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Dedication

The student, Soumia Ben ali, did not complete this paragraph... May she rest in peace.

Acknowledgement

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Introduction

In the last few decades, the study of problems related to elastic solids with voids has attracted the attention of many researchers due to the extensive practical applications of such materials in different fields, such as petroleum industry, foundation engineering, soil mechanics, power technology, biology, material science and so on. Elastic solids with voids is one of the simplest extensions of the theory of the classical elasticity. It allows the treatment of porous solids in which the matrix material is elastic and the interstices are void of material.

In 1972, Godman and Cowin [3] proposed an extension of the classical elasticity theory to porous media. They introduced the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition to their usual elastic effects, these materials have a microstructure with the property that the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction. This latter idea was introduced by Nunziato and Cowin [4] in 1979 when they developed a nonlinear theory of elastic materials with voids. This representation (i.e the mass at each point is obtained as the product of the mass density of the material matrix by the volume fraction) introduces an additional degree of kinematic freedom and was employed previously by Goodman and Cowin [3] to develop a theory for flowing granular materials.

In 1983, Cowin and Nunziato [1] developed a linear theory of elastic materials with voids to study mathematically the mechanical behavior of porous solids. We refer the reader to [1, 2] and the references therein for more details.

Recent technological advancements in the fields of robotics and space science give rise to increase interests in the dynamics of flexible structures with boundary and/or internal control forces. The Euler-Bernoulli equation had been deployed to model the dynamics of the transverse vibration of an elastic beam by neglecting the effect of its rotatory inertia since the dimension of its cross-section is negligible compared to its actual length.

The main results of this thesis:

This thesis contains three chapters.

In chapter 1, we recall some notations and we review some mathematical concepts that will be used throughout this dissertation.

In **chapter 2**, we present the results of Hassan and Tatar [6] which is the existence and stability of solution for Timoshenko system under the boundary feedback controls:

$$\begin{cases} \varphi_{tt} - (\varphi_t + \psi)_x = 0, & 0 < x < 1, t > 0 \\ \psi_{tt} - \psi_{xx} + (\varphi_x + \psi) = 0, & 0 < x < 1, t > 0 \\ \varphi(0,t) = \psi(0,t) = 0, & t > 0 \\ \varphi(1,t) + \psi(1,t) = u_1(t), \psi_x(1,t) = u_2(t), & t > 0 \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & 0 < x < 1 \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & 0 < x < 1 \end{cases}$$

We study this system under the influence of the following high adaptive boundary controls, where we detail the work of Hassan and Tatar [6]:

$$\begin{cases} u_i(t) = -k_i(t) y_i(t) \\ k_i'(t) = r_i y_i^2(t) \quad k_i(0) = k_{0i}, \quad k_{0i}, r_i > 0 \quad for \quad i = 1, 2 \end{cases}$$

The main of this memory is to analyze the existence and asymptotic stability of solution for the porous elastic system under adaptive boundary controls.

In chapter 3, we study, a new system, the porous-elastic system, which mentioned in the work of Lacheheb et al. [5], but with same boundary and initial conditions with [6] i.e the following problem:

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \quad\ 0 < x < 1, \quad t > 0 \\ j\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0 \quad\ 0 < x < 1, \quad t > 0 \\ u(0,t) = \phi(0,t) = 0 \quad\ t > 0 \\ u(1,t) + \dfrac{b}{\mu}\phi(1,t) = f_1(t), \quad \phi_x(1,t) = f_2(t), \quad\ t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad\ 0 < x < 1 \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x) \quad\ 0 < x < 1 \end{cases}$$

with the adaptive boundary controls:

$$\begin{cases} f_i(t) = -h_i(t)y_i(t) \\ h_i^{'}(t) = r_i y_i^2(t) \;\; h_i(0) = h_{0i}, \quad h_{0i}, r_i > 0 \;\; for \;\; i = 1, 2 \end{cases}$$

We use the Faedo-Galerkin method to establish the well-posedness, then we employ the multiplier and energy methods to prove an exponential decay for the equal and non equal speed of propagation.

Chapter 1

PRELIMINARIES

In this chapter, we present some preliminary results that we should know them for use in our theme

Let $\|\cdot\|$ denote the usual norm in $L^2(0;1)$,

$$V = \{w \in H^1(0,1) : w(0) = 0\}$$

equipped with the following norm

$$\|w\|_V = \|w_x\|,$$

and

$$W = \{w \in H^2(0,1) : w(0) = 0\}$$

equipped with the following norm

$$\|w\|_W^2 = \|w_{xx}\|^2 + \|w_x\|^2.$$

Lemma 1.1 *[6] For any $w \in V$, we have*

$$\|w\|^2 \leq \frac{1}{2}\|w_x\|^2 \tag{1.1}$$

and

$$w^2(1) \leq \|w_x\|^2. \tag{1.2}$$

It follows from the above lemma that $\|\cdot\|_V$ and $\|\cdot\|_W$ are equivalent norms in V and W, respectively, and

$$\frac{1}{2}(\|v_x\|^2 + \|w_x\|^2) \leq \|v_x + w\|^2 + \|w_x\|^2 \leq 2(\|v_x\|^2 + \|w_x\|^2) \tag{1.3}$$

for any $v, w \in V$

Holder inequality

Let $f \in L^p$ et $g \in L^{p'}$ with $1 \leq p \leq \infty$, then $f.g \in L^1$ and

$$\int |f.g| \leq \|f\|_{L^p}.\|g\|_{L^{p'}}$$

Cauchy-Schwarz inequality

For $p = q = 2$ the Holder inequality is none other than the Cauchy-Schwarz inequality,

$$\int_\Omega |f.g| \leq \|f\|_{L^2}.\|g\|_{L^2}$$

Young's inequality

Let a, b two real positive and $p > 1, p' < \infty$,

$$ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$$

where $\frac{1}{p} + \frac{1}{p'} = 1$, more standard inequality :

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{b^2}{2\varepsilon}$$

for $a, b \in \mathbb{R}$, and $\varepsilon > 0$

Chapter 2

Asymptotic stability of Timoshenko system by boundary feedback controls

2.1 Statement of the problem

$$\begin{cases} \varphi_{tt} - (\varphi_t + \psi)_x = 0, & 0 < x < 1, t > 0 \\ \psi_{tt} - \psi_{xx} + (\varphi_x + \psi) = 0, & 0 < x < 1, t > 0 \\ \varphi(0,t) = \psi(0,t) = 0, & t > 0 \\ \varphi(1,t) + \psi(1,t) = u_1(t), \psi_x(1,t) = u_2(t), & t > 0 \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & 0 < x < 1 \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & 0 < x < 1 \end{cases} \tag{2.1}$$

where φ_0, φ_1, ψ_0,ψ_1 are given data, u_1 and u_2 are boundary control inputs. The measured outputs of the system at the right end are given by

$$y_1(t) = \varphi_t(1,t) \;\; and \;\; y_2(t) = \psi(1,t), \quad t > 0 \tag{2.2}$$

and the adaptive boundary controls

$$\begin{cases} u_i(t) = -k_i(t)y_i(t) \\ k_i^{'}(t) = r_i y_i^2(t) \;\; k_i(0) = k_{0i}, \quad k_{0i}, r_i > 0 \;\; for \;\; i = 1,2 \end{cases} \tag{2.3}$$

the closed-loop system associated to (2.1) , (2.2) and (2.3) is given by

$$\begin{cases} \varphi_{tt} - (\varphi_t + \psi)_x = 0, & 0 < x < 1, t > 0 \\ \psi_{tt} - \psi_{xx} + (\varphi_x + \psi) = 0, & 0 < x < 1, t > 0 \\ \varphi(0,t) = \psi(0,t) = 0, & t > 0 \\ \varphi(1,t) + \psi(1,t) = -k_1(t)\varphi_t(1,t), & t > 0 \\ \psi_x(1,t) = -k_2(t)\psi_1(t), & t > 0 \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), & 0 < x < 1 \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), & 0 < x < 1 \\ k_1'(t) = r_1[\varphi_t(1,t)]^2 & k_1(0) = k_{01}, \quad k_{01}, r_1 > 0 \\ k_2'(t) = r_2[\psi_t(1,t)]^2 & k_2(0) = k_{02}, \quad k_{02}, r_2 > 0 \end{cases} \tag{2.4}$$

2.2 Global existence

In this section , we use Faedo-Galerkin method to prove our existence result.

Definition 2.1 *A pair of functions* (φ, ψ)*defined on* $(0,1) \times [0,T], T > 0$ *arbitrary , is called a global strong solution of the closed-loop system* (2.4) *if*

$$\varphi, \psi \in C([0,T];W) \bigcap C^1([0,T;V]) \bigcap C^2([0,T];L^2(0,1))$$
$$(\varphi(.,0), \varphi_t(.,0), \psi(.,0), \psi(.,0)) = (\varphi_0, \varphi_1, \psi_0, \psi_1) \in (W \times V)^2$$

and satisfies

$$\int_0^1 \varphi_{tt} u dx + \int_0^1 (\varphi_x + \psi) u_x dx + k_2(t)\varphi_t(1,t)u(1) = 0$$
$$\int_0^1 \psi_{tt} v dx + \int_0^1 \psi_x v_x dx + \int_0^1 (\varphi_x + \psi) v dx + k_2(t)\psi_t(1,t)v(t) = 0$$
$$k_1'(t) = r_1\varphi_t^2(1,t), k_1(0) = k_{01}, k_{01}, r_1 > 0$$
$$k_2'(t) = r_2\psi_t^2(1,t), k_2(0) = k_{02}, k_{02}, r_2 > 0$$

for any $u, v \in V$*and any* $t \in [0,T]$
Now , we are ready to state and prove our existence result

Theorem 2.2 *Let* $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in W \times V$*satisfy the compatibility conditions*

$$\varphi_{0x}(1) + \psi_{0x}(1) = -k_{01}\varphi_1(1) \quad and \quad \psi_{0x}(1) = -k_{02}\psi_1(1)$$

then , there exist a unique global strong solution to system (2.4)

Proof. *Let $w^j{}_{j\geq 1}$ be a complete orthogonal basis for W and V, and for each $m \geq 1$let*

$$W_m := span\{w^1, ..., w^m\}.$$

We look for a solution in the form

$$\varphi^m(x,t) = \sum_{j=1}^{m} a_{mj}(t)w^j(x) \quad and \quad \psi^m(x,t) = \sum_{j=1}^{m} b_{mj}(t)w^j(x)$$

to the approximate problem

$$\begin{cases} \int_0^1 \varphi_{tt}^m u dx + \int_0^1 (\varphi_x^m + \psi^m) u_x dx + k_2^m(t)\varphi_t^m(1,t)u(1) = 0 \\ \int_0^1 \psi_{tt}^m v dx + \int_0^1 \psi_x^m v_x dx + \int_0^1 (\varphi_x^m + \psi^m) v dx + k_2^m(t)\psi_t^m(1,t)v(t) = 0 \\ (k_1^m)'(t) = r_1[\varphi_t^m(1,t)]^2, k_1^m(0) = k_{01} \\ (k_2^m)'(t) = r_2[\psi_t^m(1,t)]^2, k_2^m(0) = k_{02} \end{cases} \tag{2.5}$$

for all $u, v \in V_m$, where

$$\begin{cases} (\varphi_0^m, \varphi_1^m) := \left(\sum_{j=1}^{m} (\varphi_0, w^j)w^j, \sum_{j=1}^{m} (\varphi_1, w^j)w^j \right) \to (\varphi_0, \varphi_1) \quad in \quad W \times V \\ \\ (\psi_0^m, \psi_1^m) := \left(\sum_{j=1}^{m} (\psi_0, w^j)w^j, \sum_{j=1}^{m} (\psi_1, w^j)w^j \right) \to (\psi_0, \psi_1) \quad in \quad W \times V \end{cases} \tag{2.6}$$

$$\varphi_{0x}^m(1) + \psi_0^m(1) = -k_{01}\varphi_1^m(1) \tag{2.7}$$

and

$$\psi_{0x}^m(1) = -k_{02}\psi_1^m(1) \tag{2.8}$$

Problem (2.5) *is a system of ordinary differential equations in t which has a local solution , in $[0, t_m)$.*

The next a priori estimates show that $t_m = \infty$ for any $m \geq 1$

First a priori estimate: *Substituting $u = \varphi_t^m$ in $(2.5)_1$ and $v = \psi_t^m$ in $(2.5)_2$, we obtain*

$$\int_0^1 \varphi_{tt}^m \varphi_t^m dx + \int_0^1 (\varphi_x^m + \psi^m)\varphi_{tx}^m dx + k_1^m(t)[\varphi_t^m(1,t)]^2 = 0$$

where

$$\int_0^1 \varphi_{tt}^m \varphi_t^m dx = \frac{1}{2}\frac{d}{dt}\int_0^1 (\varphi_t^m)^2 dx \tag{2.9}$$

and

$$\frac{d(\varphi_x^m + \psi^m)^2}{dt} = 2(\varphi_x^m + \psi^m)(\varphi_x^m + \psi^m)_t$$

$$\frac{1}{2}\frac{d(\varphi_x^m + \psi^m)^2}{dt} = (\varphi_x^m + \psi^m)\varphi_{xt}^m + (\varphi_x^m + \psi^m)\psi_t^m$$

