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Theme

Lyapunov function and global asymptotic stability

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Dedication

To everyone who supported me

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Notations

$\mathbb{R}^n$: n-dimensional real vector.

$\mathbb{N}$: The set of natural numbers.

$\mathbb{R}$: The set of real numbers.

$\mathbb{R}^n, \mathbb{R}^+, \mathbb{R}^p, \mathbb{R}^m, \mathbb{R}^j, D$: All parts of $\mathbb{R}^n$.

$P.D$: Positive definite.

μ : Control parameter.

F : Recurrence function.

X_k : Instant state vector.

x^* : Fixed point.

T : Finite set of linearly independent real numbers.

A : Open set of $\mathbb{R}^n$.

λ_i : Proper value.

V : Lyapunov function.

$|.|$: Absolute value.

$\| . \|$: The euclidean norm.

$\phi(t, x)$: Semi flow of dynamical system.

$\dot{X}$: Derivable from X.

x_e : Equilibrium point.

$\delta, \lambda, c, \gamma, \eta$: Positive constants.

$G.A.S$: Globally asymptotically stable.

$L.A.S$: Locally asymptotically stable.

$W(x)$: Positive definite function.

$\frac{\partial V}{\partial x}$: The jacobian matrix.

∂V : The boundary of a set V.

S : Set $\{x \in D | \dot{V}(x) = 0\}$.

$A(t)$: State matrix $n \times n$.

$B(t)$: Control matrix $n \times m$.

h : Continous function.

$L^+(x)$: The $\omega - limit$ set of x.

B_ϵ : The ball $\{x \in \mathbb{R}^n \ , \ \| x \| \leq \delta\}$.

ν : Non empty neighborhood of the origin in $\mathbb{R}^n$.

Γ : Set $\{x \in D \ , \ V'(x) = 0\}$.

Ω_c : Set $\{x \in \mathbb{R}^n \ , \ V(x) \leq c\}$.

$C^0(\mathbb{R}^n, \mathbb{R}^+)$: Set of continous functions of $\mathbb{R}^n$ in $\mathbb{R}^+$.

G : Set $\{x \in \mathbb{R}^n \ : \ V(g(x)) - V(x) = 0\}$.

G^* : Largest positively invariant.

$\overline{B_\delta}$: ball $\{x \in \mathbb{R}^n \ : \| x \| \leq \epsilon\}$.

A_{G^*} : Domain of attractivity relative to G^*.

ΔV : The distance.

$g(x_k, \mu)$: Universal controller function.

Introduction

The theory of dynamical systems may $[\mathbf{1}]$ be said to have begun as a special topic in the theory of ordinary diffeerential equations with the pioneering work of $\mathbf{Henri\ Poincar}$ in the late 19th century. $\mathbf{Poincar}$ followed by $\mathbf{Bendixson}$, studied topological properties of solutions of autonomous ordinary differential equations in the plane. The Poincaré-Bendixson theory is now a standard topic of discussion in courses on ordinary differential equations.
Almost simultaneously with $\mathbf{Poincar}$, $\mathbf{A}.\mathrm{M}.\mathbf{Liapunov}$ developed his theory of stability of a motion (solution) for a system of n first order ordinary differential equations. He defined in a precise form the concept of stability, asymptotic stability, and instability; and gave a " method " (the second or direct method of $\mathbf{liapunov}$). For the analysis of the stability properties of a given solution of an ordinary differential equation. Both his definition and his " method " characterize, in a strictly local setting, the stability properties of a solution of the differential equation. As such, the liapunov theory is strikingly different from the Poincaré theory, in which, on the contary, the study of the global properties of differential equations in the plane play a major role.
One of the main aspects of the Poincaré theory is the introduction of the concept of trajectory, i.e, a curve in the $\mathbf{x}\ \dot{\mathbf{x}}$ plane, parametrized by time variable $\mathbf{t}$, which can be found by eliminating the variable $\mathbf{t}$ from the given equations, thus reducing it to a first order differential equation connecting $\mathbf{x}$ and $\dot{\mathbf{x}}$. In this way, Poincaré set up a convenient geometric framework in which to study qualitative behavior of planar differential equations. $\mathbf{Poincar}$ was not interested in the integration of particular types of equations, but in classifying all possible behaviors of the class of all second order differential equations. By introducing this concept of trajectory, $\mathbf{Pioncar}$ was able to formulate and solve, as topological problems, problems in the theory of differential equations.
The " butterfly effect " has become a popular $[\mathbf{2}]$ slogan of chaos. But is it really so surprising that minor details sometimes have major impacts? sometimes the proverbial minor detail is taken to be the difference between a world with some butterfly and an alternative universe; as a result of this small difference, the worlds soon come to differ dramatically from one another. The mathematical version of this concept is known a $\mathbf{sensitvedependence}$. Chaotic systems not only exhibit sensitive dependence, but two other properties as well: they are $\mathbf{deterministic}$, and they are $\mathbf{nonlinear}$.
Chaos is important, in part, because it helps us to cope with unstable systems by improving our ability to describe, to understand, perhaps even to forecast them. Indeed, one of the myths of chaos we will debunk is that chaos makes forecasting a useless task. In an alternative but equally popular butterfly story, there is one world where a butterfly flaps its wings and another world where it does not. This small difference means a tornado appears in only one of these two worlds, linking chaos to uncertainty and prediction: in which world are we? chaos is the name given to the mechanism which allows such rapid growth of uncertainty in our mathematical models. The image of chaos amplifying uncertainty and confounding forecasts will be a recurring theme throughout this introduction.
The stability of dynamical systems (chaotic or not) has an inportance in serval sciences $[\mathbf{3}-\mathbf{4}-\mathbf{8}-\mathbf{9}]$, there are also several ways to study the stability of dynamic systems see $[\mathbf{4}-\mathbf{5}-\mathbf{6}-\mathbf{7}-\mathbf{8}-\mathbf{9}-\mathbf{10}-\mathbf{11}]$. In our study, we chose the Lyapunov approach, they are an indispensable tool in analysis and controller design of nonlinear systems $[\mathbf{3}-\mathbf{11}-\mathbf{12}]$, where we divided our not into three chapters:

In the first chapter, we defined some basic concepts related to dynamic order and chaos, the orbits, flow, trajectory, invariance and attraction, basin of attraction, the sensitivity to initial conditions, poincare section, Lyapunov exponents, phase space, bifurcation and finally bifurcation diagram.

The second chapter, we have defined everything related to Lyapunov as a Lyapunov function and settled on the Lyapunov approach, Lyapunov stability for autonomous and nonautonomous system, LaSalle's invariance principle theorem, Lyapunov theorem for global asymptotic stability finally the controlled systems.

In the final chapter, it is the most important part of our memorandum where we address our problem "Lyapunov function and global asymptotic stability" by presenting our application and reaching stability through the steps used previously and providing a numerical example to ensure the validity of the experiment.

Chapter 1

Generalities on dynamical systems and chaos

1.1 Introduction

As a mathematical discipline [**29**] , the study of dynamical systems most likely originated at the end of the 19th century through the work of **Henri Poincare** in his study of celestial mechanics . Once the equations describing the movement of the planets around the sun are formulated (that is, once the mathematical model is constructed), looking for solutions as a means to describe the planets motion and make predictions of positions in time is the next step. But when finding solutions to sets of equations is seemingly too complicated or impossible, one is left with studying the mathematical structure of the model to somehow and creatively narrow down the possible solution functions. This view of studying the nature and structure of the equations in mathematical model for clues as to the nature and structure of its solutions is general idea behind the techniques and theory of what we now call dynamical systems. Being only a $\mathbf{100+}$ years old, the mathematical concept of a dynamical system is a relatively new idea. And since it really is a focused study of the nature of functions of a single (usually), real (usually) independent variable, it is a subdiscipline of what mathematicians call real analysis. However, one can say that dynamical systems draws its theory and techniques from many areas of mathematics, from analysis to geometry, topology and dynamics second generation mathematics, since they tend to bridge other more pure areas in their theories. But as the study of what is actually means to model phenomena via functions and equations, dynamical systems is sometimes called the mathematical study of any mathematical concept that evolves over time.[**28**] There exist two essentially different approaches to the study of dynamical systems, based on the following distinction:

$$\text{time-continuous nonlinear differential equations} \rightleftharpoons \text{time-discrete maps}$$

One approach starts from time-continuous differential equations and leads to time-discrete maps, which are obtained from them by a suitable discretization of time. This path is pursued, e.g, in the book by **strogatz**. The other approach starts from the study of time-discrete maps and then gradually builds up to time-continuous differential equations. After a short motivation in terms of nonlinear differential equations, for the rest of this course we shall folow the latter route to dynamical systems theory. This allows a generally more simple way of introducing the important concepts, In this chapter we provide some important definitions about dynamical systems and chaos.

1.2 Dynamical systems

A dynamic system is a system whos state changes over time. Mathematically, a dynamical system consists of a state space and the law of dynamics that allows determining the state that corresponds to the current state defined by linear differential equations of the first order of the form :

$$x = \frac{dx}{dt} = f(x, t, \mu) \tag{1.1}$$

Where f is vector fields.
$x \in U \subseteq \mathbb{R}^n$ state vector.
$\mu \in V \subseteq \mathbb{R}^P$ parameter vector and $\mathtt{t}$ temporal variable.
When the vector field f does not depend explicitly on time, we say that the system dynamic is autonomons otherwise it is nonautonomous.

1.3 Autonomous systems

When the vector field f does not depend explicity on time, we say that the system dynamic is autonomous otherwise, it is nonautonomous.

Note

Each non-dependent dynamical system with dimension $\mathbf{n}$ can be written as an independent dynamical system whith dimension $\mathbf{n+1}$ by variable variation.

1.4 Dynamical systems solutions

Dynamic systems is general have no solutions, but linear dynamic systems can be completely solved, and linear systems can be used to understand the qualitative behavoir of systems by calculating the equilibrium points and approximating then as a linear system around each of these points.

1.5 deterministic system

A deterministic system is a system that does not include any randomness in the evolution of the state of the system over time.

1.6 discrete system

The range of dynamic systems can be extended to discrete systems. We call a discrete dynamic system each system of iterative algebraic differential equations defined by :

$$x_{k+1} = F(x_k, \mu) \tag{1.2}$$

Where $\mathbf{F}$ recurrence function , $x_k \in U \subseteq \mathbb{R}^n$ instant state vector $\mathbf{t_k}$
$\mu \in \mathbf{V} \subseteq \mathbb{R}^{\mathbf{n}}$ parameter vector and $\mathbf{k} \in \mathbb{N}$

1.7 Chaos theory

This theory is one of the latest physical mathematical theories that deals with nonlinear dynamic systems(moving sentences) that show a kind of random behavior known as chaos , this behavior results either throngh the inability to determine the initial conditions (the butterfly effect) of through the probabilistic physical nature of quantum mechanics . This theory reveals the hidden system by establishing rules for studying these systems such as weather forecasts, the solar system, the market economy $\cdots$.

1.8 The orbits

For any point x of a function $\mathbf{F}$ the discrete orbit of x defined by the set of points $\{x, f(x), f^2(x), \cdots\}$.

1.9 Equilibrium orbits

x^* is a balance solution that achieves the relationship :

$$x^* = f(x^*) = 0 \tag{1.3}$$

All points that satisfy the relation $(\mathbf{1.3})$ are called add points.

1.10 Periodic orbits

Let $x(t, x_0)$ be the orbit of the stable or unstable dynamical system representing the periodic solution if and only if :

$$\exists\, \tau > 0 \,,\ \forall\, t \qquad x(t+\tau, x_0) = x(t, x_0)$$

The periodic orbit of any dynamical system is called an isolated orbit is single and if the system is independent then the isolated orbit is called a limit cycle.

1.11 Quasi-periodic orbits

Let's $x(t, x_0)$ dynamical system orbit and $\mathbf{T} = \{\mathbf{T_1}, \mathbf{T_2}, \cdots, \mathbf{T_n}\}$ a finite set of linearly independent real numbers.
The orbital $x(t, x_0)$ is quasiperiodic if it is periodic for all $\mathbf{T_i} \in \mathbf{T}$
The orbit $x(t, x_0)$ called n-periodic.

1.12 Chaotic orbit

At the beginning of the seventies, the convergence between the work of physicists and mathematicians made it possible to discover the mathematical reality of chaos in deterministic physics systems. It is difficult to formulate the basis of the idea of a chaotic orbit, so we try to approach it on the basis of the behavioral characteristics of this type of orbit.

1.13 Chaotic attractors

Considered as the rest of the other orbits, the orbits of the chaotic solution is a steady flow that is configured in the basin of attraction that is a orbit that converges towards the chaotic attractor.

