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Theme

Stabilization of some hyperbolic type problems by boundary controls

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Before the jury composed of:

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DEDICATION

There is no way I can express how much I owe to my family for their love, generous spirit and support through the many years of my education.

I dedicate this research to my tender **mother** for her never ending-love. I will be always grateful to my **father** for his incomparable love and moral suppoet.

To my adorable sisters and brothers, each one by her/his name and all the family members

To everyone who watched my success during the school for my classmates, without exception.

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Contents

NOTATIONS

➤ $\mathbb{R}^+$:The set of positive real number.

➤ $\|.\|_V$:Norm in V.

➤ a.e : Almost everywhere.

➤ $\longrightarrow$:Strong convergence.

➤ $\rightharpoonup$:Weak convergence.

➤ span$\{w^1, w^2, ..., w^m\}$:Space spanned by $w^1, w^2, ..., w^m$.

➤ $u_t = \dfrac{\partial u}{\partial t}, \quad u_{tt} = \dfrac{\partial^2 u}{\partial t^2}, \quad u_{ttt} = \dfrac{\partial^3 u}{\partial t^3}$

➤ $\nabla f = \left(\dfrac{\partial f}{\partial x_1}, \dfrac{\partial f}{\partial x_2}, ..., \dfrac{\partial f}{\partial x_n} \right)$

Introduction

The model for Timoshenko beam was first derived in [1] in order to describe the dynamics of a "thick" beam and it consists of a system of two coupled hyperbolic equations of the form

$$\begin{aligned} \rho_1\phi_{tt} - K(\phi_x + \psi)_x = 0, \in (0,1)\times(0,+\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + K(\phi_x + \psi) = 0, \in (0,1)\times(0,+\infty), \end{aligned} \tag{1}$$

where ϕ is the transveres displacement, ψ is the rotational angle of the filament of the beam and ρ_1, ρ_2, b and K are fixed positive physical contants. Recent technological advancements in the fields of robotics and space science give rise to increase interests in the dynamics of flexible structures with boundary and/or internal control forces. The Euler-Bernoulli equation had been deployed to model the dynamics of the transverse vibration of an elastic beam by neglecting the effect of its rotatory inertia since the dimension of its cross-section is negligible compared to its actual length. when the effect of the rotational inertia of the beam is taken into consideration, then we end up with the famous Timoshenko system of equations. Therefor, the Timoshenko model describes the dynamics behavior of the beam more accurately than the Euler-Bernoulli beam model in this situation. Timoshenko system received considerable attention from various researchers and many questions related to well-posedness and long-time behavior of the equation with internal feedback controls had been investigated [2]. In the last few decades, the study of problems related to elastic solids with voids has attracted the attention of many researchers due to the extensive practical applications of such materials in different fields, such as petroleum industry, foundation engineering, soil mechanics, power technology, biology, material science and so on. Elastic solids with voids is one of the simplest extensions of the theory of the classical elasticity. It allows the treatment of porous solids in which the matrix material is elastic and the interstices are void of material.

In 1972, Godman and Cowin [3] proposed an extension of the classical elasticity theory to porous media. They introduced the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to their usual elastic effects, these materials have a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This latter idea was introduced by Nunziato and Cowin [4] in 1979 when they developed a nonlinear theory of elastic materials with voids. This representation (i.e the mass at each point is obtained as the product of the product of the mass density of the material matrix by the volume fraction) introduces an additional degree of kinematic freedom and was employed previously by Goodman and Cowin [3] to develop a theory for for flowing granular materials.

In 1983, Cowin and Nunziato [5] developed a linear theory of elastic materials with voids to study mathematically the mechanical behavior of porous solids. We refer the reader to [5, 6] and the refernces therein for more details.

The main results of this thesis:

This thesis contains three chapters.

In chapter 1, we recall some notations and we review some mathematical concepts that will be used throughout this dissertation.

In **chapter 2**, we present the results of Hassan and Tatar [2] which is the existence and stability of solution for Timoshenko system under the boundary feedback controls:

$$\begin{cases} \varphi_{tt} - (\varphi_t + \psi)_x = 0, & 0 < x < 1, t > 0 \\ \psi_{tt} - \psi_{xx} + (\varphi_x + \psi) = 0, & 0 < x < 1, t > 0 \\ \varphi(0,t) = \psi(0,t) = 0, & t > 0 \\ \varphi_x(1,t) + \psi(1,t) = u_1(t), \psi_x(1,t) = u_2(t), & t > 0 \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & 0 < x < 1 \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & 0 < x < 1 \end{cases}$$

We study this system under the influence of the following high adaptive boundary controls, where we detail the work of Hassan and Tatar [2]:

$$\begin{cases} u_i(t) = -k_i(t)y_i(t) \\ k_i'(t) = r_i y_i^2(t) \quad k_i(0) = k_{0i}, \quad k_{0i}, r_i > 0 \quad for \quad i = 1,2 \end{cases}$$

The main of this dissertation is to analyze the existence and asymptotic stability of solution for the porous elastic system under adaptive boundary controls.

In chapter 3, we study, a new system, which mentioned in the work of Lacheheb et al. [7], but with same boundary and initial conditions with [2] i.e the following problem:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 & 0 < x < 1, \quad t > 0 \\ j\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0 & 0 < x < 1, \quad t > 0 \\ u(0,1) = \phi(0,1) = 0 & t > 0 \\ u_x(1,t) + \dfrac{b}{\mu}\phi(1,t) = f_1(t), \quad \phi_x(1,t) = f_2(t), & t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & 0 < x < 1 \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x) & 0 < x < 1 \end{cases}$$

with adaptive boundary controls:

$$\begin{cases} f_i(t) = -h_i(t)y_i(t) \\ h_i'(t) = r_i y_i^2(t) \quad h_i(0) = h_{0i}, \quad h_{0i}, r_i > 0 \quad for \quad i = 1,2 \end{cases}$$

We use the Faedo-Galerkin method to establish the well-posedness, then we employ the multiplier and energy methods to prove an exponential decay.

Chapter 1

Preliminaries

In this chapter we discuss some mathematical concepts that we should know them for use in our .

Definition 1 *Let Ω be a domain in $\mathbb{R}^n$ $(n \in N)$, for $1 \leq p < \infty$, the Lebesgue space $L^p(\Omega)$ is defined by:*

$$L^p(\Omega) = \left\{ u : \Omega \longrightarrow \mathbb{R},\ u \text{ is measurable and } \int_\Omega |u(x)|^p dx < \infty \right\},$$

with the norm

$$\|u\|_p = \left(\int_\Omega |u(x)|^p dx \right)^{\frac{1}{p}}.$$

In addition, we define $L^\infty(\Omega)$ by:

$$L^\infty(\Omega) = \{ u : \Omega \longrightarrow \mathbb{R},\ u \text{ is measurable and } \exists\ c > 0 \text{ such that } |u(x)| \leq c \text{ a.e on } \Omega \},$$

equipped with the norm

$$\|u\|_\infty = ess \sup_{x \in \Omega} |u(x)| = \inf \{ c :\ |u(x)| \leq c \text{ a.e on } \Omega \}.$$

Theorem 2 ***(Hölder's Inequality)***
Let $1 \leq p, q \leq \infty$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$*, assume that* $f \in L^p(\Omega)$ *and* $g \in L^q(\Omega)$*, then* $f.g \in L^1(\Omega)$ *and*

$$\|fg\|_1 \leq \|f\|_{L^p}\|g\|_{L^q},$$

Theorem 3 ***(Cauchy-Schwarz Inequality)***
For $p = q = 2$ *the Holder inequality is none other the Cauchy-Schwarz inequality :*

$$\|fg\| \leq \|f\|_{L^2(\Omega)}\|g\|_{L^2(\Omega)},$$

Theorem 4 ***(Young's Inequality)***

For all $a, b \geq 0$*, the following inequality holds*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

where $\frac{1}{p} + \frac{1}{q} = 1$, more standard inequality:

$$ab \leq \frac{\alpha}{2}a^2 + \frac{b^2}{2\alpha}$$

for $a, b \in \mathbb{R}^+$, and $\alpha > 0$.

Theorem 5 ***(Poincaré's Inequality)***
Let Ω *is bounded open subset.Then there exists a constant* c *depending on* Ω *such that:*

$$\|f\|_{L^2(\Omega)} \leq c\|\nabla f\|_{L^2(\Omega)}$$

Theorem 6 ***(Faedo-Galerkin)***
The demonstration of the overall existence of the solution is based on the Faedo-Galerkin

method which consists of carrying out the following three steps:

Step 01: *Search for approximate solution.*

Step 02: *We establish, on these approximate solution a priori estimates.*

Step 03: *We go beyond the limit thanks to compactness properties.*

Lemma 1 ***(Aubin-Lions[8])***

Let X_0, X *and* X_1 *be three Banach spaces with* $X_0 \subseteq X \subseteq X_1$. *Assume that* X_0 *is compactly embedded in* X *and that* X *is compactly embedded in* X_1 *, assume also that* X_0 *and* X_1 *are reflexive spaces.*

For $1 < p, q < +\infty$, *let*

$W = \{u \in L^p([0,T]; X_0)/ \quad u_t \in L^q([0,T]; X_1)\}$

Then the embedding of W *into* $L^P([0,T]; X)$ *is also compact.*

Lemma 2 ***(Lyapunov Equivalence [9])***

For the stability results, we use the multiplier method based on construction of a Lyapunov function f *equivalent to energy* E *of the solution. We denote by* $f \sim g$ *to the equivalence between* f *and* g. *it means*

$$c_1 g(t) \leq f(t) \leq c_2 g(t), \quad \forall t > 0,$$

for two positive constants c_1 *and* c_2.

Lemma 3 ***(Lebesgue's dominated convergence[10])***

Let (f_n) *be a sequence of function of* L^1.

Assume that:

a)$f_n(x) \to f(x)$ *a.e on* Ω.

b)There is a function $g \in L^1$*such that for each n.* $|f_n(x)| \leq g(x)$ *a.e on* Ω

so it is $f \in L^1(\Omega)$*and* $\|f_n - f\|_{L^1} \to 0$

Lemma 4 *[2]*

In this section, we present some preliminary results, then state and the well-posedness result of the system. Let$\|.\|$ *denote the usual norm in* $L^2(0,1)$,

$$V = \{w \in H^1(0,1) : w(0) = 0\}$$

equipped with the norm

$$\|W\|_V = \|W_x\|,$$

and

$$W = \{w \in H^2(0,1) : w(0) = 0\}$$

equipped with the norm

$$\|w\|_W^2 = \|w_x\|^2 + \|w_x\|^2$$

.

Lemma 5 *[2]*

For any $w \in V$, *we have*

$$\|w\|^2 \leq \frac{1}{2}\|w_x\|^2 \tag{1.1}$$

and

$$w^2(1) \leq \|w_x\|^2. \tag{1.2}$$

It follows from above lemma that $\|.\|_V$ *and* $\|.\|_W$ *are equivalent norms in* V *and* W*, respectively, and*

$$\frac{1}{2}(\|v_x\|^2+\|w_x\|^2) \leq \|v_x + w\|^2+\|w_x\|^2\leq 2(\|v_x\|^2+\|w_x\|^2) \tag{1.3}$$

for any $v, w \in V$ *.*

Chapter 2

Asymptotic stability of a Timoshenko system by boundary feedback controls

2.1 Statement of the problem

We consider the following problems

$$\begin{cases} \varphi_{tt} - (\varphi_x + \psi)_x = 0, & 0 < x < 1, t > 0 \\ \psi_{tt} - \psi_{xx} + (\varphi_x + \psi) = 0, & 0 < x < 1, t > 0 \\ \varphi(0,t) = \psi(0,t) = 0, & t > 0 \\ \varphi_x(1,t) + \psi(1,t) = u_1(t), \psi_x(1,t) = u_2(t), & t > 0 \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & 0 < x < 1 \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x) & 0 < x < 1 \end{cases} \tag{2.1}$$

where $\varphi_0, \varphi_1, \psi_0, \psi_1$ are given data, u_1 and u_2 are boundary control inputs to be determined later. The measured outputs of the system at the right end are given by

$$y_1(t) = \varphi_t(1,t) \qquad and \qquad y_2(t) = \psi_t(1,t) \qquad\qquad t > 0. \tag{2.2}$$

with the following boundary controls

$$\begin{cases} u_i(t) = -k_i(t)y_i(t), \\ k_i'(t) = r_i y_i^2(t),\ k_i(0) = k_{0i}, \quad k_{0i}, r_i > 0, \end{cases} \tag{2.3}$$

for i =1,2.

The closed - loop system associaed to (2.1) ,(2.2) , and (2.3) is given by

$$\begin{cases} \varphi_{tt} - (\varphi_x + \psi)_x = 0, & 0 < x < 1, t > 0 \\ \psi_{tt} - \psi_{xx} + (\varphi_x + \psi) = 0, & 0 < x < 1, t > 0 \\ \varphi(0,t) = \psi(0,t) = 0, & t > 0 \\ \varphi_x(1,t) + \psi(1,t) = -k_1(t)\varphi_t(1,t), & t > 0 \\ \psi_x(1,t) = -k_2(t)\psi_t(1,t), & t > 0 \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x) & 0 < x < 1 \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x,) & 0 < x < 1 \\ k_1'(t) = r_1[\varphi_t(1,t)]^2,\ k_1(0) = k_{01}, k_{01}, r_1 > 0 \\ k_2'(t) = r_2[\psi_t(1,t)]^2,\ k_2(0) = k_{02}, k_{02}, r_2 > 0 \end{cases} \tag{2.4}$$

2.2 Global existence

In this section, we give the existence result of the system (2.4) using the Faedo-Galerkin method.

Definition 7 *A pair of functions* (φ, ψ) *defined on* $(0,1) \times [0,T]$, $T > 0$ *arbitrary, is called a local strong solution of the closed-loop system* (2.4) *if*

$$\varphi, \psi \in C([0,T];W) \cap C^1([0,T];V) \cap C^2([0,T];L^2(0,1))$$

$$(\varphi_0, \psi_0),\ (\varphi_1, \psi_1) \in W \times V$$

and it satisfies

$$\int_0^1 \varphi_{tt} u dx + \int_0^1 (\varphi_x + \psi) u_x dx + k_1(t)\varphi_t(1,t)u(1) = 0$$

$$\int_0^1 \psi_{tt} v dx + \int_0^1 \psi_x v_x dx + \int_0^1 (\varphi_x + \psi) v dx + k_2(t)\psi_t(1,t)v(1) = 0$$

$$k_1'(t) = r_1\varphi_t^2(1,t), \quad k_1(0) = k_{01}, \quad k_{01}, r_1 > 0,$$

$$k_2'(t) = r_2\psi_t^2(1,t), \quad k_2(0) = k_{01}, \quad k_{01}, r_2 > 0,$$

for any $u, v \in V$ *and any* $t \in [0,T]$.

Now, we are ready to state and prove the existence result.

Theorem 8 *[2]*
Let (φ_0, φ_1),$(\psi_0, \psi_1) \in W \times V$ *satisfy the compatibility conditions.*

$$\varphi_{0x}(1) + \psi_0(1) = -k_{01}\varphi_1(1) \qquad \textit{and} \qquad \psi_{0x}(1) = -k_{02}\psi_1(1).$$

Then, there exists a unique global strong solution to system(2.4).

Proof. Let $\{w^j\}_{j\geq 1}$ be a complete orthogonal basis for W and V, and for each $m \geq 1$ let $W_m := span\{w^1, ..., w^m\}$.

We look for a solution in the form

$$\varphi^m(x,t) = \sum_{j=1}^{m} a_{mj}(t)w^j(x) \qquad \text{and} \qquad \psi^m(x,t) = \sum_{j=1}^{m} b_{mj}(t)w^j(x)$$

to the approximate problem

$$\begin{cases} \int_0^1 \varphi_{tt}^m u dx + \int_0^1 (\varphi_x^m + \psi^m) u_x dx + k_1^m(t)\varphi_t^m(1,t)u(1) = 0 \\ \int_0^1 \psi_{tt}^m v dx + \int_0^1 \psi_x^m v_x dx + \int_0^1 (\varphi_x^m + \psi^m) v dx + k_2^m(t)\psi_t^m(1,t)v(1) = 0 \\ k_1'^m(t) = r_1[\varphi_t^m(1,t)]^2, k_1^m(0) = k_{01}, \\ k_2'^m(t) = r_2[\psi_t^m(1,t)]^2, k_2^m(0) = k_{02}, \\ \varphi^m(.,0) = \varphi_0^m, \varphi_t^m(.,0) = \varphi_1^m, \psi^m(.,0) = \psi_0^m, \psi_t^m(.,0) = \psi_1^m, \end{cases} \tag{2.5}$$

for all u,v$\in V_m$, where

$$\begin{cases} (\varphi_0^m, \varphi_1^m) := \left(\sum_{j=1}^m (\varphi_0, w^j) w^j, \sum_{j=1}^m (\varphi_1, w^j) w^j \right) \to (\varphi_0, \varphi_1) \text{ in } W \times V, \\ (\psi_0^m, \psi_1^m) := \left(\sum_{j=1}^m (\psi_0, w^j) w^j, \sum_{j=1}^m (\psi_1, w^j) w^j \right) \to (\psi_0, \psi_1) \text{in } W \times V, \end{cases} \tag{2.6}$$

$$\varphi_{0x}^m(1) + \psi_0^m(1) = -k_{01}\varphi_1^m(1). \tag{2.7}$$

and

$$\psi_{0x}^m(1) = -k_{02}\psi_1^m(1). \tag{2.8}$$

problem (2.5) is a system of ordinary differential equation in t which has a local solution , say, in $[0, t_m)$.

