KASDI MERBAH UNIVERSITY OUARGLA

Faculty of Mathematics and Sciences Material

N° d'ordre :
N° de série :

Department of : Mathematics

Master 2^{rd} cycle LMD

Path: Mathematics

Specialty: Functional and numerical analysis

Present by:WIAM HAMADA

Theme:

Existence and uniqueness results for discrete fractional differential equations

Represented in : 15/06/2023

Limb from jury:

Dr. Mzabia Mohamed El Hadi	MCA	Kasdi Merbah University-Ouargla	Chairman
Dr. Amara Abdelkader	MCA	Kasdi Merbah University-Ouargla	Supervisor
PhD. Alili Fayçal.	PhD	Kasdi Merbah University-Ouargla	Co-Supervisor
Dr. Tellab Brahim	MCA	Kasdi Merbah University-Ouargla	Examiner

Dedicate

To All Who Are Humble In Seeking Science.

With respect, Wiam

Acknowledgements

First of all, I would like to thank **Allah** *for all his blessings and favours.*

Then, I would like to thank **my family** *for the needed support they provided during my thesis journey, especially* **my fiance** *for being the source of my resolve.*

I would like to express my sincere gratitude to my supervisor **Dr.Amara Abdelkader** *for suggesting this interesting subject, as well as my Co-supervisor* ***PhD. Alili Fayçal*** *for guiding me through the process of writing my thesis, and equipping me with the necessary research skills.*

I am also grateful to **Dr. Brahim Tellab**, **Dr. Mezabia Mohammed El Hadi**,*for honouring us with their presence in the examination committee, and accepting to evaluate my thesis, without forgetting all of my teachers and lecturers who have helped me during my years in college.*

الملخص

في هذه العمل، سوف نهتم بدراسة نتائج الوجود و الوحدانية لبعض المعادلات التفاضلية الكسرية المتقطعة مع شروط حدية في فضاء بناخ، لهذا، فان التقنية المستخدمة هي تحويل دراسة مسألتنا إلى البحث عن نقطة ثابتة لمعادلات تكاملية .النتائج التي تم الحصول عليها تستند إلى نظرية النقطة الثابتة لبناخ " التقليص"

الكلمات المفتاحية: المعادلات التفاضلية الكسرية المتقطعة ، المعادلات التكاملية ، النقطة الثابتة لبناخ

Résumé

Dans ce travail, nous nous intéressons à l'étude des résultats sur l'existence et l'unicité de certaines équations différentielles fractionnaires discrète avec conditions aux limites dans les espaces de Banach. Pour cela, la technique utilisée consiste à transformer notre problème en recherche d'un point fixe pour les équations intégrales. Les résultats obtenus sont liés au théorème du point fixe de Banach

Mots et expressions clés : équations différentielles fractionnaires discrète, équations intégrales, point fixe.

Abstract

In this work, we are interested in studying results on existence and uniqueness of some discrete fractional differential equations with boundary conditions in Banach space. For this, the technique used is to transform our problem in search of a fixed point for integral equations. The results obtained are related to Banach's fixed-point theory,

Key words and phrases: Discrete fractional differential equations,, integral equations, fixed point..

Contents

Introduction

Fractional difference calculus is a tool used to explain many phenomena in physics, control problems, modelling, chaotic dynamical systems, and various fields of engineering and applied mathematics. In this direction, different kinds of methods and techniques, including numerical and analytical methods, have been utilized by researchers to discuss given fractional discrete and continuous mathematical models and boundary value problems (BVPs) [1, 2, 3, 4]. For some recent developments on the existence, uniqueness, and stability of solutions for fractional differential equations, see, for example, [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23] and the references therein.

In this work, we discuss existence and uniqueness of solutions to the discrete fractional equation that involves Caputo discrete derivatives with boundary condition, these results are determined, by applying Banach?s fixed point theory.

This work is structured as follows. The first chapter contains some basic concepts in addition to the notions of the topology discrete that play an important role in the difference fractional calculus

The second chapter is devoted to concepts and characteristics of integrals and derivatives related to the two most important approaches to fractional computation, the Riemann-Liouville approach

In the final chapter, we will study existence and uniqueness of solutions to the following discrete fractional equation that involves Caputo discrete derivative:

$$\begin{cases} {}^{c}\Delta_{\xi}^{\varrho}\chi(\xi) = \Phi(\xi+\varrho-1, \chi(\xi+\varrho-1)), \quad 2 < \varrho \leq 3, \\ \Delta\chi(\varrho-3) = 0, \chi(\varrho+T) = 0, \Delta^{2}\chi(\varrho-3) = 0, \end{cases} \tag{0.0.1}$$

for $\xi \in [0, T]_{\mathbb{N}_0} = [0, 1, 2, \ldots, T]$, $T \in \mathbb{N}$, ${}^{c}\Delta_{\xi}^{\varrho}$ is the Caputo difference operator and $\Phi : [\varrho - 3, \varrho + T]_{\mathbb{N}_{\varrho-3}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

Chapter 1

Topologie discrete

1.0.1 Topological space

This chapter will explore the concept of topologies and continuous functions, and We will focus on discrete topology, which is a particular case of a topology useful in many mathematics areas.

Definition 1.1 *Let X be a set. A topology $\mathcal{T}$ on X is collection of subsets of X, each called a open set such that*

1- $\emptyset$ and X are open sets.

2- The intersection of finitely many open sets is an open set.

3-The union of any collection of open sets is an open set.

The set X together with a topology $\mathcal{T}$ on X is called a topological space.

Definition 1.2 *If X is a topological space and $E \subset X$, we say E is* ***closed*** *iff $X - E$ is open.*

Definition 1.3 *If X is a topological space and $x \in X$, a* ***neighbourhood*** *of x is a set U which contains an open set V containing x.*

Thus, evidently, U is a neighbourhood of x iff $x \in U^\circ$. The collection $\mathcal{U}_x$ of all neighbourhoods of x is the neighbourhood system at x.

Theorem 1.4 *The neighbourhood system $\mathcal{U}_x$ at x in a topological space X has the following properties:*

1- If $U \in \mathcal{U}_x$, then $x \in U$.

2- If $U, V \in \mathcal{U}_x$, then $U \cap V \in \mathcal{U}_x$.

3- If $U \in \mathcal{U}_x$, then there is a $V \in \mathcal{U}_x$, such that $U \in \mathcal{U}_y$, for each $y \in V$.

4-If $U \in \mathcal{U}_x$ and $U \subset V$, then $V \in \mathcal{U}_x$.

4- $G \subset X$ is open iff G contains a neighbourhood of each of its points.