So

$$(\varphi_x^m + \psi^m)\varphi_{xt}^m = \frac{1}{2}\frac{d(\varphi_x^m + \psi^m)^2}{dt} - (\varphi_x^m + \psi^m)\psi_t^m \tag{2.10}$$

from (2.9) and (2.10),gives
the first equation

$$\frac{1}{2}\frac{d}{dt}[\|\varphi_t^m(t)\|^2 + \|\varphi_x^m(t) + \psi^m(t)\|^2] - \int_0^1 (\varphi_x^m(t) + \psi^m(t))\psi_t^m(t) + k_1^m(t)[\varphi_t^m(1,t)]^2 = 0 \tag{2.11}$$

and we have

$$\int_0^1 \psi_{tt}^m \psi_t^m dx + \int_0^1 \psi_x^m \psi_{tx} dx + \int_0^1 (\varphi_x^m + \psi^m)\psi_t^m dx + k_2^m(t)[\psi_t^m(1,t)]^2 = 0$$

where

$$\int_0^1 \psi_{tt}^m \psi_t^m dx = \frac{1}{2}\frac{d}{dt}\int_0^1 (\psi_t^m)^2 dx \tag{2.12}$$

and

$$\int_0^1 \psi_x^m \psi_{tx} dx = \frac{1}{2}\frac{d}{dt}\int_0^1 (\psi_x^m)^2 dx \tag{2.13}$$

from (2.12) and (2.13) ,we get
the second equation

$$\frac{1}{2}\frac{d}{dt}[\|\psi_t^m(t)\|^2 + \|\psi_x^m(t)\|^2] + \int_0^1 (\varphi_x^m(t) + \psi^m(t))\psi_t^m(t) + k_2^m(t)[\psi_t^m(1,t)]^2 = 0 \tag{2.14}$$

by adding up (2.11) *and* (2.14) *,we arrive to*

$$\frac{d}{dt}\frac{1}{2}[\|\varphi_t^m(t)\|^2+\|\varphi_x^m(t)+\psi^m(t)\|^2\|\psi_t^m(t)\|^2+\|\psi_x^m(t)\|^2]+k_1^m(t)[\varphi_t^m(1,t)]^2+k_2^m(t)[\psi_t^m(1,t)]^2=0 \tag{2.15}$$

using $(2.5)_3$ *and* $(2.5)_4$*, we obtain*

$$k_1^m(t)[\varphi_t^m(1,t)]^2=\frac{1}{r_1}k_1^m(t)[k_1^m]'(t)=\frac{d}{dt}\frac{1}{2r_1}[k_1^m(t)]^2 \tag{2.16}$$

and

$$k_2^m(t)[\psi_t^m(1,t)]^2=\frac{1}{r_2}k_2^m(t)[k_2^m]'(t)=\frac{d}{dt}\frac{1}{2r_2}[k_2^m(t)]^2 \tag{2.17}$$

Substituting (2.1) *and* (2.17) *in*(2.15) *, we get , for each* $m\geq 1$ *and for any* $0<t<t_m$

$$\frac{d}{dt}\frac{1}{2}\{\|\varphi_t^m(t)\|^2+\|\varphi_x^m(t)+\psi^m(t)\|^2\|\psi_t^m(t)\|^2+\|\psi_x^m(t)\|^2+\frac{1}{r_1}[k_1^m(t)]^2+\frac{1}{r_2}[k_2^m(t)]^2\}=0$$

integration over $(0,1)$ *and using* (2.6) *, we get*

$$\begin{aligned}&\|\varphi_t^m(t)\|^2+\|\varphi_x^m(t)+\psi^m(t)\|^2\|\psi_t^m(t)\|^2+\|\psi_x^m(t)\|^2+\frac{1}{r_1}[k_1^m(t)]^2+\frac{1}{r_2}[k_2^m(t)]^2\\&=\|\varphi_1^m\|^2+\|\varphi_{0x}^m+\psi_0^m\|^2\|\psi_1^m\|^2+\|\psi_{0x}^m(t)\|^2+\frac{1}{r_1}k_{01}^2(t)+\frac{1}{r_2}k_{02}^2\leq C\end{aligned} \tag{2.18}$$

where C *is independent of* m *and* t*, Also from* $(2.5)_3$ *and* $(2.5)_4$*, we deduce*

$$\int_0^t[\varphi_t^m(1,s)]^2ds=\frac{1}{r_1}\int_0^t[k_1^m]'(s)ds\leq\frac{1}{r_1}k_1^m(t)\leq C \tag{2.19}$$

and

$$\int_0^t[\psi_t^m(1,s)]^2ds=\frac{1}{r_2}\int_0^t[k_2^m]'(s)ds\leq\frac{1}{r_2}k_2^m(t)\leq C \tag{2.20}$$

Second a priori estimate: *Set* $t=0$*,* $u=\varphi_t^m(.,0)$ *in* (2.5) *, integration by parts, then exploit* (2.6) *and* (2.7) *to obtain*

$$\begin{aligned}\|\varphi_{tt}^m(.,0)\|^2&=\int_0^1[\varphi_{0xx}^m(x)+\psi_{0x}^m(x)]\varphi_{tt}(x,0)dx\\&\leq\|\varphi_{0xx}^m+\psi_{0x}^m\|\|\varphi_{tt}(.,0)\|\end{aligned}$$

This entails that

$$\|\varphi_{tt}^m(.,0)\| \leq \|\varphi_{0xx}^m + \psi_{0x}^m\| \leq C \tag{2.21}$$

Similarly , sitting $t = 0, v = \psi_t^m(.,0)$ *in* (2.5)*,integrate by parts,then exploit* (2.6) *and* (2.8) *to get*

$$\|\psi_{tt}^m(.,0)\| \leq \|\psi_{0xx}^m\| + \|\psi_{0x}^m + \psi_0^m\| \leq C \tag{2.22}$$

Third a priori estimate: *First, differentiating* $(2.5)_1$ *and* $(2.5)_2$ *with respect to* t *and replacing* u *and* v *by* φ_{tt}^m *and* ψ_{tt}^m *respectively, then, adding the resultants, we find*

$$\begin{aligned}\frac{d}{dt}\frac{1}{2}[\|\varphi_{tt}^m(t)\|^2 + \|\varphi_{xt}^m(t) + \psi_t^m(t)\|^2\|\psi_{tt}^m(t)\|^2 + \|\psi_{xt}^m(t)\|^2]\\ + \frac{r_1}{2}[\varphi_t^m(1,t)]^4 + \frac{r_2}{2}[\psi_t^m(1,t)]^4\\ = -k_1^m(t)[\varphi_{tt}^m(1,t)]^2 - k_2^m(t)[\psi_{tt}^m(1,t)]^2 \leq 0\end{aligned}$$

Integration over $(0,1)$ *and taking advantage of* (2.6) , (2.21) *and* (2.22) , *we arrive at*

$$\begin{aligned}&\|\varphi_{tt}^m(t)\|^2 + \|\varphi_{xt}^m(t) + \psi_t^m(t)\|^2\|\psi_{tt}^m(t)\|^2 + \|\psi_{xt}^m(t)\|^2 + \frac{r_1}{2}[\varphi_t^m(1,t)]^4 + \frac{r_2}{2}[\psi_t^m(1,t)]^4\\ &\leq \|\varphi_{tt}^m(.,0)\|^2 + \|\varphi_{xt} + \psi_1^m\|^2\|\psi_{tt}^m(.,0)\|^2 + \|\psi_{xt}^m\|^2 + \frac{r_1}{2}[\varphi_t^m(1)]^4 + \frac{r_2}{2}[\psi_t^m(1)]^4\\ &\leq C\end{aligned} \tag{2.23}$$

We conclude from (2.18) *to* (2.20) *and* (2.23) *that* $t_m = \infty$ *and for any* $T > 0$, *we have*

$$\begin{aligned}&(\varphi^m) \text{ and } (\psi^m) \text{ are bounded in } L^\infty(0,T,V),\\ &(\varphi_t^m) \text{ and } (\psi_t^m) \text{ are bounded in } L^\infty(0,T,V),\\ &(\varphi_{tt}^m) \text{ and } (\psi_{tt}^m) \text{ are bounded in } L^\infty(0,T,L^2(0,1)),\\ &(\varphi_t^m(1,0)) \text{ and } (\psi_t^m(1,0)) \text{ are bounded in } L^2(0,T)\cap L^\infty(0,T),\end{aligned} \tag{2.24}$$

So, there exists a subsequence,still denoted by (φ^m, ψ^m)*,such that*

$$\begin{aligned}&\varphi^m \to \varphi, \psi^m \to \psi \text{ weakly star in } L^\infty(0,T,V),\\ &\varphi_t^m \to \varphi_t, \psi_t^m \to \psi_t \text{ weakly star in } L^\infty(0,T,V),\\ &\varphi_{tt}^m \to \varphi_{tt}, \psi_{tt}^m \to \psi_{tt} \text{ weakly star in } L^\infty(0,T,L^2(0,1)),\end{aligned} \tag{2.25}$$

using Aubin-Lions lemma , we infer that

$$\varphi_t^m \to \varphi_t, \psi_t^m \to \psi_t \quad in \quad L^\infty(0,T,L^2(0,1))$$

from which we deduce

$$\varphi_t^m \to \varphi, \psi_t^m \to \psi \quad a.e. \quad in \quad (0,1)\times(0,T)$$

this, together with the continuity of $\varphi_t^m, \psi_t^m, \varphi_t, \psi_t$ *,leads to*

$$\varphi_t^m(1,.) \to \varphi_t(1,.), \psi_t^m(1,.) \to \psi_t(1,.) \quad pointwise \quad in \quad [0,T]$$

The boundedness of $(\varphi_t^m(1,.))$ *and* $(\psi_t^m(1,.))$ *in* $L^2(0,T)$ *follows from* $(2.24)_4$*, Owing to the Lebesgue dominated convergence theorem we deduce that*

$$\varphi_t^m(1,.) \to \varphi_t(1,.) \quad and \quad \psi_t^m(1,.) \to \psi_t(1,.) \quad in \quad L^2(0,T) \tag{2.26}$$

An exploitation of $(2.5)_3$ *,* $(2.5)_4$*, and* (2.26) *gives*

$$\begin{aligned} k_1^m(t) = k_{01} + r_1\int_0^t [\varphi_t^m(1,s)]^2 ds \to k_{01} + r_1\int_0^t [\varphi_t(1,s)]^2 ds = k_1(t) \\ k_2^m(t) = k_{02} + r_2\int_0^t [\psi_t^m(1,s)]^2 ds \to k_{02} + r_2\int_0^t [\psi_t(1,s)]^2 ds = k_2(t) \end{aligned} \tag{2.27}$$

in $L^\infty(0,T)$*. Also, from* $(2.24)_4$ *and* (2.26)*, we have*

$$\begin{aligned} (k_1^m)'(.) = r_1[\varphi_t^m(1,.)]^2 \rightharpoonup r_1[\varphi_t(1,.)]^2 = (k_1^m)'(.) \quad in \quad L^2(0,T) \\ (k_2^m)'(.) = r_2[\psi_t^m(1,.)]^2 \rightharpoonup r_2[\psi_t(1,.)]^2 = (k_2^m)'(.) \quad in \quad L^2(0,T) \end{aligned}$$

Next, we take the limit of (2.5) *as* m *goes to infinity , then use* (2.25)*-*(2.27) *to obtain*

$$\begin{aligned} &\int_0^1 \varphi_{tt} u dx + \int_0^1 (\varphi_x + \psi)u_x dx + k_2(t)\varphi_t(1,t)u(1) = 0 \\ &\int_0^1 \psi_{tt} v dx + \int_0^1 \psi_x v_x dx + \int_0^1 (\varphi_x+\psi) v dx + k_2(t)\psi_t(1,t)v(t) = 0 \end{aligned}$$

for any $u,\, v \in V$*. Using Aubin-Lions lemma again , we entail that* $\varphi, \psi \ \in C([0,T];W), \varphi_t, \psi_t \ \in C([0,T];V), \varphi_{tt}, \psi_{tt} \in C([0,T];L^2(0,1))$ *and* $\varphi(0,.) = \varphi_0, \varphi_t(0,.) = \varphi_1, \psi(0,.) = \psi_0, \psi_t(0,t) = \psi_1$

To prove the uniqueness , let $(\varphi, \psi, k_1, k_2)$ *and* $(\tilde{\varphi}, \tilde{\psi}, \tilde{k_1}, \tilde{k_2})$ *be two solutions of* (2.4) *with* *the same initial data. Then,*

$$\frac{d}{dt}\{\|\varphi_t - \tilde{\varphi_t}\|^2 + \|(\varphi_x + \psi) - (\tilde{\varphi_x} + \tilde{\psi})\|^2 + \|\psi_t - \tilde{\psi_t}\|^2 + \|\psi_x - \tilde{\psi_x}\|^2$$
$$+ \frac{1}{r_1}[k_1(t) - \tilde{k_1}(t)]^2 + \frac{1}{r_2}[k_2(t) - \tilde{k_2}(t)]\}$$
$$= -[k_1(t) + \tilde{k_1}(t)][\varphi_t(1,t) - \tilde{\varphi_t}(1,t)]^2 - [k_2(t) + \tilde{k_2}(t)][\psi_t(1,t) - \tilde{\psi_t}(1,t)]^2$$
$$\leq 0$$