The next a priori estimates show that $t_m = \infty$ for any $m \geq 1$.

First a priori estimate.

Substituting $u = \varphi_t^m$ in $(2.5)_1$ and $v = \psi_t^m$ in $(2.5)_2$, then adding the resultants, we obtain

$$\int_0^1 \varphi_{tt}^m \varphi_t^m dx + \int_0^1 (\varphi_t^m + \psi^m)\varphi_{tx}^m dx + k_1^m(t)[\varphi_t^m(1,t)]^2 = 0$$

where

$$\int_0^1 \varphi_{tt}^m \varphi_t^m dx = \int_0^1 \frac{1}{2}\frac{d}{dt}(\varphi_t^m)^2 dx = \frac{1}{2}\frac{d}{dt}\int_0^1 (\varphi_t^m)^2 dx = \frac{1}{2}\frac{d}{dt}\|\varphi_t^m\|^2 \tag{2.9}$$

we have

$$\frac{d}{dt}(\varphi_x^m + \psi^m)^2 = 2(\varphi_x^m + \psi^m)(\varphi_x^m + \psi^m)_t$$

$$\frac{1}{2}\frac{d}{dt}(\varphi_x^m + \psi^m)^2 = (\varphi_x^m + \psi^m)(\varphi_x^m + \psi^m)_t = (\varphi_x^m + \psi^m)\varphi_{xt}^m + (\varphi_x^m + \psi^m)\psi_t^m$$

and

$$(\varphi_x^m + \psi^m)\varphi_{xt}^m = \frac{1}{2}\frac{d}{dt}(\varphi_x^m + \psi^m)^2 - (\varphi_x^m + \psi^m)\psi_t^m$$

we enter integration

$$\int_0^1 (\varphi_x^m + \psi^m)\varphi_{xt}^m = \int_0^1 \frac{1}{2}\frac{d}{dt}(\varphi_x^m + \psi^m)^2 - \int_0^1 (\varphi_x^m + \psi^m)\psi_t^m$$

$$\int_0^1 (\varphi_x^m + \psi^m)\varphi_{xt}^m = \frac{1}{2}\frac{d}{dt}\|\varphi_x^m + \psi^m\|^2 - \int_0^1 (\varphi_x^m + \psi^m)\psi_t^m \tag{2.10}$$

from (2.9) and (2.10), gives

the first equation

$$\frac{1}{2}\frac{d}{dt}[\|\varphi_t^m\|^2 + \|\varphi_x^m + \psi^m\|^2] - \int_0^1 (\varphi_x^m + \psi^m)\psi_t^m + k_1^m[\varphi_t^m(1,t)]^2 = 0 \tag{2.11}$$

and we have

$$\int_0^1 \psi_{tt}^m\psi_t^m dx + \int_0^1 \psi_x^m\psi_{tx}^m dx + \int_0^1 (\varphi_x^m + \psi^m)\psi_t^m dx + k_2^m(t)[\psi_t^m(1,t)]^2 = 0$$

where

$$\int_0^1 \psi_{tt}^m\psi_t^m dx = \int_0^1 \frac{1}{2}\frac{d}{dt}(\psi_t)^2 dx = \frac{1}{2}\frac{d}{dt}\int_0^1 (\psi_t)^2 dx = \frac{1}{2}\frac{d}{dt}\|\psi_t\|^2 \tag{2.12}$$

and

$$\int_0^1 \psi_x^m\psi_{tx}^m dx = \int_0^1 \frac{1}{2}\frac{d}{dt}(\psi_x^m)^2 dx = \frac{1}{2}\frac{d}{dt}\int_0^1 (\psi_x^m)^2 dx = \frac{1}{2}\frac{d}{dt}\|\psi_x^m\|^2 \tag{2.13}$$

from (2.12) and (2.13), we get

the second equation

$$\frac{1}{2}\frac{d}{dt}[\|\psi_t^m\|^2 + \|\psi_x^m\|^2] + \int_0^1 (\varphi_x^m + \psi^m)\psi_t^m dx + k_2^m(t)[\psi_t^m(1,t)]^2 = 0 \tag{2.14}$$

by adding up (2.11) and (2.14), we arrive to

$$\frac{d}{dt}\frac{1}{2}[\|\varphi_t^m(t)\|^2+\|\varphi_x^m(t)+\psi^m(t)\|^2+\|\psi_t^m(t)\|^2+\|\psi_x^m\|^2]+k_1^m(t)[\varphi_t^m(1,t)]^2+k_2^m(t)[\psi_t^m(1,t)]^2=0. \tag{2.15}$$

using$(2.5)_3$, we obtain

$$k_1'^m(t)=r_1[\varphi_t^m(1,t)]^2,$$

$$[\varphi_t^m(1,t)]^2=\frac{1}{r_1}[k_1'^m(t)]$$

multiplication in $k_1^m(t)$

$$k_1^m(t)[\varphi_t^m(1,t)]^2=\frac{1}{r_1}k_1^m(t)[k_1'^m(t)]=\frac{d}{dt}\frac{1}{r_1}[k_1^m(t)]^2 \tag{2.16}$$

using$(2.5)_4$, we obtain

$$k_2'^m(t)=r_2[\psi_t^m(1,t)]^2,$$

$$[\psi_t^m(1,t)]^2=\frac{1}{r_2}[k_2'^m(t)]$$

multiplication in $k_2^m(t)$

$$k_2^m(t)[\psi_t^m(1,t)]^2=\frac{1}{r_2}k_2^m(t)[k_2'^m(t)]=\frac{d}{dt}\frac{1}{r_2}[k_2^m(t)]^2 \tag{2.17}$$

Substituting (2.16) and (2.17) in (2.15) we get, for each $m\geq 1$ and for any $0<t<t_m$,

$$\frac{d}{dt}\frac{1}{2}\left\{\|\varphi_t^m(t)\|^2+\|\varphi_x^m(t)+\psi^m(t)\|^2+\|\psi_t^m(t)\|^2+\|\psi_x^m(t)\|^2+\frac{1}{r_1}[k_1^m(t)]^2+\frac{1}{r_2}[k_2^m(t)]^2\right\}=0.$$

Integrating over (0,t) and using (2.6) we see that

$$\|\varphi_t^m(t)\|^2+\|\varphi_x^m(t)+\psi^m(t)\|^2+\|\psi_t^m(t)\|^2+\|\psi_x^m(t)\|^2+\frac{1}{r_1}[k_1^m(t)]^2+\frac{1}{r_2}[k_2^m(t)]^2$$

$$=\|\varphi_1^m\|^2+\|\varphi_{0x}^m+\psi_0^m\|^2+\|\psi_1^m\|^2+\|\psi_{0x}^m\|^2\frac{1}{r_1}k_{01}^2+\frac{1}{r_2}k_{02}^2\leq C, \tag{2.18}$$

where C is independent of m and t. Also, from $(2.5)_3$ and $(2.5)_4$ we deduse

$$\int_0^t[\varphi_t^m(1,s)]^2ds=\frac{1}{r_1}\int_0^t k_1^{'m}(s)ds\leq\frac{1}{r_1}k_1^m(t)\leq C. \tag{2.19}$$

and

$$\int_0^t[\psi_t^m(1,s)]^2ds=\frac{1}{r_2}\int_0^t k_2^{'m}(s)ds\leq\frac{1}{r_2}k_2^m(t)\leq C. \tag{2.20}$$

Second a priori estimate.

Set t=0 ,$u=\varphi_t^m(.,0)$ in (2.5), integrate by parts, then exploit (2.6) and (2.7) to obtain.

$$\int_0^1\varphi_{tt}^m\varphi_t^m(.,0)dx+\int_0^1(\varphi_x^m+\psi^m)\varphi_{xt}^m(.,0)dx+k_1^m(0)[\varphi_t^m(1,0)]^2=0$$

$$\|\varphi_{tt}^m(.,0)\|^2=\int_0^t[\varphi_{0xx}^m(x)+\psi_{0x}^m(x)]\varphi_{tt}(x,0)dx$$

$$\leq\|\varphi_{0xx}^m+\psi_{0x}^m\|\|\varphi_{tt}(.,0)\|.$$

This entails that

$$\|\varphi_{tt}^m(.,0)\| \leq \|\varphi_{0xx}^m + \psi_{0x}^m\| \leq C. \tag{2.21}$$

Similarly, Setting t=0 ,$v = \psi_t^m(.,0)$ in (2.5), integrate by parts, then exploit (2.6) and (2.8) we find

$$\int_0^1 \psi_{tt}^m \psi_t^m(.,0)dx + \int_0^1 \psi_x^m \psi_{xt}^m(.,0)dx + \int_0^1 (\varphi_x^m + \psi^m)\psi_t^m dx + k_2^m(0)[\psi_t^m(1,0)]^2 = 0$$

$$\|\psi_{tt}^m(.,0)\| \leq \|\psi_{0xx}^m\| + \|\varphi_{0x}^m + \psi_0^m\| \leq C. \tag{2.22}$$

Third a priori estimate.

First, differentiating $(2.5)_1$ and $(2.5)_2$ with respect to t and replacing u and v by φ_{tt}^m and ψ_{tt}^m, respectively, subsequently, addin the resultants we reach

$$\frac{d}{dt}\frac{1}{2}\left\{\|\varphi_{tt}^m(t)\|^2+\|\varphi_{xt}^m(t) + \psi_t^m(t)\|^2+\|\psi_{tt}^m(t)\|^2+\|\psi_{xt}^m(t)\|^2+\frac{r_1}{2}[\varphi_t^m(1,t)]^4 + \frac{r_2}{2}[\psi_t^m(1,t)]^4\right\}$$

$$= -k_1^m(t)[\varphi_{tt}^m(1,t)]^2 - k_2^m(t)[\psi_{tt}^m(1,t)]^2 \leq 0.$$

Integrating over (0,t) and taking advantage of (2.6), (2.21), and (2.22), we arrive at

$$\begin{aligned} &\|\varphi_{tt}^m(t)\|^2 + \|\varphi_{xt}(t) + \psi_t^m(t)\|^2 + \|\psi_{tt}^m(t)\|^2 + \|\psi_{xt}^m(t)\|^2 + \frac{r_1}{2}[\varphi_t^m(1,t)]^4 + \frac{r_2}{2}[\psi_t^m(1,t)]^4 \\ &\leq \|\varphi_{tt}^m(.,0)\|^2+\|\varphi_{1x}+\psi_1^m\|^2+\|\psi_{tt}^m(.,0)\|^2+\|\psi_{1x}^m\|^2+\frac{r_1}{2}[\varphi_1^m(1)]^4+\frac{r_2}{2}[\psi_1^m(1)]^4 \end{aligned} \tag{2.23}$$

$$\leq C.$$

We conclude from (2.18) to (2.20) and (2.23) that $t_m = \infty$ and for any $T > 0$, we have

$$\begin{aligned}
&(\varphi^m) \text{ and } (\psi^m) \text{ are bounded in } L^\infty(0,T;V),\\
&(\varphi_t^m) \text{ and } (\psi_t^m) \text{ are bounded in } L^\infty(0,T;V),\\
&(\varphi_{tt}^m) \text{ and } (\psi_{tt}^m) \text{ are bounded in } L^\infty(0,T;L^2(0,1)),\\
&(\varphi_t^m(1,.)) \text{ and } (\psi_t^m(1,.)) \text{ are bounded in } L^2(0,T)\cap L^\infty(0,T)
\end{aligned} \tag{2.24}$$

So, there exists a subsequence, still denoted by(φ^m, ψ^m), such that

$$\begin{aligned}
&\varphi^m \to \varphi, \psi^m \to \psi \text{ weakly star in } L^\infty(0,T;V),\\
&\varphi_t^m \to \varphi_t, \psi_t^m \to \psi_t \text{ weakly star in } L^\infty(0,T;V),\\
&\varphi_{tt}^m \to \varphi_{tt}, \psi_{tt}^m \to \psi_{tt} \text{ weakly star in } L^\infty(0,T;L^2(0,1)),
\end{aligned} \tag{2.25}$$

Using Aubin-Lions lemma 1 , we infer that

$$\varphi_t^m \to \varphi_t,\ \psi_t^m \to \psi_t \text{ in } L^\infty(0,T;L^2(0,1)),$$

from which we deduce

$$\varphi_t^m \to \varphi,\ \psi_t^m \to \psi \text{ a.e. in } (0,1)\times(0,T).$$

This, together with the continuity of $\varphi_t^m, \psi_t^m, \varphi_t, \psi_t$, yield

$$\varphi_t^m(1,.) \to \varphi_t(1,.), \quad \psi_t^m(1,.) \to \psi_t(1,.) \text{ pointwise in } [0,T].$$

The boundedness of $(\varphi_t^m(1,.))$ and $(\psi_t^m(1,.))$ in $L^2(0,T)$ follows from $(2.24)_4$. Owing to Lebesgue dominated convergence theorem we infer that

$$\varphi_t^m \to \varphi_t(1,.) \quad \text{and} \quad \psi_t^m \to \psi_t(1,.) \text{ in } L^2(0,T). \tag{2.26}$$

An exploitation of $(2.5)_3$, $(2.5)_4$, and (2.26) gives

$$\begin{aligned} k_1^m(t) =& k_{01} + r_1 \int_0^t [\varphi_t^m(1,s)]^2 ds \to k_{01} + r_1 \int_0^t [\varphi_t(1,s)]^2 ds = k_1(t) \\ k_2^m(t) =& k_{02} + r_2 \int_0^t [\psi_t^m(1,s)]^2 ds \to k_{02} + r_2 \int_0^t [\psi_t(1,s)]^2 ds = k_2(t) \end{aligned} \tag{2.27}$$

in $L^\infty(0,T)$. Also, from $(2.24)_4$ and (2.26), we have

$$\begin{aligned} k_1^{'m}(.) = r_1[\varphi_t(1,.)]^2 \rightharpoonup r_1[\varphi_t(1,.)]^2 = k_1^{'} \text{ in } L^2(0,T) \\ k_2^{'m}(.) = r_2[\psi_t(1,.)]^2 \rightharpoonup r_2[\psi_t(1,.)]^2 = k_2^{'} \text{ in } L^2(0,T). \end{aligned} \tag{2.28}$$

Next, we take the limit of (2.6) as m goes to infinity, then use (2.25)-(2.27) to obtain

$$\int_0^1 \varphi_{tt} u dx + \int_0^1 (\varphi_x + \psi) u_x dx + k_1(t)\varphi_t(1,t)u(1) = 0,$$

$$\int_0^1 \psi_{tt} v dx + \int_0^1 \psi_x v_x dx + \int_0^1 (\varphi_x + \psi) v dx + k_2(t)\psi_t(1,t)v(1) = 0,$$

for any $u, v \in V$. Using Aubin-Lions lemma 1 again, we entail that $\varphi, \psi \in C([0,T];W), \varphi_t, \psi_t \in C([0,T];V), \varphi_{tt}, \psi_{tt} \in C([0,T];L^2(0,1))$ and $\varphi(0,.) = \varphi_0$, $\varphi_t(0,.) = \varphi_1$, $\psi(0,.) = \psi_0$, $\psi_t(0,.) = \psi_1$.

To prove the uniqueness, let $(\varphi, \psi, k_1, k_2)$ and $(\tilde{\varphi}, \tilde{\psi}, \tilde{k_1}, \tilde{k_2})$ be two solutions of (2.4) with the same initial data. Then,

$$\begin{aligned} \frac{d}{dt}\Big\{\|\varphi_t - \tilde{\varphi}_t\|^2 &+ \|(\varphi_x + \psi) - (\tilde{\varphi}_x + \tilde{\psi})\|^2 + \|\psi_t - \tilde{\psi}_t\|^2 + \|\psi_x - \tilde{\psi}_x\|^2 \\ &+ \frac{1}{2}[\tilde{k}_1(t) - \tilde{k}_1(t)]^2 + \frac{1}{2}[\tilde{k}_2(t) - \tilde{k}_2(t)]^2\Big\} \\ &= [k_1(t) + \tilde{k}_1(t)][\varphi_t(1,t) - \tilde{\varphi}_t(1,t)]^2 - [k_2(t) + \tilde{k}_2(t)][\psi_t(1,t) - \tilde{\psi}_t(1,t)]^2 \\ &\leq 0. \end{aligned} \tag{2.29}$$

This implies that

$$(\varphi, \psi, k_1, k_2) = (\tilde{\varphi}, \tilde{\psi}, \tilde{k}_1, \tilde{k}_2).$$

■

2.3 Stability analysis

In this section, we use the energy method to prove that system (2.4) is exponentially stable ,To achieve this goal, we first establish some technical lemmas needed in the proof of exponential stability result.

