θ are respectively

$$\hat{J}_r = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\left[1 + \left(\mathbb{I} - \hat{R}_1\right)\mu_1 + \left(\mathbb{I} - \hat{R}_2\right)\mu_2 + \left(\mathbb{I} - \hat{R}_3\right)\mu_3\right]\frac{\partial}{\partial r} - \frac{m^2\omega^2}{\hbar^2}r^2, \tag{3.212}$$

and

$$\hat{J}_{\varphi}=\frac{\partial^2}{\partial\varphi^2}+2\left[\mu_1\cot\varphi+\mu_2\tan\varphi\right]\frac{\partial}{\partial\varphi}-\frac{\mu_1}{\cos^2\varphi}\left(\mathbb{I}-\hat{R}_1\right)-\frac{\mu_2}{\sin^2\varphi}\left(\mathbb{I}-\hat{R}_2\right). \tag{3.213}$$

and

$$\hat{J}_{\theta}=\frac{\partial^2}{\partial\theta^2}+\cot\theta+2\left[(\mu_2+\mu_1)\cot\theta-\mu_3\tan\theta\right]\frac{\partial}{\partial\theta}-\frac{\mu_3}{\cos^2\varphi}\left(1-\hat{R}_3\right). \tag{3.214}$$

In spherical coordinates, the reflection operator act on the wave function gives to the following transformations :

$$\hat{R}_1\Psi(r,\theta,\varphi)=\Psi(r,\theta,\pi-\varphi), \tag{3.215}$$

$$\hat{R}_2\Psi(r,\theta,\varphi)=\Psi(r,\theta,\pi-\varphi), \tag{3.216}$$

$$\hat{R}_3\Psi(r,\theta,\varphi)=\Psi(r,\pi-\theta,\varphi). \tag{3.217}$$

After this step we can use the separation method with $\Psi=F(r)\Theta(\theta)\Phi(\varphi)$. Then, Eq. (4.64) is written as

$$\left[\hat{J}_r+\frac{\hat{J}_{\varphi}}{r^2\sin^2\theta}+\frac{\hat{J}_{\theta}}{r^2}+\frac{2m}{\hbar}E\right]F(r)\Theta(\theta)\Phi(\varphi)=0 \tag{3.218}$$

then

$$\left[\Theta(\theta)\Phi(\varphi)\left(r^2\hat{J}_r+\frac{2m}{\hbar}Er^2\right)F(r)+F(r)\left(\frac{\hat{J}_{\varphi}}{\sin^2\theta}+\hat{J}_{\theta}\right)\Theta(\theta)\Phi(\varphi)\right]=0 \tag{3.219}$$

that's to say à dire

$$\left(\hat{J}_r+\frac{2m}{\hbar}E\right)F(r)=\frac{q^2}{2r^2}F(r). \tag{3.220}$$

and

$$\left(\frac{\hat{J}_{\varphi}}{\sin^2\theta}+\hat{J}_{\theta}\right)\Theta(\theta)\Phi(\varphi)=-\frac{q^2}{2}\Theta(\theta)\Phi(\varphi). \tag{3.221}$$

and the second is related to (θ,φ)-variables,

$$\left[\frac{\hat{J}_{\varphi}}{\sin^2\theta}+\hat{J}_{\theta}\right]\Theta(\theta)\Phi(\varphi)=-\varpi^2\Theta(\theta)\Phi(\varphi). \tag{3.222}$$

where q is the separation constant.

From the equation (3.222) we can re-write by :

$$\left[\hat{J}_{\varphi}+\sin^2\theta\left(\hat{J}_{\theta}+\frac{q^2}{2}\right)\right]\Theta(\theta)\Phi(\varphi)=0. \tag{3.223}$$

The first operator in Eq. (3.223) is related by $\varphi-$variable and the second operator is related by $\theta-$variable, for these considerations we devise by $\Theta(\theta)\Phi(\varphi)$, we obtain

$$\left[\frac{1}{\Phi(\varphi)}\hat{J}_{\varphi}\Phi(\varphi)+\frac{\sin^2\theta}{\Theta(\theta)}\left(\hat{J}_{\theta}-\frac{q^2}{2}\right)\Theta(\theta)\right]=0. \tag{3.224}$$

This later leads to

$$\frac{1}{\Phi(\varphi)}\hat{J}_{\varphi}\Phi(\varphi)=-\Omega^2. \tag{3.225}$$

$$\frac{\sin^2\theta}{\Theta(\theta)}\left(\hat{J}_{\theta}-\frac{q^2}{2}\right)\Theta(\theta)=\Omega^2. \tag{3.226}$$

here Ω is also the separation constant.

After these we can write the equations (3.213) and (3.214) as follow :

$$\left\{\frac{\partial^2}{\partial\varphi^2}+2\left[\mu_1\cot\varphi+\mu_2\tan\varphi\right]\frac{\partial}{\partial\varphi}-\frac{\mu_1}{\cos^2\varphi}\left(1-\hat{R}_1\right)-\frac{\mu_2}{\sin^2\varphi}\left(1-\hat{R}_2\right)+\Omega^2\right\}\Phi(\varphi)=0. \tag{3.227}$$

and

$$\left\{\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+2\left(\left(\mu_1+\mu_2\right)\cot\theta-\mu_3\tan\theta\right)\frac{\partial}{\partial\theta}-\frac{\mu_3}{\cos^2\varphi}\left(1-\hat{R}_3\right)-\frac{\Omega^2}{\sin^2\theta}+\frac{q^2}{2}\right\}\Theta(\theta)=0. \tag{3.228}$$

Using the eigenvalues of the operators of reflection, we find,

$$\left\{\frac{\partial^2}{\partial\varphi^2}+2\left[\mu_1\cot\varphi+\mu_2\tan\varphi\right]\frac{\partial}{\partial\varphi}-\frac{\mu_1}{\cos^2\varphi}\left(1-s_1\right)-\frac{\mu_2}{\sin^2\varphi}\left(1-s_2\right)+\Omega^2\right\}\Phi(\varphi)=0. \tag{3.229}$$

and

$$\left\{\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+2\left(\left(\mu_1+\mu_2\right)\cot\theta-\mu_3\tan\theta\right)\frac{\partial}{\partial\theta}-\frac{\mu_3}{\cos^2\varphi}\left(1-s_3\right)-\frac{\Omega^2}{\sin^2\theta}+\varpi\right\}\Theta(\theta)=0. \tag{3.230}$$

The solution of Eq. (3.229) is calculated in previous section

$$\Phi=\cos^k(\varphi)\sin^p(\varphi)\tilde{\Phi}(x)$$

While the second equation, we use the following anzast (see., the reference [4]),

$$\Theta(\theta)=\cos^{\nu}(\theta)\sin^{2\delta}(\theta)\tilde{\Theta}(\cos\theta). \tag{3.231}$$