This implies that

$$(\varphi, \psi, k_1, k_2) = (\tilde{\varphi}, \tilde{\psi}, \tilde{k_1}, \tilde{k_2})$$

■

2.3 Stability analysis

In this section, use the energy method to prove that system (3.4) is exponentially stable. To achieve this goal, we first establish some lemmas needs in the proof of exponential stability result

Lemma 2.3 *Let* (φ, ψ)*be the solution of* (3.4) *, then the energy functional* E *, defined by*

$$E(t) = \frac{1}{2}[\|\varphi_t\|^2 + \|\varphi_x + \psi\|^2 + \|\psi_t\|^2 + \|\psi_x\|^2] \quad \forall t > 0 \tag{2.28}$$

satisfies

$$E'(t) = -k_1(t)\varphi_t^2(1,t) - k_2(t)\psi_t^2(1,t) \leq 0 \quad \forall t > 0 \tag{2.29}$$

Proof. Multiplying $(2.4)_1$ and $(2.4)_2$ by φ_t and ψ_t respectively , integrating over $(0,1)$ and using integration by parts and the boundary conditions

The first equation:

$$\int_0^1 \varphi_{tt}\varphi_t dx - \int_0^1 (\varphi_x + \psi)\varphi_t dx = 0$$

where

$$\int_0^1 \varphi_{tt}\varphi_t dx = \frac{1}{2}\frac{d}{dt}\int_0^1 \varphi_t^2 dx = \frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 \tag{2.30}$$

and

$$\begin{aligned}
-\int_0^1 (\varphi_x + \psi)_x \varphi_t dx &= -\int_0^1 \varphi_{xx}\varphi_t dx - \int_0^1 \psi_x \varphi_t dx \\
&= -[\varphi_x\varphi_t]_0^1 + \int_0^1 \varphi_x\varphi_{tx} dx - [\psi\varphi_t]_0^1 + \int_0^1 \psi\varphi_{tx} dx \\
&= -\varphi_x(1,t)\varphi_t(1,t) + \varphi_x(0,t)\varphi_t(0,t) + \frac{1}{2}\frac{d}{dt}\int_0^1 \varphi_x^2 dx \\
&- \psi(1,t)\varphi_t(1,t) + \psi(0,t)\varphi_t(0,t) + \frac{d}{dt}\int_0^1 \psi\varphi_x dx - \int_0^1 \psi_t\varphi_x dx \\
&= \frac{1}{2}\|\varphi_x\|^2 + \frac{d}{dt}\int_0^1 \psi\varphi_x dx - \int_0^1 \psi_t\varphi_x dx - \varphi_x(1,t)\varphi_t(1,t) - \psi_x(1,t)\varphi_t(1,t)
\end{aligned} \tag{2.31}$$

from (2.30) and (2.31) , we obtain

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\varphi_x\|^2 + \frac{d}{dt}\int_0^1 \psi\varphi_x dx - \int_0^1 \psi_t\varphi_x dx - [-k_1(t)\varphi_t(1,t) - \psi(1,t)]\varphi_t(1,t) - \psi(1,t)\varphi_t(1,t) = 0$$

So

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\varphi_x\|^2 + \frac{d}{dt}\int_0^1 \psi\varphi_x dx - \int_0^1 \psi_t\varphi_x dx + k_1(t)\varphi_t^2(1,t) = 0 \tag{2.32}$$

The second equation:

$$\int_0^1 \psi_{tt}\psi_t dx - \int_0^1 \psi_{xx}\psi_t dx + \int_0^1 (\varphi_x\psi)\psi_t dx = 0$$

where

$$\int_0^1 \psi_{tt}\psi_t dx = \frac{1}{2}\frac{d}{dt}\int_0^1 \psi_t^2 dx = \frac{1}{2}\frac{d}{dt}\|\psi_t\|^2 \tag{2.33}$$

and

$$-\int_0^1 \psi_{xx}\psi_t dx = -[\psi_x\psi_t]_0^1 + \int_0^1 \psi_x\psi_{tx}dx$$
$$= -\psi_x(1,t)\psi_t(1,t) + \psi_x(0,t)\psi_t(0,t) + \frac{1}{2}\frac{d}{dt}\int_0^1 \psi_x^2 \tag{2.34}$$
$$= \frac{1}{2}\frac{d}{dt}\|\psi_x\|^2 - \psi_x(1,t)\psi_t(1,t)$$

$$\int_0^1 (\varphi_x\psi)\psi_t dx = \int_0^1 \varphi_x\psi_t dx + \int_0^1 \psi\psi_t dx$$
$$= \int_0^1 \varphi_x\psi_t dx + \frac{1}{2}\frac{d}{dt}\int_0^1 \psi^2 dx \tag{2.35}$$

from (2.33), (2.34) and (2.35), we get

$$\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\psi_x\|^2 + \frac{1}{2}\frac{d}{dt}\|\psi\|^2 + \int_0^1 \varphi_x\psi_t dx - (-k_2(t)\psi_t(1,t))\psi_t(1,t) = 0$$

So

$$\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\psi_x\|^2 + \frac{1}{2}\frac{d}{dt}\|\psi\|^2 + \int_0^1 \varphi_x\psi_t dx + k_2(t)\psi_t^2(1,t) = 0 \tag{2.36}$$

adding up (2.32) and (2.36) , we arrive at

$$\frac{1}{2}\frac{d}{dt}\|\varphi_t\|^2+\frac{1}{2}\frac{d}{dt}\int_0^1(\varphi_x^2+\psi^2+2\varphi_x\psi)dx+\frac{1}{2}\frac{d}{dt}\|\psi_t\|^2+\frac{1}{2}\frac{d}{dt}\|\psi_x\|^2 = -k_1(t)\varphi_t^2(1,t)-k_2(t)\psi_t^2(1,t)$$

So

$$\frac{1}{2}\frac{d}{dt}[\|\varphi_t\|^2 + \|\varphi_x + \psi\|^2 + \|\psi_t\|^2 + \|\psi_x\|^2] = -k_1(t)\varphi_t^2(1,t) - k_2(t)\psi_t^2(1,t)$$

this is exactly (2.29)

■

Lemma 2.4 *Let (φ, ψ)be the solution of (2.4), we define a Lyapunov functional L by*

$$L(t) = E(t) + \frac{1}{2r_1}k_1^2(t) + \frac{1}{2r_2}k_2^2(t) \quad \forall t \geq 0$$

satisfies, for any $t \geq 0$.

$$
\begin{aligned}
L^{'}(t) &= E^{'}(t) + \frac{1}{2r_1}[2k_1(t)k_1^{'}(t)] + \frac{1}{2r_2}[2k_2(t)k_2^{'}(t)] = -k_1(t)\varphi_t^2(1,t) \\
&= -k_2(t)\psi_t^2(1,t) + \frac{1}{r_1}k_1(t)[r_1\varphi_t^2(1,t)] + \frac{1}{r_2}k_2(t)[r_2\psi_t^2(1,t)] = 0
\end{aligned}
$$

As a consequence, we have

$$L(t) = L(0) \quad \forall t \geq 0$$

and this entails that

$$\sup_{t\geq 0} |E(t) + k_1^2(t) + k_2^2(t)| \leq C_1 \tag{2.37}$$

where C_1 is a positive constant that depends on the initial data

Lemma 2.5 *Let (φ, ψ)be the solution of* (2.4) *,then the functional*

$$I_1(t) = 2\int_0^1 x\varphi_x\varphi_t dx \tag{2.38}$$

satisfies

$$I_1^{'}(t) = -\|\varphi_t\|^2 - \|\varphi_x\|^2 + [\psi(1,t) + k_1(t)\varphi_t(1,t)]^2 + \varphi_t^2(1,t) + 2\int_0^1 x\varphi_x\psi_x dx \tag{2.39}$$

Proof. using (2.4) and integration by parts , gives

$$
\begin{aligned}
I_1^{'}(t) &= 2\int_0^1 x\varphi_t\varphi_{xt}dx + 2\int_0^1 x\varphi_{tt}\varphi_x dx \\
&= 2\int_0^1 x\varphi_t\varphi_{xt}dx + 2\int_0^1 x(\varphi_x + \psi)_x\varphi_x dx \\
&= 2\int_0^1 x\varphi_t\varphi_{xt}dx + 2\int_0^1 x\varphi_{xx}\varphi_x dx + 2\int_0^1 x\psi_x\varphi_x dx
\end{aligned}
$$

where

$$2\int_0^1 x\varphi_t\varphi_{xt}dx = 2\int_0^1 x(\frac{1}{2}\frac{d\varphi_t^2}{dx})dx$$
$$= \int_0^1 x\frac{d\varphi_t^2}{dx}dx \tag{2.40}$$
$$= [x\varphi_t^2]_0^1 - \int_0^1 \varphi_t^2 dx$$
$$= \varphi_t^2(1,t) - \int_0^1 \varphi_t^2 dx$$

and

$$2\int_0^1 x\varphi_{xx}\varphi_x dx = 2\int_0^1 x(\frac{1}{2}\frac{d\varphi_x^2}{dx})dx$$
$$= \int_0^1 x\frac{d\varphi_x^2}{dx}dx \tag{2.41}$$
$$= [x\varphi_x^2]_0^1 - \int_0^1 \varphi_x^2 dx$$
$$= \varphi_x^2(1,t) - \int_0^1 \varphi_x^2 dx$$

from (2.40) and (2.41) , we obtain

$$I_1^{'}(t) = -\|\varphi_t\|^2 - \|\varphi_x\|^2 + \varphi_x^2(1,t) + \varphi_t^2(1,t) + 2\int_0^1 x\varphi_x\psi_x dx$$
$$= -\|\varphi_t\|^2 - \|\varphi_x\|^2 + [\psi(1,t) + k_1(t)\varphi_t(1,t)]^2 + \varphi_t^2(1,t) + 2\int_0^1 x\varphi_x\psi_x dx$$

■

Lemma 2.6 *Let* (φ, ψ)*be the solution of* (2.4) *,then the functional*

$$I_2(t) = 2\int_0^1 x\psi_x\psi_t dx \tag{2.42}$$

satisfies

$$I_2^{'}(t) = -\|\psi_t\|^2 - \|\psi_x\|^2 + \|\psi\|^2 + [1 + k_2^2(t)]\psi_t^2(1,t) - \psi^2(1,t) - 2\int_0^1 x\varphi_x\psi_x dx \tag{2.43}$$

Proof. using (2.4) and integration by parts , we find

$$
\begin{aligned}
I_2^{'}(t) &= 2\int_0^1 x\psi_t\psi_{xt}dx + 2\int_0^1 x\psi_{tt}\psi_x dx \\
&= 2\int_0^1 x\psi_t\psi_{xt}dx + 2\int_0^1 x\psi_x\psi_{xx}dx - 2\int_0^1 x\psi_x(\varphi_x+\psi)dx \\
&= 2\int_0^1 x\psi_t\psi_{xt}dx + 2\int_0^1 x\psi_x\psi_{xx}dx - 2\int_0^1 x\psi_x\varphi_x dx - 2\int_0^1 x\psi_x\psi dx
\end{aligned}
$$

where

$$
\begin{aligned}
2\int_0^1 x\psi_t\psi_{xt}dx &= 2\int_0^1 x(\frac{1}{2}\frac{d\psi_t^2}{dx})dx \\
&= \int_0^1 x\frac{d\psi_t^2}{dx}dx \\
&= [x\psi_t^2]_0^1 - \int_0^1 \psi_t^2 dx \\
&= \psi_t^2(1,t) - \int_0^1 \psi_t^2 dx
\end{aligned}
\tag{2.44}
$$

and

$$
\begin{aligned}
2\int_0^1 x\psi_x\psi_{xx}dx &= 2\int_0^1 x(\frac{1}{2}\frac{d\psi_x^2}{dx})dx \\
&= \int_0^1 x\frac{d\psi_x^2}{dx}dx \\
&= [x\psi_x^2]_0^1 - \int_0^1 \psi_x^2 dx \\
&= \psi_x^2(1,t) - \int_0^1 \psi_x^2 dx
\end{aligned}
\tag{2.45}
$$

and

$$
\begin{aligned}
2\int_0^1 x\psi_x\psi dx &= 2\int_0^1 x(\frac{1}{2}\frac{d\psi^2}{dx})dx \\
&= \int_0^1 x\frac{d\psi^2}{dx}dx \\
&= [x\psi^2]_0^1 - \int_0^1 \psi^2 dx \\
&= \psi^2(1,t) - \int_0^1 \psi^2 dx
\end{aligned}
\tag{2.46}
$$

from (2.44) , (2.45) and (2.46) ,we get

$$
\begin{aligned}
I_2'(t) &= -\|\psi_t\|^2 - \|\psi_x\|^2 + \|\psi\|^2 + \psi_t^2(1,t) + \psi_x^2(1,t) - \psi^2(1,t) - 2\int_0^1 x\varphi_x\psi_x dx \\
&= -\|\psi_t\|^2 - \|\psi_x\|^2 + \|\psi\|^2 + \psi_t^2(1,t) + k_2^2(t)\psi_t^2(1,t) - \psi^2(1,t) - 2\int_0^1 x\varphi_x\psi_x dx \\
&= -\|\psi_t\|^2 - \|\psi_x\|^2 + \|\psi\|^2 + [1 + k_2^2(t)]\psi_t^2(1,t) - \psi^2(1,t) - 2\int_0^1 x\varphi_x\psi_x dx
\end{aligned} \tag{2.47}
$$