Lemma 6 *Let* (φ, ψ)*be the solution of* (2.4)*, Then the energy functional E, defined by*

$$E(t) := \frac{1}{2}[\|\varphi_t\|^2 + \|\varphi_x + \psi\|^2 + \|\psi_t\|^2 + \|\psi_x\|^2], \quad \forall t \geq 0 \tag{2.30}$$

satisfies

$$E^{'}(t) = -k_1(t)\varphi_t^2(1,t) - k_2(t)\psi_t^2(1,t) \leq 0 \qquad \forall t \geq 0 \tag{2.31}$$

Proof. Multiplying $(2.4)_1$ and $(2.4)_2$ by φ_t and ψ_t respectively, integrating over (0,1) and using integration by parts and the boundary conditions.

The first equation

$$\int_0^1 \varphi_{tt}\varphi_t dx - \int_0^1 (\varphi_x + \psi)_x \varphi_t dx = 0 \tag{2.32}$$

where

$$\int_0^1 \varphi_{tt}\varphi_t dx = \frac{1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2 dx = \frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 \tag{2.33}$$

and

$$-\int_0^1 (\varphi_x + \psi)_x \varphi_t dx = -\int_0^1 \varphi_{xx}\varphi_t dx - \int_0^1 \psi_x \varphi_t dx$$

$$= -[\varphi_x \varphi_t]_0^1 + \int_0^1 \varphi_x \varphi_{tx} - [\psi \varphi_t]_0^1 + \int_0^1 \psi \varphi_{tx} dx$$

$$= -\varphi_x(1,t)\varphi_t(1,t) + \varphi_x(0,t)\varphi_t(0,t) + \frac{1}{2}\frac{d}{dt}\int_0^1 \varphi_x^2$$

$$-\psi(1,t)\varphi_t(1,t) + \psi(0,t)\varphi_t(0,t) + \frac{d}{dt}\int_0^1 \psi \varphi_x dx - \int_0^1 \psi_t \varphi_x dx$$

So

$$= \frac{1}{2}\frac{d}{dt}\|\varphi_x\|^2 + \frac{d}{dt}\int_0^1 \psi \varphi_x dx - \int_0^1 \psi_t \varphi_x dx - \varphi_x(1,t)\varphi_t(1,t) - \psi(1,t)\varphi_t(1,t) \qquad (2.34)$$

Substituting (2.33) and (2.34) in (2.32) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\varphi_x\|^2 + \frac{d}{dt}\int_0^1 \psi \varphi_x dx - \int_0^1 \psi_t \varphi_x dx - \varphi_x(1,t)\varphi_t(1,t) - \psi(1,t)\varphi_t(1,t) = 0$$

Using $(2.4)_4$, we get

$$\varphi_x(1,t) + \psi(1,t) = -k_1(t)\varphi_t(1,t)$$

$$\varphi_x(1,t) = -k_1(t)\varphi_t(1,t) - \psi(1,t)$$

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\varphi_x\|^2 + \frac{d}{dt}\int_0^1 \psi \varphi_x dx - \int_0^1 \psi_t \varphi_x dx$$

$$-[-k_1(t)\varphi_t(1,t) - \psi(1,t)]\varphi_t(1,t) - \psi(1,t)\varphi_t(1,t) = 0.$$

So

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\varphi_x\|^2 + \frac{d}{dt}\int_0^1 \psi \varphi_x dx - \int_0^1 \psi_t \varphi_x dx + k_1(t)\varphi_t^2(1,t) = 0 \qquad (2.35)$$

The second equation

$$\int_0^1 \psi_{tt}\psi_t dx - \int_0^1 \psi_{xx}\psi_t dx + \int_0^1 (\varphi_x + \psi)\psi_t dx = 0 \tag{2.36}$$

The first term

$$\int_0^1 \psi_{tt}\psi_t dx = \frac{1}{2}\frac{d}{dt}\int_0^1 \psi_t^2 dx = \frac{1}{2}\frac{d}{dt}\|\psi_t\|^2 \tag{2.37}$$

The second term

$$-\int_0^1 \psi_{xx}\psi_t dx = -[\psi_x\psi_t]_0^1 + \int_0^1 \psi_x\psi_{tx} dx$$

$$= -\psi_x(1,t)\psi_t(1,t) + \psi_x(0,t)\psi_t(0,t) + \frac{1}{2}\frac{d}{dt}\int_0^1 \psi_x^2 dx$$

$$= \frac{1}{2}\frac{d}{dt}\|\psi_x\|^2 - \psi_x(1,t)\psi_t(1,t) \tag{2.38}$$

The third term

$$\int_0^1 (\varphi_x + \psi)\psi_t dx = \int_0^1 \varphi_x\psi_t dx + \int_0^1 \psi\psi_t dx$$

$$= \int_0^1 \varphi_x\psi_t dx + \frac{1}{2}\frac{d}{dt}\int_0^1 \psi^2 dx$$

$$= \int_0^1 \varphi_x\psi_t dx + \frac{1}{2}\frac{d}{dt}\|\psi\|^2 \tag{2.39}$$

Substituting (2.37) and (2.38), (2.39) in (2.36) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\psi_x\|^2 + \frac{1}{2}\frac{d}{dt}\|\psi\|^2 + \int_0^1 \varphi_x\psi_t dx - \psi_x(1,t)\psi_t(1,t) = 0.$$

Using $(2.4)_2$, we find

$$\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2+\frac{1}{2}\frac{d}{dt}\|\psi_x\|^2+\frac{1}{2}\frac{d}{dt}\|\psi\|^2+\int_0^1\varphi_x\psi_t dx-[-k_2(t)\psi_t(1,t)]\psi_t(1,t)=0.$$

So

$$\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2+\frac{1}{2}\frac{d}{dt}\|\psi_x\|^2+\frac{1}{2}\frac{d}{dt}\|\psi\|^2+\int_0^1\varphi_x\psi_t dx+k_2(t)\psi_t^2(1,t)=0. \tag{2.40}$$

from (2.35) and (2.40) , we get

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2+\frac{1}{2}\frac{d}{dt}\|\varphi_x\|^2+\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2+\frac{1}{2}\frac{d}{dt}\|\psi_x\|^2+\frac{1}{2}\frac{d}{dt}\|\psi\|^2+\frac{d}{dt}\int_0^1\psi\varphi_x dx$$

$$=-k_1(t)\varphi_t^2(1,t)-k_2(t)\psi_t^2(1,t)$$

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2+\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2+\frac{d}{dt}\frac{1}{2}\|\psi_x\|^2+\frac{1}{2}\frac{d}{dt}\int_0^1(\varphi_x^2+\psi^2+2\psi\varphi_x)dx$$

$$=-k_1(t)\varphi_t^2(1,t)-k_2(t)\psi_t^2(1,t)$$

$$=\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2+\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2+\frac{1}{2}\frac{d}{dt}\|\psi_x\|^2+\frac{1}{2}\frac{d}{dt}\|\varphi_x+\psi\|^2$$

$$=-k_1(t)\varphi_t^2(1,t)-k_2(t)\psi_t^2(1,t)$$

$$\frac{1}{2}\frac{d}{dt}[\|\varphi_t\|^2+\|\psi_t\|^2+\|\psi_x\|^2+\|\varphi_x+\psi\|^2]$$

$$=-k_1(t)\varphi_t^2(1,t)-k_2(t)\psi_t^2(1,t)$$

$$E'(t) = -k_1(t)\varphi_t^2(1,t) - k_2(t)\psi_t^2(1,t) \leq 0$$

■

Lemma 7 *Let (φ, ψ) be the solution of (2.4), We define another Lyapounov functional L by*

$$L(t) := E(t) + \frac{1}{2r_1}k_1^2(t) + \frac{1}{2r_2}k_2^2(t), \qquad \forall t \geq 0.$$

satisfies

$$L'(t) = -k_1(t)\varphi_t^2(1,t) - k_2(t)\psi_t^2(1,t) + \frac{1}{r_1}k_1(t)[r_1\varphi_t^2(1,t)] + \frac{1}{r_2}k_2(t)[r_2\psi_t^2(1,t)] = 0.$$

and

$$\sup_{t\geq 0}[E(t) + k_1^2(t) + k_2^2(t)] \leq C_1,$$

where $C_1 = L(0)(1 + 2r_1 + 2r_2)$

Proof. Then, it follows from definitions of k_1, k_2 in (2.4), (2.30) and (2.31) that, for any $t \geq 0$,

$$\begin{aligned} L'(t) &= E'(t) + \frac{1}{2r_1}[2k_1(t)k_1'(t)] + \frac{1}{2r_2}[2k_2(t)k'(t)] \\ &= -k_1(t)\varphi_t^2(1,t) - k_2(t)\psi_t^2(1,t) + \frac{1}{r_1}k_1(t)k_1'(t) + \frac{1}{r_2}k_2(t)k_2'(t) \\ &= -k_1(t)\varphi_t^2(1,t) - k_2(t)\psi_t^2(1,t) + \frac{1}{r_1}k_1(t)[r_1\varphi_t^2(1,t)] + \frac{1}{r_2}k_2(t)[r_2\psi_t^2(1,t)] = 0. \end{aligned}$$

As a consequence, we get

$$L(t) = L(0), \qquad \forall t \geq 0.$$

On other hand, we have

$$E(t) \leq L(0), \qquad \forall t \geq 0, \tag{2.41}$$

and

$$\frac{1}{2r_1}k_1^2(t) \leq L(0) \qquad \forall t \geq 0,$$

$$k_1^2(t) \leq 2r_1 L(0) \tag{2.42}$$

and

$$\frac{1}{2r_2}k_2^2(t) \leq L(0) \qquad \forall t \geq 0,$$

$$k_2^2(t) \leq 2r_2 L(0) \tag{2.43}$$

by adding up (2.41), (2.42) and (2.43), we arrive

$$E(t) + k_1^2(t) + k_2^2(t) \leq C_1$$

and this entails

$$\sup_{t\geq 0}[E(t) + k_1^2(t) + k_2^2(t)] \leq C_1. \tag{2.44}$$

■

Lemma 8 *Let (φ, ψ) be the solution of (2.4), then the functional*

$$I_1(t) := 2\int_0^1 x\varphi_x\varphi_t dx, \tag{2.45}$$

satisfies

$$I_1'(t) = -\|\varphi_t\|^2 - \|\varphi_x\|^2 + [\psi(1,t) + k_1(t)\varphi(1,t)]^2 + \varphi_t^2(1,t) + 2\int_0^1 x\varphi_x\psi_x dx,‘$$

Proof. Using (2.45) and integration by parts, gives

$$I_1'(t) = 2\int_0^1 x\varphi_{xt}\varphi_t dx + 2\int_0^1 x\varphi_x\varphi_{tt} dx$$

$$= 2\int_0^1 x\varphi_{xt}\varphi_t dx + 2\int_0^1 x(\varphi_x + \psi)_x\varphi_x dx$$

$$= 2\int_0^1 x\varphi_{xt}\varphi_t dx + 2\int_0^1 x\varphi_{xx}\varphi_x x dx + 2\int_0^1 x\psi_x\varphi_x dx$$

where

$$2\int_0^1 x\varphi_{xt}\varphi_t dx = 2\int_0^1 x(\frac{1}{2}\frac{d}{dx}\varphi_t^2)dx = \int_0^1 x\frac{d\varphi_t^2}{dx}dx$$

$$= [x\varphi_t^2]_0^1 - \int_0^1 \varphi_t^2 dx = \varphi_t^2(1,t) - \int_0^1 \varphi_t^2 dx = \varphi_t^2(1,t) - \|\varphi_t\|^2 \tag{2.46}$$

and

$$2\int_0^1 x\varphi_{xx}\varphi_x dx = 2(\frac{1}{2}\frac{d}{dx}\varphi_x^2)dx = \int_0^1 x\frac{d\varphi_t^2}{dx}dx$$

$$= [x\varphi_x^2]_0^1 - \int_0^1 \varphi_x^2 dx = \varphi_x^2(1,t) - \int_0^1 \varphi_x^2 dx = \varphi_x^2(1,t) - \|\varphi_x\|^2 \tag{2.47}$$

from (2.46) and (2.47), we obtain

$$I_1'(t) = -\|\varphi_t\|^2 - \|\varphi_x\|^2 + \varphi_x^2(1,t) + \varphi_t^2(1,t) + 2\int_0^1 x\varphi_x\psi_x dx$$

$$= -\|\varphi_t\|^2 - \|\varphi_x\|^2 + [\psi(1,t) + k_1(t)\varphi(1,t)]^2 + \varphi_t^2(1,t) + 2\int_0^1 x\varphi_x\psi_x dx, \quad (2.48)$$

■

Lemma 9 *Let (φ, ψ) be the solution of (2.4), then the functional*

$$I_2 := 2\int_0^1 x\psi_x\psi_t dx, \quad (2.49)$$

satisfies

$$I_2'(t) = -\|\psi_t\|^2 - \|\psi_x\|^2 + \|\psi\|^2 + [1 + k_2^2(t)]\psi_t^2(1,t) - 2\int_0^1 x\varphi_x\psi_x dx$$

Proof. Using (2.49) and integration by parts, gives

$$I_2'(t) = 2\int_0^1 x\psi_{xt}\psi_t dx + 2\int_0^1 x\psi_x\psi_{tt} dx$$

$$= 2\int_0^1 x\psi_{xt}\psi_t dx + 2\int_0^1 x\psi_x\psi_{xx} dx - 2\int_0^1 x\psi_x(\varphi_x + \psi) dx$$

$$= 2\int_0^1 x\psi_{xt}\psi_t dx + 2\int_0^1 x\psi_x\psi_{xx} dx - 2\int_0^1 x\psi_x\varphi_x dx - 2\int_0^1 x\psi_x\psi dx$$

where

$$2\int_0^1 x\psi_{xt}\psi_t dx = 2\int_0^1 x(\frac{1}{2}\frac{d}{dx}\psi_t^2) dx = \int_0^1 x\frac{d\varphi_t^2}{dx} dx$$

$$= [x\psi_t^2]_0^1 - \int_0^1 \psi_t^2 dx = \psi_t^2(1,t) - \|\psi_t\|^2 \quad (2.50)$$

and

$$2\int_0^1 x\psi_x\psi_{xx}dx = 2\int_0^1 x(\frac{1}{2}\frac{d}{dx}\psi_x)dx = \int_0^1 x\frac{d}{dx}\psi_x dx$$

$$= [x\psi_x^2]_0^1 - \int_0^1 \psi_x^2 dx = \psi_x^2(1,t) - \|\psi_x^2\|dx \tag{2.51}$$

and

$$-2\int_0^1 x\psi_x\psi dx = -2\int_0^1 x(\frac{1}{2}\frac{d}{dx}\psi)dx = -\int_0^1 x\frac{d}{dx}\psi dx$$

$$= -[x\psi^2]_0^1 + \int_0^1 \psi_x^2 dx = -\psi^2(1,t) + \|\psi^2\|dx \tag{2.52}$$

from (2.50), (2.51) and (2.52), we get

$$I_2'(t) = -\|\psi_t\|^2 - \|\psi_x\|^2 + \|\psi\|^2 + \psi_t^2(1,t) + \psi_x^2(1,t) - \psi^2(1,t) - 2\int_0^1 x\varphi_x\psi_x dx$$

$$= -\|\psi_t\|^2 - \|\psi_x\|^2 + \|\psi\|^2 + \psi_t^2(1,t) + k_2^2\psi_t^2(1,t) - \psi^2(1,t) - 2\int_0^1 x\varphi_x\psi_x dx$$

$$= -\|\psi_t\|^2 - \|\psi_x\|^2 + \|\psi\|^2 + \psi_t^2(1,t) + [1+k_2^2]\psi_t^2(1,t) - \psi^2(1,t) - 2\int_0^1 x\varphi_x\psi_x dx, \tag{2.53}$$

■

Lemma 10 *Let* (φ,ψ) *be the solution of* (2.4), *then the functional*

$$I_3(t) := -\frac{1}{2}\int_0^1 \varphi\varphi_t dx, \tag{2.54}$$

satisfies

$$I_3'(t) = -\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\varphi_x\|^2 + \frac{1}{2}k_1(t)\varphi_t(1,t)\varphi(1,t) + \frac{1}{2}\int_0^1 \varphi_x\psi dx,$$