After these changes we can find,

$$\tilde{\Phi}(x)=C_{\varphi}P_{n-\frac{p}{2}-\frac{k}{2}}^{\left(\mu_2+p-\frac{1}{2},\mu_1+k-\frac{1}{2}\right)}\cos(2\varphi)$$

and

$$\tilde{\Theta}(\theta) = C_\theta P_{\ell-\frac{\nu}{2}}^{\left(\mu_2+p-\frac{1}{2},\mu_1+k-\frac{1}{2}\right)} \cos(2\theta). \tag{3.232}$$

where C_φ and C_θ are the normalization constants and they are written as

$$C_\varphi = \sqrt{\frac{(2n+\mu_x+\mu_y)\,\Gamma\left(n+\mu_x+\mu_y+\frac{k}{2}+\frac{p}{2}\right)\left(n-\frac{k}{2}-\frac{p}{2}\right)!}{2\Gamma\left(n+\mu_x+\frac{k}{2}-\frac{p}{2}+\frac{1}{2}\right)\Gamma\left(n+\mu_y-\frac{k}{2}+\frac{p}{2}+\frac{1}{2}\right)}}.$$

and

$$C_\theta = \sqrt{\frac{\left(2\ell+2\delta+\mu_x+\mu_y+\mu_z+\frac{1}{2}\right)\Gamma\left(\ell+2\delta+\mu_x+\mu_y+\mu_z+\frac{1}{2}+\frac{\nu}{2}\right)\left(\ell-\frac{\nu}{2}\right)!}{\Gamma\left(\ell+2\delta+\mu_x+\mu_y+1-\frac{\nu}{2}\right)\Gamma\left(\ell+\mu_z-\frac{1}{2}+\frac{\nu}{2}\right)}}. \tag{3.233}$$

For the validity of the given solutions, the separation constants have to satisfy the following conditions :

$$\Omega^2 = 4\sigma(\sigma+\mu_1+\mu_2). \tag{3.234}$$

and

$$q^2 = 4(l+\sigma)\left(l+\sigma+\mu_1+\mu_2+\frac{1}{2}\right). \tag{3.235}$$

also for the values of k, p, δ and ν

$$k = \frac{1-s_1}{2}$$
$$p = \frac{1-s_2}{2}$$
$$\nu = \frac{1-s_3}{2}$$

and σ is a positive integer.

Following that, we seek an exact solution to the radial equation. we will start by substituting Eq. (4.86) and Eq. (4.87) into Eq. (3.220), we find

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}(1+\mu_1+\mu_2+\mu_3)\frac{\partial}{\partial r}-r^2+2E \right.$$
$$\left.-\frac{4(l+\nu)\left(l+\nu+\mu_1+\mu_2+\frac{1}{2}\right)}{r^2}\right\}F(r)=0. \tag{3.236}$$

After this later we can write the following ansazt

$$F = \rho^{2(\nu+l)}e^{-\frac{\rho}{2}}\Xi(\rho), \tag{3.237}$$

So, the function $\Xi(\rho)$ are given by

$$\Xi(\rho) = \sqrt{\frac{2n_r}{\Gamma(n_r+\alpha+1)}}L_{n_r}^{\alpha}(r), \tag{3.238}$$

with $\alpha = \left(2\left(l+\nu\right)+\mu_1+\mu_2+\mu_3+\frac{1}{2}\right)$ and correspond to the total energy

$$E = 2\left(n_r + l + \nu\right) + \mu_1 + \mu_2 + \mu_3 + \frac{3}{2}. \qquad (3.239)$$

Chapitre 4

Applications of the Dunkl-Klein-Gordon oscillator

4.1 Introduction

In this section, we will examine the Dunkl-Klein-Gordon oscillator in one dimension and three dimensions. In three dimensions we will study this poblem in representation of Cartesian and spherical coordinates.

4.2 Klein-Gordon oscillator in one dimension

The Hamiltonian of Dunkl-Klein-Gordon oscillators equation in one dimension is written as

$$\hat{H} = \left[E^2 - \left(\frac{1}{\imath} \hat{D}_x + \imath m \omega x \right) \left(\frac{1}{\imath} \hat{D} - \imath m \omega x \right) - m^2 \right]. \tag{4.1}$$

the corresponding wave equation of Eq. (4.1) is obtained by substituting the Dunkl derivative

$$\hat{H} \Psi (x) = 0 \tag{4.2}$$

By developing all operators in Eq. (4.1) we find

$$\hat{H} = E^2 + \left(\hat{D}_x^2 + m^2 x^2 \omega^2 + \hat{D}_x m \omega x - m \omega x \hat{D}_x - m^2 \right) \tag{4.3}$$

As we have

$$\hat{D}_x m \omega x - m \omega x \hat{D}_x = \left[\hat{D}_x, m \omega \hat{x} \right] \tag{4.4}$$

Substituting Eq. (4.4) into Eq. (4.3) we get

$$\begin{aligned}\hat{H} &= \partial^2 + \frac{2\mu}{x}\partial - \frac{\mu}{x^2}\left(1-\hat{R}\right) + \left[\hat{D}_x, m\omega\hat{x}\right] - m^2 + E^2 \\ &= \partial^2 + \frac{2\mu}{x}\partial - \frac{\upsilon}{x^2}\left(1-\hat{R}\right) + m^2\omega^2x^2 - im\omega\left[\hat{x},\hat{p}\right] + E^2 - m^2 \\ &= \partial^2 + \frac{2\mu}{x}\partial - \frac{\mu}{x^2}\left(1-\hat{R}\right) + m^2\omega^2x^2 + m\omega\left(1+2\mu p\right) + E^2. \end{aligned} \tag{4.5}$$

The Dunkl-Klein-Gordon Hamiltonian operator is simplified by,

$$\hat{H} = \frac{d^2}{dx^2} + \frac{2\mu}{x}\frac{d}{dx} - \frac{\mu}{x^2}\left(1-\hat{R}\right) + m^2\omega^2x^2 + m\omega\left(1+2\mu\hat{R}\right) + E^2 - m^2 \tag{4.6}$$

The reflection operator commutes with the Hamiltonian $\left[\hat{H},\hat{R}\right]=0$, they should have a common eigenbasis. Therefore, they can be diagonalized simultaneously. So, the eigenfunction, $\psi(x)$, can be selected to have a definite parity $\hat{R}\psi(x) = s\psi(x)$ with $s=\pm1$. As a result, Eq. (4.2) appears as

$$\left[\frac{d^2}{dx^2} + \frac{2\mu}{x}\frac{d}{dx} - \frac{\mu}{x^2}(1-s) + m^2\omega^2x^2 + m\omega\left(1+2\mu s\right) + E^2 - m^2)\right]\Psi^s(x) = 0. \tag{4.7}$$

or by other writing

$$\left[\frac{d^2}{dx^2} + \frac{2\mu}{x}\frac{d}{dx} - \frac{\mu}{x^2}(1-s) + m^2\omega^2x^2 + c\right]\Psi^s(x) = 0 \tag{4.8}$$

Where $c = m\omega\left(1+2\mu s\right)+E^2-m^2)$. So, to solve the wave equation. (4.8), we use a new variable $y = m\omega x^2$. For these we will calculate the following important terms presented in Eq. (4.8),

$$\frac{d\Psi^s}{dx} = \frac{d\Psi^s}{dy}\frac{dy}{dx}. \tag{4.9}$$

$$\frac{d^2\Psi^s}{dx^2} = \frac{dy}{dx}\frac{d}{dy}\left(\frac{dy}{dx}\frac{d\Psi^s}{dy}\right). \tag{4.10}$$

where $\frac{dy}{dx} = 2m\omega x$, we find

$$\frac{d\Psi^s}{dx} = 2m\omega x\frac{d\Psi^s}{dy}, \quad \frac{1}{x}\frac{d\Psi^s}{dx} = 2m\omega\frac{d\Psi^s}{dy}. \tag{4.11}$$

also

$$\begin{aligned}\frac{d^2\Psi^s}{dx^2} &= \frac{d}{dx}\left(2m\omega x\left(\frac{d\Psi^s}{dy}\right)\right) \\ &= \left(2m\omega\left(\frac{d\Psi^s}{dy}\right)\right) + \left(2m\omega x\frac{d}{dx}\left(\frac{d\Psi^s}{dy}\right)\right). \end{aligned} \tag{4.12}$$

which gives

$$\frac{d^2\Psi^s}{dx^2} = (2m\omega x)^2\left(\frac{d^2\Psi^s}{dy^2}\right) + 2m\omega\left(\frac{d\Psi^s}{dy}\right) = 4m\omega y\frac{d^2\Psi^s}{dy^2} + 2m\omega\left(\frac{d\Psi^s}{dy}\right). \tag{4.13}$$

