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Asymptotic modeling of shells with von Kármán boundary conditions

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Dedication

to my:

- mother

- father

-friends

*- all familly *Legougui**

- Our colleagues at department of mathematatique University Kasdi Merbah of Ouargla

I didecated this work.

Marwa

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Contents

Notations and conventions

Conventions

1. Latin indices and exponents: $i, j, p, ...$, take their values in the set $\{1, 2, 3\}$, unless otherwise indicated as when they are used for indexing sequences.

2. Greek indices and exponents: $\alpha, \beta, \sigma, ...$, except ε and ν in the outer normal derivative operator ∂_ν, take their values in the set $\{\alpha, \beta\}$

3. The repeated index summation convention is systematically used in conjunction with conventions 1 and 2.

4. The symbol ε designates a parameter that is > 0 and approaches zero.

Notations

$\mathbf{a}.\mathbf{b}$: Euclidean inner product of $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$.

$\mathbf{a} \wedge \mathbf{b}$: Exterior product of $\mathbf{a} \in \mathbb{R}^3$ and $\mathbf{b} \in \mathbb{R}^3$.

$\mid \mathbf{a} \mid$ Euclidean norm of $\mathbf{a} \in \mathbb{R}^3$.

$\mathbf{E}^3$: denote a three-dimensional Euclidean space.

Ω: domain in $\mathbb{R}^3$ (open, bounded, connected subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary, the set Ω being locally on one side of its boundary).

$x = (x_i)$: generic point in $\bar{\Omega}$.

dx: volume element in Ω.

$\partial_i = \dfrac{\partial}{\partial x_i}$.

Γ: boundary of Ω.

$d\Gamma$: area element along Γ.

(n_i): unit outer normal vector (defined $d\Gamma$-almost everywhere) along Γ.

$\Gamma = \Gamma_0 \cup \Gamma_1$: partition of the boundary of Ω with area $\Gamma_0 > 0$.

ω: middle surface of the shell.

2ε: thickness of the shell.

$\bar{\Omega}^\varepsilon = \bar{\omega} \times [-\varepsilon, \varepsilon]$: reference configuration of a shell.

ω: domain in $\mathbb{R}^2$ (open, boundary, connected subset with a Lipschitz-continuous boundary, th set ω being "locally on one side of its boundary").

γ or $\partial\omega$: boundary of the set ω.

$d\gamma$: length element along γ.

γ_0: measurable subset of γ with length $\gamma_0 > 0$.

γ_1: measurable subset of γ with length $\gamma_1 > 0$.

$y = (x_\alpha) = (x_1, x_2)$: generic point in the set $\bar{\omega}$, sometimes also denoted y.

$\partial_\alpha = \dfrac{\partial}{\partial x_\alpha}$, $\partial_{\alpha\beta} = \dfrac{\partial^2}{\partial_\alpha \partial x_\beta}$.

$\Omega = \omega \times]-1, 1[$.

$\gamma \times [-1, 1]$: lateral face of the set $\bar{\Omega}$.

$\Gamma_0 = \gamma_0 \times [-1, 1]$.

$\Gamma_1 = \gamma_1 \times [-1, 1]$.

$\Gamma_+ = \omega \times \{1\}$: upper face of the set $\bar{\Omega}$.

$\Gamma_- = \omega \times \{-1\}$: lower face of the set $\bar{\Omega}$.

$\gamma \times [-\varepsilon, \varepsilon]$: lateral face of the set $\bar{\Omega}^\varepsilon$.

$\Gamma_0^\varepsilon = \gamma_0 \times [-\varepsilon, \varepsilon]$: portion of the lateral face where a shell is clamped.

$\Gamma_+^\varepsilon = \omega \times \{\varepsilon\}$: upper face of the set $\bar{\Omega}^\varepsilon$.

$\Gamma_-^\varepsilon = \omega \times \{-\varepsilon\}$: lower face of the set $\bar{\Omega}^\varepsilon$.

$x^\varepsilon = (x_i^\varepsilon) = (x_1, x_2, x_3^\varepsilon) = (y, x_3^\varepsilon)$: generic point in the set $\bar{\Omega}^\varepsilon$.

$\partial_i^\varepsilon = \dfrac{\partial}{\partial x_i^\varepsilon}$.

π^ε: bijection from $\bar{\Omega}$ onto $\bar{\Omega}^\varepsilon$, defined by $\pi^\varepsilon(x_1, x_2, x_3) = (x_1, x_2, \varepsilon x_3)$.

$\Delta = \partial_{\alpha\alpha}$: Laplacian.

$\Delta^2 = \Delta\Delta = \partial_{\alpha\alpha}\partial_{\beta\beta}$: biharmonic operator.

(ν_α): unit outer normal vector along γ.

(τ_α) with $\tau_1 = -\nu_2$, $\tau_2 = \nu_1$: unit tangent vector along γ.

$\partial_\nu \theta = \nu_\alpha \partial_\alpha \theta$: outer normal derivative of θ along γ.

$\partial_\tau \theta = \tau_\alpha \partial_\alpha \theta$: tangential derivative of θ along γ.

$\rightharpoonup$: weak convergence.

$\rightarrow$: strong convergence.

Definitions

$W^{s,p}(.)$, $(s \in \mathbb{R},\ p \geq 1)$: usual Sobolev space.

$\|\ \|_{s,p,.}$: norm in $W^{s,p}(.)$.

$|\ |_{s,p,.}$: semi-norm in $W^{s,p}(.)$, $(s \in \mathbb{N})$.

$H^s(.) = W^{s,2}(.)$, $\|\ \|_{s,.} = \|\ \|_{s,2,.}$ and $|\ |_{s,.} = |\ |_{s,2,.}$.

Introduction

The advancement in science and technology have brought forward many mathematical models. Among these models involving structural mechanics. Most applications of these structures have been made to plates and shells. The mathematical formulation of these models leads to a system of partial differential equations and a set of boundary conditions with a complicated geometrical shape like that of many shells. The most important among them is the von Kármán equations, which are two-dimensional model for a nonlinearly elastic plate subjected to boundary conditions of von Kármán's type. They were initially proposed by von Kármán [1], which is originating from continuum mechanics and play an important role in applied mathematics. Next, these equations are extended to Marguerre–von Kármán equations for a nonlinearly elastic shallow shell by Marguerre [2] and von Kármán and Tsien [3].

The asymptotic methods can be used for justifying the two-dimensional models of elastic plates and shells starting from the three-dimensional models. More details about von Kármán and Marguerre–von Kármán theories, can be found in, e.g., [4]–[22] and the historical references therein. In addition, we refer to the works are due to Ghezal and the others [23]–[28] for Marguerre–von Kármán shallow shells, [29], and [30] for linear shallow shells. In this direction, numerous works have been devoted to shell theory, see, for example, [31] and the references cited therein. Especially, we refer to [32]–[39] about nonlinearly elastic shells.

Limited studies for von Kármán shells theory based on the minimization of the energy, were done in the past few years. This theory was derived by Lewicka, Mora, and Pakzad in [40] and [41], using Γ-convergence. Then Hornung and Velčić derived the homogenized von Kármán shell theory in [42]. We refer to Li and Chermisi [43] for von Kármán theory of incompressible shells, Roychowdhury and Gupta [44] for Föppl–von Kármán shells. In the same direction, the time-dependent von Kármán shells equation recently obtained by Qin and Yao [45].

In the first chapter, we review the basic notions, such as the metric tensor and covariant derivatives, arising when a three-dimensional open set is equipped with curvilinear coordinates. Next, we prove that the vanishing of the Riemann curvature tensor is sufficient for the existence of isometric immersions from a simply-connected open subset of $\mathbb{R}^n$ equipped with a Riemannian metric into a Euclidean space of the same dimension. We then study basic notions about surfaces, such as their two fundamental forms, the Gaussian curvature and covariant derivatives.

In the second chapter, we give a detailed account of recent justifications of nonlinear shell theories that are also based on an asymptotic analysis of the three-dimensional solution with the thickness as the "small" parameter. A remarkable progress in the asymptotic analysis of nonlinearly elastic shells is due to Miara [32], Miara and Lods [35], Ciarlet [31], who justified the two-dimensional equations of a nonlinearly elastic "membrane" shells and "flexural" shells, by means of the method of formal asymptotic expansions applied to the three-dimensional equations of a nonlinearly elastic shell modeled by a St Venant-Kirchhoff material. Another remarkable progress is due to Le Dret and Raoult [34], who gave the first proof of convergence of the three-dimensional solutions to a two-dimensional one as the thickness approaches zero. The purpose of this chapter is to lay the preliminary grounds for the formal approach.

In the third chapter, we give the asymptotic justification of the two-dimensional equations for membrane shells with boundary conditions of von Kármán's type. More precisely, we consider a three-dimensional model for a nonlinearly elastic membrane shell of Saint Venant–Kirchhoff material, where only a portion of the lateral face is subjected to boundary conditions of von Kármán's type. Using technics from formal asymptotic analysis with the thickness of the shell as a small parameter, we show that the scaled three-dimensional solution still leads to the two-dimensional equations of von Kármán membrane shell. This work was published in [46].

In the fourth chapter, we give the asymptotic justification of the two-dimensional equations of von Kármán flexural shell. Also, we prove an existence theorem for the minimization problem.

Realized works

Publication:

- M. Legougui and A. Ghezal, "Asymptotic justification of equations for von Kármán membrane shells," Mathematical Notes **114**, 536–552 (2023).
  DOI: 10.1134/S0001434623090237.

- M. Legougui and A. Ghezal, "Asymptotic modelling of viscoelastic von Kármán membrane shells," Analysis (2023).
  https://doi.org/10.1515/anly-2022-1106.

International conferences:

- M. Legougui and A. Ghezal, Asymptotic modeling of shells with von Kármán boundary conditions, 3^{me} colloque international sur la théorie des opérateurs, Université Echahid Hamma Lakhdar d'El Oued, Algeria, 24-25 Avril, 2019.

- M. Legougui and A. Ghezal, Justification of the two-dimensional equations of von Kármán shells, The First Online International Conference on Pure and Applied Mathematics, Ouargla University, Algeria, May 26-27, 2021.

- M. Legougui and A. Ghezal, A Justification of two-dimensional equations of a nonlinear elastic shells (with von Kármán boundary conditions), IEEE International Conference on Recent Advances in Mathematics and Informatics, Tebessa University, Algeria, September 21-22, 2021.

Chapter 1

DIFFERENTIAL GEOMETRY OF SHELLS

In this Chapter, let us briefly recall some properties of the three-dimensional differential geometry and differential geometry of surfaces, due to Ciarlet [54], which will be used here.
We begin by reviewing basic definitions and properties arising when the three-dimensional open subset $\Theta(\Omega)$ of E^3 is equipped with the coordinates of the points of Ω as its curvilinear coordinates. Of fundamental importance is the metric tensor of the set $\Theta(\Omega)$, whose covariant and contravariant components. It is shown in particular how volumes, areas, and lengths, in the set $\Theta(\Omega)$ are computed in terms of its curvilinear coordinates, by means of the functions g_{ij} and g. Covariant derivatives constitute a generalization of the usual partial derivatives of vector fields defined by means of their Cartesian components.

Consider ω is a two-dimensional open set in R^2. Then by contrast, such a two-dimensional manifold equipped with the coordinates of the points of ω as its curvilinear coordinates, requires two tensor fields for its definition (this time up to proper isometries of E^3), the first and second fundamental forms of $\hat{\omega}$. In particular, it is shown how areas and lengths, i.e., âmetric notionsâ, on the surface $\hat{\omega}$ are computed in terms of its curvilinear coordinates by means of the components $a_{\alpha\beta}$ of the first fundamental form. It is also shown how the curvature of a curve on $\hat{\omega}$ can be similarly computed, this time by means of the components of both fundamental forms.

1.1 THREE-DIMENSIONAL DIFFERENTIAL GEOMETRY

1.1.1 CURVILINEAR COORDINATES

Let there be given an open subset $\hat{\Omega}$ of $\mathbf{E}^3$ and assume that there exist an open subset $\hat{\Omega}$ of $\mathbb{R}^3$ and an injective mapping $\Theta : \Omega \to \mathbf{E}^3$ such that $\Theta(\Omega) = \hat{\Omega}$.

Then each point $\hat{x} \in \hat{\Omega}$ can be unambiguously written as

$$\hat{x} = \Theta(x), x \in \Omega,$$

and the three coordinates x_i of x are called the curvilinear coordinates of $\hat{x}$ (e.i., cylindrical coordinates and spherical coordinates).

The three coordinates x_1, x_2, x_3 of $x \in \Omega$ are the curvilinear coordinates curvilignes of $\widehat{x} = \Theta(x) \in \widehat{\Omega}$.

If the three vecteors $g_i(x) = \partial_i \Theta(x)$ are linearly independants, they form the covariant basis at $\widehat{x} = \Theta(x)$ and they are tangent to the coordinate lines passing through $\widehat{x}$.

Example 1.1 (Cylindrical coordinates)

$$\Theta : (\varphi, \rho, z) \in \Omega \longrightarrow (\rho\cos(\varphi), \rho\sin(\varphi), z) \in \mathbb{E}^3,$$

(φ, ρ, z) are the cylindrical coordinates of $\widehat{x} = \Theta(\varphi, \rho, z)$.

Example 1.2 (Spherical coordinates)

$$\Theta : (\varphi, \psi, r) \in \Omega \longrightarrow (r\cos(\psi)\cos(\varphi), r\cos(\psi)\sin(\varphi), r\sin(\psi)) \in \mathbb{E}^3,$$

(φ, ψ, r) are the spherical coordinates of $\widehat{x} = \Theta(\varphi, \psi, r)$.

1.1.2 METRIC TENSOR

Let Ω be an open subset of $\mathbb{R}^3$ and let

$$\Theta = \Theta_i \hat{e}^i : \Omega \to \mathbf{E}^3$$

be a mapping that is differentiable at a point $x \in \Omega$. If δx is such that $(x + \delta x) \in \Omega$, then

$$\Theta(x + \delta x) = \Theta(x) + \nabla\Theta(x)\delta x + o(\delta x), \tag{1.1}$$

where the 3×3 matrix $\nabla\Theta(x)$ and the column vector δx are defined by

$$\nabla\Theta(x) = \begin{pmatrix} \partial_1\Theta_1 & \partial_2\Theta_1 & \partial_3\Theta_1 \\ \partial_1\Theta_2 & \partial_2\Theta_2 & \partial_3\Theta_2 \\ \partial_1\Theta_3 & \partial_2\Theta_3 & \partial_3\Theta_3 \end{pmatrix}(x) \text{ and } \delta x = \begin{pmatrix} \delta x_1 \\ \delta x_2 \\ \delta x_3 \end{pmatrix}.$$

Let the three vectors $g_i(x) \in \mathbb{R}^3$ be defined by

$$g_i(x) = \partial_i\Theta(x) = \begin{pmatrix} \partial_i\Theta_1 \\ \partial_i\Theta_2 \\ \partial_i\Theta_3 \end{pmatrix}(x),$$

i.e., $g_i(x)$ is the i-th column vector of the matrix $\nabla\Theta(x)$. Then the expansion of Θ about x may be also written as

$$\Theta(x + \delta x) = \Theta(x) + \delta x^i g_i(x) + o(\delta x). \tag{1.2}$$

If in particular δx is of the form $\delta x = \delta t e_i$ in (1.2), where $\delta t \in \mathbb{R}$ and e_i is one of the basis vectors in $\mathbb{R}^3$, this relation reduces to

$$\Theta(x + \delta t e_i) = \Theta(x) + \delta t g_i(x) + o(\delta x). \tag{1.3}$$

Definition 1.1 *A mapping $\Theta : \Omega \to \mathbf{E}^3$ is an immersion at $x \in \Omega$ if it is differentiable at x and the matrix $\nabla\Theta(x)$ is invertible or, equivalently, if the three vectors $g_i(x) = \partial_i\Theta(x)$ are linearly independent.*

Assume from now on in this section that the mapping Θ is an immersion at x. Then the three vectors $g_i(x)$ constitute the covariant basis at the point $\hat{x} = \Theta(x)$.

In this case, the relation (1.3) thus shows that each vector $g_i(x)$ is tangent to the i-th coordinate line passing through $\hat{x} = \Theta(x)$, defined as the image by Θ of the points of Ω that lie on the line parallel to e_i passing through x.

There exist t_0 and t_1 with $t_0 < 0 < t_1$ such that the i-th coordinate line is given by

$$t \in]t_0, t_1[\to f_i(t) = \Theta(x + te_i)$$

in a neighborhood of x, hence $f_i'(0) = \partial_i\Theta(x) = g_i(x)$, since $\delta x = \delta x^i e_i$ of (1.2), we obtain

$$\begin{aligned} |\Theta(x+\delta x) - \Theta(x)|^2 &= \delta x^i g_i(x).g_j(x)\delta x^j + o(|\delta x|^2) \\ &= \delta x^T \nabla\Theta(x)^T.\nabla\Theta(x)\delta x + o(|\delta x|^2). \end{aligned}$$

In other words, the principal part with respect to δx of the length between the points $\Theta(x+\delta x)$ and $\Theta(x)$ is $\{\delta x^i g_i(x)\cdot g_j(x)\delta x^j\}^{1/2}$. This observation suggests to define a matrix $(g_{ij}(x))$ of order three, by letting

$$g_{ij}(x) = g_i(x).g_j(x) = (\nabla\Theta(x)^T\nabla\Theta(x))_{ij}. \tag{1.4}$$

The elements $g_{ij}(x)$ of this symmetric matrix are called the covariant components of the metric tensor at $\hat{x} = \Theta(x)$.

Note that the matrix $\nabla\Theta(x)$ is invertible and that the matrix $(g_{ij}(x))$ is positive definite, since the vectors $g_i(x)$ are assumed to be linearly independent.

The three vectors $g_i(x)$ being linearly independent, the nine relations

$$g^i(x)g_j(x) = \delta^i_j. \tag{1.5}$$

unambiguously define three linearly independent vectors $g_i(x)$. To see this, let a priori $g_i(x) = X^{ik}(x)g_k(x)$ in the relations $g^i(x)\cdot g_j(x) = \delta^i_j$. This gives $X^{ik}(x)g_{kj}(x) = \delta^i_j$; consequently, $X^{ik}(x) = g^{ik}(x)$, where

$$(g^{ij}(x)) = (g_{ij}(x))^{-1}.$$

Hence $g^i(x) = g^{ik}(x)g^k(x)$. These relations in turn imply that

$$g^i(x)\cdot g^j(x) = g^{ik}(x)g_k(x))\cdot(g^{jl}(x)g_l(x)) = g^{ik}(x)g^{jl}(x)g_{kl}(x) = g^{ik}(x)\delta^j_k = g^{ij}(x),$$

and thus the vectors $g^i(x)$ are linearly independent since the matrix $(g^{ij}(x))$ is positive definite. We would likewise establish that $g_i(x) = g_{ij}(x)g^j(x)$.

The three vectors $g^i(x)$ form the contravariant basis at the point $\hat{x} = \Theta(x)$ and the elements $g^{ij}(x)$ of the symmetric positive definite matrix $(g^{ij}(x))$ are the contravariant components of the metric tensor at $\hat{x} = \Theta(x)$.

Let us record for convenience the fundamental relations that exist between the vectors of the covariant and contravariant bases and the covariant and contravariant components of the metric tensor at a point $x \in \Omega$ where the mapping Θ is an immersion:

$$g_{ij}(x) = g_i(x) \cdot g_j(x) \text{ and } g^{ij}(x) = g^i(x) \cdot g^j(x),$$
$$g_i(x) = g_{ij}(x) g^j(x) \text{ and } g^i(x) = g^{ij}(x) g_j(x).$$

Definition 1.2 *A mapping $\Theta : \Omega \to \mathbf{E}^3$ is an immersion if it is an immersion at each point in Ω, i.e., if Θ is differentiable in Ω and the three vectors $g_i(x) = \partial_i \Theta(x)$ are linearly independent at each $x \in \Omega$.*

1.1.3 VOLUMES, AREAS, AND LENGTHS IN CURVILINEAR COORDINATES

Theorem 1.1 *Let Ω be an open subset of $\mathbb{R}^3$, let $\Theta : \Omega \to \mathbf{E}^3$ be an injective and smooth enough immersion, and let $\hat{\Omega} = \Theta(\Omega)$.*

(a) The volume element $d\hat{x}$ at $\hat{x} = \Theta(x) \in \hat{\Omega}$ is given in terms of the volume element dx at $x \in \Omega by$

$$d\hat{x} = |\det \nabla\Theta(x)| dx = \sqrt{g(x)} dx, \text{ where } g(x) = \det(g_{ij}(x)). \tag{1.6}$$

(b) Let D be a domain in $\mathbb{R}^3$ such that $\bar{D} \subset \Omega$. The area element $d\hat{\Gamma}(\hat{x})$ at $\hat{x} = \Theta(x) \in \partial\hat{D}$ is given in terms of the area element $d\Gamma(x)$ at $x \in \partial D$ by

$$d\hat{\Gamma}(\hat{x}) = |\mathbf{Cof} \nabla\Theta(x) n(x)| d\Gamma(x) = \sqrt{g(x)} \sqrt{n_i(x) g^{ij}(x) n_j(x)} d\Gamma(x), \tag{1.7}$$

where $n(x) = n_i(x) e^i$ denotes the unit outer normal vector at $x \in \partial D$.

(c) The length element $d\hat{l}(\hat{x})$ at $\hat{x} = \Theta(x) \in \Omega$ is given by

$$d\hat{l}(\hat{x}) = \{\delta x^T \nabla\Theta(x)^T \nabla\Theta(x) \delta x\}^{1/2} = \{\delta x^i g_{ij}(x) \delta x^j\}^{1/2}, \tag{1.8}$$

where $\delta x = \delta x^i e_i$.

Proof.

(a) Since $\widehat{x} = \Theta(x)$, then

$$d\widehat{x} = |\det \nabla\Theta(x)| dx.$$

follows

$$g(x) = \det(g_{ij}(x)) = \det(\nabla\Theta(x)^T \nabla\Theta(x)) = |\det(\nabla\Theta(x)|^2.$$

Hence

$$|\det \nabla\Theta(x)| = \sqrt{g(x)}.$$

(b) According to Theoreme 1.7-1 in [54] that

$$d\widehat{\Gamma}(\widehat{x}) = |cof \nabla\Theta(x) n(x)| d\Gamma(x),$$

hence

$$|cof \nabla\Theta(x) n(x)|^2 = n(x)^T (cof \nabla\Theta(x))^T cof \nabla\Theta(x) n(x).$$

Using the relations

$$(cof A)^T = cof A^T \text{ and } cof(AB) = (cof A)(cof B),$$

we next have

$$|cof \nabla\Theta(x) n(x)|^2 = n(x)^T cof(\nabla\Theta(x)^T \nabla\Theta(x)) n(x) = g(x) n_i(x) g^{ij}(x) n_j(x).$$

(c) Recalls that $d\hat{l}(\hat{x})$ is by definition the principal part with respect $\delta x = \delta x^i e_i$ of the length $\Theta(x + \delta x)$ and $\Theta(x)$. For more detailed, we refer to Theorem 1.3-1 in Ciarlet [54].

■

Remark 1.1 *The relations found in Theorem 2.16 are used in particular for computing volumes, areas, and lengths inside $\hat{\Omega}$ by means of integrals inside $\hat{\Omega}$, i.e., in terms of the curvilinear coordinates used in the open set $\hat{\Omega}$:*

Let D be a domain in $\mathbb{R}^3$ such that $\bar{D} \subset \Omega$ let $\hat{D} = \Theta(D)$, and let $\hat{f} \in L^1(\hat{D})$ be given. Then

$$\int_{\hat{D}} \hat{f}(\hat{x}) d\hat{x} = \int_{D} (\hat{f} \circ \Theta)(x) \sqrt{g(x)} dx.$$

In particular, the volume of $\hat{D}$ is given by

$$vol\ \hat{D} = \int_{\hat{D}} d\hat{x} = \int_{D} \sqrt{g(x)} dx.$$

Next, let $\Gamma = \partial D$, let Σ be a $d\Gamma$-measurable subset of Γ, let $\hat{\Sigma} = \Theta(\Sigma) \subset \partial\hat{D}$, and let $\hat{h} \in L^1(\hat{\Sigma})$ be given. Then

$$\int_{\hat{\Sigma}} \hat{h}(\hat{x}) d\hat{\Gamma}(\hat{x}) = \int_{\Sigma} (\hat{h} \circ \Theta)(x) \sqrt{g(x)} \sqrt{n_i(x) g^{ij}(x) n_j(x)} d\Gamma(x).$$

In particular, the area of $\hat{\Sigma}$ is given by

$$area\ \hat{\Sigma} = \int_{\hat{\Sigma}} d\hat{\Gamma}(\hat{x}) = \int_{\Sigma} \sqrt{g(x)} \sqrt{n_i(x) g^{ij}(x) n_j(x)} d\Gamma(x)$$

Finally, consider a curve $C = f(I)$ in Ω, where I is a compact interval of $\mathbb{R}$ and $f = f^i e_i : I \to \Omega$ is a smooth enough injective mapping. Then the length of the curve $\hat{C} = \Theta(C) \subset \hat{\Omega}$ is given by

$$length\ \hat{C} = \int_{I} |\frac{d}{dt}(\Theta \circ f)(t)| dt = \int_{I} \sqrt{g_{ij}(f(t)) \frac{df^i}{dt}(t) \frac{df^j}{dt}(t)} dt.$$

This relation shows in particular that the lengths of curves inside the open set $\Theta(\Omega)$ are precisely those induced by the Euclidean metric of the space $\mathbf{E}^3$. For this reason, the set $\Theta(\Omega)$ is said to be isometrically imbedded in $\mathbf{E}^3$.

1.1.4 COVARIANT DERIVATIVES OF A VECTOR FIELD

Suppose that a vector field is defined in an open subset $\hat{\Omega}$ of $\mathbf{E}^3$ by means of its Cartesian components $\hat{v}_i : \hat{\Omega} \to \mathbb{R}$, i.e., this field is defined by its values $\hat{v}_i(\hat{x})\hat{e}^i$ at each $\hat{x} \in \hat{\Omega}$ where the vectors $\hat{e}^i$ constitute the orthonormal basis of $\mathbf{E}^3$.

Suppose now that the open set $\hat{\Omega}$ is equipped with curvilinear coordinates from an open subset Ω of $\mathbb{R}^3$, by means of an injective mapping $\Theta : \Omega \to \mathbf{E}^3$ satisfying $\Theta(\Omega) = \hat{\Omega}$.

It turns out that the proper way to do so consists in defining three functions $v_i : \Omega \to \mathbb{R}$ by requiring that

$$v_i(x)g^i(x) = \hat{v}_i(\hat{x})\hat{e}^i \text{ for all } \hat{x} = \Theta(x), x \in \Omega$$

where the three vectors $g^i(x)$ form the contravariant basis at $\hat{x} = \Theta(x)$.

Using the relations $g^i(x) \cdot g_j(x) = \delta^i_j$ and $\hat{e}^i \cdot \hat{e}_j = \delta^i_j$, we immediately find how the old and new components are related,

$$\begin{aligned} v_j(x) &= v_i(x)g^i(x) \cdot g_j(x) = \hat{v}_i(\hat{x})\hat{e}^i \cdot g_j(x), \\ \hat{v}_i(\hat{x}) &= \hat{v}_j(\hat{x})\hat{e}^j \cdot \hat{e}_i = v_j(x)g^j(x) \cdot \hat{e}_i. \end{aligned}$$

The three components $v_i(x)$ are called the covariant components of the vector $v_i(x)g^i(x)$ at $\hat{x}$, and the three functions $v_i : \Omega \to \mathbb{R}$ defined in this fashion are called the covariant components of the vector field $v_i g^i : \Omega \to \mathbf{E}^3$.

Theorem 1.2 *Let $\Theta : \Omega \to \mathbb{E}^3$ be an immersion injective is also a C^2- diffemorphisme of Ω onto $\widehat{\Omega} = \Theta(\Omega)$. Given a vector field $\widehat{v}_i\widehat{e}^i : \widehat{\Omega} \to \mathbb{R}^3$ with $\widehat{v}_i \in C^1(\widehat{\Omega})$ that defined by:*

$$\widehat{v}_i(\widehat{x})\widehat{e}^i = v_i(x)g^i(x), \forall \widehat{x} = \Theta(x), x \in \Omega.$$

Then $v_i \in C^1(\widehat{\Omega})$ and for all $x \in \Omega$

$$\widehat{\partial}_j\widehat{v}_i(\widehat{x}) = (v_{k\|\ell}[g^k]_i[g^\ell]_j)(x), \forall \widehat{x} = \Theta(x), x \in \Omega,$$

where

$$v_{i\|j} = \partial_j v_i - \Gamma^p_{ij}v_p \;(\textit{ The first order covariant derivatives of the vector field } v_i g^i),$$

$$\Gamma^p_{ij} = g^p.\partial_i g_j \;(\textit{ Christoffel symbols of the second kind }),$$

$$[g^i(x)]_k = g^i(x).\widehat{e}_k \;(\textit{ Denotes the } i-th \textit{ component of } g^i(x) \textit{ over the basis } \{\widehat{e}_1, \widehat{e}_2, \widehat{e}_3\}).$$

Proof.

(i) Let $\Theta(x) = \Theta^k(x)\widehat{e}_k$ and $\widehat{\Theta} : \widehat{\Omega} \longrightarrow \mathbb{R}^3, \widehat{\Theta}(\widehat{x}) = \widehat{\Theta}^i(\widehat{x})e_i$, where $\widehat{\Theta} = \Theta^{-1}$.

Since

$$\widehat{\Theta}(\Theta(x)) = x, \forall x \in \Omega,$$

and

$$\widehat{\nabla}\widehat{\Theta}(\widehat{x})\nabla\Theta(x) = I,$$

where

$$\nabla\Theta(x) = (\partial_j\Theta^k(x)) \text{ (the row index is } k),$$
$$\widehat{\nabla}\widehat{\Theta}(\widehat{x}) = (\widehat{\partial}_k\widehat{\Theta}^i(\widehat{x})) \text{ (the row index is } i).$$

or equivalently

$$\widehat{\partial}_k\widehat{\Theta}^i(\widehat{x})\partial_j\Theta^k(x) = (\widehat{\partial}_1\widehat{\Theta}^i(\widehat{x}) \ \widehat{\partial}_2\widehat{\Theta}^i(\widehat{x}) \ \widehat{\partial}_3\widehat{\Theta}^i(\widehat{x})) \begin{pmatrix} \partial_j\Theta^1(x) \\ \partial_j\Theta^2(x) \\ \partial_j\Theta^3(x) \end{pmatrix} = \delta^i_j.$$

We deduce that

$$\widehat{\partial}_k\widehat{\Theta}^i(\widehat{x}).g_j(x) = \delta^i_j.$$

Since $g^i(x)$ is uniquely defined by $g^i(x).g_j(x) = \delta^i_j$, we obtain

$$[g^i(x)]_k = \widehat{\partial}_k\widehat{\Theta}^i(\widehat{x}).$$

(ii) Since $\Theta \in C^2(\Omega;\mathbb{E}^3)$, then

$$g^q = g^{qr}g_r \in C^1(\Omega), \ \partial_\ell g^q \in C^0(\Omega).$$

Recalling that the vectors $g^k(x)$ form a basis, we may write a priori

$$\partial_\ell g^q(x) = \Gamma^q_{\ell k}(x)g^k(x), \quad \Gamma^q_{\ell k} : \Omega \longrightarrow \mathbb{R}.$$

we observe that

$$\Gamma^q_{\ell k} = \Gamma^q_{\ell m}(x)\delta^m_k = \Gamma^q_{\ell m}g^m(x).g_k(x) = -\partial_\ell g^q(x).g_k(x).$$

Hence, noting that $\partial_\ell(g^q(x).g_k(x)) = 0$, we obtain

$$\Gamma^q_{\ell k}(x) = g^q(x).\partial_\ell g_k(x).$$

and $[g^q(x)]_p = \widehat{\partial}_p\widehat{\Theta}^q(\widehat{x})$, we obtain

$$\Gamma^q_{\ell k}(x) = \widehat{\partial}_p\widehat{\Theta}^q(\widehat{x})\partial_{\ell k}\Theta^p(x) = \Gamma^q_{lk}(x).$$

Since $\Theta \in C^2(\Omega;\mathbb{E}^3)$ and $\widehat{\Theta} \in C^1(\widehat{\Omega};\mathbb{R}^3)$, we deduce that $\Gamma^p_{lk} \in C^0(\Omega)$.

