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Theme

**Solution of the quadratic system in
non-commutative phase space (Coupled harmonic
oscillator pair in the mixed non-commutative phase
space)**

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Dedication

I dedicate this work to my parents:

May they find here the testimony of my deep gratitude and acknowledgment

*To my brothers and my sisters, my grandparents and my family who give love
and liveliness.*

*To all those who have helped me - directly or indirectly - and those who shared
with me the emotional moments during the accomplishment of this work and
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General introduction

The field of theoretical physics is continuously evolving, enhancing our understanding of the material world and the laws of nature. Among the various domains within this field, quantum mechanics and deformed quantum mechanics stand out as crucial areas that expand our knowledge of physical phenomena at the atomic and subatomic levels. Quantum mechanics, the branch of physics that studies the behavior of particles at the atomic level, has revolutionized our understanding of the microscopic world. The Schrödinger equation, in particular, has played a pivotal role in describing the state and evolution of a quantum system over time. However, this equation faces significant challenges when considering systems with topological deformations or nonlinear effects. Deformations in quantum physics can arise from various factors, including interactions with external fields, the presence of material defects, or techniques used to manipulate particles with electromagnetic fields. These deformations lead to modifications in the eigenvalues and wave functions of the quantum system, necessitating the use of advanced mathematical methods to fully understand these effects.

This study seeks to provide a comprehensive understanding of the impact of deformations on quantum systems, particularly in the context of the Schrödinger equation. By focusing on the relationship between deformation parameters (σ and θ) and the energy and probability density of the quantum system, this research aims to contribute, even if through trial and error, to the development of more accurate mathematical models for describing deformed quantum systems. Such advancements can open new horizons in technological applications, including quantum computing, nanotechnology, and the exploration of new materials.

Study Objectives:

1. Analyze the Impact of Deformations on Eigenvalues: Study how the eigenvalues,

representing the energy levels of the Schrödinger equation, change with the application of different deformations.

2. Verify the Relationship Between Deformation Parameters and Probability Density: Understand how deformations affect the probability density distribution of the quantum system.

3. Compare Two Different but Theoretically Equivalent Approaches: Analyze and compare the results obtained using the Feynman approach and the Schrödinger method for dealing with deformations in quantum systems.

Several mathematical methodologies have been employed to analyze deformed quantum systems, including Hamiltonian transformations and finding analytical solutions to the Schrödinger equation in irregular spaces. Advanced analytical tools such as eigenfunctions and eigenvalues have been applied, along with matrix diagonalization techniques, to a coupled harmonic oscillator system in a deformed mixed phase space. After obtaining the solution, it is analyzed and directly compared using Python software, specifically the Google Colab platform, to overcome the formal mathematical complexity of the solution expressions. Subsequently, the behavior of energy and probability density in terms of deformation parameters is studied.

Non-Commutative Phase Space

In this chapter, some mathematical foundations of commutative and non-commutative phase space and its representations in classical and quantum physics.

1.1 Introduction:

In classical mechanics, the phase space of a system is a specific type of mathematical structure called a symplectic manifold (M, Λ) that possesses a closed, non-degenerate 2-form Λ [1]. This mathematical structure allows for the derivation of measurable physical quantities as well as the conservation of physical laws under changes in the coordinate system, as long as it meets the criteria of the Poisson bracket. Since the measurement process does not affect the measured quantities, the Poisson bracket of the commutator is sufficient to meet the computational requirements.

The symmetry of the manifold forces the phase space to be commutative, meaning the measurement process has no impact on the measured physical quantities. This is valid in macroscopic cases. However, over time, microscopic phenomena such as blackbody radiation, the Compton effect, the photoelectric effect, and the behavior of electrons in the atom necessitated the development of another mathematical structure that is non-commutative. This new structure takes into account the deformation caused by the measurement process. Given that physical quantities are inherently related to the Poisson bracket, finding non-commutative structures that satisfy the Poisson bracket condition provides an opportunity to understand these effects and allows for the precise study of microscopic structures.

As we will detail in this chapter, Therefore, in this chapter we will:
The method transitioning from classical mechanics to non-commutative classical mechanics, known as deformation of phase space.

The method transitioning from quantum mechanics to non-commutative quantum mechanics, known as canonical quantization in the deformed phase space

1.2 Structure of Commutative Phase Space:

Definition 1.2.1 *Symplectic Manifold:* [2-5]

Let M be a smooth manifold of dimension $2n$ and $\Lambda \in \Omega^2(M)$ a differential 2-form. The pair (M, Λ) is called a symplectic manifold if the following conditions are satisfied:

1. Closedness: The form Λ is closed, i.e., $d\Lambda = 0$.
2. Strong Non-degeneracy: The form Λ is strongly non-degenerate, i.e. if $\Lambda_x(v, w) = 0 \quad \forall v \in T_xM$, then $w = 0$.

Properties 1.2.1 : [4-6]

1. Non-degeneracy: For any non-zero vector $v \in T_xM$, there exists a vector $w \in T_xM$ such that $\Lambda(v, w) \neq 0$.
2. Strong Non-degeneracy: The symplectic form Λ satisfies $d\Lambda = 0$.
3. Canonical Transformations (Darboux's Theorem: For every smooth symplectic manifold (M, Λ) , there exists a local coordinate system (q^i, p_i) called Darboux coordinates, in which the symplectic form can be expressed as:

$$\omega = \sum_{i=1}^n dq^i \wedge dp_i.$$

Definition 1.2.2 *Hamiltonian Vector Field:* [3, 6, 7]

Let $H : M \rightarrow \mathbb{R}$ be a smooth function called the Hamiltonian. The Hamiltonian vector field X_H associated with H is defined by:

$$\iota_{X_H}\omega = dH,$$

where ι_{X_H} denotes the interior product of the vector field X_H with the form ω .

In local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$, the Hamiltonian vector field X_H is given by:

$$X_H = \sum_{i=1}^n \left(\frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right)$$

Properties 1.2.2 :

1. Existence and Uniqueness: Due to the non-degeneracy of ω , for every smooth function H , there exists a unique Hamiltonian vector field X_H .
2. Symplectic Flow: The flow ϕ_t generated by X_H preserves the symplectic form: $\phi_t^* \Lambda = \Lambda$.
3. Hamilton's Equations:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Definition 1.2.3 *Poisson Bracket:*[3, 6, 7]

Let (M, Λ) be a symplectic manifold, and $F, G : M \rightarrow \mathbb{R}$ be smooth functions. The Poisson bracket $\{F, G\}$ is defined by:

$$\{F, G\} = \Lambda(X_F, X_G),$$

where X_F and X_G are the Hamiltonian vector fields associated with the functions F and G , respectively.

In local coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$, [8] this becomes:

$$\{f, g\} = \sum_{i=1}^n \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right)$$

Properties 1.2.3 [1, 3, 7]

1. Bilinearity $\{af + bg, h\} = a\{f, h\} + b\{g, h\}$ for all $a, b \in \mathbb{R}$.
2. Antisymmetry $\{f, g\} = -\{g, f\}$.
3. Jacobi Identity $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
4. Leibniz Rule $\{fg, h\} = f\{g, h\} + g\{f, h\}$.

Note 1.2.1 (1), (2), and (4) define the Lie bracket on A . [1, 9]

Definition 1.2.4 [1, 2, 10, 11] Let A be a K -vector space, and \cdot and $\{\cdot, \cdot\}$ two bilinear applications on $A \times A$. A is a Poisson algebra if:

1. \cdot is associative and commutative.
2. $\{\cdot, \cdot\}$ is a Lie bracket.
3. \cdot and $\{\cdot, \cdot\}$ are related by the Leibniz identity:

$$\forall f, g, h \in A, \{f \cdot g, h\} = f \cdot \{g, h\} + \{f, h\} \cdot g \tag{1.1}$$

Definition 1.2.5 [1, 9, 11] Let \mathfrak{g} be a vector space equipped with a binary operation called the Lie bracket $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$, we call the $(\mathfrak{g} [\cdot, \cdot])$ a Lie algebra if it was:

1. Bilinearity: The bracket operation is linear in both arguments.
2. Antisymmetry: $[x, y] = -[y, x]$ for all $x, y \in \mathfrak{g}$.
3. Jacobi Identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

Note 1.2.2 Let V be a vector space. We denote \cdot as the usual product of two functions, and $\{\cdot, \cdot\}$ as a Poisson bracket on V . Then $(F(V), \cdot, \{\cdot, \cdot\})$ is a Poisson algebra. A manifold equipped $(M, \{\cdot, \cdot\})$ with a Poisson bracket is called a Poisson manifold. [1, 4, 12]

the application $f \mapsto H_f$ is a homomorphism of Lie algebras from $(N, \{\cdot, \cdot\})$ into the Lie algebra of Hamiltonian vector fields [1, 5]. Now, we can give the properties of the Poisson bracket, there exists a unique antisymmetric contravariant 2-tensor, denoted Λ , such that

$$\{f, g\} = \Lambda(df, dg) \quad (1.2)$$

Thus, $\Lambda(z)$ is called the Poisson tensor of the Poisson manifold M or the Poisson bivector [10], and in local coordinates, we have

$$\{f, g\}(z) = \sum_{ij} \Lambda_{ij}(z) \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j}, \quad \text{where } \Lambda_{ij}(z) = \{z_i, z_j\} \quad (1.3)$$

In the case of an N -dimensional system, the base $A_0 = \{p_i, x_i\}$ of the phase space satisfies the following relations

$$\{x_i, x_j\} = 0, \quad \{x_i, p_j\} = \delta_{ij}, \quad \{p_i, p_j\} = 0; \quad i, j = 1, \dots, N \quad (1.4)$$

In the unified notation of the phase space variables: $z(t) = (x_i(t), p_i(t))$, $i = 1, \dots, N$, and $a = 1, \dots, 2N$, we have

$$\{z_a, z_b\} = \Lambda_{ab} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad (1.5)$$

Using (1.3), it is possible to rewrite (1.5) in the multidimensional case as follows:

$$\{g(z(t)), f(z(t))\} = (\nabla_z g) \Lambda (\nabla_z f) \quad (1.6)$$

We now consider a dynamic system described by the Hamiltonian $H(z)$. The Hamiltonian system thus defined on the previous symplectic manifold admits the following equation of motion:

$$\dot{z}_i(t) = \{z_i, H(z)\} = \Lambda_{ij} \partial_j H(z) \quad (1.7)$$

This structure represents the fact that no physical process affects the other value, which allows for the interchange of these values (functions that satisfy the Poisson bracket). However, on the microscopic level and due to the smallness of physical quantities, it is impossible to measure without affecting the measured quantities, as long as we rely on physical phenomena whose effects are much larger than the phenomena we want to study.

This situation forces us to develop a new structure for phase space that represents the influence of measurable quantities by the measurement process itself or even by the interaction of the system's components with each other. Thus, the concept of quantization was developed, which succeeded in explaining phenomena such as the blackbody spectrum through Planck's quantization. In this context, scientists continue to refine and develop quantization methods, among which is the quantization of deformed phase space.

Here, it is necessary to mention the most important point in any scientific theory: to be considered valid, a scientific theory must explain and predict the studied phenomenon and also be able to explain and predict the phenomena explained by previous theories.

From this perspective, we notice that the derivation of Hamilton's equations in classical phase space is merely a result and property of the Poisson bracket. From here, Moyal hypothesized that if the Poisson bracket is modified to meet the condition of non-commutativity by changing the definition of the product in the algebra of measurable functions (specifically, using the Moyal product), and if this new structure still satisfies the essential properties of the Poisson bracket, then its quantization will lead to the quantization of deformed phase space, thus accommodating the deformation in phase space. This is what we will study now.

1.3 Canonical Quantization:

The state of a physical system, as previously described, is represented by a point in phase space with coordinates of position and momentum. The symmetric structure of classical mechanics includes Poisson brackets between these variables. Thus, all transformations that remain invariant under the application of the Poisson bracket are called canonical transformations.

In quantum mechanics, the dynamic variables are replaced by operators in the Hilbert space of quantum states. The Poisson brackets are directly replaced by commutation relations, which lead to Heisenberg's uncertainty principle[11, 13].

$$\{f, g\} \rightarrow \frac{1}{i\hbar}[\hat{f}, \hat{g}] \quad (1.8)$$

This proposal can be interpreted as seeking a "quantization map," \mathcal{O} , that associates a function f on classical phase space with an operator \hat{f} in quantum Hilbert space such that:

$$\mathcal{O}\{f, g\} = \frac{1}{i\hbar}[\mathcal{O}(f), \mathcal{O}(g)] \quad (1.9)$$

If we consider the following representation for the position and momentum operators in Hilbert space:

$$\mathcal{O}(x_i) = x_i \quad \text{and} \quad \mathcal{O}(p_i) = -i\hbar \frac{\partial}{\partial x_i} \quad (1.10)$$

Applying the Poisson bracket to these operators forms the following Heisenberg algebra:

$$[x_i, x_j] = 0, \quad [x_i, p_j] = i\hbar\delta_{ij}, \quad [p_i, p_j] = 0 \quad (1.11)$$

Since canonical quantization lacks precision in its assumptions, which are restricted to a standard form that assumes no interaction between the dynamical variables, mathematicians have proposed various quantization structures considered more general, most notably Deformation quantization[1, 14].

1.4 Deformation Quantization:

In classical mechanics, a system is described by the triple $(M, \{\cdot, \cdot\}, H)$, where M is the phase space, $\{\cdot, \cdot\}$ is the Poisson bracket, and H is the Hamiltonian. Quantum mechanics, on the other hand, uses a complex Hilbert space H with an associated Hamiltonian operator \hat{H} . The correspondence between classical and quantum mechanics has been challenging due to the lack of a precise mathematical mapping. Groenewold's[14, 15] no-go theorem shows that the Poisson algebra $C^\infty(\mathbb{R}^{2n})$ cannot be quantized such that the Poisson bracket maps to the commutator of operators.

