Kasdi Merbah University of Ouargla
Faculty of Mathematics and Material Sciences
Department of Mathematics

N° d'ordre :
N° de série :

DEPARTMENT OF MATHEMATICS

MASTER

Domain: Mathematics

Option:Modelling and Numerical Analysis

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Title:

Dynamic viscoelastic problem with exponentially decaying kernel

Presented Publicly on: 25/06/2023

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Dedication

To my dear parents, for all their sacrifices, their support throughout my studies,
To my dear sisters for their constant encouragement and moral support,
To my dear brothers, for their support and encouragement,
To all my family for their support throughout my university career.
To my teachers and mentors, for their expertise, patience and guidance throughout my studies.
Finally, to all the people who have contributed directly or indirectly to the realization of this thesis, I express my deepest gratitude.

Acknowledgement

First of all, I would like to thank **Allah** for all his blessings and favours.
I would like to thank my **father** and my **mother** who have been a constant source of inspiration and support throughout my thesis journey. Their encouragement and support have been invaluable. I would also like to thank my brothers and sisters for their constant support.
I would like to extend my sincere gratitude to my thesis advisor, Professor **Merabet Ismail**, for accepting to supervise this dissertation with patience, seriousness, and expertise. His guidance, corrections, and insights have greatly contributed to the progress of my work. I sincerely appreciate his dedication.
I am also grateful to Professor **Chacha Djamal Ahmed**, Professor **Abdallah Bensayah**, and **Ilyes Lacheheb** for agreeing to be part of the examination committee and evaluating my thesis. Their presence and expertise have been highly valued.
Lastly, I would like to express my gratitude to all my professors and lecturers who have assisted me throughout my college years. Their teachings and advice have been instrumental in my academic development.
I am grateful to all the individuals mentioned above, as well as others who have contributed, directly or indirectly, to the completion of this work. Their unwavering support and encouragement have been invaluable.
Thank you Zenkhri Kouthar

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Notations

➤ $\nabla = \left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\right)$:The gradient of a vector .

➤ $\Delta = \left(\frac{\partial^2}{\partial x_1^2} + \ldots + \frac{\partial^2}{\partial^2 x_n}\right)$:Laplace operator .

➤ $(.,.)$:The scalar product and $\langle .,. \rangle$:The duality product.

➤ $H^m(\Omega) = \{v \in L^2(\Omega) / \forall \alpha \in \mathbb{N} : |\alpha| \leq m, \partial^\alpha v \in L^2(\Omega)\}$. Sobolev space of order m.

➤ $H_0^1(\Omega) = \{v \in H^1 / v = 0 \text{ dans } \partial\Omega\}$.

➤ H^{-1} : dual of space$H_0^1(\Omega)$.

➤ $\mathcal{C}^k(\Omega)$: the space of continuous, differentiable k-times functions on Ω.

➤ $C^\infty(\Omega)$: Space of infinitely differentiable functions on Ω.

➤ $L^p(\Omega)$: Space of p-th integrable functions on Ω with respect to the Lebesgue measure dx, for $p \in [1, +\infty[$.

➤$L^p(0, T; X)$: Bochner Spaces of vector-valued functions.

INTRODUCTION

The behavior of materials that exhibit both elastic and viscous responses when subjected to time-varying loads or deformations is referred to as dynamic viscoelasticity. These materials have the ability to store and dissipate energy over time. A common approach to modeling dynamic viscoelasticity is through a Volterra-type integro-differential equation [11], where the material response is described in terms of convolution integrals involving a kernel function.

In many cases, the kernel function used to describe the time-dependent behavior of viscoelastic materials follows an exponential decay model. This type of kernel is particularly suitable for materials that exhibit relaxation behavior, where the material response gradually decreases over time after the application of a load or deformation. The exponential decay reflects the decay of stress or strain in the material due to internal processes such as molecular rearrangement or diffusion. The use of an exponential decay kernel in dynamic viscoelasticity models allows for the incorporation of time-dependent effects and accurate capture of the transient behavior of the material. This type of kernel is commonly used to model viscoelastic relaxation in many materials.

The relaxation decay can be described using an exponential decay kernel. Mathematically, the viscoelastic relaxation behavior of the material can be modeled by an integral equation with an exponential kernel. For example, let's assume that the deformation of the material at a given time depends on its previous deformation and is given by the following equation: $\delta(t) = \int_0^t G(t-s)u(s)ds$,[12] in this equation, $\delta(t)$ represents the stress at time t, $u(s)$ denotes the displacement at time s, and $G(t-s)$ is an exponential decay kernel that represents the viscoelastic behavior of the material.The integral in this equation represents the convolution of the displacement history $u(s)$ with the exponential kernel $G(t-s)$. It describes how past displacement contribute to the stress at the present moment t, with the kernel $G(t-s)$ capturing the temporal decay of these contributions.

When waves propagate through a viscoelastic material, they undergo phenomena of dispersion, attenuation ,and shape deformation due to the viscoelastic properties of the material. The presence of an exponential decay kernel in the models allows for the consideration of these time-dependent effects and provides a more realistic description of wave propagation.

Wave equations with memory terms or integral equations are used to model wave propagation in viscoelastic materials. These equations take into account the viscoelastic behavior of the material by including memory terms that describe the relaxation or delay in the material's response. The exponential decay kernel is incorporated into these memory terms to capture the exponential decay of relaxation over time.

In this study, our objective is to investigate the wave equation with memory terms, which takes into account the viscoelastic effects and the influence of the past on wave behavior. The wave equation with memory terms can be formulated as follows [12]:

$$\frac{d^2u}{dt^2} - c^2\Delta u + \int_0^t G(t-s)\frac{du}{ds}ds = 0,$$

In our study, we focus on the exponential decay kernel, which can be expressed in the form $G(t-s) = e^{-\alpha(t-s)}$, For the study of existence and uniqueness, we used the spectral approach (see [4]), which is a powerful mathematical technique that represents solutions using a series expansion in terms of orthogonal basis functions, such as Fourier series or Chebyshev polynomials. This approach allows us to approximate the solution with high precision and capture the essential characteristics of the problem.

In our study, we employed numerical approaches based on finite element methods to solve the wave equation with memory terms. Specifically, we explored continuous Galerkin finite element methods to spatially discretize our problem as described in references [7,9]. We used the Crank-Nicolson and Newmark schemes to discretize our problem in the temporal domain. These approaches allowed us to obtain accurate and stable numerical solutions to investigate the properties of the wave equation with memory terms (e.g. see [8]).

A plan for this thesis is organised as follows:

- In the first chapter, we introduced preliminaries in terms of notations, general definitions, and theorems in functional analysis. Additionally, we studied the existence, uniqueness, and stability of solutions under certain conditions for the wave equation without memory term.

- In the second chapter, we investigated the existence, uniqueness, and stability of solutions under certain conditions for the wave equation with a memory term.

- In the third chapter, we numerically solved the wave equation without and with a memory term using the continuous Galerkin finite element method and, Crank-Nicolson and Newmark schemes.

Chapter 1

The wave equation without memory term

In this chapter, our objective is to establish a weak formulation of the problem at hand within a well-defined functional framework. The weak formulation allows us to relax the notion of classical solutions and consider solutions in a broader sense, which can be more suitable for certain problems.

To achieve this, we will utilize a spectral approach, which involves approximating the solution using a series of basis functions, such as orthogonal polynomials or trigonometric functions.

1.1 Wave equation

Wave propagation in a nonhomogeneous and anisotropic medium can be described by second-order hyperbolic equations, which are a generalization of the classical wave equation.These equations serve as fundamental models for describing various oscillatory phenomena in dimensions $n \geq 1$.

The classical wave equation given by :

$$u_{tt} - c^2 \Delta u = f.$$

Here $u = u(x,t)$is a displacement $x \in \mathbb{R}^n$ and, c is the speed of propagation, and f an external force. Let us examine some relevant solutions, more precisely, we can pose the following problems, determine a displacement $u = u(x,t)$ such that :

$$(\mathcal{P}) \begin{cases} u_{tt} - c^2 \Delta u = f, & \text{in } \mathbb{Q}_T \\ u(x,0) = u_0(x), & \text{in } \Omega \\ u_t(x,0) = u_1(x) & \text{in } \Omega \\ u(\sigma,t) = 0 & \text{on } \mathbb{S}_T \end{cases} \tag{1.1}$$

Remark 1.1.1 *Since we are in the context of an evolution problem in a bounded open set $\Omega \subset \mathbb{R}^n$ over a finite time interval $(0,T)$, it is convenient to define the space-time cylinder $\mathbb{Q}_T$ and the lateral part $\mathbb{S}_T$ as follows :*

The space-time cylinder $\mathbb{Q}_T$ is the set of all points in space and time within the given domain :

$$\mathbb{Q}_T = \Omega \times (0,T)$$

Here, Ω represents the spatial domain and $(0,T)$ represents the time interval. The space-time cylinder $\mathbb{Q}_T$ includes all possible combinations of points in Ω and times within $(0,T)$. It allows us to consider the behavior and evolution of the problem throughout the entire domain and time range.

The lateral part $\mathbb{S}_T$ corresponds to the boundary of the space-time cylinder $\mathbb{Q}_T$ at a given time t :

$$\mathbb{S}_T = \Gamma \times (0,T)$$

Here, Γ represents the boundary of the spatial domain Ω, and $(0,T)$ represents the time interval. The lateral part $\mathbb{S}_T$ captures the boundary of the space-time cylinder $\mathbb{Q}_T$ at each *time t within the interval $(0,T)$. It specifically includes all points on the spatial boundary Γ at different times.*

By defining the space-time cylinder $\mathbb{Q}_T$ and the lateral part $\mathbb{S}_T$, we establish a suitable framework for analyzing the evolution problem over the given spatial domain and time interval. This framework allows us to consider the solution's behavior and properties throughout the entire space and time range, including the spatial boundary at different time instances.

1.2 The Variational Problem

In this section, our objective is to determine an appropriate weak formulation to our problem. In our way to that, we will give some preliminaries and define some spaces needed to carry on our study. The preliminaries given here are not exclusively dedicated to this chapter but to the other chapters too.

1.2.1 Preliminaries

Let Ω be open bounded set in $\mathbb{R}^n$. Consider the spaces $H = L^2(\Omega)$ and $V = H_0^1(\Omega)$, equipped with the following norms respectively

$$\|u\|_{L^2(\Omega)} = \left(\int_\Omega |u(x)|^2 dx\right)^{\frac{1}{2}} \tag{1.2}$$
$$\|u\|_{H^1(\Omega)} = \left(\int_\Omega \left(|u(x)|^2 + |\nabla u(x)|^2\right) dx\right)^{\frac{1}{2}}$$

Lemma 1.2.1 ***(Gronwall's Inequality)***
Let Ψ, G be continuous in $[0,T]$, with G nondecreasing and $\gamma > 0$. If

$$\Psi(t) \le G(t) + \gamma \int_0^t \Psi(s)ds, \qquad \textit{for all } t \in [0,T]$$

then

$$\Psi(t) \le G(t)e^{\gamma t}, \qquad \textit{for all } t \in [0,T]$$

Proof. See([1] , P.587) ■

Theorem 1.2.1 ***(Young's Inequality)***
Let a and b be two non negative real numbers. if p,q $\in]1,\infty[$ with $\frac{1}{p} + \frac{1}{q} = 1$then

$$ab \le \frac{a^p}{p} + \frac{a^q}{q}$$

Proof. See([6];P.49) ■

Theorem 1.2.2 ***(Young's Inequaliy With a Parameter)***
Let a and b be two non negative real numbers.
For all $\alpha \leq 0$

$$ab \leq \frac{1}{2\alpha}a^2 + \frac{\alpha}{2}b^2$$

Theorem 1.2.3 ***(Poincaré's Inequality)***
Let Ω be a bounded Lipschitz domain. Then, there exists a constant C_P such that

$$\|u\|_{L^2(\Omega)} \leq C_P\|\nabla u\|_{L^2(\Omega)} \quad ,\forall u \in H_0^1(\Omega)$$

Proof. See ([1]; P486) ■

Lemma 1.2.2 *The space V is separable, that is: V admits a countably dense subset*

Proof. See ([2] ; P6) ■

Theorem 1.2.4 ***(the weak compactness theorem)***
Every bounded sequence in a Hilbert space H contains a sub sequence which is weakly convergent to an element $x \in H$.