1.0.2 Continuous functions:

We provide a general definition of continuity for functions that map from one topological space to another. This topological definition of continuity is very simple to state and, as we will show, is equivalent to the $\epsilon - \delta$ definition for functions that map $\mathbb{R}$ to $\mathbb{R}$.

Definition 1.5 *Let X and Y be topological spaces. A function $f : X \to Y$ is* ***continuous*** *if $f^{-1}(V)$ is open in X for every open set V in Y.*

We call this the ***open set definition of continuity****. Paraphrased, it states that f is continuous if the preimage of every open set is open.*

Theorem 1.6 *Let X and Y be topological spaces and $\mathcal{B}$ be a basis for the topology on Y. Then $f : X \to Y$ is continuous if and only if $f^{-1}(B)$ is open in X for every $B \in \mathcal{B}$.*
Proof. *Suppose $f : X \to Y$ is continuous. Then $f^{-1}(V)$ is open in X for every V open in Y. Since every basis element B is open in Y, it follows that $f^{-1}(B)$ is open in X for all $B \in \mathcal{B}$.*

Now, suupose $f^{-1}(B)$ *is open in* x *for every* $B \in \mathcal{B}$. *We show that* f *is continuous. Let* V *be an open set in* Y. *Then* V *is a union of basis elements, say* $V = \cup B_\alpha$. *Thus,*

$$f^{-1}(V) = f^{-1}(\cup B_\alpha) = \cup f^{-1}(B_\alpha)$$

By assumption, each set $f^{-1}(B_\alpha)$ *is open in* X; *therefore so is their union. Thus,* $f^{-1}(V)$ *is open in* x, *and it follows that the preimage of every open set in* Y *is open in* X. *Hence,* f *is continuous.* ∎

Example 1.7 *Let* X *be a non-empty set and let* $\mathcal{T}$ *be the collection of all subsets of* X. *Clearly this is a topology, since unions and intersections of subsets of* x *are themselves subsets of* X *and therefore are in the collection* $\mathcal{T}$.
We call this the ***discrete topology*** *on* X. *This is the largest topology that we can define on* X.

1.0.3 Metric space

Definition 1.8 *Let* X *be a set and* $d : X \times X \to \mathbb{R}$ *be a function such that*

1- $d(x, y) \geq 0$ *for all* x, y *in* X.

2- $d(x, y) = 0$ *if and only if* $x = y$.

3- $d(x, y) = d(y, x)$.

4-$d(x, z) = d(x, y) + d(y, z)$ *(triangle inequality).*

Then the pair (X, d) *is called a metric space. The function* d *is called the metric or sometimes the distance function.*

Definition 1.9 *Let* a *be a point in a metric space* (X, d), *and assume that* r *is a positive real number. The open ball centred at* a *with radius* r *is the set*

$$B(a; r) = \{x \in X : d(x, a) < r\}$$

The closed ball centred at a with radius r is the set

$$\bar{B}(a;r) = \{x \in X : d(x,a) \le r\}$$

If A is a subset of X and x is a point in X, there are three possibilities:

1- There is a ball $B(x;r)$ around x which is contained in A. In this case x is called an **interior** point of A.

2- Ther is a ball $B(x;r)$ around x which is contained in the complement A^c. In this case x is called an **exterior** point of A.

3- All balls $B(x;r)$ around x contains points in A as well as points in the complements A^c. In this case x is a **boundary** point of A.

Proposition 1.10 *A subset A of a metric space X is open if and only if its complement A^c is closed.*
Proof. *If A is open, it does not contain any of the boundary points. Hence they all belong to A^c, and A^c must be closed.*
Conversely, if A^c is closed, it contains all boundary points, and hence A can not have any. This means that A is open. ■

Definition 1.11 *Let E be a metric space, A a subset of E. One says that* ***A is open*** *if, for each $x_0 \in A$, there exists an $\epsilon > 0$ such that every point x of E satisfying $d(x_0, x) < \epsilon$ belongs to A.*

Theorem 1.12 *Let E be a metric space.*

1- The subsets ϕ and E of E are open.

2- Every union of open subsets of E is open.

3- Every finite intersection of open subsets of E is open.

Definition 1.13 *Let (X, d) be a metric space. Take $x \in X$ and $r > 0$. We define the* ***open ball*** *(or simply* ***ball****) of radius r centred at x to be the set*

$$B_r(x) = \{y \in X \quad | \quad d(x, y) < r\}.$$

A ball centred at $x \in X$ is said to be the ***unit ball*** *centred at x if $r = 1$.*
A set $N(x)$ is called a ***neighbourhood*** *of $x \in X$ if there exists an $r > 0$ such that $B_r(x) \subseteq N(x)$.*
This seems fairly straight-forward. The open ball is just the set of all points in our space within the specified distance r.

Example 1.14 *Let $X = \mathbb{R}$ be the set of real numbers equipped with usual metric, i.e. $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+ \cup \{0\}$ is defined by $u(x, y) = |x - y| \quad \forall x, y \in \mathbb{R}$.*
Let's take $x = 0$ and $r = 1$, then by definition

$$\begin{aligned} b\,[0; 1) &= \{y \in \mathbb{R} : u\,(0, y) < 1\} \\ &= \{y \in \mathbb{R} : |0 - y| < 1\} \\ &= \{y \in \mathbb{R} : |y| < 1\} \\ &= \{y \in \mathbb{R} : -1 < y < 1\} \\ &= (-1, 1) \end{aligned}$$

So, open ball with centre 0 and radius 1 is an open interval $(-1, 1)$ in $\mathbb{R}$

Theorem 1.15 *Every neighbourhood is an open set.*
That is, for any metric space X, any $p \in X$, and any $r > 0$, the set $N_r(p)$ ***(def:*** *$N_r(p) := q \in X : d(p, q) < r$****)****, could fail to be open as a subset of X.*
Proof. *We must show that for any $q \in N_r(p)$ there is an $h > 0$ such that $N_h(q) \subseteq N_r(p)$. We claim that $h = r - d(p, q)$ works.Indeed, h is positive by the definition of $N_r(p)$; and for any $s \in N_h(q)$ we have $s \in N_r(p)$ because*

$$d(p, s) \leq d(p, q) + d(q, s) < (r - h) + h = r$$

So $N_h(q)$ is a subset of $N_r(p)$ as desired. ■

Example 1.16 *1-* ***The set of natural numbers***
consider the set of natural numbers, denoted as $\mathbb{N}$. We can define a discrete metric on this

set as follows:

- For any two distinct natural numbers m *and* n*, the distance between them is* $d(m,n) = 1$
The distance between a number and itself is zero,i.e., $d(m,m) = 0$.