Substituting Eq. (4.11) and Eq. (4.13) into Eq.(4.8) and using the new variable $y = m\omega x^2$, we get

$$\left[4m\omega y\frac{d^2}{dy^2} + 2m\omega\frac{d}{dy} + 4m\omega y\mu\frac{d}{dy} - \frac{m\omega\mu}{y}(1-s) - m\omega y + c\right]\psi^s = 0. \tag{4.14}$$

or

$$\left(y\frac{d^2}{dy^2} + \left(\mu + \frac{1}{2}\right)\frac{d}{dy} - \frac{\mu}{4y}(1-s) - \frac{1}{4}y + \frac{c}{4m\omega}\right)\Psi^s = 0 \tag{4.15}$$

At this stage we use the following ansazt

$$\Psi^s = y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\Psi^s(y), \tag{4.16}$$

we will develop the $\frac{d\Psi^s}{dy}$ and $\frac{d^2\Psi^s}{dy^2}$, we find :

$$\begin{aligned}\frac{d\Psi^s}{dy} &= \frac{1-s}{4}y^{\frac{1-s}{4}-1}e^{-\frac{y}{2}}\Psi^s - \frac{1}{2}y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\Psi^s + y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\frac{d\Psi^s}{dy} \\ &= y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\frac{d\Psi^s}{dy} + \left(\frac{1-s}{4y} - \frac{1}{2}\right)y^{\frac{1-s}{4}-1}e^{-\frac{y}{2}}\Psi^s.\end{aligned} \tag{4.17}$$

Then

$$\frac{d^2\Psi^s}{dy^2} = \frac{d}{dy}\left(\frac{1-s}{4}y^{\frac{1-s}{4}-1}e^{-\frac{y}{2}}\Psi^s\right) - \frac{d}{dy}\left(\frac{1}{2}y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\Psi^s\right) + \frac{d}{dy}\left(y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\frac{d\Psi^s}{dy}\right). \tag{4.18}$$

The first term of Eq .(4.18) is developed as

$$\begin{aligned}\frac{d}{dy}\left(\frac{1-s}{4}y^{\frac{1-s}{4}-1}e^{-\frac{y}{2}}\Psi^s\right) &= \left(\frac{1-s}{4}\right)\left(\frac{1-s}{4} - 1\right)y^{\frac{1-s}{4}-2}e^{-\frac{y}{2}}\Psi^s \\ &-\frac{1}{2}\left(\frac{1-s}{4}\right)y^{\frac{1-s}{4}-1}e^{-\frac{y}{2}}\Psi^s + \frac{1-s}{4}y^{\frac{1-s}{4}-1}e^{-\frac{y}{2}}\frac{d\Psi^s}{dy}.\end{aligned} \tag{4.19}$$

For second term is developed as

$$\begin{aligned}\frac{d}{dy}\left(\frac{1}{2}y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\Psi^s\right) &= \frac{1}{2}\left(\frac{1-s}{4}\right)y^{\frac{1-s}{4}-1}e^{-\frac{y}{2}}\Psi^s \\ &-\frac{1}{4}y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\Psi^s + \frac{1}{2}y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\frac{d\Psi^s}{dy},\end{aligned} \tag{4.20}$$

and the third term is

$$\begin{aligned}\frac{d}{dy}\left(y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\frac{d\Psi^s}{dy}\right) &= \left(\frac{1-s}{4}\right)y^{\frac{1-s}{4}-1}e^{-\frac{y}{2}}\frac{d\Psi^s}{dy} \\ &-\frac{1}{2}y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\frac{d\Psi^s}{dy} + y^{\frac{1-s}{4}}e^{-\frac{y}{2}}\frac{d^2\Psi^s}{dy^2}.\end{aligned} \tag{4.21}$$

As a result we find with some simplification,

$$\frac{d^2\Psi^s}{dy^2} = \left(-\frac{(3+s)(1-s)}{16y^2} - \frac{1-s}{4y} + \frac{1}{4}\right) y^{\frac{1-s}{4}-2} e^{-\frac{y}{2}}\Psi^s + \left(\frac{1-s}{2y} - 1\right) y^{\frac{1-s}{4}-1} e^{-\frac{y}{2}} \frac{d\Psi^s}{dy} + y^{\frac{1-s}{4}} e^{-\frac{y}{2}} \frac{d^2\Psi^s}{dy^2}, \tag{4.22}$$

or

$$y\frac{d^2\Psi^s}{dy^2} = yy^{\frac{1-s}{4}} e^{-\frac{y}{2}} \frac{d^2\Psi^s}{dy^2} + \left(\frac{1-s}{2} - y\right) y^{\frac{1-s}{4}-1} e^{-\frac{y}{2}} \frac{d\Psi^s}{dy} + \left(-\frac{(3+s)(1-s)}{16y} - \frac{1-s}{4} + \frac{y}{4}\right) y^{\frac{1-s}{4}-2} e^{-\frac{y}{2}}\Psi^s \tag{4.23}$$

Substituting Eq.(4.17) and Eq.(4.23) into Eq.(4.15) and devised by $(y^{\frac{1-s}{4}-2}e^{-\frac{y}{2}})$, we obtain

$$y\frac{d^2\Psi^s}{dy^2} + \left(\frac{1-s}{2} - y\right)\frac{d\Psi^s}{dy} + \left(-\frac{(3+s)(1-s)}{16y} - \frac{1-s}{4} + \frac{y}{4}\right)\Psi^s + \left(\mu + \frac{1}{2}\right)\frac{d\Psi^s}{dy} + \left(\mu + \frac{1}{2}\right)\left(\frac{1-s}{4y} - \frac{1}{2}\right)\Psi^s - \frac{\mu}{4y}(1-s)\Psi^s - \frac{1}{4}y\Psi^s + \frac{c}{4m\omega}\Psi^s = 0. \tag{4.24}$$

After simplification

$$y\frac{d^2\Psi^s}{dy^2} + \left(1 - \frac{s}{2} + \mu - y\right)\frac{d\Psi^s}{dy} - \frac{1-s^2}{16y}\Psi^s + \left(-\frac{2\mu+1}{4} - \frac{1-s}{4} + \frac{c}{4m\omega}\right)\Psi^s = 0. \tag{4.25}$$

Since $s = \pm 1$ and $c = m\omega(1+2\mu s) + E^2 - m^2$, then Eq. (4.25) becomes as :

$$y\frac{d^2\Psi^s}{dy^2} + \left(1 - \frac{s}{2} + \mu - y\right)\frac{d\Psi^s}{dy} - \left(\frac{(2\mu+1)(1-s)}{4} - \frac{E^2 - m^2}{4m\omega}\right)\Psi^s = 0. \tag{4.26}$$

Which can be rewritten as

$$y\frac{d^2\Psi^s}{dy^2} + (b - y)\frac{d\Psi^s}{dy} - a\Psi^s = 0 \tag{4.27}$$

Where $b = 1 - \frac{s}{2} + \mu$ and $a = \frac{(2\mu+1)(1-s)}{4} - \frac{E^2 \; - m^2}{4m\omega}$.