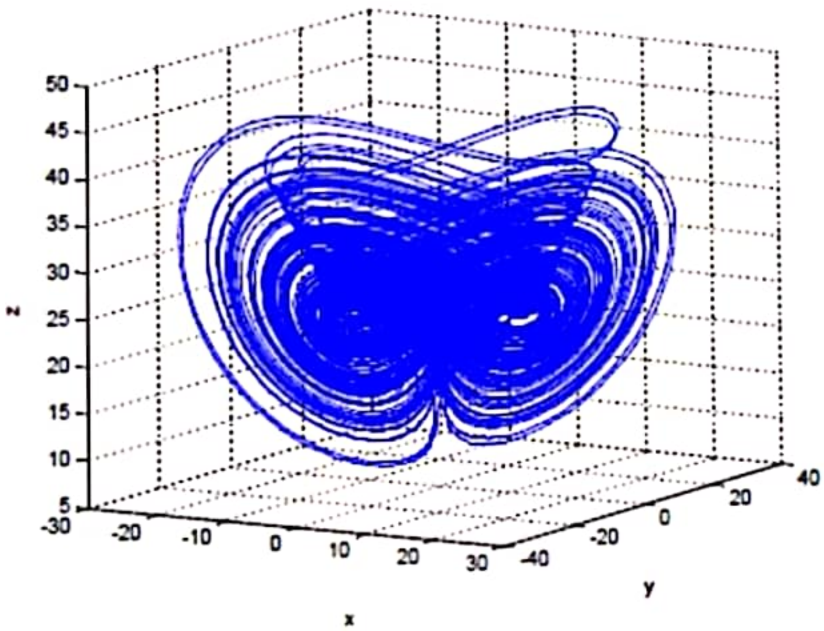

Figure 1.1: Chen's attractor

1.14 Flow and Trajectory

Consider the autonoms dynamic system :

$$\frac{dx}{dt} = f(x) \quad , \; x \in U \subset \mathbb{R}^n \tag{1.4}$$

We call flow of $(\mathbf{1.2})$ the application :

$$\Psi = \begin{cases} \mathbb{R} \times U \longrightarrow \mathbb{R}^n \\ (t, x_0) \longrightarrow \phi(t, x_0) = X(t, x_0) \end{cases} \tag{1.5}$$

Such as :
For each x_0 fixed, $t \longrightarrow \phi(t, x_0)$ is a solution of the differential equation $\phi(0, x_0) = x_0$
$\phi(t, x_0)$ the value at time $\mathbf{t}$ of the solution which goes from x_0 to $\mathbf{t} = \mathbf{0}$.
In other words $\phi(t, x_0)$ is the value at time $\mathbf{t}$ of the solution which is worth x_0 at $\mathbf{t} = \mathbf{0}$.

Let $x_0 \in U$ be an initial condition and $x(t, x_0)$ the solution to $(\mathbf{1.4})$. The set of points $\{\forall\, t \in \mathbb{R}, x(t, x_0)\}$ is the trajectory in state space passing to point x_0 at the initial instant.
Therefore, two identical trajectories necessarily emanate from the same initial state:

$$\forall\, t\ ,\ \phi(t, x_1) = \phi(t, x) \Rightarrow x_1 = x_2 \tag{1.6}$$

1.15 Invariance and attraction

1.15.1 Invariant set

Definition 1.1

$\mathbf{A}$ open set of $\mathbb{R}^n$, we say that $\mathbf{A}$ is invariant if $[\mathbf{19}]$:

$$\forall\, x \in A\ ,\ \phi^t(x) \in A$$

Where $\phi^t(x)$ system flow.

1.15.2 Attractant set

Definition 1.2

$\mathbf{A}$ a closed invariant set $[\mathbf{19}]$ is said to be attractive if there exists $\mathbf{U}$ a neighbor hood of $\mathbf{A}$ such that:

$$\forall\, x \in U, \lim_{x \to \infty} d(\phi^t(x), A) = 0$$

1.15.3 Attractor

The attractor is a geometrical $[\mathbf{17}]$ object towards which all the trajectories of the points in phase space tend, i.e a situation (or a set of situations) towards wich a dynamical system evolves what ever its initial conditions tiales.
There are two type of attractors: regular attractors and strange attractors or chaotic.

1.15.4 Basin of attraction

The basin of attraction $[\mathbf{17}]$ of an attractor is the set of points in phase space which give a trajectory evolving towards the attractor.

1.16 Sensitivity to initial conditions

Sensitivity to initial $[\mathbf{18}]$ conditions is a phenomenon first discovered at the end of the 19th century by Poincare. Then rediscovered in $\mathbf{1963}$ by $\mathbf{Lorenz\ during}$ his work in meteorology. This discovery led to a large number of important works mainly in the field of mathematics. This sensitivity explains the fact that for a chaotic system a tiny modification of the initial conditions can lead to unperdictable results over the long term. The degree of sensitivity to the initial conditions quantifies the chaotic character of the system.

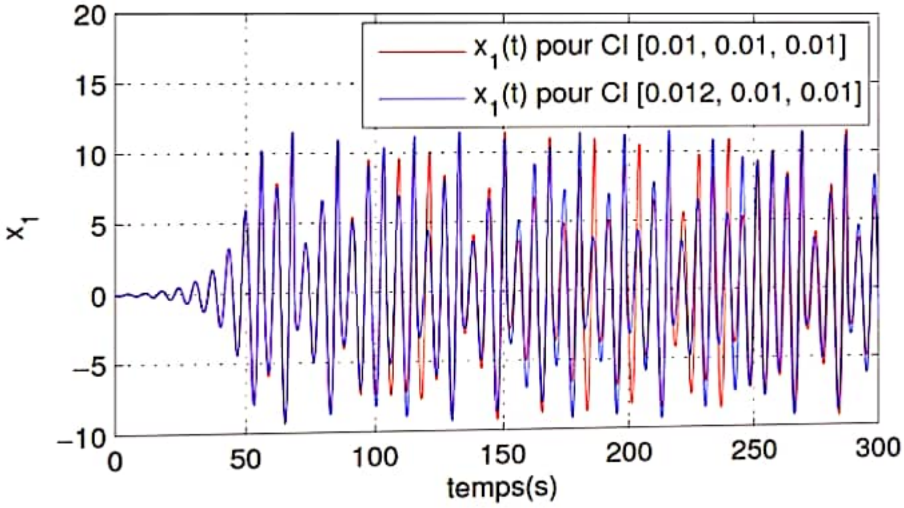

Figure 1.2: Sentivity to initial conditions

1.17 Poincare section

The Poincare section [**20**] is a simple mathematical tool very frequently used to study dynamical systems and in particular periodic trajectories. This method transforms a continuous dynamical system in to a discrete system and reduces its dimension while keeping the same topological properties. Consider now a continuous dynamical system described in a state space of dimension $\mathbf{n}$ and a surface of dimension $\mathbf{n-1}$. The Poincare map is the dynamical system in discrete time whose sequence of iterations corresponds to the coordinates of the successive points of intersection of the trajectory with this surface. The set of intersection points located on the surface represents the Poincare section.

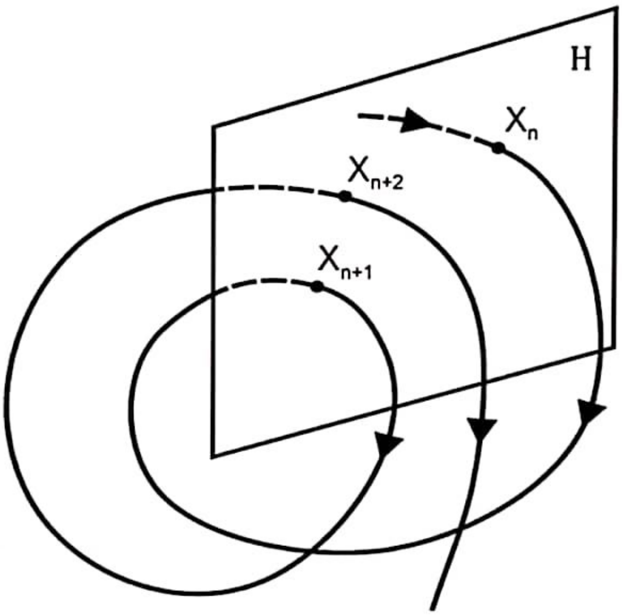

Figure 1.3: Poincare section

1.18 Asymptotic behaviors

In this component we will learn about the basic dynamic properties of dynamical systems solutions. There is no general method for the integration of nonlinear differential systems, so it is not always possible to find an exact solution and therefore we can estimate the solution obtained by numerical simulation when integration is not possible, on the other hand qualitative knowledge of solutions is very userful when it comes to indentifying potential long-term devlopments stability ramifications or even getting an overview of dynamic behaviors depending on the situation excitement or information .

1.19 Stability of dynamical systems

Stability is a specific characteristic of dynamical systrms. It is a basic term and is used frequently in discipline science and systems theory. It means that all movements of a dynamic system except for the stable ones must be controllable. It means that all motions of dynamical system beyond the stable ones have to be controllable.

1.20 Stability of discrete systems

Consider the discrete time system $[\mathbf{14}]$:

$$x(k+1) = A(k), \quad x \in \mathbb{R}^n \tag{1.7}$$

Definition 1.3

The discrete time $(\mathbf{1.7})$ is said to be

1. Stable in the Lyapunov sense, or internally stable, if for every initial condition $X(k_0) = x_0 \in \mathbb{R}^n$ the homogeneous response

$$X(k) = \phi(k, k_0)X_0, \;\; \forall\, k_0 \geq 0$$

is uniformly bounded.

2. Asymptotically stable if, in addition, for every initial condition $X_{k_0} = X_0 \in \mathbb{R}^n$, we have $X(k) \to 0 \; as \; t \to 0$.

3. Exponentially stable, if in addition there exist constants $\mathbf{c} > \mathbf{0}$, $\lambda < \mathbf{1}$ such that for any initial condition $X_{k_0} = X_0 \in \mathbb{R}^n$

$$\| \, X(k) \, \| \leq C\lambda^{k-k_0} \, \| \, X(k_0) \, \|, \forall \, k \geq k_0$$

4. Unstable if not marginally stable in the Lyapunov sense.

Theorem 1.1

The system $(\mathbf{1.7})$ is :

1. Marginally stable if and only if all the eigen values of $\mathbf{A}$ have magnitude smaller than or equal to $\mathbf{1}$ and jordan blocks corresponding to eigen values magnitude $\mathbf{1}$ are $\mathbf{1} \times \mathbf{1}$.

2. Asymptotically stable and exponentially if and only if all the eigen values of $\mathbf{A}$ have magnitude strictly less then $\mathbf{1} \times \mathbf{1}$.

3. Unstable if and only if atleast one eigen value of $\mathbf{A}$ has magnitude greater than $\mathbf{1}$ or equal to $\mathbf{1}$ with the corresponding jordan block is larger than $\mathbf{1} \times \mathbf{1}$.

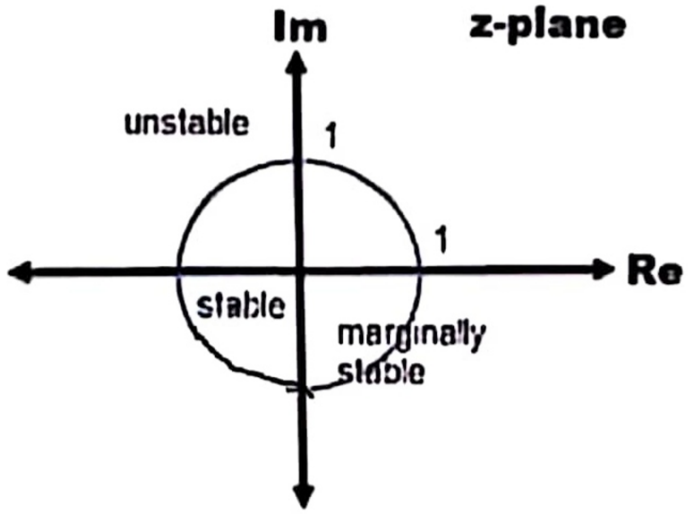

Figure 1.4: Unit circle for mapping z-plane poles

1.21 Lyapunov exponents

The divergence [**18**] speed of two initially close trajectories can be studied from the Lyapunov exponents in order to characterize the nature of the detected chaos.
Alexander Lyapunov (**1857** − **1918**) developed a parameter which allows us to calculate the rate of divergence between the evolution of trajectories resulting from close initial conditions within this bounded space which is the strange attractor.This means of control is called "Lyapunov exponent" which is a quantity allowing to characterize temporal chaos and is defined for dynamical system by:

$$\begin{aligned}\lambda(x_0) &= \lim_{n\to\infty} \frac{1}{n} \ln \mid \frac{dF^n(x_0)}{dx} \mid \\ &= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln \mid F^t(x_i) \mid\end{aligned}$$

Where $F : \mathbb{R} \longrightarrow \mathbb{R}$ is an application and x_0 is an initial condition.
The Lyapunov exponent number is equal to the species dimension of the largest exponent quantifies the degree of chaos.
The following conditions are necessary for the appearance of chaos in a dynamic system.

