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Theme

A mixed formulation and a posteriori error analysis for a hybrid formulation of a prestressed shell model

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Dedication

I dedicate this humble work to my mother and father, who have been my great supporters in every step I take in my life. Their deep love and infinite affection have been a source of strength for me. To my brothers and sisters, you have been an integral part of my journey. And to every individual, young or old, in my family, to the dear soul of my maternal uncle whom I miss dearly, but his spirit and memories always live on in my life. I present this note to express my gratitude and immense love for all of you. You are the source of my success and the reasons for my happiness.

Your daughter, Maizi Yamina

Thanks and gratitude

Praise be to **Allah**, who granted us success and guidance to complete this thesis.
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I consider my success and achievement as a result of your tremendous efforts and limitless support. I cannot express in words how grateful I am to each and every one of you.
With sincere thanks and appreciation.
The student Maizi Yamina

Abstract

In this work, we study the finite element approximation of a prestressed shell model for the hybrid formulation. The unknowns in this model, namely the displacements and the rotations,are described using Cartesian and local covariant bases, respectively. However, due to the constraints in the solution space, we cannot directly use the finite element method. Therefore, we employ a mixed formulation instead.
We study the existence and uniqueness of its solution the convergence properties both a priori and a posteriori for this formulation.

key worde: flexural prestressed model,a hybrid formulation,a mixed formulation, finit element methode,a priori and a posteriori analusi.

Résumé

Dans ce travail, nous étudions l'approximation par éléments finis d'un modèle de coque précontrainte pour la formulation hybride. Les inconnues de ce modèle, à savoir les déplacements et les rotations,sont décrites à l'aide de bases covariantes cartésiennes et locales,respectivement. Cependant,en raison des contraintes dans l'espace des solutions, nous ne pouvons pas utiliser directement la méthode des éléments finis. Par conséquent, nous utilisons plutôt une formulation mixte. Nous étudions l'existence et l'unicité de sa solution les propriétés de convergence à la fois a priori et a posteriori pour cette formulation et établissons .

les mots clés:modèle précontraint en flexion,une formulation hybride,une formulation mixte, méthode des éléments finis,analyse a priori et a posteriori.

ملخص

في هذا العمل، قمنا بدراسة تقريب العناصر المحدودة لنموذج الصدفة سابقة الإجهاد للصيغة الهجينة. يتم وصف المجهول في هذا النموذج، أي عمليات الإزاحة والدوران، باستخدام قواعد التباين الديكارتية والمحلية، على التوالي. ومع ذلك، نظرًا للقيود الموجودة في مساحة الحل، لا يمكننا استخدام طريقة العناصر المحدودة مباشرة. لذلك، نستخدم صيغة مختلطة بدلاً من ذلك.

ندرس وجود ووحدانية الحل، وخصائص التقارب المسبق والاحق لهذه الصيغة.

الكلمات المفتاحية نموذج سابق الاجهاد,صيغة هجينة ,صيغة مختلطة, طريقة العناصر المحدودة, التحليل المسبق واللاحق

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Notations

➤ Greek indices α, β, ρ take their values in the set 1,2.

➤ Latin indices$i, j, ...$and exponents take their values in the set $1, 2, 3$.

➤ $u \cdot v$ The inner product of u and v in $\mathbb{R}^3$.

➤ $u \times v$, The vector product of u and v .

➤ $\int_w A : B$ denote $\sum_{\alpha=1,2} \sum_{\beta=1,2} \int_w A_{\alpha,\beta} B_{\alpha,\beta} dx$.

➤ $A \lesssim$ B Denote A $\leq CB$.

➤w: Be a domain of $\mathbb{R}^2$.

➤ S:a midsurface of the shell.

➤ $\Gamma^{\rho}_{\alpha\beta}$: The Christoffel symbols of the surface .

➤$[G]_e$:Denotes the jump of G across e .

➤ λ ,μ : The Lame moduli of the homogeneous and isotropic material that constitutes the shell.

➤ v,E Denote respectively the poisson modulus and coefficient the Young of the material.

➤ $tr(A)$: Trace of the matrix A,($tr(A) = A_{11} + A_{22}$).

➤ $H^m(\omega)$:Sobolev space of order m .

➤ $\Delta(T)$ Is the union of triangles of τ_h that intersect T .

➤ $\Delta(e)$ Is union of triangles of τ_h that intersect e

Introduction

The prestressed shell structure is a crucial component in both mechanical engineering and civil engineering. It finds wide-ranging applications in various fields such as satellites, offshore structures, aircraft, towers, and high-rise buildings. However, analyzing and designing prestressed shells pose significant challenges due to their complex elastic and inelastic behavior.

To address these challenges, several mathematical models have been developed. These models fall into three categories: force-based models, displacement-based models, and hybrid models. Hybrid models utilize mathematical techniques to integrate force, displacement, equilibrium, and deformation into a single formulation. This enables an accurate representation of both membrane and bending effects in prestressed shells.

In this work we are then performing some error analysis of a prestressed (two dimensional) shell model which was introduced for the first time in [10]. This model is the same as the one of a parametrized shell up to the addition of a prestressed energy term. This term (as well as the flexural one) is derived from the Kirchhoff model of the bending of the nonlinear elastic plate (obtained as a limit of three dimensional nonlinear elasticity). The unknown of the problem is the couple (u, r), where u is the displacement from the reference configuration and r is the infinitesimal rotation of the cross section of the shell. In [10] both u and r are described in Cartesian coordinates and they are sought in the Sobolev space H^1 (each one is a vector field with three components). However, the bilinear form describing the model involves the first order derivative of the components $u_i, i = 1, 2, 3$ and $r_\alpha, \alpha = 1, 2$, whereas, it does not use any derivative for the component r_3. This causes a loss of coercivity of the bilinear form on the space H^1. In order to solve this issue, a larger Hilbert space was considered in [11], where the third component r_3 is sought in the L^2 space.

A hybrid formulation is considered here, i.e., the unknowns (the displacement and the rotation to the shell midsurface) are described respectively in Cartesian and local covariant basis. The use of a hybrid formulation in a similar spirit of the present paper was used in Blouza [1] for Naghdi's shell model. The aim of using hybrid formulation in [1] was to reduce the number of the unknowns (from six to five because $s \cdot a_3 = 0$) and to get rid of the tangency constraint for the rotation which was presented by Blouza and al [2]. Hybrid formulation allows to use conforming finite element methods on unconstrained functional space with a smaller number of degrees of freedom. Another hybrid formulation of general shell element involving three incremental displacements corresponding to the stationary global coordinate directions and two rotations described in a local coordinates system was used in [9].

The purpose of this work is to provide a robust a priori error analysis and a posteriori error estimator of mixed formulation of the hybrid formulation. The objective of this work is to provide robust a priori error analysis and a posteriori error estimators for the mixed formulation of the hybrid formulation. These estimators yield global upper bounds and local lower bounds for the error, measured as the energy norm distance between the exact solution and its approximation. . In this study, we perform a posteriori analysis of the residual type for the mixed formulation and prove upper and lower bounds for the error, explicitly dependent on the mixed parameter. These estimators can be used to construct adapted meshes, enabling the computation of an approximated solution with a given accuracy.

For plates and shell models, there already exist several a posteriori error estimation approaches. We refer to [[7], [8], [14], [15]] for the pioneering works concerning plate models. Up to our knowledge, the first a posteriori estimate concerning shell models formulated in global coordinate system was done in [6] for Naghdis shell model. This work is divided into

▶ **In chapter 1**, We define the flexural prestressed shell model and a hybrid formulation

▶**In Chapter 2**, In this chapter, we are going to study the approximation using the finite element method and the a poriori analysis .

▶ **In Chapter 3**, In this chapter we derive also a posteriori estimates and we prove the reliability and efficiency of our a posteriori error estimator.

Chapter 1

A Prestressed shell model

INTRODUCTION

In this chapter we present the characteristics and geometrical notion related to shell ,espicially notation ,definitions and fundamentals required for analysis of mathimatical shell models. The aim of this chapter In the first section,we define the problem of the prestressed flexural shell model.In the second and third section,we present the hybrid formula and the mixed formula of the hybrid formula with proof of existence and uniqueness of the solution.

Let(e_1, e_2, e_3) be the canonical orthogonal basis of $\mathbb{R}^3$ and let U and V be to vector of $\mathbb{R}^3$.and $U \times V$ the vector product of U and V for a given domain W of $\mathbb{R}^2$ with a lipshitz boundary we assume that the boundary ∂W is divided into two part $\tau_0$and τ_1 we thus consider a shell with a midsurface (denoted by$S$) defined by a chart φ which is an injective mapping from the closuve of a bounded open subset of $\mathbb{R}^2$
$S = \varphi(\bar{\omega})$ where $\varphi \in W^{(2,\infty)}(\omega, \mathbb{R}^3)$ such that

$$\varphi = \bar{\omega} \longrightarrow \mathbb{R}^3 \ x = (x_1, x_2) \longmapsto \varphi(x).$$

We define two tangentail vectors to the surface S by :

$$a_\alpha(x) = \frac{\partial \varphi(x)}{\partial x_\alpha}; \alpha = 1, 2.$$

in each point $P = \varphi(x)$ of S.
The unit normal vector a_3 is then efined by

$$a_3 = \frac{a_1 \times a_3}{|a_1 \times a_2|}$$

The two vectors (a_1, a_2) defined the tangent plan TpS on every point of S and the triple $(a_1, a_2, a_3)$the covariant basis on each point P of the surface S. The contravarient basis a^i are denoted by the relation $a_i a^j = \delta_i^j$ with $a_3 = a^3$ and δ_j^i beiging the Kronecher symbol($\delta_j^i = 1$ if $i = j$ and 0).