■

Lemma 2.7 *Let* (φ, ψ)*be the solution of* (2.4) *,then the functional*

$$I_3(t) = -\frac{1}{2}\int_0^1 \varphi\varphi_t dx \tag{2.48}$$

satisfies

$$I_3'(t) = -\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\varphi_x\|^2 + \frac{1}{2}k_1(t)\varphi_t(1,t)\varphi(1,t) + \frac{1}{2}\int_0^1 \varphi_x\psi dx \tag{2.49}$$

Proof. using (2.4) and integration by parts , we have

$$
\begin{aligned}
I_3'(t) &= -\frac{1}{2}\int_0^1 \varphi_t^2 - \frac{1}{2}\int_0^1 \varphi\varphi_{tt} dx \\
&= -\frac{1}{2}\int_0^1 \varphi_t^2 - \frac{1}{2}\int_0^1 \varphi(\varphi_x + \psi)_x dx \\
&= -\frac{1}{2}\int_0^1 \varphi_t^2 - \frac{1}{2}\int_0^1 \varphi\varphi_{xx} dx - \frac{1}{2}\int_0^1 \varphi\psi_x dx
\end{aligned}
$$

where

$$\frac{1}{2}\int_0^1 \varphi\varphi_{xx}dx = \frac{1}{2}[\varphi\varphi_x]_0^1 - \frac{1}{2}\int_0^1 \varphi_x^2 dx \\ = \frac{1}{2}\varphi(1,t)\varphi_x(1,t) - \frac{1}{2}\|\varphi_x\|^2 \tag{2.50}$$

and

$$\frac{1}{2}\int_0^1 \varphi\psi_x dx = \frac{1}{2}[\varphi\psi]_0^1 - \frac{1}{2}\int_0^1 \varphi_x\psi dx \\ = \frac{1}{2}\varphi(1,t)\psi(1,t) - \frac{1}{2}\int_0^1 \varphi_x\psi dx \tag{2.51}$$

from (2.50) and (2.51) , we get

$$\begin{aligned} I_3^{'}(t) &= -\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\varphi_x\|^2 - \frac{1}{2}\varphi(1,t)\varphi_x(1,t) - \frac{1}{2}\varphi(1,t)\psi(1,t) + \frac{1}{2}\int_0^1 \varphi_x\psi dx \\ &= -\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\varphi_x\|^2 - \frac{1}{2}[-k_1(t)\varphi_t(1,t) - \psi(1,t)]\varphi(1,t) - \frac{1}{2}\varphi(1,t)\psi(1,t) + \frac{1}{2}\int_0^1 \varphi_x\psi dx \\ &= -\frac{1}{2}\|\varphi_t\|^2 + \frac{1}{2}\|\varphi_x\|^2 + \frac{1}{2}k_1(t)\varphi_t(1,t)\varphi(1,t) + \frac{1}{2}\int_0^1 \varphi_x\psi dx \end{aligned}$$

■

Lemma 2.8 *Let* (φ, ψ)*be the solution of* (2.4) *,then the functional*

$$I_4(t) = \frac{1}{2}\int_0^1 \psi\psi_t dx \tag{2.52}$$

satisfies

$$I_4^{'}(t) = \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\psi_x\|^2 - \frac{1}{2}\|\psi\|^2 - \frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t) - \frac{1}{2}\int_0^1 \varphi_x\psi dx \tag{2.53}$$

Proof. using (2.4) and integration by parts , yield

$$\begin{aligned} I_4'(t) &= \frac{1}{2}\int_0^1 \psi_t^2 + \frac{1}{2}\int_0^1 \psi\psi_{tt}dx \\ &= \frac{1}{2}\int_0^1 \psi_t^2 + \frac{1}{2}\int_0^1 \psi[\psi_{xx}(\varphi_x+\psi)]dx \\ &= \frac{1}{2}\int_0^1 \psi_t^2 + \frac{1}{2}\int_0^1 \psi\psi_{xx}dx - \frac{1}{2}\int_0^1 \psi\varphi_x dx - \frac{1}{2}\int_0^1 \psi^2 dx \end{aligned}$$

where

$$\frac{1}{2}\int_0^1 \psi\psi_{xx}dx = \frac{1}{2}[\psi\psi_x]_0^1 - \frac{1}{2}\int_0^1 \psi^2 = \frac{1}{2}\psi(1,t)\psi_x(1,t) - \frac{1}{2}\int_0^1 \psi^2 dx$$

So

$$\begin{aligned} I_4'(t) &= \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\psi_x\|^2 - \frac{1}{2}\|\psi\|^2 + \frac{1}{2}\psi_x(1,t)\psi(1,t) - \frac{1}{2}\int_0^1 \varphi_x\psi dx \\ &= \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\psi_x\|^2 - \frac{1}{2}\|\psi\|^2 - \frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t) - \frac{1}{2}\int_0^1 \varphi_x\psi dx \end{aligned}$$

■

Lemma 2.9 *Let*(φ,ψ)*be the solution of* (2.4) *,for* $\varepsilon > 0$ *, we introduce another Lyapunov functional*

$$L_\varepsilon(t) = E(t) + \varepsilon I(t) \tag{2.54}$$

where

$$I(t) = \sum_{j=1}^{4} I_j(t)$$

satisfies

$$L_\varepsilon(t) \sim E$$

and the estimate

$$L_\varepsilon'(t) \le -\frac{\varepsilon}{4}E(t) - [k - \varepsilon(1 + \frac{13}{4}C_1)][\varphi_t^2(1,t) + \psi_t^2(1,t)] \tag{2.55}$$

Proof.

At first, exploiting inequalities(1.1) and (1.3) together with the Cauchy-Schwarz inequality, we get

$$|I(t)| \leq \sum_{j=1}^{4} |I_j(t)| \leq \frac{9}{2}E(t), \quad \forall t \geq 0 \tag{2.56}$$

we deduce from (2.56) that

$$|L_\varepsilon(t) - E(t)| = \varepsilon|I(t)| \leq \frac{9}{2}\varepsilon E(t), \quad \forall t \geq 0$$

which implies

$$(1 - \frac{9}{2}\varepsilon)E(t) \leq L_\varepsilon(t) \leq (1 + \frac{9}{2}\varepsilon)E(t), \quad \forall t \geq 0$$

we choose $\varepsilon \leq \frac{2}{9}$, gives

$$0 \leq (1 - \frac{9}{2}\varepsilon)E(t) \leq L_\varepsilon(t) \leq (1 + \frac{9}{2}\varepsilon)E(t), forall t \geq 0 \tag{2.57}$$

So

$$L_\varepsilon(t) \sim E$$

then using inequalities relations (2.39), (2.43), (2.49) and (2.53), we obtain

$$L_\varepsilon^{'}(t) = E^{'}(t) + \varepsilon I^{'}(t)$$

then

$$\begin{aligned}
I^{'}(t) &= \sum_{j=1}^{4} I_j^{'}(t) \\
&= -\frac{3}{2}\|\varphi_t\|^2 - \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\varphi_x\|^2 - \frac{3}{2}\|\psi_x\|^2 + \frac{1}{2}\|\psi\|^2 - \psi^2(1,t) \\
&+ \varphi_t^2(1,t) + [1 + k_2^2(t)]\psi_t^2(1,t) + [\psi(1,t) + k_1(t)\varphi_t(1,t)]^2 \\
&+ \frac{1}{2}k_1(t)\varphi_t(1,t)\varphi(1,t) - \frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t) \\
&= -\frac{3}{2}\|\varphi_t\|^2 - \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\varphi_x\|^2 - \frac{3}{2}\|\psi_x\|^2 + \frac{1}{2}\|\psi\|^2 - \psi^2(1,t) \\
&+ \varphi_t^2(1,t) + \psi^2(1,t) + k_1^2(t)\varphi_t^2(1,t) + 2k_1(t)\varphi_t(1,t)\psi(1,t) \\
&+ \frac{1}{2}k_1(t)\varphi_t(1,t)\varphi(1,t) - \frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t)
\end{aligned}$$

using inequality (1.1), we get

$$\|\psi\|^2 \leq \frac{1}{2}\|\psi_x\|^2$$

So

$$\frac{1}{2}\|\psi\|^2 \leq \frac{1}{4}\|\psi_x\|^2$$

therefore

$$-\frac{3}{2}\|\psi_x\|^2 + \frac{1}{2}\|\psi\|^2 \leq -\frac{3}{2}\|\psi_x\|^2 + \frac{1}{4}\|\psi_x\|^2 \leq -\frac{5}{4}\|\psi_x\|^2$$

and using Young's inequality, gives

$$2k_1(t)\varphi_t(1,t)\psi(1,t) \leq 2[\frac{1}{4}\psi^2(1,t) + k_1^2(t)\varphi_t^2(1,t)] \leq \frac{1}{2}\psi^2(1,t) + 2k_1^2(t)\varphi_t^2(1,t)$$

$$\frac{1}{2}k_1(t)\varphi_t(1,t)\varphi(1,t) \leq \frac{1}{2}[\frac{1}{2}\varphi^2(1,t) + \frac{1}{2}k_1^2(t)\varphi_t^2(1,t)] \leq \frac{1}{4}\varphi^2(1,t) + \frac{1}{4}k_1^2(t)\varphi_t^2(1,t)$$

$$-\frac{1}{2}k_2(t)\psi_t(1,t)\psi(1,t) \leq \frac{1}{4}k_2^2(t)\psi_t^2(1,t) + \frac{1}{4}\psi^2(1,t)$$

So

$$\begin{aligned} I^{'}(t) \leq & -\frac{3}{2}\|\varphi_t\|^2 - \frac{1}{2}\|\psi_t\|^2 - \frac{1}{2}\|\varphi_x\|^2 - \frac{5}{4}\|\psi_x\|^2 \\ & + \frac{1}{4}\varphi^2(1,t) + \frac{3}{4}\psi^2(1,t) + [1 + \frac{13}{4}k_1^2(t)]\varphi_t^2(1,t) \\ & + [1 + \frac{5}{4}k_2^2(t)]\psi_t^2(1,t) \end{aligned}$$

using inequality (1.2), leads to

$$\varphi^2(1,t) \leq \|\varphi_x\|^2$$

$$\frac{1}{4}\varphi^2(1,t) \leq \frac{1}{4}\|\varphi_x\|^2$$

So

$$\begin{aligned} -\frac{1}{2}\|\varphi_x\|^2 + \frac{1}{4}\varphi^2(1,t) \leq & -\frac{1}{2}\|\varphi_x\|^2 + \frac{1}{4}\|\varphi_x\|^2 \\ & - \leq \frac{1}{4}\|\varphi_x\|^2 \end{aligned}$$

and

$$\psi^2(1,t) \le \|\psi_x\|^2$$
$$\frac{3}{4}\psi^2(1,t) \le \frac{3}{4}\|\psi_x\|^2$$

So

$$\begin{aligned}-\frac{5}{4}\|\psi_x\|^2 + \frac{3}{4}\psi^2(1,t) &\le -\frac{5}{4}\|\psi_x\|^2 + \frac{3}{4}\|\psi_x\|^2\\ &\le -\frac{1}{2}\|\psi_x\|^2\end{aligned}$$

therefore

$$\begin{aligned}I^{'}(t) \le &-\frac{3}{2}\|\varphi_t\|^2 - \frac{1}{2}\|\psi_t\|^2 - \frac{1}{4}\|\varphi_x\|^2 - \frac{1}{2}\|\psi_x\|^2\\ &+ [1 + \frac{13}{4}k_1^2(t)]\varphi_t^2(1,t) + [1 + \frac{5}{4}k_2^2(t)]\psi_t^2(1,t)\\ \le &-\frac{1}{4}[\|\varphi_t\|^2 + \|\psi_t\|^2 + \|\varphi_x\|^2 + \|\psi_x\|^2] + [1 + \frac{13}{4}k_1^2(t)]\varphi_t^2(1,t)\\ &+ [1 + \frac{5}{4}k_2^2(t)]\psi_t^2(1,t)\\ \le &-\frac{1}{4}E(t) + [1 + \frac{13}{4}k_1^2(t)]\varphi_t^2(1,t) + [1 + \frac{5}{4}k_2^2(t)]\psi_t^2(1,t)\\ \le &-\frac{1}{4}E(t) + (1 + \frac{13}{4}C_1)[\varphi_t^2(1,t) + \psi_t^2(1,t)], \quad \forall t \ge 0\end{aligned} \tag{2.58}$$

using (2.29), (2.37), (2.57) and (2.58), we conclude, for $t \ge 0$

$$\begin{aligned}L_\varepsilon^{'}(t) &\le -k_1(t)\varphi_t^2(1,t) - k_2(t)\psi_t^2(1,t) - \frac{\varepsilon}{4}E(t) + \varepsilon(1 + \frac{13}{4}C_1)[\varphi_t^2(1,t) + \psi_t^2(1,t)]\\ &\le -k_{01}\varphi_t^2(1,t) - k_{02}\psi_t^2(1,t) - \frac{\varepsilon}{4}E(t) + \varepsilon(1 + \frac{13}{4}C_1)[\varphi_t^2(1,t) + \psi_t^2(1,t)]\\ &\le -\frac{\varepsilon}{4}E(t) - k[\varphi_t^2(1,t) + \psi_t^2(1,t) + \varepsilon(1 + \frac{13}{4}C_1)][\varphi_t^2(1,t) + \psi_t^2(1,t)]\\ &\le -\frac{\varepsilon}{4}E(t) - [k - \varepsilon(1 + \frac{13}{4}C_1)][\varphi_t^2(1,t) + \psi_t^2(1,t)]\end{aligned}$$

where $k = min\{k_{01}, k_{02}\}$ ■

Theorem 2.10 *Let* (φ, ψ)*be the solution of* (2.4) *,then there exist two positive constants* C_ε *and* λ_ε *such that the energy functional* (2.28) *satisfies*

$$E^{'}(t) \le C_\varepsilon e^{-\lambda_\varepsilon t} \quad ,\forall t \ge 0$$

Proof. From (2.55), we choose $\varepsilon < min\{\frac{2}{9}, \frac{4}{4+13C_1}\}$, so that

$$L_\varepsilon^{'}(t) \leq -\frac{\varepsilon}{4}E(t)$$

Exploiting (2.57), get

$$L_\varepsilon^{'}(t) \leq -\frac{\varepsilon}{4}(\frac{2}{2-9\varepsilon})L_\varepsilon(t)$$

