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Finite element approximation of the Timoshenko obstacle type problem

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DEDICATION

In the Name of Allah, Most Gracious, Most Merciful

From the depths of my heart, I dedicate this work to those whom I will never be able to fully express my sincere love:

To my dear father who has been a constant source of support, guidance and encouragement, enabling me to achieve my goals.

To my dear mother for the sacrifices she has made, paving the way for my success.

To my brothers **Mohammed Nadir** and **Brahim Zoheir** who have always been my biggest supporters.

To my little sister **Meriem Iman** who has always been there for me.

To my extended family who has supported me all the time.

To everyone who helped me to accomplish this work

Thank you all

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Without forgetting all the teachers at the mathematics department at Kasdi Merbah University.

CONTENTS

NOTATIONS AND CONVENTIONS

Notations

- $\partial_i = \frac{\partial}{\partial x_i}$: Partial derivative with respect to x_i.
- $C_0^\infty(\Omega)$: Space of infinitely differentiable functions with compact support in Ω.
- $H^m(\Omega) = \{v \in L^2(\Omega), \forall \alpha : |\alpha| \leq m, \partial_\alpha v \in L^2(\Omega)\}$.
- $H_0^1(\Omega) = \{v \in H^1(\Omega), v = 0 \text{ on } \partial\Omega\}$.
- $\|v\|_0 = \|v\|_{L^2}$: The norm $L^2(\Omega)$.
- $\|v\|_1 = \|v'\|_0$: The semi-norm $H^1(\Omega)$.
- $(.,.)$: The scalar product.
- $\langle .,. \rangle$: The duality product.
- $\rightarrow$: Strong convergence.
- $\rightharpoonup$: Weak convergence.
- $\overset{com}{\hookrightarrow}$: Compact embedding.
- $a \lesssim b$ (resp. $a \gtrsim b$): There exists a constant C such that $a \leq Cb$ (resp. $a \geq Cb$).

Conventions

- Greek indices $\{\alpha, \beta, \rho\}$ vary over the set $\{1, 2\}$.
- Einstein summation is used over the repeated indices , i.e.,

$$a_\beta b_\beta = \sum_{\beta=1}^{2} a_\beta b_\beta$$

- $a_{\alpha\beta} : b_{\alpha\beta}$: Inner product of matrix defined by

$$a_{\alpha\beta} : b_{\alpha\beta} = a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}$$

- $\delta_{\alpha\beta}$: Kronecker symbol.

INTRODUCTION

The Timoshenko beam theory, developed by Stephen Timoshenko and Paul Ehrenfest, is a mathematical model used to describe the deformation of beams due to shear forces that accounts for the effects of shear deformation and rotational bending [4].

Unlike the classical Euler-Bernoulli beam theory, which assumes that plane cross-section remain perpendicular to the neutral axis during deformation, the Timoshenko beam theory relaxes this assumption and allows for the shear deformation. It can be considered as the one-dimensional version of the Reissner-Mindlin plate theory [9].

When the beam is constrained by an obstacle, which can be either rigid or deformable, it leads to a variational inequality problem known as the Timoshenko obstacle problem, which makes it nonlinear. Its analysis is challenging due to the presence of the obstacle function. Furthermore, when the beam has a variable thickness, the situation becomes more complex.

In our thesis, we study the finite element approximation of the Timoshenko rigid obstacle problem with a variable thickness parameter, tends to zero, which requires advanced mathematical analysis and numerical methods.

Our work begins by a chapter presenting mathematical preliminaries which will be used in the next chapters. For the second chapter, The mathematical analysis of the Timoshenko equations, in the absence or the presence of the obstacle, involves several key aspects:

1. Derivation of the Timoshenko beam model from 2D linear elasticity by dimensional reduction and minimizing the energy [1].

2. Well-posedness, classical problem and complementarity system, regularity of the solutions.

3. Extract Euler-Bernoulli's variational formulation using penalization method and classic problem using asymptotic analysis.

4. Locking phenomena: When solving the Timoshenko equations numerically using finite element methods, the convergence between the approximate solution and the exact solution is not uniform for certain choices of parameters and finite element spaces.

Our next steps will involve strategies to avoid the locking phenomena. Instead of focusing on the primal formulation, we study an alternative variational formulation based on the method of Lagrange multipliers [2, 7]. The Lagrange multiplier formulation introduces an additional physically relevant unknown, the reaction force between the membrane and the obstacle, which in itself can be a useful tool [6].

In the third chapter, we use the mixed formulation. For this class of methods, the finite element spaces have to satisfy the " Babuška-Brezzi" condition

$$\sup_{v_h \in V_h} \frac{b(\mu_h, v_h)}{\|v_h\|_V} \gtrsim \|\mu_h\|_Q, \quad \forall \mu_h \in Q_h$$

However, when using low-order finite element spaces, locking issues may still arise.

In the fourth chapter, we define the stabilized formulation by introducing additional terms that are mesh-dependent and consistent. The resulting formulation, stability and posteriori error estimates are valid for almost any finite element pair.

In the last chapter, we do some numerical tests in FreeFem++.

CHAPTER 1

PRELIMINARIES

Theorem 1.1 (*Young's inequality*)

Let $p, q \in]1, \infty[$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$. If a and b are two positive real numbers then

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

Proof: See [5], p.706. ■

Theorem 1.2 (*Young's Inequality With a Parameter*)

Let a and b two positive real numbers. For all $\epsilon > 0$

$$ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$$

Proof: We know that

$$(a - \epsilon b)^2 \geq 0$$

Then

$$2\epsilon ab \leq a^2 + \epsilon^2 b^2 \implies ab \leq \frac{1}{2\epsilon}a^2 + \frac{\epsilon}{2}b^2$$

■

Theorem 1.3 (*Hölder's inequality*)

Let $\Omega \subset \mathbb{R}^N$ and $p, q \in]1, \infty[$ with $\frac{1}{p} + \frac{1}{q} = 1$. If $u \in L^p(\Omega), v \in L^q(\Omega)$ then $uv \in L^1(\Omega)$ and

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}$$

Proof: See [5], p.706. ■

Remark 1

For $p = q = 2$, we get the famous Cauchy-Schwarz inequality

$$\|uv\|_{L^1(\Omega)} \leq \|u\|_{L^2(\Omega)}\|v\|_{L^2(\Omega)}$$

Theorem 1.4 (*Poincaré's Inequality*)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain. There exists a strictly positive constant C_p such that, for all $v \in H_0^1(\Omega)$

$$\|v\|_{L^2(\Omega)} \leq C_p\|\nabla v\|_{L^2(\Omega)}$$

Proof: See [10], p.400. ■

Theorem 1.5 (*Green's Integration by Parts Formula*)

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a sufficiently smooth boundary Γ and n be the outward normal.
For all $u, v \in \mathcal{C}^1(\overline{\Omega})$

$$\int_\Omega \partial_i u(x) v(x)\ dx = -\int_\Omega u(x)\partial_i v(x)\ dx + \int_\Gamma u(x)v(x)n_i\ d\Gamma$$

Proof: See [5], p.712. ■

Definition 1.1

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain.

1. The deformation of an elastic solid body due to external forces is expressed by the displacement $\mathbf{u}$.
2. $\mathbf{e}(\mathbf{u})$ is the strain tensor which represent the change of shape, defined as

$$e_{\alpha\beta}(u) = \frac{\partial_\alpha u_\beta + \partial_\beta u_\alpha}{2}$$

3. $\sigma(\mathbf{u})$ is the stress tensor which represent the internal forces, defined as

$$\sigma_{\alpha\beta}(u) = 2\mu e_{\alpha\beta}(u) + \lambda e_{\rho\rho}(u)\delta_{\alpha\beta}$$

where λ and μ are the Lamé coefficients.