Here Eq.(4.27) is called kummer's differential equation ,the solution can be expressed in terms of the confluent hypergeometric function of the first kind $F(a,b,y) ={}_1 F_1(a,b,y)$,

$$\Psi_n^s(y) = cF(a,b;y) = c.F\left(\frac{(2\mu+1)(1-S)}{4} - \frac{E^2 - m^2}{4m\omega}, 1 - \frac{s}{2} + \mu, y\right) \tag{4.28}$$

As a result, the wave function reads

$$\Psi_n^s(x) = c(m\omega x)^{\frac{1-s}{2}} e^{-\frac{m\omega x^2}{2}} F\left(\frac{(2\mu+1)(1-S)}{4} - \frac{E^2 - m^2}{4m\omega}, 1 - \frac{s}{2} + \mu, m\omega x^2\right) \tag{4.29}$$

If the first input is equal to a, the confluent hypergeometric function simplifies to a polynomial of degree n in y,

$$\frac{(2\mu+1)(1-s)}{4} - \frac{E^2-m^2}{4m\omega} = -n \tag{4.30}$$

The solution of this equation is

$$\frac{E}{m} = \pm\sqrt{4\frac{\omega}{m}n + 2\frac{\omega}{m}\left(\mu+\frac{1}{2}\right)(1-s)+1}. \tag{4.31}$$

where $\frac{\omega}{m} = r$ we have

$$\frac{E_{n,s}}{m} = \pm\sqrt{4rn + 2r\left(\mu+\frac{1}{2}\right)(1-s)+1}. \tag{4.32}$$

4.3 Klein-Gordon oscillator in three dimensions

In this section, we will try to solve the Klein-Gordon oscillator in three dimensions by replacing the ordinary momentum operator by Dunkl derivation with the stationary Klein-Gordon oscillator equation

$$\left[E^2 - \left(\frac{1}{\iota}\hat{D}_j + \iota m\omega x_j\right)\left(\frac{1}{\iota}\hat{D}_j - \iota m\omega x_j\right) - m^2\right]\psi = 0 \text{ with } j = 1,2,3, \tag{4.33}$$

where m and ω are the rest mass and oscillator frequency, respectively.

By following the Dunkl algebra which is defined in the previous section, the Dunkl-Klein-Gordon oscillator equation in three dimensions is developed by,

$$\left[E^2 - \left(\hat{D}_j^2 - m\omega\hat{D}_j x_j + m\omega x_j\hat{D}_j + (m\omega x_j)^2\right) - m^2\right]\psi = 0, \tag{4.34}$$

or

$$\left(\hat{D}_j^2 - m\omega\left[\hat{D}_j, \hat{x}_j\right] - (m\omega x_j)^2\right) + E^2 - m^2)\psi = 0. \tag{4.35}$$

Multiplying both sides by a minus sign, and we have $\left[\hat{D}_j, \hat{x}_j\right] = \left(1 + 2\mu_j\hat{R}_j\right)$, we get

$$\left[-\hat{D}_j^2 - m\omega\left(1+2\mu_j\hat{R}_j\right) + (m\omega x_j)^2\right]\psi = \left(E^2-m^2\right)\psi. \tag{4.36}$$

As a result Eq. (4.36) is developed by :

$$\begin{aligned}&\Big[-\hat{D}_1^2 - \hat{D}_2^2 - \hat{D}_3^2 - m\omega\left(1+2\mu_1\hat{R}_1\right) - m\omega\left(1+2\mu_2\hat{R}_2\right)\\ &-m\omega\left(1+2\mu_3\hat{R}_3\right) + m^2\omega^2(x_1^2+x_2^2+x_3^2)\Big]\psi = \left(E^2-m^2\right)\psi.\end{aligned} \tag{4.37}$$

or

$$\Big[-\hat{D}_1^2 - \hat{D}_2^2 - \hat{D}_3^2 - 2m\omega(\frac{3}{2} + \mu_1\hat{R}_1 + \mu_2\hat{R}_2 + \mu_3\hat{R}_3) + m^2\omega^2(x_1^2 + x_2^2 + x_3^2)\Big]\,\psi = \left(E^2 - m^2\right)\psi. \tag{4.38}$$

In follow sections, the above wave equation we will calculate exactly in Cartesian and spherical coordinates.

4.3.1 Cartesian coordinates solution

In this subsection, we use the separation method variable to solve exact solution for Eq. (4.38), indeed we have,

$$\psi = \psi(x_1)\,\psi(x_2)\,\psi(x_3)\,. \tag{4.39}$$

and

$$\hat{H} = \hat{H}_1 + \hat{H}_2 + \hat{H}_3. \tag{4.40}$$

where

$$H_j = -\hat{D}_j^2 - m\omega\left(1 + 2\mu_j\hat{R}_j\right) + (m\omega x_j)^2 \quad \text{with } j = 1, 2, 3. \tag{4.41}$$

We assume the following definition

$$E^2 - m^2 = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 \tag{4.42}$$

Consequently, the Dunkl-Klein-Gordon oscillator equation (4.38) reduces by three wave equation for each value of j,

$$\left\{\frac{\partial^2}{\partial x_j^2} + 2\frac{\mu_j}{x_j}\left(1 - \hat{R}_j\right)\frac{\partial}{\partial x_j} - \frac{\mu_j}{x_j^2}(1 - \hat{R}_j) - m^2\omega^2 x_j^2 + m\omega\left(1 + 2\mu_j\hat{R}_j\right) - \varepsilon_j\right\}\psi(x_j) = 0 \tag{4.43}$$

In addition, since the commutator between $\hat{H}_j$ and $\hat{R}_j$ equals zero (i.e., $\left[\hat{H}_j, \hat{R}_j\right] = 0$) the eigenfunctions could be selected as they have a definite parity,$\hat{R}_j\psi(x_j) = s_j\psi(x_j)$ with $s_j = \pm 1$. Eq (4.43) becomes as

$$\left\{\frac{\partial^2}{\partial x_j^2} + 2\frac{\mu_j}{x_j}(1 - s_j)\frac{\partial}{\partial x_j} - \frac{\mu_j}{x_j^2}(1 - s_j) - m^2\omega^2 x_j^2 + m\omega(1 + 2\mu_j s_j) - \varepsilon_j\right\}\psi(x_j) = 0 \tag{4.44}$$

Then, we set the following transformations

$$\xi_j = m\omega x_j^2, \quad \psi(\xi_j) = \xi_j^{\frac{1-s_j}{4}} e^{-\frac{\xi_j}{2}}\phi(\xi_j)\,. \tag{4.45}$$

Thus, Eq. (4.43) becomes as

$$\left\{\xi_j \frac{\partial^2}{\partial\ \xi_j^2} + \left(1 + \mu_j - \frac{s_j}{2} - \xi_j\right)\frac{\partial}{\partial \xi_j} + n_j\right\}\phi^{s_j}\left(\xi_j\right) = 0 \tag{4.46}$$

As we know the above equation (4.46) is similar to the problem which has solved in the previous chapter, and it becomes the following solution

$$\phi_n^s\left(\xi\right) = cF\left(a, b; \xi\right) = cF\left(\frac{\left(2\mu_j + 1\right)\left(1 - s_j\right)}{4} - \frac{\varepsilon_j}{4m\omega}, 1 - \frac{s_j}{2} + \mu, \xi\right). \tag{4.47}$$