In his research paper, Bain et al. (1978)[1, 11, 14, 15] proposed a formulation of deformation quantization that provides a rigorous framework for transitioning from classical mechanics to quantum mechanics by deforming the algebraic structure of phase space[1]. Instead of directly creating operators on Hilbert space, this approach focuses on modifying the algebra of functions in classical phase space $F(M)$. By modifying the standard product between functions to a star product, non-commutativity in phase space is expressed with a deformation parameter \hbar .

Definition 1.4.1 *Star Product:*

The star product $*$ is a bilinear map on the space of smooth functions $C^\infty(M)$ defined as:

$$(f, g) \mapsto \sum_{k=0}^{\infty} C_k(f, g) \hbar^k \quad (1.12)$$

Properties 1.4.1 :

The star product satisfies the fundamental properties of the Poisson algebra [1, 14, 15], except commutativity:

1. **Associativity:** For all $p \geq 0$,

$$\sum_{p=k+l=0} C_k(C_l(f, g), h) = \sum_{p=k+l=0} C_k(f, C_l(g, h)) = 0 \quad (1.13)$$

2. $C_0(f, g) = fg$

3.

$$\frac{1}{2}(C_1(f, g) - C_1(g, f)) = \{f, g\} \quad (1.14)$$

where $\{f, g\}$ is the Poisson bracket.

4. Each map $C_k : A \times A \rightarrow A$ must be a bidifferential operator.

The formal deformation of the Poisson bracket is an antisymmetric map as follow:

$$(f, g) \mapsto \sum_{k=0}^{\infty} T_k(f, g) \hbar^k \quad (1.15)$$

It satisfies:

1. **Jacobi Identity:** For all $p \geq 0$,

$$\sum_{p=k+l=0} T_k(T_l(f, g), h) = 0 \quad (1.16)$$

where the sum is taken over cyclic permutations of the set $\{f, g, h\}$.

2. $T_0(f, g) = \{f, g\}$, where $\{f, g\}$ is the Poisson bracket.

3. Each map $T_k : A \times A \rightarrow A$ must be a bidifferential operator.

The maps C_k and T_k are bidifferential operators [1, 14]. The Moyal star product is given by:

$$f * g = \exp \left(\frac{i\hbar}{2} \sum_{i,j} \omega^{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right) (f \cdot g) \quad (1.17)$$

where ω^{ij} is a constant skew-symmetric tensor on R^n with $i, j = 1, \dots, n$,

The Moyal product is a specific example of a star product defined on the symplectic vector space \mathbb{R}^{2n} .

Definition 1.4.2 Given functions f and g , the Moyal product is:

$$\begin{aligned} (f * g)(q, p) &= f(q, p) \exp \left(\frac{i\hbar}{2} \left(\overleftarrow{\partial}_q \overrightarrow{\partial}_p - \overleftarrow{\partial}_p \overrightarrow{\partial}_q \right) \right) g(q, p) \\ &= \exp \left(\frac{i\hbar}{2} \sum_{i,j} \omega^{ij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} \right) (f \cdot g)(q, p) \end{aligned} \quad (1.18)$$

Where: ω^{ij} they are the elements of a Poisson tensor $\omega_{ij} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$, with I is the identity matrix and $i, j = 1, \dots, 2n$,

Properties 1.4.2 The Moyal product has the following properties[16, 17]:

1. **Non-commutativity:**

$$f(x, p) * g(x, p) \neq g(x, p) * f(x, p) \quad (1.19)$$

2. **Associativity:**

$$(f(x, p) * g(x, p)) * h(x, p) = f(x, p) * (g(x, p) * h(x, p)) \quad (1.20)$$

3. **Complex Conjugation:**

$$(f(x, p) * g(x, p))^* = g(x, p)^* * f(x, p)^* \quad (1.21)$$

4. **Integral Relation:**

$$\int d^D x (f * g)(x, p) = \int d^D x f(x, p)g(x, p) \quad (1.22)$$

5. **Cyclic Permutation:**

$$\int d^D x (f * g * h)(x, p) = \int d^D x (h * f * g)(x, p) \quad (1.23)$$

6. **Leibniz Rule:**

$$\partial_u (f * g) = (\partial_u f) * g + f * (\partial_u g) \quad (1.24)$$

1.4.1 Deformation of the Poisson Algebra

To prove that the new star quantization satisfies the basic properties of Poisson algebras, except for commutativity, we need to check the following properties[14, 15]:

1. **Bilinearity.**
2. **Associativity.**
3. **Antisymmetry.**
4. **Jacobi Identity.**
5. **Leibniz Rule.**

Bilinearity

The star product $*$ is defined as:

$$(f * g)(x) = f(x)g(x) + \sum_{k=1}^{\infty} \hbar^k C_k(f, g)(x) \quad (1.25)$$

For any scalars $\alpha, \beta \in \mathbb{C}$ and functions $f, g, h \in C^\infty(M)$, the product is bilinear:

$$(\alpha f + \beta g) * h = \alpha(f * h) + \beta(g * h) \quad (1.26)$$

$$f * (\alpha g + \beta h) = \alpha(f * g) + \beta(f * h) \quad (1.27)$$

Proof:

From the definition of the star product:

$$(f * (\alpha g + \beta h))(x) = f(x)(\alpha g(x) + \beta h(x)) + \sum_{k=1}^{\infty} \hbar^k C_k(f, \alpha g + \beta h)(x) \quad (1.28)$$

Since C_k are bidifferential operators, they are linear in both arguments, so:

$$\sum_{k=1}^{\infty} \hbar^k C_k(f, \alpha g + \beta h)(x) = \alpha \sum_{k=1}^{\infty} \hbar^k C_k(f, g)(x) + \beta \sum_{k=1}^{\infty} \hbar^k C_k(f, h)(x) \quad (1.29)$$

Thus,

$$(f * (\alpha g + \beta h))(x) = \alpha(f * g)(x) + \beta(f * h)(x) \quad (1.30)$$

Associativity.

The star product must satisfy:

$$(f * g) * h = f * (g * h) \quad (1.31)$$

For the Moyal product:

$$(f * g * h)(x) = f(x)g(x)h(x) + \frac{i\hbar}{2}\{f, g\}h + O(\hbar^2) \quad (1.32)$$

The associativity follows from the fact that the Moyal product is constructed to be associative through its definition involving exponential operators and the Baker-Campbell-Hausdorff formula.

Proof:

Using the Baker-Campbell-Hausdorff formula, the exponential representation of the Moyal product ensures associativity:

$$(f * g) * h = \left(f e^{\frac{i\hbar}{2} \overleftarrow{\partial} \theta \overrightarrow{\partial}} g \right) e^{\frac{i\hbar}{2} \overleftarrow{\partial} \theta \overrightarrow{\partial}} h = f e^{\frac{i\hbar}{2} \overleftarrow{\partial} \theta \overrightarrow{\partial}} \left(g e^{\frac{i\hbar}{2} \overleftarrow{\partial} \theta \overrightarrow{\partial}} h \right) = f * (g * h) \quad (1.33)$$

Antisymmetry.

For the commutator of the star product, which gives the deformed Poisson bracket, we have:

$$\{f, g\}_* = \frac{1}{i\hbar}(f * g - g * f) \quad (1.34)$$

This bracket is antisymmetric:

$$\{f, g\}_* = -\{g, f\}_* \quad (1.35)$$

Proof:

The antisymmetry is a direct consequence of the definition of the deformed Poisson bracket:

$$\{f, g\}_* = \frac{1}{i\hbar}(f * g - g * f) \quad (1.36)$$

which clearly satisfies:

$$\{f, g\}_* = -\{g, f\}_* \quad (1.37)$$

Jacobi Identity.

The deformed Poisson bracket satisfies the Jacobi identity if:

$$\{f, \{g, h\}_*\}_* + \{g, \{h, f\}_*\}_* + \{h, \{f, g\}_*\}_* = 0 \quad (1.38)$$

Proof:

The Jacobi identity for the deformed Poisson bracket follows from the associativity of the star product and the properties of the commutator. The associativity of the Moyal product ensures that:

$$\{f, \{g, h\}_*\}_* + \{g, \{h, f\}_*\}_* + \{h, \{f, g\}_*\}_* = 0 \quad (1.39)$$

Leibniz Rule.

The star product satisfies the Leibniz rule:

$$\{f, gh\}_* = \{f, g\}_* h + g\{f, h\}_* \quad (1.40)$$

Proof:

The Leibniz rule is verified by considering the action of the star product on products of functions and using the definition of the Moyal product. Using the properties of bidifferential operators C_k and their derivation properties, we find:

$$\{f, gh\}_* = \{f, g\}_* h + g\{f, h\}_* \quad (1.41)$$

When the phase space is deformed, the Poisson bracket is modified to incorporate non-commutativity. The deformed Poisson bracket is given by:

$$\{f, g\}_\theta = (\nabla_z f) \Lambda^{\text{def}} (\nabla_z g) \quad (1.42)$$

where Λ^{def} is the deformed Poisson tensor. This tensor satisfies:

$$\Lambda_{ij}^{\text{def}}(z) = \Lambda_{ij}^{(0)}(z) + \theta \Lambda_{ij}^{(1)}(z) + \theta^2 \Lambda_{ij}^{(2)}(z) + \dots \quad (1.43)$$

Where the modified Poisson tensor Λ^{def} is given by:

$$\omega_{\text{def}}^{ij} = \omega^{ij} + \theta^{ij} \quad (1.44)$$

The modified Poisson bracket is then defined using Λ_{def} :

$$\{f, g\}_{\text{def}} = \sum_{i,j} \Lambda_{\text{def}}^{ij} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} \quad (1.45)$$

where z_i represents the phase space variables (q_i, p_i) .

On a system with four coordinates $(x_i(t), p_i(t))$, defining a point in the deformed phase space. This phase space is naturally equipped with the symplectic form:

$$dx_i \wedge dp_i = \Lambda_{\text{def}} = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & \sigma \\ 0 & -1 & -\sigma & 0 \end{pmatrix} \quad (1.46)$$

where θ and σ are antisymmetric matrices. These commutation relations correspond to the deformed Poisson brackets in the classical phase space:

$$\{x_i, x_j\} = \theta_{ij}, \quad \{p_i, p_j\} = \sigma_{ij}, \quad \{x_i, p_j\} = \delta_{ij} \quad (1.47)$$

Example 1.4.1 Let us try to write the Hamiltonian for a system consisting of a constant coupled harmonic vibrating pair whose coupling is λ .

In the commutative case:

We will use a switching Poisson tensor with zero distortion parameters, which means the Poisson tensor is:

$$\Lambda_0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Using this tensor, the modified Poisson bracket becomes the standard Poisson bracket:

$$\{f, g\} = \sum_{i,j} \Lambda_0^{ij} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j}$$

The Hamiltonian for the system in this switching case is:

$$H_0(x_1, x_2; p_1, p_2) = \frac{1}{2m} p_1^2 + \frac{1}{2m} p_2^2 + \frac{m\omega^2}{2} x_1^2 + \frac{m\omega^2}{2} x_2^2 + \lambda x_1 x_2 \quad (1.48)$$

In the non-commutative case:

Using the modified Poisson tensor:

$$\Lambda_{def} = \begin{pmatrix} 0 & \theta & 1 & 0 \\ -\theta & 0 & 0 & 1 \\ -1 & 0 & 0 & \sigma \\ 0 & -1 & -\sigma & 0 \end{pmatrix}$$

The modified Poisson bracket is:

$$\{f, g\}_{def} = \sum_{i,j} \Lambda_{def}^{ij} \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j}$$

The Hamiltonian incorporating the effects of the non-commutative geometry becomes:

$$H_{\theta\sigma}(x_1, x_2; p_1, p_2) \approx \frac{1}{2\mu_1} p_1^2 + \frac{1}{2\mu_2} p_2^2 + \frac{1}{2} \omega_2^2 p_2^2 + \frac{1}{2} \omega_1^2 x_1^2 + g p_1 x_2 + \lambda x_1 x_2 + \theta \lambda p_1 x_1 \quad (1.49)$$

where, the parameters $\mu_1, \mu_2, \omega_1, \omega_2, g, \lambda, \theta$ have specific values as defined:

$$\mu_1 = \frac{m}{1 + m^2 \theta \omega^2}, \quad \mu_2 = \frac{m}{\sigma^2 + m^2 \omega^2}, \quad \omega_2 = \frac{1}{\sqrt{m}}, \quad \omega_1 = \omega \sqrt{m}, \quad g = \frac{\sigma}{m} + \theta m \omega^2 \quad (1.50)$$

The new terms represent mutual effects that express the interaction of measurable physical quantities with each other.