Remark 2

σ and e are symmetric, i.e.,

$$\sigma_{\alpha\beta} = \sigma_{\beta\alpha} \quad , \quad e_{\alpha\beta} = e_{\beta\alpha}$$

Theorem 1.6

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a sufficiently smooth boundary Γ.
For all $v \in \mathcal{C}^1(\Omega)$

$$\int_\Omega \text{div}(\sigma) v\ dx = -\int_\Omega \sigma : e(v)\ dx + \int_\Gamma \sigma v n\ dx$$

Proof: Using theorem 1.5

$$\int_\Omega \text{div}(\sigma)v\ dx = \int_\Omega \partial_\alpha \sigma_{\alpha\beta}\ v_\beta\ dx = -\int_\Omega \sigma_{\alpha\beta}\ \partial_\alpha v_\beta\ dx + \int_\Gamma \sigma_{\alpha\beta}\ v_\beta\ n_\alpha\ d\Gamma$$

In other hand, since σ is symmetric

$$\begin{aligned}
\int_\Omega \sigma_{\alpha\beta}\ \partial_\alpha v_\beta\ dx &= \frac{1}{2}\int_\Omega \sigma_{\alpha\beta}\ \partial_\alpha v_\beta\ dx + \frac{1}{2}\int_\Omega \sigma_{\alpha\beta}\ \partial_\alpha v_\beta\ dx \\
&= \frac{1}{2}\int_\Omega \sigma_{\alpha\beta}\ \partial_\alpha v_\beta\ dx + \frac{1}{2}\int_\Omega \sigma_{\beta\alpha}\ \partial_\beta v_\alpha\ dx \\
&= \frac{1}{2}\int_\Omega \sigma_{\alpha\beta}\ \partial_\alpha v_\beta\ dx + \frac{1}{2}\int_\Omega \sigma_{\alpha\beta}\ \partial_\beta v_\alpha\ dx \\
&= \int_\Omega \sigma_{\alpha\beta}\ e_{\alpha\beta}(v)\ dx
\end{aligned}$$

Hence

$$\int_\Omega \text{div}(\sigma)v\ dx = -\int_\Omega \sigma : e(v)\ dx + \int_\Gamma \sigma v n\ dx$$

■

Theorem 1.7 (*Lax-Milgram theorem*)

Let H be a Hilbert space equipped with the norm $\|.\|_H$.

Consider the variational problem

$$\begin{cases} \text{Find } u \in H \\ a(u,v) = F(v), \quad \forall v \in H \end{cases} \tag{1.1}$$

Suppose that

- The bilinear form a is continuous, i.e.,

$$\exists C_1 > 0, \quad \forall (u,v) \in H \times H, \quad |a(u,v)| \leq C_1 \|u\|_H \|v\|_H$$

- The bilinear form a is coercive (H-elliptic), i.e.,

$$\exists C_2 > 0, \quad \forall v \in H, \quad a(v,v) \geq C_2 \|v\|_H^2$$

- The linear form F is continuous, i.e.,

$$\exists C_3 > 0, \quad \forall v \in H, \quad |F(v)| \leq C_3 \|v\|_H$$

Then there exists a unique solution u of the problem (1.1).

Moreover, if a is symmetric, then u is the unique solution of the minimization problem

$$J(u) = \min_{v \in H} J(v), \quad \text{with} \quad J(v) = \frac{1}{2}a(v,v) - F(v)$$

Proof: See [3], p.140. ■

Theorem 1.8 (*Stampacchia theorem*)

Let H be a Hilbert space, $K \subset H$ be a nonempty convex closed subset.

Consider the variational problem

$$\begin{cases} \text{Find } u \in K \\ a(u, v-u) \geq F(v-u), \quad \forall v \in K \end{cases} \tag{1.2}$$

Suppose that $a(.,.)$ is a bilinear continuous coercive form, and $F(.)$ is a linear continuous form.

Then the problem (1.2) admits a unique solution.

Moreover, if a is symmetric, then u is the unique solution of the minimization problem

$$J(u) = \min_{v \in K} J(v), \quad \text{with} \quad J(v) = \frac{1}{2}a(v,v) - F(v)$$

Proof: See [3], p.138. ■

Definition 1.2

Let X be a normed vector space equipped with the norm $\|.\|$, $(x_n)_{n\geq 0}$ be a sequence in X, $x \in X$.

- $(x_n)_{n\geq 0}$ is said to be strongly convergent to x if:

  $$\|x_n - x\| \to 0 \quad \text{as} \quad n \to \infty$$

  and we note $x_n \to x$.

- $(x_n)_{n\geq 0}$ is said to be weakly convergent to x if:

  $$\langle T, x_n \rangle \to \langle T, x \rangle, \quad \text{as} \quad n \to \infty, \quad \forall T \in X' \quad (X' \text{ is the dual space of } X)$$

  and we note $x_n \rightharpoonup x$.

Definition 1.3

Let X, Y two Banach spaces.

1. It is said that X is injected continuously into Y if:
   - $X \subset Y$.
   - $\exists C > 0$ s.t. $\forall u \in X$, $\|u\|_Y \leq C\|u\|_X$.

   and we note $X \hookrightarrow Y$

2. It is said that X is injected in a compact way into Y if:

   - $X \hookrightarrow Y$.
   - The image of every bounded set in X is relatively compact in Y.

   and we note $X \overset{com}{\hookrightarrow} Y$.

Theorem 1.9 (*Rellich-Kondrachov*)

Let $\Omega \subset \mathbb{R}^n$ be a Lipschitz domain, $m \in \mathbb{N}$, $1 \leq p < \infty$.

- If $p < n$ then $W^{1,p}(\Omega) \overset{com}{\hookrightarrow} L^q(\Omega)$, $\forall q \in [1, p^\star)$, s.t., $\dfrac{1}{p^\star} = \dfrac{1}{p} - \dfrac{1}{n}$.
- If $p = n$ then $W^{1,p}(\Omega) \overset{com}{\hookrightarrow} L^q(\Omega)$, $\forall q \in [p, \infty)$.
- If $p > n$ then $W^{1,p}(\Omega) \overset{com}{\hookrightarrow} \mathcal{C}(\overline{\Omega})$.

In particular, $W^{1,p}(\Omega) \overset{com}{\hookrightarrow} L^p(\Omega)$, $\forall p$.

Proof: See [3], p.285. ■

Remark 3

If $X \overset{com}{\hookrightarrow} Y$ and $x_n \rightharpoonup x$ in X, then $x_n \to x$ in Y.

Theorem 1.10

Let X be a Banach reflexive space and $(x_n)_{n\geq 0}$ be a bounded sequence in X. Then there exists a weakly convergent subsequence of $(x_n)_{n\geq 0}$.

Proof: See [8], p.496. ■

Definition 1.4 (*The mesh*)

In one dimension, a mesh is just a subdivision of $\Omega =]a, b[$ into a finite number of sub-intervals with a non-zero measure $\{I_i = [x_i, x_{i+1}]\}_{0\leq i\leq N}$.

- $\{x_0, x_1, ..., x_{N+1}\}$: the vertices of the mesh.
- $h_i = x_{i+1} - x_i$: sub-interval size.
- $h = \max\limits_{0\leq i\leq N} h_i$: mesh size.
- I_i : elements (or cells) of the mesh.
- $\mathcal{I}_h$: the mesh.

Remark 4

If the sub-intervals are uniform, then h is defined by: $h = \dfrac{b-a}{N+1}$ and $x_i = a + ib$.