The first argument is equal to as, the confluent hypergeometric function simplifies to a polynomial of degree n in y. We conclude that,

$$\frac{\left(2\upsilon_j + 1\right)\left(1 - s_j\right)}{4} - \frac{\varepsilon_j}{4m\omega} = -n_j, \tag{4.48}$$

$$-\left(\mu_j + \frac{1}{2}\right)\left(1 - s_j\right) + \frac{\varepsilon_j}{2m\omega} = 2n_j, \tag{4.49}$$

$$\frac{\varepsilon_j}{2m\omega} = 2n_j + \left(\mu_j + \frac{1}{2}\right)\left(1 - s_j\right), \tag{4.50}$$

where n_j is non-negative integer quantum numbers. The energy eigenvalue function, which is parity dependent, is quantized as follows

$$\varepsilon_{n_j}^{s_j} = 2m\omega\left[2n_j + \left(\frac{1}{2} + \mu_j\right)\left(1 - s_j\right)\right]. \tag{4.51}$$

So the general solution to the equation (4.46) can be expressed in terms of the associated Laguerre polynomials as

$$\phi_{n_j}^{s_j}\left(\xi_j\right) = C_{s_j} L_{n_j}^{\mu_j - \frac{s_j}{2}}\left(m\omega x_j^2\right). \tag{4.52}$$

where C_{s_j} is a constant normalization term that can be calculated using Eq.(4.13). We have,

$$\int x^{\alpha} e^{-x} L_{n_j}^{\mu_j - \frac{s_j}{2}}\left(x\right) L_{m_j}^{\mu_j - \frac{s_j}{2}}\left(x\right) = \delta_{nm}\frac{\left(n - \alpha\right)!}{n!}. \tag{4.53}$$

This relation is transformed by following new variable

$$\xi = m\omega x^2 \Longrightarrow dx = \frac{d\xi}{2\sqrt{m\omega\xi}}. \tag{4.54}$$

Therefore Eq. (4.13) is simplified by

$$\langle\psi \mid \psi\rangle = \left|c_{s_j}\right|^2 \int \frac{d\xi}{2\sqrt{m\omega\xi}} \xi^{\frac{1 - s_j}{2}} e^{-\xi}\left(L_{n_j}^{\mu_j - \frac{s_j}{2}}\left(\xi\right)\right)^2 \left|\sqrt{\frac{\xi}{m\omega}}\right|^{2\mu_j} \tag{4.55}$$

$$= \frac{\left|c_{s_j}\right|^2}{2\sqrt{m\omega}} \int d\xi \xi^{\frac{-s_j}{2}} e^{-\xi}\left(L_{n_j}^{\mu_j - \frac{s_j}{2}}\left(\xi\right)\right)^2 \left|\sqrt{\frac{\xi}{m\omega}}\right|^{2\mu_j} \tag{4.56}$$

$$= \frac{\left|c_{s_j}\right|^2}{2\left(m\omega\right)^{\frac{1}{2} + \mu_j}} \int \xi^{\mu_j + \frac{s_j}{2}} e^{-\xi}\left(L_{n_j}^{\mu_j - \frac{s_j}{2}}\left(\xi\right)\right)^2 d\xi. \tag{4.57}$$

or

$$\langle \psi \mid \psi \rangle = \frac{\left|c_{s_j}\right|^2}{2\left(m\omega\right)^{\frac{1}{2}+\mu_j}} \frac{\left(n_j+\mu_j-s_j\right)!}{n_j!} = 1$$
$$\Longrightarrow \left|c_{s_j}\right|^2 = 2\left(m\omega\right)^{\frac{1}{2}+\mu_j} \frac{n_j!}{\left(n_j+\mu_j-s_j\right)!}. \tag{4.58}$$

We get the following expression for the normalization constant,

$$c_{s_j} = \sqrt{2\left(m\omega\right)^{\frac{1}{2}+\mu_j} \frac{n_j!}{\left(n_j+\mu_j-s_j\right)!}}. \tag{4.59}$$

In order to obtain the level energy for our system we use the equation (4.42). We can write,

$$E = \sqrt{\varepsilon_1+\varepsilon_2+\varepsilon_3+m^2}, \tag{4.60}$$

and ε_j are calculated in Eqs. (4.51), then we find,

$$E_n^{(s)} = \left[2m\omega\left[m^2+2\left(n_1+n_2+n_3\right)\right]\right.$$
$$\left. +2m\omega\left[\left(\frac{1}{2}+\mu_1\right)\left(1-s_1\right)+\left(\frac{1}{2}+\mu_3\right)\left(1-s_3\right)+\left(\frac{1}{2}+\mu_2\right)\left(1-s_2\right)\right]\right]^{1/2}. \tag{4.61}$$

where $n_{j=1,2,3} = 0, 1, 2, \ldots$.

Exact solutions in spherical coordinates

In spherical coordinates, the Cartesian coordinates (x_1, x_2, x_3) of a vector $\mathbf{x}$ are related to its spherical polar coordinates (r, θ, φ) by

$$x_1 = r\sin\theta\cos\varphi, \qquad x_2 = r\sin\theta\sin\varphi, \qquad x_3 = r\cos\theta. \tag{4.62}$$

From Eq. (4.43) and by substituting Eqs. (3.206)-(3.208) into Eq. (4.43) we find

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\left(1+\mu_1+\mu_2+\mu_3\right)\frac{\partial}{\partial r}+\left[\frac{1}{r^2}\cot\theta+\frac{2}{r^2}\left(\mu_1+\mu_2\right)\cot\theta-\frac{2}{r^2}\mu_3\tan\theta\right]\frac{\partial}{\partial\theta}\right.$$
$$\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}+\frac{2}{r^2\sin^2\theta}\left(\mu_2\cot\varphi+\mu_1\tan\varphi\right)\frac{\partial}{\partial\varphi}-\frac{\mu_1}{r^2\sin^2\theta\cos^2\varphi}\left(1-\hat{R}_1\right)$$
$$\frac{\mu_2}{r^2\sin^2\theta\sin^2\varphi}\left(1-\hat{R}_2\right)-\frac{\mu_3}{r^2\cos^2\varphi}\left(1-\hat{R}_3\right)++m^2\omega^2r^2\left(\sin^2\theta\cos^2\varphi+\sin^2\theta\sin^2\varphi+\cos^2\varphi\right)$$
$$\left.+2m\omega\left(\frac{3}{2}+\mu_1\hat{R}_1+\mu_2\hat{R}_2+\mu_3\hat{R}_3\right)-E^2+m^2\right\}\Psi = 0. \tag{4.63}$$

After simplification, Eq. (4.63) reduces to

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\left(1+\mu_1+\mu_2+\mu_3\right)\frac{\partial}{\partial r}+m^2\omega^2r^2+\right.$$
$$\left.2m\omega\left(\frac{3}{2}+\mu_1\hat{R}_1+\mu_2\hat{R}_2+\mu_3\hat{R}_3\right)+\frac{\hat{J}_\varphi}{r^2\sin^2\theta}+\frac{\hat{J}_\theta}{r^2}+-E^2+m^2\right\}\Psi = 0 \tag{4.64}$$

where the operators J_φ and J_θ are

$$J_\varphi = \frac{\partial^2}{\partial\varphi^2} + 2\left[\mu_1 \cot\varphi + \mu_2 \tan\varphi\right]\frac{\partial}{\partial\varphi} - \frac{\mu_1}{\cos^2\varphi}\left(1-\hat{R}_1\right) - \frac{\mu_2}{\sin^2\varphi}\left(1-\hat{R}_2\right). \tag{4.65}$$

and

$$J_\theta = \frac{\partial^2}{\partial\theta^2} + \cot\theta + 2\left[(\mu_2+\mu_1)\cot\theta - \mu_3\tan\theta\right]\frac{\partial}{\partial\theta} - \frac{\mu_3}{\cos^2\varphi}\left(1-\hat{R}_3\right). \tag{4.66}$$