Proof. See ([1] , p. 393) ■

Theorem 1.2.5 ***(Green's Integration by Parts Formula)***
Let Ω be a bounded open domain in $\mathbb{R}^3$ with a sufficiently smooth boundary Γ and $\mathbf{n}$ is the outward normal. Then for all $u, v \in \mathcal{C}^1(\bar{\Omega})$

$$\int_\Omega \partial_i u(x)v(x)dx = -\int_\Omega u(x)\partial_i v(x)dx + \int_\Gamma u(x)v(x)n_i d\Gamma.$$

Definition 1.2.1 ***(Cauchy sequence)***
Let (X, d) be a metric space and $(x_n)_n$ a sequence of elements of X. We say that the sequence $(x_n)_n$ is a Cauchy sequence in (X, d) if:

$$\forall \varepsilon \in \mathbb{R}_+^* \quad \exists N_\varepsilon \in \mathbb{N} \quad \forall (m,n) \in \mathbb{N}^2 \quad (n \geqslant m \geqslant N_\varepsilon \Rightarrow d(x_n, x_m) \leqslant \varepsilon).$$

Theorem 1.2.6 *Let (X, d) be a metric space and $(x_n)_n$ a sequence of elements of X. If the sequence $(x_n)_n$ converges in (X, d) then it is a Cauchy sequence in (X, d).*

Proof. See([13],P.400) ■

Theorem 1.2.7 *Let V and H be two Hilbert spaces such that $V \subset H$ with compact injection and V is dense in H. Let $a(\cdot,\cdot)$ be a continuous and coercive symmetric bilinear form in V. Then the eigenvalues $\{\lambda_k\}_{k\geq 1}$, which satisfy*

$$\exists w_k \neq 0, w_k \in V, \quad a(w_k, v) = \lambda_k \langle w_k, v\rangle, \quad \forall v \in V \tag{1.3}$$

form an increasing sequence of positive real numbers tends to infinity and the sequence $\{w_k\}_{k\geq 1}$ is a Hilbert basis of H. What's more the sequence $\{\frac{w_k}{\sqrt{\lambda_k}}\}_{k\geq 1}$ is a Hilbert basis of V.

Proof. See([4] p.219) ■

1.2.2 Spectral approach

The spectral approach is a mathematical method used to solve differential equations by representing the solutions as a sum of basis functions or eigenfunctions. This approach involves approximating the solution using a series of basis functions, such as orthogonal polynomials or trigonometric functions. These basis functions form a complete set in the chosen function space, allowing us to express the solution as a linear combination of these functions.

By applying the concept of weak solutions and utilizing the chosen basis functions, we can transform the original problem into an equivalent variational problem. This variational problem involves finding a solution that minimizes a suitable functional, which represents the energy or some other relevant quantity associated with the problem.

The advantage of this approach is that it allows us to work with infinite-dimensional function spaces and utilize the powerful tools of functional analysis. Moreover, by choosing an appropriate basis, we can accurately capture the behavior of the solution and obtain convergence properties.

By formulating the problem in this manner, we can establish existence and uniqueness results, analyze stability, and explore various numerical approximation techniques. This approach provides a solid mathematical foundation for studying the wave equation and the wave equation with a memory term and paves the way for further investigations and applications in the field.

1.2.3 Functional Framework

First of all, since we are dealing with evolution equations, it is convenient to adopting the following point of view of the space involving time. Assume that $t \in [0,T]$ and that for every t, or at least for a.e. t, the function $u(.,t)$ belongs to a separable Hilbert space V Then, we may consider u as a function of the real variable t with values into V :

$$u : [0,T] \to V. \tag{1.4}$$

When we adopt this convention, we write $u(t)$ and $\dot{u}(t)$ instead of $u(x,t)$ and $u_t(x,t)$. Following this, Let $1 \leq p < \infty$ and X a Banach space, if p is finite we define $L^p(0,T;X)$ the space of measurable functions on $]0,T[$ with respect to the measure dt :

$$L^p(0,T;X) = \left\{ u : [0,T] \to X \quad \text{measurable such that} \quad \int_0^T \|u\|_X^p dt < \infty \right\}$$

where $[0,T]$ is an interval of $\mathbb{R}$. We endow this space with the norm :

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u\|_X^p dt \right)^{\frac{1}{p}}.$$

And scalar products :

$$(u,v) = \int_\Omega u\,(x)\,v\,(x)\,dx, \quad \forall u,v \in L^2\,(\Omega)\,,$$

Now, we introduce the following Hilbert spaces

$$\begin{cases} \mathbb{V} = L^2(0,T;V) & \text{with} \quad V = H_0^1(\Omega) \\ \mathbb{H} = L^2(0,T;H) & \text{with} \quad H = L^2(\Omega) \\ \mathbb{V}^* = L^2(0,T;V^*) & \text{with} \quad V^* = H^{-1}(\Omega) \end{cases}$$

The extension to infinite dimensional spaces requires some care. In particular, before setting, to avoid confusion. The problem involves two Hilbert spaces : V , the space where we seek the solution, and V^*, the dual space of V . Let us introduce a third space H intermediate between V and V^* which the data f belongs to. For better clarity, it is convenient to use the symbol $<,>_*$ to denote the duality between V^* and V and while V is a Sobolev space. In practice one often encounters a pair of Hilbert spaces V ,H with the following properties:

1. $V \hookrightarrow H$, i.e. V is continuously embedded in H Recall that this simply means that the embedding operator $\mathcal{I}_V \hookrightarrow H$, from V into H, is continuous or, equivalently that there exists C such that.

$$\|u\|_H \leq C\|v\|_V \quad \forall u \in V \tag{1.5}$$

2. V is dense in H, Using Rieszs Representation Theorem we may identify H with H^*. Also, we may continuously embed H into V^* , so that any element in H can be thought as an element of V^*. To see it, observe that, for any fixed $u \in H$, the functional $T_u : V \to \mathbb{R}$, defined by :

$$\langle T_u, v\rangle_* = (u,v)_H \quad v \in V \tag{1.6}$$

is continuous in V. In fact, the Schwarz inequality and (1.5) give

$$|(u,v)_H| \leq \|u\|_H\|v\|_H \leq C\|u\|_H\|v\|_V \tag{1.7}$$

Thus, the map $u \to T_u$ is continuous from H into V^*, with $\|T_u\|_{V^*} \leq C\|u\|_H$. Moreover, if $T_u = 0$, then

$$0 = \langle T_u, v\rangle_* = (u,v)_H, \quad \forall v \in V$$

which forces $u = 0$, by the density of V in H. Thus, the map $u \to T_u$ is one to one and defines the continuous embedding $I_{H\to V^*}$ of H into V^*. This allows the identification of $u \in H$ with $T_u \in V$. In particular, instead of (1.6), we can write

$$\langle T_u, v\rangle_* = \langle u, v\rangle_* = (u,v)_H, \quad \forall v \in V$$

Finally, V (and therefore also H) is dense in V^*. Summarizing, we have

$$V \hookrightarrow H \hookrightarrow V^* \tag{1.8}$$

with dense embedding. We call V, H, V^* a Hilbert triplet.

1.2.4 Weak Formulations

As usual, to find a weak formulation, we proceed formally and multiply the wave equation by a smooth function v, vanishing at the boundary. Integrating over Q_T, we find

$$\int_{Q_T} u_{tt}(x,t)v(x,t)dxdt - c^2\int_{Q_T} \Delta u(x,t)v(x,t)dxdt = \int_{Q_T} f(x,t)v(x,t)dxdt$$

by applying Green's formula in the equation (1.1),and using the appropriate boundary condition, we get:

$$\int_{Q_T} u_{tt}(x,t)v(x,t)dxdt + c^2\int_{Q_T} \nabla u(x,t)\cdot\nabla v(x,t)dxdt = \int_{Q_T} f(x,t)v(x,t)dxdt \quad (1.9)$$

$$\int_0^T (\ddot{u}(t),v)_0 dt + c^2\int_0^T (\nabla u(t),\nabla v)_0 dt = \int_0^T (f(t),v)_0 dt$$

The variational formulation deduced from (1.1) is, therefore:
find a solution u in $\mathscr{C} = C\left([0,T];H_0^1(\Omega)\right) \cap C^1\left([0,T];L^2(\Omega)\right)$ such that:

$$\begin{cases} \dfrac{d^2}{dt^2}\langle u(t),v\rangle_H + c^2 a(u(t),v) = \langle f(t),v\rangle_H \quad \forall v\in V, 0<t<T \\ u(t=0)=u^0, \\ \dfrac{du}{dt}(t=0)=u^1 \end{cases} \quad (1.10)$$

where the symmetric bilinear form $a(.,.)$ is defined by

$$a(u,v) = (\nabla u(t),\nabla v)_0$$

1.3 The Well-Posedness of the Problem

Our purpose, in this section is to show that problem (1.1) has a unique solution, which continuously depends on the data, in appropriate norms. Once more, we are going to use here a spectral approach.

1.3.1 Existence, Uniqueness and stability of Solution

The fallowing theorem assure the existence uniqueness and stability of solution

Theorem 1.3.1 *Let V and H be two Hilbert spaces such that $V \subset H$ with a compact injection, and V is dense in H. Let $a(\cdot,\cdot)$ be a continuous and coercive symmetric bilinear form on V. Let $T>0$ be the final time, $u^0 \in V, u^1 \in H$ be the initial data, and $f \in L^2(]0,T[;H)$. So, the problem (1.10) has a unique solution satisfying the following regularities $u \in \mathscr{C} = C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$ Moreover, there is a constant $C>0$ (which only depends on Ω and T) such that*

$$\|u\|_{C([0,T];V)} + c^2\|u\|_{C^1([0,T];H)} \leq C\left(\|u_0\|_V + \|u_1\|_H + \|f\|_{L^2(]0,T[;H)}\right). \tag{1.11}$$

Remark 1.3.1 *We can weaken the assumption of Theorem 1.3.1 regarding the coercivity of the symmetric bilinear form $a(u,v)$. We can obtain the same conclusions by assuming only the existence of two positive constants $\nu>0$ and $\eta>0$ such that*

$$a(v,v) + \eta|v|_H^2 \geq \nu|v|_V^2 \text{ for all } v \in V.$$

Proof. To obtain the existence and uniqueness of solutions for hyperbolic problems, we employ a spectral approach.