Definition 1.5 (***The*** $\mathbb{P}_1$ ***Lagrange finite element space***)

Define the space of continuous, piecewise linear functions

$$V_h^1 = \{v_h \in \mathcal{C}(\overline{\Omega}); \forall i = \overline{0, N}; v_{h|I_i} \in \mathbb{P}_1\}$$

The basis of V_h^1 is the set $\{\varphi_i\}_{0 \le i \le N+1}$ where the function φ_i is defined as

$$\varphi_i(x) = \begin{cases} \dfrac{x - x_{i-1}}{h_{i-1}} & \text{if } x \in I_{i-1} \\ \dfrac{x_{i+1} - x}{h_i} & \text{if } x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

Definition 1.6 (***The*** $\mathbb{P}_2$ ***Lagrange finite element space***)

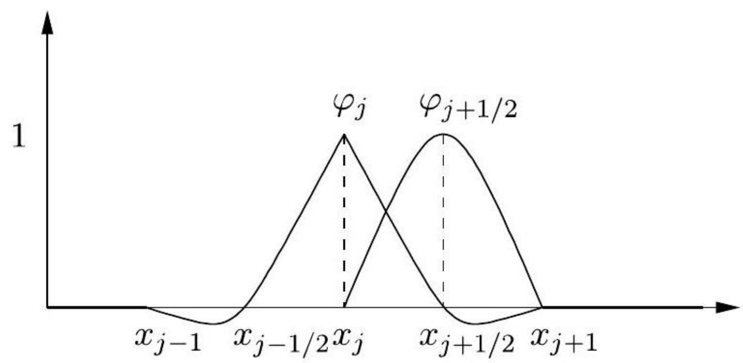
Define the space of continuous, piecewise parabolic functions

$$V_h^2 = \{v_h \in \mathcal{C}(\overline{\Omega}); \forall i = \overline{0, N}; v_{h|I_i} \in \mathbb{P}_2\}$$

and introduce the midpoints $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1})$

The basis of V_h^2 is the set $\{\varphi_i\}_{0 \le i \le N+1}$ where the function φ_i is characterized by

$$\varphi_i(x_j) = \delta_{ij} \qquad \varphi_i(x_{j+\frac{1}{2}}) = 0$$
$$\varphi_{i+\frac{1}{2}}(x_j) = 0 \qquad \varphi_{i+\frac{1}{2}}(x_{j+\frac{1}{2}}) = \delta_{ij}$$



Definition 1.7 (***The $\mathbb{P}_0$ finite element space***)

Define the space of piecewise constant functions

$$V_h^0 = \{v_h \in L^2(\Omega); \forall i = \overline{0, N}; v_{h|I_i} \in \mathbb{P}_0\}$$

The basis of V_h^0 is the set $\{\varphi_i\}_{0 \leq i \leq N+1}$ where the function φ_i is defined as

$$\varphi_i(x) = \begin{cases} 1 & \text{if } x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

Theorem 1.11 (***Inverse inequality***)

Let $V_h \subset V$ a finite element space. There exists a positive constant C such that for any $v_h \in V_h$

$$\|v_h'\|_0 \leq C h^{-1} \|v_h\|_0 \tag{1.3}$$

Propostion 1.1 (***Bubble functions***)

Let $b_e \in H_0^1(I_i)$ be a function such that:

- $\text{supp}(b_e) \subset I_i$
- $0 \leq b_e \leq 1$
- $\exists J_i \subset I_i$ s.t., mes $J_i \geq 0$ and $b_{e|J_i} \geq \dfrac{1}{2}$.

Let $m \in \mathbb{N}$. For every $\phi \in \mathbb{P}_m(I_i)$, there exist $c_1, c_2 > 0$ such that

$$\|b_e \phi\|_{0,I_i} \leq \|\phi\|_{0,I_i} \leq c_1 \|b_e^{1/2} \phi\|_{0,I_i} \tag{1.4}$$

$$\|b_e \phi\|_{1,I_i} \leq c_2 h_i^{-1} \|\phi\|_{0,I_i} \tag{1.5}$$

CHAPTER 2

MATHEMATICAL ANALYSIS OF THE TIMOSHENKO EQUATIONS

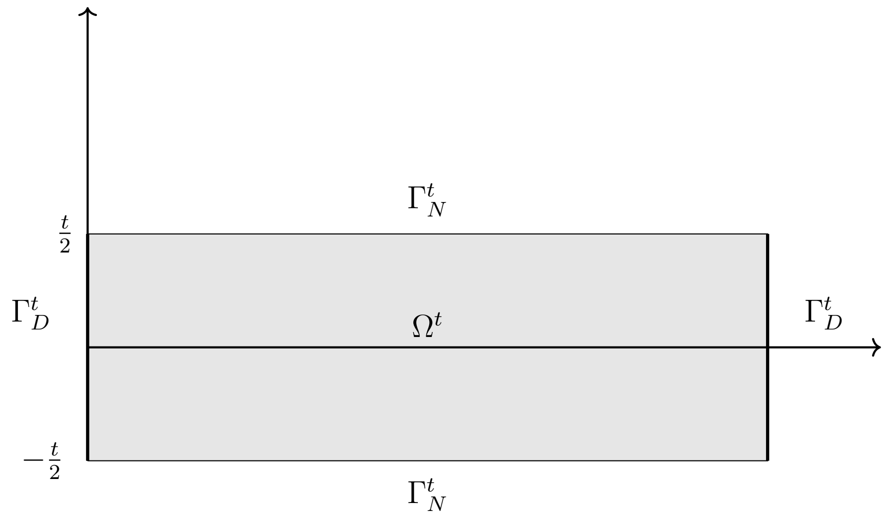
2.1 In the absence of the obstacle

2.1.1 Derivation of the Timoshenko beam model

Let us suppose that the undisplaced beam occupies the region

$$\Omega^t = (0, L) \times (-\frac{t}{2}, \frac{t}{2})$$

in the (x_1, x_2)-plane and is subject to a vertical body force $t^2 f(x_1)$ which is independent of x_2. (The factor t^2 assures that the energy has a finite, non-zero limit as $t \to 0$).



The Timoshenko model may be derived from the classical theory of plane linear elasticity by dimensional reduction. Indeed, we start from the plane linear elasticity problem:

$$\begin{cases} -\partial_\beta \sigma_{\alpha\beta}(u^t) = t^2 f_\alpha, & \text{in } \Omega^t \\ u^t = 0, & \text{on } \Gamma_D^t \\ \sigma_{\alpha\beta}(u^t) n_\beta = 0, & \text{on } \Gamma_N^t \end{cases} \tag{2.1}$$

We introduce the space

$$V^t = \{v = (v_1, v_2) \in \left(H^1(\Omega^t)\right)^2 \quad | \quad v_1 = v_2 = 0 \text{ on } \Gamma_D^t\}$$

Theorem 2.1

The variational formulation reads:

$$\begin{cases} \text{Find } u^t \in V^t \quad \text{s.t.} \quad \forall v \in V^t \\ \displaystyle\int_{\Omega^t} \big(2\mu e_{\alpha\beta}(u^t) : e_{\alpha\beta}(v) + \lambda \mathrm{div}(u^t)\mathrm{div}(v)\big)\ dx = t^2 \int_{\Omega^t} f_2 v_2\ dx \end{cases} \tag{2.2}$$

Proof: Let $v \in V^t$. We multiply each side of the first equation of (2.1) by v and integrate on Ω^t to obtain:

$$-\int_{\Omega^t} (\partial_\beta \sigma_{\alpha\beta}(u^t))v\ dx = t^2 \int_{\Omega^t} f_\alpha v_\alpha\ dx$$

Using Theorem (1.6) and the boundary conditions in (2.1), we deduce

$$\begin{aligned} \int_{\Omega^t} \sigma_{\alpha\beta}(u^t) : e_{\alpha\beta}(v)\ dx &= t^2 \int_{\Omega^t} f_2 v_2\ dx \\ \int_{\Omega^t} \Big(\big(2\mu e_{\alpha\beta}(u^t) + \lambda e_{\rho\rho}(u^t)\delta_{\alpha\beta}\big) : e_{\alpha\beta}(v)\Big)\ dx &= t^2 \int_{\Omega^t} f_2 v_2\ dx \\ \int_{\Omega^t} \Big(2\mu e_{\alpha\beta}(u^t) : e_{\alpha\beta}(v) + \lambda e_{\rho\rho}(u^t)\delta_{\alpha\beta} : e_{\alpha\beta}(v)\Big)\ dx &= t^2 \int_{\Omega^t} f_2 v_2\ dx \end{aligned}$$

On the other hand

$$\begin{aligned} e_{\rho\rho}(u^t)\delta_{\alpha\beta} : e_{\alpha\beta}(v) &= e_{\rho\rho}(u^t)(e_{11}(v) + e_{22}(v)) \\ &= e_{\rho\rho}(u^t)(\partial_1 v_1 + \partial_2 v_2) \\ &= \mathrm{div}(u^t)\mathrm{div}(v) \end{aligned}$$