Proof. Using (2.54) and integration by parts, gives

$$I_3'(t) = -\frac{1}{2}\int_0^1 \varphi_t^2 dx - \frac{1}{2}\int_0^1 \varphi\varphi_{tt} dx$$

$$= -\frac{1}{2}\int_0^1 \varphi_t^2 dx - \frac{1}{2}\int_0^1 \varphi(\varphi_x + \psi)_x dx$$

$$= -\frac{1}{2}\int_0^1 \varphi_t^2 dx - \frac{1}{2}\int_0^1 \varphi\varphi_{xx} dx - \frac{1}{2}\int_0^1 \varphi\psi_x dx$$

where

$$-\frac{1}{2}\int_0^1 \varphi_t^2 dx = -\frac{1}{2}\|\varphi_t\|^2, \tag{2.55}$$

and

$$-\frac{1}{2}\int_0^1 \varphi\varphi_{xx} dx = -\frac{1}{2}[\varphi\varphi_x]_0^1 + \frac{1}{2}\varphi_x^2 dx$$

$$= -\frac{1}{2}\varphi(1,t)\varphi_x(1,t) + \frac{1}{2}\|\varphi_x\|^2 dx, \tag{2.56}$$

and

$$-\frac{1}{2}\int_0^1 \varphi\psi_x dx = -\frac{1}{2}[\varphi\psi]_0^1 + \frac{1}{2}\int_0^1 \varphi_x\psi dx$$

$$= -\frac{1}{2}\varphi(1,t)\psi(1,t) + \frac{1}{2}\int_0^1 \varphi_x\psi dx, \tag{2.57}$$

from (2.55), (2.56) and (2.57), we get

$$I_3'(t) = -\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\varphi_x\|^2 - \frac{1}{2}\varphi(1,t)\varphi_x(1,t) - \frac{1}{2}\varphi(1,t)\psi(1,t) + \frac{1}{2}\int_0^1 \varphi_x\psi dx$$

$$= -\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\varphi_x\|^2 - \frac{1}{2}[-k_1(t)\varphi_t(1,t) - \psi(1,t)]\varphi_x(1,t) - \frac{1}{2}\varphi(1,t)\psi(1,t) + \frac{1}{2}\int_0^1 \varphi_x\psi dx$$

$$= -\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\varphi_x\|^2 - \frac{1}{2}k_1(t)\varphi_t(1,t)\varphi_x(1,t) + \frac{1}{2}\int_0^1 \varphi_x\psi dx, \tag{2.58}$$

■

Lemma 11 *Let (φ, ψ) be the solution of (2.4), then the functional*

$$I_4(t) := \frac{1}{2}\int_0^1 \psi\psi_t dx, \tag{2.59}$$

satisfies

$$I_4'(t) = \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\psi_x\|^2 - \frac{1}{2}\|\psi\|^2 - \frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t) + \frac{1}{2}\varphi_x\psi dx,$$

Proof. Using (2.59) and integration by parts, gives

$$I_4'(t) = \frac{1}{2}\int_0^1 \psi_t^2 dx + \frac{1}{2}\int_0^1 \psi\psi_{tt} dx$$

$$= \frac{1}{2}\int_0^1 \psi_t^2 dx + \frac{1}{2}\int_0^1 \psi[\psi_{xx} - (\varphi_x + \psi)]dx$$

$$= \frac{1}{2}\int_0^1 \psi_t^2 dx + \frac{1}{2}\int_0^1 \psi\psi_{xx} dx - \frac{1}{2}\int_0^1 \psi\varphi_x dx - \frac{1}{2}\int_0^1 \psi^2 dx$$

where

$$\frac{1}{2}\int_0^1 \psi_t^2 dx = \frac{1}{2}\|\psi_t\|^2, \tag{2.60}$$

and

$$\frac{1}{2}\int_0^1 \psi\psi_{xx}dx = \frac{1}{2}[\psi\psi_x]_0^1 - \frac{1}{2}\int_0^1 \psi_x^2$$

$$= \psi(1,t)\psi_x(1,t) - \frac{1}{2}\|\psi_x\|^2 \quad (2.61)$$

and

$$-\frac{1}{2}\int_0^1 \psi^2 dx = -\frac{1}{2}\|\psi\|^2 \quad (2.62)$$

from (2.60), (2.61) and (2.62), we get

$$I_4'(t) = \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\psi_x\|^2 - \frac{1}{2}\|\psi\|^2 + \psi_x(1,t)\psi(1,t) - \frac{1}{2}\int_0^1 \varphi_x\psi dx$$

$$= \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\psi_x\|^2 - \frac{1}{2}\|\psi\|^2 - \frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t) - \frac{1}{2}\int_0^1 \varphi_x\psi dx \quad (2.63)$$

■

Lemma 12 *The following function*

$$I(t) := \sum_{j=1}^{4} I_j(t)$$

satisfies

$$|I(t)| \leq \frac{9}{2}E(t), \qquad \forall t \geq 0. \quad (2.64)$$

Proof.
using Young's inequality, gives

$I_1(t) = 2\int_0^1 x\varphi_x\varphi_t dx \le 2[\frac{\varepsilon_1}{2}\int_0^1 (x\varphi_x)^2 dx + \frac{1}{2\varepsilon_1}\int_0^1 \varphi_t^2 dx]$

$I_2(t) = 2\int_0^1 x\psi_x\psi_t dx \le 2[\frac{\varepsilon_2}{2}\int_0^1 (x\psi_x)^2 dx + \frac{1}{2\varepsilon_2}\int_0^1 \psi_t^2 dx]$

$I_3(t) = -\frac{1}{2}\int_0^1 \varphi\varphi_t \le \frac{1}{2}[\frac{\varepsilon_3}{2}\int_0^1 \varphi^2 dx + \frac{1}{2\varepsilon_3}\int_0^1 \varphi_t^2 dx]$

$I_4(t) = \frac{1}{2}\int_0^1 \psi\psi_t \le \frac{1}{2}[\frac{\varepsilon_4}{2}\int_0^1 \psi^2 dx + \frac{1}{2\varepsilon_4}\int_0^1 \psi_t^2 dx]$

$\sum_{j=1}^{4} I_j(t) \le \varepsilon_1\int_0^1 (x\varphi_x)^2 dx + \frac{1}{\varepsilon_1}\|\varphi_t\|^2 + \varepsilon_2\int_0^1 (x\psi_x)^2 dx + \frac{1}{\varepsilon_2}\|\psi_t\|^2 + \frac{\varepsilon_3}{4}\|\varphi\|^2 + \frac{1}{4\varepsilon_3}\|\varphi_t\|^2 + \frac{\varepsilon_4}{4}\|\psi\|^2 + \frac{1}{4\varepsilon_4}\|\psi_t\|^2$

we have

$x^2 \le \text{Max } x^2$

so

$(x\varphi)^2 \le \text{Max } (x\varphi)^2$

$\le \varepsilon_1\|\varphi_x\|^2 + \frac{1}{\varepsilon_1}\|\varphi_t\|^2 + \varepsilon_2\|\psi_x\|^2 + \frac{1}{\varepsilon_2}\|\psi_t\|^2 + \frac{\varepsilon_3}{4}\|\varphi\|^2 + \frac{1}{4\varepsilon_3}\|\varphi_t\|^2 + \frac{\varepsilon_4}{4}\|\psi\|^2 + \frac{1}{4\varepsilon_4}\|\psi_t\|^2$

using inequality (1.1), we get

$\le \varepsilon_1\|\varphi_x\|^2 + \frac{1}{\varepsilon_1}\|\varphi_t\|^2 + \varepsilon_2\|\psi_x\|^2 + \frac{1}{\varepsilon_2}\|\psi_t\|^2 + \frac{\varepsilon_3}{8}\|\varphi_x\|^2 + \frac{1}{4\varepsilon_3}\|\varphi_t\|^2 + \frac{\varepsilon_4}{8}\|\psi_x\|^2 + \frac{1}{4\varepsilon_4}\|\psi_t\|^2$

$\le (\frac{\varepsilon_3}{8} + \varepsilon_1)\|\varphi_x\|^2 + (\frac{1}{\varepsilon_1} + \frac{1}{4\varepsilon_3})\|\varphi_t\|^2 + (\frac{\varepsilon_4}{8} + \varepsilon_2)\|\psi_x\|^2 + (\frac{1}{\varepsilon_2} + \frac{1}{4\varepsilon_4})\|\psi_t\|^2$

$(\frac{\varepsilon_3}{8} + \varepsilon_1) = (\frac{\varepsilon_4}{8} + \varepsilon_2)$

$\varepsilon_1 = \varepsilon_2$

$\varepsilon_3 = \varepsilon_4$

using inequality (1.3)

$(\frac{\varepsilon_2}{8} + \varepsilon_1)(\|\varphi_x\|^2 + \|\psi_x\|^2) \le (\frac{\varepsilon_2}{4} + 2\varepsilon_1)(\|\varphi_x + \psi\|^2 + \|\psi_x\|^2)$

$(\frac{1}{\varepsilon_1} + \frac{1}{4\varepsilon_2})(\|\varphi_t\|^2 + \|\psi_t\|^2)$

$(\frac{\varepsilon_2}{4} + 2\varepsilon_1)(\|\varphi_x + \psi\|^2 + \|\psi_x\|^2) + (\frac{1}{\varepsilon_1} + \frac{1}{4\varepsilon_2})(\|\varphi_t\|^2 + \|\psi_t\|^2)$

$\leq \frac{9}{4}[\|\varphi_x + \psi\|^2 + \|\psi_x\|^2 + \|\varphi_t\|^2 + \|\psi_t\|^2]$

$|I(t)| \leq \frac{9}{2}E(t)$

■

We introduce another Lyapunov functional

$$L_\varepsilon(t) = E(t) + \varepsilon I(t) \tag{2.65}$$

where $\varepsilon > 0$

Lemma 13 *The Lyapunov functional L_ε is equivalent E.*

Proof. We have

$$|L_\varepsilon(t) - E(t)| = \varepsilon|I(t)| \leq \frac{9}{2}\varepsilon E(t), \qquad \forall t \geq 0.$$

which implies

$$(1 - \frac{9}{2}\varepsilon)E(t) \leq L_\varepsilon(t) \leq (1 + \frac{9}{2}\varepsilon)E(t), \qquad \forall t \geq 0.$$

By choosing $\varepsilon < \frac{2}{9}$, we obtain the equivalence between L_ε and E.

$$0 \leq (1 - \frac{9}{2}\varepsilon)E(t) \leq L_\varepsilon(t) \leq (1 + \frac{9}{2}\varepsilon)E(t), \qquad \forall t \geq 0. \tag{2.66}$$

■

Lemma 14

$$I^{'}(t) \leq -\frac{1}{4}E(t) + (1 + \frac{13}{4}C_1)[\varphi_t^2(1,t) + \psi_t^2(1,t)], \quad \forall t \geq 0. \tag{2.67}$$

Proof. First, we have

$$I^{'}(t) := \sum_{j=1}^{4} I_j^{'}(t)$$

Applying integration by parts, we find

$$I^{'}(t) = -\frac{3}{2}\|\varphi_t\|^2 - \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\varphi_x\|^2 - \frac{3}{2}\|\psi_x\|^2 + \frac{1}{2}\|\psi\|^2 - \psi^2(1,t) + \varphi_t^2(1,t)$$

$$+[1+k_2^2(t)]\psi_t^2(1,t)+[\psi(1,t)+k_1(t)\varphi_t(1,t)]^2+\frac{1}{2}k_1(t)\varphi_t(1,t)\varphi(1,t)-\frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t)$$

$$= -\frac{3}{2}\|\varphi_t\|^2 - \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\varphi_x\|^2 - \frac{3}{2}\|\psi_x\|^2 + \frac{1}{2}\|\psi\|^2 + \varphi_t^2(1,t) + \psi_t^2(1,t) + k_2^2(t)\psi_t^2(1,t)$$

$$+k_1^2(t)\varphi_t^2(1,t) + 2k_1(t)\varphi_t(1,t)\psi(1,t) + \frac{1}{2}k_1(t)\varphi_t(1,t)\varphi(1,t) - \frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t)$$

using inequality (1.1), we get

$$\|\psi\|^2 \leq \frac{1}{2}\|\psi_x\|^2$$

So

$$\frac{1}{2}\|\psi\|^2 \leq \frac{1}{4}\|\psi_x\|^2$$

therefore

$$-\frac{3}{2}\|\psi_x\|^2+\frac{1}{2}\|\psi\|^2 \leq -\frac{3}{2}\|\psi_x\|^2+\frac{1}{4}\|\psi_x\|^2 \leq -\frac{5}{4}\|\psi_x\|^2$$

and using Young's inequality, gives

$$2k_1(t)\varphi_t(1,t)\psi(1,t) \leq 2[\frac{1}{4}\psi^2(1,t)+k_1^2(t)\varphi_t^2(1,t)] \leq \frac{1}{2}\psi^2(1,t)+2k_1^2(t)\varphi_t^2(1,t)$$

$$\frac{1}{2}k_1(t)\varphi_t(1,t)\varphi(1,t) \leq \frac{1}{2}[\frac{1}{2}\varphi^2(1,t)+\frac{1}{2}k_1^2(t)\varphi_t^2(1,t)] \leq \frac{1}{4}\varphi^2(1,t)+\frac{1}{4}k_1^2(t)\varphi_t^2(1,t)$$

$$-\frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t) \leq \frac{1}{4}k_2^2(t)\psi_t^2(1,t)+\frac{1}{4}\psi^2(1,t)$$

we get

$$I' \leq -\frac{3}{2}\|\varphi_t\|^2-\frac{1}{2}\|\psi_t\|-\frac{1}{2}\|\varphi_x\|-\frac{5}{4}\|\psi_x\|^2+\varphi_t^2(1,t)+\psi_t^2(1,t)+k_2^2(t)\psi_t^2(1,t)+k_1^2(t)\varphi_t^2(1,t)$$

$$+\frac{1}{2}\psi^2(1,t)+2k_1^2(t)\varphi_t^2(1,t)+\frac{1}{4}\varphi^2(1,t)+\frac{1}{4}k_1^2(t)\varphi_t^2(1,t)+\frac{1}{4}k_2^2(t)\psi_t^2(1,t)+\frac{1}{4}\psi^2(1,t)$$

$$\leq -\frac{3}{2}\|\varphi_t\|^2-\frac{1}{2}\|\psi_t\|-\frac{1}{2}\|\varphi_x\|-\frac{5}{4}\|\psi_x\|^2+\varphi_t^2(1,t)+\psi_t^2(1,t)+\frac{5}{4}k_2^2(t)\psi_t^2(1,t)$$

$$+\frac{13}{4}k_1^2(t)\varphi_t^2(1,t)+\frac{3}{4}\psi^2(1,t)+\frac{1}{4}\varphi^2(1,t)$$

So

$$I^{'}(t) \leq -\frac{3}{2}\|\varphi_t\|^2 - \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\varphi_x\| - \frac{5}{4}\|\psi_x\|^2 + \frac{1}{4}\varphi^2(1,t) + \frac{3}{4}\psi^2(1,t)$$

$$+ [1 + \frac{13}{4}k_1^2(t)]\varphi_t^2(1,t) + [1 + \frac{5}{4}k_2^2(t)]\psi_t^2(1,t)$$

using inequality (1.2)

$$\varphi^2(1,t) \leq \|\varphi_x\|^2$$

$$\frac{1}{4}\varphi^2(1,t) \leq \frac{1}{4}\|\varphi_x\|^2$$

So

$$-\frac{1}{2}\|\varphi_x\|^2 + \frac{1}{4}\varphi^2(1,t) \leq \frac{1}{2}\|\varphi_x\|^2 + \frac{1}{4}\|\varphi_x\|^2 \leq -\frac{1}{4}\|\varphi_x\|^2$$

and

$$\psi^2(1,t) \leq \|\psi_x\|^2$$

$$\frac{3}{4}\psi^2(1,t) \leq \frac{3}{4}\|\psi_x\|^2$$

So

$$-\frac{5}{4}\|\psi_x\|^2+\frac{3}{4}\psi^2(1,t)\le-\frac{5}{4}\|\psi_x\|^2+\frac{3}{4}\|\psi_x\|^2\le-\frac{1}{2}\|\psi_x\|^2$$

therefore

$$I'(t)\le-\frac{3}{2}\|\varphi_t\|^2-\frac{1}{2}\|\psi_t\|^2-\frac{1}{4}\|\varphi_x\|-\frac{1}{4}\|\psi_x\|^2$$

$$+[1+\frac{13}{4}k_1^2(t)]\varphi_t^2(1,t)+[1+\frac{5}{4}k_2^2(t)]\psi_t^2(1,t)$$

$$\le-\frac{1}{4}[\|\varphi_t\|^2+\|\psi_t\|^2+\|\varphi_x\|^2+\|\psi_x\|^2]$$

$$+[1+\frac{13}{4}k_1^2(t)]\varphi_t^2(1,t)+[1+\frac{5}{4}k_2^2(t)]\psi_t^2(1,t)$$

$$\le-\frac{1}{4}E(t)+[1+\frac{13}{4}k_1^2(t)]\varphi_t^2(1,t)+[1+\frac{5}{4}k_2^2(t)]\psi_t^2(1,t)$$

$$I'(t)\le-\frac{1}{4}E(t)+(1+\frac{13}{4}C_1)[\varphi_t^2(1,t)+\psi_t^2(1,t)],\quad\forall t\ge0.\tag{2.68}$$

■

Lemma 15

$$L_\varepsilon'(t)\le-\frac{\varepsilon}{4}E(t)-[k-\varepsilon(1+\frac{13}{4}C_1)][\varphi_t^2(1,t)+\psi_t^2(1,t)],\tag{2.69}$$

where $k=\min\{k_{01},k_{02}\}$.