The commutative base A_0 is replaced by the non-commutative base A_θ following the canonical method:

$$\{f, g\}_{\text{nc}} \rightarrow [\hat{f}, \hat{g}] \quad (1.51)$$

In our case $N = 2$, we choose for E.C.O.C. the set of operators $O_{q_1} = q_1, O_{p_2} = p_2$, and consequently, we have:

$$O_{p_1} = \frac{\hbar}{i} \frac{\partial}{\partial q_1} = -\frac{\sigma_i \hbar}{i} \frac{\partial}{\partial p_2}; \quad O_{q_2} = \frac{\hbar}{i} \frac{\partial}{\partial p_2} = \frac{\theta_i \hbar}{i} \frac{\partial}{\partial q_1} \quad (1.52)$$

These operators generate the modified Heisenberg algebra:

$$[q_1, q_2] = i\hbar\theta; \quad [q_i, p_j] = i\hbar\delta_{ij}; \quad [p_1, p_2] = i\hbar\sigma \quad (1.53)$$

To retain the spirit of the Feynman construction, let us introduce the following notations: $X^T = (q_1, p_2) = (X_1, X_2)$, $P^T = (q_2, p_1) = (P_1, P_2)$. The previous commutation relations become:

$$[X_i, P_j] = i\Lambda_{\theta, \sigma}^{-1} \delta_{ij} \quad (1.54)$$

with

$$\Lambda_{\theta, \sigma} = \frac{1}{1 - \theta\sigma} \begin{pmatrix} 1 & \theta \\ \sigma & 1 \end{pmatrix} \quad (1.55)$$

At this level, let us introduce an E.C.O.C[1]. basis on which this realization of the algebra is verified. Let $\{|X_1, X_2\rangle = |X_i\rangle\}$ be a complete set of eigenvectors of the E.C.O.C. The components of P will then be represented by:

$$\begin{aligned} \hat{P}_1 \Psi(X_1, X_2) &= i\hbar \left(\frac{\partial}{\partial X_2} \Psi(X_1, X_2) - \theta \frac{\partial}{\partial X_1} \Psi(X_1, X_2) \right) \\ \hat{P}_2 \Psi(X_1, X_2) &= i\hbar \left(\sigma \frac{\partial}{\partial X_2} \Psi(X_1, X_2) - \frac{\partial}{\partial X_1} \Psi(X_1, X_2) \right) \end{aligned} \quad (1.56)$$

where $|X_1, X_2\rangle$ is a state in Hilbert space.

In the absence of deformation, the expressions for the momentum operators will reduce to:

$$\hat{P}_1 \Psi(X_1, X_2) = i\hbar \left(\frac{\partial}{\partial X_2} \Psi(X_1, X_2) \right), \quad \hat{P}_2 \Psi(X_1, X_2) = i\hbar \left(-\frac{\partial}{\partial X_1} \Psi(X_1, X_2) \right) \quad (1.57)$$

Thus, we have come to understand the structure of non-commutative phase space and its essential properties for accurately describing microscopic phenomena by introducing the concept of deformation quantization. This is achieved by modifying the algebra of functions in classical phase space to account for the inherent non-commutativity at microscopic scales. Using the rigorous framework established by Bain et al. (1978) to transition from classical mechanics to quantum mechanics involves deforming the algebraic structure of phase space. Instead of directly creating operators on Hilbert space, their approach modifies the algebra of functions in classical phase space $F(M)$ by introducing a star product, which includes a deformation parameter \hbar . This star product retains the fundamental properties of the Poisson algebra, such as associativity, bilinearity, and the Jacobi identity, while allowing for non-commutativity. This quantization procedure, known as "deformation quantization," provides a bridge between classical and quantum descriptions and preserves the non-commutative nature of quantum mechanics.

By modifying the Poisson bracket to include non-commutativity, represented by a deformed Poisson tensor, this adjustment is essential for accurately describing physical systems at microscopic scales where two physical quantities cannot be measured without affecting each other. This provides a more general framework that accommodates the non-commutativity of phase space.

Schrödinger equation in the non-commutative case

2.1 Introduction

In 1926, Schrödinger succeeded in formulating a differential equation by generalizing de Broglie's principle, which describes the behavior of a particle through its solutions—the wave function and the energy spectrum[13, 18]. Since then, Schrödinger's wave equation has served as a strong bridge between classical and quantum descriptions of physical systems. While it has been extensively studied within the framework of commutative geometry, recent developments have expanded its applicability to non-commutative phase spaces, unveiling new horizons of scientific discoveries. These focus primarily on revealing more precise corrections to previous quantum formulations. This chapter will attempt to address aspects of this through:

- A brief review of Schrödinger's equation, the properties of the wave function, and an example within the framework of commutative phase space.
- Studying a system defined on a deformed phase space and solving the corresponding Schrödinger equation.

2.2 Review of the Time-Independent Schrödinger Equation

Definition 2.2.1 [18–20]:

The general expression for the time-independent Schrödinger equation for a physical system is given by:

$$\hat{H} |\psi(\mathbf{x})\rangle = E |\psi(\mathbf{x})\rangle \quad (2.1)$$

where:

- \hat{H} is called the Hamiltonian operator and represents the total energy of the system. It must be Hermitian to ensure that all its eigenvalues are real.
- E is the eigenvalue of the Hamiltonian operator, also known as the energy spectrum.
- $\psi(\mathbf{x})$ is the wave function.

Since we are dealing with operators in Hilbert space, it is necessary to specify the general properties of operators in quantum mechanics in general, and the Schrödinger equation in particular.

Properties 2.2.1 Operators in the Schrödinger equation[13, 21–23]:

1. All Eigenvalues of a Hermitian Operator (Self-Adjoint Operator Spectrum) are Real: Let ψ be an eigenfunction of the Hermitian operator \hat{H} , meaning there exists an E such that:

$$\hat{H} |\psi\rangle = E |\psi\rangle \quad (2.2)$$

Given that the operator is Hermitian:

$$\langle \hat{H}\psi_{n_1}, \psi_{n_2} \rangle = \langle \psi_{n_1}, \hat{H}\psi_{n_2} \rangle = E \langle \psi_{n_1}, \psi_{n_2} \rangle \iff \hat{H} = \hat{H}^\dagger \quad (2.3)$$

2. Orthogonality of Eigenfunctions: If $|\psi_1\rangle$ and $|\psi_2\rangle$ are eigenfunctions of the Hermitian operator \hat{H} with different eigenvalues, then the functions are orthogonal:

$$\langle \psi_1 | \psi_2 \rangle = 0 \quad (2.4)$$

If they are eigenfunctions with the same eigenvalue, any linear combination of these functions will also be an eigenfunction of the operator with the same eigenvalue. This is straightforward.

The Hermiticity of operators in quantum mechanics ensures that they have physical significance in reality, with each operator corresponding to a measurable physical quantity. As long as it satisfies the Hermiticity condition, like the Hamiltonian operator in our specific case, its eigenvalues represent the energy spectrum of the system described by it.

2.2.1 Wave Function

The wave function in quantum mechanics is a mathematical function that describes the quantum state of a physical system[18, 20]. It contains all the information needed to determine the physical properties of the system, most notably the probability density of finding a particle, for example. Since the wave function is not a real-valued function but rather a complex one, it cannot have a physical meaning by itself[23, 24]. However, when multiplied by its complex conjugate, it gives us the probability density of the particle's presence.

The wave function also has important conditions that enable us to calculate the eigenvalues of the Hamiltonian operator, which, as we mentioned earlier, represent the energy spectrum of the studied system through strict conditions and properties that characterize it[25, 26].

The wave function is usually denoted by $\psi(\mathbf{r}, t)$, where \mathbf{r} represents the spatial coordinates and t represents time[26]. Some of the most important properties of the wave function are:

1. **Normalization**[18]:

$$\int |\psi(\mathbf{r})|^2 d\mathbf{r} = 1 \quad (2.5)$$

The wave function must be normalized to ensure that the total probability of finding the particle in all space is one. This property is crucial for the probabilistic interpretation of quantum mechanics.

2. **Continuity**[20]:

$$\psi(\mathbf{r}) \text{ and } \frac{\partial\psi(\mathbf{r})}{\partial x_i} \text{ must be continuous.}$$

The wave function and its first derivative must be continuous to avoid infinite probabilities and ensure physical viability.

3. **Boundary Conditions**[27]: The wave function must satisfy appropriate boundary conditions depending on the physical problem. For example, it must go to zero at infinity for a bound state.

4. **Single-Valued**[23]: $\psi(\mathbf{r})$ must be single-valued.:

$$\forall x \in \text{Domain of } \psi, \quad \exists! y : y = \psi(x) \quad (2.6)$$

The wave function must have a unique value at each point in space to ensure the consistency of the physical description.

2.3 Deformed phase space and Schrödinger Equation

There is no general rule that describes how the Schrödinger equation is affected by the deformation of phase space, as quantum deformations inherently preserve the laws of differentiation and integration, as we discussed earlier when talking about Moyal quantization. This means that only the operators will change their forms depending on the altered generalized Poisson brackets, which in turn modify the corresponding Heisenberg algebra relations.

Thus, when defining operators in the deformed phase space, the wave function—which is responsible for finding the eigenvalues of the Hamiltonian operator—will retain its essential properties. Given that we have already applied the concept of deformation to the example of the coupled harmonic oscillator in the previous chapter and written the Hamiltonian in both commutative and non-commutative cases, formulating the corresponding Hamiltonian operator suffices to account for the modifications arising from the non-standard forms of the coordinate and momentum operators. These changes present a challenge when attempting to solve the Schrödinger equation for the coupled harmonic oscillator.

We will now focus on solving the Schrödinger equation for this specific example to study the effect of space deformation on the energy values and the wave function.

2.4 Coupled harmonic oscillator pair in the mixed non-commutative phas space:

Starting from writing the Hamiltonian for the coupled harmonic oscillator pair in the equation In the non-commutative case (1.49):

$$H_{\theta\sigma}(x_1, x_2; p_1, p_2) \approx \frac{1}{2\mu_1}p_1^2 + \frac{1}{2\mu_2}x_2^2 + \frac{1}{2}\omega_2^2p_2^2 + \frac{1}{2}\omega_1^2x_1^2 + gp_1x_2 + \lambda x_1x_2 + \theta\lambda p_1x_1 \quad (2.7)$$

where, the parameters $\mu_1, \mu_2, \omega_1, \omega_2, g, \lambda, \theta$ have specific values as defined:

$$\mu_1 = \frac{m}{1 + m^2\theta\omega^2}, \quad \mu_2 = \frac{m}{\sigma^2 + m^2\omega^2}, \quad \omega_2 = \frac{1}{\sqrt{m}}, \quad \omega_1 = \omega\sqrt{m}, \quad g = \frac{\sigma}{m} + \theta m\omega^2 \quad (2.8)$$

And by substituting $Q = (x_1, p_2), P = (x_2, p_1)$ we can, based on our definition of the

Poisson bracket, reformulate the Hamiltonian in the following matrix form[1, 28]:

$$H_{\theta\sigma}(Q, P) = \frac{1}{2} (P^T M P + Q^T \Omega^2 Q) + P^T G Q \quad (2.9)$$

Where:

$$M = \begin{pmatrix} \frac{1}{\mu_1} & 0 \\ 0 & \frac{1}{\mu_2} \end{pmatrix}, \quad \Omega = \begin{pmatrix} \frac{1}{\sqrt{m}} & 0 \\ 0 & \omega\sqrt{m} \end{pmatrix}, \quad G = \begin{pmatrix} \lambda & 0 \\ \theta\lambda & 0 \end{pmatrix} \quad (2.10)$$

Note that the presence of the interaction term makes the complete matrix formulation of the Hamiltonian non-diagonal. In the upcoming transformations, our goal is to diagonalize the Hamiltonian matrix to handle it conveniently and solve the corresponding Schrödinger equation accurately.

By setting the following transformation:

$$P = T_P \bar{P}, \quad Q = T_Q \bar{Q} \quad (2.11)$$

where

$$T_P = \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \\ -\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \end{pmatrix}, \quad T_Q = \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \\ -\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \end{pmatrix} \quad (2.12)$$

The Hamiltonian in terms of the transformed is given by:

$$H_{\theta\sigma}(\bar{Q}, \bar{P}) = \frac{1}{2} (\bar{P}^T T_P^T M T_P \bar{P} + \bar{Q}^T T_Q^T \Omega^2 T_Q \bar{Q}) + \bar{P}^T T_P^T G T_Q \bar{Q} \quad (2.13)$$

This gives us the general form of the transformed Hamiltonian:

$$H_{\theta\sigma}(\bar{Q}, \bar{P}) = \frac{1}{2} (\bar{P}^T \bar{\mu} \bar{P} + \bar{Q}^T \varpi \bar{Q}) + \bar{P}^T \Lambda \bar{Q} \quad (2.14)$$

Where:

$$\bar{\mu} = T_P^T M T_P, \quad \varpi = T_Q^T \Omega^2 T_Q, \quad \Lambda = T_P^T G T_Q \quad (2.15)$$

We chose this transformation because it allows us to diagonalize the momentum matrix, which is the desired goal of this transformation. We will now compute the resulting matrices for the kinetic part $\bar{\mu}$, the potential part ϖ , and the interaction part Λ in the following steps.