Step 1. Let us introduce the Hilbertian basis $(u_k)_{k\geq 1}$ of H composed of the eigenfunctions

$$w_k \in V, \text{ such as } a\left(w_k, v\right) = \lambda_k \left\langle w_k, v\right\rangle_H \quad \forall v \in V.$$

We write

$$u(t) = \sum_{k=1}^{+\infty} \alpha_k(t) w_k \quad \text{with} \quad \alpha_k(t) = \langle u(t), w_k\rangle_H. \tag{1.12}$$

Choosing $v = w_k$ in (1.10), and noting

$$\beta_k(t) = \langle f(t), w_k\rangle_H, \quad \alpha_k^0 = \langle u_0, w_k\rangle_H, \quad \text{et} \quad \alpha_k^1 = \langle w_1, w_k\rangle_H$$

we obtain

$$\begin{cases} \dfrac{d^2\alpha_k}{dt^2} + c^2\lambda_k\alpha_k = \beta_k \text{ in }]0,T[\\ \alpha_k(t=0) = \alpha_k^0, \\ \dfrac{d\alpha_k}{dt}(t=0) = \alpha_k^1 \end{cases} \tag{1.13}$$

Lemma 1.3.1 *The general solution is written:*

$$\alpha_k(t) = C_1\cos(\mu_k t) + C_2\sin(\mu_k t) + \alpha_k^p(t)$$

with $\mu_k = \sqrt{\lambda_k}$.

Proof. To determine $\alpha_k^p(t)$, we use the method of variation of parameters .

$$\begin{cases} C_1' \cos(\mu_k t) + C_2' \sin(\mu_k t) = 0 \\ -\mu_k C_1' \sin(\mu_k t) + C_2' \sin(\mu_k t) = \beta_k \end{cases} \tag{1.14}$$

So

$$C_1(t) = -\int_0^t \frac{\beta_k(s)}{\mu_k} \sin(\mu_k s) ds$$
$$C_2(t) = \int_0^t \frac{\beta_k(s)}{\mu_k} \cos(\mu_k s) ds$$

By applying this method, we find that the unique solution of (1.10) is given by:

$$\alpha_k(t) = \alpha_k^0 \cos(\mu_k t) + \frac{\alpha_k^1}{\mu_k} \sin(\mu_k t) + \frac{1}{\mu_k} \int_0^t \beta_k(s) \sin(\mu_k(t-s))\, ds \tag{1.15}$$

■

which gives an explicit formula for u. If u is a solution of (1.10) (this is what we will demonstrate in the next step) then it is unique. Indeed, if there are two solutions u and $\tilde{u}$ the difference between $\alpha_k - \tilde{\alpha}_k$ is the unique solution of (1.13) with zero data so $\alpha_k(t) = \tilde{\alpha}_k(t)$.

Step 2 To show that the series

$$\sum_{j=1}^{+\infty} \left(\alpha_k^0 \cos(\omega_j t) + \frac{\alpha_k^1}{\omega_j} \sin(\omega_j t) + \frac{1}{\omega_j} \int_0^t \beta_k(s) \sin(\omega_j(t-s))\, ds \right) u_j \tag{1.16}$$

converges in $\mathscr{C}$,we will show that the sequence z^m is Cauchy in$\mathscr{C}$ defined by:

$$z^m = \begin{cases} 0 & \text{if } m = 0 \\ \sum_{j=1}^{m} \alpha_k(t) w_k & \text{if } m \geq 1 \end{cases} \tag{1.17}$$

In V we consider the scalar product $a(u, v)$ for which the family $\{w_k\}_{k\geq 1}$ is, orthogonal, we obtain, for $m > n$, and for any time t

$$c^2 a(z^m - z^n, z^m - z^n) + \left\| \frac{d}{dt}(z^m - z^n) \right\|_H^2 = \sum_{k=n+1}^{m} \left(c^2 \lambda_k |\alpha_k(t)|^2 + \left| \frac{d\alpha_k}{dt}(t) \right|^2 \right). \tag{1.18}$$

However, by multiplying (1.13) by $\frac{d\alpha_k}{dt}$ and integrating in time, we obtain

$$\left| \frac{d\alpha_k}{dt}(t) \right|^2 + c^2 \lambda_k |\alpha_k(t)|^2 = |\alpha_k^1|^2 + c^2 \lambda_k |\alpha_k^0|^2 + 2 \int_0^t \beta_k(s) \frac{d\alpha_k}{dt}(s) ds. \tag{1.19}$$

From the formula (1.15) we infer that

$$\left| \frac{d\alpha_k}{dt}(t) \right| \leq c^2 \mu_j |\alpha_k^0| + |\alpha_k^1| + \int_0^t |\beta_k(s)|\, ds. \tag{1.20}$$

Combining these two results, we deduce

$$\left|\frac{d\alpha_k}{dt}(t)\right|^2 + c^2\lambda_k \left|\alpha_k(t)\right|^2 \leq 2\left|\alpha_k^1\right|^2 + c^2 2\lambda_k \left|\alpha_k^0\right|^2 + 2t\int_0^t \left|\beta_k(s)\right|^2 ds \tag{1.21}$$

As $u_0 \in V, u_1 \in H$ and $f \in L^2(]0,T[;H)$, we have

$$\|u_0\|_V^2 = a\left(u_0, u_0\right) = \sum_{j=1}^{+\infty} \lambda_k \left|\alpha_k^0\right|^2 < +\infty,$$

$$\|u_1\|_H^2 = \sum_{k=1}^{+\infty} \left|\alpha_k^1\right|^2 < +\infty,$$

$$\|f\|_{L^2(]0,T[;H)}^2 = \sum_{k=1}^{+\infty} \int_0^t \left|\beta_k(s)\right|^2 ds < +\infty,$$

This implies that the series, with the general term being the left-hand side of (1.21), is convergent. In other words, the sequence z^m satisfies:

$$\lim_{n,m\to+\infty} \max_{0\leq t\leq T} \left(c^2\left\|z^m(t) - z^n(t)\right\|_V^2 + \left\|\frac{d}{dt}\left(z^m(t) - z^n(t)\right)\right\|_H^2\right) = 0,$$

In other words, the sequence z^m is Cauchy in $C^1([0,T];H)$ and in $C([0,T];V)$. Since these spaces are complete, the Cauchy sequence z^m converges, and we can define its limit as u. In particular, since $\left(z^m(0), \frac{dz^m}{dt}(0)\right)$ converges to (u_0, u_1) in $V \times H$, we obtain the desired initial conditions. Moreover, it is clear that $u(t)$, as the sum of the series (1.12), satisfies the variational formulation (1.10) for each test function $v = w_k$. Since $\left(w_k/\sqrt{\lambda_k}\right)$ forms an orthonormal basis of V, $u(t)$ satisfies the variational formulation (1.10) for any $v \in V$. Therefore, $u(t)$ is indeed the desired solution of (1.10).

Furthermore, we have actually shown that:

$$c^2 a\left(z^m - z^n, z^m - z^n\right) + \left\|\frac{d}{dt}\left(z^m - z^n\right)\right\|_H^2 \leq C\left(c^2\|u_0\|_V^2 + \|u_1\|_H^2 + \|f\|_{L^2(]0,T[;H)}^2\right),$$

and the energy estimate (1.11) is then easily obtained by taking $n = 0$ and letting m tend to infinity. ■

Proposition 1.3.1 *with $f = 0$, the solution of the wave equation (1.1) satisfies, for all $t \in [0,T]$, the equality of conservation of energy.*

$$E(t) := \frac{1}{2}\int_\Omega \left(\left|\frac{\partial u}{\partial t}(x,t)\right|^2 + c^2|\nabla u(x,t)|^2\right) dx = \frac{1}{2}\int_\Omega \left(|u_1(x)|^2 + c^2\left|\nabla u_0(x)\right|^2\right) dx =: E(0) \tag{1.22}$$

Proof. By resuming the proof of Theorem **??** with $f = 0$, i.e. $\beta_k = 0$, we deduce directly from (1.13) that the energy of the harmonic oscillator is conserved, i.e.

$$\left|\frac{d\alpha_k}{dt}(t)\right|^2 + c^2\lambda_k \left|\alpha_k(t)\right|^2 = \left|\alpha_k^1\right|^2 + c^2\lambda_k \left|\alpha_k^0\right|^2$$

which gives the equality

$$c^2 a\left(z^m - z^n, z^m - z^n\right) + \left\|\frac{d}{dt}\left(z^m - z^n\right)\right\|_H^2 = \sum_{k=n+1}^{l}\left(\left|\alpha_k^1\right|^2 + c^2\lambda_k\left|\alpha_k^0\right|^2\right)$$

And (1.22) is obtained by taking $n = 0$ and letting m tend to infinity. If the solution u is regular, we can demonstrate (1.22) more directly by multiplying the wave equation (1.1) by $\frac{\partial u}{\partial t}$ and integrating by parts.
■

Chapter 2

Dynamic viscoelastic problem with exponentially decaying kernel

In this chapter, we will follow the same steps as in the previous chapter to study the existence ,uniqueness and stability of solutions for the wave equation with a memory term. Our objective is to analyze the properties of solutions and investigate under which conditions the problem has a unique solution. By considering appropriate mathematical techniques and analysis, we aim to establish the existence and uniqueness of solutions for the wave equation with a memory term. This analysis will contribute to a better understanding of the dynamics and behavior of waves with memory effects.

2.1 Wave Equation with Memory Term

The wave equation with a memory term incorporates an additional memory term that represents the influence of the past on the current state of the wave. This form of the wave equation with a memory term can be written as:

$$\ddot{u}(x,t) - \Delta u(x,t) + \int_0^t g(t-s)\Delta u(x,s)ds = f(x,t),$$

where g is a function that represents the memory term, and the integral takes into account the contributions of the wave's past history on its present behavior. In our study, we are specifically interested in the particular cases where $g(t-s) = e^{-(t-s)}$. Therefore, we are interested in solving the problem of finding the displacement u such that:

$$(\mathcal{P}) \begin{cases} \ddot{u}(x,t) - \Delta u(x,t) + \displaystyle\int_0^t e^{-(t-s)}\Delta u(x,s)ds = f(x,t), \text{ in}, \quad \mathbb{Q}_T \\ u(0) = u_0(x), \quad \dot{u}(0) = u_1(x) \qquad \text{for } x \in \Omega, \\ u(t,x) = 0, \qquad\qquad\qquad\qquad \text{on } \mathbb{S}_T \end{cases} \tag{2.1}$$

Consider a bounded open set Ω in $\mathbb{R}^n$, where n is the dimension. The time interval $[0,T]$ is finite and u_0 , u_1 are the initial conditions.

The first equation in (2.1) can be classified as an integro-differential partial equation of the hyperbolic type.