(iii) If $w : \Omega \to \mathbb{R}$ a differentaiable function, satisfies

$$\widehat{\partial}_j w(\widehat{\Theta}(\widehat{x})) = \partial_\ell w(x) \widehat{\partial}_j \widehat{\Theta}^\ell(\widehat{x}) = \partial_\ell w(x)[g^\ell(x)]_j.$$

Since $\widehat{v}_i(\widehat{x}) = v_k(x)[g^k(x)]_i$, we obtain

$$\begin{aligned}
\widehat{\partial}_j \widehat{v}_i(\widehat{x}) &= \widehat{\partial}_j v_k(\widehat{\Theta}(\widehat{x}))[g^k(x)]_i + v_q(x)\widehat{\partial}_j [g^q(\widehat{\Theta}(\widehat{x}))]_i \\
&= \partial_\ell v_k(x)[g^\ell(x)]_j [g^k(x)]_i + v_q(x)(\partial_\ell [g^q(x)]_i)[g^\ell(x)]_j \\
&= (\partial_\ell v_k(x) - \Gamma^q_{\ell k}(x) v_q(x))[g^k(x)]_i [g^\ell(x)]_j,
\end{aligned}$$

since $\partial_\ell g^q(x) = -\Gamma^q_{\ell k}(x) g^k(x)$.

Since $\partial_\ell v_k(x) - \Gamma^q_{\ell k}(x) v_q(x) = v_{k\|\ell}(x)$, then

$$\widehat{\partial}_j \widehat{v}_i(\widehat{x}) = (v_{k\|\ell}[g^k]_i [g^\ell]_j)(x).$$

For more detailed, we refer to Theorem 1.4-1 in Ciarlet [54]. ■

Theorem 1.3 *Let $\Theta : \Omega \to \mathbb{E}^3$ be an injective immersion and C^2-diffeomorphism of Ω onto $\widehat{\Omega} = \Theta(\Omega)$, and let there be given a vector field $v_i g^i : \Omega \to \mathbb{R}^3$ with $v_i \in C^1(\Omega)$.*

(a) *$v_{i\|j} \in C(\Omega)$ which are defined by:*

$$\partial_j(v_i g^i) = v_{i\|j} g^i,$$

and

$$v_{i\|j} = \{\partial_j(v_k g^k)\} g_i.$$

(b) *$\Gamma^p_{ij} = g^p.\partial_i g_i = \Gamma^p_{ji} \in C(\Omega)$ satisfy the relations*

$$\partial_i g^p = -\Gamma^p_{ij} g^j,$$

and

$$\partial_j g_q = \Gamma^p_{jq} g_p.$$

Proof.

(a) Let

$$\partial_j(v_i g^i) = (\partial_j v_i) g^i + v_i \partial_j g^i.$$

Since $\partial_j g^i = -\Gamma^i_{jk} g^k$, we obtain

$$\begin{aligned}
\partial_j(v_i g^i) &= (\partial_j v_i) g^i - v_i \Gamma^i_{jk} g^k \\
&= (\partial_j v_i) g^i - \Gamma^p_{ij} v_p g^i \\
&= v_{i\|j} g^i.
\end{aligned}$$

(b) We note that

$$\begin{aligned}
0 = \partial_j(g^p.g_q) &= \partial_j g^p.g_q + g^p.\partial_j g_q \\
&= -\Gamma^p_{ji} g^i.g_q + g^p.\partial_j g_q \\
&= -\Gamma^p_{jq} + g^p.\partial_j g_q.
\end{aligned}$$

Hence

$$g^p.\partial_j g_q = \Gamma^p_{jq},$$

then

$$\partial_j g_q = \Gamma^p_{jq} g_p.$$

For more detailed,we refer to Theorem 1.4-2 in Ciarlet [54]. ■

1.2 DIFFERENTIAL GEOMETRY OF SURFACES

1.2.1 CURVILINEAR COORDINATES ON A SURFACE

Let there be given an open subset ω of $\mathbb{R}^2$ and a smooth enough mapping $\theta : \omega \to \mathbb{E}^3$. The set

$$\widehat{\omega} = \theta(\omega),$$

is called a surface in $\mathbb{E}^3$.

If θ is injective, each point $\hat{y} \in \hat{\omega}$ can be unambiguously written as

$$\forall \hat{y} \in \hat{\omega}, \hat{y} = \theta(y), y \in \omega,$$

and the two coordinates (y_α) of y are called the curvilinear coordinates of $\widehat{y}$.

If the two vectors $a_\alpha(y) = \partial_\alpha \theta(y)$ are linearly independent, they are tangent to the coordinate lines passing through $\hat{y}$ and they form the covariant basis of the tangent plane to $\widehat{\omega}$ at $\widehat{y} = \theta(y)$.

The two vectors $a^\alpha(y)$ form this tangent plane defined by

$$a^\alpha(y).a_\beta(y) = \delta^\alpha_\beta.$$

The vecteors $a^\alpha(y)$ form its contravariant basis.

1.2.2 FIRST FUNDAMENTAL FORM

Let $\theta : \theta_i \widehat{e}^i : \omega \subset \mathbb{R}^2 \to \theta(\omega) = \widehat{\omega} \subset \mathbb{E}^3$ is differentiable at $y \in \omega$.

If $(y + \delta y) \in \omega$, then

$$\theta(y + \delta y) = \theta(y) + \nabla\theta(y)\delta y + o(\delta y)$$

where

$$\nabla\theta(y) = \begin{pmatrix} \partial_1\theta_1 & \partial_2\theta_1 \\ \partial_1\theta_2 & \partial_2\theta_2 \\ \partial_1\theta_3 & \partial_2\theta_3 \end{pmatrix}(y)$$

and

$$\delta y = \begin{pmatrix} \delta y_1 \\ \delta y_2 \end{pmatrix}.$$

Let the two vectors $a_\alpha(y) \in \mathbb{R}^3$ be defined by

$$a_\alpha(y) = \partial_\alpha\theta(y) = \begin{pmatrix} \partial_\alpha\theta_1 \\ \partial_\alpha\theta_2 \\ \partial_\alpha\theta_3 \end{pmatrix}(y).$$

Then

$$\theta(y + \delta y) = \theta(y) + \delta y^\alpha a_\alpha(y) + o(\delta y). \tag{1.9}$$

If $\delta y = \delta t e_\alpha$, where $\delta t \in \mathbb{R}$ and $\{e_\alpha\}$ is one of the basis vectors in $\mathbb{R}^2$. This relation reduces to

$$\theta(y + \delta t e_\alpha) = \theta(y) + \delta t a_\alpha(y) + o(\delta t). \tag{1.10}$$

Definition 1.3 *A mapping $\theta : \omega \to \mathbb{E}^3$ is an immersion at $y \in \omega$, if it is differentiable at y and the matrix $\nabla\theta(y)$ is of rank two, i.e., the two vecteors $a_\alpha(y)$ are linearly independent.*

Assume form now on in this section that the mapping θ is an immersion at $y \in \omega$. In this case, the last relation shows that each vecteor $a_\alpha(y)$ is tangent to the $\alpha -$ th coordinate line passing through $\widehat{y} = \theta(y)$, defiend as the image by θ of the points of ω that lie on a line parallel to e_α passing through y.

Then there exist t_0 and t_1 with $t_0 < 0 < t_1$ such that the $\alpha - th$ coordinate line is given by

$$t \in]t_0, t_1[\rightarrow f_\alpha(t) = \theta(y + te_\alpha),$$

in a neighborhood of $\widehat{y}$ hence $f'_\alpha(0) = \partial_\alpha \theta(y) = a_\alpha(y)$.
From (1.9), we obtain

$$\begin{aligned} |(\theta(y+\delta y) - \theta(y)|^2 &= \delta y^T \nabla\theta(y)^T \nabla\theta(y)\delta y + o(|\delta y|^2) \\ &= \delta y^\alpha a_\alpha(y).a_\beta(y)\delta y^\beta + o(|\delta y|^2). \end{aligned}$$

In other words, the principal part with respect to δy of the length between the points $\theta(y + \delta y)$ and $\theta(y)$ is $\sqrt{\delta y^\alpha a_\alpha(y).a_\beta(y)\delta y^\beta}$.

The define a matrix $(a_{\alpha\beta}(y))$ of order two by letting

$$a_{\alpha\beta}(y) = a_\alpha(y).a_\beta(y) = (\nabla\theta(y)^T \nabla\theta(y))_{\alpha\beta}.$$

The elements $a_{\alpha\beta}(y)$ of this symmetric matrix are called the covariant components of the first fundamental form, also called the metric tensor, of the surface $\widehat{\omega}$ at $\widehat{y} = \theta(y)$.

The two vecteors $a^\alpha(y)$ being thus defined, the four relation:

$$a^\alpha(y).a_\beta(y) = \delta^\alpha_\beta.$$

We pose $a^\alpha(y) = Y^{\alpha\sigma}(y)a_\sigma(y)$.
This gives

$$Y^{\alpha\sigma}(y)a_{\sigma\beta}(y) = \delta^\alpha_\beta.$$

Hence,

$$Y^{\alpha\sigma}(y) = a^{\alpha\sigma}(y),$$

where $(a^{\alpha\beta}(y)) = (a_{\alpha\beta}(y))^{-1}$.

Hence

$$a^\alpha(y) = a^{\alpha\sigma}(y)a_\sigma(y).$$

These relations in turn imply that

$$\begin{aligned} a^\alpha(y).a^\beta(y) &= a^{\alpha\sigma}(y)a_\sigma(y).a^{\beta\tau}(y)a_\tau(y) \\ &= a^{\alpha\sigma}(y)a^{\beta\tau}(y)a_{\sigma\tau}(y) \\ &= a^{\alpha\sigma}(y)\delta^\beta_\sigma \\ &= a^{\alpha\beta}(y). \end{aligned}$$

Since the matrix $(a^{\alpha\beta}(y))$ is positive definite, then the vecteors $a^\alpha(y)$ are linearly independant.

The two vecteors $a^\alpha(y)$ form the contravariante basis of the tangent plane to the surface $\widehat{\omega}$ at $\widehat{y} = \theta(y)$.

The elements $a^{\alpha\beta}(y)$ are called the contravariant component of the first fundamental form, or metric tensor, of the surface $\widehat{\omega}$ at $\widehat{y} = \theta(y)$.

We deduce that

$$\begin{aligned} a_\alpha(y) = a_{\alpha\beta}(y)a^\beta(y) \quad &\text{and } a^\alpha(y) = a^{\alpha\beta}(y)a_\beta(y), \\ a_{\alpha\beta}(y) = a_\alpha(y)a_\beta(y) \quad &\text{and } a^{\alpha\beta}(y) = a^\alpha(y)a^\beta(y) \end{aligned}$$

Definition 1.4 *A mapping $\theta : \omega \to \mathbb{E}^3$ is an immersion if it is an immersion at each point in ω, i.e., if θ is differentiable in ω and the two vectors $\partial_\alpha\theta(y)$ are linearly independent at each $y \in \omega$*

1.2.3 AREAS AND LENGTHS ON A SURFACE

Theorem 1.4 *Let $\theta : \omega \to \mathbb{E}^3$ be an injective and smooth enough immersion, and let $\widehat{\omega} = \theta(\omega)$.*

(a) *The area element $d\widehat{a}(\widehat{y})$ at $\widehat{y} = \theta(y) \in \widehat{\omega}$ is given in terms of the area element dy at $y \in \omega$ by*

$$d\widehat{a}(\widehat{y}) = \sqrt{a(y)}dy,$$

where $a(y) = \det(a_{\alpha\beta}(y))$.

(b) *The length element* $d\widehat{\ell}(\widehat{y})$ *at* $\widehat{y} = \theta(y) \in \widehat{\omega}$ *is given by*

$$d\widehat{\ell}(\widehat{y}) = \sqrt{\delta y^{\alpha} a_{\alpha\beta}(y)\delta y^{\beta} dy}. \tag{1.11}$$

(c) *Let* I *is a compact interval of* $\mathbb{R}$ *and* $C = f(I)$ *a curve in* ω, *with* $f = f^{\alpha} e_{\alpha} : I \to \omega$ *is a smooth enough injective mapping. Then the length of the curve* $\widehat{C} = \theta(C) \subset \widehat{\omega}$ *is given by*

$$\begin{aligned} \textit{length } \widehat{C} &= \int_I |\frac{d}{dt}(\theta \circ f)(t)| dt \\ &= \int_I \sqrt{a_{\alpha\beta}(f(t))\frac{df^{\alpha}}{dt}(t)\frac{df^{\beta}}{dt}(t)} dt. \end{aligned}$$

Proof. See proof of Theorem 2.3-1 in Ciarlet [54]). ■

1.2.4 SECOND FUNDAMENTAL FORM

Let γ be a smooth enough planar curve parametrized by its curvilinear abscissa s. Consider two points $p(s)$ and $p(s + \Delta s)$ with curvilinear abscissae s and $s + \Delta s$, let $\Delta\phi(s)$ be the algebraic angle between the two normals $\nu(s)$ and $\nu(s + \Delta s)$ to γ at those points (oriented in the usual way).

If $\lim\limits_{\Delta s \to 0} \frac{\Delta\phi(s)}{\Delta(s)}$ exist, called the curvature of γ at $p(s)$, if this limit is non zero, its inverse R is called the "algebraic radius of curvature" of γ at $p(s)$ (the sing of R depends on the orientation chosen on γ).

The point $p(s) + R\nu(s)$ is called the "center of curvature" of γ at $p(s)$.

Theorem 1.5 *Let* $\theta \in C^2(\omega; \mathbb{E}^3)$ *be an injective immersion and* $y \in \omega$ *be fixed.*

Consider a plane P *normal to* $\widehat{\omega} = \theta(\omega)$ *at the point* $\widehat{y} = \theta(y)$. *The intersection* $P \cap \widehat{\omega}$ *is a curve* $\widehat{C}$ *on* $\widehat{\omega}$, *which is the image* $C \subset \bar{\omega}$ *of a curve* C *in the set* $\bar{\omega}$. *Assume that, in a sufficiently small neighborhood of* y, *the restriction of* C *to this neighborhood is the image* $f(I)$ *of an open interval* $I \subset \mathbb{R}$, *where* $f = f^{\alpha} e_{\alpha} : I \mapsto \mathbb{R}$ *is a smooth enough injective mapping that satisfies*

$$\frac{df^{\alpha}}{dt}(t) e_{\alpha} \neq 0, t \in I, y = f(t).$$

Then, the curvature $\frac{1}{R}$ *of the planar curve* $\widehat{C}$ *at* $\widehat{y}$ *is given by the ratio*

$$\frac{1}{R} = \frac{b_{\alpha\beta}(f(t))\frac{df^\alpha}{dt}(t)\frac{df^\beta}{dt}(t)}{a_{\alpha\beta}(f(t))\frac{df^\alpha}{dt}(t)\frac{df^\beta}{dt}(t)}, \tag{1.12}$$

where

$$b_{\alpha\beta}(y) = a_3(y).\partial_\alpha a_\beta(y) = -\partial_\alpha a_3(y).a_\beta(y) = b_{\beta\alpha}(y),$$

are called the covariant components of the second fundamental form of the surface $\widehat{\omega}$ *at* $\widehat{y} = \theta(y)$ *and*

$$a_3(y) = \frac{a_1(y) \wedge a_2(y)}{|a_1(y) \wedge a_2(y)|}$$

is thus well defined, has euclidean norm one, and is normal to the surface $\widehat{\omega}$ *at* $\widehat{y}$.

The denominater in the definition of $a_3(y)$ *may be also written as*

$$|a_1(y) \wedge a_2(y) = \sqrt{a(y)},$$

where $a(y) = det(a_{\alpha\beta}(y))$.

Proof.

(i) We note that

$$\begin{aligned} \sin \Delta\phi(s) &= \nu(s).\tau(s + \Delta s) \\ &= -[\nu(s + \Delta s) - \nu(s)].\tau(s + \Delta s), \end{aligned}$$

hence

$$\begin{aligned} \frac{1}{R} &= \lim_{\Delta s \to 0} \frac{\Delta\phi(s)}{\Delta s} \\ &= \lim_{\Delta s \to 0} \frac{\sin \Delta\phi(s)}{\Delta(s)} \\ &= -\frac{d\nu(s)}{ds}.\tau(s). \end{aligned}$$

(ii) There thus exist an interval $\tilde{I} \subset I$, $J \subset \mathbb{R}$ and a mapping $p : J \to P$ such that $(\theta \circ f)(t) = p(s)$ and $(a_3 \circ f)(t) = \nu(s), \forall t \in \tilde{I}, s \in J$.

Then the curvature $\frac{1}{R}$ of $\widehat{C}$ is given by:

$$\frac{1}{R} = -\frac{d\nu(s)}{ds}.\tau(s).$$

Where

$$\begin{aligned}\frac{d\nu(s)}{ds} &= \frac{d(a_3 \circ f)}{dt}(t)\frac{dt}{ds} \\ &= \partial_\alpha a_3(f(t))\frac{df^\alpha}{dt}(t)\frac{dt}{ds}.\end{aligned}$$

$$\begin{aligned}\tau(s) &= \frac{dp(s)}{ds} \\ &= \frac{d(\theta \circ f)(t)}{dt}\frac{dt}{ds} \\ &= \partial_\beta\theta(f(t))\frac{df^\beta(t)}{dt}\frac{dt}{ds} \\ &= a_\beta(f(t))\frac{df^\beta(t)}{dt}\frac{dt}{ds}.\end{aligned}$$

Hence

$$\frac{1}{R} = -\partial_\alpha a_3(f(t)).a_\beta(f(t))\frac{df^\alpha(t)}{dt}\frac{df^\beta(t)}{dt}(\frac{dt}{ds})^2.$$

Since $b_{\alpha\beta}(f(t)) = -\partial_\alpha a_3(f(t)).a_\beta(f(t))$ and the relation (1.11) that

$$ds = \sqrt{\delta y^\alpha a_{\alpha\beta}(y)\delta y^\beta} = \sqrt{a_{\alpha\beta}(f(t))\frac{df^\alpha(t)}{dt}\frac{df^\beta(t)}{dt}}dt, \tag{1.13}$$

For more detailed, we refer to Theorem 2.4-1 in Ciarlet [54]. ■

Chapter 2

ASYMPOTOTIC ANALYSIS OF NONLINEARLY ELASTIC SHELLS

In this Chapter due to Ciarlet [31], we give a detailed account of recent justifications of nonlinear shell theories that are also based on an asymptotic analysis of the three-dimensional solution with the thickness as the "small" parameter.

A remarkable progress in the asymptotic analysis of nonlinearly elastic shells is due to B. Miara in [32], then to B. Miara and V. Lods in [35], who justified the two-dimensional equations of a nonlinearly elastic "membrane" shell and those of a nonlinearly elastic "flexural " shell, by means of the method of formal asymptotic expansions applied to the three-dimensional equations of a nonlinearly elastic shell modeled by a St Venant-Kirchhoff material.

2.1 THREE-DIMENSIONAL PROBLEMS SHELLS IN CARTESIAN COORDINATES

Let ω be a bounded, open and connected subset of $\mathbb{R}^2$, we assume that the boundary γ of ω Lipschitz-continous. Let γ_0 be a relatively open subset of γ such that $length(\gamma_0) > 0$. The unit outer normal vector (ν_α) along boundary γ, we denote by $y = (y_\alpha)$ a generic point of $\bar{\omega}$, and $\partial_\alpha = \partial/\partial y_\alpha$. Let the mapping $\theta : \bar{\omega} \to \mathbb{R}^3$ is a smooth enough injective immersion of class $\mathcal{C}^3$.

For any $\varepsilon > 0$, let

$$\Omega^\varepsilon = \omega \times] -\varepsilon, +\varepsilon[, \ \Gamma^\varepsilon_\pm = \omega \times \{\pm\varepsilon\}, \ \Gamma^\varepsilon_0 = \gamma_0 \times [-\varepsilon, +\varepsilon].$$

Let $\Theta : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ be the mapping a smooth enough immersion given through the relation

$$\Theta(x^\varepsilon) = \theta(y) + x^\varepsilon_3 a_3(y) \text{ for all } x^\varepsilon = (y, x^\varepsilon_3) \in \bar{\Omega}^\varepsilon,$$

hence $x^\varepsilon_\alpha = y_\alpha$. The three vectors

$$g^\varepsilon_i(x^\varepsilon) = \partial^\varepsilon_i \Theta(x^\varepsilon),$$

(with $\partial^\varepsilon_\alpha = \partial_\alpha$, $\partial^\varepsilon_3 = \partial/\partial x^\varepsilon_3$) are then linearly independent and they form the covariant basis at the point $\Theta(x^\varepsilon)$.

We consider a nonlinearly elastic shell whose reference configuration is $\overline{\hat{\Omega}^\varepsilon}$, we denote by $\hat{x} = \Theta(x^\varepsilon)$ a generic point in $\overline{\hat{\Omega}^\varepsilon}$, and we let $\hat{\partial}^\varepsilon_i = \partial/\partial\hat{x}^\varepsilon_i$, where $\hat{\Omega}^\varepsilon = \Theta(\Omega^\varepsilon)$,

with middle surface $\hat{\omega} = \theta(\bar{\omega})$ and thickness $2\varepsilon > 0$, we assume that the elastic material constituting the shell is a Saint Venant-Kirchhoff i.e, a homogeneous and isotropic, and that the reference configuration is natural state with Lamé constants $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$, $(\hat{n}^\varepsilon_j)$ is the unit outer normal vector along the upper and lower faces $\hat{\Gamma}^\varepsilon_\pm = \Theta(\Gamma^\varepsilon_\pm)$ and that $\hat{\Gamma}^\varepsilon_0 = \Theta(\Gamma^\varepsilon_0)$ the position of the lateral face $\hat{\Gamma}^\varepsilon = \Theta(\Gamma^\varepsilon)$ (where $\hat{\gamma}_0 = \theta(\gamma_0)$). We assume that the shell is clamped on a portion $\hat{\Gamma}^\varepsilon_0$.

The shell is subjected to body forces of density $(\hat{f}^\varepsilon_i) : \hat{\Omega}^\varepsilon \to \mathbb{R}^3$ and surface force on the upper and lower faces with density $(\hat{l}^\varepsilon_i) : \hat{\Gamma}^\varepsilon_+ \cup \hat{\Gamma}^\varepsilon_- \to \mathbb{R}^3$. We define the spaces

$$V(\hat{\Omega}^\varepsilon) = \{\hat{v}^\varepsilon = (\hat{v}^\varepsilon_i) \in W^{1,4}(\hat{\Omega}^\varepsilon; \mathbb{R}^3); \ \hat{v}^\varepsilon_i = 0 \text{ on } \hat{\Gamma}^\varepsilon_0\},$$

$$\hat{\Sigma}^\varepsilon = \{\hat{\tau}^\varepsilon = (\hat{\tau}^\varepsilon_{ij}) \in (L^2(\hat{\Omega}^\varepsilon))^9; \hat{\tau}^\varepsilon_{ij} = \hat{\tau}^\varepsilon_{ji}\}.$$

The unknown displacement field $\hat{u}^\varepsilon = (\hat{u}_i^\varepsilon)$ and stress field $\hat{\sigma}^\varepsilon = (\hat{\sigma}_{ij}^\varepsilon)$ satisfy the following three-dimensional shell problem in cartesian coordinates

$$\begin{cases} -\hat{\partial}_j^\varepsilon(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon) = \hat{f}_i^\varepsilon \text{ in } \hat{\Omega}^\varepsilon, \\ (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{n}_j^\varepsilon = \hat{l}_i^\varepsilon \text{ on } \hat{\Gamma}_-^\varepsilon \cup \hat{\Gamma}_+^\varepsilon, \\ \hat{u}_i^\varepsilon = 0 \text{ on } \hat{\Gamma}_0^\varepsilon, \end{cases}$$

such that the Piola-Kirchhoff stress tensor $(\hat{\sigma}_{ij}^\varepsilon)$ and the Green-Saint Venant strain tensor $(\hat{E}_{ij}(\hat{u}^\varepsilon))$ are given by

$$\begin{cases} \hat{\sigma}_{ij}^\varepsilon = \lambda^\varepsilon \hat{E}_{pp}^\varepsilon(\hat{u}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon \hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon), \\ \hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon) = \frac{1}{2}(\hat{\partial}_i^\varepsilon \hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon \hat{u}_i^\varepsilon + \hat{\partial}_i^\varepsilon \hat{u}_m^\varepsilon \hat{\partial}_j^\varepsilon \hat{u}_m^\varepsilon), \end{cases} \tag{2.1}$$

First, we rewrite the previous boundary value problem in the weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem

$$P(\hat{\Omega}^\varepsilon) \begin{cases} \text{Find } (\hat{u}^\varepsilon, \hat{\sigma}^\varepsilon) \in V(\hat{\Omega}^\varepsilon) \times \hat{\Sigma}^\varepsilon \text{ such that} \\ \int_{\hat{\Omega}^\varepsilon} (\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon \hat{\partial}_k^\varepsilon \hat{u}_i^\varepsilon)\hat{\partial}_j^\varepsilon \hat{v}_i^\varepsilon d\hat{x}^\varepsilon = \int_{\hat{\Omega}^\varepsilon} \hat{f}_i^\varepsilon \hat{v}_i^\varepsilon d\hat{x}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}_-^\varepsilon} \hat{l}_i^\varepsilon \hat{v}_i^\varepsilon d\hat{\Gamma}^\varepsilon \text{ for all } \hat{v}^\varepsilon \in V(\hat{\Omega}^\varepsilon). \end{cases}$$

Next, the variational problem $P(\hat{\Omega}^\varepsilon)$ may be formulated as a minimization problem

$$\hat{u}^\varepsilon \in V(\hat{\Omega}^\varepsilon) \text{ and } \hat{J}^\varepsilon(\hat{u}^\varepsilon) = \inf_{\hat{v}^\varepsilon \in V(\hat{\Omega}^\varepsilon)} \hat{J}^\varepsilon(\hat{v}^\varepsilon),$$

such that the stored energy function $\hat{J}^\varepsilon$ of a Saint Venant-Kirchhoff material given by

$$\hat{J}^\varepsilon(\hat{v}^\varepsilon) = \frac{1}{2}\int_{\hat{\Omega}} \hat{A}^{ijkl,\varepsilon} \hat{E}_{kl}^\varepsilon(\hat{v}^\varepsilon)\hat{E}_{ij}^\varepsilon(\hat{v}^\varepsilon)d\hat{x}^\varepsilon - \left\{\int_{\hat{\Omega}^\varepsilon} \hat{f}_i^\varepsilon \hat{v}_i^\varepsilon d\hat{x}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}_-^\varepsilon} \hat{l}_i^\varepsilon \hat{v}_i^\varepsilon d\hat{\Gamma}^\varepsilon\right\} \text{ for all } \hat{v}^\varepsilon \in V(\hat{\Omega}^\varepsilon),$$

where

$$\hat{A}^{ijkl,\varepsilon} = \lambda^\varepsilon \delta^{ij}\delta^{kl} + \mu^\varepsilon(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}).$$

We hereby declare that there is no conclusive result confirming the existence of solutions to the minimization problem stated above. The only proof we have is that $\hat{J}$ is coercive on $V(\hat{\Omega})$.

There are two theories of existence. The first theory is based on the implicit function theorem, which is valid for St Venant-Kirchhoff materials and is therefore restricted to specific categories of boundary conditions. The bodies are either fixed along their entire boundaries (a pure displacement problem, i.e., $\Gamma_0 = \partial\Omega$) or nowhere along their boundary (a pure traction problem, i.e., $\Gamma_1 = \partial\Omega$ because the displacements fields does not reduce to $\{0\}$). This theory does not include the conditions found here if the applied forces are not small enough.

The second theory, presented by John Ball, demonstrates the existence theory of the minimization problem of the energy for hyperelastic materials that satisfies certain physically realistic conditions of polyconvexity, coerciveness, and ad hoc growth conditions. This theory conforms to non-smooth boundaries and boundary conditions of the type found in our problem and is not limited to forces that are small enough. This theory also applies to stored energy functions of St Venant-Kirchhoff materials that are not polyconvex, as stated in Raoult [49]. Thm 4-10-1) .However, there is no solution to the variational problem that exists in our problem because the energy is not differentiable (see [50] Sect. 7.10).