So

$$L_\varepsilon^{'}(t) \leq -\frac{\varepsilon}{2(2-9\varepsilon)}L_\varepsilon(t)$$

$$\frac{L_\varepsilon^{'}(t)}{L_\varepsilon(t)} \leq -\frac{\varepsilon}{2(2-9\varepsilon)}$$

A simple integration over $(0,1)$, gives

$$\int_0^t \frac{L_\varepsilon^{'}(s)}{L_\varepsilon(s)}ds \leq -\int_0^t \lambda_\varepsilon s ds$$

$$[\ln L_\varepsilon(s)]_0^t \leq [-\lambda_\varepsilon s]_0^t$$

$$\ln(L_\varepsilon(t)) - \ln(L_\varepsilon(0)) \leq -\lambda_\varepsilon t$$

$$\ln(\frac{L_\varepsilon(t)}{L_\varepsilon(0)}) \leq -\lambda_\varepsilon t$$

So

$$L_\varepsilon(t) \leq L_\varepsilon(0)e^{-\lambda_\varepsilon t} \quad \forall t \geq 0 \tag{2.59}$$

where $\lambda_\varepsilon = \frac{\varepsilon}{2(2-9\varepsilon)}$

Finally , a combination of (2.57) and (2.59) gives

$$E^{'}(t) \leq C_\varepsilon e^{-\lambda_\varepsilon t} \quad ,\forall t \geq 0 \tag{2.60}$$

where $C_\varepsilon = \frac{2+9\varepsilon}{2-9\varepsilon}E(0)$

■

Chapter 3

Asymptotic stability of porous-elastic system by boundary feedback controls

3.1 Statement of the problem

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \quad 0 < x < 1, \quad t > 0 \\ j\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0 \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = \phi(0,t) = 0 \quad t > 0 \\ u(1,t) + \dfrac{b}{\mu}\phi(1,t) = f_1(t), \quad \phi_x(1,t) = f_2(t), \quad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad 0 < x < 1 \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x) \quad 0 < x < 1 \end{cases} \tag{3.1}$$

where u_0, u_1, ϕ_0, ϕ_1 are given data and f_1 and f_2 are boundary control inputs. The measured outputs of the system at the right end are given by

$$y_1(t) = u_t(1,t) \ \ and \ \ y_2(t) = \phi(1,t), \quad t > 0 \tag{3.2}$$

and the adaptive boundary controls

$$\begin{cases} f_i(t) = -h_i(t)y_i(t) \\ h_i^{'}(t) = r_i y_i^2(t) \ \ h_i(0) = h_{0i}, \quad h_{0i}, r_i > 0 \ \ for \ \ i = 1,2 \end{cases} \tag{3.3}$$

The closed-loop system associated to (3.1), (3.2) and (3.3)

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0 \quad 0 < x < 1, \quad t > 0 \\ j\phi_{tt} - \delta\phi_{xx} + bu_x + \xi\phi = 0 \quad 0 < x < 1, \quad t > 0 \\ u(0,t) = \phi(0,t) = 0 \quad t > 0 \\ u(1,t) + \dfrac{b}{\mu}\phi(1,t) = -h_1(t)u_t(1,t), \quad \phi_x(1,t) = -h_2(t)\phi_t(1,t), \quad t > 0 \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad 0 < x < 1 \\ \phi(x,0) = \phi_0(x), \quad \phi_t(x,0) = \phi_1(x) \quad 0 < x < 1 \\ h_1^{'}(t) = r_1[u_t(1,t)]^2 \\ h_2^{'}(t) = r_2[\phi_t(1,t)]^2 \end{cases} \tag{3.4}$$

3.2 Global existence

In this section , we use Faedo-Galerkin method to prove our existence result

Definition 3.1 *A pair of functions* (u, ϕ) *defined on* $(0,1) \times [0,T], T > 0$ *arbitrary, is called a global strong solution of the closed-loop system* (3.4) *if*

$$u, \phi \in C([0,T]; W) \bigcap C^1([0,T; V]) \bigcap C^2([0,T]; L^2(0,1))$$
$$(u(\cdot,0), u_t(\cdot,0), \phi(\cdot,0), \phi(\cdot,0)) = (u_0, u_1, \phi_0, \phi_1) \in (W \times V)^2$$

and it satisfies

$$\rho\int_0^1 u_{tt}z dx + \mu\int_0^1 u_x z + b\int_0^1 \phi z_x dx + h_1(t)u_t(1,t)z(1) = 0$$
$$J\int_0^1 \phi_{tt}v dx + \delta\int_0^1 \phi_x v_x dx + b\int_0^1 u_x v dx + \xi\phi v dx + h_2(t)\phi_t(1,t)v(1) = 0$$
$$h_1^{'}(t) = r_1 u_t^2(1,t), h_1(0) = h_{01}, h_{01}, r_1 > 0$$
$$h_2^{'}(t) = r_2 \phi_t^2(1,t), h_2(0) = h_{02}, h_{02}, r_2 > 0$$

for any $u, v \in V$ *and any* $t \in [0,T]$

Now, we are ready to state and prove our existence result.

Theorem 3.2 *Let* $(u_0,u_1),(\phi_0,\phi_1)\in W\times V$ *satisfy the compatibility conditions*

$$u_{0x}(1)+\frac{b}{\mu}\phi_0(1)=-h_{01}u_1(1)\ \ \textit{and}\ \ \phi_{0x}(1)=-h_{02}\phi_1(1)$$

then, there exist a unique global strong solution to system (2.4).

Proof. Let ${w^j}_{j\geq 1}$ be a complete orthogonal basis for $W$and V, and for each $m\geq 1$ let

$$W_m := span\{w^1,...,w^m\}.$$

We look for a solution in the form

$$u^m(x,t)=\sum_{j=1}^{m}a_{mj}(t)w^j(x)\ \ and\ \ \phi^m(x,t)=\sum_{j=1}^{m}b_{mj}(t)w^j(x)$$

to the approximate problem

$$\begin{cases} \rho\displaystyle\int_0^1 u_{tt}^m z dx+\mu\int_0^1 u_x^m z+b\int_0^1\phi^m z_x dx+h_1^m(t)u_t^m(1,t)z(1)=0\\ J\displaystyle\int_0^1\phi_{tt}^m v dx+\delta\int_0^1\phi_x^m v_x dx+b\int_0^1 u_x^m v dx+\xi\int_0^1\phi^m v dx+h_2^m(t)\phi_t^m(1,t)v(1)=0\\ (h_1^m)'(t)=r_1[u_t^m(1,t)]^2, h_1^m(0)=h_{01},\\ (h_2^m)'(t)=r_2[\phi_t^m(1,t)]^2, h_2^m(0)=h_{02},\\ u^m(\cdot,0)=u_0^m,\ \ u_t^m(\cdot,0)=u_1^m,\ \ \phi^m(\cdot,0)=\phi_0^m,\ \ \phi_t^m(\cdot,0)=\phi_1^m, \end{cases} \tag{3.5}$$

for all $z,v\in V_m$, where

$$\begin{cases} (u_0^m,u_1^m):=\left(\displaystyle\sum_{j=1}^{m}(u_0,w^j)w^j,\sum_{j=1}^{m}(u_1,w^j)w^j\right)\to(u_0,u_1)\ \ in\ \ W\times V,\\ \\ (\phi_0^m,\phi_1^m):=\left(\displaystyle\sum_{j=1}^{m}(\phi_0,w^j)w^j,\sum_{j=1}^{m}(\phi_1,w^j)w^j\right)\to(\phi_0,\phi_1)\ \ in\ \ W\times V \end{cases} \tag{3.6}$$

$$u_{0x}^m(1)+\frac{b}{\mu}\phi_0^m(1)=-h_{01}u_1^m(1) \tag{3.7}$$

and

$$\phi_{0x}^m(1) = -h_{02}\phi_1^m(1) \tag{3.8}$$

Problem (3.5) is a system of ordinary differential equations in t which has a local solution , in $[0, t_m)$.

The next a priori estimates show that $t_m = \infty$ for any $m \geq 1$

First a priori estimate: Substituting $z = u_t^m$ in $(3.5)_1$ and $v = \phi_t^m$ in $(3.5)_2$,we obtain

$$\rho \int_0^1 u_{tt}^m u_t^m dx - \mu \int_0^1 u_x^m u_{tx}^m dx - b \int_0^1 \phi^m u_{tx}^m dx + h_1^m(t)[u_t^m(1,t)]^2 = 0$$

where

$$\rho \int_0^1 u_{tt}^m u_t^m dx = \frac{\rho}{2}\frac{d}{dt}\int_0^1 (u_t^m)^2 dx \tag{3.9}$$

and

$$\mu \int_0^1 u_x^m u_{tx}^m = \frac{\mu}{2}\frac{d}{dt}\int_0^1 (u_x^m)^2 dx$$

and

$$b \int_0^1 \phi^m u_{tx}^m dx = b\frac{d}{dt}\int_0^1 \phi^m u_x^m dx - b\int_0^1 \phi_t^m u_x^m dx$$

from (3.9), (3.2) and (3.2) ,gives

The first equation:

$$\frac{1}{2}\frac{d}{dt}[\|u_t^m(t)\|^2 + \|u_x^m(t)\|^2 + 2b\int_0^1 u_x^m(t)\phi^m(t)dx] - b\int_0^1 u_x^m(t)\phi_t^m(t)dx + h_1^m(t)[u_t^m(1,t)]^2 = 0 \tag{3.10}$$

and we have

$$J\int_0^1 \phi_{tt}^m \phi_t^m dx + \delta\int_0^1 \phi_x^m \phi_{tx}^m dx + b\int_0^1 u_x^m \phi_t^m dx + \xi\int_0^1 \phi^m \phi_t^m dx + h_2^m(t)[\phi_t^m(1,t)]^2 = 0$$

where

$$J\int_0^1 \phi_{tt}^m \phi_t^m dx = \frac{J}{2}\frac{d}{dt}\int_0^1 (\phi_t^m)^2 dx \tag{3.11}$$

and

$$\delta\int_0^1 \phi_x^m \phi_{tx}^m dx = \frac{\delta}{2}\frac{d}{dt}\int_0^1 (\phi_x^m)^2 dx \tag{3.12}$$

and

$$\xi\int_0^1 \phi^m \phi_t^m dx = \frac{\xi}{2}\frac{d}{dt}\int_0^1 (\phi^m)^2 dx \tag{3.13}$$

from (3.11) , (3.12)and (3.13), we get

The second equation:

$$\frac{1}{2}\frac{d}{dt}[J\|\phi_t^m(t)\|^2+\delta\|\phi_x^m(t)\|^2+\xi\|\phi^m(t)\|^2]+b\int_0^1 u_x^m(t)\phi_t^m(t)dx+h_2^m(t)[\phi_t^m(1,t)]^2=0 \tag{3.14}$$

by adding up the first and the second equations, we arrive to

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}[\rho\|u_t^m(t)\|^2+\mu\|u_x^m(t)\|^2+J\|\psi_t^m(t)\|^2+\delta\|\psi_x^m(t)\|^2+\xi\|\psi^m(t)\|^2+2b\int_0^1 \phi^m(t)u_x^m(t)dx]\\&+h_1^m(t)[u_t^m(1,t)]^2+h_2^m(t)[\phi_t^m(1,t)]^2=0\end{aligned} \tag{3.15}$$

using $(3.5)_3$ and $(3.5)_4$, we obtain

$$h_1^m(t)[u_t^m(1,t)]^2=\frac{1}{r_1}h_1^m(t)[h_1^m]'(t)=\frac{d}{dt}\frac{1}{2r_1}[h_1^m(t)]^2 \tag{3.16}$$

and

$$h_2^m(t)[\phi_t^m(1,t)]^2=\frac{1}{r_2}h_2^m(t)[h_2^m]'(t)=\frac{d}{dt}\frac{1}{2r_2}[h_2^m(t)]^2 \tag{3.17}$$