Proof. we have from (2.65)
$L_\varepsilon'(t)=E'(t)+\varepsilon I'(t)$

$$L_\varepsilon'(t)\le-k_1(t)\varphi_t^2(1,t)-k_2(t)\psi_t^2(1,t)-\frac{\varepsilon}{4}E(t)+\varepsilon(1+\frac{13}{4}C_1)[\varphi_t^2(1,t)+\psi_t^2(1,t)]$$

$$\le-k_{01}\varphi_t^2(1,t)-k_{02}\psi_t^2(1,t)-\frac{\varepsilon}{4}E(t)+\varepsilon(1+\frac{13}{4}C_1)[\varphi_t^2(1,t)+\psi_t^2(1,t)]$$

$$\le-\frac{\varepsilon}{4}E(t)-k[\varphi_t^2(1,t)+\psi_t^2(1,t)]+\varepsilon(1+\frac{13}{4}C_1)[\varphi_t^2(1,t)+\psi_t^2(1,t)]$$

$$L_{\varepsilon}^{'}(t) \leq -\frac{\varepsilon}{4}E(t) - [k - \varepsilon(1 + \frac{13}{4}C_1)][\varphi_t^2(1,t) + \psi_t^2(1,t)],$$

■

Lemma 16 *Given any initial data* (ϕ_0, ϕ_1), $(\psi_0, \psi_1) \in W \times V$, *the solution associated to the closed-loop system* (2.4) *is exponentially stable*

$$E(t) \leq C_{\varepsilon} e^{-\lambda_{\varepsilon} t}, \qquad \forall t \geq 0, \tag{2.70}$$

where $C_{\varepsilon} = \dfrac{2 + 9\varepsilon}{2 - 9\varepsilon}E(0)$.

Proof. We choose $\varepsilon < \min\{\frac{2}{9}, \frac{4k}{4 + 13C_1}\}$ in the last lemma, so that

$$L_{\varepsilon}^{'}(t) \leq -\frac{\varepsilon}{4}E(t) \tag{2.71}$$

$$L_{\varepsilon}^{'}(t) \leq -\frac{\varepsilon}{2(2 + 9\varepsilon)}L_{\varepsilon}(t), \qquad \forall t \geq 0,$$

from (2.66),

$$L_{\varepsilon}(t) \leq (1 + \frac{9}{2}\varepsilon)E(t)$$

$$L_{\varepsilon}(t)(\frac{2}{2 + 9\varepsilon}) \leq E(t)$$

$$-\frac{\varepsilon}{4}E(t) \leq -\frac{\varepsilon}{4}(\frac{2}{2 + 9\varepsilon})L_{\varepsilon}(t)$$

so

$$-\frac{\varepsilon}{4}E(t) \leq -\frac{\varepsilon}{2(2 + 9\varepsilon)}L_{\varepsilon}(t)$$

from (2.71)

$$L_{\varepsilon}^{'}(t) \leq -\frac{\varepsilon}{2(2 + 9\varepsilon)}L_{\varepsilon}(t), \quad \forall t \geq 0.$$

A simple integration over (0,t) gives

$$\frac{L'_\varepsilon(t)}{L_\varepsilon(t)} \leq -\frac{\varepsilon}{2(2+9\varepsilon)}$$

$$\int_0^t \frac{L'_{\varepsilon(s)}}{L(s)} ds \leq -\int_0^t \frac{\varepsilon}{2(2+9\varepsilon)} ds$$

$$\int_0^t \frac{L'_{\varepsilon(s)}}{L(s)} ds \leq -\int_0^t \lambda_\varepsilon ds$$

$$[lnL_\varepsilon(s)]_0^t \leq [-\lambda_\varepsilon s]_0^t$$

$$ln(L_\varepsilon(t)) - ln(L_\varepsilon(0)) \leq -\lambda_\varepsilon t$$

$$ln(\frac{L_\varepsilon(t)}{L_\varepsilon(0)}) \leq -\lambda_\varepsilon t$$

$$\frac{L_\varepsilon(t)}{L_\varepsilon(0)} \leq e^{-\lambda_\varepsilon t}$$

So

$$L_\varepsilon(t) \leq L_\varepsilon(0) e^{-\lambda_\varepsilon t}, \qquad \forall t \geq 0, \tag{2.72}$$

where $\lambda_\varepsilon = \dfrac{\varepsilon}{2(2+9\varepsilon)}$.

Finally,a combination of this with (2.66) leads to

$$E(t) \leq C_\varepsilon e^{-\lambda_\varepsilon t}, \qquad \forall t \geq 0, \tag{2.73}$$

where $C_\varepsilon = \dfrac{2+9\varepsilon}{2-9\varepsilon}E(0)$. This completes the proof of our drsired result. ■

Chapter 3

Porous-elastic systems with boundary feedback controls

3.1 Statement of the problem

We consider the following problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & 0 < x < 1, t > 0 \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0, & 0 < x < 1, t > 0 \\ u(0,t) = \phi(0,t) = 0, & t > 0 \\ u_x(1,t) + \dfrac{b}{\mu}\phi(1,t) = f_1(t), \quad \phi_x(1,t) = f_2(t), & t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & 0 < x < 1 \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x) & 0 < x < 1 \end{cases} \tag{3.1}$$

where u_0, u_1, ϕ_0, ϕ_1 are given data, f_1 and f_2 are boundary control inputs to be determined later. The measured outputs of the system at the right end are given by

$$y_1(t) = u_t(1,t) \qquad \textit{and} \qquad y_2(t) = \phi_t(1,t) \qquad\qquad t > 0. \tag{3.2}$$

with the following boundary controls

$$\begin{cases} f_i(t) = -h_i(t)y_i(t), \\ h_i' = r_i y_i^2(t),\ h_i(0) = h_{0i}, \quad h_{0i}, r_i > 0, \end{cases} \tag{3.3}$$

for i=1,2.

The closed - loop system associaed to (3.1), (3.2), and (3.3) is given by

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, & 0 < x < 1, t > 0 \\ J\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0, & 0 < x < 1, t > 0 \\ u(0,t) = \phi(0,t) = 0, & t > 0 \\ u_x(1,t) + \dfrac{b}{\mu}\phi(1,t) = -h_1(t)u_t(1,t), & t > 0 \\ \phi_x(1,t) = -h_2(t)\phi_t(1,t), & t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) & 0 < x < 1 \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x,) & 0 < x < 1 \\ h_1'(t) = r_1[u_t(1,t)]^2,\ h_1(0) = h_{01}, h_{01}, r_1 > 0 \\ h_2'(t) = r_2[\phi_t(1,t)]^2,\ h_2(0) = h_{02}, h_{02}, r_2 > 0 \end{cases} \tag{3.4}$$

3.2 Global existence

In this section, we use Faedo-Galerkin method to prove our existence result.

Definition 9 *A pair of functions* (u, ϕ) *defined on* $(0,1) \times [0,T]$, $T > 0$ *arbitrary, is called a local strong solution of the closed-loop system* (3.4) *if*

$$u, \phi \in C([0,T]; W) \cap C^1([0,T]; V) \cap C^2([0,T]; L^2(0,1))$$

$$(u_0, \phi_0),\ (u_1, \phi_1) \in W \times V$$

and it satisfies

$$\rho\int_0^1 u_{tt}zdx + \mu\int_0^1 u_x z_x dx + b\int_0^1 \phi z_x dx + h_1(t)u_t(1,t)z(1) = 0$$

$$J\int_0^1 \phi_{tt}vdx + \delta\int_0^1 \phi_x v_x dx + b\int_0^1 u_x vdx + \xi\int_0^1 \phi vdx + h_2(t)\phi_t(1,t)v(1) = 0$$

$$h_1'(t) = r_1 u_t^2(1,t), \quad h_1(0) = h_{01}, \quad h_{01}, r_1 > 0,$$

$$h_2'(t) = r_2 \phi_t^2(1,t), \quad h_2(0) = h_{02}, \quad h_{02}, r_2 > 0,$$

for any $z, v \in V$ *and any* $t \in [0,T]$.

Now, we are ready to state and prove our existence result.

Theorem 10 *[2]*
Let (u_0, u_1),$(\phi_0, \phi_1) \in W \times V$ *satisfy the compatibility conditions.*

$$u_{0x}(1) + \frac{b}{\mu}\phi_0(1) = -h_{01}u_1(1) \qquad and \qquad \phi_{0x}(1) = -h_{02}\phi_1(1).$$

Then, there exists a unique global strong solution to system (3.4).

Proof. Let $\{w^j\}_{j\geq 1}$ be a complete oethogonal basis for W and V, and for each $m \geq 1$ let

$W_m := span\{w^1, ...w^m\}$.

We look for a solution in the form

$$u^m(x,t) = \sum_{j=1}^{m} a_{mj}(t) w^j(x) \qquad \text{and} \qquad \phi^m(x,t) = \sum_{j=1}^{m} b_{mj}(t) w^j(x)$$

to the approximate problem

$$\begin{cases} \rho \int_0^1 u_{tt}^m z dx + \mu \int_0^1 u_x^m z_x dx + b \int_0^1 \phi^m z_x dx + h_1^m(t) u_t^m(1,t) z(1) = 0 \\ J \int_0^1 \phi_{tt}^m v dx + \delta \int_0^1 \phi_x^m v_x dx + b \int_0^1 u_x^m v dx + \xi \int_0^1 \phi^m v dx + h_2^m(t) \phi_t^m(1,t) v(1) = 0 \\ h_1'^m(t) = r_1 [u_t^m(1,t)]^2, h_1^m(0) = h_{01}, \\ h_2'^m(t) = r_2 [\phi_t^m(1,t)]^2, h_2^m(0) = h_{02}, \\ u^m(.,0) = u_0^m, u_t^m(.,0) = u_1^m, \phi^m(.,0) = \phi_0^m, \phi_t^m(.,0) = \phi_1^m, \end{cases} \tag{3.5}$$

for all z,v $\in V_m$, where

$$\begin{cases} (u_0^m, u_1^m) := \left(\sum_{j=1}^{m} (u_0, w^j) w^j, \sum_{j=1}^{m} (u_1, w^j) w^j \right) \to (u_0, u_1) \text{ in } W \times V, \\ (\phi_0^m, \phi_1^m) := \left(\sum_{j=1}^{m} (\phi_0, w^j) w^j, \sum_{j=1}^{m} (\phi_1, w^j) w^j) \right) \to (\phi_0, \phi_1) \text{in } W \times V, \end{cases} \tag{3.6}$$

$$u_{0x}^m(1) + \frac{b}{\mu} \phi_0^m(1) = -h_{01} u_1^m(1). \tag{3.7}$$

and

$$\phi_{0x}^m(1) = -h_{02} \phi_1^m(1). \tag{3.8}$$

problem (3.5) is a system of ordinary differential equation in t which has a local solution , say, in $[0, t_m)$.

The next a priori estimates show that $t_m = \infty$ for any $m \geq 1$.

First a priori estimate

Substituting $z = u_t^m$ in $(3.5)_1$ and $v = \phi_t^m$ in $(3.5)_2$, then adding the resultants, we obtain

$$\rho \int_0^1 u_{tt}^m u_t^m dx + \mu \int_0^1 u_x^m u_{xt}^m dx + b \int_0^1 \phi^m u_{xt}^m dx + h_1^m(t)[u_t^m(1,t)]^2 = 0$$

where

$$\rho \int_0^1 u_{tt}^m u_t^m dx = \frac{\rho}{2} \frac{d}{dt} \int_0^1 (u_t^m)^2 dx = \frac{\rho}{2} \frac{d}{dt} \|u_t^m\|^2 \tag{3.9}$$

and

$$\mu \int_0^1 u_x^m u_{xt}^m dx = \frac{\mu}{2} \frac{d}{dt} \int_0^1 (u_x^m)^2 dx = \frac{\mu}{2} \frac{d}{dt} \|u_x^m\|^2 \tag{3.10}$$

and

$$b \int_0^1 \phi^m u_{xt}^m dx = b \frac{d}{dt} \int_0^1 \phi^m u_x^m dx - b \int_0^1 \phi_t^m u_x^m dx \tag{3.11}$$

from (3.9), (3.10) and (3.11), gives

The first equation

$$\frac{1}{2}\frac{d}{dt}[\rho\|u_t^m(t)\|^2+\mu\|u_x^m(t)\|^2+2b\int_0^1\phi^m(t)u_x^m(t)]dx-b\int_0^1\phi_t^m(t)u_x^m(t)dx+h_1^m(t)[u_t^m(1,t)]^2=0 \tag{3.12}$$

and we have

$$J\int_0^1\phi_{tt}^m\phi_t^m dx+\delta\int_0^1\phi_x^m\phi_{xt}^m dx+b\int_0^1 u_x^m\phi_t^m dx+\xi\int_0^1\phi^m\phi_t^m dx+h_2^m(t)[\phi_t^m(1,t)]^2=0$$

where

$$J\int_0^1\phi_{tt}^m\phi_t^m dx=\frac{J}{2}\frac{d}{dt}\int_0^1(\phi_t^m)^2dx=\frac{J}{2}\frac{d}{dt}\|\phi_t^m\|^2 \tag{3.13}$$

and

$$\delta\int_0^1\phi_x^m\phi_{xt}^m dx=\frac{\delta}{2}\frac{d}{dt}\int_0^1(\phi_x^m)^2dx=\frac{\delta}{2}\frac{d}{dt}\|\phi_x^m\|^2 \tag{3.14}$$

and

$$\xi\int_0^1\phi^m\phi_t^m dx=\frac{\xi}{2}\frac{d}{dt}\int_0^1(\phi^m)^2dx=\frac{\xi}{2}\frac{d}{dt}\|\phi^m\|^2 \tag{3.15}$$

from (3.13), (3.14) and (3.15), we get

The second equation

$$\frac{1}{2}\frac{d}{dt}[J\|\phi_t^m(t)\|^2+\delta\|\phi_x^m(t)\|^2+\xi\|\phi^m(t)\|^2]dx+b\int_0^1 u_x^m(t)\phi_t^m(t)dx+h_2^m(t)[\phi_t^m(1,t)]^2=0 \tag{3.16}$$

by adding up (3.12) and (3.16), we arriv to

$$\frac{d}{dt}\frac{1}{2}[\rho\|u_t^m(t)\|^2+\mu\|u_x^m(t)\|^2+J\|\phi_t^m(t)\|^2+\delta\|\phi_x^m(t)\|^2+\xi\|\phi^m(t)\|+2b\int_0^1\phi^m(t)u_x^m(t)]dx+h_1^m(t)[u_t^m(1,t)]^2 \tag{3.17}$$

using $(3.5)_3$, we obtain

$$h_1'^m(t) = r_1[u_t^m(1,t)]^2,$$

$$[u_t^m(1,t)]^2 = \frac{1}{r_1}[h_1'^m(t)],$$

multiplication in $h_1^m(t)$

$$h_1^m(t)[u_t^m(1,t)]^2 = \frac{1}{r_1}h_1^m(t)[h_1'^m(t)] = \frac{d}{dt}\frac{1}{r_1}[h_1^m(t)]^2, \tag{3.18}$$

using $(3.5)_4$, we obtain

$$h_2'^m(t) = r_2[\phi_t^m(1,t)]^2,$$

$$[\phi_t^m(1,t)]^2 = \frac{1}{r_2}[h_2'^m(t)],$$

multiplication in $h_2^m(t)$

$$h_2^m(t)[\phi_t^m(1,t)]^2 = \frac{1}{r_2}h_2^m(t)[h_2'^m(t)] = \frac{d}{dt}\frac{1}{r_2}[h_2^m(t)]^2, \tag{3.19}$$

Substituting (3.18) and (3.19) in(3.17) we get, for each $m \geq 1$ and for any $0 < t < t_m$,

$$\frac{d}{dt}\frac{1}{2}\left\{\rho\|u_t^m(t)\|^2 + \mu\|u_x^m(t)\|^2 + J\|\phi_t^m(t)\|^2 + \delta\|\phi_x^m(t)\|^2 + \xi\|\phi^m(t)\|^2\right.$$

$$+2\text{b}\int_0^1 \phi^m(t)u_x^m(t)]dx + \frac{1}{r_1}[h_1^m(t)]^2 + \frac{1}{r_2}[h_2^m(t)]^2 = 0$$

Integrating over (0,t) and using (3.6) we see that

$$\rho\|u_t^m(t)\|^2 + \mu\|u_x^m(t)\|^2 + J\|\phi_t^m(t)\|^2 + \delta\|\phi_x^m(t)\|^2 + \xi\|\phi^m(t)\|^2 + 2b\int_0^1 \phi^m(t)u_x^m(t)dx$$
$$+\frac{1}{r_1}[h_1^m(t)]^2 + \frac{1}{r_2}[h_2^m(t)]^2 = 0$$
$$= \rho\|u_1^m\|^2 + \mu\|u_{0x}^m\|^2 + J\|\phi_1^m\|^2 + \delta\|\phi_{0x}^m\|^2 + \xi\|\phi_0^m\|^2 + 2b\int_0^1 \phi_0^m u_{0x}^m dx + \frac{1}{r_1}h_{01}^{m2} + \frac{1}{r_2}h_{02}^2 \leq C \tag{3.20}$$

where C is independent of m and t. Also, from $(3.5)_3$ and $(3.5)_4$ we deduse

$$\int_0^t [u_t^m(1,s)]^2 ds = \frac{1}{r_1}\int_0^t h_1'^m(s)ds \leq \frac{1}{r_1}h_1^m(t) \leq C. \tag{3.21}$$

and

$$\int_0^t [\phi_t^m(1,s)]^2 ds = \frac{1}{r_2}\int_0^t h_2'^m(s)ds \leq \frac{1}{r_2}h_2^m(t) \leq C. \tag{3.22}$$

Second a priori estimate.