The restriction of the metric tensor to the tangent plane also called the first fundamontal form of the surface is given by component

$$a_{\alpha\beta} = a_\alpha \times a_\beta$$

The contravariant components of the metricare given by

$$a^{\alpha\beta} = a^\alpha \times a^\beta = (a_{\alpha\beta})^- 1 = \frac{1}{a} \times \begin{pmatrix} a_{22} & -a_{12} \\ -a_{12} & a_{11} \end{pmatrix} \tag{1.1}$$

with $a = det(a_{\alpha\beta}) = a_{11} \times a_{22} - (a_{12})^2$ indeed,the infinitesimal area corresponding to the differential (dx_1, dx_2)
the coordinates can be expressed as $dS = \sqrt{a} dx_1 dx_2$
We have this relation

$$a_1 \times a_3 = -\sqrt{a} a^2 \ and \ a_2 \times a_3 = \sqrt{a} a^1 \tag{1.2}$$

$$a^1 \times a^2 = det(a^{\alpha\beta})\sqrt{a}a_3 \tag{1.3}$$

$$a^1 \times a^3 = -det(a^{\alpha\beta})\sqrt{a}a_2 \tag{1.4}$$

$$a^2 \times a^3 = det(a^{\alpha\beta})\sqrt{a}a_1. \tag{1.5}$$

The components of the second fundamental form of the surface are defined by

$$b_{\alpha\beta} = a_3 \cdot \partial_\beta a_\alpha = -a_\alpha \partial_\beta a_3$$

The proof can be found in[4] .

The second fundamental form is calle the curvature bensor and the mised components are defined by

$$b_\alpha^\beta = a^{\beta p} \times b_{p\alpha}$$

the christoffel symebols of the surface$\tau_{\alpha\beta}^p$ take the forme

$$\tau_{\alpha\beta}^p = \tau_{\beta\alpha}^p = a^p \cdot \partial_\beta a_\alpha = -\partial_\beta a^p \cdot a_\alpha.$$

1.1 A flexural prestressed shell model

The concept of prestressing involves intentionally applying permanent stresses to strengthen structures. In [4], Marohnic and TambaCa developed a flexural model for prestressed shells. The objective of their study was to determine the unknowns of the problem, namely the displacement (u) from the reference configuration and the infinitesimal rotation(r) of the shell's cross section. They formulated the following variational problem:

$$\begin{cases} Find\ U = (u, r) \in \mathbb{V} such that \\ \mathsf{A}(U, V) + \mathsf{A}_p(r, s) = L(v, s) \forall V = (v, s) \in \mathbb{V} \end{cases}$$

It has been demonstrated that the bilinear form $A(.,.)$ defines a norm on the space $\mathbb{V}$. However, it should be noted that this space is not complete with respect to this norm . To address this limitation, we introduce a larger Hilbert space $\mathbb{V}$, which is the completion of the space$\mathbb{V}$with respect to the norm $\|v\| = A(v, v)^{\frac{1}{2}}$. Consequently, the existence and uniqueness of the solution can be inferred from the Lax-Milgram Lemma in this new space. In this chapter, we present a prestressed shell model proposed in [11], where a global coordinate system is utilized instead of the local coordinate system.

We assume that the shell is fixed on a part$\Gamma_0$of the bondary ofω then function space for the lineairized flexural probleme is

$$\mathbb{V} = \left\{(v, s) \in H^1(\omega, \mathbb{R}^3) \times L^2(\omega, \mathbb{R}^3) : s \cdot a_\alpha \in H^1(\omega, \mathbb{R}), s \cdot a_3 = \tilde{\gamma}_{12}(v), v|_{\Gamma 0} = 0\right\}. \quad (1.6)$$

with

$$\tilde{\gamma}_{12}(v) = \frac{1}{2}(\partial_1 v \cdot a_2 - \partial_2 v \cdot a_1) \quad (1.7)$$

the norm of $\mathbb{V}$ is defined by

$$\|(v, s)\|_{\bar{\mathbb{X}}} = \left(\|v\|^2_{H^1(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2} \|s \cdot a_\alpha\|^2_{H^1(\omega)} + \|s \cdot a_3\|^2_{L^2(\omega)}\right)^{\frac{1}{2}}, \quad (1.8)$$

Lemma 1 *The $\mathbb{V}$ equipped with the norm$\|(v, s)\|_{\bar{\mathbb{X}}}$ is a Hilbert space*

Proof. Let us introduce the Hilbert space

$$\bar{\mathbb{X}} = \left\{(v,s) \in H^1(\omega,\mathbb{R}^3) \times L^2(\omega,\mathbb{R}) \mid s_\alpha \in H^1(\omega), v|_{\Gamma 0} = s_\alpha|_{\Gamma 0} = 0\right\}$$

equipped with the natural norm (1.8) and the linear and continuous operator $q : \bar{\mathbb{X}} \longrightarrow L^2(\omega) : (v,s) \longmapsto s.a_\alpha - \tilde{\gamma}_{12}(v)$.Then $\mathbb{V}$ is a closed subspace of $\bar{\mathbb{X}}$, because$\mathbb{V}$ is simply the kernel of q see[13]. ■

the variational problem reads as follows

$$\begin{cases} Find\ U = (u,r) \in \mathbb{V} such that \\ \mathsf{A}(U,V) + \mathsf{A}_p(r,s) = L(v,s) \forall V = (v,s) \in \mathbb{V} \end{cases} \tag{1.9}$$

let $U = (u,r)$ and $V = (v,s)$,we introduce the following bilineair forms

$$\mathsf{A}(U,V) = eA_m(u,v) + eA_t(U,V) + \frac{e^3}{12} A_f(r,s) \tag{1.10}$$

The flexural term is equal to

$$A_f(r,s) = 2\mu \int_\omega \Pi(r) \cdot \Pi(s)\, dx + \frac{2\lambda\mu}{2\mu+\lambda} \int_\omega \mathrm{tr}\Pi(r) \cdot \mathrm{tr}\Pi(s)\, dx.$$

denote $\Pi(r)$ by a symmetrized linearized second fundamental form

$$\Pi(s) = \begin{pmatrix} \partial_1.a_2 & \frac{1}{2}(\partial_2 s.a_2 - \partial_1 s.a_1) \\ \frac{1}{2}(\partial_2 s.a_2 - \partial_1 s.a_1) & -\partial_2 s.a_1 \end{pmatrix}$$

The prestressed bilinear form (corresponding to the prestressed energy) reads

$$A_p(r,s) = 2\mu \int_\omega \mathrm{tr}((II_0 + II^T)\tau(r,s))\, dx + \frac{4\lambda\mu}{2\mu+\lambda} \int_\omega \mathrm{tr}(II\tau(r,s))\, dx.$$

Where

$$\tau(r,s) = \frac{1}{2} \begin{pmatrix} -\partial_1 r.a_1 & \frac{1}{2}(\partial_1 a_2 - \partial_2 r.a_1) \\ \frac{1}{2}(\partial_1 a_2 - \partial_2 r.a_1) & \partial_2 r.a_2 \end{pmatrix} (s.a_3)$$

$$+\frac{1}{2}\begin{pmatrix} -\partial_1 s.a_1 & \frac{1}{2}(\partial_1 s.a_2 - \partial_2 s.a_1) \\ \frac{1}{2}(\partial_1 s.a_2 - \partial_2 s.a_1) & \partial_2 s.a_2 \end{pmatrix}(r.a_3)$$

and

$$II_0 = \nabla\varphi^\top \nabla a_3 = \begin{pmatrix} \partial_1\varphi.\partial_1 a_3 & \partial_1\varphi.\partial_2 a_3 \\ \partial_2\varphi.\partial_1 a_3 & \partial_2\varphi.\partial_2 a_3 \end{pmatrix}.$$

The bilinear form $A_p(.,.)$ is symmetric but not necessarily positive. The linear form (the force)$L(V)$ equals

$$L(V) = \int_\omega f.vdx$$

with $f \in L^2(\omega, \mathbb{R}^3)$ that represents a given resultant force dencity.
The membrane term is equal to

$$A_m(u,v) = 4\mu \int_\omega \gamma(u)\cdot\gamma(v)dx + \frac{4\lambda\mu}{2\mu+\lambda}\int_w tr\gamma(u)tr\gamma(u)dx \tag{1.11}$$

Where $\gamma(v)$ is a liearized strain tensor.this is a standard membrane term in the theory of shells for st.Venant-Kirchhoff material.In global coordinates,Blouza and le dret showed that this term is equale [3]

$$\gamma_{\alpha\beta}(u) = (\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha) \tag{1.12}$$

$$A_t(U,V) = \mu \int_\omega a_3^T(\nabla u - r\times\nabla\varphi).\, a_3^T(\nabla v - s\times\nabla\varphi)dx \tag{1.13}$$

This term is a standard term in the theory of naghdi shells[4] but in the case that $\varphi$is isometric . The rotation in this model is different than the rotation of the Naghdi shell.
e being the thickness of the shell assumed to be constant and positive.
ans

$$\mathsf{A}_P(r,s) = \frac{e^3}{12}A_p(r,s) \tag{1.14}$$

Theorem 1 *For* $\|\nabla a_3\|_{L^\infty(\omega)}$ *small enourgh problem(1,9) admits a unique solution. Moreover,this solution satisfies*

$$\|U\|_{\bar{\mathbb{X}}} \leq C\|L\|.$$

Proof. see [13] ■

1.2 A hybrid formulation

Let us introduce the space $\mathbb{W}$ such that the displacement and the rotation are describd in cartesian and local covariant or contravariant basis respectively .

$$\mathbb{W} = \left\{ (v, s) = \sum_{i=1}^{3} s_i a_i) \in H^1(\omega, \mathbb{R}^3) \times (L^2(\omega))^3 | s_\alpha \in H^1(\omega), s_3 = \tilde{\gamma}_{12}(v), a.e.\text{in}\omega, v|_{\Gamma 0} = s_\alpha|_{\Gamma 0} = 0 \right\} \tag{1.15}$$

such that $\tilde{\gamma}(v)$ is defined by (1.7).

equipped with the norm.

$$\|(v, s)\|_{\mathbb{X}} = (\|v\|^2_{H^1(\omega,\mathbb{R}^3)} + \sum_{\alpha=1,2} \|s_\alpha\|^2_{H^1(\omega)} + \|s_3\|^2_{L^2(\omega)})^{\frac{1}{2}} \tag{1.16}$$

The difference between the definition of $\mathbb{W}$ and $\mathbb{V}$ is that the regularity of the rotation variable r and the constraint is expressed in curvilinear variables instead of cartesian ones.Let us now show that the definition are equivalent. Indeed if$r = (r_1^{ca}, r_2^{ca}, r_3^{ca})$is the expression of the rotation in cartesian coordinates,then it can also be written as

$$r = \sum_{i=1}^{3} r_i a_i,$$

where $r_i, i = 1, 2, 3$ are its curvilinear coordinates.then we get

$$r_i = r.a_i.$$

This simple means that $\mathbb{W}$ coincides whith $\mathbb{V}$,and therefore the bilinear form A and A_p are well on $\mathbb{W}$.

before going,we want to amphasize that from now on for$(u, r) \in \mathbb{W}, r_i$ always mean the curvilinear coordinates of r .