- At least one of Lyapunov exponents is positive in explaining the divergence of trajectories.
- At least one of Lyapunov exponents is negative to justify the folding of the trajectories.
- The sum of all the exponents is negative to indicate that a chaotic system is dissipative that is it loses energy.

Table:illustrates the connection between the Lyapunov exponents and the forms of attractors and give the signs of the Lyapunov exponents for each type of attractor:

attractor	Lyapunov exponents
fixe point	$0 > \lambda_1 > \lambda_2 > \cdots \geq \lambda_n$
limit cycle	$\lambda_1 = 0, 0 > \lambda_2 \geq \lambda_3 \geq \cdots \lambda_n$
tornus of order2	$\lambda_1 = \lambda_2 = 0, 0 > \lambda_3$
tornus of order K	$\lambda_1 = \lambda_2 = \cdots = \lambda_k = 0, 0 > \lambda_{k+1} \geq \cdots \geq \lambda_n$
chaotic attractor	$\lambda_1 > 0, \sum_{i=1}^{n} \lambda_i < 0$

Table 1.1: Lyapunov exponents for different attractors

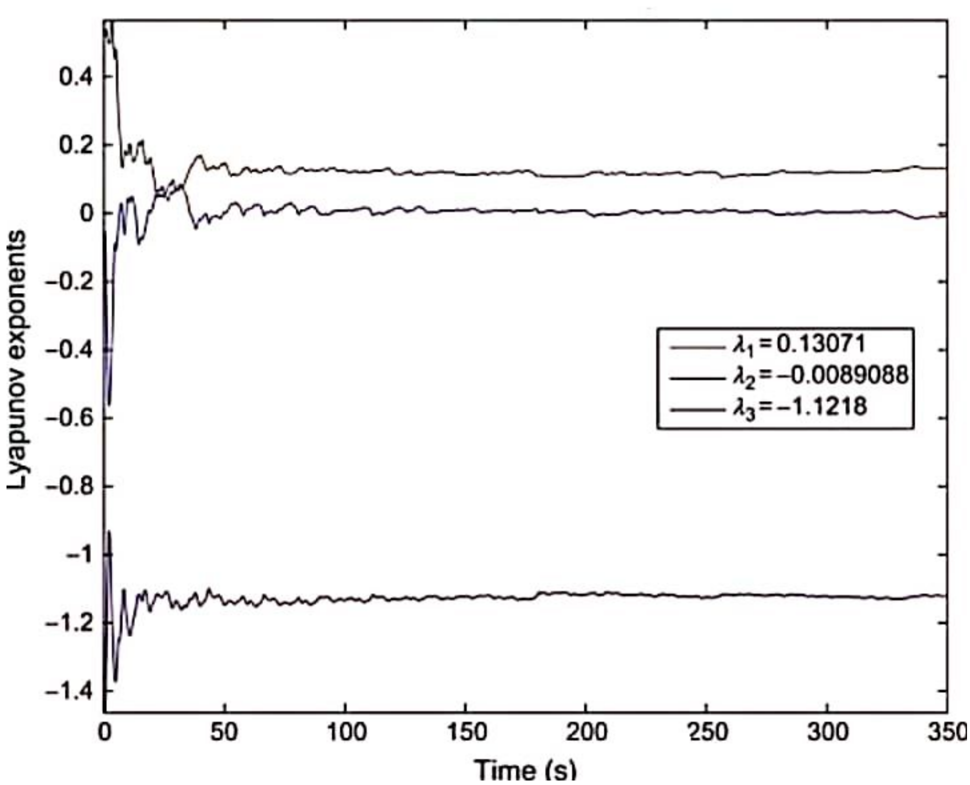

Figure 1.5: Lyapunov exponent of the Lorenz continuous system

Note

• A negative Lyapunov exponent a long one direction indicates that the trajectories get closer and consequently we lose the information on the initial conditiions. The orbit is therefore attractive towards a periodic orbit or a fixed point. It characterizes dissipative systems, this kind of system exhibits asymptotic stability.
• A positive Lyapunov exponent along one direction indicates that the divergence between two neighboring trajectories increases exponentially with time. the trajectories diverge the orbit is therefore chaotic. Intuitively it is sensitivity to initial conditions.
• A null Lyapunov exponent the orbits resulting from different initial conditions, keep a constant separation, neither convergent non divergent one compared to the other. In this case such a physical system is said to be conservative.

1.22 Phase space

In dynamical systems theory [**21**], a phase space (or state space) is a mathematical space in which all possible states of a system are represented each possible state corresponding to a unique point in phase space . For machanical systems, the phase space usually consists of all possible values of the position and momentum variables . For a particle the phase space has **6** dimensions, the position and momentum space each having **3** dimensions. The concept of phase space was devloped at the end of the 19th century by **Ludwing Boltzmann**, **Henri Poincare** and **Josiah Willard Gibbs**.

Definition

The dynamical system is characterized by a certain number of state variables at a given moment . The dynamic behavior of the system is thus linked to the evolution of each of these state variables. This space is called " phase space " where each point defines a state and the point associated with this state defines a trajectory called an orbit.

1.23 Bifurcation

The fundamental aspect [**23**] of the study of dynamical systems is the notion of bifurcation. The term bifurcation was introduced by **Henri Poincare** at the beginning of the 20th century in these works on differential systems for certain critical values of the control parameters of the system. The solution of the differential equation changes qualitatively, we say that there is bifuraction.

Definition 1

Bifuraction means [**23**] a qualitative change in the properties of a system such as stability, the number of equilibrium points or the nature of steady states duing a quantitative variation of a system parameter. The values of the parameters at the time of the change are called bifuraction values.

Definition 2

A bifuraction [**23**] is a qualitative change in the solution " or " of the system (**1.1**) and (**1.2**). When the control parameter is modified and more precisely the dispperarance or change of stability and the appearance of new solution.
The approach to study dynamic systems is as follows :

- Search for fixed points.

- Study of the stability of fixed points.

- Bifurcation diagram.

There are four type of bifurcation :

• Node-pass bifurcation.
• Fork bifurcation.
• Hopf bifurcation.
• Transcritical bifurcation.

1.24 Bifurcation diagram

The bifurcation diagram [**22**] is a plot of the points of the steady state of the system as a function of the parameter of the control. Typically a variable state is chosen and its limit value is plotted as a function of a single control parameter for discontinuous systems, one simply plots the successive valus of a variable state for continuous systems, some type of discretization is required usually using a Poincare section. A bifurcation diagram summarizes the information on the state space and the variation depending on the parameter can be visualized, the transition from a steady state to chaos can be observed.

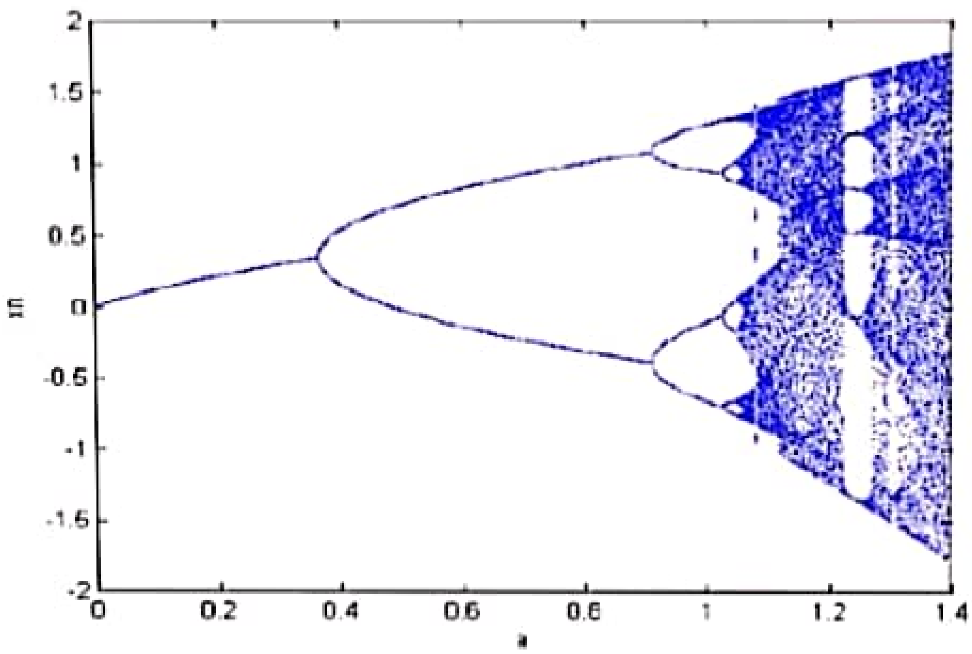

Figure 1.6: Bifurcation diagram of the henon system

Chapter 2

Stablility in the sense of Lyapunov

2.1 Introduction

Since the 1960s $[\mathbf{19}]$, Lyapunov function based approaches have been well devloped for the analysis of system stability among these a very useful criterion, called the "LaSalle invariance principle" and has been applied and extended to the study of many diverse areas in the recent literature. For instance, $\mathbf{Byrnes\ and\ Martin}$ proposed an integral invariance principle to study the stability of nonlinear time-invariant systems. However neither the LaSalle invaraince principle nor the integral invariance principle can be applied to time-varying systems directly. This is due to the fact that the $\omega - \mathbf{limit}$ set is not an invariant set in general time-varyind systems. Since the invariance principles have been proved to be important and use ful in the analysis of system dynamics, the extension of these principles to general time-varying systems has attracted much attention, results for some classes of time-varying systems such as almost periodic systems and asymptotically autonomous systems were optained using. The concept of pseudo-invariant set. However, no simple method was given for the determination of the pseudo-invariance set. Instead of using the concept of the invariance principles, two interesting results employing the concept of "limit equations" and the direct Lyapunov approach were obtained for time-varying systems. All though the stability criteria proposed in pervious literature can be used in some time-varying systems, their approaches are in general hard to check. The devlopment of simple stability criteria for easy checking remains an important issue. In this mote a simple stability criterion for time-varying systems is proposed. Instead of using the existence of $\omega - \mathbf{limit}$ set, the concept of limit systems is defined for time-varying systems. Two detectability conditions will be given in terms of limit systems. Based on these conditions and an integral inequality for the observer function, bounded solutions of system dynamics are shown to approach a pre-specified equilibrium set. The relationships between the proposed scheme and LaSalle invariance principle as well as the integral invariance principle are also studied. Finally, throgh such an application it can be see that, just like the LaSalle invariance principle being feasible to the stability study of time-invariant systems, the approach presented in this note is applicable to analyze the stability of time-varying systems.

2.2 Lyapunov's function

Lyapunov function $[\mathbf{24}]$ are one of the fundamental tools in control theory that can be used to verify stability of a dynamical system $(\mathbf{Kalman\ and\ Bertram, 1960})$.
- A Lyapunov function $\mathbf{V}$ is a continous function $V : \mathbb{R}^n \to \mathbb{R}$ that satisfies the following properties :

1. (nonnegative) $\mathbf{V}(\xi) \geq \mathbf{0}$ for all ξ,

2. (zero at fixed-point) $\mathbf{V}(\xi) = \mathbf{0}$ if and only if $\xi = \xi_*$,

3. (radially unbounded) $\mathbf{V}(\xi) \to \infty$ as $\| \xi \| \to \infty$,

4. (decreasing) $\mathbf{V}(\xi_{\mathbf{k+1}}) \leq \rho^{\mathbf{2}}\mathbf{V}(\xi_{\mathbf{k}})$ for $\mathbf{k} \geq \mathbf{N}$,

Where $\xi_k := (X_k, g_k, f_k)$ is the state of the system at iteration $\mathbf{k}$. The state at iteration $\mathbf{k}$ includes past iterates function values and gradient values from iterations $\mathbf{K} - \mathbf{N}$ up to k. If we can find such a $\mathbf{V}$, then it can be used to show that the state converges linearly to the fixed-point from any initial condition (the rate of convergence depends on both ρ and the structure of $\mathbf{V}$).
Lyapunov functions are typically found by searching over a parameterized family of functions (called Lyapunov function condidates). In the simple case where the state $\{\xi_{\mathbf{k}}\}$ is generated by a linear dynamical system, one can search over quadratic Lyapunov function condidates by solving a semidefinite program.

Definition 2.1

A Lyapunov function is a function that allows to estimate the stability of point of equilibrium (or, more generally of a movement, that is to say of a maximum solution) of a differential equation.

Theorem 2.1

Consider a dynamical system $\dot{X} = f(x)$ and x^* an fixed point. It is assumed that he exists a function $\mathbf{V}$ which are $\mathbf{C^1}$ withe the properties:

1. $V(x^*) = 0$.

2. $\forall\, x \neq x^*, V(x) > 0$ (we say thet V is positive definite).