Therefore

$$\int_{\Omega^t} \Big(2\mu e_{\alpha\beta}(u^t) : e_{\alpha\beta}(v) + \lambda \mathrm{div}(u^t)\mathrm{div}(v)\Big)\ dx = t^2 \int_{\Omega^t} f_2 v_2\ dx$$

Hence

$$\begin{cases} \text{Find } u^t \in V^t \quad \text{s.t.} \quad \forall v \in V^t \\ \displaystyle\int_{\Omega^t} \Big(2\mu e_{\alpha\beta}(u^t) : e_{\alpha\beta}(v) + \lambda \mathrm{div}\ u^t \mathrm{div} v\Big)\ dx = t^2 \int_{\Omega^t} f_2 v_2\ dx \end{cases}$$

■

Note that, thanks to Lax-Milgram theorem, the problem (2.2) can be formulated as a minimization problem

$$\begin{cases} \text{Find } u^t \in V^t \\ u^t = \underset{v \in V^t}{\arg\min}\ J(v) \end{cases} \tag{2.3}$$

where

$$J(v) = \frac{1}{2}\int_{\Omega^t} \left(2\mu e_{\alpha\beta}(v) : e_{\alpha\beta}(v) + \lambda(\mathrm{div}(v))^2\right) dx - t^2 \int_{\Omega^t} f_2 v_2 \, dx \tag{2.4}$$

We have

$$\begin{aligned} e_{\alpha\beta}(v) : e_{\alpha\beta}(v) &= (e_{11}(v))^2 + (e_{12}(v))^2 + (e_{21}(v))^2 + (e_{22}(v))^2 \\ &= (\partial_1 v_1)^2 + (\partial_2 v_2)^2 + \frac{1}{2}(\partial_1 v_2 + \partial_2 v_1)^2 \end{aligned}$$

and

$$(\mathrm{div}(v))^2 = (\partial_1 v_1 + \partial_2 v_2)^2$$

So,

$$\begin{aligned} J(v) = \frac{1}{2}\int_{\Omega^t} \left(2\mu\Big((\partial_1 v_1)^2 + (\partial_2 v_2)^2 + \frac{(\partial_2 v_1 + \partial_1 v_2)^2}{2}\Big) + \lambda(\partial_1 v_1 + \partial_2 v_2)^2\right) dx \\ - t^2 \int_{\Omega^t} f_2 v_2 \, dx \end{aligned} \tag{2.5}$$

We will use the Mindlin assumptions to prove the Timoshenko beam equation. The Mindlin assumptions are :

- The vertical displacement u_2^t is independent of x_2.
- The mid line $x_2 = 0$ is not displaced horizontally.
- The vertical fibres $x_1 = Cte$ remain linear after displacement.

Mathematically, this implies that u_1^t and u_2^t have the special form

$$\begin{cases} u_1^t(x_1, x_2) = -x_2\theta_t(x_1) \\ u_2^t(x_1, x_2) = w_t(x_1) \end{cases} \tag{2.6}$$

Then the Timoshenko model can be seen as

$$\min_{v \in V_0} J(v) \tag{2.7}$$

where $V_0 \subset V^t$ and (v_1, v_2) satisfy the analogue of (2.6).

$$J(v) = \frac{1}{2}\int_{\Omega^t} \left(2\mu\Big((\partial_1 v_1)^2 + (\partial_2 v_2)^2 + \frac{(\partial_2 v_1 + \partial_1 v_2)^2}{2}\Big) + \lambda(\partial_1 v_1 + \partial_2 v_2)^2\right) dx - t^2 \int_{\Omega^t} f_2 v_2 \, dx.$$

We have

$$\begin{cases} v_1(x) = -x_2\eta(x_1), & \partial_1 v_1 = -x_2\eta'(x_1), & \partial_2 v_1 = -\eta(x_1) \\ v_2(x) = z(x_1), & \partial_1 v_2 = z'(x_1), & \partial_2 v_2 = 0 \end{cases}$$

So $(\eta, z) \in V = H_0^1(0,L) \times H_0^1(0,L)$, and

$$J((\eta,z)) = \frac{1}{2}\int_{-t/2}^{t/2}\int_0^L \left(2\mu\Big(x_2^2\eta'(x_1)^2 + \frac{(z'(x_1)-\eta(x_1))^2}{2}\Big) + \lambda x_2^2\eta'(x_1)^2\right)\, dx_1\, dx_2 \\ - t^2\int_{-t/2}^{t/2}\int_0^L f_2(x_1)z(x_1)\ dx_1\ dx_2$$

i.e.,

$$J((\eta,z)) = \frac{1}{2}\int_{-t/2}^{t/2}\int_0^L \Big((2\mu+\lambda)x_2^2\eta'(x_1)^2 + \mu(z'(x_1)-\eta(x_1))^2\Big)\ dx_1\ dx_2 \\ - t^2\int_{-t/2}^{t/2}\int_0^L f_2(x_1)z(x_1)\ dx_1\ dx_2$$

We have,

$$\int_{-t/2}^{t/2} y^2 dy = \frac{y^3}{3}\Big|_{-t/2}^{t/2} = \frac{1}{3}\left(\frac{t^3}{8}+\frac{t^3}{8}\right) = \frac{t^3}{12}$$
$$\int_{-t/2}^{t/2} dy = t$$

Then,

$$J((\eta,z)) = \frac{1}{2}\int_0^L \left(\frac{t^3(\lambda+2\mu)}{12}\eta'(x_1)^2 + t\mu(z'(x_1)-\eta(x_1))^2\right)\ dx_1 - t^3\int_0^L f_2(x_1)z(x_1)\ dx_1$$

i.e.,

$$J((\eta,z)) = \frac{1}{2}\left(\frac{t^3(\lambda+2\mu)}{12}\int_0^L \eta'(x_1)^2\ dx_1\ + t\mu\int_0^L (z'(x_1)-\eta(x_1))^2\ dx_1\right) \\ - t^3\int_0^L f_2(x_1)z(x_1)\ dx_1 \quad (2.8)$$

And:

$$\langle \nabla J(\theta_t, w_t), (\eta, z)\rangle = 0, \quad \forall (\eta,z) \in V$$

means that, the variational problem reads:

$$\begin{cases} \text{Find } (\theta_t, w_t) \in V \\ \dfrac{\lambda+2\mu}{12}\displaystyle\int_0^L \theta_t'\eta'\ dx_1\ + \mu t^{-2}\int_0^L (w_t'-\theta_t)(z'-\eta)\ dx_1 = \int_0^L f_2 z\ dx_1,\ \forall(\eta,z)\in V \end{cases}$$

In other words

$$\begin{cases} \text{Find } (\theta_t, w_t) \in V \\ \displaystyle\int_0^L \theta_t' \eta' \, dx_1 \ + a_1 t^{-2} \int_0^L (w_t' - \theta_t)(z' - \eta) \, dx_1 = a_2 \int_0^L f_2 z \, dx_1, \ \forall (\eta, z) \in V \end{cases}$$

with $a_2 = \dfrac{12}{\lambda + 2\mu}$, $a_1 = a_2 \mu$.

For convenience and without loss of generality we shall assume that $a_1 = a_2 = 1$.

Hence

$$\begin{cases} \text{Find } (\theta_t, w_t) \in V \\ \displaystyle\int_0^L \theta_t' \eta' \, dx_1 \ + t^{-2} \int_0^L (w_t' - \theta_t)(z' - \eta) \, dx_1 = \int_0^L f_2 z \, dx_1, \ \forall (\eta, z) \in V \end{cases} \tag{2.9}$$

2.1.2 Existence and uniqueness

Consider the problem

$$\begin{cases} \text{Find } (\theta, w) \in V, \\ \boldsymbol{a}((\theta, w), (\eta, v)) = L((\eta, v)), \quad \forall (\eta, v) \in V, \end{cases} \tag{2.10}$$

where

$$\boldsymbol{a}((\theta, w); (\eta, v)) = a((\theta, w); (\eta, v)) + t^{-2} b((\theta, w); (\eta, v)) \tag{2.11}$$

$$a((\theta, w); (\eta, v)) = a(\theta, \eta) = \int_0^L \theta' \eta' \, dx \tag{2.12}$$

$$b((\theta, w); (\eta, v)) = \int_0^L (w' - \theta)(v' - \eta) \, dx \tag{2.13}$$

$$L((\eta, v)) = \int_0^L f v \, dx \tag{2.14}$$

Lemma 2.1

The bilinear form $a + b$ is coercive on V.