In spherical coordinates, the reflection operator act on the wave function gives to the following transformations :

$$\hat{R}_1\Psi(r,\theta,\varphi) = \Psi(r,\theta,\pi-\varphi), \tag{4.67}$$

$$\hat{R}_2\Psi(r,\theta,\varphi) = \Psi(r,\theta,\pi-\varphi), \tag{4.68}$$

$$\hat{R}_3\Psi(r,\theta,\varphi) = \Psi(r,\pi-\theta,\varphi). \tag{4.69}$$

After this step we can use the separation method with $\Psi = \digamma(r)\,\Theta(\theta)\,\Phi(\varphi)$. Then, Eq. (4.64) is written as

$$\begin{aligned}\Bigg\{ r^2\Bigg[&\frac{\partial^2}{\partial r^2} + \frac{2}{r}(1+\mu_1+\mu_2+\mu_3)\frac{\partial}{\partial r} + m^2\omega^2 r^2 \\ &+2m\omega\left(\frac{3}{2}+\mu_1\hat{R}_1+\mu_2\hat{R}_2+\mu_3\hat{R}_3\right) - E^2 + m^2\Bigg] \\ &+\frac{J_\varphi}{\sin^2\theta} + J_\theta\Bigg\}\digamma(r)\,\Theta(\theta)\,\Phi(\varphi) = 0\end{aligned} \tag{4.70}$$

The first and second line in Eq. (4.70) are related by $r-$variable, whereas the third line is related by θ and φ variables. That leads Eq. (4.70) to become as

$$\begin{aligned}\Bigg\{\Theta(\theta)\,\Phi(\varphi)\, r^2\Bigg[&\frac{\partial^2}{\partial r^2} + \frac{2}{r}(1+\mu_1+\mu_2+\mu_3)\frac{\partial}{\partial r} + m^2\omega^2 r^2 \\ &+2m\omega\left(\frac{3}{2}+\mu_1\hat{R}_1+\mu_2\hat{R}_2+\mu_3\hat{R}_3\right) - E^2 + m^2\Bigg]\digamma(r) \\ &+\digamma(r)\left[\frac{J_\varphi}{\sin^2\theta} + J_\theta\right]\Theta(\theta)\,\Phi(\varphi)\Bigg\} = 0\end{aligned} \tag{4.71}$$

After this we devise by $\digamma(r)\,\Theta(\theta)\,\Phi(\varphi)$ we find

$$\begin{aligned}\Bigg\{\frac{r^2}{\digamma(r)}\Bigg[&\frac{\partial^2}{\partial r^2} + \frac{2}{r}(1+\mu_1+\mu_2+\mu_3)\frac{\partial}{\partial r} + m^2\omega^2 r^2 \\ &+2m\omega\left(\frac{3}{2}+\mu_1\hat{R}_1+\mu_2\hat{R}_2+\mu_3\hat{R}_3\right) - E^2 + m^2\Bigg]\digamma(r) \\ &+\frac{1}{\Theta(\theta)\,\Phi(\varphi)}\left[\frac{\hat{J}_\varphi}{\sin^2\theta} + \hat{J}_\theta\right]\Theta(\theta)\,\Phi(\varphi)\Bigg\} = 0\end{aligned} \tag{4.72}$$

Therefore, Eq. (4.72) will be divided into two wave equations. The first is related to $r-$variable

$$\left\{r^2\left[\frac{\partial^2}{\partial r^2}+\frac{2}{r}\left(1+\mu_1+\mu_2+\mu_3\right)\frac{\partial}{\partial r}+m^2\omega^2r^2\right.\right.$$
$$\left.+2m\omega\left(\frac{3}{2}+\mu_1\hat{R}_1+\mu_2\hat{R}_2+\mu_3\hat{R}_3\right)-E^2+m^2\right]F\left(r\right)=\varpi^2F\left(r\right), \tag{4.73}$$

and the second is related to (θ,φ)-variables,

$$\left[\frac{\hat{J}_\varphi}{\sin^2\theta}+\hat{J}_\theta\right]\Theta\left(\theta\right)\Phi\left(\varphi\right)=-\varpi^2\Theta\left(\theta\right)\Phi\left(\varphi\right). \tag{4.74}$$

where ϖ is the separation constant.

From the equation (4.74) we can re-write by :

$$\left[\hat{J}_\varphi+\sin^2\theta\left(\hat{J}_\theta+\varpi^2\right)\right]\Theta\left(\theta\right)\Phi\left(\varphi\right)=0. \tag{4.75}$$

The first operator in Eq. (4.75) is related by $\varphi-$variable and the second operator is related by $\theta-$variable, for these considerations we devise by $\Theta\left(\theta\right)\Phi\left(\varphi\right)$, we obtain

$$\left[\frac{1}{\Phi\left(\varphi\right)}\hat{J}_\varphi\Phi\left(\varphi\right)+\frac{\sin^2\theta}{\Theta\left(\theta\right)}\left(\hat{J}_\theta-\varpi^2\right)\Theta\left(\theta\right)\right]=0. \tag{4.76}$$

This later leads to

$$\frac{1}{\Phi\left(\varphi\right)}\hat{J}_\varphi\Phi\left(\varphi\right)=-\Omega^2. \tag{4.77}$$

$$\frac{\sin^2\theta}{\Theta\left(\theta\right)}\left(\hat{J}_\theta-\varpi^2\right)\Theta\left(\theta\right)=\Omega^2. \tag{4.78}$$

here Ω is also the separation constant.

After these we can write the equations (4.77) and (4.78) as follow :

$$\left\{\frac{\partial^2}{\partial\varphi^2}+2\left[\mu_1\cot\varphi+\mu_2\tan\varphi\right]\frac{\partial}{\partial\varphi}-\frac{\mu_1}{\cos^2\varphi}\left(1-\hat{R}_1\right)-\frac{\mu_2}{\sin^2\varphi}\left(1-\hat{R}_2\right)+\Omega^2\right\}\Phi\left(\varphi\right)=0. \tag{4.79}$$

and

$$\left\{\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+2\left(\left(\mu_1+\mu_2\right)\cot\theta-\mu_3\tan\theta\right)\frac{\partial}{\partial\theta}-\frac{\mu_3}{\cos^2\varphi}\left(1-\hat{R}_3\right)-\frac{\Omega^2}{\sin^2\theta}+\varpi\right\}\Theta\left(\theta\right)=0. \tag{4.80}$$