2- ***The set of integers:***

Similar to the previous example, we can define a discrete metric on the set of integers, denoted as $\mathbb{Z}$*:*

- For any two distinct integers m *and* n*, the distance between them is* $d(m,n) = 1$.

- The distance between an integer and itself is zero,i.e, $d(m,m) = 0$.

*3-**Finite sets:***

Any finite set can be equipped with a discrete metric.For examples, consider the set $\mathcal{S} = \{a, b, c\}$*. We can define the following metric:*

- For any two distinct elements X *and* Y *in* $\int$*, the distance between them is* $d(x,y) = 1$.

- The distance between an element and itself is zero; $d(x,x) = 0$.

4- ***Subsets of given set:***

Let X *be a set, and consider the power set of* X*, denoted as* $P(X)$*, which is the set of all subsets of* X*. We can define a discrete metric on* $P(X)$ *as follows:*

- For any two distinct subsets A *and* B *in* $P(X)$*, the distance between them is* $d(A,B) = 1$.

- The distance between a subset and itself is zero,i.e., $d(A,A) = 0$.

These are just a few examples, but in general, any set can be turned into a discrete metric space by defining the distance between distinct elements to be 1 and the distance between an element and itself to be 0.

1.0.4 Topology discrete

The discrete topology is one of the simplest and most important examples of a topology. It is defined as follows:

Definition 1.17 *Let* X *be a set. The discrete topology on* X *is the topology* $\mathcal{T}_{disc}$ *consisting* *of all subsets of X.*

Example 1.18 *Consider the set $X = \{1, 2, 3\}$. The discrete topology on X is given by:*

$$\mathcal{T}_{discrete} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

In other words, every subset of X is considered an open set in the discrete topology. This means that any subset of X can be a neighbourhood of any of its points. Intuitively, this topology is as "fine" or "precise" as possible, because it allows us to distinguish between all possible subsets of X. Another important feature of the discrete topology is that it is always metrizable. In other words, there always exists a metric on the space that induces the discrete topology. To see why, consider the following metric:

Definition 1.19 *Let X be a set. The discrete metric on X is the function $d : X \times X \to \mathbb{R}$ defined by*

$$\begin{cases} 1, & \text{if } x \neq y, \\ 0, & \text{if } x = y. \end{cases} \tag{1.0.1}$$

It is easy to see that this metric induces the discrete topology, because every singleton set is an open ball in this metric. This means that the discrete topology is not only a useful tool in its own right, but also a building block for constructing more complex topologies. In summary, the discrete topology is a simple but powerful tool that allows us to consider all possible subsets of a space as open sets. This topology is always metrizable, and can be a useful starting point for constructing more complex topologies.

The discrete metric space is often denoted as (X, d_{disc}) or simply X when the metric is clear from the context. It is the most basic example of a metric space and has several important properties:

1. Every subset of a discrete metric space is an open set.
2. Every function defined on a discrete metric space is continuous.
3. The discrete metric space is complete, which means that every Cauchy sequence in the space converges.

The discrete metric space is commonly used as a tool in analysis and topology to provide simple counter examples and illustrate concepts. It is also foundational in studying discrete mathematics and combinatorics.

1.0.5 Functions on the discrete topology

A function $f : X \to Y$ is continuous on the discrete topology if and only if it maps every open set in X to an open set in Y. Since every subset of X is open in the discrete topology, this condition is always satisfied, and every function on the discrete topology is continuous.

Example 1.20 *Consider the set* $X = \{a, b, c\}$ *with the discrete topology. Define a function* $f : X \to \mathbb{R}$ *by* $f(a) = 1$, $f(b) = 2$, *and* $f(c) = 3$. *Since every subset of* X *is open,* f *is continuous on the discrete topology.*

Example 1.21 *Consider any set* X *with the discrete topology. The identity function* $id_X : X \to X$ *is always continuous, since every subset of* X *is open.*

Properties of functions on the discrete Topology:

Functions on the discrete topology have several important properties:

First, every function on the discrete topology is continuous, as we saw in the previous section.

Second, every function on the discrete topology is uniformly continuous, meaning that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever x and y are within δ of each other.
This is because the discrete topology has no limit points, so there is no need to worry about continuity at a point where the function is *jumping*.

Chapter 2

Difference calculus

2.1 Delta calculus:

2.1.1 Two important set:

1. For $a \in \mathbb{R}$, we define $\mathbb{N}_a := \{a, a+1, a+2, ...\}$.

2. For $a, b \in \mathbb{R}$ with $a - b \in \mathbb{Z}^+$, we define $\mathbb{N}_a^b := \{a, a+1, a+2, ...\}$.

Definition 2.1 *Let $f : \mathbb{N}_a^b \to \mathbb{R}$. If $b - 1 \geq a$, then for $t \in \mathbb{N}_a^{b-1}$. We define the forward difference operator Δ by*

$$(\Delta f)(t) := f(t+1) - f(t)$$

Definition 2.2 *For $t \in \mathbb{N}_a^{b-1}$ the forward jump operator σ is defined by $\sigma(t) = t + 1$*

Note:
For $t \in \mathbb{N}_a^{b-1}$, the composition of $f : \mathbb{N}_a^b \to \mathbb{R}$ and σ is given by :

$$f^\sigma(t) = (f \circ \sigma)(t) = f(\sigma(t)) = f(t+1) \Longrightarrow f^\sigma(t) = f(t+1)$$

2.1.2 Properities:

Let $f, g : \mathbb{N}_a^b \to \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}$.Then for $t \in \mathbb{N}_a^{b-1}$

1. $\Delta\alpha = 0$

2. $\Delta\alpha f(t) = \alpha\Delta f(t)$

3. $\Delta[f+g](t) = (\Delta f)(t) + (\Delta g)(t)$

4. $\Delta \alpha^{t+\beta} = (\alpha - 1)\alpha^{t+\beta}$

5. $\Delta[fg](t) = (\Delta f)(t)g(t) + f(\sigma(t))(\Delta g)(t)$

6. $\Delta\big[\frac{f}{g}\big](t) = \dfrac{(\Delta f)(t)g(t) - f(t)(\Delta g)(t)}{g(t)g(\sigma(t))} \qquad (g(t) \neq 0, t \in \mathbb{N}_a^b)$

Note:
If $b - a \geq n$, then for $t \in \mathbb{N}_a^{b-n}$, we define:

$$(\Delta^n f)(t) = \Delta(\Delta^{n-1} f)(t) \qquad n = 1, 2, 3...$$

$$(\Delta^0 f)(t) = f(t)$$

Definition 2.3 *For $n \in \mathbb{N}$, we define the following function $t^{\underline{n}}$ by*

$$t^{\underline{n}} := t(t-1)(t-2)...(t-n+1) = \prod_{j=0}^{n-1}(t-j) = \frac{\Gamma(t+1)}{\Gamma(t-n+1)}$$

we have

- $t^{\underline{0}} := 1$
- $\prod_{j=0}^{n-1}(t-j) = 0 \quad$ if $t - j + 1 = 0 \quad$ for some j.