Lemma 2 *The space $\mathbb{W}$ equipped with the norme (1.16) is a Hilbert space .*

Proof. We remark that $\mathbb{W}$is a closed subspace of

$$\mathbb{X} = \left\{ (v, s = \sum_{i=1}^{3} s_i a_i) \in H^1(\omega, \mathbb{R}^3) \times (L^2(\omega))^3 | s_\alpha \in H^1(\omega), v|_{\Gamma 0} = s_\alpha|_{\Gamma 0} = 0 \right\}$$

equipped with the norm (1.15) because $\mathbb{W}$ is simply the kernel of the linear and continuous operator Q defined by

$$Q : \mathbb{X} \longrightarrow L^2(\omega) : (v, s) \longrightarrow s_3 - \tilde{\gamma}_{12}(v)$$

■

Then,the new variational formulation reads

$$\begin{cases} Find\ U = (u, r) \in \mathbb{W}\ suchthat \\ \mathsf{A}(U, V) + \mathsf{A}_p(U, V) = L(V) \end{cases} \tag{1.17}$$

The bilineare forme $\mathsf{A}(.,.)$ and $\mathsf{A}_p(.,.)$ are defined by (1.9)
we can write the bilinear forms $A_m(.,.), A_f(.,.) and\ A_t(.,.)$ respectively corresponding to the membrane,flexural,and the transverse shear energies by

$$A_m(u, v) = \frac{4\lambda\mu}{\lambda + 2\mu} \int_w \mathrm{tr}\gamma(u)\mathrm{tr}\gamma(v) dx + 4\mu \int_w \gamma(u) : \gamma(v) dx, \tag{1.18}$$

$$A_f(r, s) = \frac{2\lambda\mu}{\lambda + 2\mu} \int_\omega \mathrm{tr}\Pi(r)\mathrm{tr}\Pi(s) dx + 2\mu \int_\omega \Pi(r) : \Pi(s) dx, \tag{1.19}$$

$$A_t((u, r), (v, s)) = \mu \int_w a_3^\top (\nabla\mu - r \times \nabla\varphi)[a_3^\top (\nabla v - s \times \nabla\varphi)]^\top dx, \tag{1.20}$$

Theorem 2 *If $\|\nabla a_3\|_{L^\infty(\omega)}$ is small enourgh problem hybride formulation admits a uniques solution .*
Moreover, this solution satisfies

$$\|U\|_X \lesssim \|L\|.$$

Proof. Since the bilinear form$\mathsf{A} + \mathsf{A}_p$ and the form L are clearly continuous on $\mathbb{W}$, the well-posedness of problem (1.17) will be guaranteed if $\mathsf{A} + \mathsf{A}_p$ is coercive on $\mathbb{W}$. For that purpose, we need the following lemma.

Lemma 3 *Suppose that $\varphi \in H^2(\omega, \mathbb{R}^3)$and that $\varphi(\Gamma_0)$ is not included into a straight line. Let$V = (v, s) \in \mathbb{W}$. Then $\mathsf{A}(V, V) = 0$ if and only if$V = 0$.*

Lemma 4 *Under the assumptions of lemma3,the bilinear form $\mathsf{A}(.,.)$ is coercive on $\mathbb{W}$.*

The proofs are fully similar to those given in Lemma 2 and 3 from [11] and are then omitted. and if $\|\nabla a_3\|_{L^\infty(\omega)}$ is small enough,the bilinear form $\mathsf{A}(.,.) + \mathsf{A}_p(.,.)$ remains coerciv on $\mathbb{W}$.Hence,the well-posedness of(1,17)follows the Lax-Milgram lemma ■

1.3 A mixed formulation for a hybrid formulation

In this subsection,we present a mixed formulation for problem (1.17)

Let us consider the functional space

$$\mathbb{X} = \left\{(v,s) \in H^1(\omega,\mathbb{R}^3) \times (L^2(\omega))^3 \mid s_\alpha \in H^1(\omega), v|_{\Gamma 0} = s_\alpha|_{\Gamma 0} = 0\right\} \quad (1.21)$$

equipped with the norm

$$\|(v,s)\|_{\mathbb{X}} = \left(\|v\|^2_{H^1_{(\omega,\mathbb{R}^3)}} + \textstyle\sum_{\alpha=1,2} \|s_\alpha\|^2_{H^1(\omega)} + \|s_3\|^2_{L^2(\omega)}\right)^{\frac{1}{2}} \quad (1.22)$$

and we set

$$\mathbb{M} = L^2(\omega). \quad (1.23)$$

for all $\rho > 0$ we consider the following variational problem

$$\left\{\begin{array}{c} \text{find}(U,\psi) = (u,r,\psi) \in \mathbb{X} \times \mathbb{M} \text{ sush that} \\ A(U,V) + A_p(U,V) + \rho b(U,V) + \tilde{b}(V,\psi) = L(V), \forall V \in \mathbb{X} \\ \tilde{b}(U,\phi) = 0, \forall \phi \in \mathbb{M} \end{array}\right. \quad (1.24)$$

for $V = (v,s) \in \mathbb{X}$ and $\phi \in \mathbb{M}$ the bilinear form $\tilde{b}(.\,,.)$ is defined by

$$\tilde{b}(V,\phi) = \int_\omega (s_3 - \bar{\gamma}_{12}(v))\phi dx \quad (1.25)$$

and where the bilinear form $b(.,.)$ is defined by

$$b(U,V) = \int_\omega Q(W)Q(V) \quad (1.26)$$

such that

$$Q(V) = s_3 - \bar{\gamma}_{12}(v) \text{For any} V = (v,s) \in \mathbb{X}$$

Moreover ,the flowing characterization holds:

$$\mathbb{W} = \left\{(v,s) \in \mathbb{X}, \forall \phi \in \mathbb{M}, \tilde{b}(v,\phi) = 0\right) \quad (1.27)$$

Lemma 5 *There exists a constant $C > 0$ such that*

$$\forall \phi \in \mathbb{M} \sup_{v \in X} \frac{\tilde{b}(V,\phi)}{\|V\|_X} \geq C\|\phi\|_{L^2(\omega)} \quad (1.28)$$

Proof. We prove that $b(.,.)$satisfie the inf-sup condition see [12]. We prove that $b(.,.)$ satisfie the $\inf - \sup$ condition . Let $\phi \in \mathbb{M}$ and let $\bar{V} = (\bar{v}, \bar{s}) \in \mathbb{X}$ such that $\bar{v} = 0, \bar{s} \cdot a_\alpha = 0, \bar{s} \cdot a_3 = \phi$ therefore

$$\sup_{V \in \mathbb{X}} \frac{\tilde{b}(V, \phi)}{\|V\|_{\mathbb{X}}} \geq \frac{\tilde{b}(\bar{V}, \phi)}{\|\bar{V}\|_{\mathbb{X}}}$$

$$= \frac{\|\phi\|^2_{L^2(\omega)}}{\|\phi\|_{L^2(\omega)}}$$

$$= \|\phi\|_{L^2(\omega)}$$

■

Theorem 3 *If$\|\nabla a_3\|_{L^\infty}$ is sufficiently small, the problem (1.24) has a unique solution (U, ψ),such that U is the solution of the problem (1.17).*

Proof. combining the ellipticity property for $\mathsf{A}(.,.) + \rho b(.,.) + \mathsf{A}_p(.,.)$ and the condition inf-sup .lemma(2)

Let us now check that U is the solution to the problem (1.17)this solution satisfies

$$\|U\|_{\mathbb{X}} \leq C\|L\|$$

this we apply the Lax-miligram lemma.
taking

$$\phi = r_i - \tilde{\gamma}_{12}(u)$$

In the second equation of (1.24),obtain $U \in \mathbb{W}$ then taking $V \in \mathbb{W}$ cancels the term b in the first equation of (1.24),then we have the result. ■

Chapter 2

Approximation by finit element method

Finite element method are used to numerically and approximating the solution of the mathematical models. In this chapter we use the approximation by finite element method for the mixed problem which are presented in the previous chapter.with the study of the a priori analysis of it.