Using the eigenvalues of the operators of reflection, we find,

$$\left\{\frac{\partial^2}{\partial\varphi^2}+2\left[\mu_1\cot\varphi+\mu_2\tan\varphi\right]\frac{\partial}{\partial\varphi}-\frac{\mu_1}{\cos^2\varphi}\left(1-s_1\right)-\frac{\mu_2}{\sin^2\varphi}\left(1-s_2\right)+\Omega^2\right\}\Phi\left(\varphi\right)=0. \tag{4.81}$$

and

$$\left\{\frac{\partial^2}{\partial\theta^2}+\cot\theta\frac{\partial}{\partial\theta}+2\left(\left(\mu_1+\mu_2\right)\cot\theta-\mu_3\tan\theta\right)\frac{\partial}{\partial\theta}-\frac{\mu_3}{\cos^2\varphi}\left(1-s_3\right)-\frac{\Omega^2}{\sin^2\theta}+\varpi\right\}\Theta\left(\theta\right)=0. \tag{4.82}$$

In order to determine the separation constants, and the solution of equations (4.79) and (4.80), we use the following ansazt (see., the reference [12]),

$$\Phi\left(\varphi\right)=\cos^k\left(\varphi\right)\sin^p\left(\varphi\right)\tilde{\Phi}\left(\cos\varphi\right). \tag{4.83}$$

The same remark we can find for the second equation (4.80),

$$\Theta\left(\theta\right)=\cos^\sigma\left(\theta\right)\sin^{2\nu}\left(\theta\right)\tilde{\Theta}\left(\cos\theta\right). \tag{4.84}$$

After these changes we can find (see chapter 3),

$$\tilde{\Phi}\left(x\right)=C_\varphi P^{\left(\mu_2+p-\frac{1}{2},\mu_1+k-\frac{1}{2}\right)}\cos\left(2\varphi\right)$$

and

$$\tilde{\Theta}\left(\theta\right)=C_\theta P^{\left(\mu_2+p-\frac{1}{2},\mu_1+k-\frac{1}{2}\right)}\cos\left(2\theta\right). \tag{4.85}$$

where C_φ and C_θ are the normalization constants. For the validity of the given solutions, the separation constants have to satisfy the following conditions :

$$\Omega^2=4\nu\left(\nu+\mu_1+\mu_2\right). \tag{4.86}$$

and

$$\varpi^2=4\left(l+\nu\right)\left(l+\nu+\mu_1+\mu_2+\frac{1}{2}\right). \tag{4.87}$$

also for the values of k, p, σ and ν

$$k=\frac{1-s_1}{2}$$
$$p=\frac{1-s_2}{2}$$
$$\sigma=\frac{1-s_3}{2}$$

and ν is a positive integer.

Following that, we seek an exact solution to the radial equation. we will start by substituting Eq. (4.86) and Eq. (4.87) into Eq. (3.220), we find

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\left(1+\mu_1+\mu_2+\mu_3\right)\frac{\partial}{\partial r}+m^2\omega^2r^2+2m\omega\left(\frac{3}{2}+\mu_1s_1+\mu_2s_2+\mu_3s_3\right)\right.$$
$$\left.-\frac{4\left(l+\nu\right)\left(l+\nu+\mu_1+\mu_2+\frac{1}{2}\right)}{r^2}\right\}F\left(r\right)=0. \tag{4.88}$$

After this later we can write the following ansazt

$$F=\rho^{\nu+l}e^{-\frac{\rho}{2}}\Xi\left(\rho\right), \tag{4.89}$$

and by taking the new variable $\rho=m\omega r^{2}$. We find

$$\begin{aligned}&\left\{\rho\frac{\partial^{2}}{\partial\rho^{2}}+\left(\frac{3}{2}+2\left(l+\nu\right)+\mu_{1}+\mu_{2}+\mu_{3}-\rho\right)\frac{\partial}{\partial\rho}\right.\\&\left.+\left(\mu_{1}k+\mu_{2}p+\mu_{3}\sigma\right)+\left(l+\nu\right)+\frac{E^{2}-m^{2}}{4m\omega}\right\}\Xi\left(\rho\right)=0\end{aligned} \tag{4.90}$$

The following is the solution to the above equation in terms of Laguerre polynomials :

$$\Xi\left(\rho\right)=C_{r}L_{N}^{4(l+\nu)\left(l+\nu+\mu_{1}+\mu_{2}+\frac{1}{2}\right)}\left(\rho\right). \tag{4.91}$$

where N is a quantum number given by

$$\begin{aligned}N&=-\left(\mu_{1}k+\mu_{2}p+\mu_{3}\sigma\right)+\left(l+\nu\right)+\frac{E^{2}-m^{2}}{4m\omega}\\&=-\frac{1}{2}\left[\mu_{1}\left(1-s_{1}\right)+\mu_{2}\left(1-s_{2}\right)+\mu_{3}\left(1-s_{3}\right)+2\left(l+\nu\right)\right]+\frac{E^{2}-m^{2}}{4m\omega}.\end{aligned} \tag{4.92}$$

Thus we get the level energy

$$N=-\frac{1}{2}\left[\mu_{1}\left(1-s_{1}\right)+\mu_{2}\left(1-s_{2}\right)+\mu_{3}\left(1-s_{3}\right)+2\left(l+\nu\right)\right]+\frac{E^{2}-m^{2}}{4m\omega}$$

or

$$E^{2}=4m\omega\left(N+\frac{1}{2}\left[\mu_{1}\left(1-s_{1}\right)+\mu_{2}\left(1-s_{2}\right)+\mu_{3}\left(1-s_{3}\right)+2\left(l+\nu\right)\right]\right)+m^{2}. \tag{4.93}$$

So the spectral energies are

$$E=\sqrt[\pm]{2m\omega\left[2\left(N+l+\nu\right)+\mu_{1}\left(1-s_{1}\right)+\mu_{2}\left(1-s_{2}\right)+\mu_{3}\left(1-s_{3}\right)\right]+m^{2}}. \tag{4.94}$$

As a result, the three-dimensional Dunkl-Klein-Gordon oscillator's radial eigenfunctions become

$$F=\rho^{\nu+l}e^{-\frac{\rho}{2}}\Xi\left(\rho\right)=\rho^{\nu+l}e^{-\frac{\rho}{2}}C_{r}L_{N}^{4(l+\nu)\left(l+\nu+\mu_{1}+\mu_{2}+\frac{1}{2}\right)}\left(\rho\right). \tag{4.95}$$

We observe that the energy spectrum explicitly depends not only on the quantum numbers (N,ν,ℓ) but the other parameters, (μ_{j},s_{j}), which characterize the Dunkl derivative. Therefore, we conclude that the energy spectrum is dependent on a term originating from the conventional Klein-Gordon oscillator and an additional term originating from the Dunkl derivative.