2.2 THREE-DIMENSIONAL VARIATIONAL PROBLEM SHELLS IN CURVILINEAR COOORDINATS

In view of writing problem $P(\hat{\Omega}^\varepsilon)$ in curvilinear coordinates, we define de covariant components u_m^ε of the displacement by the formula

$$\hat{u}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = u_m^\varepsilon(x^\varepsilon)g^{m,\varepsilon}(x^\varepsilon) \text{ for all } \hat{x}^\varepsilon = \Theta(x^\varepsilon) \in \{\hat{\Omega}^\varepsilon\}^-,$$

where

$$[g_i^\varepsilon(x^\varepsilon)]^j = g_i^\varepsilon(x^\varepsilon) \cdot \hat{e}^j \text{ and } [g^{i,\varepsilon}(x^\varepsilon)]_j = g^{i,\varepsilon}(x^\varepsilon) \cdot \hat{e}_j,$$

$[g_i(x)]^j$ denotes the j-th component of the vector $g_i(x)$, and $[g^i(x)]_j$ denotes the j-th component of the vector $g^i(x)$, over the basis $\{\hat{e}^1, \hat{e}^2, \hat{e}^3\} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\}$ has the following expression in terms of the inverse mapping $\hat{\Theta}$:

$$[g^i(x)]_k = \hat{\nabla}^\varepsilon\hat{\Theta}(\hat{x}^\varepsilon) = (\hat{\partial}_k^\varepsilon\hat{\Theta}^i(\hat{x}^\varepsilon)) \text{ for all } x^\varepsilon \in \bar{\Omega}^\varepsilon.$$

Using the relations $g^i(x) \cdot g_j(x) = \delta_j^i$ and $\hat{e}^i \cdot \hat{e}_j = \delta_j^i$, note that the

$$[g^{p,\varepsilon}(x^\varepsilon)]_k[g_p^\varepsilon(x^\varepsilon)]^i = \delta_k^i.$$

We likewise associate functions v_i with the functions $\hat{v}_i$ appearing in variational problem $P(\hat{\Omega}^\varepsilon)$ by letting

$$\hat{v}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = v_j^\varepsilon(x^\varepsilon)g^{j,\varepsilon}(x^\varepsilon) \text{ and } \hat{v}_i^\varepsilon(\hat{x}^\varepsilon) = v_j^\varepsilon(x^\varepsilon)g^{j,\varepsilon}(x^\varepsilon)\cdot\hat{e}_i, \text{ for all } \hat{x}^\varepsilon = \Theta(x^\varepsilon) \in \{\hat{\Omega}^\varepsilon\}^-.$$

The preceding relations thus become

$$v_i^\varepsilon(x^\varepsilon) = \hat{v}_j^\varepsilon(\hat{x}^\varepsilon)[g_i^\varepsilon(x^\varepsilon)]^j \text{ and } u_i^\varepsilon(x^\varepsilon) = \hat{u}_k^\varepsilon(\hat{x}^\varepsilon)[g_i(x)]^k, \text{ for all } \hat{x}^\varepsilon = \Theta(x^\varepsilon),\ x^\varepsilon \in \bar{\Omega}^\varepsilon.$$

We deduce that v^ε is in the following space

$$\mathbf{V}(\Omega^\varepsilon) = \{v^\varepsilon = (v_i^\varepsilon) \in W^{1,4}(\Omega^\varepsilon;\mathbb{R}^3);\ v_i^\varepsilon = 0 \text{ on } \Gamma_0^\varepsilon\},$$

$$\Sigma^\varepsilon = \{\tau^\varepsilon = (\tau_{ij}^\varepsilon) \in (L^2(\Omega^\varepsilon))^9; \tau_{ij}^\varepsilon = \tau_{ji}^\varepsilon\}.$$

The vector v^ε is in the space $\mathbf{V}(\Omega^\varepsilon)$ of theorems (1.2) and (1.3), such that

$$\begin{cases} \hat{\partial}_j^\varepsilon \hat{v}_i^\varepsilon(\hat{x}^\varepsilon) = \left(v_{k\|l}^\varepsilon[g^{k,\varepsilon}]_i[g^{l,\varepsilon}]_j\right)(x^\varepsilon), \\ v_{k\|l}^\varepsilon(x^\varepsilon) = \partial_l^\varepsilon v_k^\varepsilon(x^\varepsilon) - \Gamma_{lk}^{q,\varepsilon}(x^\varepsilon)v_q^\varepsilon(x^\varepsilon), \\ \Gamma_{lk}^{q,\varepsilon}(x^\varepsilon) = g^{q,\varepsilon}(x^\varepsilon)\cdot\partial_l^\varepsilon g_k^\varepsilon(x^\varepsilon), \end{cases} \tag{2.2}$$

for all $\hat{x}^\varepsilon = \Theta(x^\varepsilon), x^\varepsilon \in \bar{\Omega}^\varepsilon$. The symmetric matrix σ^ε is in the space Σ^ε reads

$$\hat{\sigma}_{ij}^\varepsilon(\hat{x}^\varepsilon) = \sigma_{kl}^\varepsilon(x^\varepsilon)[g^{i,\varepsilon}(x^\varepsilon)]_k[g^{j,\varepsilon}(x^\varepsilon)]_l \text{ for all } \hat{x}^\varepsilon = \Theta(x^\varepsilon),\ x^\varepsilon \in \bar{\Omega}^\varepsilon. \tag{2.3}$$

We use the relations (2.2), we obtain

$$\begin{aligned} \hat{E}_{ij}^\varepsilon(\hat{v}^\varepsilon)(\hat{x}^\varepsilon) &= \frac{1}{2}(\hat{\partial}_j^\varepsilon\hat{v}_i^\varepsilon + \hat{\partial}_i^\varepsilon\hat{v}_j^\varepsilon + \hat{\partial}_i^\varepsilon\hat{v}_m^\varepsilon\hat{\partial}_j^\varepsilon\hat{v}^{j,\varepsilon})(\hat{x}^\varepsilon) \\ &= \frac{1}{2}((v_{k\|l}^\varepsilon + v_{l\|k}^\varepsilon + g^{mn,\varepsilon}v_{m\|k}^\varepsilon v_{n\|l}^\varepsilon)[g^k]_i[g^l]_j)(x^\varepsilon) \\ &= \left(E_{k\|l}^\varepsilon(v^\varepsilon)[g^{k,\varepsilon}]_i[g^{l,\varepsilon}]_j\right)(x^\varepsilon) \text{ for all } \hat{x}^\varepsilon = \Theta(x^\varepsilon),\ x^\varepsilon \in \bar{\Omega}^\varepsilon. \end{aligned} \tag{2.4}$$

We have

$$\begin{aligned} \left((\hat{E}_{ij}^\varepsilon)'(\hat{u}^\varepsilon)\hat{v}^\varepsilon\right)(\hat{x}^\varepsilon) &= \left(\frac{1}{2}(\hat{\partial}_j^\varepsilon\hat{v}_i^\varepsilon + \hat{\partial}_i^\varepsilon\hat{v}_j^\varepsilon + \hat{\partial}_i^\varepsilon\hat{u}_m^\varepsilon\hat{\partial}_j^\varepsilon\hat{v}^{m,\varepsilon} + \hat{\partial}_j^\varepsilon\hat{u}_m^\varepsilon\hat{\partial}_i^\varepsilon\hat{v}^{m,\varepsilon})\right)(x^\varepsilon) \\ &= \left(\frac{1}{2}(v_{k\|l}^\varepsilon + v_{l\|k}^\varepsilon + g^{mn,\varepsilon}\{u_{m\|k}^\varepsilon v_{n\|l}^\varepsilon + u_{n\|l}^\varepsilon v_{m\|k}^\varepsilon\})[g^{k,\varepsilon}]_i[g^{l,\varepsilon}]_j\right)(x^\varepsilon) \\ &= \left((E_{k\|l}^\varepsilon)'(u)^\varepsilon v^\varepsilon[g^{k,\varepsilon}]_i[g^{l,\varepsilon}]_j\right)(x^\varepsilon) \text{ at all } \hat{x}^\varepsilon = \Theta(x^\varepsilon), x^\varepsilon \in \bar{\Omega}^\varepsilon. \end{aligned} \tag{2.5}$$

From (2.3) and (2.5), we show that

$$\left(\hat{\sigma}_{ij}^\varepsilon((\hat{E}_{ij}^\varepsilon)'(\hat{u}^\varepsilon)\hat{v}^\varepsilon)\right)(\hat{x}^\varepsilon) = \left(\sigma_{ij}^\varepsilon((E_{k\|l}^\varepsilon)'(u^\varepsilon)v^\varepsilon)\right)(x^\varepsilon) \text{ at all } \hat{x}^\varepsilon = \Theta(x^\varepsilon), x^\varepsilon \in \bar{\Omega}^\varepsilon,$$

where

$$\sigma_{ij}^\varepsilon = A^{ijkl,\varepsilon}E_{k\|l}^\varepsilon(u^\varepsilon).$$

From the relations (1.6) and (1.7), taking into account the following relations

$$d\hat{x}^\varepsilon = \sqrt{g^\varepsilon(x^\varepsilon)}dx^\varepsilon, \tag{2.6}$$

$$d\hat{\Gamma}^\varepsilon(\hat{x}^\varepsilon) = \sqrt{g^\varepsilon(x^\varepsilon)}\sqrt{n_k(x^\varepsilon)g^{kl,\varepsilon}(x^\varepsilon)n_l(x^\varepsilon)}d\Gamma^\varepsilon(x^\varepsilon), \tag{2.7}$$

where (n_i^ε) is the unit outer normal vector along the boundary $\Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon$.

We associate with the Cartesian components of the applied forces $\hat{f}^{i,\varepsilon} = \hat{f}_i^\varepsilon$ and $\hat{l}^{i,\varepsilon} = \hat{l}_i^\varepsilon$, the contravariant components $f^{i,\varepsilon} \in L^2(\Omega^\varepsilon)$ and $l^{i,\varepsilon} \in L^2(\Gamma_+^\varepsilon \cup \Gamma_-^\varepsilon)$ defined by

$$\hat{f}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = f^{i,\varepsilon}(x^\varepsilon)g_i^\varepsilon(x^\varepsilon), \hat{l}_i^\varepsilon(\hat{x}^\varepsilon)\hat{e}^i = \{n_k(x^\varepsilon)g^{kl,\varepsilon}(x^\varepsilon)n_l(x^\varepsilon)\}^{-\frac{1}{2}}l^{i,\varepsilon}(x^\varepsilon)g_i^\varepsilon(x^\varepsilon).$$

Then we obtain

$$\int_{\hat{\Omega}^\varepsilon} \hat{f}_i^\varepsilon \hat{v}_i^\varepsilon d\hat{x}^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon}v_i^\varepsilon\sqrt{g^\varepsilon}dx^\varepsilon, \ \int_{\hat{\Gamma}_-^\varepsilon\cup\hat{\Gamma}_+^\varepsilon} \hat{l}_i^\varepsilon\hat{v}_i^\varepsilon d\hat{\Gamma}^\varepsilon = \int_{\Gamma_-^\varepsilon\cup\Gamma_+^\varepsilon} l^{i,\varepsilon}v_i^\varepsilon\sqrt{g^\varepsilon}d\Gamma^\varepsilon.$$

Consequently, the variational problem $P(\hat{\Omega}^\varepsilon)$ is equivalent to the following variational problem in curvilinear coordinates

$$P(\Omega^\varepsilon)\begin{cases}\text{Find } u^\varepsilon \in V(\Omega^\varepsilon) \text{ such that}\\ \int_{\Omega^\varepsilon} \mathrm{A}^{ijkl,\varepsilon}E_{k\|l}^\varepsilon(u^\varepsilon)F_{i\|j}^\varepsilon(u^\varepsilon,v^\varepsilon)\sqrt{g^\varepsilon}dx^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon}v_i^\varepsilon\sqrt{g^\varepsilon}dx^\varepsilon\\ +\int_{\Gamma_-^\varepsilon\cup\Gamma_+^\varepsilon} l^{i,\varepsilon}v_i^\varepsilon\sqrt{g^\varepsilon}d\Gamma^\varepsilon, \ \forall v^\varepsilon \in V(\Omega^\varepsilon),\end{cases}$$

where

$$\begin{cases}A^{ijkl,\varepsilon} = \lambda^\varepsilon g^{ij,\varepsilon}g^{kl,\varepsilon} + \mu^\varepsilon(g^{ik,\varepsilon}g^{jl,\varepsilon} + g^{il,\varepsilon}g^{jk,\varepsilon}),\\ F_{i\|j}^\varepsilon(u^\varepsilon,v^\varepsilon) = (E_{i\|j}^\varepsilon)'(u^\varepsilon)v^\varepsilon.\end{cases}$$

Therefore, the stored energy function J^ε of a Saint Venant-Kirchhoff material in curvilinear coordinates given by

$$\begin{aligned}J^\varepsilon(v^\varepsilon) &= \tfrac{1}{2}\int_{\Omega^\varepsilon} A^{ijkl,\varepsilon}E_{k\|l}^\varepsilon(v^\varepsilon)E_{i\|j}^\varepsilon(v^\varepsilon)\sqrt{g^\varepsilon}dx^\varepsilon\\ &-\Big\{\int_{\Omega^\varepsilon} f^{i,\varepsilon}v_i^\varepsilon\sqrt{g^\varepsilon}dx^\varepsilon + \int_{\Gamma_-^\varepsilon\cup\Gamma_+^\varepsilon} l^{i,\varepsilon}v_i^\varepsilon\sqrt{g^\varepsilon}d\Gamma^\varepsilon\Big\}.\end{aligned}$$

2.3 FORMAL ASYMPTOTIC ANALYSIS

2.3.1 THE THREE-DIMENSIONAL EQUATIONS OVER A DOMAIN INDEPENDENT OF ε

Using technics from asymptotic analysis due to Ciarlet [31], we transform the problem $P(\Omega^\varepsilon)$ into asymptotically equivalent problem posed over a domain independent of ε.

More specifically, we let

$$\Omega = \omega \times]-1,1[,\ \Gamma_0 = \gamma_0 \times [-1,1],\ \Gamma_\pm = \omega \times \{\pm 1\},$$

and with any point $x^\varepsilon = (x_i^\varepsilon) \in \bar{\Omega}^\varepsilon$, we associate the point $x = (x_i) \in \bar{\Omega}$ defined by $x_\alpha = y_\alpha$ and $x_3 = \frac{1}{\varepsilon} x_3^\varepsilon$.

We define the bijective mapping π^ε from $\bar{\Omega}$ to $\bar{\Omega}^\varepsilon$ such as

$$\pi^\varepsilon : x = (x_i) \in \bar{\Omega} \longrightarrow x^\varepsilon = (x_i^\varepsilon) = (x_1, x_2, \varepsilon x_3) \in \bar{\Omega}^\varepsilon.$$

So we have

$$\partial_\alpha^\varepsilon = \partial_\alpha \text{ and } \partial_3^\varepsilon = \frac{1}{\varepsilon}\partial_3.$$

We define the space

$$V(\Omega) = \{v = (v_i) \in \mathrm{W}^{1,4}(\Omega); v_i = 0 \text{ on } \Gamma_0\}.$$

To begin the asymptotic analysis, we first make the following scalings.

To the fields u^ε, $v^\varepsilon \in V(\Omega^\varepsilon)$, we associate the scaled fields $u(\varepsilon)$, $v \in V(\Omega)$ are defined by

$$u_i(\varepsilon)(x) = u_i^\varepsilon(x^\varepsilon) \text{ and } v_i(x) = v_i^\varepsilon(x^\varepsilon), \forall x^\varepsilon = \pi^\varepsilon x \in \bar{\Omega}^\varepsilon.$$

The scaled functions $g^{ij}(\varepsilon)$, $\Gamma_{ij}^p(\varepsilon)$, $g(\varepsilon), A^{ijkl}(\varepsilon)$, $E_{i\|j}(\varepsilon; u(\varepsilon))$, $F_{i\|j}(\varepsilon; u(\varepsilon), v)$, $u_{i\|j}(\varepsilon)$, and

$v_{i\|j}(\varepsilon)$ are defined by

$$\begin{cases} g^{ij}(\varepsilon)(x) = g^{ij,\varepsilon}(x^\varepsilon),\ \Gamma_{ij}^p(\varepsilon)(x) = \Gamma_{ij}^{p,\varepsilon}(x^\varepsilon),\ g(\varepsilon)(x) = g^\varepsilon(x^\varepsilon), \\ A^{ijkl}(\varepsilon)(x) = A^{ijkl,\varepsilon}(x^\varepsilon),\ E_{i\|j}(\varepsilon; u(\varepsilon))(x) = E_{i\|j}^\varepsilon(u^\varepsilon)(x^\varepsilon), \\ F_{i\|j}(\varepsilon; u(\varepsilon), v)(x) = F_{i\|j}^\varepsilon(u^\varepsilon, v^\varepsilon)(x^\varepsilon),\ u_{i\|j}(\varepsilon)(x) = u_{i\|j}^\varepsilon(x^\varepsilon), \\ v_{i\|j}(\varepsilon)(x) = v_{i\|j}^\varepsilon(x^\varepsilon), \end{cases}$$

for all $x^\varepsilon = \pi^\varepsilon x \in \bar{\Omega}^\varepsilon$.

Next, we make the following assumptions on the data: there exists constant $\lambda > 0$ and $\mu > 0$, the functions $f^i \in L^2(\Omega)$ and $l^i \in L^2(\Gamma_+ \cup \Gamma_-)$ are independent of $\varepsilon > 0$ such that

$$\begin{cases} \lambda = \lambda^\varepsilon,\ \mu = \mu^\varepsilon, \\ f^i(\varepsilon)(x) = f^{i,\varepsilon}(x^\varepsilon)\ \forall x^\varepsilon = \pi^\varepsilon x \in \Omega^\varepsilon, \\ l^i(\varepsilon)(x) = l^{i,\varepsilon}(x^\varepsilon)\ \forall x^\varepsilon = \pi^\varepsilon x \in \Gamma_-^\varepsilon \cup \Gamma_+^\varepsilon. \end{cases} \tag{2.8}$$

We thus have the following result.

Theorem 2.1 *The scaled unknown $u(\varepsilon)$ satisfies the following variational equations*

$$P(\varepsilon;\Omega)\begin{cases} \textit{Find } u(\varepsilon)\in V(\Omega) \textit{ such that} \\ \varepsilon\int_\Omega \mathrm{A}^{ijkl}(\varepsilon)E_{k\|l}(\varepsilon,u(\varepsilon))F_{i\|j}(\varepsilon,u(\varepsilon),v)\sqrt{g(\varepsilon)}dx = \\ \varepsilon\int_\Omega f^i(\varepsilon)v_i\sqrt{g(\varepsilon)}dx+\int_{\Gamma_-\cup\Gamma_+} l^i(\varepsilon)v_i\sqrt{g(\varepsilon)}d\Gamma \qquad \forall v\in V(\Omega), \end{cases}$$

where

$$\begin{cases} A^{ijkl}(\varepsilon)=\lambda g^{ij}(\varepsilon)g^{kl}(\varepsilon)+\mu(g^{ik}(\varepsilon)g^{jl}(\varepsilon)+g^{il}(\varepsilon)g^{jk}(\varepsilon)), \\ E_{i\|j}(\varepsilon,u(\varepsilon))=\frac{1}{2}(u_{i\|j}(\varepsilon)+u_{j\|i}(\varepsilon)+g^{mn}(\varepsilon)u_{m\|i}(\varepsilon)u_{n\|j}(\varepsilon)), \\ F_{i\|j}(\varepsilon,u(\varepsilon),v)=\frac{1}{2}(v_{i\|j}(\varepsilon)+v_{j\|i}(\varepsilon)+g^{mn}(\varepsilon)\{u_{m\|i}(\varepsilon)v_{n\|j}(\varepsilon)+u_{n\|j}(\varepsilon)v_{m\|i}(\varepsilon)\}), \end{cases}$$

with

$$\begin{cases} u_{\beta\|\alpha}(\varepsilon)=\partial_\alpha u_\beta(\varepsilon)-\Gamma^p_{\alpha\beta}(\varepsilon)u_p(\varepsilon),\ v_{\beta\|\alpha}(\varepsilon)=\partial_\alpha v_\beta-\Gamma^p_{\alpha\beta}(\varepsilon)v_p, \\ u_{3\|\alpha}(\varepsilon)=\partial_\alpha u_3(\varepsilon)-\Gamma^p_{\alpha 3}(\sigma)u_\sigma(\varepsilon),\ v_{3\|\alpha}(\varepsilon)=\partial_\alpha v_3-\Gamma^\sigma_{\alpha 3}(\varepsilon)v_\sigma, \\ u_{\alpha\|3}(\varepsilon)=\frac{1}{\varepsilon}\partial_3 u_\alpha(\varepsilon)-\Gamma^p_{\alpha 3}(\sigma)u_\sigma(\varepsilon),\ v_{\alpha\|3}(\varepsilon)=\frac{1}{\varepsilon}\partial_3 v_\alpha-\Gamma^\sigma_{\alpha 3}(\varepsilon)v_\sigma, \\ u_{3\|3}(\varepsilon)=\frac{1}{\varepsilon}\partial_3 u_3(\varepsilon),\ v_{3\|3}(\varepsilon)=\frac{1}{\varepsilon}\partial_3 v_3. \end{cases}$$

Proof. See proof of Theorem 8.4-1 in [31] ■

2.3.2 FORMAL ASYMPTOTIC EXPANSIONS METHODS

The objective of the asymptotic analysis is to study the behavior of the solution $u(\varepsilon)$ of the problem $P(\varepsilon;\Omega)$ when ε approaches zero. To this end, in order to obtain a membrane model in the limit, we transform the variational problem $P(\varepsilon;\Omega)$ into the following problem

$$P^\star(\varepsilon;\Omega)\begin{cases} \text{Find } u(\varepsilon)\in V(\Omega) \text{ such that} \\ \int_\Omega \mathrm{A}^{ijkl}(\varepsilon)E_{k\|l}(\varepsilon,u(\varepsilon))F_{i\|j}(\varepsilon,u(\varepsilon),v)\sqrt{g(\varepsilon)}dx = \\ \int_\Omega f^i(\varepsilon)v_i\sqrt{g(\varepsilon)}dx+\frac{1}{\varepsilon}\int_{\Gamma_-\cup\Gamma_+} l^i(\varepsilon)v_i\sqrt{g(\varepsilon)}d\Gamma\ \forall v\in V(\Omega). \end{cases}$$

Next, we write the scaled unknown as a formal expansion in terms of powers of the thickness as follows

$$u(\varepsilon)(x)=\frac{1}{\varepsilon^N}u^{-N}(x)+\frac{1}{\varepsilon^{N-1}}u^{-N+1}(x)+\cdots+u^0(x)+\varepsilon u^1(x)+\cdots, \tag{2.9}$$

for some integer $N \geqq 0$, where each term u^q, $q \geqq -N$ is independent of ε, with $u^{-N}, u^{-N+1} \in V(\Omega)$.

Next, according to Theorems 8.5-1 and 8.5-2 in [31], for all $0 < \varepsilon \leq \varepsilon_0$, we have

$$\begin{cases} g_i(\varepsilon)(x) = a_i(x_1, x_2) + \varepsilon x_3 g_i^1(x_1, x_2) + o(\varepsilon), \\ g^j(\varepsilon)(x) = a^j(x_1, x_2) + \varepsilon x_3 g^{j,1}(x_1, x_2) + o(\varepsilon), \\ g^{ij}(\varepsilon)(x) = a^{ij}(x_1, x_2) + \varepsilon x_3 g^{ij,1}(x_1, x_2) + o(\varepsilon), \\ \sqrt{g(\varepsilon)(x)} = \sqrt{a(x_1, x_2)} + \varepsilon x_3 g^1(x_1, x_2) + o(\varepsilon), \\ \Gamma^k_{ij}(\varepsilon)(x) = \Gamma^{k,0}_{il}(x_1, x_2) + \varepsilon x_3 \Gamma^{k,1}_{ij}(x_1, x_2) + o(\varepsilon), \end{cases} \tag{2.10}$$

where

$$\begin{cases} g^{\alpha\beta,1} = 2a^{\alpha\sigma} b^\beta_\sigma,\ g^{i3,1} = 0,\ \Gamma^{\sigma,0}_{\alpha\beta} = \Gamma^\sigma_{\alpha\beta}, \Gamma^{3,0}_{\alpha\beta} = b_{\alpha\beta},\ \Gamma^{\sigma,0}_{\alpha 3} = -b^\sigma_\alpha, \\ \Gamma^{3,0}_{\alpha 3} = \Gamma^{p,0}_{33} = 0, \Gamma^{\sigma,1}_{\alpha\beta} = -b^\sigma_\beta \,|_\alpha = -(\partial_\alpha b^\sigma_\beta + \Gamma^\sigma_{\alpha\tau} b^\tau_\beta - \Gamma^\tau_{\alpha\beta} b^\sigma_\tau), \\ \Gamma^{3,1}_{\alpha\beta} = -b^\sigma_\alpha b_{\sigma\beta},\ \Gamma^{\sigma,1}_{\alpha 3} = -b^\tau_\alpha b^\sigma_\tau,\ \Gamma^{3,1}_{\alpha 3} = \Gamma^{p,1}_{33} = 0. \end{cases} \tag{2.11}$$

Also, the contravariant components $A^{ijkl}(\varepsilon)$ of the scaled three-dimensional elasticity tensor satisfy

$$A^{ijkl}(\varepsilon)(x)\sqrt{g(\varepsilon)(x)} = A^{ijkl}(0)\sqrt{a(x_1, x_2)} + \varepsilon B^{ijkl,1} + \varepsilon^2 B^{ijkl,2} + o(\varepsilon^2), \tag{2.12}$$

where

$$\begin{cases} A^{\alpha\beta\sigma\tau}(0) = \lambda a^{\alpha\beta} a^{\sigma\tau} + \mu(a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}),\ A^{\alpha\beta 33}(0) = \lambda a^{\alpha\beta}, \\ A^{\alpha 3\sigma 3}(0) = \mu a^{\alpha\sigma},\ A^{3333}(0) = \lambda + 2\mu,\ A^{\alpha\beta\sigma 3}(0) = A^{\alpha 333}(0) = 0. \end{cases} \tag{2.13}$$

Finally, we will need the following Lemmas.

Lemma 2.1

(i) Let the functions $A^{ijkl}(0)$ be defined as in (2.13). Then for any symmetric matrices (s_{kl}) and (t_{ij}),

$$\begin{aligned} A^{ijkl}(0) s_{kl} t_{ij} = (\lambda a^{\alpha\beta} a^{\sigma\tau} + \mu\{a^{\alpha\sigma} a^{\beta\tau} + a^{\alpha\tau} a^{\beta\sigma}\}) s_{\sigma\tau} t_{\alpha\beta} + 4\mu a^{\alpha\sigma} s_{\alpha 3} t_{\sigma 3} \\ + \lambda a^{\alpha\beta} s_{33} t_{\alpha\beta} + \lambda a^{\sigma\tau} s_{\sigma\tau} t_{33} + (\lambda + 2\mu) s_{33} t_{33}. \end{aligned} \tag{2.14}$$

(ii) Let $a^{ij} = a^i \cdot a^j$. Then for any $y \in \bar{\omega}$ and any matrix (t_{ij}),

$$a^{ij}(y) a^{mn}(y) t_{im} t_{jn} \geq 0 \text{ and } a^{ij}(y) a^{mn}(y) t_{im} t_{jn} = 0 \Leftrightarrow t_{ij} = 0. \tag{2.15}$$

Proof. See proof of Theorem 8.7-1, part (i)-(ii) in [31], or proof of Lemma 1 in [32].
■

Lemma 2.2 *The formal asymptotic expansions of $u_{i\|j}(\varepsilon)$, $E_{i\|j}(\varepsilon)(u(\varepsilon))$ and $F_{i\|j}(\varepsilon)(u(\varepsilon),v)$, beginning with different powers of ε, are of the form*

$$\begin{cases} u_{m\|\alpha}(\varepsilon) = \dfrac{1}{\varepsilon^N} u_{m\|\alpha}^{-N} + \cdots\,, \\ u_{m\|3}(\varepsilon) = \dfrac{1}{\varepsilon^{N+1}} u_{m\|3}^{-N-1} + \dfrac{1}{\varepsilon^N} u_{m\|3}^{-N} + \cdots\,, \\ E_{\alpha\|\beta}(\varepsilon;u(\varepsilon)) = \dfrac{1}{\varepsilon^{2N}} E_{\alpha\|\beta}^{-2N} + \cdots\,, \\ E_{\alpha\|3}(\varepsilon;u(\varepsilon)) = \dfrac{1}{\varepsilon^{2N+1}} E_{\alpha\|3}^{-2N-1} + \cdots\,, \\ E_{3\|3}(\varepsilon;u(\varepsilon)) = \dfrac{1}{\varepsilon^{2N+2}} E_{3\|3}^{-2N-2} + \cdots\,, \\ F_{\alpha\|\beta}(\varepsilon;u(\varepsilon);v) = \dfrac{1}{\varepsilon^{N}} F_{\alpha\|\beta}^{-N}(v) + \cdots\,, \\ F_{\alpha\|3}(\varepsilon;u(\varepsilon);v) = \dfrac{1}{\varepsilon^{N+1}} F_{\alpha\|3}^{-N-1}(v) + \cdots\,, \\ F_{3\|3}(\varepsilon;u(\varepsilon);v) = \dfrac{1}{\varepsilon^{N+2}} F_{3\|3}^{-N-2}(v) + \cdots\,, \end{cases} \tag{2.16}$$

where

$$\begin{cases} u_{m\|\alpha}^{-N} = \partial_\alpha u_m^{-N} - \Gamma_{\alpha m}^{p,0} u_p^{-N}, \\ u_{m\|3}^{-N} = \partial_3 u_m^{-N+1} - \Gamma_{m3}^{p,0} u_p^{-N}, \\ v_{m\|\alpha} = \partial_\alpha v_m - \Gamma_{\alpha m}^{p,0} v_p. \end{cases} \tag{2.17}$$

The expressions for the functions $E_{i\|j}$ and $F_{i\|j}$ above differ according to the value of N, for instance

$$\begin{cases} E_{\alpha\|\beta}^{-2N} & = \dfrac{1}{2} a^{mn} u_{m\|\alpha}^{-N} u_{n\|\beta}^{-N} \ \textit{if } N \geq 1, \\ & = \dfrac{1}{2}(u_{\alpha\|\beta}^0 + u_{\beta\|\alpha}^0 + a^{mn} u_{m\|\alpha}^0 u_{n\|\beta}^0) \ \textit{if } N = 0, \\ F_{\alpha\|\beta}^{-N}(v) & = \dfrac{1}{2} a^{mn} \{u_{m\|\alpha}^{-N} v_{n\|\beta} + u_{n\|\beta}^{-N} v_{m\|\alpha}\} \ \textit{if } N \geq 1, \\ & = \dfrac{1}{2}(v_{\alpha\|\beta} + v_{\beta\|\alpha} + a^{mn}\{u_{m\|\alpha}^0 v_{n\|\beta} + u_{n\|\beta}^0 v_{m\|\alpha}\}) \ \textit{if } N = 0. \end{cases}$$

Proof. The proof is as in Theorem 8.7-1, part (iii) of Ciarlet [31]. In addition to detailing the accounts. Using a formal asymptotic expansion (2.9) and the relations (2.10)-(2.11), we obtain

$$\begin{cases} u_{m\|\alpha}(\varepsilon) = \partial_\alpha u_m(\varepsilon) - \Gamma^p_{\alpha m}(\varepsilon) u_p(\varepsilon) \\ = \partial_\alpha(\frac{1}{\varepsilon^N} u_m^{-N} + \frac{1}{\varepsilon^{N-1}} u_m^{-N+1} + \cdots) - (\Gamma^{p,0}_{\alpha m} + \varepsilon x_3 \Gamma^{p,1}_{\alpha m} + \cdots)(\frac{1}{\varepsilon^N} u_p^{-N} + \cdots), \\ = \frac{1}{\varepsilon^N}(\partial_\alpha u_m^{-N} - \Gamma^{p,0}_{\alpha m} u_P^{-N}) + \frac{1}{\varepsilon^{N-1}}(\partial_\alpha u_m^{-N+1} - \Gamma^{p,0}_{\alpha m} u_P^{-N+1}) + \cdots , \\ = \frac{1}{\varepsilon^N} u_{m\|\alpha}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{m\|\alpha}^{-N+1} + \cdots , \end{cases} \tag{2.18}$$

$$\begin{cases} u_{m\|3}(\varepsilon) = \frac{1}{\varepsilon}\partial_3 u_m(\varepsilon) - \Gamma^p_{m3}(\varepsilon) u_p(\varepsilon) \\ = \frac{1}{\varepsilon}\partial_3(\frac{1}{\varepsilon^N} u_m^{-N} + \frac{1}{\varepsilon^{N-1}} u_m^{-N+1} + \cdots) - (\Gamma^{p,0}_{m3} + \varepsilon x_3 \Gamma^{p,1}_{m3} + \cdots)(\frac{1}{\varepsilon^N} u_p^{-N} + \cdots), \\ = \frac{1}{\varepsilon^{N+1}}\partial_3 u_m^{-N} + \frac{1}{\varepsilon^N}(\partial_3 u_m^{-N+1} - \Gamma^{p,0}_{m3} u_p^{-N}) + \cdots , \\ = \frac{1}{\varepsilon^{N+1}} u_{m\|3}^{-N-1} + \frac{1}{\varepsilon^N} u_{m\|3}^{-N} + \cdots , \end{cases} \tag{2.19}$$

$$\begin{cases} u_{3\|3}(\varepsilon) = \frac{1}{\varepsilon}\partial_3 u_3(\varepsilon) = \frac{1}{\varepsilon}\partial_3(\frac{1}{\varepsilon^N} u_3^{-N} + \frac{1}{\varepsilon^{N-1}} u_3^{-N+1} + \frac{1}{\varepsilon^{N-2}} u_3^{-N+2} + \cdots), \\ = \frac{1}{\varepsilon^{N+1}}\partial_3 u_3^{-N} + \frac{1}{\varepsilon^N}\partial_3 u_3^{-N+1} + \cdots , \\ = \frac{1}{\varepsilon^{N+1}} u_{3\|3}^{-N-1} + \frac{1}{\varepsilon^N} u_{3\|3}^{-N} + \cdots , \end{cases} \tag{2.20}$$

$$\begin{cases} E_{\alpha\|\beta}(\varepsilon; u(\varepsilon)) = E_{\alpha\|\beta}(\varepsilon, \frac{1}{\varepsilon^N} u^{-N} + \frac{1}{\varepsilon^{N-1}} u^{-N+1} + \cdots) \\ = \frac{1}{2}\left((\frac{1}{\varepsilon^N} u_{\alpha\|\beta}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{\alpha\|\beta}^{-N+1} + \cdots) + (\frac{1}{\varepsilon^N} u_{\beta\|\alpha}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{\beta\|\alpha}^{-N+1} + \cdots) \right) \\ + \frac{1}{2}(a^{mn} + \varepsilon x_3 g^{mn,1} + \cdots)(\frac{1}{\varepsilon^N} u_{m\|\alpha}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{m\|\alpha}^{-N+1} + \cdots)(\frac{1}{\varepsilon^N} u_{n\|\beta}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{n\|\beta}^{-N+1} \\ + \cdots) = \frac{1}{\varepsilon^{2N}}(\frac{1}{2} a^{mn} u_{m\|\alpha}^{-N} u_{n\|\beta}^{-N}) + \cdots , \\ = \frac{1}{\varepsilon^{2N}} E_{\alpha\|\beta}^{-2N} + \cdots , \end{cases} \tag{2.21}$$

$$\begin{cases} E_{\alpha\|3}(\varepsilon; u(\varepsilon)) = E_{\alpha\|3}(\varepsilon, \frac{1}{\varepsilon^N} u^{-N} + \frac{1}{\varepsilon^{N-1}} u^{-N+1} + \cdots) \\ = \frac{1}{2}\left((\frac{1}{\varepsilon^{N+1}} u_{\alpha\|3}^{-N-1} + \frac{1}{\varepsilon^N} u_{\alpha\|3}^{-N} + \cdots) + (\frac{1}{\varepsilon^N} u_{3\|\alpha}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{3\|\alpha}^{-N+1} + \cdots) \right) \\ + \frac{1}{2}(a^{mn} + \varepsilon x_3 g^{mn,1} + \cdots)(\frac{1}{\varepsilon^N} u_{m\|\alpha}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{m\|\alpha}^{-N+1} + \cdots)(\frac{1}{\varepsilon^{N+1}} u_{n\|3}^{-N-1} \\ + \frac{1}{\varepsilon^N} u_{n\|3}^{-N} + \cdots) = \frac{1}{\varepsilon^{2N+1}}(\frac{1}{2} a^{mn} u_{m\|\alpha}^{-N} \partial_3 u_n^{-N}) + \frac{1}{\varepsilon^{2N}}(\frac{1}{2} a^{mn} u_{m\|\alpha}^{-N} u_{n\|3}^{-N}) + \cdots , \\ = \frac{1}{\varepsilon^{2N+1}} E_{\alpha\|3}^{-2N-1} + \frac{1}{\varepsilon^{2N}} E_{\alpha\|3}^{-2N} + \cdots , \end{cases} \tag{2.22}$$