Note:
$\Delta t^{\underline{n}} = nt^{n-1}$ generalisation to $r \in \mathbb{R}$, we have

$$t^{\underline{r}} := \frac{\Gamma(t+1)}{t-r+1}, \qquad t^{\underline{0}} := 1$$

If $t - r + 1$ is a pole of Gamma function and $t + 1$ is not a pole, then $t^{\underline{r}} = 0$.

2.1.3 Properties:

For every $r, c \in \mathbb{R}$,we have:

1. $t^{\underline{1}} = t$

2. $r^{\underline{r}} = r^{\underline{r-1}} = \Gamma(r+1)$

3. If $t \leq s$ then for each $r > s$,we have $t^{\underline{r}} \leq s^{\underline{r}}$

4. If $0 < c < 1$, then $t^{\underline{rc}} \geq (t^{\underline{r}})^c$

5. $t^{\underline{r+1}} = (t-r)t^{\underline{r}}$, $t^{\underline{r+c}} = (t-c)^{\underline{r}} t^{\underline{c}}$

2.1.4 Generalized power rules

For every $\alpha, r \in \mathbb{R}$,we have:

1. $\Delta(t+\alpha)^{\underline{r}} = r(t+\alpha)^{r-1}$

2. $\Delta(\alpha - t)^{\underline{r}} = -r(\alpha - \sigma(t))^{r-1}$

3. $\Delta(t^{\underline{r}}) = rt^{\underline{r-1}}$

We know that when $n \geq k \geq 0$ are integers, then the binomial coefficient satisfies :

$$\binom{n}{k} = \frac{n!}{(n-k)!k!} = \frac{n(n-1)...(n-k+1)}{k!} = \frac{n^{\underline{k}}}{\Gamma(k+1)}$$

2.1.5 Generalized binomial coefficient

$$\binom{t}{r} := \frac{t^{\underline{r}}}{\Gamma(r+1)}$$

Properties:

1. $\Delta\binom{t}{r} = \binom{t}{r-1}$

2. $\Delta\binom{r+t}{t} = \binom{r+t}{t+1}$

3. $\Delta\Gamma(t) = (t-1)\Gamma(t)$

2.2 Delta integral

Let $f : \mathbb{N}_a \to \mathbb{R}$ and $c, d \in \mathbb{N}_a = \{a, a+1, a+2, ...\}$.
Then

$$\int_c^d f(t)\Delta t = \begin{cases} \sum_{t=c}^{d-1} f(t) & ; c < d \\ 0 & ; c \geq d \end{cases}$$

2.2.1 Properties

Let $f, g : \mathbb{N}_a \to \mathbb{R}$, $b, c \in \mathbb{N}_a$, $b < c < d$ and $\alpha \in \mathbb{R}$, then:

1. $\int_b^c \alpha f(t)\Delta t = \alpha \int_b^c f(t)\Delta t$
2. $\int_b^c (f(t) + g(t))\Delta t = \int_b^c f(t)\Delta t + \int_b^c g(t)\Delta t$
3. $\int_b^b f(t)\Delta t = 0$
4. $\int_b^d f(t)\Delta t = \int_b^c f(t)\Delta t + \int_c^d f(t)\Delta t$
5. $\left|\int_b^c f(t)\Delta t\right| \leq \int_b^c |f(t)|\, \Delta t$

Definition 2.4 *Let $f : \mathbb{N}_a^b \to \mathbb{R}$. We say F is an antidifference of f on $\mathbb{N}_a^b$ provided $F(t) := \int_a^t f(s)\Delta s$*

$$\Delta F(t) = f(t) \qquad t \in \mathbb{N}_a^{b-1}$$

2.3 Discrete delta fractional calculus

2.3.1 Fractional sum

Definition 2.5 *Let $\alpha > 0$ and $f : \mathbb{N}_a \to \mathbb{R}$. Then the fractional sum of the function f (based at a) is defined by $\Delta^{-\alpha} : C(\mathbb{N}_a, \mathbb{R}) \to C(\mathbb{N}_{a+\alpha}, \mathbb{R})$*

$$\begin{aligned} \Delta_a^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \int_a^{t-\alpha+1} (t - \sigma(s))^{\underline{\alpha-1}} f(s)\Delta s \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t - \sigma(s))^{\underline{\alpha-1}} f(s) \end{aligned}$$

For each $t \in \mathbb{N}_{a+\alpha}$ and $\sigma(t) = t + 1$.

- *If $n-1<\alpha\leq n$, then one can extend the domain of $\Delta_a^{-\alpha}f$ on $\mathbb{N}_{a+\alpha-n}$.*

Note:

Let $\alpha = m \in \mathbb{N}$. Then m-th fractional sum of the function $f : \mathbb{N}_0 \to \mathbb{R}$ (based at 0) is defined by

$$\Delta_0^{-m} f = \sum_{s=0}^{t-m} \binom{t-s-1}{m-1} f(s), \qquad t \in \mathbb{N}_m$$

Proposition 2.6 $\Delta_a^{-\alpha} f(t) \mid_{t=a+\alpha} = f(a)$

Proof.

$$\begin{aligned}
\Delta_a^{-\alpha} f(t) \mid_{t=a+\alpha} &= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{a+\alpha-\alpha} (a+\alpha-\sigma(s))^{\alpha-1} f(s) \\
&= \frac{1}{\Gamma(\alpha)} (a+\alpha-\sigma(a))^{\alpha-1} f(a) \\
&= \frac{1}{\Gamma(\alpha)} (a+\alpha-a-1)^{\alpha-1} f(a) \\
&= \frac{1}{\Gamma(\alpha)} (\alpha-1)^{\alpha-1} f(a) \\
&= \frac{\Gamma(\alpha)}{\Gamma(\alpha)} f(a) \\
&= f(a)
\end{aligned}$$

■

Note:

Let $t \in \mathbb{N}_a$.Then

1. $\int \frac{(t-\sigma(s))^{\underline{\alpha}}}{\Gamma(\alpha+1)} \Delta s = -\frac{(t-s)^{\underline{\alpha+1}}}{\Gamma(\alpha+2)} + c$
2. $\Delta_s\left(\frac{(t-s)^{\underline{\alpha}}}{\Gamma(\alpha+1)}\right) = -\frac{(t-\sigma(s))^{\underline{\alpha-1}}}{\Gamma(\alpha)} = -\frac{(t-s-1)^{\underline{\alpha-1}}}{\Gamma(\alpha)}$