Step 01:for $\bar{\mu} = T_P^T M T_P$

$$\bar{\mu} = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & -\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \\ \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \end{pmatrix} \right) \begin{pmatrix} \frac{1}{\mu_1} & g \\ g & \frac{1}{\mu_2} \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \\ -\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \end{pmatrix} \right) \quad (2.16)$$

Let's compute the product $T_P^T M$:

$$T_P^T M = \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & -\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \\ \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} \frac{1}{\mu_1} & g \\ g & \frac{1}{\mu_2} \end{pmatrix} \quad (2.17)$$

$$T_P^T M = \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \cdot \frac{1}{\mu_1} - \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \cdot g\right) & \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \cdot g - \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \cdot \frac{1}{\mu_2}\right) \\ \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \cdot g + \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \cdot \frac{1}{\mu_1}\right) & \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \cdot \frac{1}{\mu_2} + \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \cdot g\right) \end{pmatrix} \quad (2.18)$$

Now we need to multiply this intermediate product by T_P :

$$\bar{\mu} = \frac{1}{2} \begin{pmatrix} \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \cdot \frac{1}{\mu_1} - \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \cdot g\right) & \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \cdot g - \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \cdot \frac{1}{\mu_2}\right) \\ \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \cdot g + \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \cdot \frac{1}{\mu_1}\right) & \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \cdot \frac{1}{\mu_2} + \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \cdot g\right) \end{pmatrix} \begin{pmatrix} \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \\ -\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \end{pmatrix} \quad (2.19)$$

Now compute each element of the final matrix $\bar{\mu}$:

$$\bar{\mu}_{11} = \frac{1}{2} \left(\left(\left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} \cdot \frac{1}{\mu_1} - \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \cdot g \right) \cdot \left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} + \left(\left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} \cdot g - \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \cdot \frac{1}{\mu_2} \right) \cdot - \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \right)$$

$$\bar{\mu}_{12} = \frac{1}{2} \left(\left(\left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} \cdot \frac{1}{\mu_1} - \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \cdot g \right) \cdot \left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} + \left(\left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} \cdot g - \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \cdot \frac{1}{\mu_2} \right) \cdot \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \right)$$

$$\bar{\mu}_{21} = \frac{1}{2} \left(\left(\left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} \cdot g + \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \cdot \frac{1}{\mu_1} \right) \cdot \left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} + \left(\left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} \cdot \frac{1}{\mu_2} + \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \cdot g \right) \cdot - \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \right)$$

$$\bar{\mu}_{22} = \frac{1}{2} \left(\left(\left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} \cdot g + \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \cdot \frac{1}{\mu_1} \right) \cdot \left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} + \left(\left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} \cdot \frac{1}{\mu_2} + \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \cdot g \right) \cdot \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \right)$$

We have:

$$\begin{aligned} \left(\left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} \cdot \frac{1}{\mu_1} \pm \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \cdot g \right) \cdot \left(\frac{\mu_2}{\mu_1} \right)^{\frac{1}{4}} &= \frac{\sqrt{\mu_2}}{\sqrt{\mu_1 \mu_1}} \pm g \\ \frac{\sqrt{\mu_2}}{\sqrt{\mu_1 \mu_1}} + \frac{\sqrt{\mu_1}}{\sqrt{\mu_2 \mu_2}} &= \frac{\mu_2}{\sqrt{\mu_2 \mu_1 \mu_1}} + \frac{\mu_1}{\sqrt{\mu_1 \mu_2 \mu_2}} = \frac{2}{\sqrt{\mu_2 \mu_1}} \end{aligned} \quad (2.20)$$

so:

$$\bar{\mu}_{11} = \frac{1}{\sqrt{\mu_2 \mu_1}} - g, \quad \bar{\mu}_{12} = \bar{\mu}_{21} = 0, \quad \bar{\mu}_{22} = \frac{1}{\sqrt{\mu_2 \mu_1}} + g \quad (2.21)$$

we can write:

$$\bar{\mu} = T_P^T M T_P = \begin{pmatrix} \frac{1}{\sqrt{\mu_2 \mu_1}} - g & 0 \\ 0 & \frac{1}{\sqrt{\mu_2 \mu_1}} + g \end{pmatrix} \quad (2.22)$$

Step 02: for $\varpi = T_Q^T \Omega^2 T_Q$.

Now, let's calculate $T_Q^T \Omega^2$:

$$T_Q^T \Omega^2 = \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & -\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \\ \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} \frac{1}{m} & 0 \\ 0 & \omega^2 m \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{m} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & -\omega^2 m \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \\ \frac{1}{m} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \omega^2 m \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \end{pmatrix} \quad (2.23)$$

Finally, multiply this intermediate product by T_Q :

$$\varpi = T_Q^T \Omega^2 T_Q = \frac{1}{2} \begin{pmatrix} \frac{1}{m} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & -\omega^2 m \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \\ \frac{1}{m} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \omega^2 m \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \\ -\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \end{pmatrix} \quad (2.24)$$

Now compute each element of the final matrix ϖ :

$$\varpi = \frac{1}{2} \begin{pmatrix} \left(\frac{1}{m} \sqrt{\frac{\mu_1}{\mu_2}} + \omega^2 m \sqrt{\frac{\mu_2}{\mu_1}}\right) & \left(\frac{1}{m} \sqrt{\frac{\mu_1}{\mu_2}} + \omega^2 m \sqrt{\frac{\mu_2}{\mu_1}}\right) \\ \left(\frac{1}{m} \sqrt{\frac{\mu_1}{\mu_2}} + \omega^2 m \sqrt{\frac{\mu_2}{\mu_1}}\right) & \left(\frac{1}{m} \sqrt{\frac{\mu_1}{\mu_2}} + \omega^2 m \sqrt{\frac{\mu_2}{\mu_1}}\right) \end{pmatrix} \quad (2.25)$$

Extracting: $\frac{1}{m} \sqrt{\frac{\mu_1}{\mu_2}}$ as a common factor we find:

$$\varpi = \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \begin{pmatrix} 1 + \omega^2 \frac{\mu_2}{\mu_1} & 1 - \omega^2 \frac{\mu_2}{\mu_1} \\ 1 - \omega^2 \frac{\mu_2}{\mu_1} & 1 + \omega^2 \frac{\mu_2}{\mu_1} \end{pmatrix} \quad (2.26)$$

Step 03: for $\Lambda = T_P^T G T_Q$

First, calculate $T_P^T G$:

$$T_P^T G = \frac{1}{\sqrt{2}} \begin{pmatrix} \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & -\left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \\ \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \end{pmatrix} \begin{pmatrix} \lambda & 0 \\ \theta \lambda & 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \lambda \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} - \theta \lambda \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & 0 \\ \lambda \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} + \theta \lambda \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & 0 \end{pmatrix} \quad (2.27)$$

Next, multiply this intermediate product by Λ :

$$\Lambda = T_P^T G T_Q = \frac{1}{2} \begin{pmatrix} \lambda \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} - \theta \lambda \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & 0 \\ \lambda \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} + \theta \lambda \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & 0 \end{pmatrix} \begin{pmatrix} \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \\ -\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} & \left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} \end{pmatrix} \quad (2.28)$$

Now compute each element of the final matrix Λ :

$$\Lambda = \frac{\lambda}{2} \begin{pmatrix} \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} - \theta \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}}\right) \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} - \theta \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}}\right) \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \\ \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} + \theta \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}}\right) \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} & \left(\left(\frac{\mu_2}{\mu_1}\right)^{\frac{1}{4}} + \theta \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}}\right) \left(\frac{\mu_1}{\mu_2}\right)^{\frac{1}{4}} \end{pmatrix} = \frac{\lambda}{2} \begin{pmatrix} \sqrt{\frac{\mu_1}{\mu_2}} - \theta & \sqrt{\frac{\mu_1}{\mu_2}} - \theta \\ \theta + \sqrt{\frac{\mu_1}{\mu_2}} & \theta + \sqrt{\frac{\mu_1}{\mu_2}} \end{pmatrix} \quad (2.29)$$

After writing the final expressions for the matrices $\bar{\mu}$, ϖ , and Λ , we can write the final expression for the transformed Hamiltonian: :

$$H_{\theta\sigma}(\bar{Q}, \bar{P}) = \frac{1}{2} \left(\bar{P}^T \bar{\mu} \bar{P} + \bar{Q}^T \varpi \bar{Q} \right) + \bar{P}^T \Lambda \bar{Q} \quad (2.30)$$

Where the matrices $\bar{\mu}$, ϖ , and Λ are defined as:

$$\begin{aligned} \bar{\mu} &= T_P^T M T_P = \begin{pmatrix} \frac{1}{\sqrt{\mu_2 \mu_1}} - g & 0 \\ 0 & \frac{1}{\sqrt{\mu_2 \mu_1}} + g \end{pmatrix}, & \Lambda &= T_P^T G T_Q = \frac{\lambda}{2} \begin{pmatrix} \sqrt{\frac{\mu_1}{\mu_2}} - \theta & \sqrt{\frac{\mu_1}{\mu_2}} - \theta \\ \theta + \sqrt{\frac{\mu_1}{\mu_2}} & \theta + \sqrt{\frac{\mu_1}{\mu_2}} \end{pmatrix}, \\ \varpi &= T_Q^T \Omega^2 T_Q = \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \begin{pmatrix} 1 + \omega^2 \frac{\mu_2}{\mu_1} & 1 - \omega^2 \frac{\mu_2}{\mu_1} \\ 1 - \omega^2 \frac{\mu_2}{\mu_1} & 1 + \omega^2 \frac{\mu_2}{\mu_1} \end{pmatrix} \end{aligned} \quad (2.31)$$

We now notice that the momentum matrix is ready to separate. But unlike the interaction and position matrices. For this reason, we define the following transformation:

$$\bar{P} = \frac{1}{\sqrt{\bar{\mu}}} \Pi, \quad \bar{Q} = \sqrt{\bar{\mu}} X \quad (2.32)$$

The general form of the Hamiltonian in the new coordinates X and Π becomes:

$$H_{\theta\sigma}(X, \Pi) = \frac{1}{2} \left(\Pi^T \left(\frac{1}{\sqrt{\bar{\mu}}} \right)^T \bar{\mu} \left(\frac{1}{\sqrt{\bar{\mu}}} \right) \Pi + X^T \sqrt{\bar{\mu}}^T \varpi \sqrt{\bar{\mu}} X \right) + \Pi^T \left(\frac{1}{\sqrt{\bar{\mu}}} \right)^T \Lambda \sqrt{\bar{\mu}} X \quad (2.33)$$

This gives us the general form of the transformed Hamiltonian:

$$H_{\theta\sigma} = \frac{1}{2} (\Pi^T \Pi + X^T \bar{\varpi} X + 2 \Pi^T \bar{\Lambda} X) \quad (2.34)$$

Where:

$$\left(\frac{1}{\sqrt{\bar{\mu}}} \right)^T \bar{\mu} \left(\frac{1}{\sqrt{\bar{\mu}}} \right) = \mathbb{1}, \quad \sqrt{\bar{\mu}}^T \varpi \sqrt{\bar{\mu}} = \bar{\varpi}, \quad \left(\frac{1}{\sqrt{\bar{\mu}}} \right)^T \Lambda \sqrt{\bar{\mu}} = \bar{\Lambda} \quad (2.35)$$

We chose this canonical transformation specifically in order to ensure that there are no subsequent transformations on the momentum matrix, so that it is possible to break the coupling between momentum and position while preserving the diagonal of both, or at least one of them, as we will see in the following steps.

step 01: The transformed matrix $\bar{\mu}$:

by sitting a and b and κ :

$$a = \frac{1}{\sqrt{\mu_2\mu_1}} - g, \quad b = \frac{1}{\sqrt{\mu_2\mu_1}} + g, \quad \text{and } \kappa = \frac{\sqrt{a}}{\sqrt{b}} \quad (2.36)$$

$$\bar{\mu} = \begin{pmatrix} \frac{1}{\sqrt{\mu_2\mu_1}} - g & 0 \\ 0 & \frac{1}{\sqrt{\mu_2\mu_1}} + g \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \implies \begin{pmatrix} 1 \\ \sqrt{\bar{\mu}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix} \text{ and } \sqrt{\bar{\mu}} = \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix} \quad (2.37)$$

Using thus definitions (2.37), we find:

$$\bar{\mu}' = \begin{pmatrix} 1 \\ \sqrt{\bar{\mu}} \end{pmatrix}^T \bar{\mu} \begin{pmatrix} 1 \\ \sqrt{\bar{\mu}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix} = \begin{pmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathbb{1} \quad (2.38)$$

step 02: The transformed matrices ϖ and Λ .

for $\bar{\varpi}$

$$\bar{\varpi} = \sqrt{\bar{\mu}}^T \varpi \sqrt{\bar{\mu}} = \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix} \begin{pmatrix} (1 + \omega^2 \frac{\mu_2}{\mu_1}) & (1 - \omega^2 \frac{\mu_2}{\mu_1}) \\ (1 - \omega^2 \frac{\mu_2}{\mu_1}) & (1 + \omega^2 \frac{\mu_2}{\mu_1}) \end{pmatrix} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix} \quad (2.39)$$

$$\bar{\varpi} = \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \begin{pmatrix} a(1 + \omega^2 \frac{\mu_2}{\mu_1}) & \sqrt{ab}(1 - \omega^2 \frac{\mu_2}{\mu_1}) \\ \sqrt{ab}(1 - \omega^2 \frac{\mu_2}{\mu_1}) & b(1 + \omega^2 \frac{\mu_2}{\mu_1}) \end{pmatrix} \quad (2.40)$$

$$\bar{\varpi} = \frac{\sqrt{ab}}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \begin{pmatrix} \frac{\sqrt{a}}{\sqrt{b}}(1 + \omega^2 \frac{\mu_2}{\mu_1}) & (1 - \omega^2 \frac{\mu_2}{\mu_1}) \\ (1 - \omega^2 \frac{\mu_2}{\mu_1}) & \frac{\sqrt{b}}{\sqrt{a}}(1 + \omega^2 \frac{\mu_2}{\mu_1}) \end{pmatrix} \quad (2.41)$$

for $\bar{\Lambda}$:

$$\bar{\Lambda} = \begin{pmatrix} 1 \\ \sqrt{\bar{\mu}} \end{pmatrix}^T \Lambda \sqrt{\bar{\mu}} = \begin{pmatrix} \frac{1}{\sqrt{a}} & 0 \\ 0 & \frac{1}{\sqrt{b}} \end{pmatrix} \frac{\lambda}{2} \begin{pmatrix} \sqrt{\frac{\mu_1}{\mu_2}} - \theta & \sqrt{\frac{\mu_1}{\mu_2}} - \theta \\ \theta + \sqrt{\frac{\mu_1}{\mu_2}} & \theta + \sqrt{\frac{\mu_1}{\mu_2}} \end{pmatrix} \begin{pmatrix} \sqrt{a} & 0 \\ 0 & \sqrt{b} \end{pmatrix} \quad (2.42)$$

$$\bar{\Lambda} = \frac{\lambda}{2} \begin{pmatrix} \sqrt{\frac{\mu_1}{\mu_2}} - \theta & \frac{\sqrt{b}}{\sqrt{a}} \left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right) \\ \frac{\sqrt{a}}{\sqrt{b}} \left(\theta + \sqrt{\frac{\mu_1}{\mu_2}} \right) & \theta + \sqrt{\frac{\mu_1}{\mu_2}} \end{pmatrix} \quad (2.43)$$