$$
\begin{cases}
E_{3\|3}(\varepsilon; u(\varepsilon)) = E_{3\|3}(\varepsilon, \frac{1}{\varepsilon^N} u^{-N} + \frac{1}{\varepsilon^{N-1}} u^{-N+1} + \cdots) \\
= (\frac{1}{\varepsilon^{N+1}} u_{3\|3}^{-N-1} + \frac{1}{\varepsilon^N} u_{3\|3}^{-N} + \cdots) + \frac{1}{2}(a^{mn} + \varepsilon x_3 g^{mn,1} + \cdots)(\frac{1}{\varepsilon^{N+1}} u_{m\|3}^{-N-1} \\
+ \frac{1}{\varepsilon^N} u_{m\|3}^{-N} + \cdots)(\frac{1}{\varepsilon^{N+1}} u_{n\|3}^{-N-1} + \frac{1}{\varepsilon^N} u_{n\|3}^{-N} + \cdots) \\
= \frac{1}{\varepsilon^{2N+2}}(\frac{1}{2} a^{mn} u_{m\|3}^{-N-1} u_{n\|3}^{-N-1}) + \cdots , \\
= \frac{1}{\varepsilon^{2N+2}} E_{3\|3}^{-2N-2} + \cdots ,
\end{cases}
\tag{2.23}
$$

$$
\begin{cases}
F_{\alpha\|\beta}(\varepsilon; u(\varepsilon); v) = F_{\alpha\|\beta}(\varepsilon, \frac{1}{\varepsilon^N} u^{-N} + \frac{1}{\varepsilon^{N-1}} u^{-N+1} + \cdots , v) \\
= \frac{1}{2}\left((\partial_\beta v_\alpha - (\Gamma_{\beta\alpha}^{p,0} + \varepsilon x_3 \Gamma_{\beta\alpha}^{p,1} + \cdots) v_p) + (\partial_\alpha v_\beta - (\Gamma_{\alpha\beta}^{p,0} + \varepsilon x_3 \Gamma_{\alpha\beta}^{p,1} + \cdots) v_p)\right) \\
+ \frac{1}{2}(a^{mn} + \varepsilon x_3 g^{mn,1} + \cdots)\{(\frac{1}{\varepsilon^N} u_{m\|\alpha}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{m\|}^{-N+1} + \cdots)(\partial_\alpha v_m - (\Gamma_{\beta n}^{p,0} \\
+ \varepsilon x_3 \Gamma_{\beta n}^{p,1} + \cdots) v_p) + (\frac{1}{\varepsilon^N} u_{n\|\beta}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{n\|\beta}^{-N+1} + \cdots)(\partial_\alpha v_m - (\Gamma_{\alpha n}^{p,0} \\
+ \varepsilon x_3 \Gamma_{\alpha m}^{p,1} + \cdots) v_p)\} = \frac{1}{\varepsilon^N}(\frac{1}{2} a^{mn}\{u_{m\|\alpha}^{-N}(\partial_\beta v_n - \Gamma_{\beta n}^{p,0} v_p) + u_{n\|\beta}^{-N}(\partial_\alpha v_m - \Gamma_{\alpha m}^{p,0} v_p)\}) \\
+ \cdots , = \frac{1}{\varepsilon^N}(\frac{1}{2} a^{mn}\{u_{m\|\alpha}^{-N} v_{n\|\beta} + u_{n\|\beta}^{-N} v_{m\|\alpha}\}) + \cdots , \\
= \frac{1}{\varepsilon^N} F_{\alpha\|\beta}^{-N}(v) + \cdots ,
\end{cases}
\tag{2.24}
$$

$$
\begin{cases}
F_{\alpha\|3}(\varepsilon; u(\varepsilon); v) = F_{\alpha\|3}(\varepsilon, \frac{1}{\varepsilon^N} u^{-N} + \frac{1}{\varepsilon^{N-1}} u^{-N+1} + \cdots , v) \\
= \frac{1}{2}\left((\frac{1}{\varepsilon}\partial_3 v_\alpha - (\Gamma_{\alpha 3}^{\sigma,0} + \varepsilon x_3 \Gamma_{\alpha 3}^{\sigma,1} + \cdots) v_\sigma) + (\partial_\alpha v_3 - (\Gamma_{\alpha 3}^{\sigma,0} + \varepsilon x_3 \Gamma_{\alpha 3}^{\sigma,1} + \cdots) v_\sigma)\right) \\
+ \frac{1}{2}(a^{mn} + \varepsilon x_3 g^{mn,1} + \cdots)\{(\frac{1}{\varepsilon^N} u_{m\|\alpha}^{-N} + \frac{1}{\varepsilon^{N-1}} u_{m\|\alpha}^{-N+1} + \cdots)(\frac{1}{\varepsilon}\partial_3 v_n \\
- (\Gamma_{n3}^{\sigma,0} + \varepsilon x_3 \Gamma_{n3}^{\sigma,1} + \cdots) v_\sigma) + (\frac{1}{\varepsilon^{N+1}} u_{n\|3}^{-N-1} + \frac{1}{\varepsilon^N} u_{n\|3}^{-N} + \cdots)(\partial_\alpha v_m - (\Gamma_{\alpha m}^{p,0} \\
+ \varepsilon x_3 \Gamma_{\alpha m}^{p,1} + \cdots) v_p)\} = \frac{1}{\varepsilon^{N+1}}(\frac{1}{2} a^{mn} u_{m\|\alpha}^{-N} \partial_3 v_n) + \cdots , \\
= \frac{1}{\varepsilon^{N+1}} F_{\alpha\|3}^{-N-1}(v) + \cdots ,
\end{cases}
\tag{2.25}
$$

$$
\begin{cases}
F_{3\|3}(\varepsilon; u(\varepsilon); v) = F_{3\|3}(\varepsilon, \frac{1}{\varepsilon^N} u^{-N} + \frac{1}{\varepsilon^{N-1}} u^{-N+1} + \cdots , v) \\
= \frac{1}{\varepsilon}\partial_3 v_3 + (a^{mn} \varepsilon x_3 g^{mn,1} + \cdots)(\frac{1}{\varepsilon^{N+1}} u_{m\|3}^{-N-1} + \frac{1}{\varepsilon^N} u_{m\|3}^{-N} + \cdots)(\frac{1}{\varepsilon}\partial_3 v_n \\
- (\Gamma_{n3}^{\sigma,0} + \varepsilon x_3 \Gamma_{n3}^{\sigma,1} + \cdots) v_\sigma) \\
= \frac{1}{\varepsilon^{N+2}}(a^{mn} u_{m\|3}^{-N-1} \partial_3 v_n) + \frac{1}{\varepsilon^{N+1}}(a^{mn} u_{m\|3}^{-N} \partial_3 v_n) + \cdots , \\
= \frac{1}{\varepsilon^{N+2}} F_{3\|3}^{-N-2}(v) + \frac{1}{\varepsilon^{N+1}} F_{3\|3}^{-N-1}(v) + \cdots ,
\end{cases}
\tag{2.26}
$$

■

We now show that the expansion (2.9) begins with a term of order 0 with respect to ε

Theorem 2.2 *Assume that the scaled unknown satisfying problem $P(\varepsilon)$ admits for each $0<\varepsilon\le\varepsilon_0$ a formal asymptotic expansion (2.9) with $u^{-N}, u^{-N+1}\in\mathbf{V}(\Omega)$, $u^{-N}\neq 0$, for some integer $N\in\mathbb{Z}$. Then $N=0$.*

Proof. The proof is as in Theorem 8.7-1, part (iv) of Ciarlet [31]. In addition to detailing the accounts

(i) We substitute (2.9), (2.10), (2.11) and formal asymptotic expansions of $E_{i\|j}(\varepsilon)(u(\varepsilon))$ and $F_{i\|j}(\varepsilon)(u(\varepsilon),v)$ the found in (2.16) of the Lemma 2.2 in $P(\varepsilon)$, we obtain

$$\left\{\begin{array}{l}
\int_\Omega(A^{ijkl}(0)\sqrt{a}+\varepsilon x_3B^{ijkl,1}+\cdots)E_{k\|l}(\varepsilon,u(\varepsilon))F_{i\|j}(\varepsilon,u(\varepsilon),v)dx\\
=\int_\Omega(A^{ijkl}(0)E_{k\|l}(\varepsilon,u(\varepsilon))F_{i\|j}(\varepsilon,u(\varepsilon),v)\sqrt{a}dx\\
+\int_\Omega(\varepsilon x_3B^{ijkl,1}+\cdots)E_{k\|l}(\varepsilon,u(\varepsilon))F_{i\|j}(\varepsilon,u(\varepsilon),v)dx,\\
=\int_\Omega(A^{ijkl}(0)(\dfrac{1}{\varepsilon^{2N+2}}E^{-2N-2}_{k\|l}+\dfrac{1}{\varepsilon^{2N+1}}E^{-2N-1}_{k\|l}+\dfrac{1}{\varepsilon^{2N}}E^{-2N}_{k\|l}+\cdots)\\
(\dfrac{1}{\varepsilon^{N+2}}F^{-N-2}_{i\|j}(v)+\dfrac{1}{\varepsilon^{N+1}}F^{-N-1}_{i\|j}(v)+\dfrac{1}{\varepsilon^{N}}F^{-N}_{i\|j}(v)+\cdots)\sqrt{a}dx\\
+\int_\Omega(\varepsilon x_3B^{ijkl,1}+\cdots)(\dfrac{1}{\varepsilon^{2N+2}}E^{-2N-2}_{k\|l}+\dfrac{1}{\varepsilon^{2N+1}}E^{-2N-1}_{k\|l}+\dfrac{1}{\varepsilon^{2N}}E^{-2N}_{k\|l}+\cdots)\\
(\dfrac{1}{\varepsilon^{N+2}}F^{-N-2}_{i\|j}(v)+\dfrac{1}{\varepsilon^{N+1}}F^{-N-1}_{i\|j}(v)+\dfrac{1}{\varepsilon^{N}}F^{-N}_{i\|j}(v)+\cdots)dx\\
=\dfrac{1}{\varepsilon^{3N+4}}\int_\Omega A^{ijkl}(0)E^{-2N-2}_{k\|l}F^{-N-2}_{i\|j}(v)\sqrt{a}dx\\
+\dfrac{1}{\varepsilon^{3N+3}}\int_\Omega A^{ijkl}(0)E^{-2N-2}_{k\|l}F^{-N-1}_{i\|j}(v)\sqrt{a}dx\\
+\dfrac{1}{\varepsilon^{3N+2}}\int_\Omega A^{ijkl}(0)E^{-2N-2}_{k\|l}F^{-N}_{i\|j}(v)\sqrt{a}dx\\
+\dfrac{1}{\varepsilon^{3N+1}}\int_\Omega A^{ijkl}(0)E^{-2N-2}_{k\|l}F^{-N+1}_{i\|j}(v)\sqrt{a}dx\\
+\dfrac{1}{\varepsilon^{3N+3}}\int_\Omega A^{ijkl}(0)E^{-2N-1}_{k\|l}F^{-N-2}_{i\|j}(v)\sqrt{a}dx\\
+\dfrac{1}{\varepsilon^{3N+2}}\int_\Omega A^{ijkl}(0)E^{-2N-1}_{k\|l}F^{-N-1}_{i\|j}(v)\sqrt{a}dx\\
+\dfrac{1}{\varepsilon^{3N+2}}\int_\Omega A^{ijkl}(0)E^{-2N}_{k\|l}F^{-N-2}_{i\|j}(v)\sqrt{a}dx\\
+\dfrac{1}{\varepsilon^{3N+1}}\int_\Omega A^{ijkl}(0)E^{-2N}_{k\|l}F^{-N-1}_{i\|j}(v)\sqrt{a}dx\\
+\dfrac{1}{\varepsilon^{3N}}\int_\Omega\{A^{ijkl}(0)E^{-2N}_{k\|l}F^{-N}_{i\|j}(v)+E^{-2N+1}_{k\|l}F^{-N-1}_{i\|j}(v)\}\sqrt{a}dx\\
+\dfrac{1}{\varepsilon^{3N}}\int_\Omega x_3B^{ijk,1}E^{-2N}_{k\|l}F^{-N-1}_{i\|j}(v)dx+\cdots\\
=\int_\Omega f^i(\varepsilon)v_i\sqrt{g(\varepsilon)}dx+\dfrac{1}{\varepsilon}\int_{\Gamma_-\cup\Gamma_+}l^i(\varepsilon)v_i\sqrt{g(\varepsilon)}d\Gamma,
\end{array}\right. \tag{2.27}$$

for all $v\in V(\Omega)$. We are now in a position to start the cancellation of the factors of the successive powers of ε found in the variational equations (2.27). In what follows,

L^r designates for any integer $r \geq -3N-4$ the linear form defined by

$$L^r(v) = \int_\Omega f^{i,r} v_i \sqrt{a} dx + \int_{\Gamma_- \cup \Gamma_+} l^{i,r+1} v_i \sqrt{a} dx, \tag{2.28}$$

and it is always know that the functions $f^{i,r} \in L^2(\Omega)$, $l^{i,r+1} \in L^2(\Gamma_+ \cup \Gamma_-)$ are independent of ε.

(ii) Assume that $N \geq 0$. Since the lowest power of ε in the left-hand side in (2.27) is ε^{-3N-4}, using the relations (2.16) and (2.17) of the Lemma 2.2, we obtain

$$\begin{cases} \frac{1}{\varepsilon^{3N+4}} \int_\Omega A^{ijkl}(0) E_{k\|l}^{-2N-2} F_{i\|j}^{-N-2}(v) \sqrt{a} dx \\ = \frac{1}{\varepsilon^{3N+4}} \int_\Omega A^{\alpha\beta\sigma\tau}(0) E_{\sigma\|\tau}^{-2N-2} F_{\alpha\|\beta}^{-N-2}(v) \sqrt{a} dx \\ + \frac{1}{\varepsilon^{3N+4}} \int_\Omega 4A^{\alpha 3\sigma 3} E_{\alpha\|3}^{-2N-2} F_{\sigma\|3}^{-N-2}(v) \sqrt{a} dx \\ + \frac{1}{\varepsilon^{3N+4}} \int_\Omega A^{\alpha\beta 33} E_{3\|3}^{-2N-2} F_{\alpha\|\beta}^{-N-2}(v) \sqrt{a} dx \\ + \frac{1}{\varepsilon^{3N+4}} \int_\Omega A^{33\sigma\tau} E_{\sigma\|\tau}^{-2N-2} F_{3\|3}^{-N-2}(v) \sqrt{a} dx \\ + \frac{1}{\varepsilon^{3N+4}} \int_\Omega A^{3333} E_{3\|3}^{-2N-2} F_{3\|3}^{-N-2}(v) \sqrt{a} dx. \end{cases}$$

The gives the first assumption on the order of the applied forces: There exist $f^{-3N-4} \in L^2(\Omega)$, $l^{-3N-3} \in L^2(\Gamma_+ \cup \Gamma_-)$ independent of ε such that

$$\begin{cases} f^i(\varepsilon)(x) = \frac{1}{\varepsilon^{3N+4}} f^{i,-3N-4}(x), \\ l^i(\varepsilon)(x) = \frac{1}{\varepsilon^{3N+3}} l^{i,-3N-3}(x). \end{cases}$$

Then the cancellation of the coefficient of ε^{-3N-4} then produces the equations:

$$\int_\Omega (\lambda + 2\mu) E_{3\|3}^{-2N-2} F_{3\|3}^{-N-2}(v) \sqrt{a} dx = L^{-3N-4}(v) \text{ for all } v \in V(\Omega).$$

From the relations (2.23) and (2.26) of proof the Lemma 2.2, we obtain

$$E_{3\|3}^{-2N-2} = \frac{1}{2} a^{mn} \partial_3 u_m^{-N} \partial_3 u_n^{-N} \text{ and } F_{3\|3}^{-N-2}(v) = a^{mn} \partial_3 u_m^{-N} \partial_3 v_n,$$

we must have

$$L^{-3N-4}(v) = \int_\Omega f^{i,-3N-4} v_i \sqrt{a} dx + \int_{\Gamma_+ \cup \Gamma_-} l^{i,-3N-3} v_i \sqrt{a} d\Gamma = 0$$

for all $v \in V(\Omega)$ that are independent of x_3. According to the first requirement, implies that we must let $f^{i,-3N-4} = 0$, $l^{i,-3N-3} = 0$.

By choosing test function $v = u^{-N}$ in the resulting variational equations, we obtain

$$\int_\Omega E^{-2N-2}_{3\|3} F^{-N-2}_{3\|3}(u^{-N})\sqrt{a}dx = \frac{1}{2}\int_\Omega (a^{mn}\partial_3 u^{-N}_m \partial_3 u^{-N}_n)^2 \sqrt{a}dx = 0,$$

we observe that

$$a^{mn}\partial_3 u^{-N}_m \partial_3 u^{-N}_n = 0.$$

Since the symmetric matrix (a^{ij}) is positive definite and according the relation (2.15) from the Lemma 2.1, we infer

$$\partial_3 u^{-N} = \partial_3 u^{-N}_m = 0 \text{ in } \Omega, \tag{2.29}$$

we conclude that the first term of formal asymptotic expansion u^{-N} is independent of x_3. Then, we get

$$E^{-2N-2}_{3\|3} = 0 \text{ and } F^{-N-2}_{3\|3}(v) = 0 \text{ for all } v \in V(\Omega).$$

Let us introduce the space

$$V_M(\omega) = \{\eta = (\eta_i) \in W^{1,4}(\omega); \eta = 0 \text{ on } \gamma_0\}. \tag{2.30}$$

Since $u^{-N} \in V_M(\omega)$, then

$$E^{-2N-2}_{i\|j} = E^{-2N-1}_{i\|j} = 0 \text{ in } \Omega \text{ and } F^{-N-2}_{i\|j}(v) = 0 \text{ for all } v \in V(\Omega), \tag{2.31}$$

and to the new assumption that there exist $f^{-3N-3} \in L^2(\Omega)$ and $l^{-3N-2} \in L^2(\Gamma_+ \cup \Gamma_-)$ independent of ε such that

$$\begin{cases} f^i(\varepsilon)(x) = \dfrac{1}{\varepsilon^{3N+3}} f^{i,-3N-3}(x), \\ l^i(\varepsilon)(x) = \dfrac{1}{\varepsilon^{3N+2}} l^{i,-3N-2}(x). \end{cases}$$

The cancellation of the coefficient of ε^{-3N-3} in (2.27) then produces the equations:

$$\int_\Omega A^{ijkl}(0) E^{-2N-1}_{k\|l} F^{-N-2}_{i\|j}(v)\sqrt{a}dx = L^{-3N-3}(v) \text{ for all } v \in V(\Omega),$$

where

$$L^{-3N-3}(v) = \int_\Omega f^{i,-3N-3} v_i \sqrt{a}dx + \int_{\Gamma_+\cup\Gamma_-} l^{i,-3N-2} v_i \sqrt{a}d\Gamma.$$

Since $E^{-2N-1}_{k\|l} = 0$ in (4.2), according the first requirement imply that we must let $f^{i,-3N-3} = 0$ and $l^{i,-3N-2} = 0$ leads to

$$L^{-3N-3}(v) = 0 \text{ for all } v \in V(\Omega),$$

which leads to the assumption that there exist $f^{-3N-2} \in L^2(\Omega)$ and $l^{-3N-1} \in L^2(\Gamma_+ \cup \Gamma_-)$ independent of ε such that

$$\begin{cases} f^i(\varepsilon) = \dfrac{1}{\varepsilon^{3N+2}} f^{i,-3N-2}, \\ l^i(\varepsilon) = \dfrac{1}{\varepsilon^{3N+1}} l^{i,-3N-1}. \end{cases}$$

The cancellation of the coefficient of ε^{-3N-2} in (2.27) then produces the equations:

$$\int_\Omega A^{ijkl}(0) E^{-2N}_{k\|l} F^{-N-2}_{i\|j}(v)\sqrt{a}dx = L^{-3N-2}(v) \text{ for all } v \in V(\Omega),$$

where

$$L^{-3N-2}(v) = \int_\Omega f^{i,-3N-2} v_i \sqrt{a}dx + \int_{\Gamma_+\cup\Gamma_-} l^{i,-3N-1} v_i \sqrt{a} d\Gamma.$$

Since $F^{-N-2}_{i\|j} = 0$ in (4.2) for all $v \in V(\Omega)$, according the first requirement imply that we must let

$$f^{i,-3N-2} = 0, l^{i,-3N-1} = 0 \tag{2.32}$$

leads to $L^{-3N-2}(v) = 0$ for all $v \in V(\Omega)$.

(iii) Assume that $N \geq 1$. The same type of assumption: there exist $f^{-3N-1} \in L^2(\Omega)$ and $l^{-3N} \in L^2(\Gamma_+ \cup \Gamma_-)$ independent of ε such that

$$\begin{cases} f^i(\varepsilon)(x) = \dfrac{1}{\varepsilon^{3N+1}} f^{i,-3N-1}(x), \\ l^i(\varepsilon)(x) = \dfrac{1}{\varepsilon^{3N}} l^{i,-3N}(x). \end{cases}$$

The cancellation of the coefficient of ε^{-3N-1} in (2.27) then produces the equations:

$$\int_\Omega A^{ijkl}(0) E^{-2N}_{k\|l} F^{-N-1}_{i\|j}(v)\sqrt{a}dx = L^{-3N-1}(v) \text{ for all } v \in V(\Omega),$$

From the relations (2.21)-(2.22)-(2.23)-(2.24)-(4.3)-(2.26) of proof the Lemma 2.2, we obtain

$$\begin{cases} E^{-2N}_{\alpha\|\beta} &= \dfrac{1}{2} a^{mn} u^{-N}_{m\|\alpha} u^{-N}_{n\|\beta},\ F^{-N-1}_{\alpha\|\beta}(v) = 0, \\ E^{-2N}_{\alpha\|3} &= \dfrac{1}{2} a^{mn} u^{-N}_{m\|\alpha} u^{-N}_{n\|3},\ F^{-N-1}_{\alpha\|3}(v) = \dfrac{1}{2} a^{mn} u^{-N}_{m\|\alpha} \partial_3 v_n, \\ E^{-2N}_{3\|3} &= \dfrac{1}{2} a^{mn} u^{-N}_{m\|3} u^{-N}_{n\|3},\ F^{-N-1}_{3\|3}(v) = a^{mn} u^{-N}_{m\|3} \partial_3 v_n. \end{cases} \tag{2.33}$$

Choosing $v \in V(\Omega)$ be independent of x_3 then shows that we must let $f^{i,-3N-1} = 0$ and $l^{i,-3N} = 0$ (the first requirement), we obtain

$$\int_\Omega A^{ijkl}(0) E^{-2N}_{k\|l} F^{-N-1}_{i\|j}(v)\sqrt{a}dx = 0 \text{ for all } v \in V(\Omega).$$

Let the field $w^N = (w_m^N)$ be defined by

$$w_m^N = u_m^{-N+1} - (1+x_3)\Gamma_{m3}^{p,0} u_P^{-N} \text{ for all } (y, x_3) \in \Omega.$$

Then $w^N \in V(\Omega)$ and $\partial_3 w_m^N = \partial_3 u_m^{-N+1} - \Gamma_{m3}^{p,0} u_p^{-N} = u_{m\|3}^{-N}$. Next, from (2.33) we get

$$F_{\alpha\|3}^{-N-1}(w^N) = E_{\alpha\|3}^{-2N} \text{ and } F_{3\|3}^{-N-1}(w^N) = 2E_{3\|3}^{-2N}.$$

Choosing $v = w^N$ the test function in the last variational equations, we find that

$$\int_\Omega A^{ijkl}(0) E_{k\|l}^{-2N} F_{i\|j}^{-N-1}(w^N)\sqrt{a}dx = 0. \tag{2.34}$$

We substitute the relations (2.14) of the Lemma 2.1 and (2.33) in equations (4.19), we obtain

$$\int_\Omega (\lambda a^{\sigma\tau} E_{\sigma\|\tau}^{-2N} E_{3\|3}^{-2N} + 2\mu a^{\alpha\sigma} E_{\alpha\|3}^{-2N} E_{\sigma\|3}^{-2N} + (\lambda+2\mu) E_{3\|3}^{-2N} E_{3\|3}^{-2N})\sqrt{a}dx = 0$$

or

$$\int_\Omega (\lambda a^{mn} E_{m\|n}^{-2N} E_{3\|3}^{-2N} + 2\mu a^{mn} E_{m\|3}^{-2N} E_{n\|3}^{-2N})\sqrt{a}dx = 0.$$

According the relation (2.15) from the Lemma 2.1, we observe that

$$a^{mn} E_{m\|n}^{-2N} = \frac{1}{2} a^{ij} a^{mn} u_{i\|m}^{-N} u_{j\|n}^{-N} \geq 0 \text{ in } \Omega$$

and

$$E_{3\|3}^{-2N} = \frac{1}{2} a^{mn} u_{m\|3}^{-N} u_{n\|3}^{-N} \geq 0 \text{ and } a^{mn} E_{m\|3}^{-2N} E_{n\|3}^{-2N} \geq 0 \text{ in } \Omega,$$

so that

$$a^{mn} E_{m\|3}^{-2N} E_{n\|3}^{-2N} = 0 \text{ in } \Omega,$$

consequently that

$$E_{m\|3}^{-2N} = 0 \text{ in } \Omega,$$

in special then, $E_{3\|3}^{-2N} = \dfrac{1}{2} a^{mn} u_{m\|3}^{-N} u_{n\|3}^{-N} = 0$ and thus

$$u_{m\|3}^{-N} = 0. \tag{2.35}$$

(iv) Assume that $N \geq 2$, the same type of assumption: there exist $f^{-3N} \in L^2(\Omega)$ and $l^{-3N+1} \in L^2(\Gamma_+ \cup \Gamma_-)$ independent of ε such that

$$\begin{cases} f^i(\varepsilon)(x) = \dfrac{1}{\varepsilon^{3N}} f^{i,-3N}(x), \\ l^i(\varepsilon)(x) = \dfrac{1}{\varepsilon^{3N-1}} l^{i,-3N+1}(x). \end{cases}$$

The cancellation of the coefficient of ε^{-3N} in (2.27) then produces the equations:

$$\begin{cases} \int_\Omega A^{ijkl}(0)\{E^{-2N}_{k\|l}F^{-N}_{i\|j}(v) + E^{-2N+1}_{k\|l}F^{-N-1}_{i\|j}(v)\}\sqrt{a}dx \\ +\int_\Omega x_3 B^{ijkl,1}E^{-2N}_{k\|l}F^{-N-1}_{i\|j}(v)dx = L^{-3N}(v) \text{ for all } v\in V(\Omega), \end{cases} \tag{2.36}$$

From the relations (2.21)-(2.22)-(2.23)-(2.24)-(4.3)-(2.26) of proof the Lemma 2.2, we obtain

$$\begin{cases} E^{-2N}_{i\|3} = 0, \quad E^{-2N}_{\alpha\|\beta} = \dfrac{1}{2}a^{mn}u^{-N}_{m\|\alpha}u^{-N}_{n\|\beta}, \\ F^{-N-1}_{\alpha\|\beta}(v) \quad = 0,\ F^{-N}_{\alpha\|\beta}(v) = \dfrac{1}{2}a^{mn}(u^{-N}_{m\|\alpha}v_{n\|\beta} + u^{-N}_{n\|\beta}v_{m\|\alpha}), \\ F^{-N-1}_{\alpha\|3}(v) \quad = \dfrac{1}{2}a^{mn}u^{-N}_{m\|\alpha}\partial_3 v_n,\ F^{-N}_{\alpha\|3}(v) = \dfrac{1}{2}a^{mn}(u^{-N+1}_{m\|\alpha}\partial_3 v_n - u^{-N}_{m\|\alpha}\Gamma^{\sigma,0}_{n3}v_\sigma), \\ F^{-N-1}_{3\|3}(v) \quad = 0,\ F^{-N}_{3\|3}(v) = a^{mn}u^{-N+1}_{m\|3}\partial_3 v_n. \end{cases} \tag{2.37}$$

Noting that, from (2.37) that $F^{-N-1}_{\alpha\|3}(v) = F^{-N}_{3\|3}(v) = 0$ if $\partial_3 v = 0$ and using the relation (2.14) from the Lemma 2.1, we thus infer that (2.36) reduce to

$$\int_\Omega A^{\alpha\beta\sigma\tau}(0)E^{-2N}_{\sigma\|\tau}F^{-N}_{\alpha\|\beta}(v)\sqrt{a}dx = L^{-3N}(v) \tag{2.38}$$

for all $v\in V(\Omega)$ that are independent of x_3, according the linearization trick, implies that $L^{-3N}(v) = 0$ for all $v \in V(\Omega)$ that are independent of x_3. Hence we must let $f^{i,-3N} = 0$ and $l^{i,-3N+1} = 0$.

Since u^{-N} is independent of x_3, we may let $v = u^{-N}$ is a test function in (2.38), we get

$$\int_\Omega A^{\alpha\beta\sigma\tau}(0)E^{-2N}_{\sigma\|\tau}E^{-2N}_{\alpha\|\beta}\sqrt{a}dx = 0,$$

since $F^{-N}_{\alpha\|\beta}(u^{-N}) = 2E^{-2N}_{\alpha\|\beta}$. Then

$$E^{-2N}_{\alpha\|\beta} = \frac{1}{2}a^{mn}u^{-N}_{m\|\alpha}u^{-N}_{n\|\beta} = 0 \text{ in } \Omega.$$

Using the relation (2.17) from the Lemma 2.2 imply that $u^{-N}_{m\|\alpha} = 0$ and from (2.35), consequently

$$u^{-N}_{m\|i} = 0.$$

From the relations (2.11) and (2.17), we get

$$u^{-N}_{\beta\|\alpha} = \partial_\alpha u^{-N}_\beta - \Gamma^{p,0}_{\alpha\beta}u^{-N}_p = \partial_\alpha u^{-N}_\beta - \Gamma^\sigma_{\alpha\beta}u^{-N}_\sigma - b_{\alpha\beta}u^{-N}_3,$$
$$u^{-N}_{3\|\alpha} = \partial_\alpha u^{-N}_3 - \Gamma^{p,0}_{\alpha 3}u^{-N}_p = \partial_\alpha u^{-N}_3 + b^\sigma_\alpha u^{-N}_\sigma.$$

Let $\zeta_i = u_i^{-N}\mid_{x_3=0}$. Then $\zeta_i \in W^{1,4}(\omega)$ since $u_i^{-N} \in W^{1,4}(\Omega)$ and $\zeta_i = 0$ on γ_0 since $u_i^{-N} = 0$ on Γ_0, imply that

$$u^{-N} = 0 \text{ for all } N \geq 2.$$

(v) Assume that $N = 1$, the first four steps of cancellation rest the same, leading to $\partial_3 u^{-1} = 0$, $u^{-1}_{m\|3} = 0$ when $N = 1$ in the relations (4.3) and (2.26) of proof the Lemma 2.2 the expression of $F^{-N}_{i\|3}$, becomes

$$F^{-1}_{\alpha\|3} = \frac{1}{2}\partial_3 v_\alpha + \frac{1}{2}a^{mn}(u^0_{m\|\alpha}\partial_3 v_n - u^{-1}_{m\|\alpha}\Gamma^{\sigma,0}_{n3}v_\sigma),$$
$$F^{-1}_{3\|3} = \partial_3 v_3 + a^{mn}u^0_{m\|3}\partial_3 v_n.$$

Therefore, only required the consideration of functions $v \in V(\Omega)$ that are independent of x_3, in this case $F^{-1}_{3\|3} = 0$, the second requirement of linearization can be retained, we show that

$$u^{-1} = 0.$$

Hence, the formal asymptotic expansion $u(\varepsilon)$ becomes

$$u(\varepsilon)(x) = u^0(x) + \varepsilon u^1(x) + \cdots , \tag{2.39}$$

$\blacksquare$

Chapter 3

ASYMPTOTIC JUSTIFICATION OF EQUATIONS FOR VON KÁRMÁN MEMBRANE SHELLS

The objective of this chapter is to study the asymptotic justification of the two- dimensional equations for membrane shells with boundary conditions of von Kármán's type. More precisely, we consider a three-dimensional model for a nonlinearly elastic membrane shell of Saint VenantâKirchhoff material, where only a portion of the lateral face is subjected to boundary conditions of von Kármán's type. Using technics from formal asymptotic analysis with the thickness of the shell as a small parameter, we show that the scaled three-dimensional solution still leads to the two-dimensional equations of von Kármán membrane shell. This work was published in [46]

3.1 THREE-DIMENSIONAL PROBLEMS OF VON KÁRMÁN MEMBRANE SHELL IN CARTESIAN COORDINATES

Let ω be a connected bounded open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ and let γ_0 be a relatively open subset of γ such that $length(\gamma_0) > 0$ and $length(\gamma_1) > 0$, where $\gamma_1 = \gamma \backslash \gamma_0$. Let $\theta : \bar{\omega} \to \mathbb{R}^3$ is a smooth enough injective immersion of class $\mathcal{C}^3$ such that θ_3 constant on the boundary γ_1.