Proof: Using Young's, Poincaré's inequalities

$$\begin{aligned}
\|(\eta,v)\|_V^2 &= \|\eta'\|_0^2 + \|v'\|_0^2 \\
&= \|\eta'\|_0^2 + \|v'-\eta+\eta\|_0^2 \\
&\leq \|\eta'\|_0^2 + (\|v'-\eta\|_0 + \|\eta\|_0)^2 \\
&\leq \|\eta'\|_0^2 + \|v'-\eta\|_0^2 + \|\eta\|_0^2 + 2\|v'-\eta\|_0\|\eta\|_0 \\
&\leq \|\eta'\|_0^2 + 2\|v'-\eta\|_0^2 + 2\|\eta\|_0^2 \\
&\leq \|\eta'\|_0^2 + 2\|v'-\eta\|_0^2 + 2\|\eta'\|_0^2 \\
&\leq 3\big(\|\eta'\|_0^2 + \|v'-\eta\|_0^2\big) \\
&= 3(a+b)((\eta,v),(\eta,v))
\end{aligned}$$

$$\therefore (a+b)((\eta,v),(\eta,v)) \geq \frac{1}{3}\|(\eta,v)\|_V^2$$ ■

Theorem 2.2

The problem (2.10) admits a unique solution in V.

Proof: We apply the Lax-Milgram theorem,

- V is a Hilbert space.
- $\boldsymbol{a}$ is bilinear (evident) continuous form. Indeed, let $(\theta,w),(\eta,v)\in V$, using Cauchy-Schwarz inequality

$$\begin{aligned}
|\boldsymbol{a}((\theta,w),(\eta,v))| &= \left|\int_0^L \theta'\eta'\ dx + t^{-2}\int_0^L (w'-\theta)(v'-\eta)\ dx\right| \\
&\leq \int_0^L \big|\theta'\eta'\big|\, dx + t^{-2}\int_0^L \big|(w'-\theta)(v'-\eta)\big|\ dx \\
&\leq \|\theta'\|_0\|\eta'\|_0 + t^{-2}\|w'-\theta\|_0\|v'-\eta\|_0 \\
&\leq \|(\theta,w)\|_V\|(\eta,v)\|_V + t^{-2}(\|w'\|_0 + \|\theta\|_0)(\|v'\|_0 + \|\eta\|_0) \\
&\leq \|(\theta,w)\|_V\|(\eta,v)\|_V + t^{-2}(\|w'\|_0 + \|\theta'\|_0)(\|v'\|_0 + \|\eta'\|_0) \\
&\leq t^{-2}\|(\theta,w)\|_V|(\eta,v)\|_V
\end{aligned}$$

- $\boldsymbol{a}$ is coercive form. Indeed, let $(\eta, v) \in V$, from lemma 2.1

$$\begin{aligned}\|(\eta,v)\|_V^2 &\leq 3\left(\|\eta'\|_0^2 + \|v'-\eta\|_0^2\right)\\ &\leq 3\left(\|\eta'\|_0^2 + t^{-2}\|v'-\eta\|_0^2\right)\\ &= 3\boldsymbol{a}((\eta,v),(\eta,v))\end{aligned}$$

$\therefore \boldsymbol{a}((\eta,v),(\eta,v)) \geq \dfrac{1}{3}\|(\eta,v)\|_V^2$

- L is linear continuous form. Indeed, let $v \in V$,

$$\begin{aligned}|L((\eta,v))| = |\int_0^L fv\ dx| &\leq \|f\|_0\|v\|_0\\ &\leq \|f\|_0\|(\eta,v)\|_V\end{aligned}$$

$\therefore$ The problem (2.10) has a unique solution $(\theta, w) \in V$. ■

2.1.3 Classical problem

Theorem 2.3

The solution (θ, w) satisfies the problem:

$$\begin{cases} -\theta'' - t^{-2}(w'-\theta) = 0, & \text{in } (0,L)\\ -t^{-2}(w'-\theta)' = f, & \text{in } (0,L)\\ \theta(0) = \theta(L) = 0 & \\ w(0) = w(L) = 0 & \end{cases} \tag{2.15}$$

Proof: We have from (2.10), $(\theta, w) \in V$, then $\theta(0) = \theta(L) = w(0) = w(L) = 0$. And

$$\int_0^L \theta'\eta'\ dx + t^{-2}\int_0^L (w'-\theta)(v'-\eta)\ dx = \int_0^L fv\ dx, \quad \forall(\eta,v) \in V,$$

i.e.,

$$-\int_0^L \theta''\eta\ dx + t^{-2}\int_0^L (w'-\theta)(v'-\eta)\ dx = \int_0^L fv\ dx, \quad \forall(\eta,v) \in V. \tag{2.16}$$

First, we take $v = 0$ in (2.16), we obtain

$$- \int_0^L \theta'' \eta \, dx + t^{-2} \int_0^L (w' - \theta)(-\eta) \, dx = 0, \qquad \forall \eta \in H_0^1(0, L)$$

$$\Longrightarrow - \int_0^L \theta'' \eta \, dx - t^{-2} \int_0^L (w' - \theta)\eta \, dx = 0, \qquad \forall \eta \in H_0^1(0, L)$$

$$\Longrightarrow \int_0^L \Big(- \theta'' - t^{-2}(w' - \theta) \Big) \eta \, dx = 0, \qquad \forall \eta \in H_0^1(0, L)$$

$$\Longrightarrow - \theta'' - t^{-2}(w' - \theta) = 0, \quad \text{in } (0, L).$$

Second, we take $\eta = 0$ in (2.16), we obtain

$$t^{-2} \int_0^L (w' - \theta)(v) \, dx = \int_0^L f v \, dx, \qquad \forall v \in H_0^1(0, L)$$

$$\Longrightarrow - t^{-2} \int_0^L (w' - \theta)' v \, dx = \int_0^L f v \, dx, \qquad \forall v \in H_0^1(0, L)$$

$$\Longrightarrow \int_0^L \Big(- t^{-2}(w' - \theta)' - f \Big) v \, dx = 0, \qquad \forall v \in H_0^1(0, L)$$

$$\Longrightarrow - t^{-2}(w' - \theta)' = f, \quad \text{in } (0, L).$$

■

2.1.4 Regularity

Theorem 2.4

If $f \in L^2(0, L)$, then $(\theta, w) \in (H^2(0, L))^2$.

Proof: The two first equations of (2.15), and $(\theta, w) \in V$ implies that

$$-\theta'' + t^{-2}\theta \in L^2(0, L) \quad \text{and} \quad w'' \in L^2(0, L)$$

Hence by the theory of elliptic regularity, we deduce that $(\theta, w) \in (H^2(0, L))^2$. ■

2.1.5 Asymptotic Analysis

For the Euler-Bernoulli model, the deflection w is independent of the thickness t. By contrast, the solution of the Timoshenko model depends in a complex way on the thickness.

To understand the asymptotic behaviour as t tends to 0, we need first to understand the properties of the solution of the limit problem

$$\begin{cases} \text{Find } (\theta_t, w_t) \in V, \\ a(\theta_t, \eta) + t^{-2} b((\theta_t, w_t); (\eta, v)) = (f, v), \quad \forall (\eta, v) \in V, \end{cases} \tag{2.17}$$

where

$$a(\theta_t, \eta) = \int_0^L \theta_t' \, \eta' \; dx, \qquad b((\theta_t, w_t); (\eta, v)) = \int_0^L (w_t' - \theta_t)(v' - \eta) \; dx$$

$$\boldsymbol{a}((\theta_t, w_t); (\eta, v)) = a(\theta_t, \eta) + t^{-2}((\theta_t, w_t); (\eta, v))$$

Let $B : V \to L^2(0, L)$ be a linear continuous mapping such that $(Bu, Bv) = b(u, v)$, so

$$B(\eta, v) = v' - \eta$$

and its kernel

$$\ker B = \{(\eta, v) \in V \quad | \quad v' = \eta\}$$

Lemma 2.2

The kernel of B, is infinite dimensional.