3. $\forall\, x \neq x^*, \frac{dV}{dt} < 0$.

2.3 Lyapunov stability

By difinition, stability means that if a system is equilibrium, it will remain in that state when the weather varies. The analysis of stability in the sense of Lyapunov consists of the study of the trajectories of the system when the initial state is close to a state of equilibrium. The goal of stability is to drow conclusions about the behavior of the system without explicitly calculating its trajectories in ordes to study the classical results on the notion of stability in the sense of Lyapunov.

Note

The definition of stability in the sense of Lyapunov is closely related to that of continuinty of solutions. An equilibrium is stable if all solutions starting at nearby points stay nearby, otherwise, it is unstable. It is asymptotically stable if all solutions starting at nearby points not only stay nearby, but also tend to the equilibrium point as time approaches infinity.

2.4 Lyapunov stability definition 1

Consider a time-invariant autonomous (with no control) nonlinear system:

$$\begin{cases} \dot{X} = f(x) \\ X(0) = X_0 \end{cases} \tag{2.1}$$

• Assumption:

(i) f Lipschitz continous.

(ii) origin is an isolated equilibrium $f(0) = 0$.

• Stability definition: The equilibrium $x_e = 0$ is called stable in the sense of Lyapunov if:

$$\forall\, \epsilon > 0\ ,\ \exists\, \delta > 0\ ,\ s.t\ ,\ \| x(0) \| \leq \delta \Rightarrow \| x(t) \| \leq \epsilon \quad \forall\, t \geq 0$$

Definition 2.2

The point x_e is a point of equilibrium at the singular point of the system $(\mathbf{2.1})$ if when $x(t_0) = x_e$ then $x(t) = x_e$ for all $\mathbf{t} \geq \mathbf{t_0}$ in other words x_e verifies the equation $f(x_0) = 0$ we always consider the balance at **0** for the general case.

2.5 Lyapunov stability definition 2

$\star$ Asymptotically stable : if it is stable and δ can be chosen so that

$$\| x(0) \| \leq \delta \Rightarrow x(t) \rightarrow 0 \text{ as } t \rightarrow \infty$$

$\star$ Exponentially stable : if there exist positive constants $\delta, \lambda, \mathbf{c}$ such that

$$\| x(t) \| \leq c \| x(0) \| e^{-\lambda t}\ ,\ \forall \| x(0) \| \leq \delta$$

$\star$ Globally asymptotically (exponentially) stable : if the above conditions holds for all $\delta > \mathbf{0}$.

2.6 Some stability definition

We consider nonlinear time-invariant system [**26**]:

$$\dot{X} = f(x) \text{ where } f : \mathbb{R}^n \to \mathbb{R}^n$$

a point $x_e \in \mathbb{R}^n$ is an equilibrium point of the system if $f(x_e) = 0$
x_e is an equilibrium point $\Longleftrightarrow x(t) = x_e$ is a trajectory .
suppose x_e is an equilibrium point.
• system is globally asymptotically stable (G.A.S) if for every trajectory $x(t)$, we have $x(t) \to x_e$ as $\mathbf{t} \to \infty$ (implies x_e is the unique equilibrium point).
• system is locally asymptotically stable (L.A.S) near or at x_e if there is an $\mathbf{R} > \mathbf{0}$ s.t:

$$\| x(0) - x_e \| \leq R \Longrightarrow x(t) \to x_e \text{ as } t \to \infty.$$

Theorem 2.2

The equilibrium point $\mathbf{0}$ of $\dot{X} = f(X(t), t)$, $X(t_0) = X_0$ is stable in the sense of Lyapunov if there exists a locally positive definite function $V(x,t)$ such that $\dot{V}(x,t) \leq 0$ for all $\mathbf{t} \geq \mathbf{t_0}$ and all $\mathbf{X}$ in a local region $X : |x| < r$ for some $\mathbf{r} > \mathbf{0}$.

► such a $V(x,t)$ is called a Lyapunov function .

► i.e $V(x)$ is PD and $\dot{V}(x)$ is negative semidefinite in a local region $|x| < r$.

Theorem 2.3

The equilibrium point $\mathbf{0}$ of $\dot{X} = f(X(t), t)$, $X(t_0) = X_0$ is locally asymptotically stable if there exists a Lyapunov function $V(x)$ such that $\dot{\mathbf{V}}$ is locally negative definite.

Theorem 2.4

The equilibrium point $\mathbf{0}$ of $\dot{X} = f(X(t), t)$, $X(t_0 = X_0)$ is globally asymptotically stable if there exists a Lyapunov function $V(x)$ such that $V(x)$ is positive definite and $\dot{\mathbf{V}}$ is negative definite.

2.7 Basic theorem of Lyapunov

Theorem 2.5

Let $V(x,t)$ be a nonegative function [**25**] with derivative $\dot{\mathbf{V}}$ along the trajectories of the system.
1. If $V(x,t)$ is locally positive definite and $\dot{V}(x,t) \leq 0$ locally in x and for all $\mathbf{t}$, then the origin of the system is locally stable (in the sense of Lyapunov).
2. If $V(x,t)$ is locally positive definite and decrescent and $\dot{\mathbf{V}} \leq \mathbf{0}$ locally in x and for all $\mathbf{t}$, then the origin of the system is uniformly locally stable (in the sense of Lyapunov).
3. If $V(x,t)$ is locally positive definite and decrescent, and $-\dot{V}(x,t)$ is locally positive definite, then the origin of the system is uniformly locally asymptotically stable.
4. If $V(x,t)$ is positive definit and decrescent, and $-\dot{V}(x,t)$ is positive definite, then the origin of the system is globally uniformly asymptotically stable.

2.8 Lyapunov stability theorem for autonomous system

Consider the autonomous system [**3**]:

$$\dot{X} = f(x) \tag{2.2}$$

Where $f : D \to \mathbb{R}^n$ is locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$. Suppose $\overline{\mathbf{x}} \in \mathbf{D}$ is an equilibrium point of $(\mathbf{2.2})$ that is $f(\overline{x}) = 0$. Our goal is to characterize and study the stability of $\overline{\mathbf{x}}$. For convonionce, we state all definitions and theorems for the case when the equilibrium point is at the origin of $\mathbb{R}^n$ that is $\overline{\mathbf{x}} = \mathbf{0}$. There is no loss of generality in doing so because any equilibrium point can be shifted to the origin via a change of variables. Suppose $\overline{\mathbf{x}} \neq \mathbf{0}$ and consider the change of variables $y = x - \overline{x}$ the derivative of $\mathbf{y}$ is given by

$$\dot{y} = \dot{x} = f(y + \overline{x}) = g(y) \;\; , \text{ where } g(0) = 0$$

In the new variable $\mathbf{y}$, the system has equilibrium at the origin. Therefore, without loss of generality, we will always assume that $f(x)$ satisfies $f(0) = 0$ and study the stability of the origin $x = 0$

Definition 2.3

The equilibrium point $x = 0$ of $(\mathbf{2.2})$ is
• stable if, for each $\epsilon > \mathbf{0}$, there is $\delta = \delta(\epsilon) > \mathbf{0}$ such that

$$\| x(0) \| < \delta \Rightarrow \| x(t) \| < \epsilon \; , \; \forall \, t \geq 0$$

• unstable if it is not stable.
• asymptotically stable if it is stable and δ can be chosen such that

$$\| \mathbf{x(0)} \| < \delta \Rightarrow \lim_{\mathbf{t} \to \infty} \mathbf{x(t)} = \mathbf{0}$$

For autonomous system, uniform (asymptotic) stability is the same as (asymptotic) stability.

2.9 Positive definite (semi-definite) function

Definition 2.4

A continuously differentiable function $[\mathbf{26}]$ $V : \mathbb{R}^n \to \mathbb{R}$ is called positive definite in a region $U \subset \mathbb{R}^n$ containing the origin if :

a. $V(0) = 0$

b. $V(x) > 0 \; , \; x \in U$ and $x \neq 0$

A function is called positive semi-definite if condition "b" is replaced by $V(x) \geq 0$.

2.10 Lyapunov theorem

Theorem 2.4

For autonomous systems $[\mathbf{26}]$. Let $D \subset \mathbb{R}^n$ be a domain containing the equilibrium point of origin. If there exists a continuously differentiable positive definite function $V : D \to \mathbb{R}$ such that :

$$\dot{V} = \frac{\partial V}{\partial x}\frac{dx}{dt} = \frac{\partial V}{\partial x} f(x) = -W(x) \tag{2.3}$$

is negative semi-definite in $\mathbf{D}$, the equilibrium point $\mathbf{0}$ is stable. Moreover, if $W(x)$ is positive definite, then the equilibrium is asymptotically stable.
In addition, if $D = \mathbb{R}^n$ and $\mathbf{V}$ is radially unbounded, i.e

$$\| x \| \to \infty \Rightarrow V(x) \to \infty \tag{2.4}$$

Then, the origin is globally asymptotically stable.
For autonomous systems, when $W(x)$ in the above theorem is only positive semi-definite, asymptotic stability may still be obtained by applying the folowing simplified version of LaSalle's theorem.

2.11 LaSalle's invariance principle theorem

Theorem 2.5

For autonomous systems $[\mathbf{26}]$. Let $D \subset \mathbb{R}^n$ be a domain containing the equilibrium point of origin. If there exists a continuously differentiable positive definite function $V : D \to \mathbb{R}$ such that :

$$\dot{V} = \frac{\partial V}{\partial x} f(x) = -W(x) \leq 0 \tag{2.5}$$

in D Let

$$S = \{x \in D \mid \dot{V}(x) = 0\} \tag{2.6}$$

and suppose that no solution can stay identically in $\mathbf{S}$, other then the origin. Then, the origin is asymptotically stable.
In addition, if $D = \mathbb{R}^n$ and $\mathbf{V}$ is radially unbounded, the origin is globally asymptotically stable.

2.12 Stability of nonautonomous systems

Consider the nonautonomous system [$\mathbf{3}$]:

$$\dot{x} = f(t, x) \tag{2.7}$$

Where $f : [\,0, \infty) \times D \rightarrow \subset \mathbb{R}^n$ is piecewise continuous in $\mathbf{t}$ and locally Lipshitz in x on $[\,0, \infty) \times D$, and $D \subset \mathbb{R}^n$ is a domain that contains the origin $x = 0$. The origin is an equilibrium point for $(\mathbf{2.7})$ at $\mathbf{t} = \mathbf{0}$ if

$$f(t, 0) = 0 \ \ \forall\, t \geq 0$$

An equilibrium point at the origin could be a translation of a nonzero equilibrium point or, more generally, a translation of a nonzero solution of the system. To see the latter point, suppose $\overline{\mathbf{y}}(\tau)$ is a solution of the system

$$\tfrac{dy}{d\tau} = y(\tau, y)$$

defined for all $\tau \geq \mathbf{a}$. The change of variables

$$x = y = -\overline{y}(\tau)\ ;\ t = \tau - a$$

transforms the system into the form :

$$\dot{x} = g(\tau, y) - \dot{\overline{y}}(\tau) = g(t + a, x + \overline{y}(t + a)) - \dot{\overline{y}}(t + a) = f(t, x)$$

Since

$$\dot{\overline{y}}(t + a) = g(t + a, \overline{y}(t + a)),\ \ \forall\, t \geq 0$$

the origin $x = 0$ is an equilibrium point of the transformed system at $\mathbf{t} = \mathbf{0}$. Therefore, by examining the stability behavior of the origin as an equilibrium point for the transformed system, we determine the stability behavior of the solution $\overline{\mathbf{y}}(\tau)$ of the original system. Notice that if $\overline{\mathbf{y}}(\tau)$ is not constant, the transformed system will be nonautonomous even when the original system is autonomous, that is, even when $\mathbf{g}(\tau, \mathbf{y}) = \mathbf{g}(\mathbf{y})$. This is why studying the stability behavior of solutions in the sense of Lyapunov can be done only in the context of studying the stability behavior of the equilibria of nonautonomous systems.
The notions of stability and asymptotic stability of equilibrium points of nonautonomous systems are basically the same as those introduced in definition for autonomous system depends only on $(\mathbf{t} - \mathbf{t_0})$. The solution of a nonautonolous system may depend on the both $\mathbf{t}$ and t_0. Therefore, the stability behavior of the equilibrium point will, in general be dependent on $\mathbf{t_0}$. The origin $x = 0$ is a stable equilibrium point for $(\mathbf{2.7})$ if for each $\epsilon > \mathbf{0}$, an any $\mathbf{t_0} \geq \mathbf{0}$ there is $\delta = \delta(\epsilon, \mathbf{t_0}) > \mathbf{0}$ such that

$$\parallel x(t_0) \parallel \leq \delta \Rightarrow \parallel x(t) \parallel < \epsilon\ ,\ \forall\, t \geq t_0$$

The constant δ is in general dependent on the initial time $\mathbf{t_0}$. The existence of δ for every $\mathbf{t_0}$ does not necessarily guarantee that there is one constant δ, dependent only on ϵ, that would for all $\mathbf{t_0}$.