We let $\hat{\Omega}^\varepsilon = \Theta(\Omega^\varepsilon)$, $\hat{\gamma}_1 = \theta(\gamma_1)$, $\hat{\gamma}_0 = \theta(\gamma_0)$, $(\hat{n}_i^\varepsilon)$ is the unit outer normal vector along the upper and lower faces $\hat{\Gamma}_\pm^\varepsilon = \Theta(\Gamma_\pm^\varepsilon)$. We let $\hat{\Gamma}_0^\varepsilon = \Theta(\Gamma_0^\varepsilon)$ and $\hat{\Gamma}_1^\varepsilon = \Theta(\Gamma_1^\varepsilon)$ the portions of the lateral face. Since θ_3 is constant on the boundary γ_1 of ω, the portion $\hat{\Gamma}_1^\varepsilon$ is vertical. We denote by $\hat{x}^\varepsilon = \Theta(x^\varepsilon)$ a generic point in $\overline{\hat{\Omega}^\varepsilon}$ and we let $\hat{\partial}_i^\varepsilon = \partial/\partial\hat{x}_i^\varepsilon$.

Consider a nonlinearly elastic membrane shell occupying in its reference configuration the set $\overline{\hat{\Omega}^\varepsilon}$, with middle surface $\hat{\omega} = \theta(\bar{\omega})$ and thickness $2\varepsilon > 0$, the material constituting the shell is a Saint Venant–Kirchhoff material, i.e., a homogeneous, isotropic material, whose Lamé constants are denoted by $\lambda^\varepsilon > 0$ and $\mu^\varepsilon > 0$. We assume that the reference configuration is a natural state. The shell is assumed to be clamped on the portion $\hat{\Gamma}_0^\varepsilon$.

The shell is subjected to body forces of density $(\hat{f}_i^\varepsilon) : \hat{\Omega}^\varepsilon \to \mathbb{R}^3$ and surface force on the upper and lower faces with density $(\hat{l}_i^\varepsilon) : \hat{\Gamma}_+^\varepsilon \cup \hat{\Gamma}_-^\varepsilon \to \mathbb{R}^3$. On the portion $\hat{\Gamma}_1^\varepsilon$, the shell is subjected to forces of von Kármán's type, which are horizontal, only the resultant $(\hat{h}_1^\varepsilon, \hat{h}_2^\varepsilon, 0)$ after integration across the thickness is given along $\hat{\gamma}_1$. We call this a "von Kármán membrane shell", see Fig. 3.1, cf. Examples of nonlinearly elastic membrane shells given in [31, Section 9.1].

Finally, we define the spaces

$$V(\hat{\Omega}^\varepsilon) = \begin{Bmatrix} \hat{v}^\varepsilon = (\hat{v}_i^\varepsilon) \in W^{1,4}(\hat{\Omega}^\varepsilon; \mathbb{R}^3); \ \hat{v}_i^\varepsilon = 0 \quad \text{on} \quad \hat{\Gamma}_0^\varepsilon, \\ \hat{v}_\alpha^\varepsilon \quad \text{independent of} \quad \hat{x}_3^\varepsilon \qquad \text{and} \quad \hat{v}_3^\varepsilon = 0 \quad \text{on} \quad \hat{\Gamma}_1^\varepsilon \end{Bmatrix},$$

$$\hat{\Sigma}^\varepsilon = \Big\{ \hat{\tau}^\varepsilon = (\hat{\tau}_{ij}^\varepsilon) \in (L^2(\hat{\Omega}^\varepsilon))^9; \ \hat{\tau}_{ij}^\varepsilon = \hat{\tau}_{ji}^\varepsilon \Big\}.$$

The unknown displacement field $\hat{u}^\varepsilon = (\hat{u}_i^\varepsilon)$ and stress field $\hat{\sigma}^\varepsilon = (\hat{\sigma}_{ij}^\varepsilon)$ satisfy the

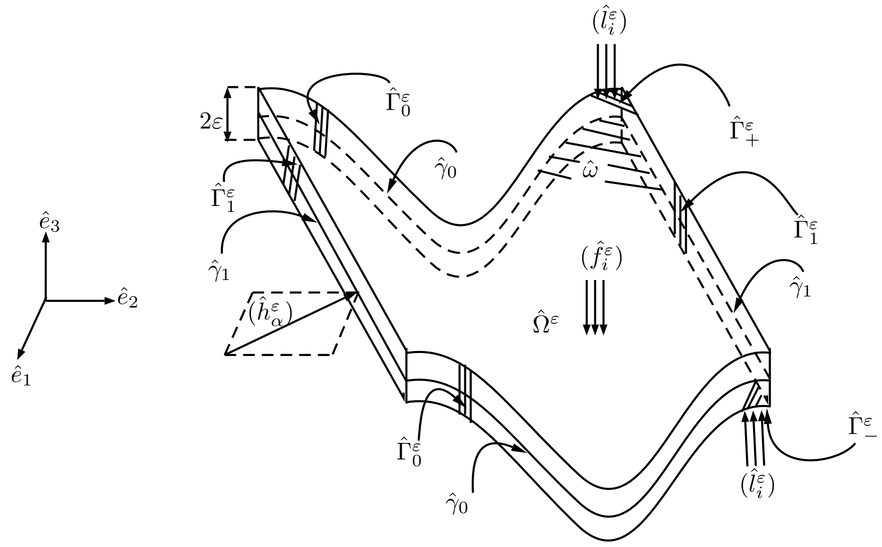

Figure 3.1: A von Kármán membrane shell.

following three-dimensional von Kármán shell problem in cartesian coordinates

$$\begin{cases} -\hat{\partial}_j^\varepsilon\Big(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon\hat{\partial}_k^\varepsilon\hat{u}_i^\varepsilon\Big) = \hat{f}_i^\varepsilon \quad \text{in} \quad \hat{\Omega}^\varepsilon, \\ \Big(\hat{\sigma}_{ij}^\varepsilon + \hat{\sigma}_{kj}^\varepsilon\hat{\partial}_k^\varepsilon\hat{u}_i^\varepsilon\Big)\hat{n}_j^\varepsilon = \hat{l}_i^\varepsilon \quad \text{on} \quad \hat{\Gamma}_-^\varepsilon \cup \hat{\Gamma}_+^\varepsilon, \\ \hat{u}_i^\varepsilon = 0 \quad \text{on} \quad \hat{\Gamma}_0^\varepsilon, \\ \begin{cases} \frac{1}{2\varepsilon}\displaystyle\int_{-\varepsilon}^{+\varepsilon}\Big(\hat{\sigma}_{\alpha\beta}^\varepsilon + \hat{\sigma}_{k\beta}^\varepsilon\hat{\partial}_k^\varepsilon\hat{u}_\alpha^\varepsilon\Big)\nu_\beta\, dx_3^\varepsilon = \hat{h}_\alpha^\varepsilon \quad \text{on} \quad \hat{\gamma}_1, \\ \hat{u}_\alpha^\varepsilon \quad \text{independent of} \quad \hat{x}_3^\varepsilon \quad \text{on} \quad \hat{\Gamma}_1^\varepsilon, \\ \hat{u}_3^\varepsilon = 0 \quad \text{on} \quad \hat{\Gamma}_1^\varepsilon, \end{cases} \end{cases}$$

such that the Piola-Kirchhoff stress tensor $(\hat{\sigma}_{ij}^\varepsilon)$ and the Green–Saint Venant strain tensor $(\hat{E}_{ij}(\hat{u}^\varepsilon))$ are given by

$$\begin{cases} \hat{\sigma}_{ij}^\varepsilon = \lambda^\varepsilon\hat{E}_{pp}^\varepsilon(\hat{u}^\varepsilon)\delta_{ij} + 2\mu^\varepsilon\hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon), \\ \hat{E}_{ij}^\varepsilon(\hat{u}^\varepsilon) = \frac{1}{2}\Big(\hat{\partial}_i^\varepsilon\hat{u}_j^\varepsilon + \hat{\partial}_j^\varepsilon\hat{u}_i^\varepsilon + \hat{\partial}_i^\varepsilon\hat{u}_m^\varepsilon\hat{\partial}_j^\varepsilon\hat{u}_m^\varepsilon\Big), \end{cases}$$

where δ_{ij} is the Kronecker's symbol.

First, we rewrite the previous boundary value problem in the weak form, by using Green's formula, we show that any smooth solution of the boundary value problem also satisfies the following variational problem

$$P(\hat{\Omega}^\varepsilon)\begin{cases}\text{Find } (\hat{u}^\varepsilon, \hat{\sigma}^\varepsilon) \in V(\hat{\Omega}^\varepsilon) \times \hat{\Sigma}^\varepsilon \quad \text{such that} \\ \displaystyle\int_{\hat{\Omega}^\varepsilon} \Big(\hat{\sigma}^\varepsilon_{ij} + \hat{\sigma}^\varepsilon_{kj}\hat{\partial}^\varepsilon_k \hat{u}^\varepsilon_i\Big)\hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_i \, d\hat{x}^\varepsilon = \int_{\hat{\Omega}^\varepsilon} \hat{f}^\varepsilon_i \hat{v}^\varepsilon_i \, d\hat{x}^\varepsilon + \int_{\hat{\Gamma}^\varepsilon_+ \cup \hat{\Gamma}^\varepsilon_-} \hat{l}^\varepsilon_i \hat{v}^\varepsilon_i \, d\hat{\Gamma}^\varepsilon \\ \displaystyle + \int_{\hat{\gamma}_1} \hat{h}^\varepsilon_\alpha \Big(\int_{-\varepsilon}^{\varepsilon} (\hat{v}^\varepsilon_\alpha \circ \Theta)\, dx^\varepsilon_3\Big)\, d\hat{\gamma}, \qquad \forall\, \hat{v}^\varepsilon \in V(\hat{\Omega}^\varepsilon).\end{cases}$$

Next, the variational problem $P(\hat{\Omega}^\varepsilon)$ may be formulated as a minimization problem

$$\hat{u}^\varepsilon \in V(\hat{\Omega}^\varepsilon) \quad \text{and} \quad \hat{J}^\varepsilon(\hat{u}^\varepsilon) = \inf_{\hat{v}^\varepsilon \in V(\hat{\Omega}^\varepsilon)} \hat{J}^\varepsilon(\hat{v}^\varepsilon),$$

such that the stored energy function $\hat{J}^\varepsilon$ of a Saint Venant–Kirchhoff material given by

$$\begin{aligned}\hat{J}^\varepsilon(\hat{v}^\varepsilon) = \frac{1}{2}\int_{\hat{\Omega}} \hat{A}^{ijkl,\varepsilon} \hat{E}^\varepsilon_{kl}(\hat{v}^\varepsilon)\hat{E}^\varepsilon_{ij}(\hat{v}^\varepsilon)\, d\hat{x}^\varepsilon \\ - \Big\{\int_{\hat{\Omega}^\varepsilon} \hat{f}^\varepsilon_i \hat{v}^\varepsilon_i \, d\hat{x}^\varepsilon + \int_{\hat{\Gamma}^\varepsilon_+ \cup \hat{\Gamma}^\varepsilon_-} \hat{l}^\varepsilon_i \hat{v}^\varepsilon_i \, d\hat{\Gamma}^\varepsilon + \int_{\hat{\gamma}_1} \hat{h}^\varepsilon_\alpha \Big(\int_{-\varepsilon}^{\varepsilon} (\hat{v}^\varepsilon_\alpha \circ \Theta)\, dx^\varepsilon_3\Big)\, d\hat{\gamma}\Big\},\end{aligned}$$

where

$$\hat{A}^{ijkl,\varepsilon} = \lambda^\varepsilon \delta^{ij}\delta^{kl} + \mu^\varepsilon \Big(\delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\Big),$$

Because of the material constituting the shell and its boundary conditions, we cannot use the implicit function theorem (valid for a Saint Venant–Kirchhoff material with smooth boundaries) and existence theory is due to Ball [48] for polyconvex stored energy (the stored energy function of a Saint Venant–Kirchhoff material is not polyconvex, see [49]). For a more detailed survey, see [50], [31], and we refer to [51] for some open problems in elasticity. Recently, some new existence results found in [52], [53].

3.2 THREE-DIMENSIONAL VARIATIONAL PROBLEM OF VON KÁRMÁN MEMBRANE SHELL IN CURVILINEAR COORDINATES

As previously stated in Sect 2.2, we follow the same method, clearly that, $v^\varepsilon_i = 0$ on Γ^ε_0.

In order to find a guaranteeing of boundary conditions on Γ^ε_1, we compute the components $g^\varepsilon_i(x^\varepsilon)$ on Γ^ε_1.

Note that (see, for example, part (iv) of the proof of Theorem 2.8-1 in [54])

$$g_\alpha^\varepsilon(x^\varepsilon) = a_\alpha(y) + x_3^\varepsilon a_3(y) \quad \text{and} \quad g_3^\varepsilon(x^\varepsilon) = a_3(y).$$

Since θ_3 is constant on γ_1 by assumption, then it is easy to obtain, for all $y \in \gamma_1$,

$$a_1(y) = \begin{pmatrix} \partial_1\theta_1 \\ \partial_1\theta_2 \\ 0 \end{pmatrix}(y), \qquad a_2(y) = \begin{pmatrix} \partial_2\theta_1 \\ \partial_2\theta_2 \\ 0 \end{pmatrix}(y),$$

$$(a_1 \wedge a_2)(y) = \begin{pmatrix} 0 \\ 0 \\ \partial_1\theta_1.\partial_2\theta_2 - \partial_1\theta_2.\partial_2\theta_1 \end{pmatrix}(y).$$

Then we have v_α^ε is independent of x_3^ε and $v_3^\varepsilon = 0$ on Γ_1^ε.

We deduce that if $\hat{v}^\varepsilon$ is in $V(\hat{\Omega}^\varepsilon)$, then v^ε is in the following space

$$V(\Omega^\varepsilon) = \left\{ \begin{matrix} v^\varepsilon = (v_i^\varepsilon) \in W^{1,4}(\Omega^\varepsilon;\mathbb{R}^3); \; v_i^\varepsilon = 0 \quad \text{on} \quad \Gamma_0^\varepsilon, \\ v_\alpha^\varepsilon \quad \text{independent of} \quad x_3^\varepsilon \qquad \text{and} \quad v_3^\varepsilon = 0 \quad \text{on} \quad \Gamma_1^\varepsilon \end{matrix} \right\}.$$

The length element $d\gamma(y) = \left\{dy^T\,dy\right\}^{1/2}$ is transformed through the components $a_{\alpha\beta}(y)$ into $d\hat{\gamma}(\hat{y})$ of the form (see, for example,[The relation (1.13), and Theorem 1.5])

$$d\hat{\gamma}(\hat{y}) = \left\{dy^\alpha a_{\alpha\beta}(y)\,dy^\beta\right\}^{1/2}, \qquad \forall\, \hat{y} = \theta(y), \quad y \in \gamma_1.$$

The length element $d\hat{\gamma}$ cannot be expressed in terms of $d\gamma$, like the formulas found in the relations (2.6) and (2.7). For simplicity, we assume that there exist a smooth function $\rho(y) : \gamma_1 \to \mathbb{R}$ such that

$$d\hat{\gamma}(\hat{y}) = \rho(y)\,d\gamma(y), \qquad \forall\, \hat{y} = \theta(y), \quad y \in \gamma_1.$$

Particularly, in the case of shallow shell, where the initial shell curvature is assumed to be small, the function θ is defined by

$$\theta(y) = (y_1, y_2, O(\varepsilon)),$$

then we obtain $\rho(y) = 1$, see, for example, [55].

Now we associate with the Cartesian components of the von Kármán forces $\hat{h}^{i,\varepsilon} = \hat{h}_i^\varepsilon$, the contravariant components $h^{i,\varepsilon} \in L^2(\gamma_1)$ defined by

$$\hat{h}_i^\varepsilon(\hat{y})\hat{e}^i = h^{i,\varepsilon}(y) g_i^\varepsilon(y,0), \qquad \forall\, \hat{y} = \theta(y), \quad y \in \gamma_1.$$

In particular, we see that $h^{3,\varepsilon} = 0$.

Hence that for all $v^\varepsilon \in V(\Omega^\varepsilon)$, we have

$$\begin{aligned}
\int_{\hat{\gamma}_1} \hat{h}^\varepsilon_\alpha \Big(\int_{-\varepsilon}^{+\varepsilon} (\hat{v}^\varepsilon_\alpha \circ \Theta)\, dx^\varepsilon_3 \Big)\, d\hat{\gamma} &= \int_{\gamma_1} h^{\beta,\varepsilon} [g^\varepsilon_\beta]^\alpha \Big(\int_{-\varepsilon}^{+\varepsilon} v^\varepsilon_\varsigma [g^{\varsigma,\varepsilon}]_\alpha \, dx^\varepsilon_3 \Big) \rho \, d\gamma \\
&= \int_{\gamma_1} \rho h^{\beta,\varepsilon} [g^\varepsilon_\beta]^\alpha v^\varepsilon_\varsigma [g^{\varsigma,\varepsilon}]_\alpha \Big(\int_{-\varepsilon}^{+\varepsilon} dx^\varepsilon_3 \Big)\, d\gamma \\
&= 2\varepsilon \int_{\gamma_1} \rho h^{\alpha,\varepsilon} v^\varepsilon_\alpha \, d\gamma.
\end{aligned}$$

We indicate if the curve γ_1 be parameterized by its arc length through the mapping ϱ, i.e.,

$$\gamma_1 = \Big\{ \varrho(t);\ t \in I \Big\},$$

where ϱ is a smooth enough injective mapping and I is a compact interval. Then the length element $d\hat{\gamma}$ is given by

$$d\hat{\gamma}(\hat{y}) = \Big\{ a_{\alpha\beta}(\varrho(t)) \frac{d\varrho^\alpha}{dt}(t) \frac{d\varrho^\beta}{dt}(t) \Big\}^{1/2} dt.$$

For more details about this, see, for example, [1.13, Theorem 1.5].

Consequently, the variational problem $P(\hat{\Omega}^\varepsilon)$ is equivalent to the following variational problem in curvilinear coordinates

$$P(\Omega^\varepsilon) \begin{cases} \text{Find } u^\varepsilon \in V(\Omega^\varepsilon) \quad \text{such that} \\ \displaystyle\int_{\Omega^\varepsilon} \mathrm{A}^{ijkl,\varepsilon} E^\varepsilon_{k\|l}(u^\varepsilon) F^\varepsilon_{i\|j}(u^\varepsilon, v^\varepsilon) \sqrt{g^\varepsilon} dx^\varepsilon = \int_{\Omega^\varepsilon} f^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon} dx^\varepsilon + \int_{\Gamma^\varepsilon_- \cup \Gamma^\varepsilon_+} l^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon} d\Gamma^\varepsilon \\ \displaystyle +2\varepsilon \int_{\gamma_1} \rho h^{\alpha,\varepsilon} v^\varepsilon_\alpha d\gamma, \qquad \forall\, v^\varepsilon \in V(\Omega^\varepsilon), \end{cases}$$

where

$$\begin{cases} A^{ijkl,\varepsilon} = \lambda^\varepsilon g^{ij,\varepsilon} g^{kl,\varepsilon} + \mu^\varepsilon \Big(g^{ik,\varepsilon} g^{jl,\varepsilon} + g^{il,\varepsilon} g^{jk,\varepsilon} \Big), \\ F^\varepsilon_{i\|j}(u^\varepsilon, v^\varepsilon) = (E^\varepsilon_{i\|j})'(u^\varepsilon) v^\varepsilon. \end{cases}$$

Therefore, the stored energy function J^ε of a Saint Venant–Kirchhoff material in curvilinear coordinates given by

$$\begin{aligned}
J^\varepsilon(v^\varepsilon) = \frac{1}{2} \int_{\Omega^\varepsilon} & A^{ijkl,\varepsilon} E^\varepsilon_{k\|l}(v^\varepsilon) E^\varepsilon_{i\|j}(v^\varepsilon) \sqrt{g^\varepsilon}\, dx^\varepsilon \\
& - \Big\{ \int_{\Omega^\varepsilon} f^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon}\, dx^\varepsilon + \int_{\Gamma^\varepsilon_- \cup \Gamma^\varepsilon_+} l^{i,\varepsilon} v^\varepsilon_i \sqrt{g^\varepsilon}\, d\Gamma^\varepsilon + 2\varepsilon \int_{\gamma_1} \rho h^{\alpha,\varepsilon} v^\varepsilon_\alpha \, d\gamma \Big\}.
\end{aligned}$$

3.3 FORMAL ASYMPTOTIC ANALYSIS

3.3.1 SCALED THREE-DIMENSIONAL VARIATIONAL PROBLEM OF VON KÁRMÁN MEMBRANE SHELL

Using technics from asymptotic analysis in Chapter 2, we transform the problem $P(\Omega^\varepsilon)$ into asymptotically equivalent problem posed over a domain independent of ε.

More specifically, we let

$$\Omega = \omega \times]-1, 1[, \qquad \Gamma_0 = \gamma_0 \times [-1, 1], \qquad \Gamma_1 = \gamma_1 \times [-1, 1], \qquad \Gamma_\pm = \omega \times \{\pm 1\},$$

We define the space

$$V(\Omega) = \begin{Bmatrix} v = (v_i) \in \mathrm{W}^{1,4}(\Omega; \mathbb{R}^3);\ v_i = 0 \quad \text{on} \quad \Gamma_0, \\ v_\alpha \quad \text{independent of} \quad x_3 \qquad \text{and} \quad v_3 = 0 \quad \text{on} \quad \Gamma_1 \end{Bmatrix}.$$

Next, we make the following assumptions the function $h^\alpha \in L^2(\gamma_1)$ are independent of $\varepsilon > 0$ such that

$$h^\alpha(\varepsilon)(y) = h^{\alpha,\varepsilon}(y) \qquad \forall\, y \in \gamma_1.$$

We thus have the following result.

Theorem 3.1 *The scaled unknown $u(\varepsilon)$ satisfies the following variational equations*

$$P(\varepsilon;\Omega) \begin{cases} \textit{Find } u(\varepsilon) \in V(\Omega) \quad \textit{such that} \\ \varepsilon \displaystyle\int_\Omega \mathrm{A}^{ijkl}(\varepsilon) E_{k\|l}(\varepsilon, u(\varepsilon)) F_{i\|j}(\varepsilon, u(\varepsilon), v) \sqrt{g(\varepsilon)}\, dx = \varepsilon \int_\Omega f^i(\varepsilon) v_i \sqrt{g(\varepsilon)}\, dx \\ + \displaystyle\int_{\Gamma_- \cup \Gamma_+} l^i(\varepsilon) v_i \sqrt{g(\varepsilon)}\, d\Gamma + 2\varepsilon \int_{\gamma_1} \rho h^\alpha(\varepsilon) v_\alpha\, d\gamma, \qquad \forall\, v \in V(\Omega), \end{cases}$$

3.3.2 FORMAL ASYMPTOTIC EXPANSIONS METHODS

The objective of the asymptotic analysis is to study the behavior of the solution $u(\varepsilon)$ of the problem $P(\varepsilon;\Omega)$ when ε approaches zero. To this end, in order to obtain a membrane model in the limit, we transform the variational problem $P(\varepsilon;\Omega)$ into the following singular *perturbation problem*

$$P^{\star}(\varepsilon;\Omega)\begin{cases} \textit{Find } u(\varepsilon)\in V(\Omega) \quad \textit{such that} \\ \displaystyle\int_{\Omega} \mathrm{A}^{ijkl}(\varepsilon)E_{k\|l}(\varepsilon,u(\varepsilon))F_{i\|j}(\varepsilon,u(\varepsilon),v)\sqrt{g(\varepsilon)}\,dx = \int_{\Omega} f^{i}(\varepsilon)v_i\sqrt{g(\varepsilon)}\,dx \\ +\frac{1}{\varepsilon}\displaystyle\int_{\Gamma_-\cup\Gamma_+} l^{i}(\varepsilon)v_i\sqrt{g(\varepsilon)}\,d\Gamma + 2\int_{\gamma_1}\rho h^{\alpha}(\varepsilon)v_{\alpha}\,d\gamma, \qquad \forall\, v\in V(\Omega). \end{cases}$$

Next, we write the scaled unknown as a formal expansion in terms of powers of the thickness as relation 2.9.

We now show that the expansion (2.9) *begins with a term of order 0 with respect to* ε.

Theorem 3.2 *Assume that the scaled unknown satisfying problem* $P^{\star}(\varepsilon;\Omega)$ *admits for each* $0<\varepsilon\leq\varepsilon_0$ *a formal asymptotic expansion of the form* (2.9) *with* $u^{-N}, u^{-N+1}\in V(\Omega)$, *and* $u^{-N}\neq 0$ *for some integer* $N\in\mathbb{Z}$. *Then* $N=0$.

Proof. *According the relations* (2.27), *then the problem* $P^{\star}(\varepsilon;\Omega)$ *is rewritten as follows:*

$$\begin{cases} \displaystyle\int_{\Omega} A^{ijkl}(0)E_{k\|l}(\varepsilon,u(\varepsilon))F_{i\|j}(\varepsilon,u(\varepsilon),v)\sqrt{a}\,dx \\ +\displaystyle\int_{\Omega}\Big(\varepsilon B^{ijkl,1}+\varepsilon^2 B^{ijkl,2}+o(\varepsilon^2)\Big)E_{k\|l}(\varepsilon,u(\varepsilon))F_{i\|j}(\varepsilon,u(\varepsilon),v)\,dx, = \int_{\Omega} f^{i}(\varepsilon)v_i\sqrt{g(\varepsilon)}\,dx \\ +\frac{1}{\varepsilon}\displaystyle\int_{\Gamma_-\cup\Gamma_+} l^{i}(\varepsilon)v_i\sqrt{g(\varepsilon)}\,d\Gamma + 2\int_{\gamma_1}\rho h^{\alpha}(\varepsilon)v_{\alpha}\,d\gamma, \qquad \forall\, v\in V(\Omega). \end{cases} \tag{3.1}$$

The proof is long, and similar to the proof of Theorem 2.2. The only extra term appearing here, comes from the functions $h^{\alpha}(\varepsilon)$. *Taking into account the two basic requirements systematized by Ciarlet [31], the first one asserts that no restriction can be put on the applied forces and the second is the linearization requirement.*

In conclusion, first, we show that the first term of formal asymptotic expansion u^{-N} *is independent of* x_3, *i.e., that satisfies*

$$\partial_3 u^{-N} = 0 \quad in \quad \Omega. \tag{3.2}$$

Then we have

$$E_{3\|3}^{-2N-2} = 0 \quad in \quad \Omega \quad and \quad F_{3\|3}^{-N-2}(v) = 0, \qquad \forall\, v \in V(\Omega).$$

Therefore, u^{-N} belongs to the space

$$V_M(\omega) = \Big\{\eta = (\eta_i) \in W^{1,4}(\omega);\ \eta = 0 \quad on \quad \gamma_0, \qquad \eta_3 = 0 \quad on \quad \gamma_1\Big\}. \tag{3.3}$$

We also have

$$E_{\alpha\|\beta}^{-2N-2} = E_{i\|j}^{-2N-1} = 0 \quad in \quad \Omega \quad and \quad F_{\alpha\|\beta}^{-N-2}(v) = 0, \qquad \forall\, v \in V(\Omega). \tag{3.4}$$

Next, we obtain

$$\begin{cases} E_{i\|3}^{-2N} = 0, & E_{\alpha\|\beta}^{-2N} = \frac{1}{2} a^{mn} u_{m\|\alpha}^{-N} u_{n\|\beta}^{-N}, \\ F_{\alpha\|\beta}^{-N-1}(v) = 0, & F_{\alpha\|\beta}^{-N}(v) = \frac{1}{2} a^{mn}\Big(u_{m\|\alpha}^{-N} v_{n\|\beta} + u_{n\|\beta}^{-N} v_{m\|\alpha}\Big), \\ F_{\alpha\|3}^{-N-1}(v) = \frac{1}{2} a^{mn} u_{m\|\alpha}^{-N} \partial_3 v_n, & F_{\alpha\|3}^{-N}(v) = \frac{1}{2} a^{mn}\Big(u_{m\|\alpha}^{-N+1} \partial_3 v_n - u_{m\|\alpha}^{-N} \Gamma_{n3}^{\sigma,0} v_\sigma\Big), \\ F_{3\|3}^{-N-1}(v) = 0, & F_{3\|3}^{-N}(v) = a^{mn} u_{m\|3}^{-N+1} \partial_3 v_n. \end{cases}$$

Ultimately, we conclude that the formal asymptotic expansion $u(\varepsilon)$ becomes

$$u(\varepsilon)(x) = u^0(x) + \varepsilon u^1(x) + \cdots\ . \tag{3.5}$$

■

3.4 TWO-DIMENSIONAL MODEL OF VON KÁRMÁN MEMBRANE SHELL

3.4.1 TWO-DIMENSIONAL VARIATIONAL EQUATION OF VON KÁRMÁN MEMBRANE SHELL

Before giving the limiting two-dimensional model of von Kármán membrane shell, we will need the following Lemma.

Lemma 3.1 *Let $u \in L^2(\Omega)$ such that $\displaystyle\int_\Omega u \cdot \partial_3 v dx = 0$ for all $v \in V(\Omega)$, then $u = 0$.*

Proof. *See proof of Theorem 3.4-1 in [31].* ■

Theorem 3.3 *Assume that the scaled unknown $u(\varepsilon)$ satisfying the three-dimensional variational problem $P^\star(\varepsilon;\Omega)$ admits a formal asymptotic expansion of the form (3.5).*

Under the two basic requirements, the components of the applied forces must be scaled as follows:

$$\begin{cases} f^\varepsilon(x^\varepsilon) = f(\varepsilon)(x) = f^0(x), & \forall\, x \in \Omega, \\ l^\varepsilon(x^\varepsilon) = l(\varepsilon)(x) = \varepsilon l^1(x), & \forall\, x \in \Gamma_+ \cup \Gamma_-, \\ h^\varepsilon(y) = h(\varepsilon)(y) = h^0(y), & \forall\, y \in \gamma_1, \end{cases}$$

where the scaled functions $f^0 \in L^2(\Omega)$, $l^1 \in L^2(\Gamma_+ \cup \Gamma_-)$, and $h^0 \in L^2(\gamma_1)$ are independent of ε.