2.1 Approximation by finit element method

As we have mentioned, the constrained problem (1.17) cannot be approximated by robust conforming methods for a general shell, hence we purpose the approximation of a mexed formulation . Note that in this section we need not to assume that the bilinear form of the right hand side is coercive, we only suppose that both problem the contrained and the relaxed one has a unique solution which supposed to be sufficiently regular. we introduce the finite dimensionel space

$$\mathbb{X}_h = \left\{ V_h = (v_h, s_h = \sum_{i=1}^{3} s_{ih} a_i) \in \mathbb{X} | v_{h \setminus T} \in \mathbb{P}_k(T)^3, s_{ih} \in \mathbb{P}_k(T), \nabla T \in \mathcal{T}_h, k \geq 1 \right\} \tag{2.1}$$

$$\tilde{\mathbb{M}}_h = \left\{ \mu_h \in C^0(\bar{\omega}) / \mu_{h|T} \in \mathbb{P}_1(T), \forall T \in \mathcal{T}_h \right\}$$

and consider the following discrete probleme:for all $\rho > 0$,

$$\left\{ \begin{array}{c} \text{Find } \ (U_h, \psi_h) = (u_h, r_h, \psi_h) \in \mathbb{X}_h \text{ such that} \\ \mathsf{A}(U_h, V_h) + \mathsf{A}_p(U_h, V_h) + \rho b(U, V) + \tilde{b}(V_h, \psi_h) = L(V_h), \forall V_h \in \mathbb{X}_h \\ \tilde{b}(U_h, \phi_h) = 0 \ \forall \phi \in \tilde{\mathbb{M}}_h \end{array} \right. \tag{2.2}$$

Proposition 4 *The discrete probleme(2.2)has a unique solution*

Proof. the existence and uniqueness of a $\inf - \sup$ condition given in lemma(4) ■

Lemma 6 *for all $\mu_h \in \tilde{\mathbb{M}}_h, V_h = (0, R_h(\mu_h))$such that $R_h(\mu_h) = \pi_h(\mu_h a_3)$($\pi_h$donote the vector valued $\mathbb{P}_1$ lagronge interpolation operator).Then,there exists a constant$C > 0$ such that*

$$\tilde{b}(V_h, \mu_h) \geq C \|\mu_h\|^2_{\mathbb{M}}$$

Proof. We note that μ_h is a scalar piecewise $\mathbb{P}_1$ function, μ_{ha3} is vector-valued, and $R_h(\mu_h)$ is a vector-valued piecewise $P1$ function. Let us set $\delta_h = R_h(\mu_h) \cdot a_3 - \mu_h$ and $V_h = (0, R_h(\mu_h))$. Then,

$$\widetilde{b}(V_h, \mu_h) = \int_\omega (R_h(\mu_h) \cdot a_3) \mu_h \, dx = \|\mu_h\|^2_{L^2(\omega)} + \int_\omega \delta_h \mu_h \, dx,$$

with

$$| \int_\omega \delta_h \mu_h \, dx | \leq \|\mu_h\|_{L^2(\omega)} \|\delta_h\|_{L^2(\omega)}.$$

Now, let us estimate $\|\delta_h\|_{L^2(\omega)}$. By Lagrange interpolation, we get

$$\mu_h(x) = \sum_{s_j} s_j \mu_h(s_j) \theta_j^h(x),$$

such that $\theta_j^h(x)$ is the shape function associated with the vertex s_j, and

$$R_h(\mu_h)(x) = \sum_{s_j} \mu_h(s_j)\theta_j^h(x)a_3(s_j).$$

Then,

$$\delta_h(x) = \sum_{s_j} \mu_h(s_j)[a_3(s_j) - a_3(x)]a_3(x)\theta_h^j(x),$$

where $a_3(x)$ is a unit vector. It holds that

$$\|\delta_h(x)\|_{L^\infty(\omega)} \leq 3\|\mu_h\|_{L^\infty(\omega)} \max_j \max_{T_j}[\frac{C}{h}|(a_3(s_j) - a_3(x)) \cdot a_3(x)|],$$

where T_j stands for the set of triangles sharing the vertex s_j. Then, using a lamma 4, we have

$$\|\delta_h(x)\|_{L^\infty(\omega)} \leq Ch\|\mu_h\|_{L^\infty(\omega)},$$

By classical discrete Sobolev estimate , we deduce that

$$\|\delta_h(x)\|_{L^2(\omega)} \leq C\|\delta_h(x)\|_{L^\infty(\omega)} \leq Ch\|\mu_h\|_{L^\infty(\omega)} \leq Ch(\ln(h)^{1/2})\|\mu_h\|_{L^2(\omega)}.$$

Taking h small enough so that $Ch(\ln(h))^{1/2} \leq 1/2$.

■

Lemma 7 *there existe $B_h > 0$dependent of h such that*

$$\inf_{\mu_h \in \mathbb{M}_h} \sup_{V_h \in \mathbb{X}_h} \frac{\tilde{b}(V_h, \mu_h)}{\|V_h\|_{\mathbb{X}}\|\mu_h\|_{L^2(\omega)}} \geq B_h$$

Proof. Let

$$\tilde{B}_h = \inf_{\mu_h \in \mathbb{M}_h} \sup_{V_h \in \mathbb{X}_h} \frac{\tilde{b}(V_h, \mu_h)}{\|V_h\|_{\mathbb{X}}\|\mu_h\|_{L^2(\omega)}}$$

see that $V_h = (0, R_h(\mu_h)) \in \mathbb{X}_h$then by lemma(3)$b(V_h, \mu_h) \geq C\|\mu_h\|_M^2$ then

$$\|V_h\|_{\mathbb{X}_h} = \|v_h\|_{H^1}^2 + \sum_{\alpha=1,2} \|s_h.a_\alpha\|_{H^1}^2 + \|s_h.a_3\|_{L^2}^2$$

$$\leq \|s_h\|_{H^1}^2$$

we have

$$\|V_h\|_{\mathbb{X}_h} \leq \|s_h\|_{H^1}$$

then $\|V_h\|_{\mathbb{X}_h} \leq \|R_h(\mu_h)\|_{H^1}$ we get

$$\tilde{B}_h \geq C \inf_{\mu_h \in \mathbb{M}_h} \frac{\|\mu_h\|_{\mathbb{M}}}{\|R_h(\mu_h)\|_{H^1}}$$

we put $R_h(\mu_h) = R_h(\mu_h) - \mu_h a_3 + \mu_h a_3$ we have

$$\|R_h(\mu_h)\|_{H^1} \leq \|R_h(\mu_h) - \mu_h a_3\|_{H^1} + \|\mu_h a_3\|_{H^1}$$

$$\leq c_1 \|\nabla(\mu_h a_3)\|_{L^2(\omega,M)} + \|\mu_h a_3\|_{H^1}$$
$$\leq c_1 \|\mu_h a_3\|_{H^1} + \|\mu_h a_3\|_{H^1}$$
$$\leq C h^{-1} \|\mu_h\|_{L^2}$$

then we obtain

$$\|R_h(\mu_h)\|_{H^1} \leq c_h \|\mu_h\|_{L^2}$$

which completes the proof

■

2.2 A priori analysis

In this subsection we derive a non robust a priori error analysis of the mexed formulation

Theorem 5 *Let* (U,ψ)*be a solution of the problem (1,16) and* (U_h,ψ_h) *be a solution of the problem (1,23) then this following estimate is hold*

$$\|U-U_h\|_{\mathbb{X}} \leq c_{1h} \inf_{V_h\in\mathbb{X}} \|U-V_h\|_{\mathbb{X}} + c_2 \inf_{\phi_h\in\mathbb{M}_h} \|\psi-\phi_h\|_{\mathbb{M}} \tag{2.3}$$

$$\|\psi-\psi_h\|_{\mathbb{X}} \leq c_{3h} \inf_{V_h\in\mathbb{X}} \|U-V_h\|_{\mathbb{X}} + c_{4h} \inf_{\phi_h\in\mathbb{M}_h} \|\psi-\phi_h\|_{\mathbb{M}} \tag{2.4}$$

such that c_{1h}, c_{3h} *and* c_{4h} *dependent on* $1/B_h$ *and* c_2 *independent on* h

Proof. Firstly,we prove (2,3) ,because of $\mathbb{X}_h \subset \mathbb{X}$ we have

$$C_1\|U_h-W_h\|_{\mathbb{X}} \leq \sup_{y_h\in\mathbb{X}_h} \frac{\mathsf{A}(U_h-W_h,y_h)+\rho b(U_h-W_h,y_h)+\mathsf{A}_p(U_h-W_h,y_h)}{\|y_h\|}$$

then

$$C_1\|U_h-W_h\|_{\mathbb{X}} \leq \sup_{y_h\in\mathbb{X}_h} \frac{\tilde{b}(y_h,\phi-\psi)+\mathsf{A}(U-W_h,y_h)+\rho b(U-W_h,y_h)+\mathsf{A}(U-W_h,y_h)}{\|y\|_h}$$

implaying

$$\|U_h-W_h\|_{\mathbb{X}} \leq \frac{\tilde{c_1}}{C_1}\|U-W_h\|_{\mathbb{X}} + \frac{\tilde{c_2}}{C_1}\|\psi-\phi_h\|_{\mathbb{M}}$$

by the triangle inequality we have

$$\|U-U_h\| \leq (1+\frac{\tilde{c_1}}{C_1})\|U-W_h\|_{\mathbb{X}} + \frac{\tilde{c_2}}{C_1}\|\psi-\phi_h\|_{\mathbb{M}} \tag{2.5}$$

the Inf-Sup condition (Lemma 5)is satisfied , there existe $r_h \in \mathbb{X}_h$ and let $V_h \in \mathbb{X}_h$ such that

$$\forall\phi\in\mathbb{M}_h\ \tilde{b}(r_h,\phi) = b(U-V_h,\phi_h) and B_h\|r_h\|_{\mathbb{X}} \leq C\|U-U_h\|_{\mathbb{X}},\ C>0,$$

then we estimat the term $\|U-W_h\|_{\mathbb{X}}$,we have

$$\|U-W_h\|_{\mathbb{X}} \leq \|U-W_h\|_{\mathbb{X}} + \|r_h\|_{\mathbb{X}} \tag{2.6}$$

$$\leq \left(1+\frac{c}{B_h}\right)\|U-V_h\|_{\mathbb{X}} \tag{2.7}$$

Now we prove the estimat (2,4)subtracting the first equation of (2,2)from the first equation(1,24),then we obtain

$$\mathsf{A}(U-U_h,V_h)+\rho b(U-U_h,V_h)+\mathsf{A}_p(U-U_h,V_h)+\tilde{b}(V_h,\psi-\psi_h)=0\text{for all}V_h\in\mathbb{X}_h$$

then for $\phi_h\in\mathbb{M}_h$ we have

$$\mathsf{A}(U-U_h,V_h)+\rho b(U-U_h,V_h)+\mathsf{A}_p(U-U_h,V_h)+\tilde{b}(V_h,\psi-\psi_h)+\tilde{b}(V_h,\phi)-\tilde{b}(V_h,\phi_h)=0$$

then to obtain

$$\tilde{b}(V_h,\phi_h-\psi_h)=\mathsf{A}(U_h-U,V_h)+\rho b(U_h-U,V_h)+\mathsf{A}_p(U_h-U,V_h)+\tilde{b}(V_h,\phi_h-\psi)$$