It is worth noting that, as in the previous section, the maximal contribution of the Dunkl term is obtained for $s_1 = s_2 = s_3 = -1$, while the minimal contribution is achieved from $s_1 = s_2 = s_3 = +1$. In addition such a correction term, which depends explicitly on sj , lifts the degeneracy of energy levels. Before we conclude this section, we briefly would like to introduce the orthogonality relation of the angular and radial parts of the wavefunction. Using the following orthogonality relation

$$\int \psi_{N,\nu,l}^{(s_1,s_2,s_3)} \psi_{N,\nu,l}^{(s_1,s_2,s_3)} |r \sin\theta \cos\phi|^{2\mu_1} |r \sin\theta \sin\phi|^{2\mu_2} |r \cos\theta|^{2\mu_3} r^2 dr \sin\theta d\theta d\phi = 1 \tag{4.96}$$

Chapitre 5

Application of the Dunkl-Dirac oscillator

5.1 Dirac oscillator in one dimension

In this section we will present the Dunkl-Dirac oscillator equation in one dimension, which is expressed as„

$$\left[\alpha_x\left(\frac{1}{\imath}\hat{D}-\imath\beta m\omega x\right)+\beta m\right]\Psi=E\Psi. \tag{5.1}$$

where the spinor wave function is defined in a two-component spinor $\Psi\binom{\Phi_1}{\Phi_2}$, and the Dirac matrices are written in 2×2 matrix

$$\alpha_x=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\qquad \beta=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \tag{5.2}$$

In matrix form Eq. (5.1) is analytic given by

$$\left[\begin{pmatrix}0 & \frac{1}{\imath}D_x\\ \frac{1}{\imath}D_x & 0\end{pmatrix}-\begin{pmatrix}0 & -\imath m\omega x\\ -\imath m\omega x & 0\end{pmatrix}+\begin{pmatrix}m & 0\\ 0 & -m\end{pmatrix}\right]\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=E\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}. \tag{5.3}$$

After simplifications, we find

$$\begin{pmatrix}0 & \frac{1}{\imath}D_x+im\omega x\\ \frac{1}{\imath}D_x-im\omega x & 0\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}=\begin{pmatrix}E-m & 0\\ 0 & E+m\end{pmatrix}\begin{pmatrix}\Phi_1\\ \Phi_2\end{pmatrix}. \tag{5.4}$$

Finally, we get the two equations below

$$\begin{cases}\left(\frac{1}{\imath}\hat{D}_x+\imath m\omega x\right)\Phi_2=(E-m)\,\Phi_1\\ \left(\frac{1}{\imath}\hat{D}_x-\imath m\omega x\right)\Phi_1=(E+m)\,\Phi_2\end{cases}. \tag{5.5}$$

From the second equation for the set of (5.5) we write

$$\Phi_2 = \frac{\left(\frac{1}{\imath}\hat{D}_x - \imath m\omega x\right)\Phi_1}{(E+m)}. \tag{5.6}$$

By substituting this Eq. (5.6) in the first equation of the set (5.5), we obtain

$$\left(\frac{1}{\imath}D_x + \imath m\omega x\right)\frac{\left(\frac{1}{\imath}D_x - im\omega x\right)\Phi_1}{(E+m)} = (E-m)\,\Phi_1. \tag{5.7}$$

or

$$\left(\frac{1}{\imath}D_x + im\omega x\right)\left(\frac{1}{\imath}D_x - im\omega x\right)\Phi_1 = (E-m)\,(E+m)\,\Phi_1. \tag{5.8}$$

As a result, we write

$$\left[\left(\hat{D} - m\omega x\right)\left(\hat{D} + m\omega x\right) + E^2 - m^2\right]\Phi_1 = 0. \tag{5.9}$$

At this stage, we can express the solution for the upper component of the spinor as the same solution of Eq. (18) which presented in previous chapter of Klein-Gordon oscillator. So the solution of Equation. (5.9) becomes as

$$\Phi_1^s = N_1\,(m\omega x)^{\frac{1-x}{2}}\,e^{\frac{-m\omega x^2}{2}}F\left(-n, 1-\frac{s}{2}+\mu; m\omega x^2\right). \tag{5.10}$$

Whereas to obtain the componentΦ_2, we have

$$\Phi_2 = \frac{\left(\frac{1}{\imath}\left[\frac{d}{dx} + \frac{\mu}{x}(1-s) - \imath m\omega x\right]\right)}{(E+m)}\Phi_1. \tag{5.11}$$

Using some simplifications we find

$$\Phi_2 = \frac{-\imath}{(E+m)}\left[\left(\frac{1-s}{2}\right)\Phi_1 - m\omega x\Phi_1 + N\,(m\omega x)^{\left(\frac{1-s}{2}\right)}\,e^{\frac{-m\omega x^2}{2}}\frac{dF}{dx} + \frac{\mu}{x}(1-s)\,\Phi_1 + m\omega x\Phi_1\right]. \tag{5.12}$$

According to the spinor solution,

$$\Psi_n^s = N_s\,(m\omega x)^{\frac{1-x}{2}}\,e^{\frac{-m\omega x^2}{2}}\begin{pmatrix} 1 \\ \frac{-\imath}{(E+m)}\left[\frac{d}{dx} + \frac{\left(\mu+\frac{1}{2}\right)(1-s)}{x}\right] \end{pmatrix} F\left(-n, 1-\frac{s}{2}+\mu; m\omega x^2\right). \tag{5.13}$$

Chapitre 6

General Conclusion

In this wok, we have examined in nonrelativistic case the isotropic harmonic oscillator and a particle in box in one, two and three dimensions using Cartesian coordinates. As well as in two and three dimensions we have determined the exact solution of isotropic harmonic oscillator potential in polar and spherical representations. In addition, we generalize this problem in relativistic case for Dunkl oscillator model in three-dimensional using Cartesian and spherical coordinates. With theses choices, we found the exact spectrum energy and the normalized radial wave functions in coordinates space. For spinorial particle we examined the problem Dunkl Dirac oscillator in one dimension.

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Abstract:

In this work, we adapt the mathematical foundations of quantum mechanics in the presence of the Dunkl derivative. First, we got to know its mathematical expressions and effect on various quantum systems. Through our treatment of some of them: for example, in non-relativistic quantum mechanics, we treat using Cartesian coordinates an infinite quantum well and harmonic oscillator, then using the polar and spherical coordinates we discussed the issue of the isotropic harmonic oscillator. Finally, in relativistic quantum mechanics, we applied this derivative to the Klein-Gordon oscillator and the Dirac oscillator equations respectively.

Key words: The Dunkl derivative, Particle in a box, Harmonic oscillator, The Dunkl-Klein- Gordon oscillator, The Dunkl-Dirac oscillator**.**

***Résumé*:**

Dans ce travail, nous adaptons les fondements mathématiques de la mécanique quantique en présence de la dérivée de Dunkl. Tout d'abord, nous avons appris à connaître ses expressions mathématiques et l'effet sur divers systèmes quantiques. A travers notre traitement de certains d'entre eux: par exemple, en mécanique quantique non relativiste, on traite en coordonnées cartésiennes un puits quantique infini et un oscillateur harmonique, puis en coordonnées polaires et sphériques on aborde le problème de l'oscillateur harmonique isotrope. Enfin, en mécanique quantique relativiste, nous avons appliqué cette dérivée respectivement aux équations de l'oscillateur de Klein-Gordon et de l'oscillateur de Dirac.

Mots clés : Le dérivé de Dunkl, Particule dans une boîte, Oscillateur harmonique, L'oscillateur de Dunkl-Klein-Gordon, L'oscillateur de Dunkl-Dirac.

:الملخص

في هذا العمل قمنا بتكييف الاسس الرياضية لميكانيك الكم في وجود مشتق Dunkl، حيث تعرفنا في البداية على عبارته الرياضية وتأثيره في مختلف الأنظمة الكمية من خلال معالجتنا لبعض منها، مثلا: في ميكانيك الكم الغير نسبي، عالجنا باستخدام الاحداثيات الديكارتية بئر كمومي لانهائي و هزاز توافقي، ثم باستخدام الاحداثيات القطبية و الكروية ناقشنا مسألة هزاز توافقي . وفي الأخير في ميكانيك الكم النسبي قمنا بتطبيق هذا المشتق على معادلتي هزاز كلاين جوردن و هزاز ديراك.

الكلمات المفتاحية: مشتق Dunkl ، جسيم في صندوق ، هزاز توافقي ، هزاز Dunkl-Klein- Gordon ، هزاز Dunkl-Dirac.