Set $t = 0$,$z = u_t^m(.,0)$ in (3.5), integrate by parts, then exploit (3.6) and (3.7) to obtain.

$$\rho\int_0^1 u_{tt}^m u_t^m(.,0)dx + \mu\int_0^1 u_x^m u_{xt}^m(.,0)dx + b\int_0^1 \phi^m u_{xt}^m(.,0)dx + h_1^m(0)[u_t^m(1,0)]^2 = 0$$

$$\rho\|u_{tt}^m(.,0)\|^2 = \mu\int_0^1 u_{0xx}^m(x)u_{tt}(x,0)dx + b\int_0^1 \phi_{0x}^m(x)u_{tt}(x,0)dx$$

$$\leq [\mu\|u^m_{0xx}\| + b\|\phi^m_{0x}\|]\|u_{tt}(.,0)\|$$

This entails that

$$\rho\|u^m_{tt}(.,0)\|\leq \mu\|u^m_{0xx}\| + b\|\phi^m_{0x}\| \leq C. \tag{3.23}$$

Similarly, Setting t=0, $v = \phi^m_t(.,0)$ in (3.5), integrate by parts, then exploit (3.6) and (3.8) we find

$$J\int_0^1 \phi^m_{tt}\phi^m_t(.,0)dx+\delta\int_0^1 \phi^m_x\phi^m_{xt}(.,0)dx+b\int_0^1 u^m_x\phi^m_t(.,0)dx+\xi\int_0^1 \phi^m\phi^m_t(.,0)dx+h^m_2(t)[\phi^m_t(1,t)]^2 = 0$$

$$J\|\phi^m_{tt}(.,0)\| \leq \delta\|\phi^m_{0xx}\| + b\|u^m_{0x}\| + \xi\|\phi^m_0\| \leq C. \tag{3.24}$$

Third a priori estimate.

First, differentiating $(3.5)_1$ and $(3.5)_2$ with respect to t and replacing z and v by u^m_{tt} and ϕ^m_{tt}, respectively, subsequently, addin the resultants we reach

$$\frac{1}{2}\frac{d}{dt}\{\rho\|u^m_{tt}(t)\|^2 + \mu\|u^m_{xt}(t)\|^2 + J\|\phi^m_{tt}(t)\|^2 + \delta\|\phi^m_{xt}(t)\|^2 + \xi\|\phi^m_t(t)\|^2$$
$$+2\mathrm{b}\int_0^1 \phi^m_t(t)u^m_{xt}(t)]dx + \frac{r_1}{2}[h^m_1(t)]^4 + \frac{r_2}{2}[h^m_2(t)]^4$$

$$=-h^m_1(t)[u^m_{tt}(1,t)]^2 - h^m_2(t)[\phi^m_{tt}(1,t)]^2 \leq 0.$$

Integrating over (0,t) and taking advantage of (3.6), (3.23), and (3.24), we arrive at

$$\rho\|u^m_{tt}(t)\|^2 + \mu\|u^m_{xt}(t)\|^2 + J\|\phi^m_{tt}(t)\|^2 + \delta\|\phi^m_{xt}(t)\|^2 + \xi\|\phi^m_t(t)\|^2$$

$$+2\mathrm{b}\int_0^1 \phi_t^m(t)u_{xt}^m(t)]dx + \frac{r_1}{2}[u_1^m(t)]^4 + \frac{r_2}{2}[\phi_1^m(t)]^4$$

$$\rho\|u_{tt}^m(.,0)\|^2 + \mu\|u_{1x}^m(t)\|^2 + J\|\phi_{tt}^m(.,0)\|^2 + \delta\|\phi_{1x}^m(t)\|^2 + \xi\|\phi_1^m(t)\|^2$$

$$+ 2b\int_0^1 \phi_1^m(t)u_{1x}^m(t)]dx + \frac{r_1}{2}[u_1^m(t)]^4 + \frac{r_2}{2}[\phi_1^m(t)]^4 \tag{3.25}$$

$$\leq C.$$

We conclude from (3.20) to (3.22) and (3.25) that $t_m = \infty$ and for any $T > 0$, we have

$$\begin{aligned}
&(u^m) \text{ and } (\phi^m) \text{ are bounded in } L^\infty(0,T;V),\\
&(u_t^m) \text{ and } (\phi_t^m) \text{ are bounded in } L^\infty(0,T;V),\\
&(u_{tt}^m) \text{ and } (\phi_{tt}^m) \text{ are bounded in } L^\infty(0,T;L^2(0,1)),\\
&(u_t^m(1,.)) \text{ and } (\phi_t^m(1,.)) \text{ are bounded in } L^2(0,T)\cap L^\infty(0,T)
\end{aligned} \tag{3.26}$$

So, there exists a subsequence, still denoted by(u^m, ϕ^m), such that

$$\begin{aligned}
&u^m \to u, \phi^m \to \phi \text{ weakly star in } L^\infty(0,T;V),\\
&u_t^m \to u_t, \phi_t^m \to \phi_t \text{ weakly star in } L^\infty(0,T;V),\\
&u_{tt}^m \to u_{tt}, \phi_{tt}^m \to \phi_{tt} \text{ weakly star in } L^\infty(0,T;L^2(0,1)),
\end{aligned} \tag{3.27}$$

Using Aubin-Lions lemma 1, we infer that

$$u_t^m \to u_t,\ \phi_t^m \to \phi_t \text{ in } L^\infty(0,T;L^2(0,1)),$$

from which we deduce

$$u_t^m \to u, \phi_t^m \to \phi \text{ a.e. in } (0,1)\times(0,T).$$

This, together with the continuity of u_t^m, ϕ_t^m, u_t, ϕ_t, yield

$$u_t^m(1,.) \to u_t(1,.), \quad \phi_t^m(1,.) \to \phi_t(1,.) \text{ pointwise in } [0,T].$$

The boundedness of $(u_t^m(1,.))$ and $(\phi_t^m(1,.))$ in $L^2(0,T)$ follows from $(3.26)_4$. Owing to Lebesgue dominated convergence theorem we infer that

$$u_t^m \to u_t(1,.) \textit{ and } \phi_t^m \to \phi_t(1,.) \textit{ in } L^2(0,T). \tag{3.28}$$

An exploitation of $(3.5)_3$, $(3.5)_4$, and (3.28) gives

$$\begin{aligned} h_1^m(t) &= h_{01} + r_1 \int_0^t [u_t^m(1,s)]^2 ds \to h_{01} + r_1 \int_0^t [u_t(1,s)]^2 ds = h_1(t) \\ h_2^m(t) &= h_{02} + r_2 \int_0^t [\phi_t^m(1,s)]^2 ds \to h_{02} + r_2 \int_0^t [\phi_t(1,s)]^2 ds = h_2(t) \end{aligned} \tag{3.29}$$

in $L^\infty(0,T)$. Also, from $(3.26)_4$ and (3.28), we have

$$h_1'^m(.) = r_1[u_t(1,.)]^2 \rightharpoonup r_1[u_t(1,.)]^2 = h_1'(.) \text{ in } L^2(0,T).$$

$$h_2'^m(.) = r_2[\phi_t(1,.)]^2 \rightharpoonup r_2[\phi_t(1,.)]^2 = h_2'(.) \text{ in } L^2(0,T).$$

Next, we take the limit of (3.5) as m goes to infinity, then use (3.27)-(3.29) to obtain

$$\rho \int_0^1 u_{tt} z dx + \mu \int_0^1 u_x z_x dx + b \int_0^1 \phi z_x dx + h_1(t) u_t(1,t) z(1) = 0$$

$$J \int_0^1 \phi_{tt} v dx + \delta \int_0^1 \phi_x v_x dx + b \int_0^1 u_x v dx + \xi \int_0^1 \phi v dx + h_2(t) \phi_t(1,t) v(1) = 0$$

for any $z, v \in V$. Using Aubin-Lions lemma 1 again, we entail that $u, \phi \in C([0,T];W), u_t, \phi_t \in C([0,T];V), u_{tt}, \phi_{tt} \in C([0,T];L^2(0,1))$ and $u(0,.) = u_0,\ u_t(0,.) = u_1,\ \phi(0,.) = \phi_0,$ $\phi_t(0,.) = \phi_1$.

To prove the uniqueness, let (u, ϕ, h_1, h_2) and $(\tilde{u}, \tilde{\phi}, \tilde{h_1}, \tilde{h_2})$ be two solutions of (2.4) with the same initial data-Then,

$$\frac{d}{dt}\Big\{\rho\|u_t-\tilde{u}_t\|^2+\mu\|u_x-\tilde{u}_x\|^2+J\|\phi_t-\tilde{\phi}_t\|^2+\delta\|\phi_x-\tilde{\phi}_x\|^2+\xi\|\phi-\tilde{\phi}\|^2$$

$$+2\mathrm{b}\int_0^1[\phi u_x-\tilde{\phi}\tilde{u}_x]dx+\frac{1}{2}[h_1(t)-\tilde{h_1}(t)]+\frac{1}{2}[h_2(t)-\tilde{h_2}(t)]$$

$$=-[h_1(t)+\tilde{h_1}(t)][u_t(1,t)-\tilde{u}_t(1,t)]^2-[h_2(t)+\tilde{h_2}(t)][\phi_t(1,t)-\tilde{\phi}_t(1,t)]^2$$

$$\leq 0$$

This implies that

$$(u, \phi, h_1, h_2) = (\tilde{u}, \tilde{\phi}, \tilde{h}_1, \tilde{h}_2).$$

■

3.3 Stability analysis

In this section, we use the energy method to prove that system (3.4) is exponentially stable ,To achieve this goal, we first estabish some technical lemmas needed in the proof of exponential stability result.

Lemma 17 *Let (u, ϕ) be the solution of* (3.4), *then the energy functional E, defined by*

$$E(t) := \frac{1}{2}\int_0^1 [\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi]dx \qquad \forall t \geq 0 \tag{3.30}$$

and satisfies

$$E^{'}(t) = -\mu h_1(t)u_t^2(1,t) - \delta h_2(t)\phi_t^2(1,t) \leq 0 \qquad \forall t \geq 0 \tag{3.31}$$

Proof. Muliplying $(3.4)_1$ and $(3.4)_2$ by u_t and ϕ_t respectively, integrating over (0,1) and using integration by parts and the boundary conditions.

the first equation

$$\rho\int_0^1 u_{tt}u_t dx - \mu\int_0^1 u_{xx}u_t dx - b\int_0^1 \phi_x u_t dx = 0$$

where

$$\rho\int_0^1 u_{tt}u_t dx = \frac{\rho}{2}\frac{d}{dt}\int_0^1 u_t^2 dx = \frac{\rho}{2}\frac{d}{dt}\|u_t\|^2 \tag{3.32}$$

and

$$-\mu\int_0^1 u_{xx}u_t dx = -\mu[u_x u_t]_0^1 + \mu\int_0^1 u_x u_{tx} dx$$
$$= -\mu u_x(1,t)u_t(1,t) + \mu u_x(0,t)u_t(0,t) + \frac{\mu}{2}\frac{d}{dt}\int_0^1 u_x^2 dx$$

$$= \frac{\mu}{2}\frac{d}{dt}\|u_x\|^2 dx - \mu u_x(1,t)u_t(1,t) \tag{3.33}$$

and

$$-b\int_0^1 \phi_x u_t dx = -b[\phi u_t]_0^1 + b\int_0^1 \phi u_{tx} dx$$

$$= -b\phi(1,t)u_t(1,t) + b\phi(0,t)u_t(0,t) + b\frac{d}{dt}\int_0^1 \phi u_x dx - b\int_0^1 \phi t u_x dx$$

$$= -b\phi(1,t)u_t(1,t) + b\frac{d}{dt}\int_0^1 \phi u_x dx - b\int_0^1 \phi_t u_x dx \tag{3.34}$$

from (3.32),(3.33) and (3.34), we obtain

$$\frac{1}{2}\frac{d}{dt}[\rho\|u_t\|^2 + \mu\|u_x\|^2] + b\frac{d}{dt}\int_0^1 \phi u_x dx - b\int_0^1 \phi_t u_x dx - \mu u_x(1,t)u_t(1,t) - b\phi(1,t)u_t(1,t) =$$

$$\frac{1}{2}\frac{d}{dt}[\rho\|u_t\|^2 + \mu\|u_x\|^2] + b\frac{d}{dt}\int_0^1 \phi u_x dx - b\int_0^1 \phi_t u_x dx - \mu(-h_1(t)u_t(1,t) - \frac{b}{\mu}\phi(1,t)u_t(1,t) - b\phi(1,t)u_t(1,t)$$

$$\frac{1}{2}\frac{d}{dt}[\rho\|u_t\|^2 + \mu\|u_x\|^2] + b\frac{d}{dt}\int_0^1 \phi u_x dx - b\int_0^1 \phi_t u_x dx + \mu h_1(t)u_t^2(1,t) = 0 \tag{3.35}$$

the second equation

$$J\int_0^1 \phi_{tt}\phi_t dx - \delta\int_0^1 \phi_{xx}\phi_t dx + b\int_0^1 u_x\phi_t + \xi\int_0^1 \phi\phi_t dx = 0$$

where

$$J\int_0^1 \phi_{tt}\phi_t dx = \frac{J}{2}\frac{d}{dt}\int_0^1 \phi_t^2 dx = \frac{J}{2}\frac{d}{dt}\|\phi_t\|^2 \tag{3.36}$$

and

$$\begin{aligned}
-\delta\int_0^1 \phi_{xx}\phi_t dx &= -\delta[\phi_x\phi_t]_0^1 + \delta\int_0^1 \phi_x\phi_{tx} dx \\
&= -\delta\phi_x(1,t)\phi_t(1,t) + \delta\phi_x(0,t)\phi_t(0,t) + \frac{\delta}{2}\frac{d}{dt}\int_0^1 \phi_x^2 dx \\
&= \frac{\delta}{2}\frac{d}{dt}\|\phi_x\|^2 - \delta\phi_x(1,t)\phi_t(1,t) \qquad (3.37)
\end{aligned}$$

and

$$\xi\int_0^1 \phi\phi_t dx = \frac{\xi}{2}\frac{d}{dt}\int_0^1 \phi^2 dx = \frac{\xi}{2}\frac{d}{dt}\|\phi\|^2 \tag{3.38}$$

from (3.36),(3.37) and (3.38), we get

$$\frac{1}{2}\frac{d}{dt}[J\|\phi_t\|^2 + \delta\|\phi_x\|^2 + \xi\|\phi\|^2] + b\int_0^1 u_x\phi_t dx - \delta\phi_x(1,t)\phi_t(1,t) = 0$$

$$\frac{1}{2}\frac{d}{dt}[J\|\phi_t\|^2 + \delta\|\phi_x\|^2 + \xi\|\phi\|^2] + b\int_0^1 u_x\phi_t dx - \delta(-h_2(t)\phi_t(1,t))\phi_t(1,t) = 0$$

$$\frac{1}{2}\frac{d}{dt}[J\|\phi_t\|^2 + \delta\|\phi_x\|^2 + \xi\|\phi\|^2] + b\int_0^1 u_x\phi_t dx + \delta h_2(t)\phi_t^2(1,t) = 0 \tag{3.39}$$

adding up (3.35) and (3.39), leads to

$$\frac{1}{2}\frac{d}{dt}[\rho\|u_t\|^2 + \mu\|u_x\|^2 + J\|\phi_t\|^2 + \delta\|\phi_x\|^2 + \xi\|\phi\|^2] + b\frac{d}{dt}\int_0^1 \phi u_x dx$$
$$+ \mu h_1(t)u_t^2(1,t) + \delta h_2(t)\phi_t^2(1,t) = 0$$

So

$$\frac{d}{dt}\frac{1}{2}\int_0^1 [\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + \delta\phi_x^2 + \xi\phi^2 + 2b\phi u_x]dx = -\mu h_1(t)u_t^2(1,t) - \delta h_2(t)\phi_t^2(1,t)$$

$$E^{'}(t) = -\mu h_1(t)u_t^2(1,t) - \delta h_2(t)\phi_t^2(1,t)$$

■

Lemma 18 *Let* (u, ϕ) *be the solution of* (3.4), *We define another Lyapounov functional* L *by*

$$L(t) := E(t) + \frac{\mu}{2r_1}h_1^2(t) + \frac{\delta}{2r_2}h_2^2(t), \qquad \forall t \geq 0.$$

satisfies

$$L^{'}(t) = -\mu h_1(t)u_t^2(1,t) - \delta h_2(t)\phi_t^2(1,t) + \frac{\mu}{r_1}h_1(t)[r_1 u_t^2(1,t)] + \frac{\delta}{r_2}h_2(t)[r_2\phi_t^2(1,t)] = 0.$$

Proof.