By the Inf-Sup condition

$$\|\phi-\psi_h\|_{\mathbb{M}}\leq\frac{1}{B_h}\sup_{V_h\in\mathbb{X}_h}\frac{\tilde{b}(V_h,\phi_h-\psi_h)}{\|V_h\|_{\mathbb{X}}}$$

$$=\frac{1}{B_h}\sup_{V_h\in\mathbb{X}_h}\frac{\tilde{b}(V_h,\phi_h-\psi_h)=\mathsf{A}(U_h-U,V_h)+\rho b(U_h-U,V_h)+\mathsf{A}_p(U_h-U,V_h)+\tilde{b}(V_h,\phi_h-\psi)}{\|V_h\|_{\mathbb{X}}}$$

One obtain therefore

$$\|\phi_h-\psi_h\|_{\mathbb{M}}\leq\frac{C_1}{B_h}\|U-U_h\|_{\mathbb{X}}+\left(1+\frac{C_2}{B_h}\right)\|\psi-\phi_h\|_{\mathbb{M}}$$

Then we use the triangle inequality , hence the result. ■

2.3 The strong formulation

Usually, a posteriori estimator is computed by element-wise integration by parts starting from the classical formulation . Hence in this section we give the strong formulation of problem (1.24), We find the working steps in detail in[4].

using the definition of the bilinear form $A_m(.,.)$ we have

$$eA_m(u,r) = -\int_\omega Div(T(u)A) - vdx + \int_{\Gamma_1} nT(u)A.vd\sigma(x) \tag{2.8}$$

Hence if we set

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

with

$$S(u,r) := e\mu((\nabla u)^\top a_3 + J\hat{r})$$

we get

$$eA_t((u,r),(v,s)) = -\int_\omega \mathrm{Div}(S(u,r)a_3).vdx + \int_{\Gamma_1} nS(u,r)a_3.vd\sigma(x) + \int_\omega J^\top S(u,r).\hat{s}dx \tag{2.9}$$

considernig the bilinear form $A_f(r;s)$. Due to the definition of the tonsor Π and the definition of the tensor $\mathbb{A}$,Hence if we set

$$M(r) = \tfrac{e^3}{24} A\Pi(r)$$

we get

$$\frac{e^3}{12}A_f(u,r) = -\int_\omega \mathrm{Div}(M(r)).\hat{s}dx + \int_{\Gamma_1} J^\top M(r)n^\top.\hat{s}d\sigma(x) - \int_\omega \begin{pmatrix} M(r) : \bar{\Gamma}^1 \\ M(r) : \bar{\Gamma}^2 \end{pmatrix}.\hat{s} + (\bar{B} : M(r)s_3)dx \tag{2.10}$$

Now we give the contribution of the prestressed term $\mathsf{A}_p(.,.)$. First as II_0 and $\tau(r,s)$are symmetric,we get

$$\mathsf{A}_p(r,s) = -\int_\omega \frac{1}{2}\tilde{J}\,\mathrm{Div}(P(r)).\hat{s}dx + \int_{\Gamma_1} \frac{1}{2}\tilde{J}P(r)n^\top.\hat{s}d\sigma(x) + \int_\omega (k(r) + \frac{1}{2}\tilde{B} : P(r))s_3 dx$$

$$+\int_{\omega}\frac{1}{2}\begin{pmatrix}P(u):\tilde{\Gamma}^1\\P(u):\tilde{\Gamma}^2\end{pmatrix}.\hat{s}dx \tag{2.11}$$

for the bilinear form $b(.,.)$ as $\tilde{\gamma}_{12}(v)=\frac{1}{2}(\partial_1 v.\partial_2\varphi-\partial_2 v.\partial_1\varphi)$ if $Q(U)$ is suffisiently regular we find

$$\rho b(U,V)=\frac{1}{2}\rho\int_{\omega}Q(U)(s_3-\tilde{\gamma}_{12}(v))dx$$

$$=\frac{1}{2}\rho\int_{\omega}Div(Q(U)JA).vdx-\frac{1}{2}\rho\int_{\Gamma_1}Q(U)A^{\top}Jn^{\top}.vd\sigma(x)+\int_{\omega}Q(U)s_3dx \tag{2.12}$$

and

$$\tilde{b}(U,\phi)=\int_{\omega}(r.a_3)\phi dx \tag{2.13}$$

using the indentities (2,8)(2,9)(2,10)(2,11)(2,12)(2,13) we see the solution $U=(u,r)\in\mathbb{X}$ of problem (1,24) satisfies

$$\begin{cases}-\operatorname{Div}(T(u)A)-\operatorname{Div}(S(U)a_3)+\frac{1}{2}\rho\operatorname{Div}(Q(U)JA) & =f \quad \text{in } \omega,\\ -J^{\top}\operatorname{Div}M(r)-\begin{pmatrix}M(r):\bar{\Gamma}^1\\M(r):\bar{\Gamma}^2\end{pmatrix}+J^{\top}S(U)-\frac{1}{2}\tilde{J}\operatorname{Div}(P(r))+\frac{1}{2}\begin{pmatrix}P(u):\tilde{\Gamma}^1\\P(u):\tilde{\Gamma}^2\end{pmatrix} & =0 \quad \text{in } \omega,\\ -\left(\bar{B}:M(r)\right)+\kappa(r)+\frac{1}{2}\tilde{B}:P(r)+\rho Q(U) & =0 \quad \text{in } \omega,\\ r.a_3 & =0 \quad \text{in}\omega\\ u=r, & =0 \quad \text{on } \Gamma_0\\ nT(u)A+nS(U)a_3-\frac{1}{2}\rho Q(U)A^{\top}Jn^{\top} & =0 \quad \text{on } \Gamma_1,\\ \frac{1}{2}\bar{J}P(r)n^{\top}+J^{\top}M(r)n^{\top} & =0 \quad \text{on } \Gamma_1.\end{cases} \tag{2.14}$$

Remark 6 *For more details on how to find both (2,8) (2,9),(2,10),(2,11)and (2,12) see [13]*

Chapter 3

A posteriori error analysis for a hybrid formulation

In this chapter, we stady the a posteriori analysis with proof of its reliability through the upper and lower bounds

3.1 A posteriori analysis

We introduce the approximation space $\tilde{\mathbb{M}}_h^{(i)}$ with $i \in N$ and $\mathbb{Z}_h$ as follows

$$\tilde{\mathbb{M}}_h^{(i)} = \left\{ \mathbb{X}_h \in L^2(\omega) \forall T \in \mathcal{T}_h, \mathcal{X}_{h|T} \in \mathbb{P}_i(T) \right\}$$

$$\mathbb{Z}_h = \left\{ j_h \in L^2(\omega)^3; \forall T \in \mathcal{T}_h, j_{h|T} \in \mathbb{P}_0(T)^3 \right\}$$

we consider an approximation f_h of f in $\mathbb{Z}_h$ and an approximation $b_{\alpha\beta}^h$ of the coefficient $b_{\alpha\beta}$ in $\tilde{\mathbb{M}}_h^{(1)}$.similarly , we consider approximations a_k^h of the vectors a_k and $d_{\alpha\beta}^h$ of$\partial_\alpha a_\beta$ in $(\tilde{\mathbb{M}}_h^{(2)})^3$ respectively.Obviously we assume that these approximation coefficients are uniformly bounded in h. We introduce the approximations $\mathsf{A}_h(.,.), \mathsf{A}_p^h(.,.)$and $b_h(.,.)$ of the bilinear forms $\mathsf{A}(.,.), \mathsf{A}_p(.,.)$and $b(.,.)$respectively where a_i,$\partial_\alpha a_\beta$,and $b_{\alpha\beta}$ are replaced by their approximations. More precisely, for$U = (u, \sum_i r_i a_i) \in \mathbb{X}$ we set

$$\gamma^h_{\alpha\beta}(u) = \frac{1}{2}\left(\partial_\alpha u.a^h_\beta + \partial_\beta u.a^h_\alpha\right)$$

$$\tilde{\gamma}^h_{12}(u) = \frac{1}{2}(\partial_1 u.a^h_2 - \partial_2 u.a^h_1)$$

$$\Pi^h(s) = \begin{pmatrix} \partial_1 s.a_2 & \frac{1}{2}\left(\partial_2 s.a_2 - \partial_1 s.a_1\right. \\ \frac{1}{2}\left(\partial_2 s.a_2 - \partial_1 s.a_1\right. & -\partial_2 s.a_1 \end{pmatrix}$$

$$\theta^h(s) = \frac{1}{2}\begin{pmatrix} -\gamma_{11}(s) & \tilde{\gamma}_{12}(s) \\ \tilde{\gamma}_{12}(s) & \gamma_{22}(s) \end{pmatrix}$$

$$II_0 = (\nabla\varphi)^T.\nabla a_3 = \begin{pmatrix} \partial_1\varphi.\partial_1 a_3 & \partial_1\varphi.\partial_2 a_3 \\ \partial_2\varphi.\partial_1 a_3 & \partial_2\varphi.\partial_2 a_3 \end{pmatrix}$$

$$Q^h(U) = s_3 - \tilde{\gamma}^h_{12}(U)$$

Not that II_0 is symmetric and therefore in $\mathsf{A}_p(.,.)$ the factor $II_0 + II_0^t$ may by $2II_0$.