Using our definition of the values a, b and κ from (2.36), we find:

$$\bar{\varpi} = \left(\sqrt{\frac{1}{\mu_1\mu_2} - g^2} \right) \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \begin{pmatrix} \kappa(1 + \omega^2 \frac{\mu_2}{\mu_1}) & (1 - \omega^2 \frac{\mu_2}{\mu_1}) \\ (1 - \omega^2 \frac{\mu_2}{\mu_1}) & \kappa^{-1}(1 + \omega^2 \frac{\mu_2}{\mu_1}) \end{pmatrix} \quad (2.44)$$

$$\bar{\Lambda} = \frac{\lambda}{2} \begin{pmatrix} \sqrt{\frac{\mu_1}{\mu_2}} - \theta & \kappa^{-1} \left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right) \\ \kappa \left(\theta + \sqrt{\frac{\mu_1}{\mu_2}} \right) & \theta + \sqrt{\frac{\mu_1}{\mu_2}} \end{pmatrix} \quad (2.45)$$

Thus, we can write the Hamiltonian expression after the second transformation with the following expression:

$$H_{\theta\sigma} = \frac{1}{2} \left(\Pi^T \Pi + X^T \bar{\omega} X + 2\Pi^T \bar{\Lambda} X \right) \quad (2.46)$$

Where:

$$\begin{aligned} \bar{\omega} &= \left(\sqrt{\frac{1}{\mu_1 \mu_2} - g^2} \right) \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \begin{pmatrix} \kappa(1 + \omega^2 \frac{\mu_2}{\mu_1}) & (1 - \omega^2 \frac{\mu_2}{\mu_1}) \\ (1 - \omega^2 \frac{\mu_2}{\mu_1}) & \kappa^{-1}(1 + \omega^2 \frac{\mu_2}{\mu_1}) \end{pmatrix} \\ \bar{\Lambda} &= \frac{\lambda}{2} \begin{pmatrix} \sqrt{\frac{\mu_1}{\mu_2}} - \theta & \kappa^{-1} \left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right) \\ \kappa \left(\theta + \sqrt{\frac{\mu_1}{\mu_2}} \right) & \theta + \sqrt{\frac{\mu_1}{\mu_2}} \end{pmatrix}, \quad \text{With: } \kappa = \frac{\sqrt{\frac{1}{\sqrt{\mu_1 \mu_2}} - g}}{\sqrt{\frac{1}{\sqrt{\mu_1 \mu_2}} + g}} \end{aligned} \quad (2.47)$$

After reducing the kinetic part within the new coordinate system by second transformation, we can now search for a transformation that decouples the interaction part and integrates it into either the kinetic or potential part.

Let the following transformation be: $\Pi \rightarrow \Pi' = \Pi - \bar{\Lambda} X$:

Substitute Π' into the Hamiltonian form (2.46), we can write:

$$H_{\theta\sigma} = \frac{1}{2} \left(\Pi'^T \Pi' + X^T \bar{\omega} X + 2\Pi'^T \bar{\Lambda} X \right) \quad (2.48)$$

First, compute $\Pi'^T \Pi'$:

$$\begin{aligned} \Pi'^T \Pi' &= (\Pi - \bar{\Lambda} X)^T (\Pi - \bar{\Lambda} X) = (\Pi^T - X^T \bar{\Lambda}^T) (\Pi - \bar{\Lambda} X) \\ &= \Pi^T \Pi - \Pi^T \bar{\Lambda} X - X^T \bar{\Lambda}^T \Pi + X^T \bar{\Lambda}^T \bar{\Lambda} X \end{aligned} \quad (2.49)$$

Next, compute the term $2\Pi'^T \bar{\Lambda} X$:

$$\begin{aligned} 2\Pi'^T \bar{\Lambda} X &= 2(\Pi - \bar{\Lambda} X)^T \bar{\Lambda} X = 2(\Pi^T - X^T \bar{\Lambda}^T) \bar{\Lambda} X \\ &= 2\Pi^T \bar{\Lambda} X - 2X^T \bar{\Lambda}^T \bar{\Lambda} X \end{aligned} \quad (2.50)$$

Now, let's substitute these results back into the Hamiltonian (2.48):

$$\begin{aligned}
 H_{\theta\sigma} &= \frac{1}{2} \left(\Pi^T \Pi - \Pi^T \bar{\Lambda} X - X^T \bar{\Lambda}^T \Pi + X^T \bar{\Lambda}^T \bar{\Lambda} X + X^T \bar{\omega} X + 2\Pi^T \bar{\Lambda} X - 2X^T \bar{\Lambda}^T \bar{\Lambda} X \right) \\
 &= \frac{1}{2} \left(\Pi^T \Pi + X^T \bar{\omega} X - \Pi^T \bar{\Lambda} X - X^T \bar{\Lambda}^T \Pi + X^T \bar{\Lambda}^T \bar{\Lambda} X + 2\Pi^T \bar{\Lambda} X - 2X^T \bar{\Lambda}^T \bar{\Lambda} X \right) \\
 &= \frac{1}{2} \left(\Pi^T \Pi + X^T \bar{\omega} X + \Pi^T \bar{\Lambda} X - X^T \bar{\Lambda}^T \Pi + X^T \bar{\Lambda}^T \bar{\Lambda} X - 2X^T \bar{\Lambda}^T \bar{\Lambda} X \right) \\
 &= \frac{1}{2} \left(\Pi^T \Pi + X^T \bar{\omega} X + \Pi^T \bar{\Lambda} X - X^T \bar{\Lambda}^T \Pi - X^T \bar{\Lambda}^T \bar{\Lambda} X \right)
 \end{aligned} \tag{2.51}$$

Notice that $\Pi^T \bar{\Lambda} X$ and $-X^T \bar{\Lambda}^T \Pi$ are transpose terms of each other and they cancel out when combined with their negative counterparts:

$$H_{\theta\sigma} = \frac{1}{2} \left(\Pi^T \Pi + X^T (\bar{\omega} - \bar{\Lambda}^T \bar{\Lambda}) X \right) \tag{2.52}$$

Notice now that the interaction term has disappeared and has been incorporated into the position matrix. For simplicity and accuracy, we will compute the difference $\bar{\omega} - \bar{\Lambda}^T \bar{\Lambda}$ and write it as a single matrix, which we will call \mathcal{A} , where $\mathcal{A} = \bar{\omega} - \bar{\Lambda}^T \bar{\Lambda}$

The transformed matrix $\bar{\omega}$ is:

$$\bar{\omega} = \left(\sqrt{\frac{1}{\mu_1 \mu_2} - g^2} \right) \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \begin{pmatrix} \kappa(1 + \omega^2 \frac{\mu_2}{\mu_1}) & 1 - \omega^2 \frac{\mu_2}{\mu_1} \\ 1 - \omega^2 \frac{\mu_2}{\mu_1} & \kappa^{-1}(1 + \omega^2 \frac{\mu_2}{\mu_1}) \end{pmatrix}$$

The matrix $\bar{\Lambda}^T \bar{\Lambda}$ is:

$$\bar{\Lambda}^T \bar{\Lambda} = \frac{\lambda^2}{4} \begin{pmatrix} \left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right)^2 + \kappa^2 \left(\sqrt{\frac{\mu_1}{\mu_2}} + \theta \right)^2 & \kappa^{-1} \left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right)^2 + \kappa \left(\sqrt{\frac{\mu_1}{\mu_2}} + \theta \right)^2 \\ \kappa^{-1} \left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right)^2 + \kappa \left(\sqrt{\frac{\mu_1}{\mu_2}} + \theta \right)^2 & \kappa^{-2} \left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right)^2 + \left(\sqrt{\frac{\mu_1}{\mu_2}} + \theta \right)^2 \end{pmatrix}$$

Now, we compute the elements of \mathcal{A} :

1. The element (1,1):

$$\mathcal{A}_{11} = \left(\sqrt{\frac{1}{\mu_1 \mu_2} - g^2} \right) \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \kappa \left(1 + \omega^2 \frac{\mu_2}{\mu_1} \right) - \frac{\lambda^2}{4} \left[\left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right)^2 + \kappa^2 \left(\sqrt{\frac{\mu_1}{\mu_2}} + \theta \right)^2 \right]$$

2. The elements (1,2) and (2,1):

$$\mathcal{A}_{12} = \mathcal{A}_{21} = \left(\sqrt{\frac{1}{\mu_1 \mu_2} - g^2} \right) \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \left(1 - \omega^2 \frac{\mu_2}{\mu_1} \right) - \frac{\lambda^2}{4} \left[\kappa^{-1} \left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right)^2 + \kappa \left(\sqrt{\frac{\mu_1}{\mu_2}} + \theta \right)^2 \right]$$

3. The element (2,2):

$$\mathcal{A}_{22} = \left(\sqrt{\frac{1}{\mu_1 \mu_2} - g^2} \right) \frac{1}{2m} \sqrt{\frac{\mu_1}{\mu_2}} \kappa^{-1} \left(1 + \omega^2 \frac{\mu_2}{\mu_1} \right) - \frac{\lambda^2}{4} \left[\kappa^{-2} \left(\sqrt{\frac{\mu_1}{\mu_2}} - \theta \right)^2 + \left(\sqrt{\frac{\mu_1}{\mu_2}} + \theta \right)^2 \right]$$

So the matrix \mathcal{A} is:

$$\mathcal{A} = \varpi - \bar{\Lambda}^T \bar{\Lambda} = \begin{pmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ \mathcal{A}_{12} & \mathcal{A}_{22} \end{pmatrix}$$

The final Hamiltonian (2.52) in terms of the matrix \mathcal{A} is:

$$H_{\theta\sigma} = \frac{1}{2} \left(\Pi_1^2 + \Pi_2^2 + A_{11}X_1^2 + A_{22}X_2^2 + 2A_{12}X_1X_2 \right) \quad (2.53)$$

where the matrix \mathcal{A} has elements as defined above. This simplifies the Hamiltonian and allows for easier analysis and solving of the Schrödinger equation in the transformed coordinates.

Since all the transformations we applied are orthogonal transformations This ensures that it is legal and normalized at the same time, which means that the momentum operator statement does not change its form therefore We can write:

$$\hat{\Pi}_2 = i\hbar \left(\sigma \frac{\partial}{\partial X_2} - \frac{\partial}{\partial X_1} \right), \quad \hat{\Pi}_1 = i\hbar \left(\frac{\partial}{\partial X_2} - \theta \frac{\partial}{\partial X_1} \right)$$

From the expression and by substituting the expressions for the momentum operators into the equation , we find:

$$\hat{H}_{\theta\sigma} = \frac{1}{2} \left(-\hbar^2 \left((1 + \sigma^2) \frac{\partial^2}{\partial X_2^2} + (\theta^2 + 1) \frac{\partial^2}{\partial X_1^2} - 2(\theta + \sigma) \frac{\partial^2}{\partial X_1 \partial X_2} \right) + A_{11}X_1^2 + 2A_{12}X_1X_2 + A_{22}X_2^2 \right) \quad (2.54)$$

We convert the linear writing to matrix form by setting $-\hbar^2 = 1$:

$$\hat{H}_{\theta\sigma} = \frac{1}{2} \begin{pmatrix} X_1 & X_2 & \frac{\partial}{\partial X_1} & \frac{\partial}{\partial X_2} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 \\ 0 & 0 & (1 + \sigma^2) & -(\theta + \sigma) \\ 0 & 0 & -(\theta + \sigma) & (\theta^2 + 1) \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ \frac{\partial}{\partial X_1} \\ \frac{\partial}{\partial X_2} \end{pmatrix} \quad (2.55)$$

Note that:

Kinetic part B

$$\mathbf{B} = \begin{pmatrix} (1 + \sigma^2) & -(\theta + \sigma) \\ -(\theta + \sigma) & (\theta^2 + 1) \end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} \\ B_{12} & B_{22} \end{pmatrix} \quad (2.56)$$

Potonsial part A

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{pmatrix} \quad (2.57)$$

Note that:

$$\begin{aligned}
 [\mathbf{A}, \mathbf{B}] &= \begin{pmatrix} A_{11}B_{11} + A_{12}B_{12} - (B_{11}A_{11} + B_{12}A_{12}) & A_{11}B_{12} + A_{12}B_{22} - (B_{11}A_{12} + B_{12}A_{22}) \\ A_{12}B_{11} + A_{22}B_{12} - (B_{12}A_{11} + B_{22}A_{12}) & A_{12}B_{12} + A_{22}B_{22} - (B_{12}A_{12} + B_{22}A_{22}) \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
 \end{aligned} \tag{2.58}$$

This means they can have common eigenvectors[29, 30], which implies that there is a common linear transformation that allows us to preserve the structure of derivation and distillation at the same time.

Find Common Eigenvectors and Eigenvalues Solve the characteristic equation for \mathbf{A} and \mathbf{B} :

$$\det(\mathbf{A} - \gamma_A \mathbf{I}) = 0 \Rightarrow \begin{vmatrix} A_{11} - \gamma_A & A_{12} \\ A_{12} & A_{22} - \gamma_A \end{vmatrix} = (A_{11} - \gamma_A)(A_{22} - \gamma_A) - A_{12}^2 = 0. \tag{2.59}$$

$$\det(\mathbf{B} - \gamma_B \mathbf{I}) = 0 \Rightarrow \begin{vmatrix} (1 + \sigma^2) - \gamma_B & -(\theta + \sigma) \\ -(\theta + \sigma) & (\theta^2 + 1) - \gamma_B \end{vmatrix} = 0 \tag{2.60}$$

The eigenvalues are:

$$\gamma_{A_{1,2}} = \frac{(A_{11} + A_{22}) \pm \sqrt{(A_{11} + A_{22})^2 - 4(A_{11}A_{22} - A_{12}^2)}}{2} \tag{2.61}$$

$$\gamma_{B_{1,2}} = \frac{((1 + \sigma^2)) + (\theta^2 + 1) \pm \sqrt{((1 + \sigma^2) + (\theta^2 + 1))^2 - 4((1 + \sigma^2)(\theta^2 + 1) - (-\theta + \sigma)^2)}}{2} \tag{2.62}$$

Eigenvectors Calculation For each eigenvalue γ_A , solve $(\mathbf{A} - \gamma_A \mathbf{I})\mathbf{v} = 0$.