Then, it follows from definitions of h_1, h_2 in (3.4),(3.30) and (3.31) that, for any $t \geq 0$,

$$L^{'}(t) = E^{'}(t) + \frac{\mu}{2r_1}[2h_1(t)h_1^{'}(t)] + \frac{\delta}{2r_2}[2h_2(t)h^{'}(t)]$$

$$= -\mu h_1(t)u_t^2(1,t) - \delta h_2(t)\phi_t^2(1,t) + \frac{\mu}{r_1}h_1(t)h_1^{'}(t) + \frac{\delta}{r_2}h_2(t)h_2^{'}(t)$$

$$= -\mu h_1(t)u_t^2(1,t) - \delta h_2(t)\phi_t^2(1,t) + \frac{\mu}{r_1}h_1(t)[r_1 u_t^2(1,t)] + \frac{\delta}{r_2}h_2(t)[r_2\phi_t^2(1,t)] = 0.$$

■

As a consequence, we have

$$L(t) = L(0), \qquad \forall t \geq 0,$$

we have

$$E(t) \leq L(0) \tag{3.40}$$

and

$$\mu h_1^2(t) \leq L(0)2r_1 \tag{3.41}$$

and

$$\delta h_2^2(t) \leq L(0)2r_1 \tag{3.42}$$

by adding up (3.40), (3.41) and (3.42), we arrive

$$E(t) + \mu h_1^2(t) + \delta h_2^2(t) \leq \lambda_1,$$

and this entails that

$$\sup_{t\geq 0}[E(t) + \mu h_1^2(t) + \delta h_2^2(t)] \leq \lambda_1, \tag{3.43}$$

where $\lambda_1 = L(0)(1 + 2r_1 + 2r_2)$

Lemma 19 *Let (u, ϕ) be the solution of* (3.4)*, then the functional*

$$F_1(t) := \alpha \int_0^1 x u_x u_t dx, \tag{3.44}$$

satisfies

$$F_1^{'}(t) = -\rho\|u_t\|^2 - \mu\|u_x\|^2 + \mu[\frac{b}{\mu}\phi(1,t) + h_1(t)u_t(1,t)]^2 + \rho u_t^2(1,t) + 2\int_0^1 xu_x\phi_x dx,$$

Proof. Using (3.44) and integration by parts, gives

$$F_1^{'}(t) = \alpha\int_0^1 xu_{xt}u_t dx + \alpha\int_0^1 xu_x u_{tt} dx$$

$$= \alpha\int_0^1 xu_{xt}u_t dx + \alpha\int_0^1 x(\frac{\mu}{\rho}u_{xx} + \frac{b}{\rho}\phi_x)u_x dx$$

$$= \alpha\int_0^1 xu_{xt}\varphi_t dx + \frac{\alpha\mu}{\rho}\int_0^1 xu_{xx}u_x dx + \frac{\alpha b}{\rho}\int_0^1 x\phi_x u_x dx$$

where

$$\alpha\int_0^1 xu_{xt}u_t dx = \alpha\int_0^1 x(\frac{1}{2}\frac{d}{dx}u_t^2)dx = \frac{\alpha}{2}\int_0^1 x\frac{du_t^2}{dx}dx$$

$$= \frac{\alpha}{2}[xu_t^2]_0^1 - \frac{\alpha}{2}\int_0^1 u_t^2 dx = \frac{\alpha}{2}u_t^2(1,t) - \frac{\alpha}{2}\int_0^1 u_t^2 dx$$

$$= \frac{\alpha}{2}u_t^2(1,t) - \frac{\alpha}{2}\|u_t\|^2 \tag{3.45}$$

and

$$\frac{\alpha\mu}{\rho}\int_0^1 xu_{xx}u_x dx = \frac{\alpha\mu}{\rho}\int_0^1 x(\frac{1}{2}\frac{d}{dx}u_x^2)dx = \frac{\alpha\mu}{2\rho}\int_0^1 x\frac{du_x^2}{dx}dx$$

$$= \frac{\alpha\mu}{2\rho}[xu_x^2]_0^1 - \frac{\alpha\mu}{2\rho}\int_0^1 u_x^2 dx = \frac{\alpha\mu}{2\rho}u_x^2(1,t) - \frac{\alpha\mu}{2\rho}\int_0^1 u_x^2 dx$$

$$= \frac{\alpha\mu}{2\rho}u_x^2(1,t) - \frac{\alpha\mu}{2\rho}\|u_x\|^2 \tag{3.46}$$

from (3.45) and (3.46), we obtain

$$F_1'(t) = -\frac{\alpha}{2}\|u_t\|^2 - \frac{\alpha\mu}{2\rho}\|u_x\|^2 + \frac{\alpha}{2}u_t^2(1,t) + \frac{\alpha\mu}{2\rho}u_x^2(1,t) + \frac{\alpha b}{\rho}\int_0^1 x\phi_x u_x dx$$

$$= -\frac{\alpha}{2}\|u_t\|^2 - \frac{\alpha\mu}{2\rho}\|u_x\|^2 + \frac{\alpha\mu}{2\rho}[\frac{b}{\mu}\phi(1,t) + h_1(t)u_t(1,t)]^2 + \frac{\alpha}{2}u_t^2(1,t) + \frac{\alpha b}{+\rho}\int_0^1 xu_x\phi_x dx, \tag{3.47}$$

■

Lemma 2 Let (φ, ψ) be the solution of (3.4), then the functional

$$F_2(t) := \beta\int_0^1 x\phi_x\phi_t dx, \tag{3.48}$$

satisfies

$$F_2(t) = -J\|\phi_t\|^2 - \delta\|\phi_x\|^2 + \xi\|\phi\|^2 + [J + \delta h_2^2(t)]\phi_t^2(1,t) - \xi\phi^2(1,t) - 2b\int_0^1 xu_x\phi_x dx$$

Proof. Using (3.48) and integration by parts, gives

$$F_2'(t) = \beta\int_0^1 x\phi_{xt}\phi_t dx + \beta\int_0^1 x\phi_x\phi_{tt} dx$$

$$= \beta \int_0^1 x\phi_{xt}\phi_t dx + \beta \int_0^1 x(\frac{\delta}{J}\phi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\phi)\phi_x dx$$

$$= \beta \int_0^1 x\phi_{xt}\phi_t dx + \frac{\beta\delta}{J} \int_0^1 x\phi_{xx}\phi_x dx - \frac{\beta b}{J} \int_0^1 xu_x\phi_x dx - \frac{\beta\xi}{J} \int_0^1 x\phi\phi_x dx$$

where

$$\beta \int_0^1 x\phi_{xt}\phi_t dx = \beta \int_0^1 x(\frac{1}{2}\frac{d}{dx}\phi_t^2)dx = \frac{\beta}{2} \int_0^1 x\frac{d\phi_t^2}{dx}dx$$

$$= \frac{\beta}{2}[x\phi_t^2]_0^1 - \frac{\beta}{2} \int_0^1 \phi_t^2 dx = \frac{\beta}{2}\phi_t^2(1,t) - \frac{\beta}{2} \int_0^1 \phi_t^2 dx$$

$$= \frac{\beta}{2}\phi_t^2(1,t) - \frac{\beta}{2}\|\phi_t\|^2 \quad (3.49)$$

and

$$\frac{\beta\delta}{J} \int_0^1 x\phi_{xx}\phi_x dx = \frac{\beta\delta}{J} \int_0^1 x(\frac{1}{2}\frac{d}{dx}\phi_x^2)dx = \frac{\beta\delta}{2J} \int_0^1 x\frac{d\phi_x^2}{dx}dx$$

$$= \frac{\beta\delta}{2J}[x\phi_x^2]_0^1 - \frac{\beta\delta}{2J} \int_0^1 \phi_x^2 dx = \frac{\beta\delta}{2J}\phi_x^2(1,t) - \frac{\beta\delta}{2J} \int_0^1 \phi_x^2 dx$$

$$= \frac{\beta\delta}{2J}\phi_x^2(1,t) - \frac{\beta\delta}{2J}\|\phi_x\|^2 \quad (3.50)$$

and

$$-\frac{\beta\xi}{J} \int_0^1 x\phi\phi_x dx = -\frac{\beta\xi}{J} \int_0^1 x(\frac{1}{2}\frac{d}{dx}\phi^2)dx = -\frac{\beta\xi}{2J} \int_0^1 x\frac{d\phi^2}{dx}dx$$

$$= -\frac{\beta\xi}{2J}[x\phi^2]_0^1 + \frac{\beta\xi}{2J}\int_0^1 \phi^2 dx = -\frac{\beta\xi}{2J}\phi^2(1,t) + \frac{\beta\xi}{2J}\int_0^1 \phi^2 dx$$

$$= -\frac{\beta\xi}{2J}\phi^2(1,t) + \frac{\beta\xi}{2J}\|\phi\|^2 \tag{3.51}$$

from (3.49) ,(3.50) and (3.51), we obtain

$$F_2'(t) = -\frac{\beta}{2}\|\phi_t\|^2 - \frac{\beta\delta}{2J}\|\phi_x\|^2 + \frac{\beta\xi}{2J}\|\phi\| + \frac{\beta}{2}\phi_t^2(1,t) + \frac{\beta\delta}{2J}\phi_x^2(1,t) - \frac{\beta\xi}{2J}\phi^2(1,t) - \frac{\beta b}{J}\int_0^1 xu_x\phi_x dx$$

$$= -\frac{\beta}{2}\|\phi_t\|^2 - \frac{\beta\delta}{2J}\|\phi_x\|^2 + \frac{\beta\xi}{2J}\|\phi\| + \frac{\beta}{2}\phi_t^2(1,t) + \frac{\beta\delta}{2J}h_2^2(t)\phi_t^2(1,t) - \frac{\beta\xi}{2J}\phi^2(1,t) - \frac{\beta b}{J}\int_0^1 xu_x\phi_x dx$$

$$= -\frac{\beta}{2}\|\phi_t\|^2 - \frac{\beta\delta}{2J}\|\phi_x\|^2 + \frac{\beta\xi}{2J}\|\phi\| + [\frac{\beta}{2} + \frac{\beta\delta}{2J}h_2^2(t)]\phi_t^2(1,t) - \frac{\beta\xi}{2J}\phi^2(1,t) - \frac{\beta b}{J}\int_0^1 xu_x\phi_x dx, \tag{3.52}$$

■

Lemma 3 Let (φ, ψ) be the solution of (3.4), then the functional

$$F_3(t) := \gamma \int_0^1 uu_t dx, \tag{3.53}$$

satisfies

$$F_3(t) = -\frac{\rho}{2}\|u_t\|^2 - \frac{\mu}{2}\|u_x\|^2 + \frac{\mu}{2}h_1(t)u_t(1,t)u(1,t) + \frac{b}{2}\int_0^1 u_x\phi dx$$

Proof. Using (3.53) and integration by parts, gives

$$F_3'(t) = \gamma \int_0^1 u_t^2 dx + \gamma \int_0^1 uu_{tt} dx$$

$$= \gamma \int_0^1 u_t^2 dx + \gamma \int_0^1 (\frac{\mu}{\rho} u_{xx} + \frac{b}{\rho}\phi_x) u dx$$

$$= \gamma \int_0^1 u_t^2 dx + \frac{\gamma\mu}{\rho} \int_0^1 u_{xx} u dx + \frac{\gamma b}{\rho} \int_0^1 \phi_x u dx$$

where

$$\begin{aligned} \frac{\gamma\mu}{\rho} \int_0^1 u_{xx} u dx &= \frac{\gamma\mu}{\rho} [u_x u]_0^1 - \frac{\gamma\mu}{\rho} \int_0^1 u_x^2 dx \\ &= \frac{\gamma\mu}{\rho} u_x(1,t) u(1,t) - \frac{\gamma\mu}{\rho} \|u_x\| \end{aligned} \tag{3.54}$$

and

$$\begin{aligned} \frac{\gamma b}{\rho} \int_0^1 \phi_x u dx &= \frac{\gamma b}{\rho} [\phi u]_0^1 - \frac{\gamma b}{\rho} \int_0^1 u_x \phi dx \\ &= \frac{\gamma b}{\rho} u(1,t)\phi(1,t) - \frac{\gamma b}{\rho} \int_0^1 u_x \phi dx \end{aligned} \tag{3.55}$$

from (3.54) and (3.55), we get

$$F_3'(t) = \gamma \|u_t\| - \frac{\gamma\mu}{\rho}\|u_x\| + \frac{\gamma\mu}{\rho} u_x(1,t)u(1,t) + \frac{\gamma b}{\rho} u(1,t)\phi(1,t) - \frac{\gamma b}{\rho} \int_0^1 u_x \phi dx$$

$$= \gamma \|u_t\| - \frac{\gamma\mu}{\rho}\|u_x\| + \frac{\gamma\mu}{\rho}[-h_1(t)u_t(1,t) - \frac{b}{\mu}\phi(1,t)]u(1,t) + \frac{\gamma b}{\rho} u(1,t)\phi(1,t) - \frac{\gamma b}{\rho} \int_0^1 u_x \phi dx$$

$$= \gamma \|u_t\| - \frac{\gamma\mu}{\rho}\|u_x\| - \frac{\gamma\mu}{\rho} h_1(t) u_t(1,t) u(1,t) - \frac{\gamma b}{\rho} \int_0^1 u_x \phi dx \tag{3.56}$$

■

Lemma4 Let (φ, ψ) be the solution of (3.4), then the functional

$$F_4(t) := \sigma \int_0^1 \phi\phi_t dx, \tag{3.57}$$

satisfies

$$F_4'(t) = \frac{J}{2}\|\phi_t\|^2 - \frac{\delta}{2}\|\phi_x\|^2 - \frac{\xi}{2}\|\phi\|^2 - \frac{\delta}{2}h_2(t)\phi_t(1,t)\phi(1,t) - \frac{b}{2}\int_0^1 u_x\phi dx,$$

Proof. Using (3.57) and integration by parts, gives

$$F_4'(t) = \sigma\int_0^1 \phi_t^2 dx + \sigma\int_0^1 \phi\phi_{tt}dx$$

$$= \sigma\int_0^1 \phi_t^2 dx + \sigma\int_0^1 \phi[\frac{\delta}{J}\phi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\phi]dx$$

$$= \sigma\int_0^1 \phi_t^2 dx + \frac{\sigma\delta}{J}\int_0^1 \phi\phi_{xx}dx - \frac{\sigma b}{J}\int_0^1 \phi u_x dx - \frac{\sigma\xi}{J}\int_0^1 \phi^2 dx$$

where

$$\sigma\int_0^1 \phi_t^2 dx = \sigma\|\phi_t\|^2 \tag{3.58}$$

and

$$\frac{\sigma\delta}{J}\int_0^1 \phi_{xx}\phi dx = \frac{\sigma\delta}{J}[\phi_x\phi]_0^1 - \frac{\sigma\delta}{J}\int_0^1 \phi_x^2$$

$$= \frac{\sigma\delta}{J}\phi(1,t)\phi_x(1,t) - \frac{\sigma\delta}{J}\|\phi_x\|^2 \tag{3.59}$$

and

$$-\frac{\sigma\xi}{J}\int_0^1 \phi^2 dx = -\frac{\sigma\xi}{J}\|\phi\|^2 \tag{3.60}$$

from (3.58), (3.59) and (3.60), we get

$$F_4'(t) = \sigma\|\phi_t\|^2 - \frac{\sigma\delta}{J}\|\phi_x\|^2 - \frac{\sigma\xi}{J}\|\phi\|^2 + \frac{\sigma\delta}{J}\phi(1,t)\phi_x(1,t) - \frac{\sigma b}{J}\int_0^1 \phi u_x dx$$

$$= \sigma\|\phi_t\|^2 - \frac{\sigma\delta}{J}\|\phi_x\|^2 - \frac{\sigma\xi}{J}\|\phi\|^2 - \frac{\sigma\delta}{J}h_2(t)\phi_t(1,t)\phi(1,t) - \frac{\sigma b}{J}\int_0^1 \phi u_x dx \tag{3.61}$$

■

Lemma

We introduce another Lyapunov functional

$$L_\varepsilon(t) = E(t) + \varepsilon F(t) \tag{3.62}$$

where $\varepsilon > 0$ and F is given by

$$F(t) := \sum_{j=1}^{4} F_j(t)$$

Therefore, inequalities (1.1) and (1.3) together with the Cauchy-Schwarz inequality gives

$$|F(t)| \leq \sum_{j=1}^{4} |F_j(t)| \leq \frac{9}{2}E(t), \qquad \forall t \geq 0. \tag{3.63}$$

We deduce from (3.63) that

$$|L_\varepsilon(t) - E(t)| = \varepsilon|F(t)| \leq \frac{9}{2}\varepsilon E(t), \qquad \forall t \geq 0.$$

which implies

$$(1 - \frac{9}{2}\varepsilon)E(t) \leq L_\varepsilon(t) \leq (1 + \frac{9}{2}\varepsilon)E(t), \qquad \forall t \geq 0.$$