Substituting (3.16) and (3.17) in (3.15), we get, for each $m\geq 1$ and for any $0<t<t_m$

$$\begin{aligned}&\frac{d}{dt}\frac{1}{2}\{\rho\|u_t^m(t)\|^2+\mu\|u_x^m(t)\|^2+J\|\psi_t^m(t)\|^2+\delta\|\psi_x^m(t)\|^2+\xi\|\psi^m(t)\|^2+2b\int_0^1 \phi^m(t)u_x^m(t)dx\\&+\frac{1}{r_1}[h_1^m(t)]^2+\frac{1}{r_2}[h_2^m(t)]^2\}=0\end{aligned}$$

integration over $(0,1)$ and using (3.6), we get

$$\rho\|u_t^m(t)\|^2+\mu\|u_x^m(t)\|^2+J\|\psi_t^m(t)\|^2+\delta\|\psi_x^m(t)\|^2+\xi\|\psi^m(t)\|^2+2b\int_0^1\phi^m(t)u_x(t)dx$$
$$+\frac{1}{r_1}[h_1^m(t)]^2+\frac{1}{r_2}[h_2^m(t)]^2$$
$$=\rho\|u_1^m\|^2+\mu\|u_{0x}^m\|^2+J\|\psi_1^m\|^2+\delta\|\psi_{0x}^m\|^2+\xi\|\psi_0^m\|^2+2b\int_0^1\phi_0^m u_{0x}^m dx+\frac{1}{r_1}h_{01}^2+\frac{1}{r_2}h_{02}^2\le C \tag{3.18}$$

where C is independent of m and t. Also from $(3.5)_3$ and $(3.5)_4$, we deduce

$$\int_0^t[u_t^m(1,s)]^2ds=\frac{1}{r_1}\int_0^t[h_1^m]'(s)ds\le\frac{1}{r_1}h_1^m(t)\le C \tag{3.19}$$

and

$$\int_0^t[\phi_t^m(1,s)]^2ds=\frac{1}{r_2}\int_0^t[h_2^m]'(s)ds\le\frac{1}{r_2}h_2^m(t)\le C \tag{3.20}$$

Second a priori estimate: Set $t=0$, $z=\varphi_t^m(.,0)$ in (3.5), integration by parts, then exploit (3.6) and (3.7) to obtain

$$\begin{aligned}\rho\|u_{tt}^m(\cdot,0)\|^2&=\mu\int_0^1 u_{0xx}^m(x)u_{tt}(x,0)dx+b\int_0^1\phi_{0x}^m(x)u_{tt}(x,0)dx\\&\le[\mu\|u_{0xx}^m\|+b\|\phi_{0x}^m\|]\|u_{tt}(\cdot,0)\|\end{aligned}$$

This entails that

$$\rho\|u_{tt}^m(\cdot,0)\|\le\mu\|u_{0xx}^m\|+b\|\phi_{0x}^m\|\le C \tag{3.21}$$

Similarly, sitting $t=0, v=\phi_t^m(\cdot,0)$ in (3.5), integrate by parts, then exploit (3.6) and (3.8) to get

$$J\|\phi_{tt}^m(.,0)\|\le\delta\|\phi_{0xx}^m\|+b\|u_{0x}^m\|+\xi\|\phi_0^m\|\le C \tag{3.22}$$

Third a priori estimate: First, differentiating $(3.5)_1$ and $(3.5)_2$ with respect to t and replacing z and v by u_{tt}^m and ϕ_{tt}^m respectively, then, adding the resultants, we find

$$\frac{d}{dt}\frac{1}{2}\{\rho\|u_{tt}^m(t)\|^2 + \mu\|u_{xt}^m(t)\|^2 + J\|\psi_{tt}^m(t)\|^2 + \delta\|\psi_{xt}^m(t)\|^2 + \xi\|\psi_t^m(t)\|^2 + 2b\int_0^1 \phi_t^m(t)u_{xt}^m(t)dx$$
$$+ \frac{r_1}{2}[u_t^m(1,t)]^4 + \frac{r_2}{2}[\phi_t^m(1,t)]^4\}$$
$$= -h_1^m(t)[u_{tt}^m(1,t)]^2 - h_2^m(t)[\phi_{tt}^m(1,t)]^2 \leq 0$$

Integration over $(0,1)$ and taking advantage of (3.6), (3.21) and (3.22), we arrive at

$$\begin{aligned}
&\rho\|u_{tt}^m(t)\|^2 + \mu\|u_{xt}^m(t)\|^2 + J\|\psi_{tt}^m(t)\|^2 + \delta\|\psi_{xt}^m(t)\|^2 + \xi\|\psi_t^m(t)\|^2 + 2b\int_0^1 \phi_t^m(t)u_{xt}^m(t)dx\\
&+ \frac{r_1}{2}[u_t^m(1,t)]^4 + \frac{r_2}{2}[\phi_t^m(1,t)]^4\\
&\leq \rho\|u_{tt}^m(\cdot,0)\|^2 + \mu\|u_{1x}^m\|^2 + J\|\psi_{tt}^m(\cdot,0)\|^2 + \delta\|\psi_{1x}^m\|^2 + \xi\|\psi_1^m\|^2 + 2b\int_0^1 \phi_1 u_{1x}^m dx\\
&+ \frac{r_1}{2}[u_1^m(1)]^4 + \frac{r_2}{2}[\phi_1^m(1)]^4\\
&\leq C
\end{aligned} \tag{3.23}$$

We conclude from (3.18) to (3.20) and (3.23) that $t_m = \infty$ and for any $T > 0$, we have

$$\begin{aligned}
&(u^m) \ \ and \ \ (\phi^m) \ \ are \ \ bounded \ \ in \ \ L^\infty(0,T,V),\\
&(u_t^m) \ \ and \ \ (\phi_t^m) \ \ are \ \ bounded \ \ in \ \ L^\infty(0,T,V),\\
&(u_{tt}^m) \ \ and \ \ (\phi_{tt}^m) \ \ are \ \ bounded \ \ in \ \ L^\infty(0,T,L^2(0,1)),\\
&(u_t^m(1,0)) \ \ and \ \ (\phi_t^m(1,0)) \ \ are \ \ bounded \ \ in \ \ L^2(0,T)\cap L^\infty(0,T),
\end{aligned} \tag{3.24}$$

So, there exists a subsequence,still denoted by (u^m, ϕ^m), such that

$$
\begin{aligned}
&u^m \to u, \phi^m \to \phi \ \ weakly \ \ star \ \ in \ \ L^\infty(0,T,V), \\
&u_t^m \to u_t, \phi_t^m \to \phi_t \ \ weakly \ \ star \ \ in \ \ L^\infty(0,T,V), \\
&u_{tt}^m \to u_{tt}, \phi_{tt}^m \to \phi_{tt} \ \ weakly \ \ star \ \ in \ \ L^\infty(0,T,L^2(0,1)),
\end{aligned} \tag{3.25}
$$

using Aubin-Lions lemma , we infer that

$$u_t^m \to u_t, \phi_t^m \to \phi_t \ \ in \ \ L^\infty(0,T,L^2(0,1))$$

from which we deduce

$$u_t^m \to u, \phi_t^m \to \phi \ \ a.e. \ \ in \ \ (0,1) \times (0,T)$$

this, together with the continuity of $u_t^m, \phi_t^m, u_t, \phi_t$, leads to

$$u_t^m(1,\cdot) \to u_t(1,\cdot), \phi_t^m(1,\cdot) \to \phi_t(1,\cdot) \ \ pointwise \ \ in \ \ [0,T]$$

The boundedness of $(u_t^m(1,\cdot))$ and $(\phi_t^m(1,\cdot))$ in $L^2(0,T)$ follows from $(3.24)_4$, Owing to the Lebesgue dominated convergence theorem we deduce that

$$u_t^m(1,\cdot) \to u_t(1,\cdot) \ \ and \ \ \phi_t^m(1,\cdot) \to \phi_t(1,\cdot) \ \ in \ \ L^2(0,T) \tag{3.26}$$

An exploitation of $(3.5)_3$, $(3.5)_4$, and (3.26) gives

$$
\begin{aligned}
h_1^m(t) &= h_{01} + r_1 \int_0^t [u_t^m(1,s)]^2 ds \to h_{01} + r_1 \int_0^t [u_t(1,s)]^2 ds = h_1(t) \\
h_2^m(t) &= h_{02} + r_2 \int_0^t [\phi_t^m(1,s)]^2 ds \to h_{02} + r_2 \int_0^t [\phi_t(1,s)]^2 ds = h_2(t)
\end{aligned} \tag{3.27}
$$

in $L^\infty(0,T)$. Also, from $(3.24)_4$ and (3.26), we have

$$
\begin{aligned}
&(h_1^m)'(\cdot) = r_1[u_t^m(1,\cdot)]^2 \rightharpoonup r_1[u_t(1,\cdot)]^2 = (h_1^m)'(\cdot) \ \ in \ \ L^2(0,T) \\
&(h_2^m)'(h) = r_2[\phi_t^m(1,\cdot)]^2 \rightharpoonup r_2[\phi_t(1,\cdot)]^2 = (h_2^m)'(\cdot) \ \ in \ \ L^2(0,T)
\end{aligned}
$$

Next, We take the limit of (3.5) as m goes to infinity, then use (3.25) – (3.27) to obtain

$$\rho\int_0^1 u_{tt}zdx + \mu\int_0^1 u_x z + b\int_0^1 \phi z_x dx + h_1(t)u_t(1,t)z(1) = 0$$
$$J\int_0^1 \phi_{tt}vdx + \delta\int_0^1 \phi_x v_x dx + b\int_0^1 u_x vdx + \xi\phi vdx + h_2(t)\phi_t(1,t)v(1) = 0$$

for any $z, v \in V$. Using Aubin-Lions lemma again , we entail that $u, \phi \ \in C([0,T];W), u_t, \phi_t \ \in C([0,T];V), u_{tt}, \phi_{tt} \in C([0,T];L^2(0,1))$ and $u(0,\cdot) = u_0, u_t(0,\cdot) = u_1, \phi(0,\cdot) = \phi_0, \phi_t(0,t) = \phi_1$

To prove the uniqueness , let (u, ϕ, h_1, h_2) and $(\tilde{u}, \tilde{\phi}, \tilde{h_1}, \tilde{h_2})$ be two solutions of (3.4) with the same initial data. Then,

$$\frac{d}{dt}\{\rho\|u_t - \tilde{u_t}\|^2 + \mu\|u_x - \tilde{u_x}\|^2 + J\|\psi_t - \tilde{\phi_t}\|^2 + \delta\|\psi_x - \tilde{\phi_x}\|^2 + \xi\|\psi - \tilde{\phi}\|^2 + 2b\int_0^1 [\phi u_x - \tilde{\phi}\tilde{u_x}]dx$$
$$+ \frac{1}{2}[h_1(t) - \tilde{h_1}(t)]^2 + \frac{1}{2}[h_2(t) - \tilde{h_2}(t)]^2\}$$
$$= -[h_1 + \tilde{h_1}(t)][u_t(1,t) - \tilde{u_t}(1,t)]^2 - [h_2(t) + \tilde{h_2}(t)][\phi_t(1,t) - \tilde{\phi_t}(1,t)]^2$$
$$\leq 0$$

This implies that

$$(u, \phi, h_1, h_2) = (\tilde{u}, \tilde{\phi}, \tilde{h_1}, \tilde{h_2})$$

■

3.3 Stability analysis

In this section, use the energy method to prove that system (3.4) is exponentially stable. To achieve this goal, we first establish some lemmas needs in the proof of exponential stability result

Lemma 3.3 *Let (u,ϕ) be the solution of* (3.4) *, then the energy functional E , defined by*

$$E(t) = \frac{1}{2}\int_0^1 [\rho u_t^2 + \ j\phi_t^2 + \mu u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi]dx \quad \forall t > 0 \tag{3.28}$$

satisfies

$$E^{'}(t) = -\mu h_1(t)u_t^2(1,t) - \delta h_2(t)\phi_t^2(1,t) \leq 0 \quad \forall t > 0 \tag{3.29}$$