2.2 The Variational Problem

we introduce the operator $\mathcal{A}$ defined by:

$$\mathcal{A}(t)u = -\Delta u(t) + \int_0^t e^{-(t-s)}\Delta u(s)ds$$

of problem (2.1) :

$$(\mathcal{P}) \begin{cases} \ddot{u}(t) + \mathcal{A}(t)u = f(x;t), \quad \text{for } x \in \Omega, \quad \forall t > 0 \\ u(0) = u_0(x), \quad \dot{u}(0) = u_1(x) \qquad \text{for } x \in \Omega, \end{cases} \tag{2.2}$$

Lemma 2.2.1 *the operator $\mathcal{A}(t)$ is an elliptic operator on V*

Proof. For all $v \in V$ there exist α such that :

$$\int_0^t e^{-(t-s)} ds = e^{-t}[e^t]_0^t = 1 - e^{-t}$$

So,

$$a_t(v, v) = e^{-t}\|v\|_V^2$$

so for $t \in [0, T]$ then

$$a_t(v, v) \geq e^{-t}\|v\|_V^2, \quad \forall v \in V \tag{2.3}$$

■

2.2.1 Weak formulation

Lemma 2.2.2 *the problem (2.2) is formally equivalent to the following variational problem :*

$$(\mathcal{P}_v) \begin{cases} \textit{Find } u \in \mathscr{C} = C\left([0, T]; V\right) \cap C^1\left([0, T]; L^2(\Omega)\right). \\ (\ddot{u}(t), v) + a_t(u, v) = f(x; v) \\ u(0) = u_0(x), \quad \dot{u}(0) = u_1(x) \end{cases} \tag{2.4}$$

where the symmetric bilinear form $a(.;.)$ is defined by

$$a_t(w, v) = (\nabla w, \nabla v) - (\nabla w(s), \nabla v) \int_0^t e^{-(t-s)} ds, \quad \forall w, v \in H_0^1(\Omega)$$

Proof. To find a weak formulation, we multiply the first equation of (2.1) by a test-function $v = v(x, t) \in L^2(0, T; H_0^1)$, vanishing at the boundary of Ω. and integrate over Ω :

$$\int_\Omega \ddot{u}(x, t)v(x, t)\ dx\ + \int_\Omega \Delta u(x, t)v(x, t)\ dx\ + \int_\Gamma \int_0^t e^{-(t-s)} \Delta u(x, s) ds v(x, t) dx$$
$$= \int_\Omega f(x, t)v(x, t)\ dx \tag{2.5}$$

By applying Green's on the integral equation (2.5), and using the appropriate boundary condition , we get:

$$\int_\Omega \ddot{u}(x, t)v(x, t)\ dx\ + \int_\Omega \nabla u(x, t)\nabla v(x, t)\ dx\ - \int_\Omega (\int_0^t e^{-(t-s)} \nabla u(x, s))\ ds \nabla v(x, t)\ dx$$
$$= \int_\Omega f(x, t)v(x, t)\ dx \tag{2.6}$$

so we get the variational problem (2.4) ■

2.2.2 Existence, uniqueness and stability of solutions

In this section under the assumptions the existence and uniqueness of the weak solution will be obtained according to the same steps of the previous chapter.

2.2.3 A new variational formulation

In the context of existence and uniqueness of solution, we introduce a new variable for the problem (2.1). This approach allows us to formulate the problem in an equivalent form that is more suitable to the method used.

Lemma 2.2.3 *For $u \in \mathscr{C}$ we introduce a new variable*

$$p(t) = \int_0^t e^{-(t-s)} u(s) ds$$

. Then

$$\dot{p}(t) + p(t) = u(t)$$

Proof. The proof depends on the statement:

$$\int_0^t g(t-s)u'(s)\, ds = \frac{d}{dt}\int_0^t g(t-s)(u(s)-u(0))ds \tag{2.7}$$

For

$$\begin{aligned} p(t) &= \int_0^t e^{-(t-s)} u(s)\, ds \\ &= \int_0^t e^{-(t-s)} (u(s)-u(0))\, ds + \int_0^t e^{-(t-s)} u(0)\, ds \end{aligned}$$

then we have,

$$\begin{aligned} \dot{p}(t) &= \frac{d}{dt}\int_0^t e^{-(t-s)} (u(s)-u(0))\, ds + \frac{d}{dt}\int_0^t e^{-(t-s)} u(0)\, ds \\ &= \int_0^t e^{-(t-s)} u'(s)\, ds + e^{-t}u(0) \end{aligned}$$

On the other hand,

$$\begin{aligned} p(t) = \int_0^t e^{-(t-s)} u(s)\, ds &= \left[e^{-(t-s)}u(s)\right]_0^t - \int_0^t e^{-(t-s)} u'(s)\, ds \\ &= u(t) - e^{-t}u(0) - \int_0^t e^{-(t-s)} u'(s)\, ds \end{aligned}$$

hence,

$$\dot{p}(t) + p(t) = u(t)$$

■

The resulting system of equations can be written as follows:

$$\begin{cases} \ddot{u}(x,t) - \Delta u(x,t) + \Delta p(x,t) = f(t,v), & \text{for } x \in \Omega, \quad \forall t > 0 \\ u(t,x) - p(x,t) - \dot{p}(x,t) = 0 & \text{for } x \in \Omega, \quad \forall t > 0 \\ u(0) = u^0(x), \ \dot{u}(0) = u^1(x), p(0) = 0 & \text{for } x \in \Omega, \quad \forall t > 0 \\ u(t,x) = 0, & \text{for } x \in \Gamma, \end{cases} \tag{2.8}$$

and **the variational formulation** becomes :

$$\begin{cases} \text{Find } (u,p) \in \mathscr{C} \text{ such that} \\ (\ddot{u}(t), v)_{L_2(\Omega)} + a(u(t), v) - a\left(p(t), v\right) = f(x; v) \quad \forall v \in V \\ a\left(\dot{p}(t), v\right) + a\left(p(t), v\right) = a(u(t), v), \quad \forall v \in V \end{cases} \tag{2.9}$$

where

$$a(u,v) = (\nabla u, \nabla v)_{L_2(\Omega)}$$

with the initial conditions

$$u(0) = u^0, \quad \dot{u}(0) = u^1 \quad \text{and} \quad p(0) = 0,$$

. The fallowing theorem assure the existence uniqueness and stability of solution

Theorem 2.2.1 *Let V and H be two Hilbert spaces such that $V \subset H$ with a compact injection, and V is dense in H. Let $a(\cdot,\cdot)$ be a continuous and coercive symmetric bilinear form on V. Let $T > 0$ be the final time, $u^0 \in H, u^1 \in V$ be the initial data, and $f \in L^2(]0,T[;H)$.*

So, the problem (2.9) has a unique solution satisfying the following regularities

$$(u,p) \in \mathscr{C} = C\left([0,T];V\right) \cap C^1\left([0,T];L^2(\Omega)\right)$$

Proof. To obtain the existence and uniqueness of solutions for problem (2.9), we introduce the same steps from the previous chapter.

Step 1. Let us introduce the Hilbertian basis $(w_k)_{k\geq 1}$ of V composed of the eigenfunctions

$$w_k \in V, \text{ such as } a\left(w_k, v\right) = \lambda_k \left\langle w_k, v\right\rangle_H \quad \forall v \in V.$$

where λ_k are the eigenvalues corresponding to the eigenfunctions $w_k(x)$.
We write

$$u(t,x) = \sum_{k=1}^{+\infty} \alpha_k(t) w_k(x) \text{ with } \quad \alpha_k(t) = \left\langle u(t), w_k\right\rangle_H . \tag{2.10}$$

$$p(t,x) = \sum_{k=1}^{\infty} \gamma_k(t) w_k(x) \text{ with } \quad \gamma_k(t) = \left\langle p(t), w_k\right\rangle_H . \tag{2.11}$$

where $\alpha_k(t)$ and $\gamma_k(t)$ are the time-dependent coefficients and w_k are the eigenfunctions satisfying appropriate boundary conditions.
Choosing $v = w_k$ in (2.9), and noting

$$\beta_k(t) = \langle f(t), w_k \rangle_H \quad w_k \in V.$$

$$\alpha_k^0 = \langle u_0, w_k \rangle_H \,, \quad \text{et} \quad \alpha_k^1 = \langle u_1, w_k \rangle_H$$

we obtain

$$\begin{cases} \dfrac{d^2\alpha_k}{dt^2} + \lambda_k\alpha_k - \lambda_k\gamma_k = \beta_k \text{ in }]0,T[\\ \dfrac{d\gamma_k}{dt} + \lambda_k\gamma_k = \lambda_k\alpha_k \\ \alpha_k(t=0) = \alpha_k^0, \quad \dfrac{d\alpha_k}{dt}(t=0) = \alpha_k^1 \\ \gamma_k(t=0) = 0 \end{cases} \tag{2.12}$$

Step 2 we will show that the sequences z^m and P^m are Cauchy sequences in $\mathscr{C}$

$$z^m = \left\{ \begin{array}{ll} 0 & \text{if } m=0 \\ \sum_{j=1}^m \alpha_k(t) w_k & \text{if } m \geq 1 \end{array} \right. \qquad P^m = \left\{ \begin{array}{ll} 0 & \text{if } m=0 \\ \sum_{j=1}^m \gamma_k(t) w_k & \text{if } m \geq 1. \end{array} \right. \tag{2.13}$$

In V we consider the scalar product $a(u,v)$ for which the family $\{w_k\}_{k\geq 1}$ is, orthogonal, we obtain, for $m > n$, and for any time t

$$\begin{aligned} a\left(z^m - z^n, z^m - z^n\right) + \left\| \frac{d}{dt}\left(z^m - z^n\right) \right\|_H^2 + a(P^m - P^n, P^m - P^n) + \int_0^t a(\dot{P}^m - \dot{P}^n, \dot{P}^m - \dot{P}^n) ds \\ = \sum_{k=n+1}^{m} \left(\left| \frac{d\alpha_k}{dt}(t) \right|^2 + |\alpha_k(t)|^2 - \lambda_k |\gamma_k(s)|^2 + \int_0^t \lambda_k |\dot{\gamma}_k|^2 ds \right). \end{aligned} \tag{2.14}$$

by multiplying the first equitation of (2.12) by $\frac{d\alpha_k}{dt}$, we obtain , we obtain

$$\begin{cases} \ddot{\alpha}_k\dot{\alpha}_k + \lambda_k\alpha_k\dot{\alpha}_k - \lambda_k\gamma_k\dot{\alpha} = \beta_k\dot{\alpha}_k \\ \lambda_k\dot{\gamma}_k^2 + \lambda_k\gamma_k\dot{\gamma}_k = \lambda_k\alpha_k\dot{\gamma}_k \end{cases} \tag{2.15}$$

By summing up, we obtain :

$$\frac{d}{dt}\dot{\alpha}_k^2 + \lambda_k\frac{d}{dt}\alpha_k^2 - 2\lambda_k\frac{d}{dt}\left(\alpha_k\gamma_k\right) + 2\lambda_k\dot{\gamma}_k^2 + 2\lambda_k\gamma_k\dot{\gamma}_k = 2\beta_k\dot{\alpha}_k$$

Thus, from integration with respect to time from 0 to t where $t \in [0;T]$,

$$\begin{aligned} \left| \frac{d\alpha_k}{dt}(t) \right|^2 + \lambda_k|\alpha_k(t)|^2 + 2\int_0^t \lambda_k\gamma_k(s)^2 \, ds + \lambda_k\gamma_k(t)^2 = |\alpha_k^1|^2 + \lambda_k|\alpha_k^0|^2 + 2\lambda_k\alpha_k(t)\gamma_k(t) \\ + 2\int_0^t \beta_k(s)\dot{\alpha}_k(s) \, ds. \end{aligned}$$

We use Young's inequality

$$\left|\frac{d\alpha_k}{dt}(t)\right|^2 + \lambda_k|\alpha_k(t)|^2+2\int_0^t \lambda_k\dot{\gamma}_k(s)^2\ ds + \lambda_k\gamma_k(t)^2 \leq |\alpha_k^1|^2 + \lambda_k|\alpha_k^0|^2 + 2\int_0^t (\frac{1}{2\varepsilon}|\beta_k(s)|^2 + \frac{\varepsilon}{2}|\dot{\alpha}_k(s)|^2)\ ds + \frac{\lambda_k}{\varepsilon}|\alpha_k(t)|^2 + \lambda_k\varepsilon|\gamma_k(t)|^2.$$

$$\left|\frac{d\alpha_k}{dt}(t)\right|^2 + (\lambda_k - \frac{\lambda_k}{\varepsilon})|\alpha_k(t)|^2+2\lambda_k\int_0^t |\dot{\gamma}_k(s)|^2\ ds + (\lambda_k - \lambda_k\varepsilon)|\gamma_k(t)|^2 \leq |\alpha_k^1|^2 + \lambda_k|\alpha_k^0|^2 + \int_0^t \frac{1}{\varepsilon}|\beta_k(s)|^2 ds + \int_0^t \varepsilon|\dot{\alpha}_k(s)|^2\ ds \tag{2.16}$$