Then the leading term u^0 is independent of the transverse variable x_3 and its average

$$\zeta^0 = (\zeta_i^0) = \frac{1}{2}\int_{-1}^{1} u^0 dx_3,$$

satisfies the following scaled two-dimensional variational equation

$$P_M(\omega) \begin{cases} \textit{Find } \zeta^0 \in V_M(\omega) \quad \textit{such that} \\ \frac{1}{2}\displaystyle\int_\omega a^{\alpha\beta\sigma\tau} E^0_{\sigma\|\tau}(\zeta^0) F^0_{\alpha\|\beta}(\eta)\sqrt{a}\, dy = \int_\omega p^{i,0}\eta_i \sqrt{a}\, dy \\ +2\displaystyle\int_{\gamma_1} \rho h^{\alpha,0}\eta_\alpha\, d\gamma, \qquad \forall\, \eta \in V_M(\omega), \end{cases}$$

where

$$\begin{cases} E^0_{\alpha\|\beta} = \frac{1}{2}\Big(\zeta^0_{\alpha\|\beta} + \zeta^0_{\beta\|\alpha} + a^{mn}\zeta^0_{m\|\alpha}\zeta^0_{n\|\beta}\Big), \\ F^0_{\alpha\|\beta}(\eta) = \frac{1}{2}\Big(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\Big\{\zeta^0_{m\|\alpha}\eta_{n\|\beta} + \zeta^0_{n\|\beta}\eta_{m\|\alpha}\Big\}\Big), \\ \eta_{\alpha\|\beta} = \partial_\beta\eta_\alpha - \Gamma^\sigma_{\alpha\beta}\eta_\sigma - b_{\alpha\beta}\eta_3, \\ \eta_{3\|\beta} = \partial_\beta\eta_3 + b^\sigma_\beta\eta_\sigma, \\ a^{\alpha\beta\sigma\tau} = \frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau} + 2\mu\Big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\Big), \\ p^{i,0} = \displaystyle\int_{-1}^{1} f^{i,0}dx_3 + l^{i,1}_+ + l^{i,1}_- \qquad \text{with} \quad l^{i,1}_\pm = l^{i,1}(.,\pm 1). \end{cases}$$

Proof. *For clarity, the proof is divided into three parts.*

(i) The first part (i) of the proof of Theorem 3.2 remains valid in case $N = 0$ *(i.e., the cancellation of the factors of* ε^{-4}*,* ε^{-3} *and* ε^{-2}*). It follows from* (3.2) *that*

$$\partial_3 u^0 = 0 \quad in \quad \Omega,$$

which implies $\zeta^0 \in V_M(\omega)$ *(where the space* $V_M(\omega)$ *is defined as in* (3.3)*).*

For any integer $r \geq -1$*,* L^r *denotes the linear form, be defined as follows:*

$$L^r(v) = \int_\Omega f^{i,r}v_i\sqrt{a}dx + \int_{\Gamma_-\cup\Gamma_+} l^{i,r+1}v_i\sqrt{a}d\Gamma + 2\int_{\gamma_1}\rho h^{\alpha,r}v_\alpha d\gamma,$$

where the functions $f^{i,r} \in L^2(\Omega)$*,* $l^{i,r+1} \in L^2(\Gamma_+ \cup \Gamma_-)$*, and* $h^{\alpha,r} \in L^2(\gamma_1)$ *are independent of* ε*.*

(ii) We assume that there exist $f^{-1} \in L^2(\Omega)$*,* $l^0 \in L^2(\Gamma_+ \cup \Gamma_-)$*, and* $h^{-1} \in L^2(\gamma_1)$ *are independent of* ε *such that*

$$\begin{cases} f^i(\varepsilon)(x) = \frac{1}{\varepsilon}f^{i,-1}(x), \\ l^i(\varepsilon)(x) = l^{i,0}(x), \\ h^\alpha(\varepsilon)(y) = \frac{1}{\varepsilon}h^{\alpha,-1}(y). \end{cases}$$

The cancellation of the factor of ε^{-1} *in* (3.1)*, it follows from* (3.4) *that*

$$\int_\Omega A^{ijkl}(0)E^0_{k\|l}F^{-1}_{i\|j}(v)\sqrt{a}\,dx = L^{-1}(v), \qquad \forall\, v \in V_M(\omega).$$

Using the relations (2.16) *and* (2.17) *in the case* $N = 0$*, we have that*

$$\begin{cases} E^0_{\alpha\|\beta} = \frac{1}{2}\Big(u^0_{\alpha\|\beta} + u^0_{\beta\|\alpha} + a^{mn}u^0_{m\|\alpha}u^0_{n\|\beta}\Big), & F^{-1}_{\alpha\|\beta}(v) = 0, \\ E^0_{\alpha\|3} = \frac{1}{2}\Big(u^{(0)}_{\alpha\|3} + u^0_{3\|\alpha} + a^{mn}u^0_{m\|\alpha}u^{(0)}_{n\|3}\Big), & F^{-1}_{\alpha\|3}(v) = \frac{1}{2}\Big(\partial_3 v_\alpha + a^{mn}u^0_{m\|\alpha}\partial_3 v_n\Big), \\ E^0_{3\|3} = u^{(0)}_{3\|3} + \frac{1}{2}a^{mn}u^{(0)}_{m\|3}u^{(0)}_{n\|3}, & F^{-1}_{3\|3}(v) = \partial_3 v_3 + a^{mn}u^{(0)}_{m\|\alpha}\partial_3 v_n, \end{cases} \tag{3.6}$$

with

$$\begin{cases} u^0_{m\|\alpha} = \partial_\alpha u^0_m - \Gamma^{p,0}_{\alpha m}u^0_p, \\ u^{(0)}_{m\|3} = \partial_3 u^1_m - \Gamma^{p,0}_{m3}u^0_p. \end{cases}$$

The special notation $u^{(0)}_{m\|3}$ *is due to Ciarlet [31], indicates that* $u^{(0)}_{m\|3}$ *also depend on* u^1*.*

The expressions of the functions $F^{-1}_{i\|j}(v)$ *are found in* (3.6) *imply that* $L^{-1}(v) = 0$ *for all* $v \in V(\Omega)$ *that are independent of* x_3*.*

The first requirement implies that $f^{i,-1} = 0$*,* $l^{i,0} = 0$*, and* $h^{\alpha,-1} = 0$*.*

Hence we obtain

$$\int_\Omega A^{ijkl}(0)E^0_{k\|l}F^{-1}_{i\|j}(v)\sqrt{a}\,dx = 0, \qquad \forall\, v \in V(\Omega).$$

Using the relations (2.13)*,* (2.14)*, and* (3.6)*, we obtain*

$$\begin{aligned} &\int_\Omega A^{ijkl}(0)E^0_{k\|l}F^{-1}_{i\|j}(v)\sqrt{a}\,dx \\ &= \int_\Omega 2\mu a^{\alpha\sigma}E^0_{\alpha\|3}\Big(\partial_3 v_\sigma + a^{mn}u^0_{m\|\sigma}\partial_3 v_n + u^0_{3\|\sigma}\partial_3 v_3\Big)\sqrt{a}\,dx \\ &\quad + \int_\Omega \Big(\lambda a^{\alpha\beta}E^0_{\alpha\|\beta} + \Big(\lambda + 2\mu\Big)E^0_{3\|3}\Big)\Big(\Big(1 + u^{(0)}_{3\|3}\Big)\partial_3 v_3 + a^{mn}u^{(0)}_{m\|3}\partial_3 v_n\Big)\sqrt{a}\,dx \\ &= \int_\Omega \Big(2\mu E^0_{\alpha\|3}\Big(a^{\alpha\tau} + a^{\alpha\sigma}a^{\beta\tau}u^0_{\beta\|\sigma}\Big) + \Big(\lambda a^{\alpha\beta}E^0_{\alpha\|\beta} + \Big(\lambda + 2\mu\Big)E^0_{3\|3}\Big)a^{\sigma\tau}u^{(0)}_{\sigma\|3}\Big)\partial_3 v_\tau\sqrt{a}\,dx \\ &\quad + \int_\Omega \Big(2\mu a^{\alpha\sigma}E^0_{\alpha\|3}u^0_{3\|\sigma} + \Big(\lambda a^{\alpha\beta}E^0_{\alpha\|\beta} + \Big(\lambda + 2\mu\Big)E^0_{3\|3}\Big)\Big(1 + u^{(0)}_{3\|3}\Big)\Big)\partial_3 v_3\sqrt{a}\,dx \\ &= 0. \end{aligned} \tag{3.7}$$

The last integral in (3.7) *takes the form* $\Big(w^\tau\partial_3 v_\tau + w^3\partial_3 v_3\Big)$*.*

Applying Lemma 3.1, shows that

$$\Big(\lambda a^{\alpha\beta}E^0_{\alpha\|\beta}+\Big(\lambda+2\mu\Big)E^0_{3\|3}\Big)a^{\sigma\tau}u^{(0)}_{\sigma\|3}+2\mu E^0_{\alpha\|3}\Big(a^{\alpha\tau}+a^{\alpha\sigma}a^{\beta\tau}u^0_{\beta\|\sigma}\Big)=0 \quad in \quad \Omega,$$
$$\Big(\lambda a^{\alpha\beta}E^0_{\alpha\|\beta}+\Big(\lambda+2\mu\Big)E^0_{3\|3}\Big)\Big(1+u^{(0)}_{3\|3}\Big)+2\mu a^{\alpha\sigma}E^0_{\alpha\|3}u^0_{3\|\sigma}=0 \quad in \quad \Omega.$$

This nonlinear system, has the obvious solution

$$\lambda a^{\alpha\beta}E^0_{\alpha\|\beta}+\Big(\lambda+2\mu\Big)E^0_{3\|3}=0 \quad and \quad E^0_{\alpha\|3}=0 \quad in \quad \Omega. \tag{3.8}$$

In order to recover the linear model suggested by the linearization requirement of Section 3.3.2, we consider only this obvious solution in the sequel, further details may be found in part (ii) of the proof of Theorem 8.8-1 in [31].

(iii) We assume that there exist $f^0\in L^2(\Omega)$, $l^1\in L^2(\Gamma_+\cup\Gamma_-)$, and $h^0\in L^2(\gamma_1)$ are independent of ε such that

$$\begin{cases} f^i(\varepsilon)(x)=f^{i,0}(x),\\ l^i(\varepsilon)(x)=\varepsilon l^{i,1}(x),\\ h^\alpha(\varepsilon)(y)=h^{\alpha,0}(y).\end{cases}$$

The cancellation of the factor of ε^0 in (3.1), it follows from (3.4) that

$$\int_\Omega A^{ijkl}(0)\Big\{E^0_{k\|l}F^0_{i\|j}(v)+E^1_{k\|l}F^{-1}_{i\|j}(v)\Big\}\sqrt{a}\,dx+\int_\Omega B^{ijkl,1}E^0_{k\|l}F^{-1}_{i\|j}(v)\,dx=L^0(v), \qquad \forall\, v\in V(\Omega).$$

The expressions of the functions $F^{-1}_{i\|j}(v)$ are found in (3.6) imply that $F^{-1}_{i\|j}(v)=0$ for all $v\in V(\Omega)$ that are independent of x_3.

Obviously, we must have that

$$\int_\Omega A^{ijkl}(0)E^0_{k\|l}F^0_{i\|j}(\eta)\sqrt{a}\,dx=L^0(\eta), \qquad \forall\,\eta\in V_M(\omega),$$

where $L^0(\eta)=\displaystyle\int_\omega p^{i,0}\eta_i\sqrt{a}\,d\omega+2\int_{\gamma_1}\rho h^{\alpha,0}\eta_\alpha\,d\gamma.$

Using the relations (2.13), (2.14), and (3.8), we obtain

$$
\begin{aligned}
&\int_{\Omega} A^{ijkl}(0) E^0_{k\|l} F^0_{i\|j}(\eta)\sqrt{a}\,dx \\
&= \int_{\Omega} \Big(\lambda a^{\alpha\beta} a^{\sigma\tau} + \mu\Big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\Big)\Big) E^0_{\sigma\|\tau} F^0_{\alpha\|\beta}(\eta)\sqrt{a}\,dx \\
&\quad + \int_{\Omega} \Big(4\mu a^{\alpha\sigma} E^0_{\alpha\|3} F^0_{\sigma\|3}(\eta) + \lambda a^{\alpha\beta} E^0_{3\|3} F^0_{\alpha\|\beta}(\eta)\Big)\sqrt{a}\,dx \\
&\quad + \int_{\Omega} \Big(\lambda a^{\sigma\tau} E^0_{\sigma\|\tau} + \Big(\lambda + 2\mu\Big) E^0_{3\|3}\Big) F^0_{3\|3}(\eta)\sqrt{a}\,dx \\
&= \int_{\Omega} \Big(\Big(\lambda a^{\alpha\beta} a^{\sigma\tau} + \mu\Big(a^{\alpha\sigma}a^{\beta\tau} + a^{\alpha\tau}a^{\beta\sigma}\Big)\Big) E^0_{\sigma\|\tau} F^0_{\alpha\|\beta}(\eta) + \lambda a^{\alpha\beta} E^0_{3\|3} F^0_{\alpha\|\beta}(\eta)\Big)\sqrt{a}\,dx \\
&= \frac{1}{2}\int_{\Omega} a^{\alpha\beta\sigma\tau} E^0_{\sigma\|\tau} F^0_{\alpha\|\beta}(\eta)\sqrt{a}\,dx \\
&= L^0(\eta).
\end{aligned}
\tag{3.9}
$$

Since $u^0 \in V(\Omega)$ *is independent of* x_3*, we can identify it with a function* $\zeta^0 \in V_M(\omega)$*. In this sense, we have*

$$
\begin{aligned}
E^0_{\alpha\|\beta} &= \frac{1}{2}\Big(\zeta^0_{\alpha\|\beta} + \zeta^0_{\beta\|\alpha} + a^{mn}\zeta^0_{m\|\alpha}\zeta^0_{n\|\beta}\Big) \in L^2(\omega), \\
F^0_{\alpha\|\beta}(\eta) &= \frac{1}{2}\Big(\eta_{\alpha\|\beta} + \eta_{\beta\|\alpha} + a^{mn}\Big\{\zeta^0_{m\|\alpha}\eta_{n\|\beta} + \zeta^0_{n\|\beta}\eta_{m\|\alpha}\Big\}\Big) \in L^2(\omega).
\end{aligned}
$$

We define the nonlinear manifold of inextensional displacements by

$$
\mathcal{M}_0(\omega) = \Big\{\eta \in W^{1,4}(\omega);\ \eta = 0 \quad on \quad \gamma_0, \quad \eta_3 = 0 \quad on \quad \gamma_1, \quad a_{\alpha\beta}(\eta) - a_{\alpha\beta} = 0 \quad in \quad \omega\Big\},
$$

where $a_{\alpha\beta}(\eta)$ *are the covariant components of the first fundamental form of the deformed surface* $\Big(\theta + \eta_i a^i\Big)(\bar{\omega})$ *be defined by*

$$
a_{\alpha\beta}(\eta) = a_\alpha(\eta).a_\beta(\eta), \qquad a_\alpha(\eta) = \partial_\alpha(\theta + \eta_i a^i).
$$

Extending the definition given in [32, 31], we conclude that if the manifold $\mathcal{M}_0(\omega)$ *reduces to* $\{0\}$*, then the variational problem* $P_M(\omega)$ *represents the so-called two-dimensional variational equation of von Kármán membrane shell.* ■

In addition, an application of [31, Theorem 9.2-1] shows that the problem $P_M(\omega)$ can be written as

$$P_M^{\#}(\omega)\begin{cases}\textit{Find } \zeta^0 \in V_M(\omega) \quad \textit{such that} \\ \displaystyle\int_\omega a^{\alpha\beta\sigma\tau} G_{\sigma\tau}(\zeta^0)\Big(G'_{\alpha\beta}(\zeta^0)\eta\Big)\sqrt{a}\,dy = \int_\omega p^{i,0}\eta_i\sqrt{a}\,dy \\ \displaystyle +2\int_{\gamma_1} \rho h^{\alpha,0}\eta_\alpha\,d\gamma, \qquad \forall\,\eta \in V_M(\omega),\end{cases}$$

where $G_{\alpha\beta}(\eta) = \frac{1}{2}\Big(a_{\alpha\beta}(\eta) - a_{\alpha\beta}\Big)$, $G_{\alpha\beta}(\zeta^0) = E^0_{\alpha\|\beta}$, *and* $G'_{\alpha\beta}(\zeta^0)\eta = F^0_{\alpha\|\beta}(\eta)$.

Finally, the variational problem $P_M(\omega)$ may be formulated as a minimization problem

$$\zeta \in V_M(\omega) \qquad \textit{and} \quad j_M(\zeta) = \inf_{\eta\in V_M(\omega)} j_M(\eta),$$

where the scaled two-dimensional energy of von Kármán membrane shells given by

$$j_M(\eta) = \frac{1}{8}\int_\omega a^{\alpha\beta\sigma\tau}\Big(a_{\sigma\tau}(\eta) - a_{\sigma\tau}\Big)\Big(a_{\alpha\beta}(\eta) - a_{\alpha\beta}\Big)\sqrt{a}\,dy - \int_\omega p^{i,0}\eta_i\sqrt{a}\,dy$$
$$- 2\int_{\gamma_1}\rho h^{\alpha,0}\eta_\alpha\,d\gamma.$$

The energy j_M is coercive on the space $V_M(\omega)$, but it is not weakly lower semi-continuous on $V_M(\omega)$. Therefore, we do not guarantee the existence of a solution to this minimization problem, referring to [31, Theorem 9.3-1], or [32, Section 1.4] for details.

Remark 3.1 *We note that "membrane shells" and "flexural shells" represents a general terminology about shells that is commonly used in the Western literature, as in e.g., Ciarlet [31]. Other terminologies are used, as "geometrically rigid shells" and "geometrically bendable shells".*

3.4.2 TWO-DIMENSIONAL EQUATIONS OF VON KÁRMÁN MEMBRANE SHELL

Now we write the two-dimensional variational problem $P_M^{\#}(\omega)$ as an equivalent boundary value problem.

Theorem 3.4 *Assume that the functions $n^{\alpha\beta}$ are in $H^1(\omega)$. Then any smooth solution ζ^0 of the variational problem $P_M^{\#}(\omega)$, is also a solution of the following equations of von Kármán membrane shell*

$$\bar{P}_M(\omega)\begin{cases} -\Big(n^{\alpha\beta}+n^{\sigma\beta}a^{\alpha\tau}\zeta^0_{\tau\|\sigma}\Big)\Big|_\beta + b^\alpha_\beta n^{\sigma\beta}\zeta^0_{3\|\sigma} = p^{\alpha,0} & \text{in} \quad \omega, \\ -b_{\alpha\beta}\Big(n^{\alpha\beta}+n^{\sigma\beta}a^{\alpha\tau}\zeta^0_{\tau\|\sigma}\Big) - \Big(n^{\alpha\beta}\zeta^0_{3\|\alpha}\Big)\Big|_\beta = p^{3,0} & \text{in} \quad \omega, \\ \zeta^0_i = 0 & \text{on} \quad \gamma_0, \\ \sqrt{a}\Big(n^{\alpha\beta}+n^{\sigma\beta}a^{\alpha\tau}\zeta^0_{\tau\|\sigma}\Big)\nu_\beta = 2\rho h^{\alpha,0} & \text{on} \quad \gamma_1, \\ \zeta^0_3 = 0 & \text{on} \quad \gamma_1, \end{cases}$$

where

$$\begin{cases} n^{\alpha\beta} = a^{\alpha\beta\sigma\tau}G_{\sigma\tau}(\zeta^0), \\ \eta^\alpha|_\beta = \partial_\beta\eta^\alpha + \Gamma^\alpha_{\beta\sigma}\eta^\sigma, \\ n^{\alpha\beta}|_\sigma = \partial_\sigma n^{\alpha\beta} + \Gamma^\alpha_{\sigma\tau}n^{\beta\tau} + \Gamma^\beta_{\sigma\tau}n^{\alpha\tau}. \end{cases}$$

Proof. *We recall that*

$$\partial_\alpha\sqrt{a} = \sqrt{a}\Gamma^\sigma_{\sigma\alpha}. \tag{3.10}$$

Taking into account $n^{\alpha\beta} = n^{\beta\alpha}$, we replace $n^{\alpha\beta}$ in the variational problem $P_M^{\#}(\omega)$ with its expression, we find that

$$\begin{aligned} &\int_\omega a^{\alpha\beta\sigma\tau}G_{\sigma\tau}(\zeta^0)\Big(G'_{\alpha\beta}(\zeta^0)\eta\Big)\sqrt{a}\,dy \\ &= \int_\omega n^{\alpha\beta}\Big(G'_{\alpha\beta}(\zeta^0)\eta\Big)\sqrt{a}\,dy \\ &= \int_\omega \sqrt{a}n^{\alpha\beta}\Big\{\frac{1}{2}\Big(\eta_{\alpha\|\beta}+\eta_{\beta\|\alpha}\Big) + \frac{1}{2}a^{mn}\Big(\zeta^0_{m\|\alpha}\eta_{n\|\beta}+\zeta^0_{n\|\beta}\eta_{m\|\alpha}\Big)\Big\}\,dy \\ &= \int_\omega \sqrt{a}n^{\alpha\beta}\eta_{\alpha\|\beta}\,dy + \int_\omega \sqrt{a}n^{\alpha\beta}a^{mn}\zeta^0_{m\|\alpha}\eta_{n\|\beta}\,dy. \end{aligned}$$

Using Green's formula and taking into account (3.10), *we obtain*

$$
\begin{aligned}
&\int_\omega \sqrt{a} n^{\alpha\beta} \eta_{\alpha\|\beta}\, dy \\
&= \int_\omega \sqrt{a} n^{\alpha\beta} \Big(\partial_\beta \eta_\alpha - \Gamma^\sigma_{\alpha\beta} \eta_\sigma - b_{\alpha\beta} \eta_3 \Big)\, dy \\
&= \int_\omega \sqrt{a} n^{\alpha\beta} \partial_\beta \eta_\alpha\, dy - \int_\omega \sqrt{a} n^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \eta_\sigma\, dy - \int_\omega \sqrt{a} n^{\alpha\beta} b_{\alpha\beta} \eta_3\, dy \\
&= -\int_\omega \partial_\beta \Big(\sqrt{a} n^{\alpha\beta} \Big) \eta_\alpha\, dy - \int_\omega \sqrt{a} n^{\alpha\beta} \Gamma^\sigma_{\alpha\beta} \eta_\sigma\, dy - \int_\omega \sqrt{a} n^{\alpha\beta} b_{\alpha\beta} \eta_3\, dy + \int_{\gamma_1} \sqrt{a} n^{\alpha\beta} \nu_\beta \eta_\alpha\, d\gamma \\
&= -\int_\omega \sqrt{a} \Big(\partial_\beta n^{\alpha\beta} + \Gamma^\alpha_{\beta\tau} n^{\beta\tau} + \Gamma^\beta_{\beta\tau} n^{\alpha\tau} \Big) \eta_\alpha\, dy - \int_\omega \sqrt{a} n^{\alpha\beta} b_{\alpha\beta} \eta_3\, dy + \int_{\gamma_1} \sqrt{a} n^{\alpha\beta} \nu_\beta \eta_\alpha\, d\gamma \\
&= -\int_\omega \sqrt{a} \Big(n^{\alpha\beta}|_\beta \Big) \eta_\alpha\, dy - \int_\omega \sqrt{a} n^{\alpha\beta} b_{\alpha\beta} \eta_3\, dy + \int_{\gamma_1} \sqrt{a} n^{\alpha\beta} \nu_\beta \eta_\alpha\, d\gamma \\
&= -\int_\omega \sqrt{a} \Big\{ \Big(n^{\alpha\beta}|_\beta \Big) \eta_\alpha + b_{\alpha\beta} n^{\alpha\beta} \eta_3 \Big\}\, dy + \int_{\gamma_1} \sqrt{a} n^{\alpha\beta} \nu_\beta \eta_\alpha\, d\gamma,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_\omega \sqrt{a} n^{\alpha\beta} a^{mn} \zeta^0_{m\|\alpha} \eta_{n\|\beta}\, dy \\
&= \int_\omega \sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} \eta_{\alpha\|\beta}\, dy + \int_\omega \sqrt{a} n^{\alpha\beta} \zeta^0_{3\|\alpha} \eta_{3\|\beta}\, dy \\
&= \int_\omega \sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} \Big(\partial_\beta \eta_\alpha - \Gamma^\varsigma_{\alpha\beta} \eta_\varsigma - b_{\alpha\beta} \eta_3 \Big)\, dy + \int_\omega \sqrt{a} n^{\alpha\beta} \zeta^0_{3\|\alpha} \Big(\partial_\beta \eta_3 + b^\sigma_\beta \eta_\sigma \Big)\, dy \\
&= \int_\omega \sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} \partial_\beta \eta_\alpha\, dy - \int_\omega \sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} \Gamma^\varsigma_{\alpha\beta} \eta_\varsigma\, dy \\
&\qquad - \int_\omega \sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} b_{\alpha\beta} \eta_3\, dy + \int_\omega \sqrt{a} n^{\alpha\beta} \zeta^0_{3\|\alpha} \partial_\beta \eta_3\, dy + \int_\omega \sqrt{a} n^{\alpha\beta} \zeta^0_{3\|\alpha} b^\sigma_\beta \eta_\sigma\, dy \\
&= -\int_\omega \partial_\beta \Big(\sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} \Big) \eta_\alpha\, dy - \int_\omega \sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} \Gamma^\varsigma_{\alpha\beta} \eta_\varsigma\, dy \\
&\qquad - \int_\omega \sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} b_{\alpha\beta} \eta_3\, dy - \int_\omega \partial_\beta \Big(\sqrt{a} n^{\alpha\beta} \zeta^0_{3\|\alpha} \Big) \eta_3\, dy \\
&\qquad + \int_\omega \sqrt{a} n^{\alpha\beta} \zeta^0_{3\|\alpha} b^\sigma_\beta \eta_\sigma\, dy + \int_{\gamma_1} \sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} \nu_\beta \eta_\alpha\, dy \\
&= -\int_\omega \sqrt{a} \Big\{ \Big(n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} \Big)\Big|_\beta - b^\alpha_\beta n^{\sigma\beta} \zeta^0_{3\|\sigma} \Big\} \eta_\alpha\, dy - \int_\omega \sqrt{a} \Big\{ b_{\alpha\beta} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} + \Big(n^{\alpha\beta} \zeta^0_{3\|\alpha} \Big)\Big|_\beta \Big\} \eta_3\, dy \\
&\qquad + \int_{\gamma_1} \sqrt{a} n^{\sigma\beta} a^{\alpha\tau} \zeta^0_{\tau\|\sigma} \nu_\beta \eta_\alpha\, d\gamma.
\end{aligned}
$$

Hence the variational problem $P_M^{\#}(\omega)$ reads as follows:

$$
\begin{aligned}
&-\int_\omega \sqrt{a}\Big\{\Big\{\Big(n^{\alpha\beta}+n^{\sigma\beta}a^{\alpha\tau}\zeta^0_{\tau\|\sigma}\Big)\Big|_\beta - b^\alpha_\beta n^{\sigma\beta}\zeta^0_{3\|\sigma}\Big\}+p^{\alpha,0}\Big\}\eta_\alpha\,dy\\
&-\int_\omega \sqrt{a}\Big\{\Big\{b_{\alpha\beta}\Big(n^{\alpha\beta}+n^{\sigma\beta}a^{\alpha\tau}\zeta^0_{\tau\|\sigma}\Big)+\Big(n^{\alpha\beta}\zeta^0_{3\|\alpha}\Big)\Big|_\beta\Big\}+p^{3,0}\Big\}\eta_3\,dy\\
&\int_{\gamma_1}\Big\{\sqrt{a}\Big(n^{\alpha\beta}+n^{\sigma\beta}a^{\alpha\tau}\zeta^0_{\tau\|\sigma}\Big)\nu_\beta-2\rho h^{\alpha,0}\Big\}\eta_\alpha\,d\gamma=0,
\end{aligned}
$$

for all $\eta\in V_M(\omega)$. The equating to zero all the factors of η_α and η_3 in their respective domains of integration, yields the boundary value problem $\bar{P}_M(\omega)$. ■

Remark 3.2 *It is remarkable that the functions $n^{\alpha\beta}$ are stated in $\bar{P}_M(\omega)$ do not satisfy the equations $\partial_\beta n^{\alpha\beta}=0$ in ω, even if the functions $p^{\alpha,0}$ vanish in ω. Hence we can't associate to this model, another equivalent model which involves an Airy function, such as plates and shallow shells.*

Chapter 4

ASYMPTOTIC JUSTIFICATION OF EQUATIONS FOR VON KÁRMÁN FLEXURAL SHELLS

The purpose of this chapter is to study the asymptotic justification of the two- dimensional equations for flexural shells with boundary conditions of von Kármán's type. More precisely, we consider a three-dimensional model for a nonlinearly elastic flexural shell of Saint VenantâKirchhoff material, where only a portion of the lateral face is subjected to boundary conditions of von Kármán's type. Using technics from formal asymptotic analysis with the thickness of the shell as a small parameter, we show that the scaled three-dimensional solution still leads to the two-dimensional equations of von Kármán flexural shell and we prove an existence theorem for the minimization problem.

4.1 TWO-DIMENSIONAL VARIATIONAL PROBLEM OF VON KÁRMÁN FLEXURAL SHELL

Let ω be a connected bounded open subset of $\mathbb{R}^2$ with a Lipschitz-continuous boundary γ and let γ_0 and γ_1 be a relatively open subsets of γ such that length$(\gamma_0) > 0$ and length$(\gamma_1) > 0$. Let $\theta : \bar{\omega} \to \mathbb{R}^3$ is a smooth enough injective immersion of class $\mathcal{C}^3$ such that θ_3 constant on the boundary γ_1.

Extending the definition given in [35, 31], we say that a nonlinearly elastic shell, is a flexural, if the manifold

$$\mathcal{M}_F(\omega) = \{\eta \in W^{2,4}(\omega); E^0_{\alpha\|\beta}(\eta) = 0 \text{ in } \omega; \eta = \partial_\nu \eta = 0 \text{ on } \gamma_0, \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_1\},$$

and its tangent space

$$\mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega) = \{\eta \in W^{2,4}(\omega); F^0_{\alpha\|\beta}(\zeta^0, \eta) = 0 \text{ in } \omega; \eta = \partial_\nu \eta = 0 \text{ on } \gamma_0, \eta_3 = \partial_\nu \eta_3 = 0 \text{ on } \gamma_1\},$$

contains nonzero functions, i.e.,

$$\mathcal{M}_F(\omega) \neq \{0\} \text{ and } \mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega) \neq \{0\} \text{ for all } \zeta^0 \in \mathcal{M}_F(\omega).$$

In this section, we need the following Lemmas.