Definition 2.4

The equilibrium point $x = 0$ of $(\mathbf{2.7})$ is:

- stable if for each $\epsilon > \mathbf{0}$, there is $\delta = \delta(\epsilon, \mathbf{t_0}) > \mathbf{0}$ such that

$$\parallel x(t_0) \parallel < \delta \Rightarrow \parallel x(t) \parallel < \epsilon\ ,\ \forall\, t \geq t_0 \geq 0 \tag{2.8}$$

- uniformly stable if for each $\epsilon > \mathbf{0}$, there is $\delta = \delta(\epsilon > \mathbf{0})$ independent of $\mathbf{t_0}$, such that $(\mathbf{2.8})$ is satisfied.

- unstable if not stable.

- asymptotically stable if it is stable and there is $\mathbf{c} = \mathbf{c}(\mathbf{t_0}) > \mathbf{0}$ such that $x(t) \rightarrow 0$ as $t \rightarrow \infty$, for all $\parallel x(t_0) \parallel < c$.

- uniformly asymptotically stable if it is uniformly stable and there is a positive constant $\mathbf{c}$ independent of $\mathbf{t_0}$, such that for all $\parallel x(t_0) \parallel < c$, $x(t) \rightarrow 0$ as $\mathbf{t} \rightarrow \infty$, uniformly in t_0, that is for each $\epsilon > \mathbf{0}$, there is $\mathbf{T} = \mathbf{T}(\epsilon) > \mathbf{0}$ such that

$$\parallel x(t) \parallel < \epsilon\ ,\ \forall\, t \geq t_0 + T(\epsilon)\ ,\ \forall \parallel x(t_0) \parallel < c \tag{2.9}$$

• globally uniformly asymptotically stable if it is uniformly stable and, for each pair of positive numbers ϵ and c, there is $\mathbf{T} = \mathbf{T}(\epsilon, \mathbf{c}) > \mathbf{0}$ such that

$$\parallel x(t) \parallel < \epsilon \ , \ \forall t \geq t_0 + T(\epsilon, c) \ , \ \forall \parallel x(t_0) \parallel < c \tag{2.10}$$

Lemma 2.1

The equilibrium point $x = 0$ of $(\mathbf{2.7})$ is

• Uniformly stable if and only if there exist a class $\mathbf{K}$ function α an a positive constant $\mathbf{c}$, independent of $\mathbf{t_0}$ such that

$$\parallel x(t) \parallel \geq \alpha(\parallel x(t_0) \parallel) \ , \ \forall \, t \geq t_0 \geq 0 \ , \ \forall \parallel x(t_0) \parallel < c \tag{2.11}$$

Theorem 2.6

Let $x = 0$ be an equilibrium point for $(\mathbf{2.9})$ and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : [0, \infty) \times D \to \mathbb{R}$ be continuously differentiable function such that

$$W_1(x) \leq V(t, x) \leq W_2(x) \tag{2.12}$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -W_3(x) \tag{2.13}$$

$\forall \, \mathbf{t} \geq \mathbf{0}$ and $\forall x \in D$, where $W_1(x), W_2(x)$, and $W_3(x)$ are continuous positive definite functions on $\mathbf{D}$. Then, $x = 0$ is uniformly asymptotically stable.

condition on $V(x)$	condition on $-\dot{V}(x)$	conclusion
$V(x) > 0$, $x \neq 0$ locally	$-V(x) \geq 0$ locally	stable
$V(x) > 0$, $x \neq 0$ locally	$-V(x) > 0$, $\forall \, x \neq 0$ locally	asymptotically stable
$V(x) > 0$, $x \neq 0$, $V(x) \to \infty$ as $\parallel x \parallel \to \infty$	$-V(x) > 0$, $\forall x \neq 0$	globally asymptotically stable
$\alpha \parallel x \parallel^p \leq V(x) \leq \beta \parallel x \parallel^p$ locally	$-V(x) \geq \epsilon V(x)$ locally	locally exponentially stable
$\alpha \parallel x \parallel^p \leq V(x) \leq \beta \parallel x \parallel^p$ globally	$-V(x) \geq \epsilon V(x)$ globally	globally exponentially stable

Table 2.1: Basic Lyapunov stability for autonomous system

2.13 Lyapunov theorem for global asymptotic stability

The region in the state space for which our earlier results hold is determined by the region over which $V(x)$ serves as a Lyapunov function. It is of special interest to determine the " basin of attraction " of an asymptotically stable equilibrium point, i.e. the set of initial conditions whose subesquent trajectories and up at this equilibrium point. An equilibrium point is globally asymptotically stable (or asymptotically stable "in the large") if its basin of attraction is the entire state space. If a function $V(x)$ is positive definite on the entire state space, and has the additional property that $|\mathbf{V}(\mathbf{x})| \nearrow \infty$ as $\parallel \mathbf{x} \parallel \nearrow \infty$, and if its derivative $\dot{\mathbf{V}}$ is negative definite on the entire state space, then the equilibrium point at the origin is globally asymptotically stable.

Exemple 2.1

Cosider the system :

$$\overline{x} = -c(x)$$

With the property that $\mathbf{c(0)} = \mathbf{0}$ and $x'c(x) > 0$ if $x \neq 0$. the unique equilibrium point of the system is at $\mathbf{0}$. New consider the condidate Lyapunov function

$$V(x) = x'x$$

which satisfies all the desired properties including $\mid V(x) \mid \nearrow \infty$ as $\parallel x \parallel \nearrow \infty$.
Evaluating its derivative along trajectories, we get

$$\dot{V}(x) = 2x\prime x = -2x\prime C(x) < 0 \ , \ \forall \, x \neq 0$$

Hence, the system is globally asymptotically stable.

2.14 Global stability definitions

2.14.1 Global asymptotic stability

Definition 2.5

The equilibrium point $\mathbf{0}$ is globally asymptotically stable if it is stable and

$$\lim_{t \to \infty} s(t, t_0, x_0) = 0 \ , \ \forall \, x_0 \in \mathbb{R}^n$$

.

Definition 2.6

The equilibrium point $\mathbf{0}$ is said to be globally uniformly asymptotically stable, if it is uniformly stable and for each pair of positive numbers $\mathbf{M}, \epsilon$ withe $\mathbf{M}$ arbitrarily large and ϵ arbitrarily small, there exists a finite number $\mathbf{T} = \mathbf{T}(\mathbf{M}, \epsilon)$ such that

$$\parallel \mathbf{x_0} \parallel < \mathbf{M}, \ \mathbf{t_0} \geq \mathbf{0} \Rightarrow \parallel \mathbf{s(t + t_0, t_0, x_0)} \parallel < \epsilon \ , \ \forall \, \mathbf{t} \geq \mathbf{T}(\mathbf{M}, \epsilon)$$

Definition 2.7

The equilibrium point $\mathbf{0}$ is said to be globally exponentially stable if there exists constants $\mathbf{c}, \gamma$ such that

$$\parallel s(t_0 + t, t_0, X_0) \geq c \parallel X_0 \parallel e^{-\gamma t} \ , \ \forall t, t_0 \geq 0 \ , \ \forall x_0 \in \mathbb{R}^n \tag{2.14}$$

For an equilibrium point to be either globally uniformly asymptotically stable or globally exponentially stable, a necessary condition is that it is the only equilibrium point.

Lemma 2.2

Suppose a function $V : \mathbb{R}^n \to \mathbb{R}$ satisfies $\mathbf{V(0)} = \mathbf{0}$ and $V(x) > 0$ for all $x \neq 0$.
If $\mathbf{V}$ is convex, then it is radially unbounded.

2.15 Global asymptotic stability using LaSalle

Corollary 2.1

Let $x = 0$ be an equilibrium point of $\dot{X} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable radially unbounded positive definite function on $\mathbb{R}^n \backslash \{0\}$ such that $\dot{\mathbf{V}} \leq \mathbf{0}$ on $\mathbb{R}^n$. Let $\mathbf{E}$ be the set of points where $\dot{V}(x) = 0$ and suppose that no solution can stay identically in E other then the trivial solution $x = 0$ then, the origin is globally asymptotically stable.

2.16 Controlled systems

A controlled nonlinear system (one controlled) is a differential system of the form :

$$\dot{x} = f(x, u, t), \quad t \in [0, +\infty[, \quad x(t) \in M\ ,\ u(.) \in U \tag{2.15}$$

In general, the vector of states $x(t)$ belongs to a differential manifold $\mathbf{M}$ of dimension $\mathbf{n}$ (we will assume here that $\mathbf{M}$ is a connected open of $\mathbb{R}^n$), the controls $\mathbf{u}(.)$ belong to an admissible set $\mathbf{U}$, which is a set of locally integrable functions defined on $[\mathbf{0}, +\infty)$, a values in $U \subset \mathbb{R}^m$, and $f(., ., .)$ is nonlinear function with values in$\mathbb{R}^n$ (wich we will assume to be sufficiently regular, we will specify this regularity later). The natural number $\mathbf{n}$ is the order of the system. A special case of such systems is the linear system of the form

$$\dot{x} = A(t)x + B(t)u \tag{2.16}$$

With $\mathbf{A}(\mathbf{t})$ an $n \times n$ matrix function called the state matrix, and $\mathbf{B}(\mathbf{t})$ an $\mathbf{n} \times \mathbf{m}$ matrix function called the control matrix.

Chapter 3

Global asymptotic stability of discrete map

3.1 Introduction

The essential motivation [**28**] of this section is to eliminate any bounded behavior (chaotic or not) is some forms of $\mathbf{n-D}$ discrete map and drives it globally asymptotically to their fixed points, via the controller of a universal continous function. by constructing explicitly a nonnegative semi-definite Lyapunov function that guaranties the stabilization of such system.

We consider the general $\mathbf{n-D}$ discrete map of the forme :

$$g(x_k,\mu)=\begin{cases} x_{k+1}^1 = x_k^i f_1(x_k,\mu)\ ; \\ x_{k+1}^2 = x_k^i f_2(x_k,\mu)\ ; \\ \qquad\cdots \\ x_{k+1}^n = x_k^i f_n(x_k,\mu)\ ; \end{cases} \tag{3.1}$$

Where $\mu = (\mu_1, \mu_2, \ldots, \mu_j) \in \mathbb{R}^j$ are bifuraction parameters and $x_k = (x_k^1, x_k^2, \ldots, x_k^n) \in \mathbb{R}^n$ is the state variable and $(f_k)_{1\leq k\leq n}$ are real continouus functions defined as follow: $f_k : \mathbb{R}^n \times \mathbb{R}^j \to \mathbb{R}$.
The origin $(\mathbf{0}, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) \in \mathbb{R}^\mathbf{n}$ is a fixed point for the map $(\mathbf{3.1})$
i.e, $\mathbf{g}(\mathbf{0}, \mathbf{0}, \mathbf{0}, \ldots, \mu) = (\mathbf{0}, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$.
On the other hand, assume that for all μ in subset $\Omega_\mu \in \mathbb{R}^j$, the solutions of system $(\mathbf{3.1})$ are bounded.

the controlled system associated with system $(\mathbf{3.1})$ is given by :

$$g(x_k,\mu)=\begin{cases} x_{k+1}^1 = x_k^i f_1(x_k,\mu) \\ x_{k+1}^2 = x_k^i f_2(x_k,\mu) \\ \qquad\cdots \\ x_{k+1}^n = x_k^i f_n(x_k,\mu) \end{cases} + U(x_k,\mu) \tag{3.2}$$

Where ther controller $\mathbf{U}$ given by :

$$U(x_k,\mu)=\begin{pmatrix} 0 \\ 0 \\ \cdots \\ -x_k^i f_i(x_k,\mu) + h(x_k)x_k^i \\ \cdots \\ 0 \end{pmatrix} \tag{3.3}$$

Where the function $f_i(x_k, \mu)$ is given in the $\mathbf{i^{th}}$ component of the uncontrolled map $(\mathbf{3.1})$, and all the components of the controller $\mathbf{U}$ are zero, expect the $\mathbf{i^{th}}$ component, and $h : \mathbb{R}^n \to \mathbb{R}$ is any real continuous function satisfying $|\mathbf{h}(\mathbf{x})| < \mathbf{1}$, for all $x \in \mathbb{R}^n$.

Hence, the controlled system $(\mathbf{3.2})$ is now given explicitly by :

$$x_{k+1} = g_c(x_k, \mu) = \begin{cases} x_{k+1}^1 = x_k^i f_1(x_k, \mu) \ ; \\ x_{k+1}^2 = x_k^i f_2(x_k, \mu) \ ; \\ \qquad \cdots \\ x_{k+1}^i = h(x_k) x_k^i \quad ; \\ \qquad \cdots \\ x_{k+1}^n = x_k^i f_n(x_k, \mu) \ ; \end{cases} \tag{3.4}$$

The origin $(0, 0, 0, \cdots, 0) \in \mathbb{R}^n$ is a fixed point for the controlled map $(\mathbf{3.4})$, and it has the same form given in $(\mathbf{3.1})$.