Finally, Gronwall's inequality implies

$$\left|\frac{d\alpha_k}{dt}(t)\right|^2 + (\lambda_k - \frac{\lambda_k}{\varepsilon})|\alpha_k(t)|^2+2\int_0^t \lambda_k|\dot{\gamma}_k(s)|^2\ ds + (\lambda_k - \lambda_k\varepsilon)\gamma_k(t)|^2 \leq C(|\alpha_k^1|^2 + \lambda_k|\alpha_k^0|^2 + \int_0^t |\beta_k(s)|^2 ds) \tag{2.17}$$

$$\left|\frac{d\alpha_k}{dt}(t)\right|^2 + |\alpha_k(t)|^2 + \int_0^t \lambda_k|\dot{\gamma}_k(s)|^2\ ds + |\gamma_k(t)|^2 \leq C(|\alpha_k^1|^2 + \lambda_k|\alpha_k^0|^2 + \int_0^t |\beta_k(s)|^2 ds) \tag{2.18}$$

If u and p are solutions of (2.9) then it is unique. Indeed, if there are two solutions u and $\tilde{u}$, p and $\tilde{p}$ according to the formula (2.18) the difference between $\alpha_k - \tilde{\alpha}_k$ and $\gamma_k - \tilde{\gamma}$ are the unique solution of (2.12) with zero data so $\alpha_k(t) = \tilde{\alpha}_k(t)$ and $\gamma_k = \tilde{\gamma}$.
In V we consider the scalar product $a(u,v)$ for which the family $\{w_k\}_{k\geq 1}$ is, orthogonal, as $u_0 \in V, u_1 \in H$ and $f \in L^2(]0,T[;H)$, we have

$$\|u_0\|_V^2 = a\left(u_0, u_0\right) = \sum_{j=1}^{+\infty} \lambda_k \left|\alpha_k^0\right|^2 < +\infty,$$

$$\|u_1\|_H^2 = \sum_{k=1}^{+\infty} \left|\alpha_k^1\right|^2 < +\infty,$$

$$\|f\|_{L^2(]0,T[;H)}^2 = \sum_{k=1}^{+\infty} \int_0^T \left|\beta_k(s)\right|^2 ds < +\infty,$$

This implies that the series, with the general term being the left-hand side of (2.18), is convergent. In other words, the sequence z^m and P^m satisfies:

$$\lim_{n,m\to+\infty}\max_{0\le t\le T}\left(a\left(z^m-z^n,z^m-z^n\right)+\left\|\frac{d}{dt}\left(z^m-z^n\right)\right\|_H^2\right.$$
$$\left.+a(P^m-P^n,P^m-P^n)+\int_0^t a(\dot P^m-\dot P^n,\dot P^m-\dot P^n)ds\right)=0$$

In other words, the sequences z^m and P^m are Cauchy in $C^1([0,T];H)$ and in $C([0,T];V)$. Since these spaces are complete, the Cauchy sequences z^m and P^m converge, and we can define there limit as u and p respectively. In particular, since $\left(z^m(0),\frac{dz^m}{dt}(0)\right)$ converges to (u_0,u_1) in $V\times H$, we get the initial conditions. Moreover, it is clear that $u(t)$ and $p(t)$, as the sum of the series (2.13), satisfy the variational formulation (2.9) for each test function $v=w_k$. As (w_k) is a hilbertian basis of V, therefore, $u(t)$ and $p(t)$ are indeed the desired solution of (2.9).

■

2.2.4 Energy of the System

We recall that the energy at time t and the initial energy are defined by:

$$E(t):=\frac{1}{2}\int_\Omega\left(\left|\frac{\partial u}{\partial t}(x,t)\right|^2+|\nabla u(x,t)|^2\right)dx$$
$$E(0):=\frac{1}{2}\int_\Omega\left(|u_1(x)|^2+|\nabla u_0(x)|^2\right)dx$$

Lemma 2.2.4 *Suppose the weak solution $u\in H^2\left(0,T;L^2(\Omega)\right)\cap H^1(0,T;V)$. Then for any $0\le t\le T$*

$$E(t)=E(0)+\int_0^t(f(s),\dot u(s))\ ds+\int_0^t a(p(s),\dot u(s))\,ds \tag{2.19}$$

Proof. Choosing $v=\dot u\in V$ in (2.9) gives

$$(\ddot u(t),\dot u)_{L_2(\Omega)}+a(u(t),\dot u)-a\left(p(t),\dot u\right)=(f(t),\dot u)$$

Note that by Leibniz's integral rule, for any differentiable u we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left\|u\left(t\right)\right\|_{L_2(\Omega)}^2&=\frac{1}{2}\frac{d}{dt}\int_\Omega u^2\left(t\right)d\Omega,\\&=\frac{1}{2}\int_\Omega\frac{\partial}{\partial t}\left(u^2\left(t\right)\right)d\Omega,\\&=\int_\Omega\dot u\left(t\right)u\left(t\right)d\Omega=\left(\dot u\left(t\right),u\left(t\right)\right)_{L_2(\Omega)}.\end{aligned}$$

and similarly, $\frac{1}{2}\frac{d}{dt}\|u(t)\|_V^2 = a(\dot{u}(t), u(t))$. Hence it yields

$$\frac{1}{2}\frac{d}{dt}\|\dot{u}(t)\|_{L_2(\Omega)}^2 + \frac{1}{2}\frac{d}{dt}\|u(t)\|_V^2 - a(p(t), \dot{u}) = (f(t), \dot{u})$$

Thus, from integration with respect to time from 0 to t where $t \in [0;T]$,

$$\frac{1}{2}\|\dot{u}(t)\|_{L_2(\Omega)}^2 + \frac{1}{2}\|u(t)\|_V^2 = \frac{1}{2}\|u_0(t)\|_{L_2(\Omega)}^2 + \frac{1}{2}\|u_1(t)\|_V^2 + \int_0^t a(p(s), \dot{u})\, ds + \int_0^t (f(s), \dot{u}) ds \tag{2.20}$$

■

Lemma 2.2.5 *Assume that $p \in H^1(0, T, V)$. Then for any $0 \le t \le T$,*

$$\int_0^t a(p(s), \dot{u}(s))\, ds = a(u, p) - \frac{1}{2}\|p\|_V^2 - \int_0^t \|\dot{p}\|_V^2 ds$$

Proof. Set $v = \dot{p}$ then 2.9 yields

$$a(\dot{p}(t), \dot{p}) + a(p(t), \dot{p}) = a(u(t), \dot{p})$$

$$\|\dot{p}\|_V^2 + \frac{1}{2}\frac{d}{dt}\|p\|_V^2 = a(u(t), \dot{p})$$

Integration by parts yields

$$\int_0^t \|\dot{p}(s)\|_V^2\, ds + \frac{1}{2}(\|p(t)\|_V^2 - \|p(0)\|_V^2) = a(u(t), p(t)) - a(u(0), p(0)) - \int_0^t a(\dot{u}(s), p(s))\, ds$$

Since $p(0) = 0$, we have

$$\int_0^t \|\dot{p}(s)\|_V^2\, ds + \frac{1}{2}\|p(t)\|_V^2 = a(u(t), p(t)) - \int_0^t a(\dot{u}(s), p(s))\, ds$$

and therefore,

$$\int_0^t a(p(s), \dot{u}(s))\, ds = a(u(t), p(t)) - \frac{1}{2}\|p(t)\|_V^2 - \int_0^t \|\dot{p}(s)\|_V^2\, ds. \tag{2.21}$$

We replace the expression (2.21) in (2.20) we obtain (2.19) ■

Theorem 2.2.2 *If $u \in H^2(0,T;L^2(\Omega)) \cap H^1(0,T;V)$, then we have the following stability bound: for any $t \in [0;T]$*

$$\|\dot{u}\|_0^2 + \|\nabla u\|_0^2 + \|\nabla p\|_0^2 + 2\int_0^t \|\nabla \dot{p}\|_0^2 \leq C\left(\left\|u^1\right\|_0 + \|u\|_0^2 + \int_0^t \|f\|_0^2\right) \tag{2.22}$$

Proof. According to the lemma (2.20)and (2.21) we have

$$\|\dot{u}\,(t)\|_{L_2(\Omega)}^2 + \|u\,(t)\|_V^2 = \|u_0\,(t)\|_{L_2(\Omega)}^2 + \|u_1\,(t)\|_V^2 + 2a(u,p) - \|p\|_V^2 - 2\int_0^t \|\dot{p}\|_V^2 ds$$
$$+ 2\int_0^t (f(t),\dot{u})ds$$

By applying Cauchy-Schwarz inequality and Young's inequality we obtain

$$\|\dot{u}\,(t)\|_{L_2(\Omega)}^2 + \|u\,(t)\|_V^2 + \|p\|_V^2 + 2\int_0^t \|\dot{p}\|_V^2 ds \leq \|u_0\,(t)\|_{L_2(\Omega)}^2 + \|u_1\,(t)\|_V^2 + 2a(u,p)$$
$$+ \frac{1}{\varepsilon}\int_0^t \|f(s)\|^2 ds + \varepsilon\int_0^t \|\dot{u}\|^2 ds$$

On the other hand, in the same sense, Cauchy-Schwarz inequality and Young's inequality allow us to have

$$a(u,p) \leq \|u\|_V \|p\|_V \leq \frac{1}{2\varepsilon}\|u\|_V^2 + \frac{\varepsilon}{2}\|p\|_V^2$$

$$\|\dot{u}\,(t)\|_{L_2(\Omega)}^2 + (1-\frac{1}{\varepsilon})\,\|u\,(t)\|_V^2 + (1-\varepsilon)\|p\|_V^2 + 2\int_0^t \|\dot{p}\|_V^2 ds \leq \|u_0\,(t)\|_{L_2(\Omega)}^2 + \|u_1\,(t)\|_V^2$$
$$+ \frac{1}{\varepsilon}\int_0^t \|f(s)\|^2 ds + \varepsilon\int_0^t \|\dot{u}\|^2 ds$$

$$\|\dot{u}\,(t)\|_{L_2(\Omega)}^2 + c_1\,\|u\,(t)\|_V^2 + c_2\|p\|_V^2 + 2\int_0^t \|\dot{p}\|_V^2 ds \leq \|u_0\,(t)\|_{L_2(\Omega)}^2 + \|u_1\,(t)\|_V^2$$
$$+ \frac{1}{\varepsilon}\int_0^t \|f(s)\|^2 ds + \varepsilon\int_0^t \|\dot{u}\|^2 ds$$

Gronwall's inequality implies :

$$\|\dot{u}\,(t)\|_{L_2(\Omega)}^2 + \|u\,(t)\|_V^2 + \|p\|_V^2 + \int_0^t \|\dot{p}\|_V^2 ds \leq C(\|u_0\,(t)\|_{L_2(\Omega)}^2 + \|u_1\,(t)\|_V^2$$
$$+ \int_0^t \|f(s)\|^2 ds) \tag{2.23}$$

■

Theorem 2.2.3 *Then the energy of (2.1) decays to zero exponentially, that is, there exist positive constants C and φ such that*

$$E(t) \leqslant Ce^{-\varphi t}, \quad t \in [0,T] \tag{2.24}$$

Proof. see [14] theorem 2. ■

Chapter 3

Finite element approximation

In this chapter, our focus is on solving problems (1.1) and (2.1) using the approximation method. We aim to employ the finite element approach to approximate the solutions to these problems.