For Eigenvalue γ_{A_1} :

$$\mathbf{A} - \gamma_{A_1} \mathbf{I} = \begin{pmatrix} A_{11} - \gamma_{A_1} & A_{12} \\ A_{12} & A_{22} - \gamma_{A_1} \end{pmatrix}. \tag{2.63}$$

Solving $(\mathbf{A} - \gamma_{A_1} \mathbf{I})\mathbf{v}_1 = 0$:

$$(A_{11} - \gamma_{A_1})v_{1x} + A_{12}v_{1y} = 0 \Rightarrow v_{1x} = \frac{A_{12}}{\gamma_{A_1} - A_{11}}v_{1y}. \tag{2.64}$$

Normalize the eigenvector \mathbf{v}_1 :

$$\mathbf{v}_1 = v_1 = \frac{1}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_1} - A_{11}}\right)^2}} \begin{pmatrix} \frac{A_{12}}{\gamma_{A_1} - A_{11}} \\ 1 \end{pmatrix}. \quad (2.65)$$

For Eigenvalue γ_{A_2} :

$$\mathbf{A} - \gamma_{A_2} \mathbf{I} = \begin{pmatrix} A_{11} - \gamma_{A_2} & A_{12} \\ A_{12} & A_{22} - \gamma_{A_2} \end{pmatrix}. \quad (2.66)$$

Solving $(\mathbf{A} - \gamma_{A_2} \mathbf{I})\mathbf{v}_2 = 0$:

$$(A_{11} - \gamma_{A_2})v_{2x} + A_{12}v_{2y} = 0 \Rightarrow v_{2x} = \frac{A_{12}}{\gamma_{A_2} - A_{11}}v_{2y}. \quad (2.67)$$

Normalize the eigenvector \mathbf{v}_2 :

$$\mathbf{v}_2 = \frac{1}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_2} - A_{11}}\right)^2}} \begin{pmatrix} \frac{A_{12}}{\gamma_{A_2} - A_{11}} \\ 1 \end{pmatrix}. \quad (2.68)$$

$$\mathbf{P} = \begin{pmatrix} v_{1x} & v_{2x} \\ v_{1y} & v_{2y} \end{pmatrix} = \begin{pmatrix} \frac{\frac{A_{12}}{\gamma_{A_1} - A_{11}}}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_1} - A_{11}}\right)^2}} & \frac{\frac{A_{12}}{\gamma_{A_2} - A_{11}}}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_2} - A_{11}}\right)^2}} \\ \frac{1}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_1} - A_{11}}\right)^2}} & \frac{1}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_2} - A_{11}}\right)^2}} \end{pmatrix} \quad (2.69)$$

Transform the Hamiltonian:

$$\hat{H}_{\theta\sigma} = \frac{1}{2} \begin{pmatrix} y_1 & y_2 & \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} \end{pmatrix} \begin{pmatrix} \gamma_{A_1} & 0 & 0 & 0 \\ 0 & \gamma_{A_2} & 0 & 0 \\ 0 & 0 & -\hbar^2 \gamma_{B_1} & 0 \\ 0 & 0 & 0 & -\hbar^2 \gamma_{B_2} \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{pmatrix}, \quad (2.70)$$

Solving the Schrödinger Equation with the Transformed Hamiltonian

The Hamiltonian in Operator Form:

$$\hat{H}_{\theta\sigma} = \frac{1}{2} \left(\gamma_{A_1} y_1^2 + \gamma_{A_2} y_2^2 - \hbar^2 \gamma_{B_1} \frac{\partial^2}{\partial y_1^2} - \hbar^2 \gamma_{B_2} \frac{\partial^2}{\partial y_2^2} \right) \quad (2.71)$$

Schrödinger Equation The time-independent Schrödinger equation is:

$$\hat{H}_{\theta\sigma} \Psi(y_1, y_2) = E \Psi(y_1, y_2) \quad (2.72)$$

Substituting the Hamiltonian H :

Separate Variables Assume a solution of the form:

$$\Psi(y_1, y_2) = \psi_1(y_1)\psi_2(y_2) \quad (2.73)$$

Separate the equations for ψ_1 and ψ_2 :

$$\frac{1}{2} \left(\gamma_{A_1} y_1^2 - \hbar^2 \gamma_{B_1} \frac{d^2}{dy_1^2} \right) \psi_1(y_1) = E_1 \psi_1(y_1) \quad (2.74)$$

$$\frac{1}{2} \left(\gamma_{A_2} y_2^2 - \hbar^2 \gamma_{B_2} \frac{d^2}{dy_2^2} \right) \psi_2(y_2) = E_2 \psi_2(y_2) \quad (2.75)$$

Each equation resembles the form of a harmonic oscillator. The solutions are known:

Wave functions

$$\psi_{n_1}(y_1) = \left(\frac{\bar{\Omega}_1}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} H_{n_1} \left(\sqrt{\frac{\bar{\Omega}_1}{\hbar}} y_1 \right) e^{-\frac{\bar{\Omega}_1 y_1^2}{2\hbar}} \quad (2.76)$$

$$\psi_{n_2}(y_2) = \left(\frac{\bar{\Omega}_2}{\pi \hbar} \right)^{1/4} \frac{1}{\sqrt{2^{n_2} n_2!}} H_{n_2} \left(\sqrt{\frac{\bar{\Omega}_2}{\hbar}} y_2 \right) e^{-\frac{\bar{\Omega}_2 y_2^2}{2\hbar}} \quad (2.77)$$

Energies eigenvalues

$$E_1 = \hbar \bar{\Omega}_1 \left(n_1 + \frac{1}{2} \right), \quad n_1 = 0, 1, 2, \dots \quad (2.78)$$

$$E_2 = \hbar \bar{\Omega}_2 \left(n_2 + \frac{1}{2} \right), \quad n_2 = 0, 1, 2, \dots \quad (2.79)$$

From the transformations , we have:

$$\alpha_1 = \frac{1}{\sqrt{2}} \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \sqrt{\frac{1}{\sqrt{\mu_2 \mu_1}} - g}, \quad \alpha_2 = \frac{1}{\sqrt{2}} \left(\frac{\mu_1}{\mu_2} \right)^{\frac{1}{4}} \sqrt{\frac{1}{\sqrt{\mu_2 \mu_1}} + g} \quad (2.80)$$

$$X_1 = \frac{1}{2\alpha_1} (Q_1 - Q_2), \quad X_2 = \frac{1}{2\alpha_2} (Q_1 + Q_2) \quad (2.81)$$

$$y_1 = v_{1x} \frac{1}{2\alpha} (Q_1 - Q_2) + v_{2x} \frac{1}{2\alpha} (Q_1 + Q_2) \quad (2.82)$$

$$y_2 = v_{1y} \frac{1}{2\alpha_1} (Q_1 - Q_2) + v_{2y} \frac{1}{2\alpha_2} (Q_1 + Q_2) \quad (2.83)$$

So, we can write the solution to the Schrödinger equation for the studied system in terms of the original coordinates as follows:

$$\begin{aligned} \Psi_{n_1, n_2}(Q_1, Q_2) = & \sqrt{\frac{\sqrt{\bar{\eta}_1 \bar{\Omega}_1 \bar{\eta}_2 \bar{\Omega}_2}}{\pi \hbar 2^{n_1+n_2} n_1! n_2!}} \exp\left\{-\frac{\bar{\eta}_1 \bar{\Omega}_1}{2\hbar} \left[v_{1x} \frac{1}{2\alpha_1} (Q_1 - Q_2) + v_{2x} \frac{1}{2\alpha_2} (Q_1 + Q_2)\right]^2\right\} \\ & \exp\left\{-\frac{\bar{\eta}_2 \bar{\Omega}_2}{2\hbar} \left[v_{1y} \frac{1}{2\alpha_1} (Q_1 - Q_2) + v_{2y} \frac{1}{2\alpha_2} (Q_1 + Q_2)\right]^2\right\} \\ & H_{n_1} \left(\sqrt{\frac{\bar{\eta}_1 \bar{\Omega}_1}{\hbar}} \left[v_{1x} \frac{1}{2\alpha_1} (Q_1 - Q_2) + v_{2x} \frac{1}{2\alpha_2} (Q_1 + Q_2)\right] \right) \\ & H_{n_2} \left(\sqrt{\frac{\bar{\eta}_2 \bar{\Omega}_2}{\hbar}} \left[v_{1y} \frac{1}{2\alpha_1} (Q_1 - Q_2) + v_{2y} \frac{1}{2\alpha_2} (Q_1 + Q_2)\right] \right) \end{aligned} \quad (2.84)$$

Energy eigenvalues

$$E = E_1 + E_2 = \hbar \bar{\Omega}_1 \left(n_1 + \frac{1}{2}\right) + \hbar \bar{\Omega}_2 \left(n_2 + \frac{1}{2}\right), \quad n_1, n_2 = 0, 1, 2, \dots \quad (2.85)$$

where H_{n_1} and H_{n_2} are Hermite Polynomials, with:

$$\bar{\Omega}_1 = \sqrt{\gamma_{A_1} \gamma_{B_1}}, \quad \bar{\Omega}_2 = \sqrt{\gamma_{A_2} \gamma_{B_2}}, \quad \bar{\eta}_1 = \frac{1}{\gamma_{B_1}}, \quad \bar{\eta}_2 = \frac{1}{\gamma_{B_2}} \quad (2.86)$$

$$v_{1x} = \frac{\frac{A_{12}}{\gamma_{A_1} - A_{11}}}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_1} - A_{11}}\right)^2}}, \quad v_{2x} = \frac{\frac{A_{12}}{\gamma_{A_2} - A_{11}}}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_2} - A_{11}}\right)^2}} \quad (2.87)$$

$$v_{1y} = \frac{1}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_1} - A_{11}}\right)^2}}, \quad v_{2y} = \frac{1}{\sqrt{1 + \left(\frac{A_{12}}{\gamma_{A_2} - A_{11}}\right)^2}} \quad (2.88)$$

And $\gamma_{A_{1,2}}, A_{12}, A_{11}, A_{22}$ and $\gamma_{B_{1,2}}$: Transformation parameters based on $\mu_1, \mu_2, g, \theta, \sigma$ and, λ .

Analysis of the results

3.1 Introduction

The computational power of many programming languages provides an appropriate environment for conducting comparisons and numerical and graphical analyses of methods for processing complex physical systems, especially those involving numerous variables and complex mathematical expressions, such as the solution and energy expression in the deformed state we solved in the previous chapter. Among the most important software for physics, particularly for computational analysis, are Maple, Mathematica, and recently, Python. The choice of suitable software depends on the required comparison accuracy, the complexity of the mathematical expressions to be processed, financial resources, computational power available to the user, and other conditions.

In this chapter, we will use Python to analyze the results obtained from solving the Schrödinger equation for the coupled harmonic oscillator system in both commutative and non-commutative phase spaces, using the Google Colab environment. We take advantage of the technical support availability, widespread use, and open-source nature of Python, although it may not match Mathematica or Maple in computational accuracy. This consideration should be kept in mind when discrepancies or uncertainty in mathematical expressions arise during the discussion of solutions. Also, we will compare the solution with Feynman's approach to the same system, published by Khayari and others in the scientific paper titled "Time-Independent Coupled Harmonic Oscillator via Path Integral Method in a Deformed Phase Space." This comparison aims to highlight the impact of phase space deformation on the energy levels and wave functions of the system, providing insights into the fundamental quantum mechanical behaviors under phase space deformation.

3.2 Range of comparison parameters

We choose an angular frequency of 10^{16} Hz for our system, aligning with the typical range of frequencies associated with electronic transitions in atoms and molecules, which spans up to 10^{16} Hz. This range corresponds to phenomena involving weak coupling between two electrons modeled as harmonic oscillators, making it particularly suitable for our analysis due to its relevance to our quantum mechanical system. Additionally, we adopt the Hartree unit system to simplify quantum mechanical equations by setting fundamental constants (electron mass, reduced Planck constant, and elementary charge) to unity. This standardization facilitates easier manipulation and comparison of quantum states and their properties. The time unit in Hartree units corresponds to the period it takes for an electron to orbit the hydrogen nucleus in the ground state, providing a natural timescale for atomic and subatomic processes.

Given the Hartree energy unit of approximately 27.2114 eV, and the time unit derived from the electron's orbital period in a hydrogen atom of approximately 2.42×10^{-17} seconds, we can convert the chosen frequency to Hartree units as follows:

$$\omega_H = 2\pi\nu \times t_H = 2\pi \times 10^{16} \text{ Hz} \times 2.42 \times 10^{-17} \text{ s} \approx 1.52$$

Given this context, we can set the angular frequency parameter within the order of unity for comparison purposes.

Regarding the coupling constant, we need to consider the conditions defining the weak coupling regime in relation to the terms in the static part of our Hamiltonian(1.48), often referred to as:

$$\lambda \ll m\omega^2$$

By using the Hartree unit system, and based on the acceptable angular frequency range in this system, we find that: $\lambda \ll 1$, so we can set a range for the λ as follows $0 < \lambda < 0.1$

Given that we are studying small deformations, it is logical that their range should be less than the range of classical parameters that affect the system's behavior. In our specific case, this means the coupling constant. Therefore, since the coupling range λ has been set $0 < \lambda < 0.1$, the range of deformation parameters should θ and σ be less than that. For simplicity, we will set it $\theta, \sigma < 0.01$.

Thus, we have defined the necessary range for all variables related to the system under study.