Proof. Multiplying $(3.4)_1$ and $(3.4)_2$ by u_t and ϕ_t respectively , integrating over $(0,1)$ and using integration by parts and the boundary conditions , we obtain

$$\begin{cases} \rho\int_0^1 u_{tt}u_t - \mu\int_0^1 u_{xx}u_t - b\int_0^1 \phi_x u_t = 0 \\ J\int_0^1 \phi_{tt}\phi_t - \delta\int_0^1 \phi_{xx}\phi_t + b\int_0^1 u_x\phi_t + \xi\int_0^1 \phi\phi_t = 0 \end{cases}$$

where

$$\rho\int_0^1 u_{tt}u_t dx = \frac{\rho}{2}\frac{d}{dt}\int_0^1 u_t^2 dx \tag{3.30}$$

and

$$\begin{aligned} -\mu\int_0^1 u_{xx}u_t dx &= -\mu[u_x u_t]_0^1 + \mu\int_0^1 u_x u_{tx}dx \\ &= -\mu u_x(1,t)u_t(1,t) + \mu u_x(0,t)u_t(0,t) + \frac{\mu}{2}\frac{d}{dt}\int_0^1 u_x^2 dx \\ &= \frac{\mu}{2}\frac{d}{dt}\int_0^1 u_x^2 dx - \mu u_x(1,t)u_t(1,t) \end{aligned} \tag{3.31}$$

and

$$\begin{aligned} -b\int_0^1 \phi_x u_t dx &= -b[\phi u_t]_0^1 + b\int_0^1 \phi u_{tx}dx \\ &= -b\phi(1,t)u_t(1,t) + b\phi(0,t)u_t(0,t) + b\frac{d}{dt}\int_0^1 \phi u_x dx - b\int_0^1 \phi_t u_x dx \\ &= -b\phi(1,t)u_t(1,t) + b\frac{d}{dt}\int_0^1 \phi u_x dx - b\int_0^1 \phi_t u_x dx \end{aligned} \tag{3.32}$$

from (3.30) ,(3.31) and (3.32) , we obtain

$$\frac{\rho}{2}\frac{d}{dt}\int_0^1 u_t^2dx + \frac{\mu}{2}\frac{d}{dt}\int_0^1 u_x^2dx + b\frac{d}{dt}\int_0^1 \phi u_x dx - b\int_0^1 \phi_t u_x dx - \mu u_x(1,t)u_t(1,t) - b\phi(1,t)u_t(1,t) =$$
$$\frac{\rho}{2}\frac{d}{dt}\int_0^1 u_t^2dx + \frac{\mu}{2}\frac{d}{dt}\int_0^1 u_x^2dx + b\frac{d}{dt}\int_0^1 \phi u_x dx - b\int_0^1 \phi_t u_x dx - \mu(-h_1(t)u_t(1,t) - \frac{b}{\mu}\phi_t u_t(1,t))u_t(1,t)$$
$$- b\phi(1,t)u_t(1,t) = 0 \tag{3.33}$$

and we have

$$J\int_0^1 \phi_{tt}\phi_t dx = \frac{J}{2}\frac{d}{dt}\int_0^1 \phi_t^2 dx \tag{3.34}$$

and

$$\begin{aligned} -\delta\int_0^1 \phi_{xx}\phi_t dx &= -\delta[\phi_x\phi_t]_0^1 + \delta\int_0^1 \phi_x\phi_{tx}dx \\ &= -\delta\phi_x(1,t)\phi_t(1,t) + \delta\phi_x(0,t)\phi_t(0,t) + \frac{\delta}{2}\frac{d}{dt}\int_0^1 \phi_x^2 \\ &= \frac{\delta}{2}\frac{d}{dt}\|\phi_x\|^2 - \delta\phi_x(1,t)\phi_t(1,t) \end{aligned} \tag{3.35}$$

and

$$\xi\int_0^1 \phi\phi_t dx = \frac{\xi}{2}\frac{d}{dt}\int_0^1 \phi^2 dx \tag{3.36}$$

from (3.34), (3.35) and (3.36) , we get

$$\frac{J}{2}\frac{d}{dt}\int_0^1 \phi_t^2dx + \frac{\delta}{2}\frac{d}{dt}\int_0^1 \phi_x^2dx + \frac{\xi}{2}\frac{d}{dt}\phi^2dx + b\int_0^1 u_x\phi_t dx - \delta\phi_x(1,t)\phi_t(1,t) =$$
$$\frac{J}{2}\frac{d}{dt}\int_0^1 \phi_t^2dx + \frac{\delta}{2}\frac{d}{dt}\int_0^1 \phi_x^2dx + \frac{\xi}{2}\frac{d}{dt}\phi^2dx + b\int_0^1 u_x\phi_t dx - \delta(-h_2(t)\phi_t(1,t))\phi_t(1,t) = 0 \tag{3.37}$$

adding up (3.33) and (3.37) , leads to

$$\frac{1}{2}\int_0^1 [\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi]dx = \mu[-h_1(t)u_t(1,t) - \frac{b}{\mu}\phi(1,t)]u_t(1,t)$$
$$+ b\phi(1,t)u_t(1,t) + \delta(-h_2(t)\phi_t(1,t))\phi_t(1,t)$$

So

$$\frac{1}{2}\int_0^1 [\rho u_t^2 + J\phi_t^2 + \mu u_x^2 + \delta\phi_x^2 + \xi\phi^2 + 2bu_x\phi]dx$$
$$= -\mu h_1(t)u_t^2(1,t) - \delta h_2(t)\phi_t^2(1,t)$$

this is exactly (3.29) ■

Lemma 3.4 *Let* (u,ϕ)*be the solution of* (3.4) *, we define a Lyapunov functional L by*

$$L(t) = E(t) + \frac{\mu}{2r_1}h_1^2(t) + \frac{\delta}{2r_2}h_2^2(t) \quad \forall t \geq 0$$

satisfies , for any $t \geq 0$

$$\begin{aligned} L^{'}(t) &= E^{'}(t) + \frac{\mu}{2r_1}[2h_1(t)h_1^{'}(t)] + \frac{\delta}{2r_2}[2h_2(t)h_2^{'}(t)] \\ &= -\mu h_1(t)u_t^2(1,t) - \delta h_2(t)\phi_t^2(1,t) + \frac{\mu}{r_1}h_1(t)[r_1 u_t^2(1,t)] \\ &+ \frac{\delta}{r_2}h_2(t)[r_2\phi_t^2(1,t)] = 0. \end{aligned}$$

As a consequence , we have

$$L(t) = L(0) \quad \forall t \geq 0,$$

and this entails that

$$\sup_{t\geq 0}|E(t) + \mu h_1^2(t) + \delta h_2^2(t)| \leq \lambda_1$$

where λ_1 *is a positive constant that depends on the initial data.*

Lemma 3.5 *Let* (u,ϕ)*be the solution of* (3.4) *,then the functional*

$$F_1(t) = 2\rho\int_0^1 x u_x u_t dx \tag{3.38}$$

satisfies

$$F_1^{'}(t) = -\rho\|u_t\|^2 - \mu\|u_x\|^2 + \mu[\frac{b}{\mu}\phi(1,t) + h_1(t)u_t(1,t)]^2 + \rho u_t^2(1,t) + 2b\int_0^1 x u_x\phi_x dx \tag{3.39}$$

Proof. using (3.4) and integration by parts , we get

$$\begin{aligned} F_1^{'}(t) &= 2\rho\int_0^1 x u_t u_{xt} dx + 2\rho\int_0^1 x u_{tt} u_x dx \\ &= 2\int_0^1 x u_t u_{xt} dx + 2\rho\int_0^1 x(\frac{\mu}{\rho}u_{xx} + \frac{b}{\rho}\phi_x)u_x dx \\ &= 2\rho\int_0^1 x u_t u_{xt} dx + 2\mu\int_0^1 x u_{xx} u_x dx + 2b\int_0^1 x\phi_x u_x dx \end{aligned}$$

where

$$
\begin{aligned}
2\rho \int_0^1 x u_t u_{xt} dx &= 2\rho \int_0^1 x(\frac{1}{2}\frac{du_t^2}{dx})dx \\
&= \rho \int_0^1 x \frac{du_t^2}{dx} dx \\
&= \rho [x u_t^2]_0^1 - \rho \int_0^1 u_t^2 dx \\
&= \rho u_t^2(1,t) - \rho \int_0^1 u_t^2 dx
\end{aligned}
\tag{3.40}
$$

and

$$
\begin{aligned}
2\mu \int_0^1 x u_{xx} u_x dx &= 2\mu \int_0^1 x(\frac{1}{2}\frac{du_x^2}{dx})dx \\
&= \mu \int_0^1 x \frac{du_x^2}{dx} dx \\
&= \mu [x u_x^2]_0^1 - \mu \int_0^1 u_x^2 dx \\
&= \mu u_x^2(1,t) - \mu \int_0^1 u_x^2 dx
\end{aligned}
\tag{3.41}
$$

from (3.40) and (3.41) , we obtain

$$
\begin{aligned}
F_1^{'}(t) &= -\rho\|U_t\|^2 - \mu\|U_x\|^2 + \rho u_t^2(1,t) + \mu u_x^2(1,t) + 2b\int_0^1 x u_x \phi_x dx \\
&= -\rho\|u_t\|^2 - \mu\|u_x\|^2 + \mu[\frac{b}{\mu}\phi(1,t) + h_1(t)u_t(1,t)]^2 + \rho u_t^2(1,t) + 2b\int_0^1 x u_x \phi_x dx
\end{aligned}
$$

■

Lemma 3.6 *Let* (u, ϕ) *be the solution of* (3.4) *,then the functional*

$$
F_2(t) = 2J\int_0^1 x\phi_x\phi_t dx \tag{3.42}
$$

satisfies

$$
F_2^{'}(t) = -J\|\phi_t\|^2 - \delta\|\phi_x\|^2 + \xi\|\phi\|^2 + [J + h_2^2(t)]\phi_t^2(1,t) - \xi\phi^2(1,t) - 2b\int_0^1 x u_x \phi_x dx \tag{3.43}
$$

Proof. using (3.4) and integration by parts , gives

$$
\begin{aligned}
F_2^{'}(t) &= 2J\int_0^1 x\phi_t\phi_{xt}dx + 2J\int_0^1 x\phi_{tt}\phi_x dx \\
&= 2J\int_0^1 x\phi_t\phi_{xt}dx + 2J\int_0^1 x(\frac{\delta}{J}\phi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\phi)\phi_x dx \\
&= 2J\int_0^1 x\phi_t\phi_{xt}dx + 2\delta\int_0^1 x\phi_x\phi_{xx}dx - 2b\int_0^1 x\phi_x u_x dx - 2\xi\int_0^1 x\phi_x\phi dx
\end{aligned} \tag{3.44}
$$

where

$$
\begin{aligned}
2J\int_0^1 x\phi_t\phi_{xt}dx &= 2J\int_0^1 x(\frac{1}{2}\frac{d\phi_t^2}{dx})dx \\
&= J\int_0^1 x\frac{d\phi_t^2}{dx}dx \\
&= J[x\phi_t^2]_0^1 - J\int_0^1 \phi_t^2 dx \\
&= J\phi_t^2(1,t) - J\int_0^1 \phi_t^2 dx
\end{aligned} \tag{3.45}
$$

and

$$
\begin{aligned}
2\delta\int_0^1 x\phi_x\phi_{xx}dx &= 2\delta\int_0^1 x(\frac{1}{2}\frac{d\phi_x^2}{dx})dx \\
&= \delta\int_0^1 x\frac{d\phi_x^2}{dx}dx \\
&= \delta[x\phi_x^2]_0^1 - \delta\int_0^1 \phi_x^2 dx \\
&= \delta\phi_x^2(1,t) - \delta\int_0^1 \phi_x^2 dx
\end{aligned} \tag{3.46}
$$

and

$$
\begin{aligned}
2\xi\int_0^1 x\phi_x\phi dx &= 2\xi\int_0^1 x(\frac{1}{2}\frac{d\phi^2}{dx})dx \\
&= \xi\int_0^1 x\frac{d\phi^2}{dx}dx \\
&= \xi[x\phi^2]_0^1 - \xi\int_0^1 \phi^2 dx \\
&= \xi\phi^2(1,t) - \xi\int_0^1 \phi^2 dx
\end{aligned} \tag{3.47}
$$

from (3.45) , (3.46) and (3.47) ,we find

$$\begin{aligned}
F_2'(t) &= -J\|\phi_t\|^2 - \delta\|\phi_x\|^2 + \xi\|\phi\|^2 + J\phi_t^2(1,t) + \delta\phi_x^2(1,t) - \xi\phi^2(1,t) - 2b\int_0^1 xu_x\phi_x dx \\
&= -J\|\phi_t\|^2 - \delta\|\phi_x\|^2 + \xi\|\phi\|^2 + J\phi_t^2(1,t) + \delta h_2^2(t)\phi_t^2(1,t) - \xi\phi^2(1,t) - 2b\int_0^1 xu_x\phi_x dx \\
&= -J\|\phi_t\|^2 - \delta\|\phi_x\|^2 + \xi\|\phi\|^2 + [J + h_2^2(t)]\phi_t^2(1,t) - \xi\phi^2(1,t) - 2b\int_0^1 xu_x\phi_x dx
\end{aligned} \tag{3.48}$$

■

Lemma 3.7 *Let* (u, ϕ)*be the solution of* (3.4)*, then the functional*

$$F_3(t) = -\frac{\rho}{2}\int_0^1 uu_t dx \tag{3.49}$$

satisfies

$$F_3'(t) = -\frac{\rho}{2}\|u_t\|^2 - \frac{\mu}{2}\|u_x\|^2 + \frac{\mu}{2}h_1(t)u_t(1,t)u(1,t) + \frac{b}{2}\int_0^1 u_x\phi dx \tag{3.50}$$