Notations

a) The $\omega - \mathbf{limit}$ set of vector $x \in \mathbb{R}^n$ is defined by:

$$f^+(x) = \{f^k(x), k \in \mathbb{N}\}$$

b) A ball in $\mathbb{R}^n$ of radius δ is given by $B_\delta = \{x \in \mathbb{R}^n, \| x \| \leq \delta\}$.

Definition 3.1

A set $\mathbf{G}^*$ is positively invariant under the action of a map $\mathbf{g}$ if and only if: $\mathbf{g}(\mathbf{G}^*) \subset \mathbf{G}^*$.

Definition 3.2

Let $\mathbf{G}^*$ be the largest positively invariant set contained in $\mathbf{G}$, i.e, $\mathbf{g_c}(\mathbf{G}^*) \subset \mathbf{G}^*$.

Definition 3.3

Let $\mathbf{G}^*$ be a closed positively invariant set such that $(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) \in \mathbf{G}^*$, then the origin is said to be :

1. $\mathbf{G}^*$ stable if $\forall\, \epsilon > \mathbf{0}, \exists\, \delta > \mathbf{0}\ ,\ \mathbf{g}^+(\mathbf{B}_\delta \cap \mathbf{G}^*) \subset \mathbf{B}_\epsilon$.

2. $\mathbf{G}^*$ asymptotically stable if it is $\mathbf{G}^*$ stable and there exists $\delta > \mathbf{0}$ such that:

$$\lim_{\mathbf{n} \to \infty} \mathbf{g^n}(\mathbf{x}) = \mathbf{0}\ ,\ \forall \mathbf{x} \in \mathbf{B}_\delta \cap \mathbf{G}^*$$

In the squel, if the system $(\mathbf{3.1})$ has a nonnegative Lyapunov function defined in a neighbourhood of the origin $\nu \subset \mathbf{U}$, we will denote by $\mathbf{G_0}$ the set where $\mathbf{V}$ vanishes, $\mathbf{G}$ the set where the difference of $\mathbf{V}$ along the trajectories of the system vanishes, $\mathbf{G}$ the set where the difference of $\mathbf{V}$ along the trajectories of the system vanishes, and $\mathbf{G}^*$ the largest invariant set contained in $\mathbf{G}$.
One can easily show that $\mathbf{G_0}$, G^* and $\mathbf{G}$ are closed sets. $\mathbf{G_0}$ is positively invariant and $\mathbf{G_0} \subseteq \mathbf{G}^* \subseteq \mathbf{G}$.

3.2 LaSalle's Invariance Principle

LaSalle's invariance principle $[\mathbf{16}]$ is a result about the asymptotic stability $[\mathbf{15}]$ of the solutions to a system of differential equation in $\mathbb{R}^n$. The notion of asymptotic stability is expressed as "remain[ing] near the equilibrium state and in addition tend[ing] to return to the equilibrium". In fact, LaSalle proves that under some conditions the solutions approach a given (bounded) region of space when time goes to infinity and he uses this result on examples where the properties of this region imply that it is the equilibrium.$[\mathbf{29}]$ LaSalle's theorem enables one to conclude asymptotic stability of an equilibrium point ever when one can't find a function $V(x)$ such that $V'(x, t)$ is locally negative definite. However, it applies only to time-invariant or periodic systems. We will deal with the time-invariant case and begin by introducing a few more definitions.

Definition 3.4

The set $S \subset \mathbb{R}^n$ is the limit set [**29**] of a trajectory $x(t, x_0)$ if for every $\mathbf{p} \in \mathbf{S}$, there exists a strictly increasing sequence of time $\mathbf{t_n}$ such that $x(t_n, x_0) \to p$ as $\mathbf{t_n} \to \infty$.

Definition 3.5

A set $\mathbf{M}$ is said to be a positive invariant set if:

$$x_0 \in M \ \ implies \ \ x(t, x_0) \in M, for \ \ all \ \ t \geq 0.$$

A set $\mathbf{M}$ is said to be an invariant set if :

$$x_0 \in M \ \ implies \ \ x(t, x_0) \in M, for \ \ all \ \ t \ \in \mathbb{R}.$$

Lemma 3.1

Suppose $\Omega \subseteq \mathbb{R}^n$ is open, $f : \Omega \to \mathbb{R}^n$ is locally Lipschiptz function and $V : \Omega \to \mathbb{R}$ is a $\mathbf{C^1}$ function and bounded from below. Suppose that $\mathbf{p} \in \mathbf{\Omega}$ and for all $\mathbf{t} \geq \mathbf{0}$, $x(t, p)$ is defined and is contained in $\mathbf{\Omega}$. Then [**30**]

$$for \ \ all \ \ q \in L^+ \ , \ V'(q) = 0.$$

Proof

if $\mathbf{L^+} = \mathbf{0}$ then there is nothing to prove. Suppose $\mathbf{q_1} = \mathbf{q_3} \ \in \ \mathbf{L^+}$, $q_2 \ \in \ L^+$ and fix any $\epsilon > \mathbf{0}$.
Since $\mathbf{V}$ is continuous at $\mathbf{q_1}$ and at $\mathbf{q_2}$, there exists $\delta > \mathbf{0}$ such that for all $\mathbf{x} \in \ \mathbf{\Omega}$, for $\mathbf{i} = \mathbf{1}, \mathbf{2}$, if

$$\parallel x - q_i \parallel < \delta \ \ then \ \ |V(x) - V(q_i)| < \frac{\epsilon}{4}.$$

Since $\mathbf{q_i} \in \mathbf{L^+}$, $i = 1, 2$, there exists $\mathbf{0} \leq \mathbf{t_1} \leq \mathbf{t_2} \leq \mathbf{t_3}$ such that for $\mathbf{i} = \mathbf{1}, \mathbf{2}, \mathbf{3}$, $\parallel \mathbf{x}(\mathbf{t}, \mathbf{p}) - \mathbf{q_i} \parallel < \delta$, and hence for

$$i = 1, 2, 3, |V(x(t, p)) - V(q_i)| < \frac{\epsilon}{4}.$$

This implies that

$$|V(x(t_3, p)) - V(x(t_1, p))| \leq |V(x(t_3, p)) - V(q_1)| + |V(q_1) - V(x(t_1, p))| < \frac{\epsilon}{2}.$$

Since $\mathbf{V'}$ does not change the sign this implies

$$\begin{aligned} |V(x(t_2, p)) - V(x(t_1, p))| &= |\int_{t_1}^{t_2} V'(x(s, p))ds| \\ &\leq |\int_{t_1}^{t_3} V(x(s, p))ds| \\ &\leq |V(x(t_3, p)) - V(x(t_1, p))| < \frac{\epsilon}{2}. \end{aligned}$$

Therefore,

$$\begin{aligned} |V(q_2) - V(q_1)| &\leq |V(q_2) - V(x(t_2, p))| + |V(x(t_2, p)) - V(x(t_1, p))| + |V(x(t_1, p))| \\ &\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$

Since $\epsilon > \mathbf{0}$ was arbitrary, this establishes that $\mathbf{V}$ is constant on $\mathbf{L^+}$.

3.3 LaSalle's Theorem

Theorem 3.1

Let f be a locally Lipschiptz function $[\mathbf{30}]$ defined over a domain $D \subset \mathbb{R}^n$ and $\mathbf{\Omega} \subset \mathbf{D}$ be a a compact set that is positively invariant with respect to $x' = f(x)$.
Let $V(x)$ be a $\mathbf{C^1}$ function defined over $\mathbf{D}$ such that $V'(x) \leq 0$ in $\mathbf{\Omega}$. Let $\mathbf{E}$ be the set of all points in $\mathbf{\Omega}$ where $V'(x) = 0$, and $\mathbf{M}$ be the largest invariant set in $\mathbf{E}$. Then, every solution starting in $\mathbf{\Omega}$ approaches $\mathbf{M}$ as $\mathbf{t} \to \infty$, means that $d(x(t, x_0), M) \to 0$ as $\mathbf{t} \to \infty$, for all $x_0 \in \ \Omega$.

Proof

Suppose $\mathbf{p} \in \ \mathbf{\Omega}$ and consider the curve $t \to x(t,p)$. Since $\mathbf{V}$ is continuous it is bounded from below from on the compact set $\mathbf{\Omega}$. Since $\mathbf{\Omega}$ is invariant by hypothesis,

$$for \ \ all \ \ t \geq 0, x(t,p) \in \Omega.$$

More ever, using that $\mathbf{\Omega}$ is compact, and hence bounded, the positive limit set $\mathbf{L^+}$ is non-empty.
Since Ω is closed,

$$L^+ \subseteq \Omega.$$

Using the Lemma above,

$$for \ \ all \ \ q \ \in L^+, \ \ V'(q) = 0$$

and therefore,

$$L^+ \subseteq \{y : V'(y) = 0\} = E \subseteq M.$$

Since the positive limit sets are invariant,

$$L^+ \subseteq M.$$

Since

$$L^+ \subseteq M,$$

it follows that

$$q \ \in M.$$

Thus, the solution curve starting at an arbitrary $\mathbf{p} \ \in \mathbf{\Omega}$ approaches $\mathbf{M}$ as $\mathbf{t} \to \infty$.

Theorem 3.2

Let $\mathbf{f}$ be a locally Lipschitz function $[\mathbf{31}]$ defined over a domain $D \subset \mathbb{R}^n \ ; \ 0 \ \in D$.

Let $V(x)$ be a $\mathbf{C^1}$ positive definite function defined over $\mathbf{D}$ such that $V'(x) \leq 0$ in $D - \{0\}$. Let

$$\Gamma = \{x \ \in D \ : \ V'(x) = 0\}$$

(i) If no solution can stay indentically in $\mathbf{\Gamma}$, other then the trivial solution $x(t) = 0$, then the origin is asymptotically stable.

(ii) Morever, if $\mathbf{\Gamma} \subset \mathbf{D}$ is compact and positively invariant, then it is a subset of the region of attraction.

(iii) Furthermore, if $D = \mathbb{R}^n$ and $V(x)$ is radially unbounded, then the origin is globally asymptotically stable.

LaSalle's theorem can also extend the Lyapunov theeorem in three different directions.

(1) It gives an estimate of the **RoA** not necessarily in the form of

$$\Omega_c = \{x \ \in \mathbb{R}^n \ : \ V(x) \leq c\}.$$

the set can be a positively invariant set which leads to less conservative estimate.

(2) Can determine stability of equilibrium set, rather then isolated equilibrium point.

(3) The function $V(x)$ does not have to be positive definite.

3.4 Relation betwen Lyapunov theory and LaSalle's theorem

If $\dot{V}(x)$ is negative definite, then the global asymptotic stability of the origin is a consequence of Lyapunov's second theorem. The invariance principle gives a criterion for asymptotic stability in the case when $\dot{V}(x)$ is only negative semi-definite.

Theorem 3.3

If there exists a function $V \in C^0(\mathbb{R}^n, \mathbb{R}^+)$ satisfying $[\mathbf{11}]$:

(1) $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $\mathbf{V(0) = 0}$.

(2) $\Delta V(x) = V(g(x)) - V(x) \leq 0$ for all $x \in \mathbb{R}^n$.

(3) $\mathbf{0}$ is $\mathbf{G}^*$ globally asymptotically stable where $\mathbf{G}^*$ is the largest positively invariant set contained in $G = \{x \in \mathbb{R}^n \ : \ V(g(x)) - V(x) = 0\}$.

(4) All the solutions of system $(\mathbf{3.1})$ are bounded.

Then the origin is globally asymptotically stable.

Proof

The set $G_0 = \{x \in \mathbb{R}^n \ : \ v(x) = 0\}$ is positively invariant so it is contained in $\mathbf{G}^*$.
All the assumptions of theorem $(\mathbf{3.1})$ are satisfied which implies that the origin is stable, that it is for any positive δ there exists a positive number γ such that any solution of $(\mathbf{3.1})$ which starts in $\mathbf{B}_\gamma$ remains in $\mathbf{B}_\delta$ for all integer $\mathbf{n}$.