Proposition 2.7 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\alpha, \beta > 0$. Then for every $t \in \mathbb{N}_{a+\alpha+\beta}$ we have:*

$$\Delta_{a+\beta}^{-\alpha} \Delta_a^{-\beta} f(t) = \Delta_{a+\alpha}^{-\beta} \Delta_a^{-\alpha} f(t) = \Delta_a^{-(\alpha+\beta)} f(t)$$

If $f:\mathbb{N}_0 \to \mathbb{R}$ and $m,n \in \mathbb{N}$, then for $t \in \mathbb{N}_{m+n}$, we have

$$\Delta_n^{-m}\Delta^{-n}f(t) = \Delta_m^{-n}\Delta^{-m}f(t) = \Delta^{-(n+m)}f(t)$$

Lemma 2.8 *Let $\alpha > 0$ and $\beta \in \mathbb{R} - \{-1,-2,-3,...\}$. Then for every $t \in \mathbb{N}_{a+\alpha+\beta}$, we have*

$$\Delta_{a+\beta}^{-\alpha}(t-a)^{\underline{\beta}} = \beta^{-\underline{\alpha}}(t-a)^{\underline{\alpha+\beta}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1+\alpha)}(t-a)^{\underline{\beta+\alpha}}$$

Note:
Let $\alpha, \beta > 0$ .Then

$$Dom(\Delta_a^{-\alpha}f) = \mathbb{N}_{a+\alpha} \qquad , \qquad Dom(\Delta_{a+\beta}^{-\alpha}\Delta_a^{-\beta}f = \mathbb{N}_{a+\beta+\alpha})$$

Now, we can define the fractional difference in terms of the fractional sum .

2.4 Fractional Difference

2.4.1 Riemann-Liouville fractional difference

Definition 2.9 *Let $f:\mathbb{N}_a \to \mathbb{R}$ and $\alpha > 0$ such that $n-1 < \alpha \le n$. Then the α-th Riemann-Liouville fractional difference of the function f (based at 0) is defined by*

$$\Delta_a^{\alpha}f(t) := \Delta^n\Delta_a^{-(n-\alpha)}f(t) \qquad , \qquad t \in \mathbb{N}_{a+n-\alpha}$$

Note that for any $\alpha = n$, $\quad \Delta_a^{\alpha}f(t) = \Delta^n f(t)$.

Definition 2.10 *Let $f:\mathbb{N}_a \to \mathbb{R}$ and $\alpha > 0$ with $n-1 < \alpha \le n$. Then for $t \in \mathbb{N}_{a+n-\alpha}$;*

$$\Delta_a^{\alpha}f(t) := \begin{cases} \int_a^{t+\alpha+1} \frac{(t-\sigma(s))^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(s)\Delta s & , n-1 < \alpha < n \\ \Delta^n f(t) & , \alpha = n \end{cases}$$

or

$$\Delta_a^{\alpha}f(t) := \begin{cases} \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{\underline{-\alpha-1}} f(s) & , n-1 < \alpha < n \\ \Delta^n f(t) & , \alpha = n \end{cases}$$

where $\sigma(t) = t + 1$

Lemma 2.11 *Let* $\alpha > 0$ *and* $\beta \in \mathbb{R} - \{-1, -2, -3, ...\}$*. Then for every* $t \in \mathbb{N}_{a+\beta+m-\alpha}$*, we have*

$$\Delta_{a+\beta}^{\alpha}(t-a)^{\underline{\beta}} = \beta^{\underline{\alpha}}(t-a)^{\underline{\beta-\alpha}} = \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}(t-a)^{\underline{\beta-\alpha}}$$

2.4.2 Properties

Let $f : \mathbb{N}_a \to \mathbb{R}, \alpha > 0$ with $n - 1 < \alpha \leq n$.Then $(0 < k < \alpha)$

1. $\Delta^k(\Delta_a^{-\alpha} f)(t) = (\Delta_a^{k-\alpha} f)(t) \qquad , t \in \mathbb{N}_{a+\alpha}$
2. $\Delta^k(\Delta_a^{\alpha} f)(t) = (\Delta_a^{k+\alpha} f)(t) \qquad , t \in \mathbb{N}_{a+n-\alpha}$
3. $\Delta_{a+\beta}^{\alpha}(\Delta_a^{-\beta} f)(t) = (\Delta_a^{\alpha-\beta} f)(t) \qquad , t \in \mathbb{N}_{a+\beta+n-\alpha}$

Remark 2.12 *Let* $f : \mathbb{N}_a \to \mathbb{R}$ *,* $n - 1 < \alpha \leq n$ *and* $m - 1 < \beta \leq m$*. Then*

1. $Dom(\Delta_a^{\alpha} f) = \mathbb{N}_{a+n-\alpha}$
2. $Dom(\Delta_{a+\beta}^{\alpha} \Delta_a^{-\beta} f) = \mathbb{N}_{a+\beta+n-\alpha}$
3. $Dom(\Delta_{a+m-\beta}^{-\alpha} \Delta_a^{\beta} f) = \mathbb{N}_{a+m-\beta+\alpha}$
4. $Dom(\Delta_{a+m-\beta}^{\alpha} \Delta_a^{\beta} f) = \mathbb{N}_{a+m-\beta+n-\alpha}$

Theorem 2.13 *Let* $f : \mathbb{N}_a \to \mathbb{R}$ *and* $\alpha > 0$ *with* $n - 1 < \alpha \leq n$*. Then for any constant* a*, the general solution of the fractional difference equation* $\Delta_{a+\alpha-n}^{\alpha} U(t) = 0$ *is given by*

$$U(t) \;=\; c_1(t-a)^{\underline{\alpha-1}} + c_2(t-a)^{\underline{\alpha-2}} + ... + c_n(t-a)^{\underline{\alpha-n}}, \qquad t \in \mathbb{N}_{a+\alpha-n}$$

where $c_1, c_2, ..., c_n \in \mathbb{R}$ *are arbitrary.*

2.4.3 Caputo fractional difference

Definition 2.14 *Let $f : \mathbb{N}_a \to \mathbb{R}$ and $\alpha > 0$ with $n-1 < \alpha \leq n$. Then the $\alpha - th$ caputo fractional difference of the function f (based at a) is defined by* $\Delta_c^\alpha : C(\mathbb{N}_a, \mathbb{R}) \to C(\mathbb{N}_{a+(n-\alpha)}, \mathbb{R})$

$$\Delta_c^\alpha f(t) := \Delta_a^{-(n-\alpha)}\Delta^n f(t) = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} (t-\sigma(s))^{\underline{n-\alpha-1}} \Delta^n f(s) & , n-1 < \alpha < n \\ \Delta^n f(t) & , \alpha = n \end{cases}$$

for any $t \in \mathbb{N}_{a+n-\alpha}$, and $\sigma(t) = t+1$

Theorem 2.15 *Let $\alpha > 0$ with $n-1 < \alpha \leq n$. Then the general solution of the Caputo difference equation $\Delta_c^\alpha U(t) = f(t)$ is given by*

$$U(t) = \Delta^{-\alpha} f(t) + c_0 + c_1 (t-a)^{\underline{1}} + c_2 (t-a)^{\underline{2}} + ... + c_{n-1}(t-a)^{\underline{n-1}}$$

where $c_0, c_1, c_2, ..., c_{n-1} \in \mathbb{R}$.