Lemma 4.1 *The term of order one in the formal ezpansion of $u(\varepsilon)$ is of the form*

$$u^1 = \zeta^1 - x_3 \psi^0,$$

with $\zeta^1 \in V(\omega)$ and $\psi^0 = (\psi^0_i) \in V(\omega)$ is define by

$$\begin{cases} \psi^0_1 = b^\alpha_1 \zeta^0_\alpha + (1 + a^{\alpha 2}\zeta^0_{\alpha\|2})\zeta^0_{3\|1} - a^{\alpha 2}\zeta^0_{\alpha\|1}\zeta^0_{3\|2}, \\ \psi^0_2 = b^\alpha_2 \zeta^0_\alpha + (1 + a^{\alpha 1}\zeta^0_{\alpha\|1})\zeta^0_{3\|2} - a^{\alpha 1}\zeta^0_{\alpha\|2}\zeta^0_{3\|1}, \\ \psi^0_3 = -a^{\alpha\beta}\zeta^0_{\alpha\|\beta} - a^{\alpha 1}a^{\beta 2}(\zeta^0_{\alpha\|1}\zeta^0_{\beta\|2} - \zeta^0_{\alpha\|2}\zeta^0_{\beta\|1}), \end{cases}$$

Proof. *See part (i) of the proof of Theorem 10.1-2 in [31].* ■

Lemma 4.2 *For $\zeta^0 \in \mathcal{M}_F(\omega)$, the tensor $E^1_{\alpha\|\beta}$ in (2.16) is given by*

$$E^1_{\alpha\|\beta} = F^0_{\alpha\|\beta}(\zeta^0, \zeta^1) - x_3 \hat{E}^1_{\alpha\|\beta}(\zeta^0), \tag{4.1}$$

where $\hat{E}^1_{\alpha\|\beta}(\zeta^0)$ is independent of x_3 and takes the form

$$\hat{E}^1_{\alpha\|\beta}(\zeta^0) = -F^0_{\alpha\|\beta}(\zeta^0, \psi^1) + \Gamma^{k,1}_{\alpha\beta}\zeta^0_k + \frac{1}{2}a^{ij}(\Gamma^{k,1}_{i\|\alpha}\zeta^0_{j\|\beta} + \zeta^0_{i\|\alpha}\Gamma^{k,1}_{j\|\beta})\zeta^0_k - \frac{1}{2}g^{\sigma\tau,1}\zeta^0_{\sigma\|\alpha}\zeta^0_{\tau\|\beta}.$$

Proof. *See part (iv) of the proof of Theorem 10.1-2 in [31], or proof of Lemma 3 in [35].* ■

Lemma 4.3 *Let $\zeta^0 \in V(\omega)$ be given. For any two-dimensional vector field $\eta \in \mathbb{T}_{\zeta^0}\mathcal{M}_0(\omega)$, there exist $v(\eta) \in V(\Omega)$ such that*

$$F^{-1}_{i\|j}(v(\eta)) = F^0_{i\|j}(\eta), \tag{4.2}$$

the vector $v(\eta)$ takes the form

$$v(\eta) = \hat{\tau} + x_3\tau(\eta),$$

where the vectors $\hat{\tau}$ and $\tau(\eta)$ belong to $V(\omega)$ and $\tau(\eta)$ is uniquely defined by the relations

$$-\tau_1(\eta) = -b^\alpha_1\eta^0_\alpha - (1 + a^{\alpha 2}\zeta^0_{\alpha\|2})\eta^0_{3\|1} - (1 + a^{\alpha 2}\eta^0_{\alpha\|2})\zeta^0_{3\|1} + a^{\alpha 2}\zeta^0_{\alpha\|1}\eta^0_{3\|2} + a^{\alpha 2}\eta^0_{\alpha\|1}\zeta^0_{3\|2},$$

$$-\tau_2(\eta) = -b^\alpha_2\eta^0_\alpha - (1 + a^{\alpha 1}\zeta^0_{\alpha\|1})\eta^0_{3\|2} - (1 + a^{\alpha 1}\eta^0_{\alpha\|1})\zeta^0_{3\|2} + a^{\alpha 1}\zeta^0_{\alpha\|2}\eta^0_{3\|1} + a^{\alpha 1}\eta^0_{\alpha\|2}\zeta^0_{3\|1},$$

$$-\tau_3(\eta) = a^{\alpha\beta}\eta^0_{\alpha\|\beta} + a^{\alpha 1}a^{\beta 2}(\zeta^0_{\alpha\|1}\eta^0_{\beta\|2} + a^{\alpha 1}a^{\beta 2}(\eta^0_{\alpha\|1}\zeta^0_{\beta\|2} - \zeta^0_{\alpha\|2}\eta^0_{\beta\|1}) - \eta^0_{\alpha\|2}\zeta^0_{\beta\|1}).$$

Proof. *See proof of Theorem 10.1-3 in [31], or proof of Lemma 4 in [35].* ■

Lemma 4.4 *For all $\eta \in \mathbb{T}_{\zeta^0}\mathcal{M}_0(\omega)$,*

$$F^1_{\alpha\|\beta}(\eta) - F^0_{\alpha\|\beta}(v(\eta)) = \hat{F}^1_{\alpha\|\beta}(\eta) - x_3\hat{F}^0_{\alpha\|\beta}(\eta), \tag{4.3}$$

where $\hat{F}^1_{\alpha\|\beta}(\eta), \hat{F}^0_{\alpha\|\beta}(\eta) \in L^2(\omega)$ are independent of x_3 and

$$\begin{aligned}\hat{F}^0_{\alpha\|\beta}(\eta) = F^0_{\alpha\|\beta}(\varphi^0(\eta)) &- \frac{1}{2}g^{\sigma\tau,1}\{\zeta^0_{\sigma\|\alpha}\eta_{\tau\|\beta} + \zeta^0_{\tau\|\beta}\eta_{\sigma\|\alpha}\} \\ &+ (\Gamma^{p,1}_{\alpha\beta} + \frac{1}{2}a^{mn}\{\Gamma^{p,1}_{m\alpha}\zeta^0_{n\|\beta} + \Gamma^{p,1}_{n\beta}\zeta^0_{m\|\alpha}\})\eta_p \\ &+ \frac{1}{2}a^{mn}\{(\psi^0_{m\|\alpha} + \Gamma^{p,1}_{m\alpha}\zeta^0_p)\eta_{n\|\beta} + (\psi^0_{n\|\beta} + \Gamma^{q,1}_{n\beta}\zeta^0_q)\eta_{m\|\alpha}\},\end{aligned}$$

with $\psi^0_i = \psi_i(\zeta^0)$, where $\psi(\eta)$ is define by

$$\begin{cases}\psi_1(\eta) = b^\alpha_1\eta_\alpha + \eta_{3\|1} + a^{\alpha 2}(\zeta^0_{\alpha\|2}\eta_{3\|1} + \zeta^0_{3\|1}\eta_{\alpha\|2} - \zeta^0_{3\|2}\eta_{\alpha\|1} - \zeta^0_{\alpha\|1}\eta_{3\|2}),\\ \psi_2(\eta) = b^\alpha_2\eta_\alpha + \eta_{3\|2} + a^{\alpha 1}(\zeta^0_{\alpha\|1}\eta_{3\|2} + \zeta^0_{3\|2}\eta_{\alpha\|1} - \zeta^0_{3\|1}\eta_{\alpha\|2} - \zeta^0_{\alpha\|2}\eta_{3\|1}),\\ \psi_3(\eta) = -a^{\alpha\beta}\eta_{\alpha\|\beta} - a^{\alpha 1}a^{\beta 2}(\zeta^0_{\alpha\|1}\eta_{\beta\|2} + \zeta^0_{\beta\|1}\eta_{\alpha\|1} - \zeta^0_{\alpha\|2}\eta_{\beta\|1} - \zeta^0_{\beta\|1}\eta_{\alpha\|2}),\end{cases}$$

Proof. *See proof of Theorem 10.1-4 in [31], or proof of Lemma 5 in [35].* ■

Theorem 4.1 *If the space $\mathcal{M}_0(\omega)$ of inextensional displacements does not reduce to $\{0\}$, then to get a limiting problem, it is necessary to assume that there exist $f^2 \in L^2(\Omega)$, $l^3 \in L^2(\Gamma_+ \cup \Gamma_-)$ and $h^2 \in L^2(\gamma_1)$ independent of ε such that*

$$\begin{cases} f^\varepsilon(x^\varepsilon) = f(\varepsilon)(x) = \varepsilon^2 f^2(x) \text{ for all } x \in \Omega, \\ l^\varepsilon(x^\varepsilon) = l(\varepsilon)(x) = \varepsilon^3 l^3(x) \text{ for all } x \in \Gamma_+ \cup \Gamma_-, \\ h^\varepsilon(y) = h(\varepsilon)(y) = \varepsilon^2 h^2(y) \text{ for all } y \in \gamma_1. \end{cases}$$

In this case, ζ^0 satisfies the following two-dimensional nonlinear limit variational problem

$$P_F(\omega) \begin{cases} \text{Find } \zeta^0 \in \mathcal{M}_F(\omega) \text{ such that} \\ \dfrac{1}{3}\displaystyle\int_\omega a^{\alpha\beta\sigma\tau} \hat{E}^0_{\sigma\|\tau} \hat{F}^0_{\alpha\|\beta}(\eta)\sqrt{a}dy = \int_\omega \left(\int_{-1}^{+1} f^{i,2} dx_3 + l^{i,3}_+ + l^{i,3}_- \right) \eta_i \sqrt{a} dy \\ +2\displaystyle\int_{\gamma_1} \rho h^{\alpha,2} \eta_\alpha d\gamma, \text{ for all } \eta \in \mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega), \end{cases} \tag{4.4}$$

where

$$\begin{cases} \hat{E}^0_{\alpha\|\beta} = F^0_{\alpha\|\beta}(\psi^0) - \dfrac{1}{2} g^{\sigma\tau,1} \zeta^0_{\sigma\|\alpha}\zeta^0_{\tau\|\beta} + (\Gamma^{p,1}_{\alpha\beta} + \dfrac{1}{2} a^{mn}\{\Gamma^{p,1}_{m\alpha}\zeta^0_{n\|\beta} + \Gamma^{p,1}_{n\beta}\zeta^0_{m\|\alpha}\})\zeta^0_p, \\ \hat{F}^0_{\alpha\|\beta}(\eta) = F^0_{\alpha\|\beta}(\varphi^0(\eta)) - \dfrac{1}{2} g^{\sigma\tau,1}\{\zeta^0_{\sigma\|\alpha}\eta_{\tau\|\beta} + \zeta^0_{\tau\|\beta}\eta_{\sigma\|\alpha}\} \\ +(\Gamma^{p,1}_{\alpha\beta} + \dfrac{1}{2} a^{mn}\{\Gamma^{p,1}_{m\alpha}\zeta^0_{n\|\beta} + \Gamma^{p,1}_{n\beta}\zeta^0_{m\|\alpha}\})\eta_p \\ +\dfrac{1}{2} a^{mn}\{(\psi^0_{m\|\alpha} + \Gamma^{p,1}_{m\alpha}\zeta^0_p)\eta_{n\|\beta} + (\psi^0_{n\|\beta} + \Gamma^{q,1}_{n\beta}\zeta^0_q)\eta_{m\|\alpha}\}, \end{cases}$$

Proof. *This proof complements what we have come up in proof the Theorem 3.3 with the use of some notations frequently used below. We are now in a position to start the cancellation of the factors of the successive powers of ε found in the variational equations* (3.1). *In what follows, L^r designates for any integer $r \geq 0$ the linear form defined by*

$$L^r = \int_\Omega f^{i,r} v_i \sqrt{a} dx + \int_{\Gamma_+ \cup \Gamma_-} l^{i,r+1} v_i \sqrt{a} d\Gamma + 2\int_{\gamma_1} \rho h^{\alpha,r} v_\alpha d\gamma, \tag{4.5}$$

where the functions $f^{i,r} \in L^2(\Omega)$, $l^{i,r+1} \in L^2(\Gamma_+ \cup \Gamma_-)$ and $h^{\alpha,r} \in L^2(\gamma_1)$ and that they are independent of ε.

(i) Our point of departure is the cancellation of the coefficients of ε^{-1}, we showed in proof the Theorem 3.3 in relations 3.8 that

$$E^0_{\alpha\|3} = 0 \text{ and } \lambda a^{\alpha\beta} E^0_{\alpha\|\beta} + (\lambda + 2\mu) E^0_{3\|3} = 0 \text{ in } \Omega,$$

and that the assumptions $\mathcal{M}_0(\omega) \neq \{0\}$ *and* $\mathbb{T}_\zeta\mathcal{M}_0(\omega) \neq \{0\}$ *at all* $\zeta \in \mathcal{M}_0(\omega)$ *imply that*

$$E^0_{\alpha\|\beta} = 0 \text{ in } \omega. \tag{4.6}$$

Then, we get

$$E^0_{i\|3} = 0 \text{ in } \Omega. \tag{4.7}$$

The equations $2E^0_{i\|3} = 0$ *take the form of a nonlinear system*

$$\begin{cases} 2E^0_{1\|3} = u^{(0)}_{1\|3} + \zeta^0_{3\|1} + a^{\alpha\beta}\zeta^0_{\alpha\|1}u^{(0)}_{\beta\|3} + \zeta^0_{3\|1}u^{(0)}_{3\|3}, \\ 2E^0_{2\|3} = u^{(0)}_{2\|3} + \zeta^0_{3\|2} + a^{\alpha\beta}\zeta^0_{\alpha\|2}u^{(0)}_{\beta\|3} + \zeta^0_{3\|2}u^{(0)}_{3\|3}, \\ 2E^0_{3\|3} = 2u^{(0)}_{3\|3} + a^{\alpha\beta}u^{(0)}_{\alpha\|3}u^{(0)}_{\beta\|3} + u^{(0)}_{3\|3}u^{(0)}_{3\|3}, \end{cases} \tag{4.8}$$

where the unknowns are the functions $u^{(0)}_{i\|3}$. *We note that the solution of this system determines the value of* $\partial_3 u^1$ *are given by*

$$\partial_3 u^1_\alpha = u^{(0)}_{\alpha\|3} - b^\sigma_\alpha\zeta^0_\sigma = -\psi^0_\alpha \text{ and } \partial_3 u^1_3 = u^{(0)}_{3\|3} = -\psi^0_3. \tag{4.9}$$

We consider the following solution (we return to [31]-[35] for more details):

$$\begin{cases} u^{(0)}_{1\|3} = -(1 + a^{\alpha 2}\zeta^0_{\alpha\|2})\zeta^0_{3\|1} + a^{\alpha 2}\zeta^0_{\alpha\|1}\zeta^0_{3\|2}, \\ u^{(0)}_{2\|3} = -(1 + a^{\alpha 2}\zeta^0_{\alpha\|1})\zeta^0_{3\|2} + a^{\alpha 1}\zeta^0_{\alpha\|2}\zeta^0_{3\|1}, \\ u^{(0)}_{3\|3} = a^{\alpha\beta}\zeta^0_{\alpha\|\beta} + a^{\alpha 1}a^{\beta 2}(\zeta^0_{\alpha\|1}\zeta^0_{\beta\|2} - \zeta^0_{\alpha\|2}\zeta^0_{\beta\|1}), \end{cases} \tag{4.10}$$

From (4.9) that $\partial_3 u^1$ *is independent of* x_3, *then the term of order one in the formal expansion (3.5) is of the form*

$$u^1 = \zeta^1 - x_3\psi^0 \text{ with } \zeta^1 \in V(\omega) \text{ and } \psi^0 \in V(\omega)$$

and the condition $u^1 \in V(\omega)$ *implies that*

$$\mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega) = \{\eta \in W^{2,4}(\omega); F^0_{\alpha\|\beta}(\zeta^0, \eta) = 0 \text{ in } \omega; \eta = \partial_\nu\eta = 0 \text{ on } \gamma_0, \eta_3 = \partial_\nu\eta_3 \text{ on } \gamma_1\},$$

(ii) Cancellation of the coefficient of ε^0. *From the relations (4.6) and (4.7). Then, we get* $E^0_{i\|j} = 0$ *and since* $L^0 = 0$, *the inspection of the coefficient of* ε^0 *in (3.1) leads to the variational problem:*

$$\int_\Omega A^{ijkl}E^1_{k\|l}F^{-1}_{i\|j}(v)\sqrt{a}dx = 0 \text{ for all } v \in V(\Omega) \tag{4.11}$$

where
$$\begin{cases} F^{-1}_{\alpha\|\beta}(v) = 0, \\ F^{-1}_{\alpha\|3}(v) = \dfrac{1}{2}(\partial_3 v_\alpha + a^{\beta\sigma} u^0_{\beta\|\alpha}\partial_3 v_\sigma) + \dfrac{1}{2} u^0_{3\|\alpha}\partial_3 v_3, \\ F^{-1}_{3\|3}(v) = a^{\alpha\beta} u^0_{\alpha\|3}\partial_3 v_\beta + (1 + u^0_{3\|3})\partial_3 v_3. \end{cases}$$
The problem (4.11) reduces to three decoupled problems (see the similar treatment of the coefficient of ε^{-1} in proof the Theorem 3.3)
$$\begin{cases} \displaystyle\int_\Omega A^{ijkl}(0) E^1_{k\|l} F^{-1}_{i\|j}(v)\sqrt{a}dx \\ = \displaystyle\int_\Omega (4A^{\alpha 3\sigma 3}(0) E^1_{\alpha\|3} F^{-1}_{\sigma\|3}(v) + (A^{\alpha\beta 33}(0) E^1_{\alpha\|\beta} + A^{3333}(0) E^1_{3\|3}) F^{-1}_{3\|3}(v))\sqrt{a}dx \\ = \displaystyle\int_\Omega (2\mu a^{\alpha\sigma} E^1_{\alpha\|3}(\partial_3 v_\sigma + a^{mn} u^0_{m\|\sigma}\partial_3 v_n + u^0_{3\|\sigma}\partial_3 v_3) \\ +(\lambda a^{\alpha\beta} E^1_{\alpha\|\beta} + (\lambda+2\mu) E^1_{3\|3})((1 + u^{(0)}_{3\|3})\partial_3 v_3 + a^{mn} u^{(0)}_{m\|3}\partial_3 v_n))\sqrt{a}dx \\ = \displaystyle\int_\Omega ((2\mu E^1_{\alpha\|3}(a^{\alpha\tau} + a^{\alpha\sigma} a^{\beta\tau} u^0_{\beta\|\alpha}) + (\lambda a^{\alpha\beta} E^1_{\alpha\|\beta} + (\lambda+2\mu) E^1_{3\|3}) a^{\sigma\tau} u^{(0)}_{\sigma\|3})\partial_3 v_\tau \\ +(2\mu a^{\alpha\sigma} E^1_{\alpha\|3} u^0_{3\|\sigma} + (\lambda a^{\alpha\beta} E^1_{\alpha\|\beta} + (\lambda+2\mu) E^1_{3\|3})(1 + u^{(0)}_{3\|3}))\partial_3 v_3)\sqrt{a}dx = 0. \end{cases} \tag{4.12}$$
The integral (4.12) takes the form $(u^\tau \partial_3 v_\tau + u^3 \partial_3 v_3)$. Then, we get, that
$$2\mu E^1_{\alpha\|3}(a^{\alpha\tau} + a^{\alpha\sigma} a^{\beta\tau} u^0_{\beta\|\alpha}) + (\lambda a^{\alpha\beta} E^1_{\alpha\|\beta} + (\lambda+2\mu) E^1_{3\|3}) a^{\sigma\tau} u^{(0)}_{\sigma\|3} = 0 \text{ in } \Omega,$$
$$2\mu a^{\alpha\sigma} E^1_{\alpha\|3} u^0_{3\|\sigma} + (\lambda a^{\alpha\beta} E^1_{\alpha\|\beta} + (\lambda+2\mu) E^1_{3\|3})(1 + u^{(0)}_{3\|3}) = 0 \text{ in } \Omega.$$
This is a linear system with respect to the three variables $(\lambda a^{\alpha\beta} E^1_{\alpha\|\beta} + (\lambda+2\mu) E^1_{3\|3})$, $E^1_{1\|3}$ and $E^1_{2\|3}$. For a displacement field ζ^0 that vanishes, this system is invertible, therefore we assume that it has a unique solution, at least in a suitable neighborhood of $\zeta^0 = 0$. Then we have a system of three nonlinear equations, has the trivial solution
$$E^1_{\alpha\|3} = 0 \text{ and } \lambda a^{\alpha\beta} E^1_{\alpha\|\beta} + (\lambda+2\mu) E^1_{3\|3} = 0 \text{ in } \Omega. \tag{4.13}$$
(iii) We assume that there exist $f^1 \in L^2(\Omega)$, $l^2 \in L^2(\Gamma_+ \cup \Gamma_-)$ and $h^1 \in L^2(\gamma_1)$
$$\begin{cases} f^i(\varepsilon)(x) = \varepsilon f^{i,1}(x), \\ l^i(\varepsilon)(x) = \varepsilon^2 l^{i,2}(x), \\ h^\alpha(\varepsilon)(y) = \varepsilon h^{\alpha,1}(y). \end{cases}$$

The cancellation of the coefficient of ε^1 in the variational problem (3.1) reads

$$\int_\Omega A^{ijkl}(0)\{E^1_{k\|l}F^0_{i\|j}(v)+E^2_{k\|l}F^{-1}_{i\|j}(v)\}\sqrt{a}dx+\int_\Omega x_3B^{ijkl,1}E^1_{k\|l}F^{-1}_{i\|j}(v)dx=L^1(v), \tag{4.14}$$

for all $v\in V(\Omega)$, where

$$L^1(v)=\int_\Omega f^{i,1}v_i\sqrt{a}dx+\int_{\Gamma_+\cup\Gamma_-}l^{i,2}v_i\sqrt{a}d\Gamma+2\int_{\gamma_1}\rho h^{\alpha,1}v_\alpha d\gamma.$$

Choosing test functions $v=\eta$ independent of x_3. Then, we get $F^{-1}_{i\|j}(v)=0$ and from (2.14), we obtain

$$\int_\Omega A^{ijkl}(0)E^1_{k\|l}F^0_{i\|j}(\eta)\sqrt{a}dx=L^1(\eta) \text{ for all } \eta\in V(\omega). \tag{4.15}$$

Using the expressions of the functions $A^{ijkl}(0)$ in 2.13 and the relation (2.14), we obtain

$$\left\{\begin{aligned}
&\int_\Omega A^{ijkl}(0)E^1_{k\|l}F^0_{i\|j}(\eta)\sqrt{a}dx\\
&=\int_\Omega(\lambda a^{\alpha\beta}a^{\sigma\tau}+\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}))E^1_{\sigma\|\tau}F^0_{\alpha\|\beta}(\eta)\sqrt{a}dx\\
&+\int_\Omega(4\mu a^{\alpha\sigma}E^1_{\alpha\|3}F^0_{\sigma\|3}(\eta)+\lambda a^{\alpha\beta}E^1_{3\|3}F^0_{\alpha\|\beta}(\eta))\sqrt{a}dx\\
&+\int_\Omega(\lambda a^{\sigma\tau}E^1_{\sigma\|\tau}+(\lambda+2\mu)E^1_{3\|3})F^0_{3\|3}(\eta)\sqrt{a}dx\\
&=\int_\Omega((\lambda a^{\alpha\beta}a^{\sigma\tau}+\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}))E^1_{\sigma\|\tau}F^0_{\alpha\|\beta}(\eta)+\lambda a^{\alpha\beta}E^1_{3\|3}F^0_{\alpha\|\beta}(\eta))\sqrt{a}dx\\
&=\frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}E^1_{\sigma\|\tau}F^0_{\alpha\|\beta}(\eta)\sqrt{a}dx=L^1(\eta) \text{ for all } \eta\in V(\omega),
\end{aligned}\right. \tag{4.16}$$

$$a^{\alpha\beta\sigma\tau}=\frac{4\lambda\mu}{\lambda+2\mu}a^{\alpha\beta}a^{\sigma\tau}+2\mu(a^{\alpha\sigma}a^{\beta\tau}+a^{\alpha\tau}a^{\beta\sigma}).$$

For the relation (4.1) (see Lemma 4.2), where the functions $F^0_{\alpha\|\beta}(\zeta^1)$ and $\hat{E}^0_{\alpha\|\beta}$ are independent of the transverse variable x_3, we obtain

$$\int_\omega a^{\alpha\beta\sigma\tau}F^0_{\sigma\|\tau}(\zeta^1)F^0_{\alpha\|\beta}(\eta)\sqrt{a}dx=0 \text{ for all } \eta\in V(\omega).$$

Choosing test functions $\eta=\zeta^1\in V(\omega)$ in these equations, shows that $F^0_{\alpha\|\beta}(\zeta^1)=0$. Hence $\zeta^1\in\mathbb{T}_{\zeta^0}\mathcal{M}_0(\omega)$, consequently $E^1_{\alpha\|\beta}$ is independent of ζ^1 and reads

$$E^1_{\alpha\|\beta}=-x_3\hat{E}^0_{\alpha\|\beta}. \tag{4.17}$$

(iii) We assume that there exist $f^2 \in L^2(\Omega)$, $l^3 \in L^2(\Gamma_+ \cup \Gamma_-)$ and $h^2 \in L^2(\gamma_1)$

$$\begin{cases} f^i(\varepsilon)(x) = \varepsilon^2 f^{i,2}(x), \\ l^i(\varepsilon)(x) = \varepsilon^3 l^{i,3}(x), \\ h^\beta(\varepsilon)(y) = \varepsilon^2 h^{\alpha,2}(y). \end{cases}$$

The cancellation of the coefficient of ε^2 in the variational problem (3.1) reads

$$\int_\Omega A^{ijkl}(0)\{E^1_{k\|l}F^1_{i\|j}(v) + E^2_{k\|l}F^0_{i\|j}(v) + E^3_{k\|l}F^{-1}_{i\|j}(v)\}\sqrt{a}dx$$
$$+ \int_\Omega x_3 B^{ijkl,1}\{E^1_{k\|l}F^0_{i\|j}(v) + E^2_{k\|l}F^{-1}_{i\|j}(v)\}dx + \int_\Omega x_3^2 B^{ijkl,2}E^1_{k\|l}F^{-1}_{i\|j}(v) = L^2(v),$$

for all $v \in V(\Omega)$. For test function $v = \eta$ independent of x_3 since $F^{-1}_{i\|j}(v) = 0$, we get

$$\int_\Omega A^{ijkl}(0)\{E^1_{k\|l}F^1_{i\|j}(\eta) + E^2_{k\|l}F^0_{i\|j}(\eta)\}\sqrt{a}dx + \int_\Omega x_3 B^{ijkl,1}E^1_{k\|l}F^0_{i\|j}(\eta)dx = L^2(\eta) \tag{4.18}$$

for all $\eta \in V(\omega)$, equations that we compare to equations (4.14), which now read

$$\int_\Omega A^{ijkl}(0)\{E^1_{k\|l}F^0_{i\|j}(v) + E^2_{k\|l}F^{-1}_{i\|j}(v)\}\sqrt{a}dx + \int_\Omega x_3 B^{ijkl,1}E^1_{k\|l}F^{-1}_{i\|j}(\eta)dx = 0 \tag{4.19}$$

for all $v \in V(\Omega)$. Used the relation (4.2) in Lemma 4.3, then by computing the difference between equations (4.18) and (4.19), we obtain

$$\int_\Omega A^{ijkl}(0)E^1_{k\|l}\{F^1_{i\|j}(\eta) - F^0_{i\|j}(v(\eta))\}\sqrt{a}dx = L^2(\eta) \textit{ for all } \eta \in \mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega),$$

Used the relations (4.2) and (4.13), we obtain

$$\frac{1}{2}\int_\Omega a^{\alpha\beta\sigma\tau}E^1_{\sigma\|\tau}\{F^1_{\alpha\|\beta}(\eta) - F^0_{\alpha\|\beta}(v(\eta))\}\sqrt{a}dx = L^2(\eta) \textit{ for all } \eta \in \mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega),$$

For the relations (4.3) and (4.17),we get that

$$\frac{1}{3}\int_\omega a^{\alpha\beta\sigma\tau}\hat{E}^0_{\sigma\|\tau}\hat{F}^0_{\alpha\|\beta}(\eta)\sqrt{a}dy = L^2(\eta) \textit{ for all } \eta \in \mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega).$$

■

Remark 4.1 *From to two-dimensional variational problem of a nonlinearly elastic flexural shell due to Ciarlet ([31], Sect. 10.3), we show that the two-dimensional von Kármán* *flexural shell problem* $P_F(\omega)$ *can be written as the following*

$$P_F^{\#}(\omega)\begin{cases} \text{Find } \zeta^0 \in \mathcal{M}_F(\omega) \text{ such that} \\ \dfrac{1}{3}\displaystyle\int_\omega a^{\alpha\beta\sigma\tau} R^\flat_{\sigma\tau}(\zeta^0)((R^\flat_{\alpha\beta})'(\zeta^0)\eta)\sqrt{a}dy = \int_\omega \left(\int_{-1}^{+1} f^{i,2}dx_3 + l_+^{i,3} + l_-^{i,3}\right)\eta_i\sqrt{a}dy \\ +2\displaystyle\int_{\gamma_1} \rho h^{\alpha,2}\eta_\alpha d\gamma, \text{ for all } \eta = (\eta_i) \in \mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega), \end{cases} \tag{4.20}$$

where

$$\hat{E}^0_{\alpha\|\beta} = R^\flat_{\alpha\beta}(\zeta^0) = (b_{\alpha\beta}(\eta) - b_{\alpha\beta}) \text{ for all } \eta \in \mathcal{M}_F(\omega),$$

$$\hat{F}^0_{\alpha\|\beta}(\eta) = (R^\flat_{\alpha\beta})'(\zeta^0)\eta \quad \text{for all } \eta \in \mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega),$$

4.2 EXISTENCE OF SOLUTIONS TO THE MINIMIZATION PROBLEM

Let the functional $j_F^\flat : W^{2,4}(\omega) \to \mathbb{R}$ *be defined by*

$$\begin{aligned} j_F^\flat(\eta) = \; & \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau} R^\flat_{\sigma\tau}(\eta) R^\flat_{\alpha\beta}(\eta)\sqrt{a}dy - \int_\omega \left(\int_{-1}^{+1} f^{i,2}dx_3 + l_+^{i,3} + l_-^{i,3}\right)\eta_i\sqrt{a}dy \\ & -2\int_{\gamma_1} \rho h^{\alpha,2}\eta_\alpha d\gamma, \forall \eta \in W^{2,4}(\omega). \end{aligned} \tag{4.21}$$

Then the functional $j_F^\flat$ *is differentiable over the space* $W^{2,4}(\omega)$ *and* $\zeta^0 \in \mathcal{M}_F(\omega)$ *is a solution to the variational problem (4.20) if and only if it is a stationary point of the functional* $j_F^\flat$ *over the manifold* $\mathcal{M}_F(\omega)$*, i.e., it satisfies* $(j_F^\flat)'(\zeta^0)\eta = 0$ *for all* η *in the tangent space* $\mathbb{T}_{\zeta^0}\mathcal{M}_F(\omega)$ *to the manifold* $\mathcal{M}_F(\omega)$ *at* ζ^0*. Since the functions* $b_{\alpha\beta}(\eta)$ *are well defined for all* $\eta \in \mathcal{M}_F(\omega)$*, particular solutions to problem (4.20) are obtained by solving the minimization problem:*

$$\begin{aligned} j_F(\eta) = \; & \frac{1}{6}\int_\omega a^{\alpha\beta\sigma\tau}(b_{\sigma\tau}(\eta) - b_{\sigma\tau})(b_{\alpha\beta}(\eta) - b_{\alpha\beta})\sqrt{a}dy \\ & -\int_\omega \left(\int_{-1}^{+1} f^{i,2}dx_3 + l_+^{i,3} + l_-^{i,3}\right)\eta_i\sqrt{a}dy \\ & -2\int_{\gamma_1} \rho h^{\alpha,2}\eta_\alpha d\gamma \end{aligned} \tag{4.22}$$

The functional j_F *is caled two-dimensional energy of a nonlinearly elastic von Kármán flexural shell.*

Theorem 4.2 *Let there be given a mapping $\theta \in W^{2,p}(\omega;\mathbb{R}^3)$ with $p>2$, let there be given a mapping $\varphi_0 : \gamma_0 \to \mathbb{R}^3$ and a mapping $\varphi_1 : \gamma_1 \to \mathbb{R}^3$ such that the manifold of admissible inextensional deformations*

$$\Phi_F(\omega) = \{\psi \in H^2(\omega); \psi = \varphi_0 \text{ on } \gamma_0, \psi_3 = \varphi_{1,3} \text{ on } \gamma_1, a_{\alpha\beta}(\psi) - a_{\alpha\beta} = 0 \text{ in } \omega\}.$$

is not empty.

Then if $\psi \in \Phi_F(\omega)$, the vectors $a_\alpha(\psi) = \partial_\alpha \psi$ are linearly independent a.e. in ω and the functions $b_{\alpha\beta}(\psi)$ are in $L^2(\omega)$. Given a continuous linear form $\mathbf{L}$ on $H^2(\omega)$, define the two-dimensional energy $\mathbf{I}_F : \Phi_F(\omega) \to \mathbb{R}$ of a nonlinearly elastic von Kármán flexural shell by

$$\mathbf{I}_F(\psi) = \frac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau}(b_{\sigma\tau}(\psi) - b_{\sigma\tau})(b_{\alpha\beta}(\psi) - b_{\alpha\beta})\sqrt{a}dy - \mathbf{L}(\psi)$$

for all $\psi \in \Phi_F(\omega)$.

Then there is at least one φ such that

$$\varphi \in \Phi_F(\omega) \text{ and } \mathbf{I}_F(\varphi) = \inf_{\psi \in \Phi_F(\omega)} \mathbf{I}_F(\psi).$$

Proof. *For the purpose of clarity, the demonstration is divided into seven numbered parts, labeled (i) to (vii).*

In the first five parts, we establish different characteristics of the manifold $\Phi_F(\omega)$. While, in the remaining two parts, we establish properties of the functional $\mathbf{I}_F$ over this manifold.

The demonstration follows a common pattern in the calculus of variations:

First, we prove that the manifold $\Phi_F(\omega)$ is sequentially weakly closed (part (i)).

Next, we demonstrate that the functional $\mathbf{I}_F$ is sequentially weakly lower semi-continuous and coercive over $\Phi_F(\omega)$ (parts (vi) and (vii)), with all these properties being applicable to the topology of the space $H^2(\omega)$.

Minimization technics can then be applied to show the existence of a minimizer of $\mathbf{I}_F$ over $\Phi_F(\omega)$ based on these properties.

Note that the coerciveness of the functional hinges on the crucial property that the manifold $\Phi_F(\omega)$ lies in a bounded subset of $W^{1,\infty}(\omega)$ (part (iii)).