$$A^h_m(u,v) = 4\mu\int_\omega \gamma^h(u) : \gamma^h(v)dx + \frac{4\lambda\mu}{2\mu+\lambda}\int_\omega tr\gamma^h(u)tr\gamma^h(v)dx$$

$$A^h_f(r,s) = 2\mu\int_\omega \Pi^h(r) : \Pi^h(s)dx + \frac{2\mu\lambda}{2\mu+\lambda}\int_\omega tr\Pi^h(r)tr\Pi^h(s)dx$$

$$A^h_t((u,r),(v,s)) = [(a^h_3)^T(\nabla v - s\times\nabla\varphi)]^T dx\mu\int_\omega (a^h_3)^T(\nabla u - r\times\nabla\varphi)$$

$$\mathsf{A}^h_p(r,s) = (\frac{4\mu\lambda}{\lambda+2\mu}\int_\omega trII^h_0 tr\tau^h(r,s)dx + (2\mu\int_\omega tr((II^h_0 + II^h_0)\tau^h(r,s))dx)$$

where

$$\tau^h(r,s) = \theta^h(r)(s.a_3) + \theta^h(s)(r,a_3)$$

we also introduce the approximation L_h of the linear form L namely

$$L_h(V,S) = \int_\omega f_h.vdx$$

withe

$$f \in L^2(\omega, \mathbb{R}^3)$$

Then for any $V \in \mathbb{X}, V_h \in \mathbb{X}_h$,and $\forall \phi \in \mathbb{M}, \forall \phi_h \tilde{\mathbb{M}}_h^i$ we may write the residuale as

$$\mathcal{R}_{u_h} = L(V - V_h) - A(U_h, V - V_h) - \mathsf{A}_p(U_h, V - V_h) - \rho b(U_h, V - V_h) - \tilde{b}(V_h, \psi - \psi_h)$$

$$= (L - L_h)(V - V_h) - (\mathsf{A} - \mathsf{A}_h)(U_h, V - V_h) - (\mathsf{A}_p - \mathsf{A}_p^h)(U_h, V - V_h) - \rho(b - b_h)(U_h, V - V_h) - (\tilde{b} - \tilde{b}_h)(U_h, \psi - \psi$$

$$-\mathsf{A}_h(U_h, V - V_h) - \mathsf{A}_p(U_h, V - V_h) - \rho b_h(U_h, V - V_h) - \tilde{b}(V_h, \psi - \psi_h) + L_h(V - V_h)$$

$$\tilde{b}(U - U_h, \phi) = -\tilde{b}(U_h, \phi - \phi_h) \tag{3.1}$$

We first observe that the bilinear forms $\mathsf{A}(.,.), \mathsf{A}_p(.,.)$ and$b(.,.)$have variable coefficients. In such a case, in order to construct error indicators we need to approximate the data and the coefficients by piecewise polynomials, see [6] we again recall the propeties of the clement operator C_h for$0 \leq m \leq l \leq 1$

$$\forall h, \forall T \in \tau_h, \forall \omega \in H^1(\omega)\ \|\omega - C_h\omega\|_{H^\omega(T)} \lesssim h_T^{l-m}\|\omega\|_{H^1(\Delta(T))} \tag{3.2}$$

$$\forall h, \forall n \in \varepsilon_h, \forall \omega \in H^l(\omega)\ \|\omega - C_h\omega\|_{H^m(n)} \lesssim h_n^{l-m-\frac{1}{2}}\|\omega\|_{H^l(\Delta(n))}, \tag{3.3}$$

where $\Delta(T) = \bigcup_{T' \in \tau_h : T' \bigcap T \neq 0} T'$(resp . $\Delta(n) = \bigcup_{T' \in \tau_h : n \subset T' \neq 0} T'$) is the patch associated with the element T (resp . the edge n) andε_h is the set of edges of the triangulation

Lemma 8 *let* $V = (v, \sum_i s_i a_i)$*and* $V_h = (v_h, s_h) = (C_h v, \sum_i (C_h s_i) a_i)$,*then we have the following estimate*

$$|(L - L_h)(V - V_h) - (\mathsf{A} - \mathsf{A}_h)(U_h, V - V_h) - (\mathsf{A}_p - \mathsf{A}_p^h)(U_h, V - V_h) - \rho(b - b_h)(U_h, V - V_h) - (\tilde{b} - \tilde{b}_h)(V_h, \psi - \psi_h)$$

$$\lesssim (\varepsilon_h^d + \varepsilon_h^c)\|V\|_X$$

where

$$\varepsilon_h^c = (\varepsilon \max_{k=1,2,3} \|a_k - a_k^h\|_{L^\infty(\omega)} + \max_{\alpha,\beta=1,2} \|\partial_\alpha a_\beta - d_{\alpha\beta}^h\|_{L^\infty(\omega)} + \max_{\rho,\sigma=1,2} \|b_{\rho\sigma} - b_{\rho\sigma}^h\|_{L^\infty(\omega)})\|L\|,$$

$$\varepsilon_T^d = h_T\|f - f_h\|_{L^2(T)^3}$$

and

$$\varepsilon_h^d = (\sum_T (\varepsilon_T^d)^2)^{\frac{1}{2}}$$

Proof. First one estimates the term $(L - L_h)(V - V_h)$.As we have

$$(L - L_h)(V - V_h) = \int_\omega f.(v - C_h v)dx - \int_\omega f_h.(v - C_h v)dx = \int_\omega (f - f_h).(v - C_h v)dx$$

$$\sum_{T\in\tau_h} \int_T (f - f_h)(v - C_h v)dx$$

cauchy-schwartz's inequality and the property (2,9)of C_h yield

$$|(L - L_h)(V - V_h)| \leq \varepsilon_h^d \|V\|_{\mathbb{X}}$$

secondly we estimate

$$(\mathsf{A} - \mathsf{A}_h)(U_h, V_{Vh}) + (\mathsf{A}_p - \mathsf{A}_p^h)(U_h, V - V_h) + \rho(b - b_h)(U_h, V_{Vh}) + (\tilde{b} - \tilde{b}_h)(V_h, \psi - \psi_h)$$

we only give an abridged proof of this tecnical rfsult,we first estimate

$$(\mathsf{A}{-}\mathsf{A}_p)(U_h, V{-}V_h) = e(A_m{-}A_m^h)(u_h, v{-}v_h){+}e(A_t{-}A_t^h)(U_h, V{-}V_h)\frac{e^3}{12}(A_f{-}A_f^h)(r_h, s{-}s_h)$$

to estimate the term $(A_m - A_m^h)(U_h, V - V_h)$,we typically have to estimate a term like

$$\mathcal{A}_h(u_h, v - v_h) = 4\mu \int_\omega \gamma_{11}(u_h)\gamma_{11}(v - v_h)dx + \frac{4\lambda\mu}{2\mu + \lambda} \int_\omega tr\gamma_{11}^h(u_h)tr\gamma_{11}^h(v - v_h)dx$$

that we tranform as

$$\mathsf{A}_h(u_h, v - v_h) = \int_\omega 4\mu\gamma_{11}(u_h)\gamma_{11}(v - v_h) + \int_\omega \frac{4\lambda\mu}{2\mu + \lambda} tr\gamma_{11}^h(u_h)tr\gamma_{11}^h(v - v_h)dx$$

$=\int_\omega 4\mu\gamma_{11}(u_h)\gamma_{11}(v - v_h) + \frac{4\lambda\mu}{2\mu+\lambda} tr\gamma_{11}^h(u_h)tr\gamma_{11}^h(v - v_h)dx$

$=\int_\omega 4\mu{+}\frac{4\lambda\mu}{2\mu+\lambda}[\gamma_{11}(u_h)(\gamma_{11}(v{-}v_h){+}tr\gamma_{11}^h(v{-}v_h)){+}(\gamma_{11}(u_h){-}tr\gamma_{11}^h(u_h)tr\gamma_{11}^h(v{-}v_h))]dx$

for the first term we use the identity $\gamma_{11}(u) - tr\gamma_{11}^h(u) = (\partial_1 u.a_1 - \partial_1 u a_1^h)$ and apply cauch-schwarz's inequality and $\|U_h\|_X \lesssim \|L\|$ to get

$$|\int_\omega \gamma_{11}(u_h)(\gamma_{11}(v - v_h) - tr\gamma_{11}^h(v - v_h))dx| \lesssim \|L\|\|\partial_1(v - v_h).(a - a_1^h)\|_{L^2(w)}$$

As

$$\|\partial_1(v - v_h).(a_1 - a_1^h)\|_{L^2(w)} \leq \|a_1 - a_1^h\|_{L^\infty(w)}\|\partial_1(v - v_h)\|_{L^2(w)}$$

$$\|\partial_1(v-v_h).a_1 - \partial_1(v-v_h).a_1\| \le \|a_1 - a_1^h\|_{L^\infty(w)}\|\partial_1(v-v_h)\|_{L^2(w)}$$

the second term is estimated in the same manner ,which leads to

$$|\mathcal{A}_h(u_h, v-v_h)| \lesssim \varepsilon_h^c\|L\|\|V\|_X$$

the last $\rho(b-b_h)$ requires a more specifie attention firste it is split up as follows

$$\rho(b-b_h)(U_h, V-V_h) = \rho\int_\omega Q(U_h)Q(V-V_h) - Q^h(U_h)Q^h(V-V_h)dx$$

$$=\rho\int_\omega Q(U_h)(Q(V-V_h) - Q^h(V-V_h))dx + \rho\int_\omega Q^h(V-V_h)(Q(U_h) - Q^h(U_h))dx$$

hence using cauchy-schwarz's inequality and the property

$$Q(u,r) - Q^h(u,r) = -\frac{1}{2}(\partial_1 u(a_2 - a_2^h) - \partial_2 u(a_1 - a_1^h))$$

$$\rho|(b-b_h)(U_h, V-V_h)| \lesssim \rho \sup_{K=1,2,3}\|a_i - a_i^h\|_{L^\infty(\omega)}\|U_h\|_{\mathbb{X}}\|V-V_h\|_{\mathbb{X}}$$

using the bound($\|U_h\|_X \lesssim \|L\|$)and the estimate (3,2) we find

$$\rho|(b-b_h)(U_h, V-V_h)| \lesssim \rho \sup_{K=1,2,3}\|a_i - a_i^h\|_{L^\infty(w)}\|f\|_{\mathbb{X}}\|V\|_{\mathbb{X}}$$