3.3 Summary of Solutions for the Coupled Harmonic Oscillator System

3.3.1 Schrödinger approach

Based on the work we did in the previous chapter, we can summarize the Schrödinger solution for the commutative and non-commutative cases as follows:

Wave function

$$\begin{aligned} \Psi_{n_1, n_2}(Q_1, Q_2) = & \sqrt{\frac{\sqrt{\bar{\eta}_1 \bar{\Omega}_1 \bar{\eta}_2 \bar{\Omega}_2}}{\pi \hbar 2^{n_1+n_2} n_1! n_2!}} \exp \left\{ -\frac{\bar{\eta}_1 \bar{\Omega}_1}{2\hbar} \left[v_{1x} \frac{1}{2\alpha} (Q_1 - Q_2) + v_{2x} \frac{1}{2\alpha} (Q_1 + Q_2) \right]^2 \right\} \\ & \exp \left\{ -\frac{\bar{\eta}_2 \bar{\Omega}_2}{2\hbar} \left[v_{1y} \frac{1}{2\alpha} (Q_1 - Q_2) + v_{2y} \frac{1}{2\alpha} (Q_1 + Q_2) \right]^2 \right\} \\ & H_{n_1} \left(\sqrt{\frac{\bar{\eta}_1 \bar{\Omega}_1}{\hbar}} \left[v_{1x} \frac{1}{2\alpha} (Q_1 - Q_2) + v_{2x} \frac{1}{2\alpha} (Q_1 + Q_2) \right] \right) \\ & H_{n_2} \left(\sqrt{\frac{\bar{\eta}_2 \bar{\Omega}_2}{\hbar}} \left[v_{1y} \frac{1}{2\alpha} (Q_1 - Q_2) + v_{2y} \frac{1}{2\alpha} (Q_1 + Q_2) \right] \right) \end{aligned} \quad (3.1)$$

Energy eigenvalues

$$E = E_1 + E_2 = \hbar \bar{\Omega}_1 \left(n_1 + \frac{1}{2} \right) + \hbar \bar{\Omega}_2 \left(n_2 + \frac{1}{2} \right), \quad n_1, n_2 = 0, 1, 2, \dots \quad (3.2)$$

where H_{n_1} and H_{n_2} are Hermite Polynomials, with:

$$\bar{\Omega}_1 = \sqrt{\gamma_{A_1} \gamma_{B_1}}, \quad \bar{\Omega}_2 = \sqrt{\gamma_{A_2} \gamma_{B_2}}, \quad \bar{\eta}_1 = \frac{1}{\gamma_{B_1}}, \quad \bar{\eta}_2 = \frac{1}{\gamma_{B_2}} \quad (3.3)$$

$$v_{1x} = \frac{1}{\sqrt{1 + \left(-\frac{A_{11} - \gamma_{A_1}}{A_{12}} \right)^2}}, \quad v_{2x} = \frac{1}{\sqrt{1 + \left(-\frac{A_{11} - \gamma_{A_2}}{A_{12}} \right)^2}} \quad (3.4)$$

$$v_{1y} = \frac{-\frac{A_{11} - \gamma_{A_1}}{A_{12}}}{\sqrt{1 + \left(-\frac{A_{11} - \gamma_{A_1}}{A_{12}} \right)^2}}, \quad v_{2y} = \frac{-\frac{A_{11} - \gamma_{A_2}}{A_{12}}}{\sqrt{1 + \left(-\frac{A_{11} - \gamma_{A_2}}{A_{12}} \right)^2}} \quad (3.5)$$

And $\gamma_{A_{1,2}}, A_{12}, A_{11}, A_{22}$ and $\gamma_{B_{1,2}}$: Transformation parameters based on $\mu_1, \mu_2, g, \theta, \sigma$ and, λ .

3.3.2 Feynman approach

Wave function

$$\begin{aligned} \Psi_{n_1, n_2} = & \left(\frac{\sqrt{\bar{\Omega}_1 \bar{\Omega}_2}}{\pi \hbar 2^{n_1 + n_2} n_1! n_2!} \right)^{1/2} \exp \left[-\frac{1}{4} \left(\bar{\Omega}_1 (Q_1 - Q_2)^2 + \bar{\Omega}_2 (Q_1 + Q_2)^2 \right) \right] \\ & \times H_{n_1} \left(\sqrt{\frac{\bar{\Omega}_1}{2}} (Q_1 - Q_2) \right) H_{n_2} \left(\sqrt{\frac{\bar{\Omega}_2}{2}} (Q_1 + x_Q) \right) \end{aligned} \quad (3.6)$$

Energy eigenvalues

$$E_{n_1, n_2} = \hbar \bar{\Omega}_1 \left(n_1 + \frac{1}{2} \right) + \hbar \bar{\Omega}_2 \left(n_2 + \frac{1}{2} \right) \quad n_1, n_2 = 0, 1, 2, \dots \quad (3.7)$$

Non-Commutative state: where:

$$\bar{\Omega}_1 = \sqrt{\sqrt{\Omega_1 \Omega_2} + \Omega_3}, \quad \bar{\Omega}_2 = \sqrt{\sqrt{\Omega_1 \Omega_2} - \Omega_3} \quad (3.8)$$

$$\begin{aligned} \Omega_1 = & (\kappa a - \kappa^2 b^2 - c^2) \left(\frac{1}{2} + \frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} \right) + \left(\frac{a}{\kappa} - b^2 - \frac{c^2}{\kappa^2} \right) \left(\frac{1}{2} - \frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} \right) \\ & + 2\kappa b^2 + \frac{2c^2}{\kappa} + \left(\frac{\sqrt{\frac{1}{\mu_1 \mu_2} - g^2}}{m} \right) \sqrt{\frac{\mu_1}{\mu_2}} \left(\frac{\omega^2}{\mu_1} \mu_2 - 1 \right) \end{aligned} \quad (3.9)$$

$$\begin{aligned} \Omega_2 = & (\kappa a - \kappa^2 b^2 - c^2) \left(\frac{1}{2} - \frac{1}{1\theta} \sqrt{\frac{\mu_1}{\mu_2}} \right) + \left(\frac{a}{\kappa} - b^2 - \frac{c^2}{\kappa^2} \right) \left(\frac{1}{2} + \frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} \right) \\ & - 2\kappa b^2 - \frac{2c^2}{\kappa} - \left(\frac{\sqrt{\frac{1}{\mu_1 \mu_2} - g^2}}{m} \right) \sqrt{\frac{\mu_1}{\mu_2}} \left(\frac{\omega^2}{\mu_1} \mu_2 - 1 \right) \end{aligned} \quad (3.10)$$

$$\begin{aligned} \Omega_3 = & (\kappa a - \kappa^2 b^2 - c^2) \left(\kappa \left(\frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} - \frac{1}{2} \right) - \frac{\left(\frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} + \frac{1}{2} \right)}{\kappa} \right) \\ & + \left(\frac{a}{\kappa} - b^2 - \frac{c^2}{\kappa^2} \right) \left(\kappa \left(\frac{1}{2} - \frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} \right) + \frac{\left(\frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} + \frac{1}{2} \right)}{\kappa} \right) \\ & + \left(2\kappa b^2 + \frac{2c^2}{\kappa} + \left(\frac{\sqrt{\frac{1}{\mu_1 \mu_2} - g^2}}{m} \right) \sqrt{\frac{\mu_1}{\mu_2}} \left(\frac{\omega^2}{\mu_1} \mu_2 - 1 \right) \right) \times \left(\frac{\kappa \left(\frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} - \frac{1}{2} \right)^2}{\left(\frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} + \frac{1}{2} \right)} + \kappa \left(\frac{1}{2\theta} \sqrt{\frac{\mu_1}{\mu_2}} + \frac{1}{2} \right) \right) \end{aligned} \quad (3.11)$$

$$\begin{aligned}
 a &= \left(\frac{\sqrt{\frac{1}{\mu_1\mu_2} - g^2}}{2m} \right) \sqrt{\frac{\mu_1}{\mu_2}} \left(\frac{\omega^2}{\mu_1} \mu_2 + 1 \right), & b &= \frac{1}{2} \lambda \left(\theta - \sqrt{\frac{\mu_1}{\mu_2}} \right) \\
 c &= \frac{1}{2} \lambda \left(\theta + \sqrt{\frac{\mu_1}{\mu_2}} \right), & \kappa &= \frac{\sqrt{\frac{1}{\sqrt{\mu_1\mu_2}} + g}}{\sqrt{\frac{1}{\sqrt{\mu_1\mu_2}} - g}}
 \end{aligned} \tag{3.12}$$

Commutative state:

Where:

$$\bar{\Omega}_1 = \sqrt{\omega^2 + \frac{\lambda}{m}}, \quad \bar{\Omega}_2 = \sqrt{\omega^2 - \frac{\lambda}{m}} \tag{3.13}$$

These simplified expressions highlight the changes in the system's behavior due to the coupling constant λ and the deformation parameters θ and σ .

It is observed from the complex expressions of both solutions that it is exceedingly difficult to study their convergence through symbolic comparison. This necessitates resorting to numerical comparison, which we will adopt, having previously set the parameter range in Hartree units.

3.4 Comparison of Results:

Numerical analysis and the exploitation of computational capabilities have greatly facilitated researchers' study of complex phenomena with intricate mathematical forms. These capabilities allow us to explore the wave function shapes and energy expressions for the example under study (Schrödinger's approach and Feynman's approach in the case of a coupled harmonic oscillator pair, which we have previously examined and provided a mathematical summary of the results of both treatments). This approach provides an understanding that cannot be easily achieved through purely mathematical treatment, although this perspective remains numerical and approximate rather than being on par with abstract mathematical analysis, except in cases where experimental results are available.

To ensure comprehensive coverage, we will first study the commutative limit of Schrödinger compared to Feynman, which has been previously defined. Then, we will study the behavior of both energy and probability density in the case of small deformations.

3.4.1 The commutative limit:

To study the commutative limit of the solution, we will set the values of θ and σ to the smallest possible values and plot the energy for different levels as a function of λ , which we vary according to the range previously defined. This will give us the output in the form of a table as well as a graphical curve through which we compare the behavior of the energy and the probability density, which represents the behavior of the wave function as it equals the function multiplied by its conjugate. Let's proceed:

3.4.1.1 State Energies

λ	Commutative Feynman	Shrodenger : $\sigma = \theta = 0$
0.00	1.00000	1.00000
0.02	0.99995	0.99995
0.04	0.99980	0.99980
0.06	0.99954	0.99955
0.08	0.99919	0.99919
0.10	0.99874	0.99874

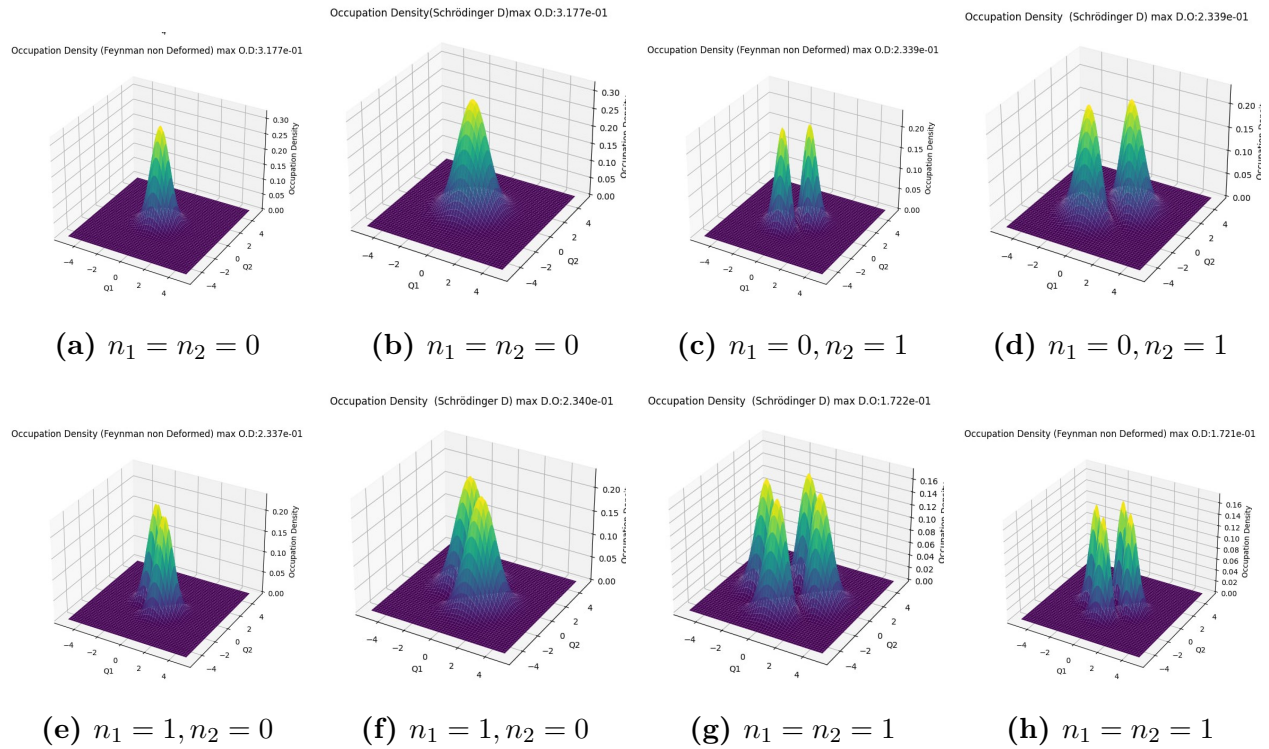
Table 3.1: State Energies $n_1 = n_2 = 0$

3.4.1.2 Probability Density

λ	Commutative Feynman	Schrödinger : $\sigma = \theta \rightarrow 0$
0.00	0.31669	0.31669
0.02	0.31665	0.31667
0.04	0.31659	0.31659
0.06	0.31645	0.31645
0.08	0.31624	0.3162
0.10	0.31597	0.31597

Table 3.2: Max Probability Density Values for Schrödinger and Feynman Approaches

Figure 3.1: Probability Density of Feynman Commutative and Schrödinger $\sigma = \theta \rightarrow 0$



3.4.1.3 Notes

- Both approaches show a gradual increase in energy with increasing deformation parameters Theta and Sigma.
- The energy surfaces are very similar, with no significant differences between the Schrödinger and Feynman approaches.
- Probability density remains relatively stable with increasing Theta and Sigma, showing only slight variations.
- The maximum probability density values are nearly identical in both approaches, indicating a high degree of similarity.