2.4.4 Some important relations

1. $\sum_{s=0}^{t-\alpha} (t-\sigma(s))^{\underline{\alpha-1}} = \sum_{s=0}^{t-\alpha} (t-s-1)^{\alpha-1} = \frac{\Gamma(t+1)}{\alpha\Gamma(t-\alpha+1)} = \frac{1}{\alpha} t^{\underline{\alpha}}$

2. $\sum_{s=0}^{T} (T+\alpha-s-1)^{\underline{\alpha-1}} = \frac{\Gamma(T+\alpha+1)}{\alpha\Gamma(T+1)} = \frac{1}{\alpha}(T+\alpha)^{\underline{\alpha}}$

3. $\sum_{s=0}^{T+1} (T+\alpha-s-1)^{\alpha-2} = \frac{\Gamma(T+\alpha+1)}{(\alpha-1)\Gamma(T+2)} = \frac{1}{\alpha-1}(T+\alpha)^{\underline{\alpha-1}}$

4. $\sum_{s=0}^{T+2} (T+\alpha-s-1)^{\underline{\alpha-3}} = \frac{\Gamma(T+\alpha+1)}{(\alpha-2)\Gamma(T+3)} = \frac{1}{(\alpha-2)}(T+\alpha)^{\underline{\alpha-2}}$

5. $\sum_{s=0}^{T-1} (T+\alpha-s-2)^{\underline{\alpha-2}} = \frac{1}{(\alpha-1)}\left(\frac{\Gamma(T+\alpha)}{\Gamma(T+1)} - \Gamma(\alpha)\right) = \frac{\Gamma(T+\alpha)-\Gamma(\alpha)\Gamma(T+1)}{(\alpha-1)\Gamma(T+1)}$

Proof.

$$\sum_{s=0}^{t-\alpha}\Big(t-\sigma(s)\Big)^{\underline{\alpha-1}} = \int_0^{t-\alpha+1}\Big(t-s\Big)^{\underline{\alpha-1}}\Delta s$$

$$= \frac{-1}{\alpha}\Big(t-s\Big)^{\underline{\alpha}}$$

$$= \frac{-1}{\alpha}\left[\Big(t-t+\alpha-1\Big)^{\underline{\alpha}} - \Big(t\Big)^{\underline{\alpha}}\right] = \frac{-1}{\alpha}\left[\Big(\alpha-1\Big)^{\underline{\alpha}} - t^{\underline{\alpha}}\right]$$

$$= \frac{-1}{\alpha}\left[\frac{\Gamma(\alpha)}{\Gamma(0)} - \frac{\Gamma(t+1)}{\Gamma(t-\alpha+1)}\right]$$

$$= \frac{\Gamma(t+1)}{\alpha\Gamma(t-\alpha+1)} = \frac{1}{\alpha}t^{\underline{\alpha}}$$

■

Chapter 3

The existence and uniqueness of solution for the discrete fractional problem

In 2020, Selvam et al. [33] proved the existence of a solution to a discrete fractional difference equation formulated as

$$\begin{cases} {}^{c}\Delta_{\xi}^{\varrho}\chi(\xi) = \Phi(\xi+\varrho-1, \chi(\xi+\varrho-1)), \quad 1<\varrho\leq 2, \\ \Delta\chi(\varrho-2) = M_1, \qquad\qquad \chi(\varrho+T) = M_2, \end{cases} \tag{3.0.1}$$

for $\xi \in [0,T]_{\mathbb{N}_0} = [0,1,2,\ldots,T]$, $T \in \mathbb{N}$, $\eta \in [\varrho-1, T+\varrho-1]_{\mathbb{N}_{\varrho-1}}$, M_1 and M_2 constants, $\Phi : [\varrho-2, \varrho+T]_{\mathbb{N}_{\varrho-2}} \times \mathbb{R} \longrightarrow \mathbb{R}$ continuous, and where ${}^{c}\Delta_{\xi}^{\varrho}$ denotes the ϱth-Caputo difference. Here, motivated by the discrete model (3.0.1), we shall consider two generalized discrete problems.

Our first goal consists to study existence and uniqueness of solutions to the following discrete fractional equation that involves Caputo discrete derivatives:

$$\begin{cases} {}^{c}\Delta_{\xi}^{\varrho}\chi(\xi) = \Phi(\xi+\varrho-1, \chi(\xi+\varrho-1)), \quad 2<\varrho\leq 3, \\ \Delta\chi(\varrho-3) = 0, \chi(\varrho+T) = 0, \Delta^2\chi(\varrho-3) = 0, \end{cases} \tag{3.0.2}$$

for $\xi \in [0,T]_{\mathbb{N}_0} = [0,1,2,\ldots,T]$, $T \in \mathbb{N}$, ${}^{c}\Delta_{\xi}^{\varrho}$ is the Caputo difference operator and $\Phi : [\varrho-3, \varrho+T]_{\mathbb{N}_{\varrho-3}} \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous.