3.1 Discretization

There are two commonly used approaches to discretize an evolution problem:

The first approach involves a semi-discretization in space using methods such as the finite element method. In this approach, only the spatial domain is discretized, meaning that the solution of the problem is approximated on a spatial mesh. This transforms the continuous problem into a system of ordinary differential equations (ODEs) in time. Then, a time discretization method such as the Newmark method or the Crank-Nicolson method can be applied to numerically solve the obtained ODEs.

The second approach involves a semi-discretization in time using methods like the Newmark method or the Crank-Nicolson method. In this approach, only the time domain is discretized, meaning that the solution of the problem is approximated at discrete time instances. This leads to a system of partial differential equations (PDEs) in space. Then, a spatial discretization method such as the finite element method can be applied to numerically solve the obtained PDEs.

This diagram summarizes what has been previously mentioned.

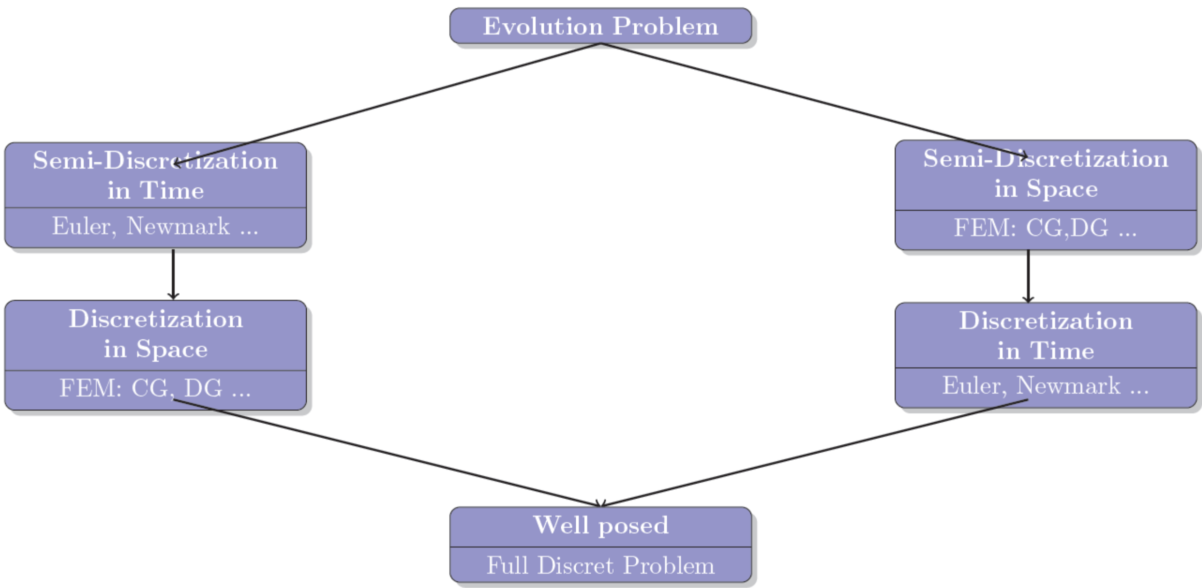


Figure 3.1: discretization diagram

3.2 Finite element approximation to wave equation

3.2.1 Semi-discretization in space

Let us define V_h such that consists of continuous local basis functions with respect to Lagrange finite elements. Hence we can define $V_h \subset V$ with its global basis functions $(\phi_i)_{1 \le i \le n_h}$(which does not depend on time) by

$$V_h = \langle \{\phi_1, \phi_2, \dots, \phi_k\} \rangle$$

We discretize in space the variational formulation (1.10) of the wave equation. The semi-discretization of (1.10) is, therefore, the following variational approximation:
find $u_h(t)$ function of $]0, T[$ with values in V_h such that:

$$\begin{cases} \dfrac{d^2}{dt^2} \langle u_h(t), v_h \rangle_{L^2(\Omega)} + a\left(u_h(t), v_h\right) = \langle f(t), v_h \rangle_{L^2(\Omega)} & \forall v_h \in V_h, 0 < t < T, \\ u_h(t=0) = u_{0,h}, \\ \dfrac{\partial u_h}{\partial t}(t=0) = u_{1,h} \end{cases} \tag{3.1}$$

where $u_{0,h} \in V_h$ and $u_{1,h} \in V_h$ are approximations of the initial data u_0 and u_1.
To show that (3.1) admits a unique solution and to calculate it in a practical way, we introduce a basis $(\phi_i)_{1 \le i \le n_h}$ of V_h , and we seek $u_h(t)$ in the form

$$u_h(t) = \sum_{i=1}^{n_h} U_i^h(t)\phi_i$$

with $U^h = \left(U_i^h\right)_{1 \le i \le n_h}$ the coordinate vector of u_h. Letting

$$u_{0,h} = \sum_{i=1}^{n_h} U_i^{0,h}\phi_i, \quad u_{1,h} = \sum_{i=1}^{n_h} U_i^{1,h}\phi_i, \quad b_i^h(t) = \langle f(t), \phi_i \rangle_{L^2(\Omega)}, 1 \le i \le n_h,$$

the variational approximation problem (3.1) is equivalent to the linear system of ordinary differential equations of second-order with constant coefficients

$$\begin{cases} \mathcal{M}_h \dfrac{d^2 U^h}{dt^2}(t) + \mathcal{K}_h U^h(t) = b^h(t), \quad 0 < t < T \\ U^h(t=0) = U^{0,h}, \quad \dfrac{dU^h}{dt}(t=0) = U^{1,h} \end{cases} \tag{3.2}$$

where we find the matrices of mass $\mathcal{M}_h$ and stiffness $\mathcal{K}_h$ as

$$(\mathcal{M}_h)_{ij} = \langle \phi_i, \phi_j \rangle_{L^2(\Omega)}, (\mathcal{K}_h)_{ij} = a\left(\phi_i, \phi_j\right) \quad 1 \le i, j \le n_h.$$

The existence and uniqueness, as well as an explicit formula, of the solution of (3.2) are easily obtained by simple simultaneous diagonalization of the matrices $\mathcal{M}_h$ and $\mathcal{K}_h$. As it is difficult and expensive to diagonalize (3.2), in practice we solve numerically (3.2) by discretization and walking in time.

Theorem 3.2.1 *The mass matrix $\mathcal{M}$ and the stiffness matrix $\mathcal{K}$ are symmetric positive definite. Thus, they are invertible.*

Proof. Note that L^2 inner product and the bilinear form $a(\cdot,\cdot)$ are symmetric hence $\mathcal{M}$ and $\mathcal{K}$ are symmetric. Let $v \in \mathbb{R}^{n^h}$. Then

$$v^\top \mathcal{M} v = \sum_{i,j=1}^{n_h} v_j \mathcal{M}_{ij} v_i = \sum_{i,j=1}^{n_h} v_j \left(\phi_j, \phi_i\right)_{L_2(\Omega)} v_i = \sum_{i,j=1}^{n_{Vh}} \left(v_j\phi_j, v_i\phi_i\right)_{L_2(\Omega)} = \|v\|^2_{L_2(\Omega)} \geq 0$$

where $v = \sum_{i=1}^{n_h} v_i\phi_i \in V^h$. By the norm axiom, $v^\top \mathcal{M} v = 0$ if and only if $v = 0$. Thus $\mathcal{M}$ is symmetric positive definite and hence $\mathcal{M}$ is invertible. On the other hand, $a(\cdot,\cdot)$ is coercive, so

$$v^\top \mathcal{K} v = \sum_{i,j=1}^{n_h} v_j \mathcal{K}_{ij} v_i = \sum_{i,j=1}^{n_h} v_j a\left(\phi_j, \phi_i\right) v_i = \sum_{i,j=1}^{n_h} a\left(v_j\phi_j, v_i\phi_i\right) = a(v,v) \geq \kappa \|v\|^2_{H^1(\Omega)} \geq 0$$

for some positive constant κ. It implies that also $v^\top \mathcal{K} v = 0$ if and only if $v = 0$, therefore $\mathcal{K}$ is symmetric positive definite and so invertible. ■

3.2.2 Fully Discrete Formulation in time

We use a finite time difference method to solve the system of ordinary differential equations (3.2). To simplify the notations, we rewrite the system (3.2) without mentioning the spatial dependence in h

$$\begin{cases} \mathcal{M}\dfrac{d^2U}{dt^2}(t) + \mathcal{K} \qquad U(t) = b(t) \\ U(t=0) = U_0, \\ \dfrac{dU}{dt}(t=0) = U_1, \end{cases} \tag{3.3}$$

where we assume that $b(t)$ is continuous on $[0,T]$. We divide the time interval $[0,T]$ into n_0 time steps $\tau = T/n_0$, we set $t_n = n\tau \quad 0 \leq n \leq n_0$, and we denote by U^n the approximation of $U\left(t_n\right)$ calculated by a scheme. For $0 \leq \theta \leq 1/2$ we propose the θ-scheme

$$\begin{aligned} \mathcal{M}\frac{U^{n+1} - 2U^n + U^{n-1}}{\tau^2} &+ \mathcal{K}\left(\theta U^{n+1} + (1-2\theta)U^n + \theta U^{n-1}\right) \\ &= \theta b\left(t_{n+1}\right) + (1-2\theta) b\left(t_n\right) + \theta b\left(t_{n-1}\right). \end{aligned} \tag{3.4}$$

When $\theta = 0$, we call (3.4) an explicit scheme (it is in fact really explicit only if the mass matrix $\mathcal{M}$ is diagonalisable). To start the scheme, we need to know U^0 and U^1, which we nà get thanks to the initial conditions

$$U^0 = U_0 \quad \text{et} \quad \frac{U^1 - U^0}{\tau} = U_1.$$

A more frequently used scheme because it is more general is the Newmark scheme. To solve the system

$$\mathcal{M}\frac{d^2U}{dt^2}(t)+\mathcal{K}U(t)=b(t)$$

we approach $U(t), dU/dt(t), d^2U/dt^2(t)$ by three sequence $U^n, \dot{U}^n, \ddot{U}^n$

$$\begin{cases} \mathcal{M}\ddot{U}^{n+1}+\mathcal{K}U^{n+1}=b\left(t_{n+1}\right) \\ \dot{U}^{n+1}=\dot{U}^n+\tau\left(\delta\ddot{U}^{n+1}+(1-\delta)\ddot{U}^n\right) \\ U^{n+1}=U^n+\tau\dot{U}^n+\frac{\tau^2}{2}\left(2\theta\ddot{U}^{n+1}+(1-2\theta)\ddot{U}^n\right) \end{cases} \tag{3.5}$$

with $0\leq\delta\leq 1$ and $0\leq\theta\leq 1/2$.
(3.5) is equivalent to

$$\mathcal{M}\frac{U^{n+1}-2U^n+U^{n-1}}{\tau^2}+\mathcal{K}\left(\theta U^{n+1}+\left(\frac{1}{2}+\delta-2\theta\right)U^n+\left(\frac{1}{2}-\delta+\theta\right)U^{n-1}\right)$$
$$=\theta b\left(t_{n+1}\right)+\left(\frac{1}{2}+\delta-2\theta\right)b\left(t_n\right)+\left(\frac{1}{2}-\delta+\theta\right)b\left(t_{n-1}\right). \tag{3.6}$$

The following lemma studies the stability of these schemes.