(i) As a subset of $H^2(\omega)$, the manifold $\Phi_F(\omega)$ is sequentially weakly closed, a.e.,

$$\psi^l \in \Phi_F(\omega), l \geq 1, \text{ and } \psi^l \rightharpoonup \psi \text{ in } H^2(\omega) \Rightarrow \psi \in \Phi_F(\omega)$$

Let $\psi^l \in \Phi_F(\omega)$, $l > 1$, be such that $\psi^l \rightharpoonup \psi$ in $H^2(\omega)$. Since the trace operator tr from $H^2(\omega)$ into $L^2(\gamma_0)$ is continuous and also tr from $H^2(\omega)$ into $L^2(\gamma_1)$ is continuous with respect to the strong topologies of both spaces, it remains so with respect to the weak topologies of both spaces. Hence $tr\psi^l \rightharpoonup tr\psi$ in $L^2(\gamma_0)$ and thus $tr\psi = \varphi_0$ on γ_0 since $tr\psi^l = \varphi_0$ on γ_0 for all $l \geq 1$, and $tr\psi_3^l \rightharpoonup tr\psi_3$ in $L^2(\gamma_1)$ and thus $tr\psi_3 = \varphi_{1,3}$ on γ_1 since $tr\psi_3^l = \varphi_{1,3}$ on γ_1 for all $l \geq 1$.

By the Rellich-Kondrašov imbedding theorem (see, e.g., Theorem 6.1-5 in [50]), $\psi^l \rightharpoonup \psi$ in $H^1(\omega)$; hence

$$a_{\alpha\beta}(\psi^l) = a_\alpha(\psi^l) \cdot a_\beta(\psi^l) \to a_\alpha(\psi) \cdot a_\beta(\psi) = a_{\alpha\beta}(\psi) \text{ in } L^1(\omega).$$

Since $a_{\alpha\beta}(\psi^l) = a_{\alpha\beta}$ a.e. in ω for all l, we conclude that $a_{\alpha\beta}(\psi) = a_{\alpha\beta}$ a.e. in ω; hence $\psi \in \Phi_F(\omega)$ as was to be proved.

Should the manifold $\Phi_F(\omega)$ include a second boundary conditions of the form $\partial_\nu\psi = \bar{\varphi}_0$ on γ_0 and $\partial_\nu\psi_3 = \bar{\varphi}_{1,3}$ on γ_1 (recall that the manifold $\mathcal{M}_F(\omega)$ comprises a boundarys conditions of the form $\partial_\nu\eta = 0$ on γ_0 and $\partial_\nu\eta_3 = 0$ on γ_1), a similar argument shows that such a manifold is again sequentially weakly closed.

(ii) There exists C_1 such that, for all vector fields $\psi \in H^2(\omega)$ satisfying $a_{\alpha\beta}(\psi) = a_{\alpha\beta}$ a.e. in ω

$$0 < C_1 \leq |a_1(\psi) \wedge a_2(\psi)| \text{ a.e. in } \omega,$$

$$C_1^{-1} \leq |a_\alpha(\psi)| \leq C_1 \text{ a.e. in } \omega.$$

Consequently, the vectors $a_i(\psi)$ and $a^i(\psi)$ associated with such vector fields ψ are well defined and "uniformly" linearly independent a.e. in ω, the corresponding functions $b_{\alpha\beta}(\psi)$ are in $L^2(\omega)$, and the functional $\mathbf{I}_F$ is well defined over the manifold $\Phi_F(\omega)$ (that the functions $b_{\alpha\beta}(\psi)$ are indeed well-defined when ψ belongs to the manifold $\Phi_F(\omega)$ was already observed in Theorem 10.3-1 in [31]).

Since the set $\bar{\omega}$ is compact, the vectors $a_\alpha = \partial_\alpha\theta$ are "uniformly" linearly independent in $\bar{\omega}$ (they belong to the space $W^{1,p}(\omega;\mathbb{R}^3)$, which is continuously imbedded into the space $C^0(\bar{\omega};\mathbb{R}^3)$ since $p > 2$), in the sense that there exist c_1 and c_2 such that

$$0 \leq c_1 < 1 \text{ and } |a_1 \cdot a_2| \leq c_1|a_1||a_2| \text{ in } \bar{\omega},$$

$$0 < c_2 \leq 1 \text{ and } c_2 \leq |a_\alpha| \leq c_2^{-1} \text{ in } \bar{\omega},$$

Furthermore

$$a_1(\psi)\cdot a_2(\psi) = a_{12}(\psi) = a_{12} = a_1 \cdot a_2 \ \textit{a.e. in } \omega,$$

$$|a_1(\psi)|^2 = a_{11}(\psi) = a_{11} = |a_1|^2 \ \textit{a.e. in } \omega,$$

$$|a_2(\psi)|^2 = a_{22}(\psi) = a_{22} = |a_2|^2 \ \textit{a.e. in } \omega,$$

consequently

$$|a_1(\psi)\cdot a_2(\psi)| \le c_1 |a_1(\psi)||a_2(\psi)| \ \textit{a.e. in } \omega.$$

This inequality shows that the vectors $a_1(\psi)$ and $a_2(\psi)$ are likewise "uniformly" linearly independent a.e. in ω, since $c_1 < 1$. Hence there exists $c_3 > 0$ such that

$$c_3|a_1(\psi)\wedge a_2(\psi)| \ge |a_1(\psi)||a_2(\psi)| = |a_1||a_2| \ \textit{a.e. in } \omega,$$

and thus there exists a constant C_1 such that the two announced inequalities hold. The vector

$$a_3(\psi) = a^3(\psi) = \frac{a_1(\psi)\wedge a_2(\psi)}{|a_1(\psi)\wedge a_2(\psi)|},$$

is thus well defined a.e. in ω. Consequently,

$$b_{\alpha\beta}(\psi) = \partial_{\alpha\beta}\psi \cdot a_3(\psi) \in L^2(\omega),$$

since $|a_3(\psi)| = 1$ a.e. in ω. The vectors $a^\alpha(\psi)$ are likewise well defined and "uniformly" linearly independent a.e. in ω.

(iii) Let $\psi \in H^2(\omega)$ be such that $a_{\alpha\beta}(\psi) = a_{\alpha\beta}$ a.e. in ω. $\psi \in W^{1,\infty}(\omega)$ and there exists C_2 such that

$$|\partial_\alpha \psi|_{0,\infty,\omega} \le C_2 \ \textit{for all } \psi \in H^2(\omega).$$

In addition, there exists C_3 such that

$$\|\psi\|_{1,\infty,\omega} \le C_3 \ \textit{for all } \psi \in \Phi_F(\omega).$$

Let $\psi = (\psi_i) \in H^2(\omega)$ be such that $a_{\alpha\beta}(\psi) = a_{\alpha\beta}$ a.e. in ω. We already noticed that there exists $c_2 > 0$ independent of such fields ψ such that, for almost all $y \in \omega$,

$$|\partial_\alpha \psi(y)|^2 = |a_\alpha(\psi)(y)|^2 = |a_\alpha(y)|^2 \le c_2^{-2}.$$

This shows that $\partial_\alpha\psi \in L^\infty(\omega)$, hence that $\psi \in W^{1,\infty}(\omega)$ since in addition $\psi \in H^2(\omega) \hookrightarrow C^0(\bar{\omega};\mathbb{R}^3)$; it also shows that there exists C_2 independent of such fields

ψ such that $|\partial_\alpha \psi|_{0,\infty,\omega} \leq C_2$. Let $q > 2$ be fixed. By assumption, length $\gamma_0 > 0$; hence, by the generalized Poincard inequality found in [50, Thm. 6.1-8], there exists $c_4 = c_4(q, \gamma_0)$ such that, for all $\psi \in W^{1,q}(\omega)$

$$\int_\omega |\psi|^q dy \leq c_4\{\int_\omega \sum_\alpha |\partial_\alpha \psi|^q dy + |\int_{\gamma_0} \psi d\gamma|^q\}.$$

Let $\psi \in \Phi_F(\omega)$. then

$$\int_\omega \sum_\alpha |\partial_\alpha \psi|^q dy \leq 2C_2^q \int_\omega dy \text{ and } |\int_{\gamma_0} \psi d\gamma|^q = |\int_{\gamma_0} \varphi_0 d\gamma|^q.$$

Since the field $\varphi_0 : \gamma_0 \to \mathbb{R}^3$ is continuous on γ_0 (as the trace on γ_0 of a function in $H^2(\omega)$; the set $\Phi_F(\omega)$ is not empty by assumption), there exists $c_5 = c_5(c_4, C_2, \varphi_0, \varphi_{0,3})$ such that, for all $\psi \in \Phi_F(\omega)$,

$$\|\psi\|^q_{W^{1,q}(\omega)} = \int \{|\psi|^q + \sum_\alpha |\partial_\alpha \psi|^q\} dy \leq c_5.$$

The Sobolev imbedding $W^{1,p}(\omega) \hookrightarrow C^0(\bar{\omega})$ then implies the existence of $c_6 = c_6(c_5)$ such that, for all $\psi \in \Phi_F(\omega)$,

$$|\psi|_{0,\infty,\omega} \leq c_6,$$

and the second assertion is proved. If $\psi \in \Phi_F(\omega)$, the components $\partial_{\alpha\beta}\psi \cdot a_\sigma(\psi)$ of the vector fields $\partial_{\alpha\beta}\psi = \partial_\alpha a_\beta(\psi) \in L^2(\omega)$ over the vectors $a^\sigma(\psi)$ of the contravariant basis of the tangent plane to the deformed surface $\psi(\bar{\omega})$ are in $L^2(\omega)$, since $a_\sigma(\psi) = \partial_\sigma \psi \in L^\infty(\omega)$ by (iii). We next show that these components remain in a bounded subset of the space $L^2(\omega)$ when varies in the set $\Phi_F(\omega)$.

(iv) There exists C_4 such that

$$|\partial_{\alpha\beta}\psi \cdot a_\sigma(\psi)|_{0,\omega} \leq C_4 \text{ for all } \psi \in \Phi_F(\omega).$$

By assumption, $\theta \in W^{2,P}(\omega; \mathbb{R}^3)$ with $p > 2$; as a consequence, $\partial_{\alpha\beta}\theta \cdot \partial_\sigma \theta \in L^p(\omega) \subset L^2(\omega)$. Differentiating the relations

$$\partial_\alpha \psi \cdot \partial_\beta \psi = a_{\alpha\beta}(\psi) = a_{\alpha\beta} = \partial_\alpha \theta \cdot \partial_\beta \theta,$$

in the sense of distributions (which is licit, as is immediately verified) then shows *that there exists cz such that, for all $\psi \in \Phi_F(\omega)$*

$$|\partial_{11}\psi \cdot \partial_1\psi|_{0,\omega} \leq c_7, |\partial_{12}\psi \cdot \partial_1\psi|_{0,\omega} \leq c_7,$$

$$|\partial_{12}\psi \cdot \partial_2\psi|_{0,\omega} \leq c_7, |\partial_{22}\psi \cdot \partial_2\psi|_{0,\omega} \leq c_7,$$

$$|\partial_{11}\psi \cdot \partial_2\psi + \partial_{12}\psi \cdot \partial_1\psi|_{0,\omega} \leq c_7,$$

$$|\partial_{22}\psi \cdot \partial_1\psi + \partial_{12}\psi \cdot \partial_2\psi|_{0,\omega} \leq c_7.$$

The relations

$$\partial_{11}\psi \cdot \partial_2\psi = (\partial_{11}\psi \cdot \partial_2\psi + \partial_{12}\psi \cdot \partial_1\psi) - \partial_{12}\psi \cdot \partial_1\psi,$$

$$\partial_{22}\psi \cdot \partial_1\psi = (\partial_{22}\psi \cdot \partial_1\psi + \partial_{12}\psi \cdot \partial_2\psi) - \partial_{12}\psi \cdot \partial_2\psi,$$

then imply that

$$|\partial_{11}\psi \cdot \partial_2\psi|_{0,\omega} \leq 2c_7, |\partial_{22}\psi \cdot \partial_1\psi|_{0,\omega} \leq 2c_7.$$

Thanks to parts (ii) to (iv), a lower bound for the norms $|b_{\alpha\beta}(\psi)|_{0,\omega}$ when $\psi \in \mathbf{\Phi}_F(\omega)$ can now be established. This lower bound will be essential for proving in part (vii) the coerciveness of the functional $\mathbf{I}_F$ over the manifold $\mathbf{\Phi}_F(\omega)$.

(v) There exists C_5 such that

$$\sum_{\alpha,\beta} |b_{\alpha\beta}(\psi)|^2_{0,\omega} \geq \| \psi \|^2_{2,\omega} + C_5 \text{ for all } \psi \in \Phi_F(\omega).$$

Let $\psi \in \Phi_F(\omega)$. For almost all $y \in \omega$, the vectors $a_i(\psi)(y)$ are linearly independent by (ii), so that the vectors $a^i(\psi)(y)$ are well defined by the relations $a^i(\psi)(y) \cdot a_j(\psi)(y) = \delta^i_j$ for almost all $y \in \omega$. We can then expand $\partial_{\alpha\beta}\psi$ as

$$\partial_{\alpha\beta}\psi = \{\partial_{\alpha\beta}\psi \cdot a_\sigma(\psi)\}a^\sigma(\psi) + \{\partial_{\alpha\beta}\psi \cdot a_3(\psi)\}a^3(\psi) \text{ a.e. in } \omega.$$

Since $b_{\alpha\beta}(\psi) = \partial_{\alpha\beta}\psi \cdot a_3(\psi)$, $|a^3(\psi)| = 1$, and $a^3(\psi) \cdot a^\sigma(\psi) = 0$, we have, for almost all $y \in \omega$,

$$|\partial_{\alpha\beta}\psi(y)|^2 = |\{\partial_{\alpha\beta}\psi(y) \cdot a_\sigma(\psi)(y) \cdot a_\sigma(\psi)(y)\}a^\sigma(\psi)(y)|^2 + |b_{\alpha\beta}(\psi)(y)|^2.$$

Then

$$|\partial_{\alpha\beta}\psi(y)|^2 - |b_{\alpha\beta}(\psi)(y)|^2 = |\{\partial_{\alpha\beta}\psi(y) \cdot a_\sigma(\psi)(y) \cdot a_\sigma(\psi)(y)\}a^\sigma(\psi)(y)|^2.$$

Since the vector fields $a^\sigma(\psi)$ lie in a bounded subset of $L^\infty(\omega)$ when ψ varies in the set $\Phi_F(\omega)$ (parts (ii) and (iii)) and since the functions $\partial_{\alpha\beta}\psi \cdot a_\sigma(\psi)$ lie in a bounded subset of $L^2(\omega)$ when ψ varies in $\Phi_F(\omega)$ (part (iv)), there exists c_8 such that

$$|\{\partial_{\alpha\beta}\psi \cdot a_\sigma(\psi)\}a^\sigma(\psi)|_{0,\omega} \leq c_8 \text{ for all } \psi \in \Phi_F(\omega).$$

Consequently, there exists c_9 such that, for all $\psi \in \Phi_F(\omega)$,

$$0 \leq \sum_{\alpha,\beta} |\partial_{\alpha\beta}\psi|^2_{0,\omega} - \sum_{\alpha,\beta} |b_{\alpha\beta}(\psi)|^2_{0,\omega} \leq c_9.$$

Since there exists c_{10} such that $\|\psi\|_{1,\omega} \leq c_{10}$ for all $\psi \in \Phi_F(\omega)$ (see part (iii)), we finally have

$$\sum_{\alpha,\beta} |b_{\alpha\beta}(\psi)|^2_{0,\omega} \geq \sum_{\alpha,\beta} |\partial_{\alpha\beta}\psi|^2_{0,\omega} + \|\psi\|^2_{1,\omega} - c_9 - \|\psi\|^2_{1,\omega}$$

$$\geq \|\psi\|^2_{2,\omega} - c_9 - c_{10}^2 \text{ for all } \psi \in \Phi_F(\omega).$$

We now turn our attention to the functional $\mathbf{I}_F$.

(vi) The functional $\mathbf{I}_F$ is sequentially weakly lower semi-continuous over the manifold $\Phi_F(\omega)$, i.e.,

$$\psi^l \in \Phi_F(\omega), l \geq 1, \text{ and } \psi^l \rightharpoonup \psi \in \Phi_F(\omega) \text{ in } H^2(\omega),$$

implies that

$$\mathbf{I}_F(\psi) \leq \liminf_{l \to \infty} \mathbf{I}_F(\psi^l).$$

The weak convergence $\psi^l \rightharpoonup \psi$ in $H^2(\omega)$ clearly implies that

$$\partial_{\alpha\beta}\psi^l \rightharpoonup \partial_{\alpha\beta}\psi \text{ in } L^2(\omega) \text{ and } a_\alpha(\psi^l) \to a_\alpha(\psi) \text{ in } L^2(\omega).$$

The last convergences being consequences of the Rellich-Kondrašov imbedding theorem. We first show that it also implies that

$$a_3(\psi^l) \to a_3(\psi) \text{ in } L^2(\omega).$$

To this end, we observe that $|a_3(\psi^l)| = 1$ a.e. in ω and that there such that exists a subsequence $(\psi^m)_{m=1}^\infty$ of $(\psi^l)_{l=1}^\infty$ such that

$$a_\alpha(\psi^m)(y) \to a_\alpha(\psi)(y) \text{ for allmost all } y \in \omega,$$

since $a_\alpha(\psi^l) \to a_\alpha(\psi)$ *in* $L^2(\omega)$.

The definition

$$a_3(\psi^m)(y) = \frac{a_1(\psi^m)(y) \wedge a_2(\psi^m)(y)}{|a_1(\psi^m)(y) \wedge a_2(\psi^m)(y)|},$$

thus shows that

$$a_3(\psi^m)(y) \to \frac{a_1(\psi)(y) \wedge a_2(\psi)(y)}{|a_1(\psi)(y) \wedge a_2(\psi)(y)|} = a_3(\psi)(y) \text{ as } m \to \infty,$$

for almost all $y \in \omega$ *(by (ii), the vectors* $a_\alpha(\psi^m)$, $m \geq 1$, *and* $a_\alpha(\psi)$ *are well defined and "uniformly" linearly independent a.e. in* ω*). Therefore,* $a_3(\psi^m) \to a_3(\psi)$ *in* $L^2(\omega)$ *by Lebesgue's dominated convergence theorem. Since the limit* $a_3(\psi)$ *is unique, the whole sequence* $(a_3(\psi^l))_{l=1}^\infty$ *strongly converges in* $L^2(\omega)$ *to this limit.*

Using these properties, we next show that

$$b_{\alpha\beta}(\psi^l) = \partial_{\alpha\beta}\psi^l \cdot a_3(\psi^l) \rightharpoonup \partial_{\alpha\beta}\psi \cdot a_3(\psi) = b_{\alpha\beta}(\psi) \text{ in } L^2(\omega).$$

To this end, fix α *and* β*, let* $f^l \in L^2(\omega)$ *denote one component of* $\partial_{\alpha\beta}\psi^l$ *(the same for all* $l \geq 1$*), let* $g^l \in L^\infty(\omega)$ *denote the same component of* $a_3(\psi^l)$*, and finally, let* $f \in L^2(\omega)$ *and* $g \in L^\infty(\omega)$ *likewise denote the corresponding components of* $\partial_{\alpha\beta}\psi$ *and* $a_3(\psi)$.

In this fashion, the two sequences $(f^l)_{l=1}^\infty$ *and* $(g^l)_{l=1}^\infty$ *satisfy:*

$$f^l \rightharpoonup f \text{ in } L^2(\omega),$$

$$g^l \to f \text{ in } L^2(\omega) \text{ and } |g^l|_{0,\infty,\omega} \leq 1 \text{ for all } l.$$

It then follows that $fg \in L^2(\omega)$ *and*

$$f^l g^l \rightharpoonup fg \text{ in } L^2(\omega).$$

Although these implications are standard, we provide a proof for completeness. For any $\varphi \in \mathcal{D}(\omega)$*, the bilinear form*

$$(f, g) \in L^2(\omega) \times L^2(\omega) \to \int_\omega fg\varphi dy,$$

is strongly continuous; hence

$$f^l \rightharpoonup f \text{ in } L^2(\omega) \text{ and } g^l \to g \text{ in } L^2(\omega) \Rightarrow \int_\omega f^l g^l \varphi dy \to \int_\omega fg\varphi dy.$$

Let $(f^m g^m)_{m=1}^{\infty}$ *be an arbitrary subsequence of* $(f^l g^l)_{l=1}^{\infty}$. *Since* $|f^m g^m|_{0,\omega} \leq |f^m|_{0,\omega}$ *and the weakly convergent sequence* $(f^m)_{m=1}^{\infty}$ *is bounded in* $L^2(\omega)$, *there is a subsequence* $(f^n g^n)_{l=1}^{\infty}$ *of* $(f^m g^m)_{l=1}^{\infty}$ *that weakly converges in* $L^2(\omega)$ *to some* $h \in L^2(\omega)$.

Therefore,

$$\int_{\omega} f^n g^n \varphi dy \to \int_{\omega} h\varphi dy = \int_{\omega} fg\varphi dy \text{ for all } \varphi \in \mathcal{D}(\omega).$$

Thus $h = fg$.

Since the limit fg *of the subsequence* $(f^n g^n)_{n=1}^{\infty}$ *is unique, the whole sequence* $(f^l g^l)_{l=1}^{\infty}$ *weakly converges in* $L^2(\omega)$ *to this limit.*

In particular then, we have established that $b_{\alpha\beta}(\psi^l) \rightharpoonup b_{\alpha\beta}(\psi)$ *in* $L^2(\omega)$.

We are now in a position to establish the sequential weak lower semi-continuity of $\mathbf{I}_F$ *over* $\Phi_F(\omega)$.

Let $\mathbf{L}_s^2(\omega)$ *denote the space of all fields of symmetric matrices of order two with components in* $L^2(\omega)$.

The symmetric bilinear form $\mathrm{B} : \mathbf{L}_s^2(\omega) \times \mathbf{L}_s^2(\omega) \to \mathbb{R}$ *defined by*

$$\mathrm{B}(\mathbf{S}, \mathbf{T}) = \int_{\omega} a^{\alpha\beta\sigma\tau} s_{\sigma\tau} t_{\alpha\beta} \sqrt{a} dy,$$

for all $(\mathbf{S}, \mathbf{T}) = ((s_{\alpha\beta}), (t_{\alpha\beta})) \in \mathbf{L}_s^2(\omega) \times \mathbf{L}_s^2(\omega)$ *is strongly continuous and positive definite. We note, first that*

$$a^{\alpha\beta}(y) a^{\sigma\tau}(y) t_{\sigma\tau} t_{\alpha\beta} = (a^{\alpha\beta}(y) t_{\alpha\beta})^2 \geq 0,$$

for all $y \in \bar{\omega}$ *and all matrices, next that*

$$(a^{\alpha\sigma}(y) a^{\beta\tau}(y) + a^{\alpha\tau}(y) a^{\beta\sigma}(y)) t_{\sigma\tau} t_{\alpha\beta} = 2t^T \mathbf{A}(y) t,$$

for all symmetric matrices $(t_{\alpha\beta})$, *where*

$$\mathbf{A}(y) = \begin{pmatrix} a^{11}a^{11} & 2a^{11}a^{12} & a^{12}a^{12} \\ 2a^{12}a^{12} & 2(a^{12}a^{12} + a^{11}a^{22}) & 2a^{12}a^{22} \\ a^{12}a^{12} & 2a^{12}a^{22} & a^{22}a^{22} \end{pmatrix} (y),$$

and

$$t = \begin{pmatrix} t_{11} \\ t_{12} \\ t_{22} \end{pmatrix}.$$

Since $a^{11}a^{11} > 0$ *in* $\bar{\omega}$

$$\det \begin{pmatrix} a^{11}a^{11} & 2a^{11}a^{12} \\ 2a^{12}a^{12} & 2(a^{12}a^{12} + a^{11}a^{22}) \end{pmatrix} = 2\frac{a^{11}a^{11}}{a} > 0 \ \textit{in}\ \bar{\omega},$$

$$\det \mathbf{A} = \frac{2}{a^3} > 0 \ \textit{in}\ \bar{\omega},$$

where $a = \det(a_{\alpha\beta})$, *we infer from a well-known characterization that the symmetric matrix* $\mathbf{A}(y)$ *is positive definite at all* $y \in \bar{\omega}$, *since there exist constant* c_{11} *such that*

$$0 < c_{11} \ \textit{and}\ a^{\alpha\beta\sigma\tau}(y)t_{\sigma\tau}t_{\alpha\beta} \geq c_{11} \sum_{\alpha,\beta} |t_{\alpha\beta}|^2,$$

for all $y \in \bar{\omega}$ *and all symmetric matrices* $(t_{\alpha\beta})$ *and there exist constant* c_{12}

$$0 < c_{12} \leq 1 \ \textit{and}\ 0 < c_{12} \leq \sqrt{a(y)} \leq c_{12}^{-1}.$$

for all $y \in \bar{\omega}$. *Being strongly continuous and (strictly) convex, the mapping*

$$\mathbf{S} \in \mathbf{L}_s^2(\omega) \to \mathrm{B}(\mathbf{S}, \mathbf{S}),$$

is thus weakly lower semi-continuous.

Let

$$s_{\alpha\beta}^l = b_{\alpha\beta}(\psi^l) - b_{\alpha\beta} \ \textit{and}\ s_{\alpha\beta} = b_{\alpha\beta}(\psi) - b_{\alpha\beta}.$$

Then

$$\mathbf{S}^l = (s_{\alpha\beta}^l) \rightharpoonup \mathbf{S} = (s_{\alpha\beta}) \ \textit{in}\ L^2(\omega),$$

since $b_{\alpha\beta}(\psi^l) \rightharpoonup b_{\alpha\beta}(\psi)$ *in* $L^2(\omega)$, *and thus*

$$\mathrm{B}(\mathbf{S}, \mathbf{S}) \leq \liminf_{l \to \infty} \mathrm{B}(\mathbf{S^l}, \mathbf{S}^l).$$

This shows that the functional $\mathbf{I}_F$, *which is defined by*

$$\mathbf{I}_F(\psi) = \frac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau}(b_{\sigma\tau}(\psi) - b_{\sigma\tau})(b_{\alpha\beta}(\psi) - b_{\alpha\beta})\sqrt{a}dy - \mathbf{L}(\psi),$$

for all $\psi \in \Phi_F(\omega)$ *is sequentially weakly lower semi-continuous on* $\mathbf{I}_F(\omega)$ *(recall that* $\mathbf{L}$ *is by assumption a continuous linear form on* $H^2(\omega)$*).*

(vii) There exist constants C_6 and C_7 such that

$$C_6 > 0 \text{ and } \mathbf{I}_F(\psi) \geq C_6 \|\psi\|^2_{2,\omega} + C_7 \text{ for all } \psi \in \Phi_F(\omega).$$

Consequently, the functional $\mathbf{I}_F$ is coercive on the manifold $\Phi_F(\omega)$. By definition of the functional $\mathbf{I}_F$, we have (the constants c_{11} and c_{12} appeared in the proof of part (vi))

$$\mathbf{I}_F(\psi) = \frac{\varepsilon^3}{6} \int_\omega a^{\alpha\beta\sigma\tau}(b_{\sigma\tau}(\psi) - b_{\sigma\tau})(b_{\alpha\beta}(\psi) - b_{\alpha\beta})\sqrt{a} dy - \mathbf{L}(\psi),$$

$$\mathbf{I}_F(\psi) \geq \frac{\varepsilon^3}{6} c_{11} c_{22} \sum_{\alpha,\beta} |b_{\alpha\beta}(\psi) - b_{\alpha\beta}|^2_{0,\omega} - c_{13} \|\psi\|_{2,\omega},$$

where c_{13} denotes the norm of the continuous linear form $\mathbf{L}$.

Since

$$|b_{\alpha\beta}(\psi) - b_{\alpha\beta}|^2_{0,\omega} \geq \frac{1}{2} |b_{\alpha\beta}(\psi)|^2_{0,\omega} - |b_{\alpha\beta}|^2_{0,\omega},$$

and since, by (v),

$$\sum_{\alpha,\beta} |b_{\alpha\beta}(\psi)|^2_{0,\omega} \geq \|\psi\|^2_{2,\omega} + C_5 \text{ for all } \psi \in \Phi_F(\omega).$$

We find

$$\mathbf{I}_F(\psi) \geq C_6 \|\psi\|^2_{2,\omega} + C_7 \text{ for all } \psi \in \Phi_F(\omega).$$

the assertion follows, and the existence of a minimizer of the functional $\mathbf{I}_F$ over the manifold $\Phi_F(\omega)$ is thus established.

■

Conclusions and perspectives

The major conclusions of these studies are:

1. *An application of the technics from formal asymptotic analysis to the three-dimensional model of nonlinearly elastic shells with a specific class of boundary conditions of von Kármán's type, made of a Saint Venant–Kirchhoff material shows that the leading term of the expansion is characterized by a two-dimensional model of von Kármán membrane shell, under specific assumptions on the geometry of the middle surface of the shell and on the components of the applied forces. We found in particular that the forces of von Kármán's type should be of order* $O(\varepsilon^0)$.

2. *An application the same technics to the three-dimensional model of von Kármán flexural shell shows that the leading term of the expansion is characterized by a two-dimensional model of von Kármán flexural shell. But we found in particular that the forces of von Kármán's type should be of order* $O(\varepsilon^2)$. *Also we establish the existence of solution to the two-dimensional model.*

As future work, we plan to:

1. *Extend these studies to viscoelastic materials.*

2. *Extend these studies to generalized von Kármán's type.*

3. *Justification of two-dimensional energy of von Kármán membrane shell using* Γ*-convergence theory.*

4. *Justification of two-dimensional energy of von Kármán flexural shell using* Γ*-convergence theory.*

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Title: Asymptotic modeling of shells with von Kármán boundary conditions

Abstract: *The objective of this work is to study the asymptotic justification of the two-dimensional equations for membrane and flexural shells with boundary conditions of von Kármán's type. More precisely, we consider a three-dimensional models for a nonlinearly elastic membrane and flexural shells of Saint Venant–Kirchhoff material, where only a portion of the lateral face is subjected to boundary conditions of von Kármán's type. Using technics from formal asymptotic analysis with the thickness of the shell as a small parameter, we show that the scaled three-dimensional solution still leads to the so-called two-dimensional equations of von Kármán membrane and flexural shells.*

Key words: *Asymptotic analysis, nonlinear elasticity, shell theory, von Kármán boundary conditions.*

Titre: Modélisation asymptotique des coques avec des conditions aux limites de von Kármán

Résumé: *L'objectif de ce travail est d'étudier la justification asymptotique des équations bidimensionnelles pour les membranes et les coques flexibles avec des conditions aux limites de type von Kármán. Plus précisément, nous considérons des modèles tridimensionnels pour des membranes élastiques non linéaires et des coques flexibles en matériau de Saint Venant-Kirchhoff, où seule une partie de la face latérale est soumise à des conditions aux limites de type von Kármán. En utilisant des techniques d'analyse asymptotique formelle avec l'épaisseur de la coque comme petit paramètre, nous montrons que la solution tridimensionnelle mise à l'échelle conduit toujours aux équations bidimensionnelles dites de von Kármán pour les membranes et les coques flexibles.*

Mots clés: *Analyse asymptotique, élasticité non linéaire, théorie des coques, conditions aux limites de von Kármán.*

العنوان: النمذجة المقاربة لهياكل مع شروط حدية لفون كارمان

ملخص: هدف هذا العمل هو دراسة النمذجة المقاربة للمعادلات ثنائية الأبعاد للهياكل مع شروط حدية من نوع فون كارمان. على وجه التحديد، نعتبر نماذج ثلاثية الأبعاد لهياكل ذات مرونة غير خطية مصنوعة من مادة سانت فينانت - كيرشوف، حيث يتعرض جزء فقط من الوجه الجانبي لشروط حدية من نوع فون كارمان. باستخدام تقنيات التحليل المقارب وباعتبار سمك الهيكل معاملاً صغيراً، أثبتنا أن الحل ثلاثي الأبعاد يؤدي إلى حل ما يسمى بالمعادلات ثنائية الأبعاد للهياكل الغشائية و المنحنية لفون كارمان.

الكلمات المفتاحية : تحليل مقارب، مرونة غير خطية، نظرية الهياكل، شروط حدية لفون كارمان.