3.1 Integral equation

Lemma 3.1 *Let* $2 < \varrho \leq 3$ *and* $\Phi : [\varrho - 3, \varrho + T]_{\mathbb{N}_{\varrho-3}} \longrightarrow \mathbb{R}$*. A function* $\chi(\xi)$ $(\xi \in [\varrho - 3, \varrho + T]_{\mathbb{N}_{\varrho-3}})$ *that satisfies the discrete FBVP*

$$\begin{cases} {}^{c}\Delta_{\xi}^{\varrho}\chi(\xi) = \Phi(\xi + \varrho - 1), \quad 2 < \varrho \leq 3, \\ \Delta\chi(\varrho - 3) = 0, \chi(\varrho + T) = 0, \Delta^{2}\chi(\varrho - 3) = 0, \end{cases} \tag{3.1.1}$$

is given by

$$\chi(\xi) = \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi - \rho(l))^{(\varrho-1)} \Phi(l + \varrho - 1) - \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{T} (\varrho + T - \rho(l))^{(\varrho-1)} \Phi(l + \varrho - 1) \tag{3.1.2}$$

Proof. Let $\chi(\xi)$ be a solution to (3.1.1). Applying Lemma **??** and Definition **??**, we find that

$$\chi(\xi) = \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi - \rho(l))^{(\varrho-1)} \Phi(l + \varrho - 1) + C_0 + C_1\xi^{(1)} + C_2\xi^{(2)}, \tag{3.1.3}$$

for $\xi \in [\varrho - 3, \varrho + T]_{\mathbb{N}_{\varrho-3}}$, where $C_0, C_1, C_2 \in \mathbb{R}$. By using the difference of order 1 for (3.1.3), we have

$$\Delta\chi(\xi) = \frac{1}{\Gamma(\varrho - 1)} \sum_{l=0}^{\xi-\varrho+1} (\xi - \rho(l))^{(\varrho-2)} \Phi(l + \varrho - 1) + C_1 + 2C_2\xi^{(1)},$$

and

$$\Delta^{2}\chi(\xi) = \frac{1}{\Gamma(\varrho - 2)} \sum_{l=0}^{\xi-\varrho+2} (\xi - \rho(l))^{(\varrho-3)} \Phi(l + \varrho - 1) + 2C_2.$$

Now, from conditions $\Delta\chi(\varrho - 3) = 0$ and $\Delta^{2}\chi(\varrho - 3) = 0$, we obtain that

$$C_1 = 0,$$
$$C_2 = 0.$$

Therefore,

$$\chi(\xi) = \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi - \rho(l))^{(\varrho-1)} \Phi(l + \varrho - 1) + C_0, \tag{3.1.4}$$

for $\xi \in [\varrho-3, \varrho+T]_{\mathbb{N}_{\varrho-3}}$. The other condition of (3.1.1) gives

$$\chi(\eta+\beta) = \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1) + C_0 = 0.$$

We have

$$C_0 = -\frac{1}{\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1),$$

one obtains (3.1.2) by substituting the value of C_0 into (3.1.4). ■

Now, let us consider the operator $H : C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R}) \to C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$ defined by

$$\begin{aligned}(H\chi)(\xi) &= \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{\xi-\varrho}(\xi-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1, \chi(l+\varrho-1)) \\ &\quad - \frac{1}{\Gamma(\varrho)}\sum_{l=0}^{T}(\varrho+T-\rho(l))^{(\varrho-1)}\Phi(l+\varrho-1, \chi(l+\varrho-1))\end{aligned} \tag{3.1.5}$$

3.2 Existence and uniqueness of solutions

In this section, we prove the existence and uniqueness of solution for the Caputo discrete fractional problem (3.0.2). To accomplish this, we denote by $C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$ the collection of all continuous functions χ with the norm

$$\|\chi\| = \max\{|\chi(\xi)| : \xi \in \mathbb{N}_{\varrho-3,\varrho+T}\}.$$

Theorem 3.2 *Assume that function Φ satisfies*

$$|\Phi(\xi,\chi_1) - \Phi(\xi,\chi_2)| \le K|\chi_1-\chi_2|,$$

where $K > 0, \forall \xi \in \mathbb{N}_{\varrho-3,\varrho+T}$ and $\chi_1, \chi_2 \in C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$. The discrete FBVP (3.1.1) has a unique solution on $C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$ provided

$$\frac{\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)} \le \frac{1}{2K} \tag{3.2.1}$$

Proof. Let $\chi_1, \chi_2 \in C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$. Then, for each $\xi \in \mathbb{N}_{\varrho-3,\varrho+T}$, we have

$$\begin{aligned}\big|(H\chi_1)(\xi) - (H\chi_2)(\xi)\big| \leq & \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi - \rho(l))^{(\varrho-1)} \\ & \times \big|\Phi(l+\varrho-1, \chi_1(l+\varrho-1)) - \Phi(l+\varrho-1, \chi_2(l+\varrho-1))\big| \\ & + \frac{1}{\Gamma(\varrho)} \sum_{l=0}^{T} (\varrho + T - \rho(l))^{(\varrho-1)} \\ & \times \big|\Phi(l+\varrho-1, \chi_1(l+\varrho-1)) - \Phi(l+\varrho-1, \chi_2(l+\varrho-1))\big|.\end{aligned}$$

It follows that

$$\begin{aligned}\big\|(H\chi_1)(\xi) - (H\chi_2)(\xi)\big\| \leq & \frac{K\|\chi_1 - \chi_2\|}{\Gamma(\varrho)} \sum_{l=0}^{\xi-\varrho} (\xi - \rho(l))^{(\varrho-1)} \\ & + \frac{K\|\chi_1 - \chi_2\|}{\Gamma(\varrho)} \sum_{l=0}^{T} (\varrho + T - \rho(l))^{(\varrho-1)} \\ \leq & \frac{K\|\chi_1 - \chi_2\|}{\Gamma(\varrho)} \frac{\Gamma(\varrho+T+1)}{\varrho\Gamma(T+1)} \\ & + \frac{K\|\chi_1 - \chi_2\|}{\Gamma(\varrho)} \frac{\Gamma(\varrho+T+1)}{\varrho\Gamma(T+1)} \\ \leq & 2K \frac{\Gamma(\varrho+T+1)}{\Gamma(\varrho+1)\Gamma(T+1)} \|\chi_1 - \chi_2\|.\end{aligned}$$

From (3.2.1), we conclude that H is a contraction. Then, by the Banach contraction principle, the discrete problem (3.1.1) has a unique solution on $C(\mathbb{N}_{\varrho-3,\varrho+T}, \mathbb{R})$. ■

3.3 Example

Bibliography

[1] T. Abdeljawad, R. Mert, D. F. M. Torres, Variable order Mittag-Leffler fractional operators on isolated time scales and application to the calculus of variations, In: *Fractional derivatives with Mittag-Leffler kernel*, Springer, 2019, 35–47.

[2] A. Chidouh, D. F. M. Torres, Existence of positive solutions to a discrete fractional boundary value problem and corresponding Lyapunov-type inequalities, *Opuscula Math.*, **38** (2018), 31–40.

[3] R. A. C. Ferreira, D. F. M. Torres, Fractional h-difference equations arising from the calculus of variations, *Appl. Anal. Discr. Math.*, **5** (2011), 110–121.

[4] D. F. M. Torres, Cauchy's formula on nonempty closed sets and a new notion of Riemann-Liouville fractional integral on time scales, *Appl. Math. Lett.*, **121** (2021) 107407.