The previous estimates yield the conclusion

■ Now we need to estimate the term

$$L_h(V-V_h) - \mathsf{A}_h(U_h, V-V_h) - \mathsf{A}_p^h(U_h, V-V_h) - \rho b_h(U_h, V-V_h) - \tilde{b}(V_h, \psi-\psi_h)$$

in order to define appropriately the indicators,we introduce

$$T_h(u) = e\mathbb{A}\gamma^h(u),$$

$$A_h = (a_1^h, a_2^h)^\top,$$

$$S_h(u,r) = e\mu((\nabla u)^\top a_3 + J\hat{r}) \ ,$$

$$M_h(r) = \tfrac{e^3}{24}\mathbb{A}\Pi^h(r),$$

$$P_h(r) = \tfrac{e^3}{12}\mathbb{A}II_0^h r_3,$$

$$K_h(r) = \tfrac{e^3}{12}(II_0^h : \mathbb{A}\theta^h(r))$$

Now for all$T \in \mathcal{T}_h$, we can define the following indicators (compare with problem (2,14))

$$\eta_T^{(1)} = h_T\|f_h + \mathrm{Div}(T_h(u_h)A_h) + \mathrm{Div}(S_h(U_h)a_3^h) - \frac{1}{2}\rho\,\mathrm{Div}(Q^h(U_h)JA_h)\|_{L^2_{(T,\mathbb{R}^3)}}$$
$$+ \sum_{e\in\varepsilon_h^i\bigcap\partial T} \frac{1}{2}h_e^{\frac{1}{2}}\|[nT_h(u_h)A_h + nS_h(U_h)a_3^h - \frac{1}{2}\rho Q(U_h)A^\top Jn^\top]_e\|_{L^2_{e,\mathbb{R}^3}}$$
$$+ \sum_{e\in\varepsilon_h^b\bigcap\bar{\Gamma}_1\bigcap\partial T} h_e^{\frac{1}{2}}\|nT_h(u_h)A_h + nS_h(U_h)a_3^h - \frac{1}{2}\rho Q(U_h)A^\top Jn^\top]_e\|_{L^2_{e,\mathbb{R}^3}}$$

$$\eta_T^{(2)} = h_T\|\frac{1}{2}\tilde{J}\,\mathrm{Div}(P_h(r_h)) - \frac{1}{2}\begin{pmatrix}P_h(u_h) : \tilde{\Gamma}_h^1\\ P_h(u_h) : \tilde{\Gamma}_h^2\end{pmatrix} + J^\top Div(M(r)) + \begin{pmatrix}M_h(r_h) : \bar{\Gamma}_h^1\\ M_h(r_h) : \bar{\Gamma}_h^2\end{pmatrix}$$
$$-J^\top S_h(U_h)\|_{L^2(T)^2} + \sum_{e\in\varepsilon_h^i\bigcap\partial T} h_e^{\frac{1}{2}}\|[\frac{1}{2}\tilde{J}P_h(r_h)n^\top - J^\top M_h(r_h)n^\top]_e\|_{L^2(e)^2}$$
$$+ \sum_{e\in\varepsilon_h^b\bigcap\bar{\Gamma}_1\bigcap\partial T} h_e^{\frac{1}{2}}\|\frac{1}{2}\tilde{J}P_h(r_h)n^\top + J^\top M_h(r_h)n^\top\|_{L^2(e)^2}$$
$$\eta_T^{(3)} = \| - \rho Q^h(U_h) - \frac{1}{2}\tilde{B}_h : P_h(r_h) - K_h(r_h) - \bar{B}_h : M_h(r_h)\|_{L^2(T)}$$
$$\eta_T^4 = h_T\|r_h.a_3\|_{L^2_{(T,\mathbb{R}^3)}}$$

Where ε_h^b the set of adges of the triangulation included into the boundary of ω ,while $\varepsilon_h^i = \varepsilon_h \setminus \varepsilon_h^b$ we forther introduce local indicator

$$\eta_T = \eta_T^{(1)} + \eta_T^{(2)} + \eta_T^{(3)} + \eta_T^{(4)}$$

and the global one

$$\eta_h = (\sum_{T\in\mathcal{T}_h}\eta_T^2)^{\frac{1}{2}}$$

Proposition 7 *Let $V = (v, \sum_i s_i a_i) \in \mathbb{X}$ and let $V_h = (C_h v, \sum_i (C_h s_i) a_i)$ be the clement interpolant of V then*

$$|\mathsf{A}_h(U_h, V - V_h) + \mathsf{A}_p^h(U_h, V - V_h) + \rho b_h(U_h, V - V_h) - L_h(V - V_h) + \tilde{b}(V_h, \psi - \psi_h)| \lesssim \eta_h \|V\|_{\mathbb{X}}$$
$$|\tilde{b}(U_h, \phi - \phi_h)| \lesssim \eta_h$$

(3.4)

Proof.

$$L_h(V - V_h) - \mathsf{A}_h(U_h, V - V_h) - \mathsf{A}_h^p(U_h, V - V_h) - \rho b_h(U_h, V - V_h) - \tilde{b}(V_h, \psi - \psi_h)$$

$$= \mathrm{A}_1(U_h, V - V_h) + A_2(U_h, V - V_h) + A_3(U_h, V - V_h)\tilde{b}(U_h, \phi_{\phi h}) = A_4(U_h, \phi - \phi_h)$$

where

$$A_1(U_h, V - V_h) = L_h(v - C_h v) - a_p(U_h, (v - C_h v, 0)) - \rho b_h(U_h, (v - C_h v, 0))$$
$$-\tilde{b}((C_h v, \textstyle\sum_\alpha (C_h s_\alpha) a_\alpha), \psi - \psi_h)$$

$$A_2(U_h, V - V_h) = -\mathsf{A}_h(U_h, (0, \sum_\alpha (s_\alpha - C_h s_\alpha) a_\alpha)) - \mathsf{A}_p^h(U_h, (0, \sum_\alpha (s_\alpha - C_h s_\alpha) a_\alpha))$$
$$-\rho b(U_h, (0, \textstyle\sum_\alpha (s_\alpha - C_h s_\alpha) a_\alpha)) - \tilde{b}((C_h v, \textstyle\sum_\alpha (C_h s_\alpha) a_\alpha), \psi - \psi_h)$$

$$A_3(U_h, V - V_h) = -\mathsf{A}_h(U_h, (0, (s_3 - C_h s_3) a_3)) - \mathsf{A}_p^h(U_h, (0, (s_3 - C_h s_3) a_3))$$
$$-\rho b_h(U_h, (0, (s_3 - C_h s_3) a_3)) - \tilde{b}((C_h v, (C_h s_3) a_3), \psi - \psi_h)$$

$$A_4(U_h, \phi - \phi_h) = \int_\omega (r_h.a_3) \phi_h dx$$

We have by green's formula

$$A_1(U_h, V - V_h) = \sum_{T \in \mathcal{T}_h} \int_T (f_h + \mathrm{Div}(T(U_h) A_h) + \mathrm{Div}(S_h(U_h) a_3^h) - \frac{1}{2} \rho \, \mathrm{Div}(Q^h(U_h) J^\top A_h).(v - C_h v) dx$$

$+\sum_{T\in\mathcal{T}_h}+\sum_{e\in\tilde{\Gamma}_1\partial T}\int_e(\frac{1}{2}\rho Q^h(U_h)A_h^T Jn^\top - nT_h(U_h)A_h - nS_h(U_h)a_3^h).(v-C_hv)d\sigma(x)$(3.5)

Cauchy-Schwarz' inequality and the properties of the Clément interpolant C_h yield

$$|A_1(U_h,V-V_h)|\lesssim(\sum_{T\in\mathcal{T}_h}(\eta_T^{(1)})^2)^{\frac{1}{2}}\|V\|_{\mathbb{X}}$$

In a fully similar manner, we have

$$|A_2(U_h,V-V_h)|\lesssim(\sum_{T\in\mathcal{T}_h}(\eta_T^{(2)})^2)^{\frac{1}{2}}\|V\|_{\mathbb{X}}$$

we directly check that

$$A_3(U_h,V-V_h)=\sum_T\int_T(\bar{B}_h:M_h(r_h)-K(r_h)-\frac{1}{2}\tilde{B}_h:P_h(r_h)-\rho Q_h(U_h))(s_3-C_hs_3)dx \tag{3.6}$$

we get

$$|A_3(U_h,V-V_h)|\lesssim(\sum_{T\in\mathcal{T}_h}(\eta_T^{(3)})^2)^{\frac{1}{2}}\|V\|_{\mathbb{X}}$$

finally we get

$$|A_4(U_h,V-V_h)|\lesssim(\sum_{T\in\mathcal{T}_h}(\eta_T^{(4)})^2)^{\frac{1}{2}}\|V\|_{\mathbb{X}}$$

then ,combining

$$|(\tilde{b}-\tilde{b}_h)(V_h,\psi-\psi_h)|\lesssim\sup_{K=1,2,3}\|a_i-a_i^h\|_{L^\infty(\omega)}\|V_h\|_{\mathbb{X}}\|\psi-\psi_h\|_{\mathbb{X}}$$

and the above inegualities leads to the global upper bound (3.7) [5].