Proof: For an arbitrary function $g \in H_0^1(0, L)$, we define

$$\theta(x) = g(x) - \frac{6}{L^3} x(L - x) \int_0^L g(\xi) \; d\xi$$
$$w(x) = \int_0^x \theta(\xi) \; d\xi$$

It is clear that $\theta \in H_0^1(0, L)$ and $w \in \{H^1(0, L), w(0) = 0\}$, we need to prove that $w(L) = 0$.
Indeed ,

$$\begin{aligned} w(L) &= \int_0^L \theta(\xi) \; d\xi \\ &= \int_0^L \left(g(\xi) - \frac{6}{L^3} \xi(L - \xi) \int_0^L g(\zeta) d\zeta \right) \; d\xi \\ &= \int_0^L g(\xi) \; d\xi - \int_0^L g(\xi) d\xi \times \frac{6}{L^3} \int_0^L \xi(L - \xi) \; d\xi \\ &= \int_0^L g(\xi) \; d\xi - \int_0^L g(\xi) \; d\xi \times \frac{6}{L^3} \left[\frac{L\xi^2}{2} - \frac{\xi^3}{3} \right]_{\xi=0}^L \\ &= \int_0^L g(\xi) \; d\xi - \int_0^L g(\xi) \; d\xi \times \frac{6}{L^3} \left(\frac{L^3}{2} - \frac{L^3}{3} \right) \\ &= 0. \end{aligned}$$

$\theta \in H_0^1(0, L)$ and $w \in H_0^1(0, L)$, $w' - \theta = 0$, i.e., $B(\theta, w) = 0$ and since g is arbitrary function in the infinite dimensional space $H_0^1(0, L)$, $\ker B$ is also infinite dimensional. ■

Propostion 2.1

The solution (θ_t, w_t) of (2.17) converges strongly as t tends to 0 to the solution of the following problem:

$$\begin{cases} \text{Find } (\theta_0, w_0) \in \ker B, \text{such that} \\ a(\theta_0, \eta) = (f, v), \quad \forall (\eta, v) \in \ker B \end{cases} \tag{2.18}$$

Proof: The proof is divided into several steps:

(i) $\exists C > 0$, $\|(\theta_t, w_t)\|_V < C$, $\forall t > 0$.

(ii) $\exists (\theta_0, w_0) \in \ker B$, $\exists ((\theta_{t'}, w_{t'})) \subset ((\theta_t, w_t))$ s.t. $(\theta_{t'}, w_{t'}) \rightharpoonup (\theta_0, w_0)$ as $t' \to 0$.

(iii) (θ_0, w_0) is solution of the problem (2.18).

(iv) $\|(\theta_{t'}, w_{t'}) - (\theta_0, w_0)\|_V \to 0$, as $t' \to 0$.

(v) $\|(\theta_t, w_t) - (\theta_0, w_0)\|_V \to 0$, as $t \to 0$.

Proof of (i):

We take $(\eta, v) = (\theta_t, w_t)$ in (2.17). With the coercivity of the form $\boldsymbol{a}$ and the inequalities of Cauchy-Schwarz and Poincaré, we obtain

$$\frac{1}{3}\|(\theta_t, w_t)\|_V^2 \leq \boldsymbol{a}((\theta_t, w_t); (\theta_t, w_t)) = (f, w_t) \leq \|f\|_0 \|(\theta_t, w_t)\|_V$$

Then

$$\|(\theta_t, w_t)\|_V \leq 3\|f\|_0 = C.$$

Proof of (ii):

(i) and V is Hilbert space i.e., V is a reflexive space $\Longrightarrow$ $\exists (\theta_0, w_0) \in V$, $\exists ((\theta_{t'}, w_{t'})) \subset ((\theta_t, w_t))$ s.t. $(\theta_{t'}, w_{t'}) \rightharpoonup (\theta_0, w_0)$ in V, as $t' \to 0$. It remains to show that $(\theta_0, w_0) \in \ker B$.

We have

$$(\theta_{t'}, w_{t'}) \rightharpoonup (\theta_0, w_0) \quad \text{in } V$$

So

$$\begin{aligned} (\theta_{t'}, w_{t'}) &\rightharpoonup (\theta_0, w_0) \quad \text{in } (L^2(0, L))^2 \\ (\theta'_{t'}, w'_{t'}) &\rightharpoonup (\theta'_0, w'_0) \quad \text{in } (L^2(0, L))^2 \end{aligned}$$

Then

$$w'_{t'} - \theta_{t'} \rightharpoonup w'_0 - \theta_0 \quad \text{in } (L^2(0, L))^2$$

Now, taking $(\eta, v) = (\theta_{t'}, w_{t'})$ in (2.17), we get

$$a(\theta_{t'}, \theta_{t'}) + t'^{-2} b((\theta_{t'}, w_{t'}); (\theta_{t'}, w_{t'})) = (f, w_{t'})$$

i.e.,

$$\begin{aligned} b((\theta_{t'}, w_{t'}); (\theta_{t'}, w_{t'})) &= t'^2 (f, w_{t'}) - t'^2 a(\theta_{t'}, \theta_{t'}) \\ &\leq t'^2 (f, w_{t'}) \end{aligned}$$

In other hand,

$$\begin{aligned} b((\theta_0, w_0); (\theta_0, w_0)) &= \|w_0' - \theta_0\|_0^2 \\ &\leq \liminf_{t' \to 0} \left(\|w_{t'}' - \theta_{t'}\|_0^2 \right) \\ &= \liminf_{t' \to 0} \left(b((\theta_{t'}, w_{t'}); (\theta_{t'}, w_{t'})) \right) \\ &\leq \liminf_{t' \to 0} \left(t'^2 (f, w_{t'}) \right) \\ &= 0 \end{aligned}$$

Thus $b((\theta_0, w_0); (\theta_0, w_0)) = 0 \implies B(\theta_0, w_0) = 0$. Hence, $(\theta_0, w_0) \in \ker B$.

Proof of (iii):

Take $(\eta, v) \in \ker B$ in (2.17), i.e., $v' = \eta$, then we deduce

$$a(\theta_{t'}, \eta) = (f, v), \quad \forall (\eta, v) \in \ker B$$

Thus

$$a(\theta_0, \eta) = (f, v), \quad \forall (\eta, v) \in \ker B$$

Proof of (iv):

From the coercivity of the form $\boldsymbol{a}$

$$\begin{aligned} \frac{1}{3}\|(\theta_{t'}, w_{t'}) - (\theta_0, w_0)\|_V^2 &\leq \boldsymbol{a}((\theta_{t'} - \theta_0, w_{t'} - w_0); (\theta_{t'} - \theta_0, w_{t'} - w_0)) \\ &= \boldsymbol{a}((\theta_{t'}, w_{t'}); (\theta_{t'} - \theta_0, w_{t'} - w_0)) - \boldsymbol{a}((\theta_0, w_0); (\theta_{t'} - \theta_0, w_{t'} - w_0)) \\ &= (f, w_{t'} - w_0) - \boldsymbol{a}((\theta_0, w_0); (\theta_{t'} - \theta_0, w_{t'} - w_0)) \\ &= (f, w_{t'} - w_0) - a(\theta_0, \theta_{t'} - \theta_0) \to 0, \quad \text{as} \quad t' \to 0 \end{aligned}$$

Thus

$$\|(\theta_{t'}, w_{t'}) - (\theta_0, w_0)\|_V \to 0, \quad \text{as} \quad t' \to 0$$

Proof of (v):

We can easily prove that the problem (2.18) admits a unique solution (θ_0, w_0), hence $\|(\theta_t, w_t) - (\theta_0, w_0)\|_V \to 0$, as $t \to 0$.