[5] M. M. Matar, M. I. Abbas, J. Alzabut, M. K. A. Kaabar, S. Etemad, S. Rezapour, Investigation of the p-Laplacian nonperiodic nonlinear boundary value problem via generalized Caputo fractional derivatives, *Adv. Differ. Equ.*, **2021** (2021), 68.

[6] M. Ahmad, J. Jiang, A. Zada, S. O. Shah, J. Xu, Analysis of implicit coupled system of fractional differential equations involving Katugampola-Caputo fractional derivative, *Complexity*, **2020** (2020), 9285686.

[7] D. Baleanu, S. Etemad, S. Rezapour, On a fractional hybrid integro-differential equation with mixed hybrid integral boundary value conditions by using three operators, *Alex. Eng. J.*, **59** (2020), 3019–3027.

[8] M. Ahmad, A. Zada, J. Alzabut, Stability analysis of a nonlinear coupled implicit switched singular fractional differential system with p-Laplacian, *Adv. Differ. Equ.*, **2019** (2019), 436.

[9] H. Mohammadi, S. Kumar, S. Rezapour, S. Etemad, A theoretical study of the Caputo-Fabrizio fractional modeling for hearing loss due to Mumps virus with optimal control, *Chaos Soliton. Fract.*, **144** (2021), 110668.

[10] S. Etemad, I. Avci, P. Kumar, D. Baleanu, S. Rezapour, Some novel mathematical analysis on the fractal-fractional model of the AH1N1/09 virus and its generalized Caputo-type version, *Chaos Soliton. Fract.*, **162** (2022), 112511.

[11] R. Rizwan, A. Zada, X. Wang, Stability analysis of non linear implicit fractional Langevin equation with noninstantaneous impulses, *Adv. Differ. Equ.*, **2019** (2019), 85.

[12] A. Amara, Existence results for hybrid fractional differential equations with three-point boundary conditions, *AIMS Math.*, **5** (2020), 1074–1088.

[13] S. Etemad, S. Rezapour, On the existence of solutions for fractional boundary value problems on the ethane graph, *Adv. Differ. Equ.*, **2020** (2020), 276.

[14] F. S. Bachir, S. Abbas, M. Benbachir, M. Benchohra, Hilfer-Hadamard fractional differential equations: Existence and attractivity, *Adv. Theor. Nonlinear Anal. Appl.*, **5** (2021), 49–57.

[15] Z. Baitiche, C. Derbazi, M. Benchohra, ψ-Caputo fractional differential equations with multi-point boundary conditions by topological degree theory, *Res. Nonlinear Anal.*, **3** (2020), 167–178.

[16] S. Ben Chikh, A. Amara, S. Etemad, S. Rezapour, On Hyers-Ulam stability of a multi-order boundary value problems via Riemann-Liouville derivatives and integrals, *Adv. Differ. Equ.*, **2020** (2020), 547.

[17] S. Ben Chikh, A. Amara, S. Etemad, S. Rezapour, On Ulam-Hyers-Rassias stability of a generalized Caputo type multi-order boundary value problem with four-point mixed integro-derivative conditions, *Adv. Differ. Equ.*, **2020** (2020), 680.

[18] M. Ismail, U. Saeed, J. Alzabut, M. Rehman, Approximate solutions for fractional boundary value problems via Green-CAS wavelet method, *Mathematics*, **7** (2019), 1164.

[19] M. Iswarya, R. Raja, G. Rajchakit, J. Cao, J. Alzabut, C. Huang, Existence, uniqueness and exponential stability of periodic solution for discrete-time delayed bam neural networks based on coincidence degree theory and graph theoretic method, *Mathematics*, **7** (2019), 1055.

[20] Y. Guo, X. Shu, Y. Li, F. Xu, The existence and Hyers-Ulam stability of solution for an impulsive Riemann-Liouville fractional neutral functional stochastic differential equation with infinite delay of order $1 < \beta < 2$, *Bound. Value Probl.*, **2019** (2019), 59.

[21] A. Salem, F. Alzahrani, L. Almaghamsi, Fractional Langevin equations with nonlocal integral boundary conditions, *Mathematics*, **7** (2019), 402.

[22] R. A. Yan, S. R. Sun, Z. L. Han, Existence of solutions of boundary value problems for Caputo fractional differential equations on time scales, *B. Iran. Math. Soc.*, **42** (2016), 247–262.

[23] A. Zada, J. Alzabut, H. Waheed, I. L. Popa, Ulam-Hyers stability of impulsive integrodifferential equations with Riemann-Liouville boundary conditions, *Adv. Differ. Equ.*, **2020** (2020), 64.

[24] T. Abdeljawad, On Riemann and Caputo fractional differences, *Comput. Math. Appl.*, **62** (2011), 1602–1611.

[25] J. Alzabut, T. Abdeljawad, D. Baleanu, Nonlinear delay fractional difference equations with applications on discrete fractional Lotka-Volterra competition model, *J. Comput. Anal. Appl.*, **25** (2018), 889–898.

[26] F. M. Atici, P. M. Eloe, Two-point boundary value problems for finite fractional difference equations, *J. Differ. Equ. Appl.*, **17** (2011), 445–456.

[27] F. Chen, Y. Zhou, Existence and Ulam stability of solutions for discrete fractional boundary value problem, *Discrete Dyn. Nat. Soc.*, **2013** (2013), 459161.

[28] C. Chen, M. Bohner, B. Jia, Ulam-Hyers stability of Caputo fractional difference equations, *Math. Method. Appl. Sci.*, **42** (2019), 7461–7470.

[29] Y. Pan, Z. Han, S. Sun, C. Hou, The existence of solutions to a class of boundary value problems with fractional difference equations, *Adv. Differ. Equ.*, **2013** (2013), 275.

[30] A. G. M. Selvam, R. Dhineshbabu, Existence and uniqueness of solutions for a discrete fractional boundary value problem, *Int. J. Appl. Math.*, **33** (2020), 283–295.

[31] A. G. M. Selvam, R. Dhineshbabu, Hyers-Ulam stability results for discrete antiperiodic boundary value problem with fractional order $2 < \delta \leq 3$, *IJEAT*, **9** (2019), 4997–5003.

[32] A. G. M. Selvam, R. Dhineshbabu, Ulam stability results for boundary value problem of fractional difference equations, *Adv. Math.*, **9** (2020), 219–230.

[33] A. G. M. Selvam, J. Alzabut, R. Dhineshbabu, S. Rashid, M. Rehman, Discrete fractional order two-point boundary value problem with some relevant physical applications, *J. Inequal. Appl.*, **2020** (2020), 221.