Lemma 3.2.1 *We consider the Newnmark scheme (3.6). If $\delta<1/2$ is still unstable. Suppose now that $\delta\geq 1/2$. The necessary condition of Von Neumann stability is always verified if $\delta\leq 2\theta\leq 1$, while, if $0\leq 2\theta<\delta$ it is satisfied only under the CFL (Courant-Friedrichs-Lewy) condition*

$$\max_i \lambda_i\tau^2<\frac{2}{\delta-2\theta} \tag{3.7}$$

where the λ_i are the eigenvalues of $\mathcal{K}U=\lambda\mathcal{M}U$

Proof. We decompose U^n and the second member of (3.6) in the orthonormal basis for $\mathcal{M}$ and orthogonal for $\mathcal{K}$. Therefore, (3.6) is equivalent, component by component, to

$$\frac{U_i^{n+1}-2U_i^n+U_i^{n-1}}{\tau^2}+\lambda_i\left(\theta U_i^{n+1}+\left(\frac{1}{2}+\delta-2\theta\right)U_i^n+\left(\frac{1}{2}-\delta+\theta\right)U_i^{n-1}\right)=b_i^n, \tag{3.8}$$

with obvious notations (the λ_i are the eigenvalues of the matrix system $\mathcal{K}V_i=\lambda_i\mathcal{M}V_i$). As the schema (3.8) is three level we introduce an iteration matrix A_i such that

$$\begin{pmatrix} U_i^{n+1} \\ U_i^n \end{pmatrix}=A_i\begin{pmatrix} U_i^n \\ U_i^{n-1} \end{pmatrix}+\frac{\tau^2}{1+\theta\lambda_i\tau^2}\begin{pmatrix} b_i^n \\ 0 \end{pmatrix}$$

The necessary Von Neumann stability condition is $\rho(A_i) \leq 1$. We therefore calculate the eigenvalues of A_i which are the roots of the polynomial in μ following

$$\mu^2 - a_{11}\mu - a_{12} = 0$$

whose discriminant is

$$\Delta = \frac{-4\lambda_i\tau^2 + \lambda_i^2\tau^4\left(\left(\frac{1}{2}+\delta\right)^2 - 4\theta\right)}{\left(1+\theta\lambda_i\tau^2\right)^2}$$

We easily check that the roots of this polynomial have a modulus less than or equal to 1 if and only if we are in one of the two following cases:

1. either $\Delta \leq 0$ and $a_{12} \geq -1$,
2. either $\Delta > 0$ and $1 - a_{12} \geq |a_{11}|$.

A tedious but simple calculation in principle then leads to the condition (3.7). ∎

Finally, we can state a convergence result of this discretization method.

Proposition 3.2.1 *Let u be the "sufficiently regular" solution of the wave equation (1.1). Let $(\mathcal{T}_h)_{h>0}$ be a sequence of regular triangular meshes of Ω. Let V_h be the subspace of $H_0^1(\Omega)$, defined by the finite element method P_k. Let τ be a sequence of time steps tending to zero. Let $u_h^n \in V_h$ be the function whose coordinates U^n in the finite element base of V_h are calculated by the Newmark scheme. If $\lim_{h\to 0} u_h^0 = u_0$ in $L^2(\Omega)$, $\lim_{h\to 0} u_h^1 = u_1$ in $L^2(\Omega)$, and if h and τ tend to 0 respecting the stability condition (3.7), then we have*

$$\lim_{h\to 0,\tau\to 0}\ \max_{0\leq n\leq n_0} \|u(t_n) - u_h^n\|_{L^2(\Omega)} = 0.$$

Proof. (see Theorme 8.7.2 [4].) ∎

3.3 Finite element approximation to wave equation whit memory term

3.3.1 Semi-discretization in space

Let's define a finite-dimensional subspace V_h that consists of continuous local basis functions with respect to Lagrange finite elements. This means that V_h is constructed using a set of basis functions $(\phi_i)_{1\leq i\leq n_h}$ that are continuous piecewise polynomials .Hence we can definde the subspace $V_h \subset V$ as follows:

$$V_h = \langle\{\phi_1, \phi_2, \dots, \phi_k\}\rangle$$

We discretize in space the variational formulation (2.9) of the wave equation withe a memory term. The semi-discretization of (2.9) is, therefore, the following variational approximation:
Find $u_h(t)$ and p_h functions of $]0, T[$, with values in V_h such that: So the approximate solution are given as

$$u_h(t) = \sum_{i=1}^{n_h} U_i^h(t)\phi_i \tag{3.9}$$

$$p_h(t) = \sum_{i=1}^{n_h} p_i^h(t)\phi_i \tag{3.10}$$

Then we define the discrete variational problem:

$$\begin{cases} (\ddot{u_h}(t), v_h)_{L_2(\Omega)} + a(u_h(t), v_h) - a\left(p_h(t), v_h\right) = (f(t), v_h) \\ \quad a\left(\dot{p_h}(t), q_h\right) + a\left(p_h(t), q_h\right) = a(u_h(t), q_h) \\ u_h(0) = u_h^0, \quad \dot{u_h}(0) = u_h^1 \\ p_h(0) = 0 \end{cases} \tag{3.11}$$

where

$$a(u_h, v_h) = (\nabla u_h, \nabla v_h)_{L_2(\Omega)}$$

where $u_{0,h} \in V_h$ and $u_{1,h} \in V_h$ are approximations of the initial data u_0 and u_1 . defined by:

$$u_{0,h} = \sum_{i=1}^{n_h} U_i^{0,h}\phi_i, \quad u_{1,h} = \sum_{i=1}^{n_h} U_i^{1,h}\phi_i$$

$$\begin{cases} \mathcal{M}_h \ddot{U}^h(t) + \mathcal{K}_h U^h(t) - \mathcal{K}_h P^h = b^h(t), \quad 0 < t < T \\ \mathcal{K}_h \dot{P}^h(t) + \mathcal{K}_h P^h = \mathcal{M}_h U^h(t) \\ U^h(t=0) = U^{0,h}, \quad \dot{U}^h(t=0) = U^{1,h} \\ P^h(t=0) = 0 \end{cases} \tag{3.12}$$

where the mass matrix $\mathcal{M}_h$ and the stiffness matrix $\mathcal{K}_h$ are defined by

$$(\mathcal{M}_h)_{ij} = \langle \phi_i, \phi_j \rangle_{L^2(\Omega)}, (\mathcal{K}_h)_{ij} = a(\phi_i, \phi_j) \quad 1 \le i, j \le n_h.$$

Considering the semidiscrete formulation, we can express the second equation of (3.11) as follows:

$$\dot{p}_h(t) + p_h(t) = u_h(t)$$

with invertible $\mathcal{K}$ with the initial condition

$$\mathcal{K}u(0) = U_0 \quad \mathcal{M}\dot{u}(0) = U_1$$

where

$$U_0 = a(u_0, \phi) \quad \text{and} \quad U_1 = (\dot{u}_1, \phi)$$

Lemma 3.3.1 *According to the theorem (3.2.1), the mass matrix $\mathcal{M}$ and the stiffness matrix $\mathcal{K}$ are symmetric positive definite. Thus, they are invertible.*

3.3.2 Fully discretization in space-time

for $t_n = n\tau$, where $\tau \ge 0$ such that $\tau = T/N, N \in \mathbb{N}$, for $n = 0, \dots, N$. With this in mind, the fully discrete formulation is determined by Crank-Nicolson method. Suppose$\dot{U}_h^n$ denotes the approximation to first derivative in time at $t = t_n$ with the relation

$$\frac{\dot{U}_h^{n+1}(x) + \dot{U}_h^n(x)}{2} = \frac{U_h^{n+1}(x) - U_h^n(x)}{\tau} \quad \text{for } n = 0, \dots, N-1. \tag{3.13}$$

With applying Crank-Nicolson method, the fully discrete formulation for (3.11) can defined as follows:
Find $U_h^n(x), \dot{U}_h^n(x)$ and $P_h^n(x) \in V_h,$ for $n = 0, \dots, N$ such that

$$\left(\frac{\dot{U}_h^{n+1}-\dot{U}_h^n}{\tau}, v_h\right)+a\left(\frac{U_h^{n+1}+U_h^n}{2}, v_h\right)-a\left(\frac{P_h^{n+1}+P_h^n}{2}, v_h\right)=\left(f^n, v_h\right) \tag{3.14}$$

$$a\left(\frac{P_h^{n+1}-P_h^n}{\tau}, v_h\right)+a\left(\frac{P_h^{n+1}+P_h^n}{2}, v_h\right)=a\left(\frac{U_h^{n+1}+U_h^n}{2}, v_h\right) \tag{3.15}$$

$$a\left(U_h^0, v_h\right)=a\left(u_0, v_h\right) \tag{3.16}$$

$$\left(\dot{U}_h^0, v_h\right)=\left(u_1, v_h\right) \tag{3.17}$$

$$P_h^0=0 \tag{3.18}$$

Using a similar approach to the semidiscrete formulation,We obtain

$$u_h^0=\mathcal{K}^{-1}U_0$$

with

$$\mu_h^1=\mathcal{M}^{-1}U^1$$

$\mathcal{K}$ is an invertabel matrix and $p_h^0(0)=0$ for $n=0$, (3.15) provides

$$\left(\frac{1}{\tau}+\frac{1}{2}\right)p_h^1=\frac{u_h^0+u_h^1}{2} \tag{3.19}$$

and (3.14) implies

$$\frac{1}{\tau}\mathcal{M}(u_1^1-u_0^1)+\frac{1}{2}\mathcal{K}(u_1^0-u_0^0)-\frac{1}{2}p_h^1\mathcal{K}=\frac{1}{2}(f^1+f^0) \tag{3.20}$$

from the relation (3.13)

$$u_1^1=\frac{2}{\tau}(u_1^0-u_0^0)-u_0^1 \tag{3.21}$$

and so (3.19) yields

$$\frac{1}{\tau}\mathcal{M}\left(\frac{2}{\tau}(u_1^0-u_0^0)-2u_0^1\right)+\frac{1}{2}\mathcal{K}\left(u_1^0-u_0^0\right)-\left(\frac{1}{\tau}+\frac{1}{2}\right)^{-1}\frac{1}{2}\mathcal{K}\left(u_1^0-u_0^0\right)=\left(f^0+f^1\right) \tag{3.22}$$

$$\begin{aligned}&\left(\frac{2}{\tau^2}\mathcal{M}+\frac{1}{2}\left(1-\frac{1}{2}\left(\frac{1}{\tau}+\frac{1}{2}\right)^{-1}\right)\mathcal{K}\right)u_0^1\\&=\frac{2}{\tau}\mathcal{M}u_0^1+\left(\frac{2}{\tau^2}\mathcal{M}-\frac{1}{2}\left(1-\frac{1}{2}\left(\frac{1}{\tau}+\frac{1}{2}\right)^{-1}\right)\mathcal{K}\right)u_0^0+\frac{1}{2}\left(f^1+f^0\right).\end{aligned} \tag{3.23}$$

So the matrix $\mathcal{A}$ is defined by

$$\mathcal{A} = \left(1 - \frac{1}{2}\left(\frac{1}{\tau} + \frac{1}{2}\right)^{-1}\right)\mathcal{K}$$

if the matrix $\frac{2}{\tau^2}\mathcal{M} + \mathcal{A}$ is invertible, and since u^0 , $\mathfrak{u}^1$ and f^n are known ,we can obtain u^1 Eventually, we can also derive u_0^1 and p^1; by (3.21) and (3.19). In this manner, we can solve the following system for $n = 0, \ldots, N-1$.

$$\begin{cases} u^{n+1} = \left(\frac{2}{\tau^2}\mathcal{M} + \mathcal{A}\right)^{-1}\left[\frac{2}{\tau}\mathcal{M}\dot{u}^n + \left(\frac{2}{\tau^2}M - \mathcal{A}\right)u^n + \frac{2}{2+\tau}\mathcal{K}p_h^n + \frac{1}{2}\left(f^{n+1} + f^n\right)\right] \\ \dot{u}^{n+1} = \frac{2}{\tau}\left(u^{n+1} - u^n\right) - \dot{u}^n, \\ p^{n+1} = \frac{2}{2+\tau}\left(\frac{2-\tau}{2\tau}p^n + \frac{1}{2}\left(u^{n+1} + u^n\right)\right). \end{cases}$$

3.3.3 Numerical Tests

In this section, we will discuss the numerical results obtained for problems (1.1) and (2.1) using the FreeFem++ software. We conducted numerical simulations to solve these problems and analyze their behavior.