Let $\mathbf{A}_{\mathbf{G}^*}$ be the domain of attractivity relative to $\mathbf{G}^*$. We choose $\delta > \mathbf{0}$, such that $\overline{\mathbf{B}_\delta} \cap \mathbf{G}^* \subset \mathbf{A}_{\mathbf{G}^*}$. To show the attractivity of the origin, we shall prove that $\mathbf{B}_\gamma$ is contained in the domain of attractivity, i.e,

$$\forall\, x \in B_\gamma \ : \ \lim_{k \to +\infty} g^k(x) = 0 \tag{3.5}$$

Let $x_0 \in B_\gamma$ and let ϵ be any positive real number. Thanks to the stability of the origin there exists $\eta > \mathbf{0}$ such that

$$g^n(B_\eta) \subset B_\epsilon \quad , \quad \forall\, n \in \mathbb{N} \tag{3.6}$$

Since

$$\overline{B_\delta} \cap G^* \subset A_{G^*}$$

there exists $\mathbf{N} \in \mathbb{N}$ such that

$$\parallel g^n(y) \parallel < \frac{\eta}{2} \quad , \quad \forall\, n \geq N \quad , \quad \forall y \in \overline{B_\delta} \cap G^* \tag{3.7}$$

The continuity of the solutions ensures the existence of $\alpha > \mathbf{0}$ such that

$$\forall (x, y) \in \overline{B_\delta} \times \overline{B_\delta} \quad , \quad \parallel x - y \parallel < \alpha \Rightarrow \parallel g^n(x) - g^n(y) \parallel < \frac{\eta}{2} \quad , \quad \forall\, n \leq \mathbb{N} \tag{3.8}$$

Now, let $\mathbf{y}$ be an element of $\mathbf{L^+(x_0)}$. According to LaSalle Invariance Principle, $\mathbf{y}$ belongs to $\overline{\mathbf{B}_\delta} \cap \mathbf{G}^*$ so by $(\mathbf{3.7})$

$$\| g^n(y) \| < \frac{\eta}{2} \quad , \quad \forall\, n \geq \mathbb{N} \tag{3.9}$$

On the other hand $\mathbf{y} \in \mathbf{L^+(x_0)}$ hence

$$\exists\, p \;\in \mathbb{N} \;: \| \, g^p(x_0) - y \, \| < \alpha \tag{3.10}$$

Using $(\mathbf{3.10})$, $(\mathbf{3.8})$ and $(\mathbf{3.9})$ we get

$$\| \, g^{N+p}(x_0) \, \| < \frac{\eta}{2} + \frac{\eta}{2} = \eta \tag{3.11}$$

and from $(\mathbf{3.6})$, it follows

$$\| \, g^n(g^{N+p}(x_0)) \, \| < \epsilon \quad , \quad \forall\, n \in \mathbb{N}$$

Wich proves that $\lim_{k \to +\infty} g^k(x_0) = 0$. So we have shown that there are exists a positive real number η such that $\mathbf{B}_\eta$ is contained in the domain of attractivity, and thus Theorem $(\mathbf{3.1})$ is established.
New suppose that the system $(\mathbf{3.1})$ is defined on $\mathbb{R}^n$ and there exists a nonnegative function $V \in C^0(\mathbb{R}^n, \mathbb{R}^+)$ which is Lyapunov function for $(\mathbf{3.1})$ that is $\Delta V(x) = V(g(x)) - V(x) \leq 0$ for all $x \in \mathbb{R}^n$. As above $\mathbf{G}^*$ denotes the largest invariant set contained in $G = \{x \in \mathbb{R}^n : \Delta V(x) = V(g(x)) - V(x) = 0\}$. One can ask the following: Does the global asymptotic stability of the system reduced to the invariant set $\mathbf{G}^*$ imply the global asymptotic stability of system $(\mathbf{3.1})$.

3.5 An universal controller function approach

In this section, we state the main result of this contribution, where the proof needs the construction of a nonnegative semi-definite Lyapunov function for the controlled system $(\mathbf{3.4})$.

Theorem 3.4

Considered the map $(\mathbf{3.1})$ and assume that for all $\mu \in \mathbf{\Omega}_\mu \subset \mathbb{R}^{\mathbf{j}}$, all its solutions are bounded.
Then, for any continuous function $h : \mathbb{R}^n \to \mathbb{R}$ verifying :

$$|h(x)| < 1, \;\; \forall\, x \in \mathbb{R}^n \tag{3.12}$$

the universal controller function $(\mathbf{3.3})$ makes the map $(\mathbf{3.1})$ globally asymptotically stable.

Proof

The idea of our proof is based on the construction of a nonnegative semi-definite Lyapunov function $V(x)$ for which the origin is glpbally asymptotically stable as shown in $[\mathbf{11}]$, that is :

1) $V(x) \in C^0(\mathbb{R}^n, \mathbb{R}^+)$,and $V(x) \geq 0$ for all $x \in \mathbb{R}^n - \{(0, 0, \ldots, 0)\}$, and $\mathbf{V(0, 0, \ldots, 0) = 0}$.

2) $\Delta V(x) = V(g(x)) - V(x) \leq 0$, for all $x \in \mathbb{R}^n - \{(0, 0, \ldots, 0)\}$, and $\mathbf{\Delta V(0, 0, \ldots, 0) = 0}$.

3) the origin $(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$ is $\mathbf{G}^*$ globally asymptotically stable where $\mathbf{G}^*$ is the largest positively invariant set contained in $G = \{x \in \mathbb{R}^n \ : \ V(g(x)) - V(x) = 0\}$.

4) All the solutions of the controlled system $(\mathbf{3.4})$ are bounded for all $\mu \in \mathbf{\Omega}_\mu$.

This result show that the global asymptotic stability of the controlled system $(\mathbf{3.5})$ reduced to the invariant set $\mathbf{G}^*$ imply its global asymptotic stability in $\mathbb{R}^n$, wich is a consequence of LaSalle Invariance Principle $[\mathbf{12}]$.

Let us consider the folowing scalar function :

$$V(x) = (x_k^i)^2 \tag{3.13}$$

And the universal controller function given by :

$$g(x_k, \mu) = \begin{pmatrix} x_k^i f_1(x_k, \mu) \\ x_k^i f_2(x_k, \mu) \\ \cdots \\ x_k^i h(x_k) \\ \cdots \\ x_k^i f_n(x_k, \mu) \end{pmatrix} \tag{3.14}$$

Assume that system $(\mathbf{3.1})$ is defined on $\mathbb{R}^n$ and possesses $\mathbf{K}$ continous first integrals $\mathbf{h}$ that are defined on $\mathbb{R}^n$ and denoted by $\mathbf{h}$ the continous function :

$$h : \mathbb{R}^n \to \mathbb{R}$$

Verifying $|\mathbf{h}(\mathbf{x})| < \mathbf{1}$ for all $\mathbf{x} \in \mathbb{R}^\mathbf{n}$.

- Now suppose that there exists a nonnegative function $V \in C^0(\mathbb{R}^n, \mathbb{R}^+)$ which is a Lyapunov function for $(\mathbf{3.1})$.

On has

$$\Delta V(x) = V(g(x)) - V(x) = (x_k^i)^2 h^2(x_k^i) - (x_k^i)^2 = (h^2(x_k^i) - 1)(x_k^i)^2$$

We have $V(x) = (x_k^i)^2 \geq 0$ for all $x \in -\{(0, 0, \ldots, 0)\}$ and $\mathbf{V(0, 0, \ldots, 0) = 0}$, then $\mathbf{V}$ is a nonnegative semi-definite Lyapunov function on $\mathbb{R}^n$. $(\mathbf{1verify})$

so $\quad h^2(x_k^i) - 1 \leq 0 \quad$ (because $|h(x)| < 1$)

Thus $\quad \Delta V(x) = V(g(x)) - V(x) \leq 0$ and $\mathbf{\Delta V(0, 0, \ldots, 0) = 0}$.Thus, $\mathbf{\Delta V}$ is semi negative-definite on $\mathbb{R}^n$. $(\mathbf{2verify})$

- In other hand, on suppose that if $\lambda = (x_1, x_2, \cdots, x_i = 0, \cdots, x_n)$ we have :

$$G = \{\lambda \ : \ \lambda \in \mathbb{R}^n\} = \{(x_1, x_2, \cdots, x_i = 0, \cdots, x_n) \in \mathbb{R}^n\}$$

Let $\mathbf{G}^*$ be the largest positively invariant set contained in $\mathbf{G}$. i.e. $(\mathbf{g_c}(\mathbf{G}^*) \subset \mathbf{G}^*)$

For all $\lambda \in \mathbf{G}$ we have $\quad \mathbf{g_c}(\lambda) = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})$. So $\quad (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) \subset \mathbf{G}^*$ thus $\mathbf{G}^* = \{(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})\}$

★ $\forall\, \epsilon > \mathbf{0}\ ,\ \exists\, \delta > \mathbf{0}\ \ :\ \ \mathbf{g_c^+}(\mathbf{B}_\delta \cap \mathbf{G}^*) \subset \mathbf{B}_\epsilon$

$\forall\, \mathbf{x} \in \mathbf{B}_\delta\ \ (\| \mathbf{x} \| \leq \delta)\ \ \Rightarrow \mathbf{g_c^+}(\mathbf{x} \cap \mathbf{G}^*) \Rightarrow \mathbf{g_c^+}(\mathbf{x} \cap \{(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})\}) = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) \subset \mathbf{B}_\epsilon$

The origin $\mathbf{G}^*$ stable.

There exists $\delta > 0$ such that $B_\delta \cap G^* \subset B_\epsilon$

$$\forall\, x\ \in B_\delta \cap G^*\ \ :\ \ \lim_{n \to +\infty} g_c^n(x) = 0$$

★ Let $x\ \in B_\delta \cap G^*$, and let $\epsilon > \mathbf{0}$. There exists $\eta > \mathbf{0}$ such that :

$$g_c^+(B_\eta) \subset B_\epsilon \ \ \forall\, n \in \mathbb{N} \tag{3.15}$$

Since $\quad \mathbf{B}_\delta \cap \mathbf{G}^* \subset \mathbf{B}_\epsilon$

There exists $N \in \mathbb{N}$ such that

$$\| g_c^n(y) \| < \frac{\eta}{2}\quad ,\quad \forall n \geq N\quad ,\quad \forall y \in B_\delta \cap G^*$$

$\exists\, \alpha > \mathbf{0}$ such that

$$\forall (x, y) \in B_\delta \times B_\delta\ ,\ \| x - y \| < \alpha \Rightarrow \| g_c^n(x) - g_c^n(y) \| \leq \frac{\eta}{2}\quad ,\quad \forall\, n \leq N \tag{3.16}$$

Let $\mathbf{y} \in \mathbf{L}^+(\mathbf{x_0})$ hence

$$\| g_c^n(y) \| < \frac{\eta}{2}\quad ,\quad \forall\, n \geq N \tag{3.17}$$

on the other hand $\mathbf{y} \in \mathbf{L}^+(\mathbf{x_0})$ hance

$$\forall p\ \in \mathbb{N}\ \ :\ \| g_c^p(x_0) - y \| < \alpha \tag{3.18}$$

Using $(\mathbf{3.18}), (\mathbf{3.16})$ and $(\mathbf{3.17})$ we get

$$\| g_c^{n+p}(x_0) \| \leq \frac{\eta}{2} + \frac{\eta}{2} = \eta$$

and from $(\mathbf{3.16})$, it follows

$$\| g_c^n(g_c^{n+p}(x_0)) \| < \epsilon\quad ,\quad \forall\, n \in \mathbb{N}$$

Which approve that $\lim_{\mathbf{n}\to+\infty} \mathbf{g_c^n} = \mathbf{0}$. The origin $\mathbf{G}^*$is asymptotically syable.
Since $\mathbf{B}_\delta \cap \mathbf{G}^* = \{(\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0})\}$ for all $\delta > \mathbf{0}$ thus, $\mathbf{0}$ is globally asymptotically stable. $(\mathbf{3verify})$

- All the solution of the controlled system $(\mathbf{3}.\mathbf{4})$ are bounded for all $\mu \in \mathbf{\Omega}_\mu$. $(\mathbf{4verify})$

From $(\mathbf{1})$, $(\mathbf{2})$, $(\mathbf{3})$ and $(\mathbf{4})$ we conclude that the origin $(\mathbf{0}, \mathbf{0}, \cdots, \mathbf{0})$ of the controlled system $(\mathbf{3}.\mathbf{4})$ is globally asymptotically stable.

Example

Let us consider the new rational chaotic map given by :

$$g(x, y, \mu = a) = \begin{pmatrix} x(\dfrac{-a}{1+y^2} + 1) \\ x \end{pmatrix} \tag{3.19}$$

Where $\mu = a \in \mathbb{R}$ is the bifurcation parameter.
We have $\mathbf{f_1}(\mathbf{x}, \mathbf{y}, \mathbf{a}) = (\dfrac{-\mathbf{a}}{\mathbf{1} + \mathbf{y^2}} + \mathbf{1})$ and $f_2 = x$, $i = 1$, for $\mathbf{a} = \mathbf{4}$.
applying the proposed control law $(\mathbf{3}.\mathbf{3})$ with $h(x, y) = \dfrac{1}{2}$ to drive map $(\mathbf{3}.\mathbf{19})$ globally asymptotically to its fixed point $(\mathbf{0}.\mathbf{0})$.