■

Propostion 2.2

The solution w_0 verify the Euler-Bernoulli variational formulation

$$\begin{cases} \text{Find } w_0 \in H_0^2(0, L), \text{such that} \\ \displaystyle\int_0^L w_0'' v'' dx = \int_0^L f v dx, \quad \forall v \in H_0^2(0, L) \end{cases} \tag{2.19}$$

Proof: From (2.18), we can take $\theta_0 = w_0'$ and $\eta = v'$, we deduce

$$\int_0^L w_0'' v'' dx = \int_0^L f v dx$$

Since $(\theta_0, w_0) \in (H_0^1(0,L))^2$, $w_0 \in H^2(0,L)$ and $w_0' = \theta_0$, we deduce that $w_0 \in H_0^2(0,L)$. ■

Propostion 2.3

The solution w_0 achieves the Euler-Bernoulli problem

$$\begin{cases} w_0^{(4)} = f \\ w_0(0) = w_0'(0) = 0 \\ w_0(L) = w_0'(L) = 0 \end{cases}$$

Proof: We shall develop asymptotic expansions with respect to t for θ and w. The expansions take the form

$$\theta = \theta_0 + t\theta_1 + t^2\theta_2 + t^3\theta_3 + ...$$
$$w = w_0 + tw_1 + t^2 w_2 + t^3 w_3 + ...$$

Inserting the expansion of θ and w in the equations of (2.15) and equating like powers of t gives the equations:

$$(\text{For } i = 0) \begin{cases} \theta_0 - w_0' = 0, \\ \theta_0' - w_0'' = 0, \end{cases}$$

$$(\text{For } i = 1) \begin{cases} \theta_1 - w_1' = 0, \\ \theta_1' - w_1'' = 0, \end{cases}$$

$$(\text{For } i = 2) \begin{cases} \theta_2 - w_2' = \theta_0'', \\ \theta_2' - w_2'' = f, \end{cases}$$

$$(\text{For } i = 3) \begin{cases} \theta_3 - w_3' = \theta_1'', \\ \theta_3' - w_3'' = 0, \end{cases}$$

$$\vdots$$

which can be written

$$(\text{For } i = 0, 1, ...) \begin{cases} \theta_i - w_i' = \theta_{i-2}'', \\ \theta_i' - w_i'' = \delta_{i2} f, \end{cases}$$

In particular , for $i = 2$

$$\begin{cases} \theta_2 - w_2' = \theta_0'', \\ \theta_2' - w_2'' = f, \end{cases}$$

it means

$$\begin{cases} \theta_2 - w_2' = \theta_0'', \\ (\theta_2 - w_2')' = f, \end{cases}$$

so

$$\theta_0^{(3)} = f \implies w_0^{(4)} = f$$

Since $(\theta_0, w_0) \in V$, $w_0 \in (H^2(0,L))$) and $w_0' = \theta_0$, we deduce that $w'(0) = w'(L) = 0$, ■

2.1.6 Locking phenomena

Let assume that the interval $(0, L)$ is divided into subintervals of length h and that $\eta_h, v_h \in V_h^1$, we consider the discrete problem

$$\begin{cases} \text{Find } (\theta_t^h, w_t^h) \in V_h^1 \times V_h^1 \text{ such that} \\ a(\theta_t^h, \eta_h) + t^{-2} b((\theta_t^h, w_t^h), (\eta_h, v_h)) = \langle f, v_h \rangle, \quad \forall (\eta_h, v_h) \in V_h^1 \times V_h^1 \end{cases} \tag{2.20}$$

and

$$B(\eta_h, v_h) = v_h' - \eta_h$$

Suppose there exists $(\theta_0, w_0) \in \ker B$ with

$$d := \langle f, w_0 \rangle > 0, \quad a\left((\theta_0, w_0), (\theta_0, w_0)\right) = a\left((\theta_0, \theta_0)\right) \leq \langle f, w_0 \rangle \tag{2.21}$$

In particular, the energy of the minimal solution satisfies

$$J\left((\theta_t, w_t)\right) \leq J\left((\theta_0, w_0)\right) = \frac{1}{2} a\left((\theta_0, \theta_0)\right) - \langle f, w_0 \rangle \leq -\frac{1}{2} d \tag{2.22}$$

with a bound that is independent of t. Thus,

$$\frac{1}{2} d \leq -J\left((\theta_t, w_t)\right) \leq \langle f, w_t \rangle \leq \|f\|_0 \|(\theta_t, w_t)\|_V$$

and so for all $t > 0$

$$\frac{1}{2} d \leq \|f\|_0 \|(\theta_t, w_t)\|_V \iff \|(\theta_t, w_t)\|_V \geq \frac{d}{2} \|f\|_0^{-1} > 0 \tag{2.23}$$

Now, note that the inverse inequality claims that:

$$\|\phi_h\|_1 \leq C h^{-1} \|\phi_h\|_0, \quad \phi_h \in \mathbb{P}_k$$

So

$$\int_{x_i}^{x_i+h} (\alpha x + \beta)^2 \, dx \geq Ch^2 \int_{x_i}^{x_i+h} \alpha^2 \, dx$$

whenever $\alpha, \beta \in \mathbb{R}$. From this inequality, it follows that on each subinterval of the partition

$$\int_{x_i}^{x_i+h} (v_h' - \eta_h)^2 \, dx \geq Ch^2 \int_{x_i}^{x_i+h} (\eta_h')^2 \, dx$$

After summing over all subintervals we get

$$\|v_h' - \eta_h\|_0 \geq Ch\|\eta_h\|_1 \tag{2.24}$$

The triangle inequality and Poincaré's inequality yield

$$\|v_h\|_1 \leq \|v_h' - \eta_h\|_0 + \|\eta_h\|_0 \leq \|v_h' - \eta_h\|_0 + \|\eta_h\|_1 \tag{2.25}$$

then,

$$h\|v_h\|_1 \leq \|v_h' - \eta_h\|_0 + h\|\eta_h\|_1 \leq C\|v_h' - \eta_h\|_0 \tag{2.26}$$

So, by (2.24) we get

$$\|B(\eta_h, v_h)\|_0 = \|v_h' - \eta_h\|_0 \geq C \; h \left(\|v_h\|_1 + \|\eta_h\|_1\right) \tag{2.27}$$

Hence

$$\|B(\eta_h, v_h)\|_0 \geq C \; h\|(\eta_h, v_h)\|_V \tag{2.28}$$

We have also (cf. part (i) of the proof of Proposition 2.1)

$$\|(\theta_t^h, w_t^h)\|_V \leq C'. \tag{2.29}$$

Taking $(\eta_h, v_h) = (\theta_t^h, w_t^h)$ in (2.20), Cauchy-Schwarz inequality yields

$$t^{-2}\|B(\theta_t^h, w_t^h)\|_0^2 \leq \|f\|_0 \, \|(\theta_t^h, w_t^h)\|_V \tag{2.30}$$

it means

$$\|B(\theta_t^h, w_t^h)\|_0^2 \leq t^2\|f\|_0 \, \|(\theta_t^h, w_t^h)\|_V \tag{2.31}$$

by using (2.28) and (2.29) , we deduce

$$\|(\theta_t^h, w_t^h)\|_V \leq Cte\frac{t}{h} \tag{2.32}$$

For h fixed and t tends to 0 ,

$$\lim_{t \to 0} \|(\theta_t^h, w_t^h)\|_V = 0$$

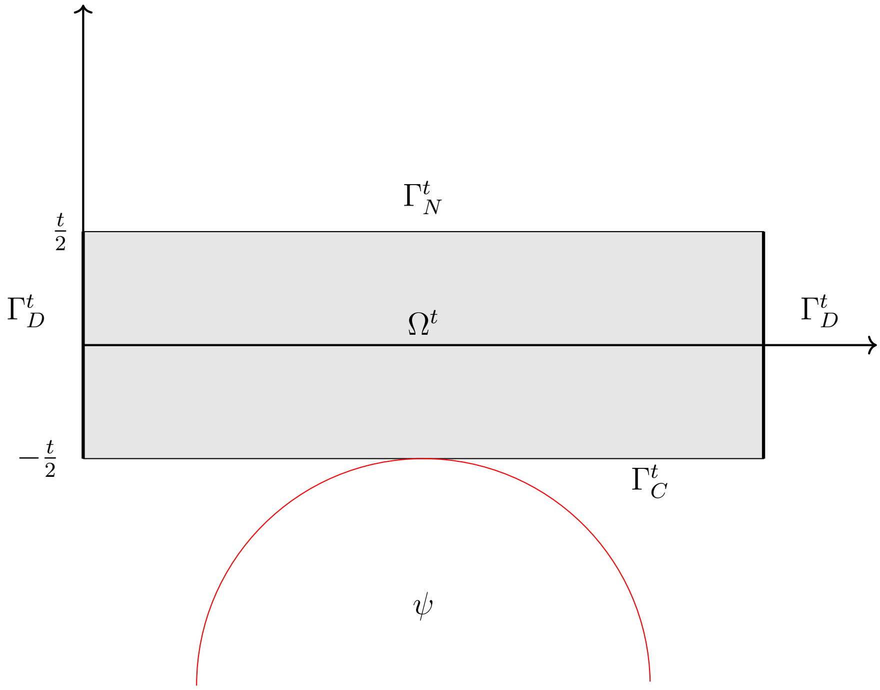
However,

$$\lim_{t \to 0} \|(\theta_t, w_t)\|_V \neq 0$$

As a result , we come to the conclusion that there is a locking phenomena.