The estimates on $|A_i(U_h,V-V_h)|$ directly yield the conclusion ■

3.2 Upper and lower bounds

Athis point we can demonstrate the following

Theorem 8 *the solution U of problem (1,24)and the solution $U_h$of problem (2,2) satisfy the following aposteriori error estimate*

$$\|U - U_h\|_{\mathbb{X}} + \|\psi - \psi_h\|_{\mathbb{X}} \lesssim \eta_h + \varepsilon_h^d + \varepsilon_h^c \tag{3.7}$$

Proof. The estimate (3,6) can be derived from the coercivity of the founctional$\mathsf{A}(.,.) + \mathsf{A}_p(.,.) + \rho b(.,.) + \tilde{b}(.,.)$ which has acoercivity constant equivalent to 1,this result is obtained by applying the identity (3,1),lemma(8)and proposition(7)

■ Let us go with the lower bond

Theorem 9 *Let U repressent the solution toproblem (1,24) and $U_h$repressent the solution to problem (2,2)consequently we can establish the following constraint*

$$\eta_T^{(i)} \leq \rho\|U - U_h\|_{\mathbb{X}(\omega_T)} + \|\psi - \psi_h\|_{\mathbb{X}(\omega_T)} + \varepsilon_{\omega_T}^d + \varepsilon_{\omega_T}^c \; i = 1,2,3,4 \tag{3.8}$$

The subscript ω_T in the index indicates that the quantity is considered exclusively within the domainω_T, while the norm $\mathbb{X}(\omega_T)$ refers to the norm of$\mathbb{X}$ calculated using integrals limited toω_T.

Proof. We will focus on proving inequality (3.8) for $\eta_T^{(1)}$, as the proof for the case of$\eta_T^{(2)}$ and $\eta_T^{(3)}$ and $\eta_T^{(4)}$ is essentially the same. To simplify the notation,we express $\eta_T^{(1)}$ in a compact form as follows

$$\eta_T^{(1)} = h_T\|F_h\|_{L^2(T,\mathbb{R}^3)} + \sum_{e\in\varepsilon_h^i \bigcap \partial T} + h^{\frac{1}{2}}_{L^2(e,\mathbb{R})} + \|[G_h]_e\|_e + \sum_{e\in\varepsilon_h^b \bigcap \partial T} h_e^{\frac{1}{2}}\|G_h\|_{L^2(2,\mathbb{R}^3)}$$

First of all, let us fix the standard bubble function Ψ_T associated with$T$ and set

$$v = \begin{cases} F_h\Psi_T \text{in} T \\ 0 \text{in} \omega\backslash T \end{cases}$$

According to the definition of Ψ_T we can observe that $v \in H_0^1(\omega, \mathbb{R}^3)$ thus implying that $(v,0)$ belongs to $\mathbb{X}$ this can be deduced from equtions (3,4)with $V_h = 0$ that

$$L_h(v,0) - \mathsf{A}_h(U_h,(v,0)) - \rho b_h(U_h,(v,0)) - \tilde{b}((v,0),\psi_h)$$

$$= \int_T (f_h + \mathrm{Div}(T_h(U_h)A_h) + \mathrm{Div}(S_h(U_h)a_3^h) - \frac{1}{2}\rho\,\mathrm{Div}(Q^h(U_h)JA_h)).vdx$$

$$= \|F_h\psi_T^{\frac{1}{2}}\|$$

Using the identity (3,1) we may write

$$\mathsf{A}(U - U_h,(v,0)) + \rho b(U - U_h,(v,0)) + \tilde{b}((v,0),\psi_h)$$

$$=(\mathsf{L}\text{-}\mathrm{L}_h)((v,0)) - (\mathsf{A} - \mathsf{A}_h)(U_h,(v,0)) - \rho(b - b_h)(U_h,(v,0)) - (\tilde{b} - \tilde{b}_h)((v,0),\psi_h) -$$

$$\text{-}\mathrm{A}_p(U_h,(v,0)) - \rho b_h(U_h,(v,0)) - \tilde{b}_h((v,0),\psi_h) + L_h((v,0))$$

Hence

$$L_h(v,0) - \mathsf{A}_h(U_h,(v,0)) - \rho b_h(U_h,(v,0)) - \tilde{b}_h((v,0),\psi_h)$$

$$=\mathrm{A}(\mathrm{u}\text{-}\mathrm{u}_h,(v,0)) + \rho b(U - U_h,(v,0)) - (L - L_h)((v,0)) - (\mathsf{A} - \mathsf{A}_h)(U_h,(v,0))$$

$$\text{-}\rho(b - b_h)(U_h,(v,0)) - (\tilde{b} - \tilde{b}_h)((v,0),\psi_h)$$

By the previous identities we get

$$\|F_h\Psi_T^{\frac{1}{2}}\|^2_{L^2(T)^3} = a(U - U_h,(v,0)) - +\rho b(U - U_h,(v,0)) - (L - L_h)(v,0)$$

$$+(\mathsf{A} - \mathsf{A}_h)(U_h,(v,0)) - \rho(b - b_h)(U_h,(v,0)) - (\tilde{b} - \tilde{b}_h)((v,0),\psi_h)$$

Applying the cauchy-schwarz inequality and leveraging the arguments of lemma (8)we can deduce that

$$\|F_h\Psi_T^{\frac{1}{2}}\|^2_{L^2(T,\mathbb{R}^3)} \lesssim (\rho\|U - U_h\|_{\mathbb{X}(T)} + \|\psi - \psi_h\|_{\mathbb{X}(T)} + \varepsilon_T^d + \varepsilon_c^h)\|v\|_{H^1(T,\mathbb{R}^3)}, \tag{3.9}$$

Utilizing the inverse inequality provided, we can conclude that

$$\|v\|_{H^1(T,\mathbb{R}^3)} \lesssim h_T^{-1}\|v\|_{L^2(T,\mathbb{R}^3)}, \tag{3.10}$$

By utilizing the fact that the function ψ_T takes values between 0 and 1, we can deduce that

$$\|v\|_{H^1(T,\mathbb{R}^3)} \lesssim h_T^{-1}\|F_h\|_{L^2,(T,\mathbb{R}^3)}, \tag{3.11}$$

Additionally ,we have

$$\|F_h\|_{L^2(T,\mathbb{R}^3)} \leq C\|F_h\Psi_T^{\frac{1}{2}}\|^2_{L^2(T,\mathbb{R}^3)}, \tag{3.12}$$

combining (3,9)(3,11)and (3,12)we get

$$h_T\|F_T\|_{L^2(T,\mathbb{R}^3)} \lesssim \rho\|U-U_h\|_{\mathbb{X}(T)} + \|\psi-\psi_h\|_{\mathbb{X}(T)} + \varepsilon_T^d + \varepsilon_T^c$$

The second step involves bounding the second term of $\eta_T^{(1)}$for all edges $e$of$T$with the element T'in this case we select fonction v in equation (3,5)as follows

$$v = \begin{cases} \mathcal{M}_{e,k}([G_h]_e)\Psi_e fork = [T,T'] \\ 0\ in\omega\backslash(T\bigcup T') \end{cases} \tag{3.13}$$

where Ψ_e is the standard edge bubble function associated with e and $\mathcal{M}_{e,k(q)}$ is an extension operator that maps apolynomial $q$in the edge coordinate of e to a polynomial in cartesian coordinates in k As before we see that

$$\|[G_h]_e\|_0\Psi_e\|^2_{L^2(e,\mathbb{R}^3)} = \mathsf{A}_h(U_h,(v,0,0,0)) + \rho b_h(U_h,(v,0,0,0)) - L_h(v,0,0,0) + \tilde{b}_h((v,0,0,0),\Psi_h)$$

$$+\int_{\Delta(e)}((f_h + \mathrm{Div}(T_h(U_h)A_h) + \mathrm{Div}(S_h(U_h)a_3^h) - \tfrac{1}{2}\rho\,\mathrm{Div}(Q^h(U_h)JA_h)).vdx)$$

By employing the identity (3.1) and leveraging the arguments presented in Lemma 5, we can conclude that

$$\|[G_h]_e\|_0\Psi_e\|^2_{L^2(e,\mathbb{R}^3)} \leq \rho\|U-U_h\|_{\mathbb{X}(\Delta(e))} + \|v\|_{\mathbb{X}(\Delta(e))} + (\varepsilon^d_{\Delta(e)} + \varepsilon^c_h)\|v\|_{L^2(e,\mathbb{R}^3)} + \|F_h\|_{L^2(\Delta(e),\mathbb{R}^3)}\|v\|_{\mathbb{X}(\Delta(e))}$$

using a standard inverse inequality we can reach the conclusion that

$$h_e^{\frac{1}{2}}\|[G_h]_e\Psi_e\|^2_{L^2(e,\mathbb{R}^3)} \lesssim \rho\sum_{k\in\{T,T'\}}\|U-U_h\|_{\mathbb{X}(\Delta(e))} + \|\psi-\psi_h\|_{\mathbb{X}(\Delta(e))} + \varepsilon^d_{\Delta(e)} + \varepsilon^c_h$$

the third and fourth term is bounded in a similar manner to the second term ,similarly we bound the remaining $\eta^{(i)}, i = 2, 3, 4$

Hence the proof is now complete.

■

Chapter 4

Conclusion

In conclusion, it can be inferred that utilizing the mixed formulation and a posteriori error analysis represents a strong and effective approach for modeling prestressed shell structures. The mixed formulation allows for an accurate estimation of the mixed variables and other variables in the model, contributing to improved accuracy and numerical efficiency in the analysis.

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