Numerical Tests for wave equation

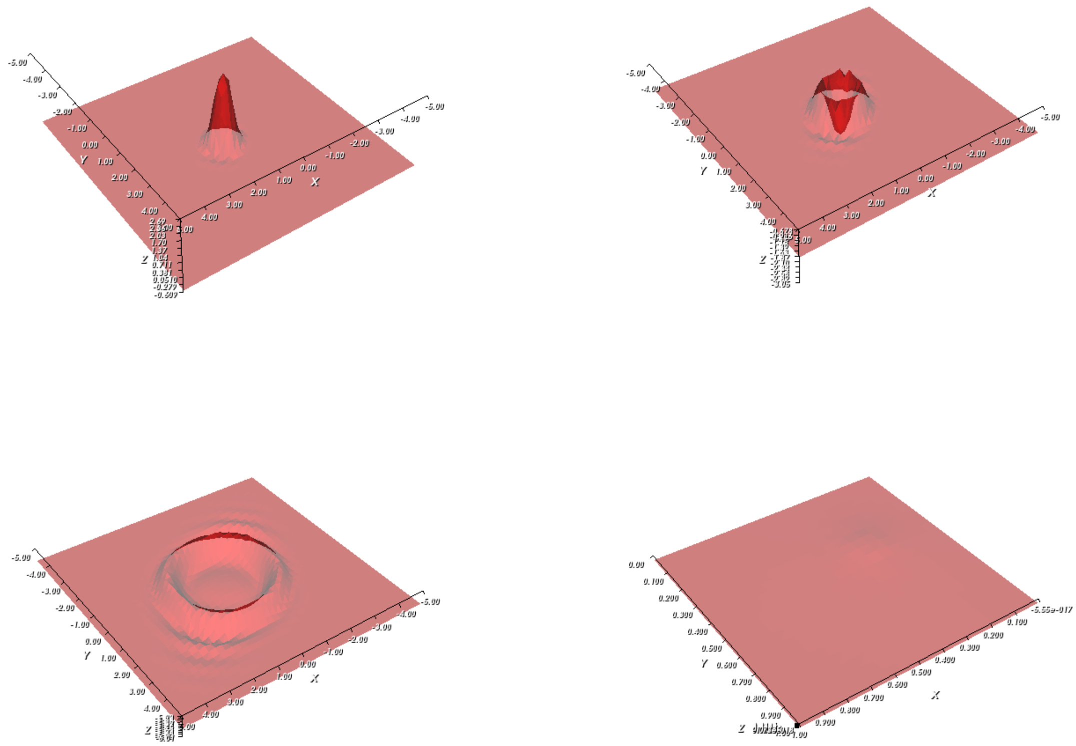


Figure 3.2: Numerical Tests for wave equation

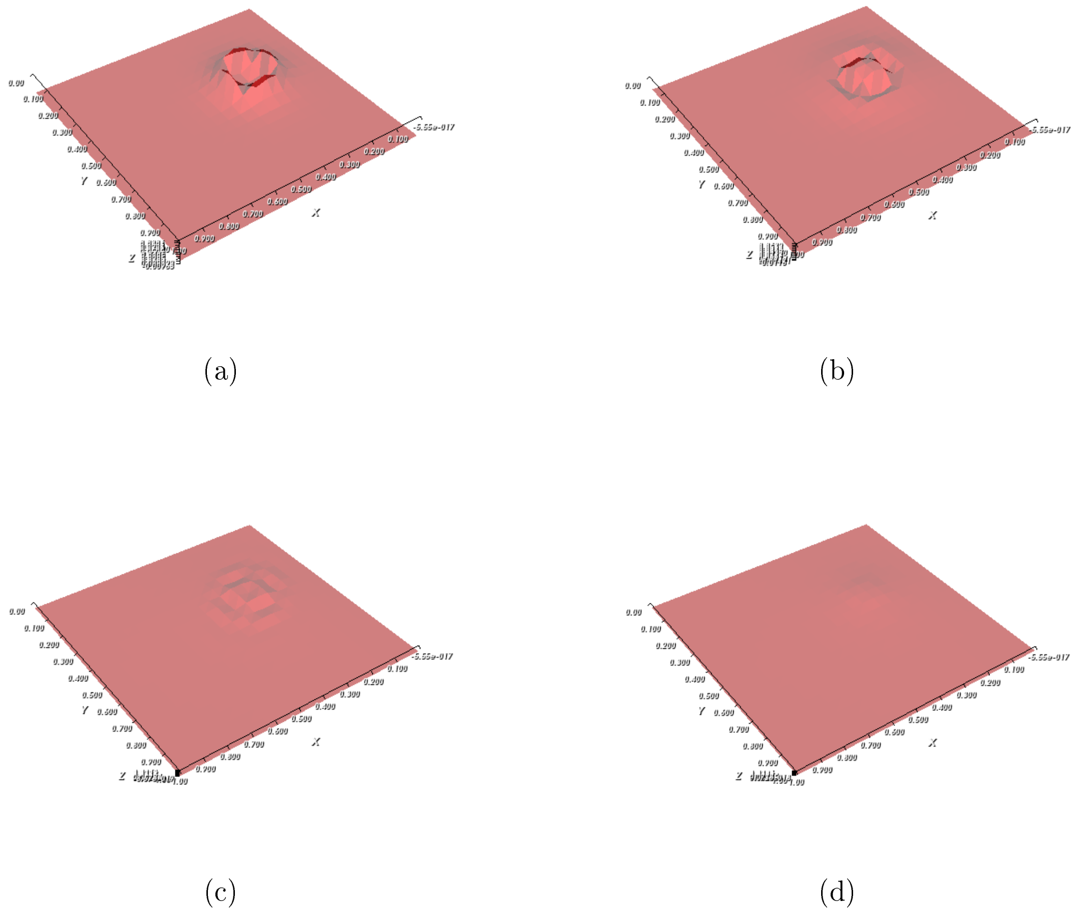
(a) displacement graph in the initial moment (b) displacement graph in the final instant

Figure 3.3: The displacement graph

Figures (3.3a) and (3.3b) provide visual confirmation of the validity of formula (1.22), which expresses energy conservation in the context of the wave equation. These figures present the results of our numerical simulations where we studied the evolution of the total energy of our system over time.

By analyzing the results, we can observe that the total energy remains constant over time, which is in agreement with formula (1.22). This means that the initial energy injected into the system is conserved over time, without significant loss or gain. This conservation of energy is an important property of the wave equation and serves as further validation of our numerical results.

Numerical Tests for wave equation with memory term



(a) (b)

(c) (d)

Figure 3.4: wave equation with memory term

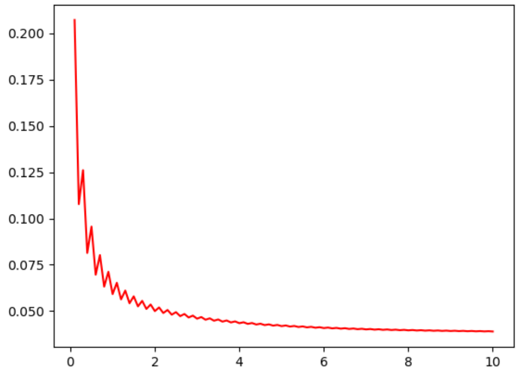


Figure 3.5: Energy function for the Exponential decay

Observing the results obtained from Figure(3.5), it can be concluded that the energy of the wave problem with memory term decreases exponentially. This decrease is influenced by the presence of the memory term in the equation, which contributes to dissipating the energy of the system.
Subsequently, vibrations or oscillations are observed in the energy, which is also attributed to these results confirm the validity of the theorem (2.2.3) of decrease exponentially. This observation of the rapid exponential decrease of energy demonstrates the importance of the term memory in the problem of waves and its influence on the evolution of energy.

Conclusion

In the conclusion, this thesis explored in the problems of wave equations with memory term , focusing on the existence, uniqueness and stability of solutions. Using a spectral approach, we were able to demonstrate the existence and uniqueness of solutions and analyze their behavior. Our numerical results confirmed the significant influence of the memory term on the properties of the solutions. We observed an exponential decrease in the energy of the system over time, which testifies to the stability and the dissipativeness of the studied model. This research opens the way to new perspectives in the field of modeling viscoelastic dynamic systems.

- we can changing the operator to another operator like the bi-laplacien operator.
- we can change the memory function by another function g which satisfies well-pressed conditions.
- In the discritization we can take the ultra weak formulation so we can take the initial data $u^0 \in H$ and $u^1 \in V^*$.

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هدفنا في هذه الأطروحة هو دراسة وجود وواحدانية ستقرار حلول مسائل معادلات الموجات بوجود و غياب حد الذاكرة. نعتمد نهجًا طيفيًا لتحليل هذه المسائل وإثبات وجود و وحدانية الحلول. ولحل هذه

المسائل عددياً نستخدم طريقة العناصر المتهية. نقوم بتقريبات عددية لتقدير المعادلات في المجال الزمني.
الكلمات المفتاحية معادلات الموجة ، حد الذاكرة ، النهج الطيفي ، اضمحلال الطاقة

Résumé :
Dans ce mémoire, notre objectif est d'étudier l'existence, l'unicité et la stabilité des solutions des problèmes d'équations des ondes sans et avec terme mémoire. Nous adoptons une approche spectrale pour analyser ces problèmes et démontrer l'existence et l'unicité des solutions. Nous examinons également la stabilité des solutions. Pour résoudre ces problèmes numériquement, nous utilisons la méthode des éléments finis. Nous mettons en œuvre les schémas de Crank-Nicolson et Newmark pour discrétiser les équations dans le domaine temporel.
mots clés: équations des ondes, terme mémoire, approche spectrale,
décroissance de l'énergie, schémas de Crank-Nicolson et Newmark.

Summary:
In this thesis, our objective is to study the existence, the uniqueness and the stability of the solutions of the problems of wave equations without and with memory term. We adopt a spectral approach to analyze these problems and demonstrate the existence and uniqueness of the solutions. We also examine the stability of the solutions. To solve these problems numerically, we use the finite element method. We implement the Crank-Nicolson and Newmark schemes to discretize the equations in the time domain.
Key words: wave equations, memory term, spectral approach, energy decay, Crank-Nicolson and Newmark schemes.