Proof. using (3.4) and integration by parts , leads

$$\begin{aligned}
F_3'(t) &= -\frac{\rho}{2}\int_0^1 u_t^2 - \frac{\rho}{2}\int_0^1 uu_{tt}dx \\
&= -\frac{\rho}{2}\int_0^1 u_t^2 - \frac{\rho}{2}\int_0^1 u(\frac{\mu}{\rho}u_{xx} + \frac{b}{\rho}\phi)_x dx \\
&= -\frac{\rho}{2}\int_0^1 u_t^2 - \frac{\mu}{2}\int_0^1 uu_{xx}dx - \frac{b}{2}\int_0^1 u\phi_x dx
\end{aligned}$$

where

$$\begin{aligned}
\frac{\mu}{2}\int_0^1 uu_{xx}dx &= \frac{\mu}{2}[uu_x]_0^1 - \frac{\mu}{2}\int_0^1 u_x^2 dx \\
&= \frac{\mu}{2}u(1,t)u_x(1,t) - \frac{\mu}{2}\int_0^1 u_x^2 dx
\end{aligned} \tag{3.51}$$

and

$$\begin{aligned}
\frac{b}{2}\int_0^1 u\phi_x dx &= \frac{b}{2}[u\phi]_0^1 - \frac{b}{2}\int_0^1 u_x\phi dx \\
&= \frac{b}{2}u(1,t)\phi(1,t) - \frac{b}{2}\int_0^1 u_x\phi dx
\end{aligned} \tag{3.52}$$

from (3.51) and (3.52) , we get

$$\begin{aligned}
F_3^{'}(t) &= -\frac{\rho}{2}\|u_t\|^2 + \frac{\mu}{2}\|u_x\|^2 - \frac{\mu}{2}u(1,t)u_x(1,t) - \frac{b}{2}u(1,t)\phi(1,t) + \frac{b}{2}\int_0^1 u_x\phi dx \\
&= -\frac{\rho}{2}\|u_t\|^2 + \frac{\mu}{2}\|u_x\|^2 - \frac{\mu}{2}[-h_1(t)u_t(1,t) - \frac{b}{\mu}\phi(1,t)]u(1,t) - \frac{b}{2}u(1,t)\phi(1,t) + \frac{b}{2}\int_0^1 u_x\phi dx \\
&= -\frac{\rho}{2}\|u_t\|^2 + \frac{\mu}{2}\|u_x\|^2 + \frac{\mu}{2}h_1(t)u_t(1,t)u(1,t) + \frac{b}{2}\int_0^1 u_x\phi dx
\end{aligned}$$

■

Lemma 3.8 *Let* (u,ϕ) *be the solution of* (3.4)*, then the functional*

$$F_4(t) = \frac{J}{2}\int_0^1 \phi\phi_t dx \tag{3.53}$$

satisfies

$$F_4^{'}(t) = \frac{J}{2}\|\phi_t\|^2 - \frac{\delta}{2}\|\phi_x\|^2 - \frac{\xi}{2}\|\phi\|^2 - \frac{\delta}{2}h_2(t)\phi_t(1,t)\phi(1,t) - \frac{b}{2}\int_0^1 u_x\phi dx \tag{3.54}$$

Proof. using (3.4) and integration by parts, gives

$$\begin{aligned}
F_4^{'}(t) &= \frac{J}{2}\int_0^1 \phi_t^2 + \frac{J}{2}\int_0^1 \phi\phi_{tt}dx \\
&= \frac{J}{2}\int_0^1 \phi_t^2 + \frac{J}{2}\int_0^1 \phi[\frac{\delta}{J}\phi_{xx} - \frac{b}{J}u_x - \frac{\xi}{J}\phi]dx \\
&= \frac{J}{2}\int_0^1 \phi_t^2 + \frac{\delta}{2}\int_0^1 \phi\phi_{xx}dx - \frac{b}{2}\int_0^1 \phi u_x dx - \frac{\xi}{2}\int_0^1 \phi^2 dx
\end{aligned}$$

where

$$\begin{aligned}
\frac{\delta}{2}\int_0^1 \phi\phi_{xx}dx &= \frac{\delta}{2}[\phi\phi_x]_0^1 - \frac{\delta}{2}\int_0^1 \phi^2 dx \\
&= \frac{\delta}{2}\phi(1,t)\phi_x(1,t) - \frac{\delta}{2}\int_0^1 \phi^2 dx
\end{aligned}$$

So

$$\begin{aligned}
F_4^{'}(t) &= \frac{J}{2}\|\phi_t\|^2 - \frac{\delta}{2}\|\phi_x\|^2 - \frac{\xi}{2}\|\phi\|^2 + \frac{\delta}{2}\phi_x(1,t)\phi(1,t) - \frac{b}{2}\int_0^1 u_x\phi dx \\
&= \frac{J}{2}\|\phi_t\|^2 - \frac{\delta}{2}\|\phi_x\|^2 - \frac{\xi}{2}\|\psi\|^2 - \frac{\delta}{2}h_2(t)\phi_t(1,t)\phi(1,t) - \frac{b}{2}\int_0^1 u_x\phi dx
\end{aligned}$$

■

Theorem 3.9 *Let(u, ϕ)be the solution of* (3.4)*, for $\varepsilon > 0$, we introduce another Lyapunov functional*

$$L_\varepsilon(t) = E(t) + \varepsilon F(t) \tag{3.55}$$

where

$$F(t) = \sum_{j=1}^{4} F_j(t)$$

satisfies

$$L_\varepsilon(t) \sim E,$$

and the energy functional is exponentially stable:

$$E(t) \leq Ce^{-ct}, \ \forall t \geq 0. \tag{3.56}$$

Proof. using inequalities, relations to and Young's inequality, we obtain

$$L_\varepsilon^{'}(t) = E^{'}(t) + \varepsilon F^{'}(t)$$

then

$$F^{'}(t)=\sum_{j=1}^{4}F_j^{'}(t)=-\frac{3}{2}\rho\|U_t\|^2-\frac{J}{2}\|\phi_t\|^2-\frac{\mu}{2}\|u_x\|^2-\frac{3}{2}\delta\|\phi_x\|^2+\frac{\xi}{2}\|\phi\|^2-\xi\phi^2(1,t)+\rho u_t^2(1,t)$$
$$+[J+\delta h_2^2(t)]\phi_t^2(1,t)+\mu[\frac{b}{\mu}\phi(1,t)+h_1(t)u_t(1,t)]^2+\frac{\mu}{2}h_1(t)u_t(1,t)u(1,t)$$
$$-\frac{\delta}{2}h_2(t)\phi_t(1,t)\phi(1,t)$$
$$=-\frac{3}{2}\rho\|U_t\|^2-\frac{J}{2}\|\phi_t\|^2-\frac{\mu}{2}\|u_x\|^2-\frac{3}{2}\delta\|\phi_x\|^2+\frac{\xi}{2}\|\phi\|^2-\xi\phi^2(1,t)+\rho u_t^2(1,t)$$
$$+[J+\delta h_2^2(t)]\phi_t^2(1,t)+\mu[\frac{b^2}{\mu^2}\phi^2(1,t)+h_1^2(t)u_t^2(1,t)+2\frac{b}{\mu}h_1(t)u_t(1,t)\phi(1,t)]$$
$$+\frac{\mu}{2}h_1(t)u_t(1,t)u(1,t)-\frac{\delta}{2}h_2(t)\phi_t(1,t)\phi(1,t)$$
$$=-\frac{3}{2}\rho\|U_t\|^2-\frac{J}{2}\|\phi_t\|^2-\frac{\mu}{2}\|u_x\|^2-\frac{3}{2}\delta\|\phi_x\|^2+\frac{\xi}{2}\|\phi\|^2-\xi\phi^2(1,t)+\rho u_t^2(1,t)$$
$$+[J+\delta h_2^2(t)]\phi_t^2(1,t)+\frac{b^2}{\mu}\phi^2(1,t)+\mu h_1^2(t)u_t^2(1,t)+2bh_1(t)u_t(1,t)\phi(1,t)]$$
$$+\frac{\mu}{2}h_1(t)u_t(1,t)u(1,t)-\frac{\delta}{2}h_2(t)\phi_t(1,t)\phi(1,t)$$
$$=-\frac{3}{2}\rho\|U_t\|^2-\frac{J}{2}\|\phi_t\|^2-\frac{\mu}{2}\|u_x\|^2-\frac{3}{2}\delta\|\phi_x\|^2+\frac{\xi}{2}\|\phi\|^2+(\frac{b^2}{\mu}-\xi)\phi^2(1,t)$$
$$+[\rho+\mu h_1(t)]u_t^2(1,t)+[J+\delta h_2^2(t)]\phi_t^2(1,t)+2bh_1(t)u_t(1,t)\phi(1,t)]$$
$$+\frac{\mu}{2}h_1(t)u_t(1,t)u(1,t)-\frac{\delta}{2}h_2(t)\phi_t(1,t)\phi(1,t)$$

using Young's inequality ,gives

$$2bh_1(t)u_t(1,t)\phi(1,t)\le 2[\frac{1}{4}b^2\psi^2(1,t)+h_1^2(t)u_t^2(1,t)]\le\frac{1}{2}b^2\phi^2(1,t)+2h_1^2(t)u_t^2(1,t)$$

$$\frac{\mu}{2}h_1(t)u_t(1,t)u(1,t)\le\frac{\mu}{2}[\frac{1}{2}u^2(1,t)+\frac{1}{2}h_1^2(t)u_t^2(1,t)]\le\frac{\mu}{4}u^2(1,t)+\frac{\mu}{4}h_1^2(t)u_t^2(1,t)$$

$$-\frac{\delta}{2}h_2(t)\phi_t(1,t)\phi(1,t)\le\frac{\delta}{4}h_2^2(t)\phi_t^2(1,t)+\frac{\delta}{4}\phi^2(1,t)$$

So

$$
\begin{aligned}
F^{'}(t) &\leq -\frac{3}{2}\rho\|U_t\|^2 - \frac{J}{2}\|\phi_t\|^2 - \frac{\mu}{2}\|u_x\|^2 - \frac{3}{2}\delta\|\phi_x\|^2 + \frac{\xi}{2}\|\phi\|^2 + \frac{\mu}{4}u^2(1,t) + (\frac{b^2}{\mu} - \xi)\phi^2(1,t) \\
&+ [\rho + (\mu + \frac{\mu}{4} + 2)h_1(t)]u_t^2(1,t) + [J + (\delta + \frac{\delta}{4})h_2^2(t)]\phi_t^2(1,t) + (\frac{b^2}{2} + \frac{\delta}{4})\phi^2(1,t) \\
&\leq -E(t) + c\left[\mu u_t^2(1,t) + \delta\phi_t^2(1,t)\right],
\end{aligned} \tag{3.57}
$$

■

Conclusion

The aim of this work is to study the existence and uniqueness and the stability of solution for the porous-elastic system by boundary feedback controls. In the second chapter, we presented the existence and the uniqueness and the exponential stability of the Timoshenko system by boundary feedback controls which is the work of Hassan and Tatar [6]. In the third chapter, we proved, using the Faedo-Galarkin method, the existence and uniqueness of solution for porous-elastic system. Then, by using the multiplier and energy methods, we established the exponential stability.

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ملخص

الهدف من هذا العمل هو دراسة بعض الأنظمة المكونة من المعادلات التفاضلية الجزئية. اول نظام هو نظام تيموشنكو, Timoshenko حيث سنبرهن على وجود و وحدانية الحل بالاعتماد على نظرية فايدو-غالركين و دراسة استقراره باستخدام تقنية الجداءات الذي يقوم على بناء دالة ليابونوف المكافئة للطاقة.
ثاني نظام, سندرس مشكلة تتعلق بعلم المواد المرنة ذات الفراغات , نثبت أولا ان المشكلة لها حل عن طريق التحكم في الحدود.

الكلمات المفتاحية: نظام المرونة المسامية – الاستقرار الاسى – طريقة دالة الطاقة – دالة ليابونوف.

Résumé

Le but de ce travail est d'étudier certains systèmes constitués d'équations aux dérives partielles. Le premier système est le système de Timoshenko, nous démontrerons l'existence et l'unité de la solution basée sur la théorie Faedo-Galerkin , et l'étude de sa stabilité à l'utilisation de la technique du multiplicateur, que appuie à l'aide de la fonction Lyapounov équivalente à l'énergie.
D'autre parte, nous étudierons un problème lié à la science des matériaux et à la mécanique, spécifiquement les matériaux élastiques avec des vides. Nous prouverons que le problème a une solution en contrôlant les frontières.

Mots clés : le système élastique poreaux – stabilité exponentiel – méthode de la fonction énergétique

Abstract

The aim of this work is to study some systems consisting of partial differential equations. The first problem is the Timoshenko system, we will prove the existence and the uniqueness of the solution based on Faedo-Galerkin method, and the study of its stability is done using the multiplier technique which is based on constructing the Lyapunov function. The second system is that we will study a problem related to machanics specifically elastic materials with voids. First, we prove that the problem has a solution by controlling the boundaries. **Keywords: Porous-elastic system, Exponential stability, Energy function method, Lyapunov functional.**