Thus, the controlled map given by :

$$g_c(x, a) = \begin{pmatrix} \dfrac{x}{2} \\ x \end{pmatrix} \tag{3.20}$$

Let $V(x, y) = x$ and $G_0 = \{(x, y) \in \mathbb{R}^2 \;:\; V(x, y) = 0\} = G$ and so $\mathbf{G_0} = \mathbf{G}^*$ the origin $\mathbf{0}$ is globally asymptotically stable.

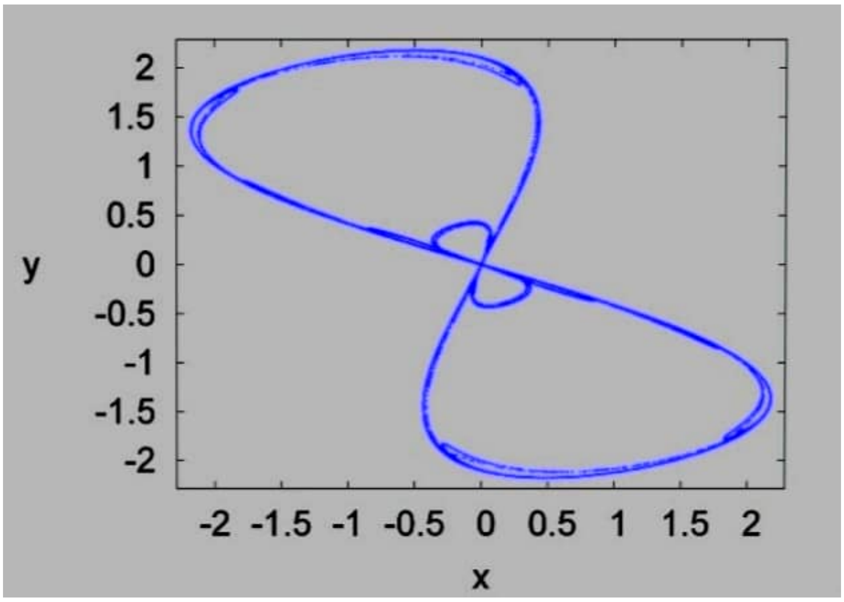

Figure 3.1: Chaotic attractor of the map (3.19) for a=3.5.

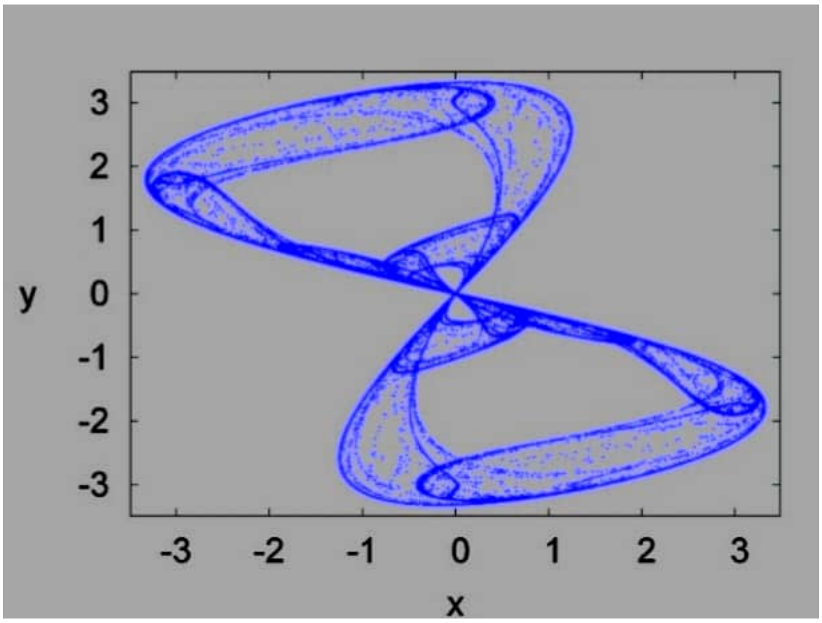

Figure 3.2: Chaotic attractor of the map (3.19) for a=4.

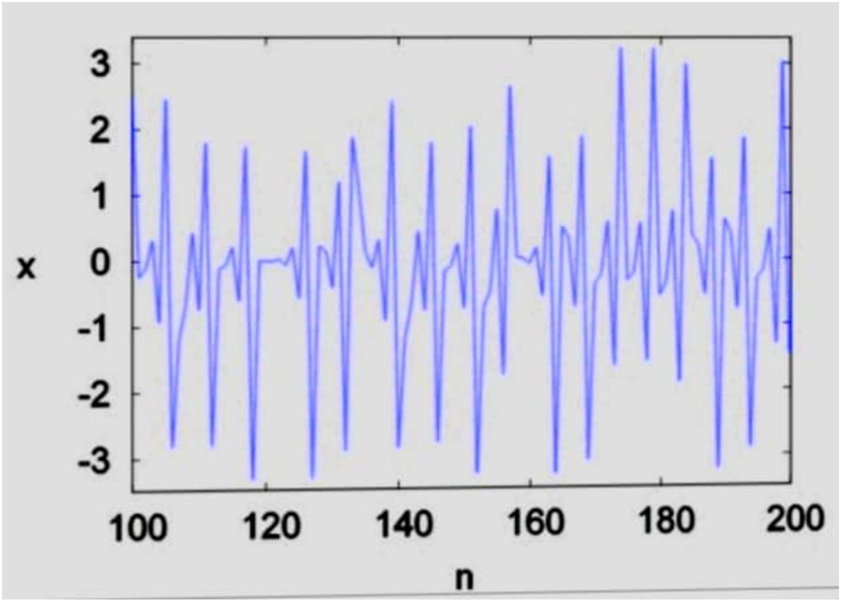

Figure 3.3: Time series of the map (3.19).

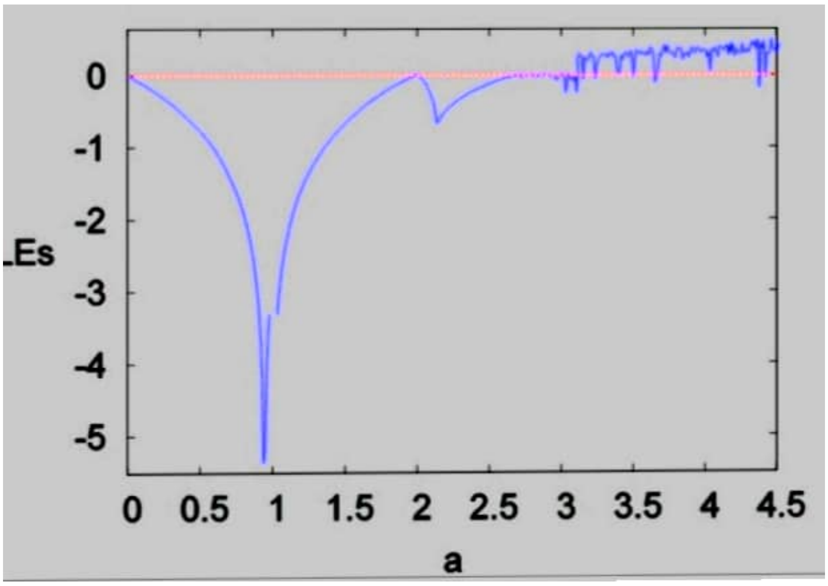

Figure 3.4: Lyapunov exponent of the map (3.19) versus $0 \leq a \leq 4.5$.

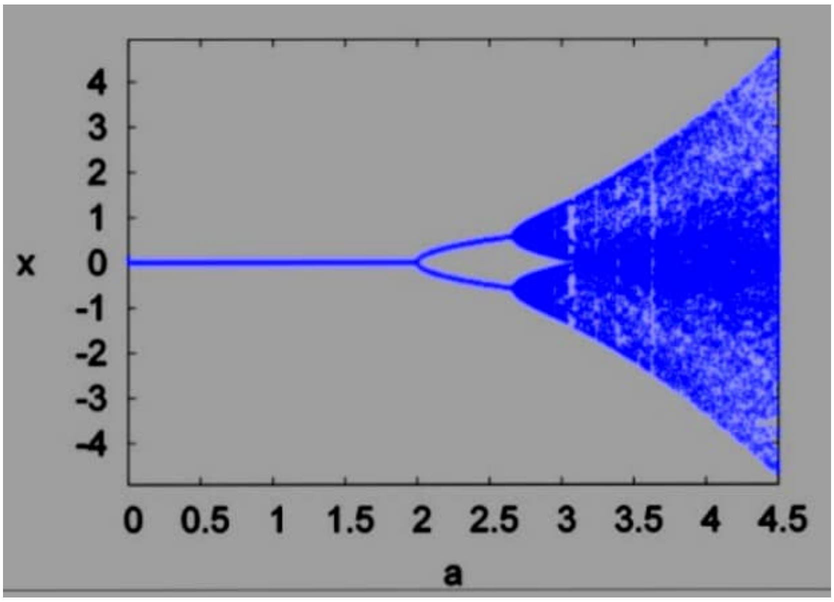

Figure 3.5: Bifurcation diagram of the map (3.19) versus $0 \leq a \leq 4.5$.

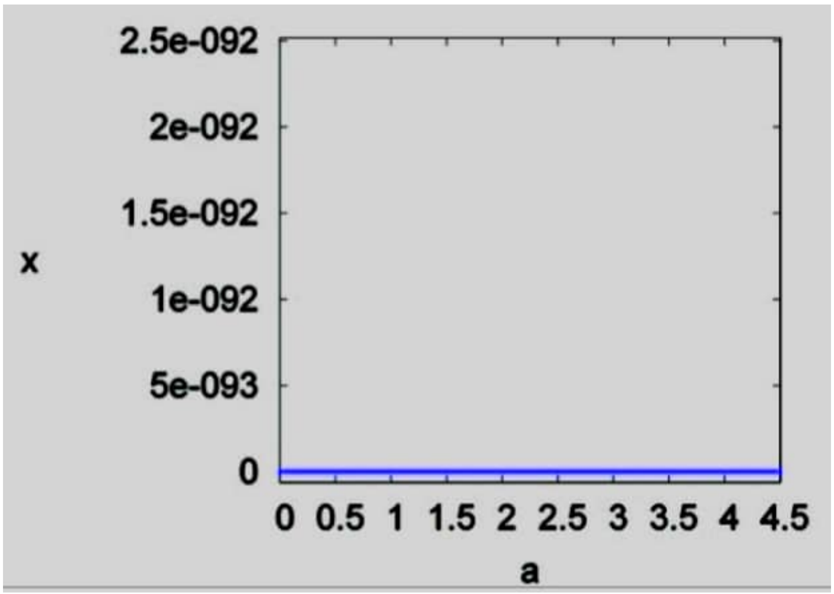

Figure 3.6: Bifurcation diagram of the map (3.20) shows the map (3.20) is globally asymptotically stable to its fixed point (0,0).

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Conclusion

In this distrcation , we have described the underlying theorem which enables us to eliminate any bounded behavior (chaotic or not) in some froms of continuous $\mathbf{n}-\mathbf{D}$ discrete maps and drives it globally asymptotically to their fixed points, vio the controller of a universal function. An elementary example for stabilizing a new $\mathbf{2}-\mathbf{D}$ rational chaotic map is also given and discussed.

الملخص

في هذه المذكرة سنتطرق الى دراسة استقرار بعض الانظمة الديناميكية (الفوضوية او غير فوضوية) وذلك باستخدام دالة ليابونوف، انها اداة لا غنى عنها في تصميم التحليل و التحكم للانظمة غير الخطية، الدافع الاساسي لهذه المساهمة هو القضاء على اي سلوك محدود (فوضوي او لا) في بعض التطبيقات المنفصلة $n-D$ تم تنفيد هذا العمل في اطار اهتمامات هذه المذكرة.

Résumé

Dans cette mémoire, nous discuterons de la stabilité de certains systèmes dynamiques (chaotiques ou non chaotiques) en utilisant la fonction de Lyapunov, c'est un outil indispensable dans l'analyse de la conception et la controle des systèmes non linéaires, la principale motivation de cette contribution est l'élimination du comportement fini (chaotique ou non) dans certaines formes de cartographie discrète $\mathbf{n}-\mathbf{D}$. Ce travail a été réalisé dans le cadre des intérets de ce mémorandum "Lyapunov function and global asymptotic stability".

Abstract

In this memorandum, we will discuss the stability of some dynamical systems (chaotic or ont chaotic) using the Lyapunov function it is an indispensable tool in the desing analysis and control of non linear systems, the primary motivation for this contribution is the elimination of finite behavior (chaotic or not) in some forms of $\mathbf{n}-\mathbf{D}$ discrete map. This work was canied ont within the frame work of the mterests of this memorandum "Lyapunov function and global asymptotic stability".