By choosing $\varepsilon < \frac{2}{9}$, we obtain

$$0 < (1 - \frac{9}{2}\varepsilon)E(t) \leq L_\varepsilon(t) \leq (1 + \frac{9}{2}\varepsilon)E(t), \qquad \forall t \geq 0. \tag{3.64}$$

Applying integration by parts, we find

$$F'(t) := \sum_{j=1}^{4} F_j'(t)$$

Proof.

$$F'(t) = -\frac{\alpha}{2}\|u_t\|^2 - \frac{\alpha\mu}{2\rho}\|u_x\|^2 + \frac{\alpha}{2}u_t^2(1,t) + \frac{\alpha b^2}{2\rho\mu}\phi^2(1,t) + \frac{\alpha\mu}{2\rho}h_1^2(t)u_t^2(1,t) + \frac{2b}{\rho}\phi(1,t)h_1(t)u_t(1,t)$$

$$+\frac{\alpha b}{\rho}\int_0^1 x\phi_x u_x dx - \frac{\beta}{2}\|\phi_t\|^2 - \frac{\beta\delta}{2J}\|\phi_x\|^2 + \frac{\beta\xi}{2J}\|\phi\|^2 + (\frac{\beta}{2} + \frac{\beta\delta}{2J}h_2^2(t))\phi_t^2(1,t) - \frac{\beta\xi}{2J}\phi^2(1,t)$$

$$-\frac{\beta b}{J}\int_0^1 xu_x\phi_x dx + \gamma\|u_t\|^2 - \frac{\gamma\mu}{\rho}\|u_x\|^2 - \frac{\gamma\mu}{\rho}h_1(t)u_t(1,t)u(1,t) - \frac{\gamma b}{\rho}\int_0^1 \phi u_x dx$$

$$+\sigma\|\phi_t\|^2 - \frac{\sigma\delta}{J}\|\phi_x\|^2 - \frac{\sigma\xi}{J}\|\phi\|^2 - \frac{\sigma\delta}{J}h_2(t)\phi_t(1,t)\phi(1,t) - \frac{\sigma b}{J}\int_0^1 u_x\phi dx$$

$$= (-\frac{\alpha}{2} + \gamma)\|u_t\|^2 + (\frac{-\alpha\mu}{2\rho} - \frac{\gamma\mu}{\rho})\|u_x\|^2 + (\frac{-\beta}{2} + \sigma)\|\phi_t\|^2 + (\frac{-\beta\delta}{2J} - \frac{\sigma\delta}{J})\|\phi_x\|^2$$

$$+(\frac{\beta\xi}{2J} - \frac{\sigma\xi}{J})\|\phi\|^2 + (\frac{\alpha}{2} + \frac{\alpha\mu}{2\rho}h_1^2(t))u_t^2(1,t) + (\frac{\beta}{2} + \frac{\beta\delta}{2J}h_2^2(t))\phi_t^2(1,t) + (\frac{\alpha b^2}{2\rho\mu} - \frac{\beta\xi}{2J})\phi^2(1,t)$$

$$-\frac{\gamma\mu}{\rho}h_1(t)u_t(1,t)u(1,t) - \frac{\sigma\delta}{J}h_2(t)\phi_t(1,t)\phi(1,t) + \frac{\alpha b}{\rho}\phi(1,t)h_1(t)u_t(1,t)$$

$$+\frac{\alpha b}{\rho}\int_0^1 x\phi_x u_x dx - \frac{\beta b}{J}\int_0^1 xu_x\phi_x dx - \frac{\gamma b}{\rho}\int_0^1 \phi u_x dx - \frac{\sigma b}{J}\int_0^1 u_x\phi dx$$

Where $\beta = \frac{\alpha J}{\rho}$, $\gamma = \frac{-\sigma\rho}{J}$

$$= (-\frac{\alpha}{2} - \frac{\sigma\rho}{J})\|u_t\|^2 + (\frac{-\alpha\mu}{2\rho} + \frac{\sigma\mu}{J})\|u_x\|^2 + (\frac{-\alpha J}{2\rho} + \sigma)\|\phi_t\|^2 + (\frac{-\alpha\delta}{2\rho} - \frac{\sigma\delta}{J})\|\phi_x\|^2$$

$$+(\frac{\alpha\xi}{2\rho}-\frac{\sigma\xi}{J})\|\phi\|^2+(\frac{\alpha}{2}+\frac{\alpha\mu}{2\rho}h_1^2(t))u_t^2(1,t)+(\frac{\alpha J}{2\rho}+\frac{\alpha\delta}{2\rho}h_2^2(t))\phi_t^2(1,t)+(\frac{\alpha b^2}{2\rho\mu}-\frac{\alpha\xi}{2\rho})\phi^2(1,t)$$

$$+\frac{\sigma\mu}{J}h_1(t)u_t(1,t)u(1,t)-\frac{\sigma\delta}{J}h_2(t)\phi_t(1,t)\phi(1,t)+\frac{\alpha b}{\rho}\phi(1,t)h_1(t)u_t(1,t)$$

Where $\alpha=\rho$, $\sigma=\frac{J}{4}$

$$F^{'}(t)=-\frac{3}{4}\rho\|u_t\|^2-\frac{\mu}{4}\|u_x\|^2-\frac{J}{4}\|\phi_t\|^2-\frac{3\delta}{4}\|\phi_x\|^2+\frac{\xi}{4}\|\phi\|^2$$

$$+(\frac{\rho}{2}+\frac{\mu}{2}h_1^2(t))u_t^2(1,t)+(\frac{J}{2}+\frac{\delta}{2}h_2^2(t))\phi_t^2(1,t)+(\frac{b^2}{2\mu}-\frac{\xi}{2})\phi^2(1,t)$$

$$+\frac{\mu}{4}h_1(t)u_t(1,t)u(1,t)-\frac{\delta}{4}h_2(t)\phi_t(1,t)\phi(1,t)+b\phi(1,t)h_1(t)u_t(1,t)$$

and using Young's inequality, gives

$$b\phi(1,t)h_1(t)u_t(1,t)\le[\frac{\varepsilon_1}{2}h_1^2(t)u_t^2(1,t)+\frac{b^2}{2\varepsilon_1}\phi^2(1,t)]\le\frac{\varepsilon_1}{2}h_1^2(t)u_t^2(1,t)+\frac{b^2}{2\varepsilon_1}\phi^2(1,t)$$

$$\frac{\mu}{4}h_1(t)u_t(1,t)u(1,t)\le\frac{\mu}{4}[\frac{\varepsilon_2}{2}h_1^2(t)u_t^2(1,t)+\frac{1}{2\varepsilon_2}u^2(1,t)]\le\frac{\mu\varepsilon_2}{8}h_1^2(t)u_t^2(1,t)+\frac{\mu}{8\varepsilon_2}u^2(1,t)$$

$$-\frac{\delta}{4}h_2(t)\phi_t(1,t)\phi(1,t)\le\frac{\delta}{4}[\frac{\varepsilon_3}{2}h_2^2(t)\phi_t^2(1,t)+\frac{1}{2\varepsilon_3}\phi^2(1,t)]\le\frac{\delta\varepsilon_3}{8}h_2^2(t)\phi_t^2(1,t)+\frac{\delta}{8\varepsilon_3}\phi^2(1,t)$$

we get

$$F'(t)\le-\frac{3}{4}\rho\|u_t\|^2-\frac{\mu}{4}\|u_x\|^2-\frac{J}{4}\|\phi_t\|^2-\frac{3\delta}{4}\|\phi_x\|^2+\frac{\xi}{4}\|\phi\|^2$$

$$+(\frac{\rho}{2}+\frac{\mu}{2}h_1^2(t))u_t^2(1,t)+(\frac{J}{2}+\frac{\delta}{2}h_2^2(t))\phi_t^2(1,t)+(\frac{b^2}{2\mu}-\frac{\xi}{2})\phi^2(1,t)$$

$$+\frac{\mu\varepsilon_2}{8}h_1^2(t)u_t^2(1,t)+\frac{\mu}{8\varepsilon_2}u^2(1,t)+\frac{\delta\varepsilon_3}{8}h_2^2(t)\phi_t^2(1,t)+\frac{\delta}{8\varepsilon_3}\phi^2(1,t)+\frac{\varepsilon_1}{2}h_1^2(t)u_t^2(1,t)+\frac{b^2}{2\varepsilon_1}\phi^2(1,t)$$

$$\leq -\frac{3}{4}\rho\|u_t\|^2 + (-\frac{\mu}{4} + \frac{\mu}{8\varepsilon_2})\|u_x\|^2 - \frac{J}{4}\|\phi_t\|^2 - \frac{3\delta}{4}\|\phi_x\|^2 - \frac{\xi}{4}\|\phi\|^2 + (\frac{b^2}{2\mu} - \frac{\xi}{2})\phi^2(1,t)$$

$$[\frac{\rho}{2} + ((\frac{\mu}{2} + \frac{\mu\varepsilon_2}{8}) + \frac{\varepsilon_1}{2})h_1^2(t)]u_t^2(1,t) + [\frac{J}{2} + (\frac{\delta}{2} + \frac{\delta\varepsilon_3}{8})h_2^2(t)]\phi_t^2(1,t) + (\frac{\delta}{8\varepsilon_3} + \frac{b^2}{2\varepsilon_1})\phi^2(1,t).$$

$$\leq -\frac{3}{4}\rho\|u_t\|^2 - \frac{\mu}{8}\|u_x\|^2 - \frac{J}{4}\|\phi_t\|^2 - \frac{3\delta}{4}\|\phi_x\|^2 + \frac{\xi}{4}\|\phi\|^2 + (\frac{b^2}{2\mu} - \frac{\xi}{2})\phi^2(1,t)$$

$$[\frac{\rho}{2} + (\frac{5\mu}{8} + \frac{\varepsilon_1}{2})h_1^2(t)]u_t^2(1,t) + [\frac{J}{2} + (\frac{\delta}{2} + \frac{\delta\varepsilon_3}{8})h_2^2(t)]\phi_t^2(1,t) + (\frac{\delta}{8\varepsilon_3} + \frac{b^2}{2\varepsilon_1})\phi^2(1,t).$$

We choose $\varepsilon_1 = \dfrac{4b^2}{\xi - \frac{b^2}{\mu}}$ and $\varepsilon_3 = \dfrac{\delta}{\xi - \frac{b^2}{\mu}}$ we get

$$\leq -\frac{3}{4}\rho\|u_t\|^2 - \frac{\mu}{8}\|u_x\|^2 - \frac{J}{4}\|\phi_t\|^2 - \frac{3\delta}{4}\|\phi_x\|^2 + \frac{\xi}{4}\|\phi\|^2 + \frac{1}{4}(\frac{b^2}{\mu} - \xi)\phi^2(1,t)$$

$$[\frac{\rho}{2} + (\frac{5\mu}{8} + \frac{\varepsilon_1}{2})h_1^2(t)]u_t^2(1,t) + [\frac{J}{2} + (\frac{\delta}{2} + \frac{\delta\varepsilon_3}{8})h_2^2(t)]\phi_t^2(1,t).$$

using inequality(1.1)
$\frac{\xi}{4}\|\phi\|^2 \leq \frac{\xi}{8}\|\phi_x\|^2$

$$\leq -\frac{3}{4}\rho\|u_t\|^2 - \frac{\mu}{8}\|u_x\|^2 - \frac{J}{4}\|\phi_t\|^2 - \frac{3\delta}{4}\|\phi_x\|^2 + \frac{\xi}{8}\|\phi_x\|^2 + \frac{1}{4}(\frac{b^2}{\mu} - \xi)\phi^2(1,t)$$

$$[\frac{\rho}{2} + (\frac{5\mu}{8} + \frac{\varepsilon_1}{2})h_1^2(t)]u_t^2(1,t) + [\frac{J}{2} + (\frac{\delta}{2} + \frac{\delta\varepsilon_3}{8})h_2^2(t)]\phi_t^2(1,t).$$

Where $\xi < 6\delta$

$$\leq -\frac{3}{4}\rho\|u_t\|^2 - \frac{\mu}{8}\|u_x\|^2 - \frac{J}{4}\|\phi_t\|^2(-\frac{3\delta}{4} + \frac{\xi}{8})\|\phi_x\|^2 + \frac{1}{4}(\frac{b^2}{\mu} - \xi)\phi^2(1,t)$$

$$[\frac{\rho}{2} + (\frac{5\mu}{8} + \frac{\varepsilon_1}{2})h_1^2(t)]u_t^2(1,t) + [\frac{J}{2} + (\frac{\delta}{2} + \frac{\delta\varepsilon_3}{8})h_2^2(t)]\phi_t^2(1,t).$$

$$\leq -\frac{3}{4}\rho\|u_t\|^2 - \frac{\mu}{8}\|u_x\|^2 - \frac{J}{4}\|\phi_t\|^2(-\frac{3\delta}{4} + \frac{\xi}{8})\|\phi_x\|^2$$

$$[\frac{\rho}{2} + (\frac{5\mu}{8} + \frac{\varepsilon_1}{2})h_1^2(t)]u_t^2(1,t) + [\frac{J}{2} + (\frac{\delta}{2} + \frac{\delta\varepsilon_3}{8})h_2^2(t)]\phi_t^2(1,t).$$

from(3.30), we have

using Young's inequality

$$2bu_x\phi \leq 2[\frac{\varepsilon_1}{2}u_x^2 + \frac{1}{2\varepsilon_1}b^2\phi^2] \leq \mu\|u_x\|^2 + \frac{b^2}{\mu}\|\phi\|^2$$

$$E(t) \leq \frac{1}{2}[\rho\|u_t\|^2 + J\|\phi_t\|^2 + 2\mu\|u_x\|^2 + \delta\|\phi_x\|^2 + (\xi + \frac{b^2}{\mu})\|\phi\|^2]$$

using inequality(1.1)

$$E(t) \leq \frac{1}{2}[\rho\|u_t\|^2 + J\|\phi_t\|^2 + 2\mu\|u_x\|^2 + (\delta + \frac{1}{2}(\xi + \frac{b^2}{\mu}))\|\phi_x\|^2]$$

we have $\frac{\xi}{2} \leq \frac{1}{2}(\frac{b^2}{\mu} + \xi)$

$$E(t) \leq \frac{1}{2}[\rho\|u_t\|^2 + J\|\phi_t\|^2 + 2\mu\|u_x\|^2 + (\delta + \frac{\xi}{2})\|\phi_x\|^2]$$

■

Conclusion

In this work, we prove the existence and uniqueness of solution for the porous-elastic system by boundary feedback controls. In the first chapter, we presented some basic definitions and theories that we will need in this work. In the second chapter, we prove the existence and the uniqueness and the exponential stability of the Timoshenko system by boundary feedback control which is the work of Hassan and Tatar[2]. In the third chapter, we proved using the Faedo-Galerkin method the existence and uniqueness of solution for porous-elastic system then by using the multiplier and energy methods we established the exponential stability (the study was not completed).

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ملخص

الهدف من هذا العمل هو دراسة بعض الأنظمة المكونة من المعادلات التفاضلية الجزئية. اول نظام هو نظام Timoshenko تيموشنكو, حيث سنبرهن على وجود و وحدانية الحل بالاعتماد على نظرية Faedo-Galerkin فايدو-غاليركين و دراسة استقراره باستخدام تقنية الجداءات الذي يقوم على بناء دالة Lyapunov ليابونوف المكافئة للطاقة.

ثاني نظام, سندرس مشكلة تتعلق بعلم المواد والميكانيك وبالتحديد المواد المرنة ذات الفراغات , نثبت أولا ان المشكلة لها حل عن طريق التحكم في الحدود ثم ندرس الاستقرار الاسي للحل (لم يتم اكمال الدراسة).

الكلمات المفتاحية: نظام المرونة المسامية – الاستقرار الاسي – طريقة دالة الطاقة – دالة ليابونوف.

Résumé

Le but de ce travail est d'étudier certains systèmes constitués d'équations aux dérives partielles. Le premier système est le système de Timoshenko, nous démontrerons l'existence et l'unité de la solution basée sur la théorie Faedo-Galerkin , et l'étude de sa stabilité à l'utilisation de la technique du multiplicateur, que appuie à l'aide de la fonction Lyapunov équivalente à l'énergie.

D'autre parte, nous étudierons un problème lié à la science des matériaux et à la mécanique, spécifiquement les matériaux élastiques avec des vides. Nous prouverons que le problème a une solution en contrôlant les frontières, puis nous étudierons la stabilité exponentielle de la solution (L'étude n'est pas terminée).

Mots clés : le système élastique poreaux – stabilité exponentiel – méthode de la fonction énergétique – la fonction de Lyapunov.

Abstract

The aim of this work is to study some systems consisting of partial differential equations.

The first problem is the Timoshenko system, we will prove the existence and the uniqueness of the solution based on Faedo-Galerkin method, and the study of its stability is done using the multiplier technique which is based on constructing the Lyapunov function.

The second system is that we will study a problem related to machanics specifically elastic materials with voids . First, we prove that the problem has a solution by controlling the boundaries, then we will study the exponential stability of the solution (the study was not completed).

Keywords : Porous-elastic system – Exponential stability – Energy function method – Lyapunov functional.