Overall, the Schrödinger and Feynman approaches exhibit similar trends in energy and probability density behavior under deformation, with no significant differences.

This indicates that the reciprocal limit of both the Feynman and Schrödinger approaches agrees within the bounds of errors of user software approximations with the standard solution

of a coupled harmonic vibrating pair. Which confirms the principle of equivalence between the Schroedtger and Feynman approaches and also confirms that the physical meaning of the mixed elimination option is a correct principle, firstly, because it relies on the basic rules, as we previously showed, for Heismarberg algebra and the rules of quantization, and also because it matches the previously known results. It now remains to compare the behavior of each approach when theta and sigma change Let's continue

3.4.2 Non-Commutative case

We will now pay special attention to 3D graphics because theta and sigma change together in the range of weak distortions. In particular, we will study the behavior of energy and the highest values of probability densities for different energy levels to observe the effect of distortion on both approaches, as shown in the following images.

Figure 3.2: Observations of Probability Density Behavior with Respect to σ and θ

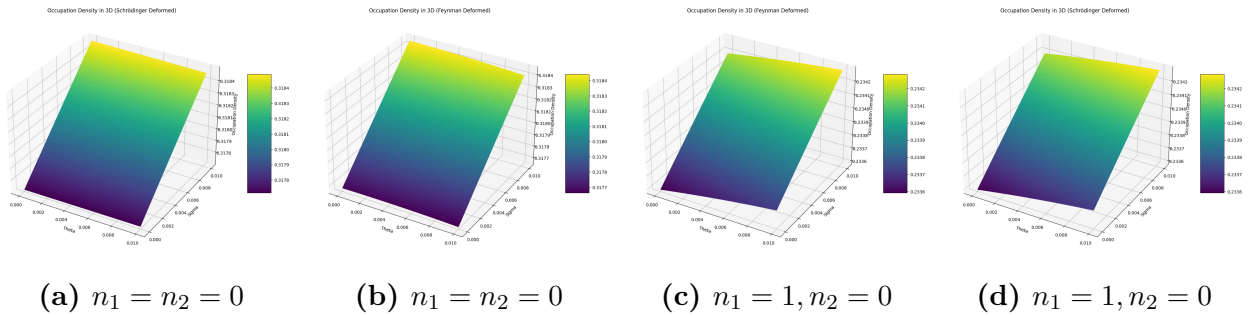
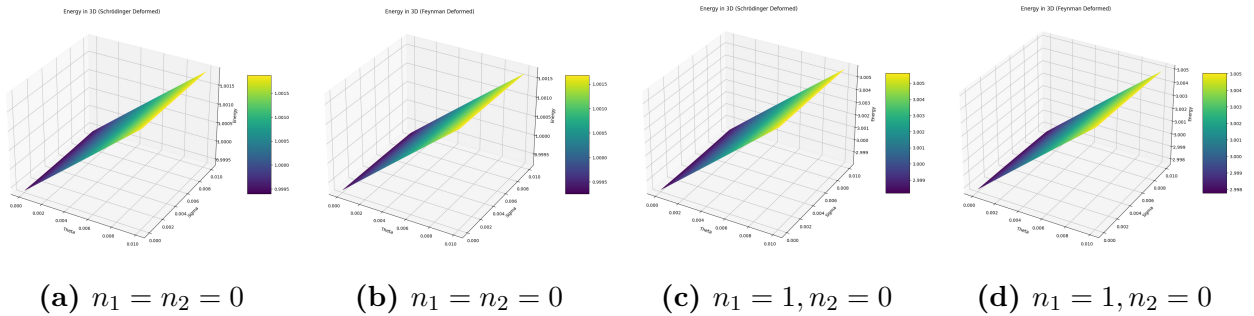


Figure 3.3: Observations of Energy Behavior with Respect to σ and θ



3.4.2.1 Nots

From the curves provided we found the following:

For energy

- **Overall Behavior:** Both the Schrödinger and Feynman approaches exhibit a gradual increase in energy as the deformation parameters, Theta and Sigma, increase. The energy surfaces show a consistent trend across both methods.
- **Similarity:** The energy values and their corresponding surfaces are highly similar between the two approaches, indicating consistent behavior and minimal discrepancies.
- **Differences:** Any differences observed in the energy values between the Schrödinger and Feynman approaches are minor and do not significantly impact the overall trend. These minor differences suggest that both approaches are reliable and can be used interchangeably for analyzing energy behavior under deformation.

For Probability Density

- **Overall Behavior:** The probability density remains relatively stable with increasing Theta and Sigma, showing only slight variations in both approaches. This indicates that the deformation parameters have a minimal impact on the overall probability density distribution.
- **Similarity:** The maximum probability density values are nearly identical in both the Schrödinger and Feynman approaches, highlighting a high degree of similarity in their results. This suggests that both methods provide a consistent and reliable measure of probability density under deformation.
- **Differences:** Any observed differences in the probability density values between the two approaches are insignificant. These minor variations do not affect the overall stability and reliability of the probability density distributions provided by either method.

Thus we can conclude that: The results confirm that the phase space deformation impacts the energy levels and wave functions in predictable ways. The equivalence between the Schrödinger and Feynman approaches in the commutative limit is maintained under small deformations, reinforcing the validity of using either method for analyzing such systems. As we point out, the key aspect of phase space deformation is the introduction of nonlinear

effects, especially in the context of non-commutative geometries; while Python is an excellent tool for numerical analysis, it may not match the computational accuracy of specialized software like Mathematica or Maple. This is important to consider when interpreting results, as numerical approximations can introduce discrepancies. However, for our purposes, the consistency observed between the Schrödinger and Feynman approaches indicates that Python provides sufficient accuracy for our analysis.

General conclusion

In this study, we focused on solving the Schrödinger equation for a coupled harmonic oscillator system in a non-commutative phase space. Our primary goal was to understand how phase space distortion affects the system's dynamics and derive analytical solutions for the modified Hamiltonian. The harmonic oscillator was chosen as a fundamental model applicable to various physical systems in quantum mechanics, allowing for a deep comparison between solutions in commutative and non-commutative phase spaces, highlighting the impact of distortion on system dynamics.

We began with an extensive bibliographic study on non-commutative phase space, enabling us to redefine momentum operators and use modified Poisson brackets, which express the effect of distortion in contexts such as non-commutative geometry and quantum gravity. We constructed a modified Hamiltonian using deformation parameters θ and σ , and coupling constant λ , as corrections to the conventional Hamiltonian.

Key Steps Included:

1. **Redefining Momentum:** Using modified Poisson brackets, we redefined the momentum operators to account for the effects of phase space distortion. This redefinition was crucial for accurately representing the dynamics in a non-commutative setting.
2. **Hamiltonian Diagonalization:** By removing cross terms and mixed differentials, we facilitated variable separation in the Schrödinger equation. This step was essential for simplifying the problem and making it tractable for analytical solutions.
3. **Deriving Analytical Solutions:** We derived an expression for the corrected frequency ($\bar{\Omega}$) and solved the Schrödinger equation in the deformed phase space. This allowed us to obtain energy levels and wave functions that reflect the impact of phase space distortion.
4. **Comparative Analysis:** Using Python software, we compared solutions in deformed and non-deformed phase spaces. This numerical approach enabled us to visualize and quantify

the differences in system behavior due to deformation.

Results and Discussion:

Our analysis revealed similar behavior between Schrödinger and Feynman solutions except for specific values of the deformation parameters. These discrepancies were attributed to Python's numerical approximations, suggesting that both approaches are robust but may yield slightly different results under certain conditions.

We summarize the results as follows

1. Eigenvalue Modifications:
 - Energy levels decrease as deformation parameters increase, indicating stronger stabilization effects due to phase space distortion.
2. Probability Density Distribution:
 - Distortion leads to increased density concentration, reflecting constrained particle dynamics.
3. Method Comparison:
 - Schrödinger and Feynman methods produce comparable results, though numerical methods highlight slight differences due to approximations.

Recommendations for Future Research:

1. Expanding the Study:
 - Investigate more complex quantum systems, including multi-particle interactions and non-linear effects, to understand the broader implications of phase space distortion.
2. Experimental Verification:
 - Conduct experiments to validate theoretical findings and provide empirical evidence for the impact of deformations on quantum systems.
3. Advanced Mathematical Tools:
 - Develop new mathematical frameworks and tools to better analyze and solve problems in non-commutative quantum mechanics, facilitating more accurate and comprehensive models.

In conclusion, this study offers significant insights into the effects of phase space distortion on quantum systems. By combining analytical and numerical methods, we have provided a detailed understanding of the modified dynamics and highlighted the importance of considering such deformations in theoretical and applied quantum mechanics. This work paves the way for future research aimed at further unraveling the complexities of deformed quantum systems and their potential technological applications.

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ملخص

في هذه الدراسة، ركزنا على حل معادلة شرودنجر لنظام زوج هزاز توافقي في فضاء الطور غير التبادلي. هدفنا الأساسي كان فهم تأثير تشويه مساحة الطور على ديناميكيات النظام واستخلاص حلول تحليلية للهاملتونيان المعدل.

بدأنا بدراسة بيبليوغرافية مكثفة حول فضاء الطور غير التبادلي، مما مكننا من إعادة تعريف مشغلي الزخم واستخدام أقواس بواسون المعدلة، التي تعبر عن أثر التشويه في سياقات الهندسة غير التبادلية والجاذبية الكمية. قمنا ببناء هاميلتونيان معدل باستخدام معلمات التشوه θ و σ ، وثابت الاقتران λ ، كتصحیحات للهاملتونيان التقليدي. ثم حللنا معادلة شرودنجر واشتقينا تعبيراً للتردد المصحح $(\bar{\Omega})$. بعدها اجرينا مقارنة رقمية بين نهجي شرودنجر وفايمان لذات النظام في الحالة المشوهة والحالة القياسية.

أظهرت النتائج تطابقاً تاماً بين النهجين عندما تؤول معلمات التشوه إلى الصفر، بينما تظهر ايضاً مستويات الطاقة وكثافة الاحتمال للنظام في حالة التشوهات الضعيفة نفس السلوك ايضاً لكلا النهجين اعتماداً على معاملات التشوه، ما يبرز دور التشوه في احداث فروق دالة في قيم الطاقة وكثافة الاحتمال. **كلمات مفتاحية:** معادلة شرودنجر، زوج هزاز توافقي، معترن، فضاء الطور غير التبادلي، امبلتونيان معدل، فواس بواسون المعدل، الكثافة الاحتمالية.

Abstract

In this study, we focused on solving the Schrödinger equation for a pair of harmonic oscillators in a non-commutative phase space. Our primary goal was to understand the effect of phase space deformation on the system dynamics and derive analytical solutions for the modified Hamiltonian.

We began with an extensive literature review on non-commutative phase space, which allowed us to redefine momentum operators and utilize modified Poisson brackets. These brackets express the impact of deformation in non-commutative geometry and quantum gravity contexts. We constructed a modified Hamiltonian using deformation parameters θ and σ , and a coupling constant λ , as corrections to the traditional Hamiltonian. We then solved the Schrödinger equation and derived an expression for the corrected frequency $(\bar{\Omega})$. Next, we conducted a numerical comparison between the Schrödinger and Feynman approaches for the same system under deformed and standard conditions.

The results showed perfect agreement between the approaches when deformation parameters approach zero. Additionally, energy levels and probability density for the system under weak deformations exhibited similar behaviors for both approaches, depending on the deformation parameters. This highlights the role of deformation in generating noticeable differences in energy values and probability density.

Key words: Schrödinger equation, Coupled harmonic oscillator pair, Non-commutative phase space, Modified Hamiltonian, Modified Poisson brackets, Probability density.

Résumé

Dans cette étude, nous nous sommes concentrés sur la résolution de l'équation de Schrödinger pour un système de deux oscillateurs harmoniques dans un espace de phase non commutatif. Notre objectif principal était de comprendre l'effet de la déformation de l'espace de phase sur la dynamique du système et d'en déduire des solutions analytiques pour l'hamiltonien modifié.

Nous avons commencé par une revue de littérature approfondie sur l'espace de phase non commutatif, ce qui nous a permis de redéfinir les opérateurs de momentum et d'utiliser des crochets de Poisson modifiés. Ces crochets expriment l'impact de la déformation dans les contextes de géométrie non commutative et de gravité quantique. Nous avons construit un hamiltonien modifié en utilisant les paramètres de déformation θ et σ , ainsi qu'une constante de couplage λ , en tant que corrections à l'hamiltonien traditionnel. Ensuite, nous avons résolu l'équation de Schrödinger et dérivé une expression pour la fréquence corrigée $(\bar{\Omega})$. Ensuite, nous avons effectué une comparaison numérique entre les approches de Schrödinger et de Feynman pour le même système dans des conditions déformées et standard.

Les résultats ont montré un accord parfait entre les approches lorsque les paramètres de déformation tendent vers zéro. De plus, les niveaux d'énergie et la densité de probabilité du système sous de faibles déformations ont présenté des comportements similaires pour les deux approches, en fonction des paramètres de déformation. Cela met en évidence le rôle de la déformation dans la génération de différences notables dans les valeurs d'énergie et la densité de probabilité.

Mots clés : Équation de Schrödinger, Paire d'oscillateurs harmoniques couplés, Espace de phase non commutatif, Hamiltonien modifié, Crochets de Poisson modifiés, Densité de probabilité.