2.2 In the presence of the obstacle

2.2.1 Derivation of the model



In this case, The Timoshenko model may also be derived from the classical two-dimensional linear elasticity problem

$$\begin{cases} -\partial_\beta \sigma_{\alpha\beta}(u^t) = t^2 f_\alpha, & \text{in } \Omega^t \\ u^t = 0, & \text{on } \Gamma_D^t \\ \sigma_{\alpha\beta}(u^t) n_\beta = 0, & \text{on } \Gamma_N^t \\ u_2^t \geq \psi, \quad \sigma_{22}(u^t) n_2 \geq 0, \quad \sigma_{22}(u^t) n_2 (u_2^t - \psi) = 0, & \text{on } \Gamma_C^t \end{cases} \tag{2.33}$$

We introduce the set

$$K^t = \{v \in V^t \quad | \quad v_2 \geq \psi \text{ on } \Gamma_C^t\}$$

Theorem 2.5

The variational formulation reads

$$\begin{cases} \text{Find } u^t \in K^t \quad \text{s.t.} \quad \forall v \in K^t \\ \int_{\Omega^t} \left(2\mu e_{\alpha\beta}(u^t) : e_{\alpha\beta}(v - u^t) + \lambda \text{div}(u^t)\text{div}(v - u^t)\right) dx \geq t^2 \int_{\Omega^t} f_2(v_2 - u_2^t)\, dx \end{cases} \tag{2.34}$$

Proof: Let $v \in K^t$. Then $v_2 \geq \psi$ on $\Gamma_C^t \implies v_2 - \psi \geq 0$ on Γ_C^t.

We have

$$\begin{cases} \sigma_{22}(u^t) n_2 (v_2 - \psi) \geq 0, & \text{on} \quad \Gamma_C^t \\ \sigma_{22}(u^t) n_2 (u_2^t - \psi) = 0, & \text{on} \quad \Gamma_C^t \end{cases} \tag{2.35}$$

by subtraction we obtain

$$\sigma_{22}(u^t) n_2 (v_2 - u_2^t) \geq 0, \quad \text{on} \quad \Gamma_C^t$$

then

$$\int_{\Gamma_C^t} \sigma_{22}(u^t) n_2 (v_2 - u_2^t) \geq 0 \tag{2.36}$$

Now, we multiply each side of the first equation of (2.33) by $(v - u^t)$ and integrate on Ω^t to obtain:

$$-\int_{\Omega^t} (\partial_\beta \sigma_{\alpha\beta}(u^t))(v - u^t)\ dx = t^2 \int_{\Omega^t} f_\alpha (v - u^t)\ dx$$

Using Theorem (1.6) and the boundary conditions in (2.33), we deduce

$$\int_{\Omega^t} \sigma_{\alpha\beta}(u^t) : e_{\alpha\beta}(v - u^t)\ dx = t^2 \int_{\Omega^t} f_2 (v_2 - u_2^t)\ dx + \int_{\Gamma_C^t} \sigma_{22}(u^t) n_2 (v_2 - u_2^t)$$

From (2.36), we obtain

$$\int_{\Omega^t} \Big(\big(2\mu e_{\alpha\beta}(u^t) + \lambda e_{\rho\rho}(u^t)\delta_{\alpha\beta}\big) : e_{\alpha\beta}(v - u^t) \Big)\ dx \geq t^2 \int_{\Omega^t} f_2 (v_2 - u_2^t)\ dx$$

$$\int_{\Omega^t} \Big(2\mu e_{\alpha\beta}(u^t) : e_{\alpha\beta}(v - u^t) + \lambda e_{\rho\rho}(u^t)\delta_{\alpha\beta} : e_{\alpha\beta}(v - u^t) \Big)\ dx \geq t^2 \int_{\Omega^t} f_2 (v_2 - u_2^t)\ dx$$

We have proven

$$e_{\rho\rho}(u^t)\delta_{\alpha\beta} : e_{\alpha\beta}(v) = \text{div}(u^t)\text{div}(v)$$

Therefore

$$\int_{\Omega^t} \Big(2\mu e_{\alpha\beta}(u^t) : e_{\alpha\beta}(v - u^t) + \lambda \text{div}(u^t)\text{div}(v - u^t) \Big)\ dx \geq t^2 \int_{\Omega^t} f_2 (v_2 - u_2^t)\ dx$$

Hence

$$\begin{cases} \text{Find } u^t \in K^t \quad \text{s.t.} \quad \forall v \in K^t \\ \displaystyle\int_{\Omega^t} \Big(2\mu e_{\alpha\beta}(u^t) : e_{\alpha\beta}(v - u^t) + \lambda \text{div}(u^t)\text{div}(v - u^t) \Big)\ dx \geq t^2 \int_{\Omega^t} f_2 (v_2 - u_2^t)\ dx \end{cases} \tag{2.37}$$

■

Note that, thanks to Stampacchia theorem, the problem (2.34) can be formulated as a minimization problem

$$\begin{cases} \text{Find } u^t \in K^t \\ u^t = \underset{v \in K^t}{\arg} \min J(v) \end{cases} \tag{2.38}$$

where

$$J(v) = \frac{1}{2} \int_{\Omega^t} \Big(2\mu e_{\alpha\beta}(v) : e_{\alpha\beta}(v) + \lambda(\mathrm{div}(v))^2 \Big) dx - t^2 \int_{\Omega^t} f_2 v_2 dx \tag{2.39}$$

Therefore, by adding Mindlin's assumptions given in the previous section, the contact problem of the Timoshenko beam takes the form:

$$\begin{cases} \text{Find } (\theta_t, w_t) \in K \\ \qquad (\theta_t, w_t) = \underset{(\eta,v) \in K}{\arg} \min J((\eta, v)) \end{cases} \tag{2.40}$$

where,

$$K := \{(\eta, v) \in V; v \geq \psi\}$$

and J is the same functional defined by (2.8).

For the same reason, the simplified Timoshenko problem consists of finding (θ, w) as the minimum solution for the following energy:

$$\begin{aligned} J((\eta, v)) &= \frac{1}{2} \left(\int_0^L (\eta')^2 \, dx + t^{-2} \int_0^L (v' - \eta)^2 \, dx \right) - (f, v) \\ &= \frac{1}{2} \boldsymbol{a}((\eta, v); (\eta, v)) - (f, v) \end{aligned}$$

Note that $J((\theta_t, w_t)) \leq J((\eta, v))$, $\forall (\eta, v) \in K$.

Since K is convex, $\rho(\eta, v) + (1-\rho)(\theta_t, w_t) \in K$, $\rho \in (0, 1)$. We put $U = (\theta_t, w_t)$ and $W = (\eta, v)$,

$$\begin{aligned} J(U) &\leq J(U + \rho(W - V)) \\ &\leq \frac{1}{2} \boldsymbol{a}(U + \rho(W-U); U + \rho(W-U)) - (f, w_t + \rho(v - w_t)) \\ &= \frac{1}{2} \boldsymbol{a}(U;U) + \rho \boldsymbol{a}(U; W-U) + \frac{\rho^2}{2} \boldsymbol{a}(W-U; W-U) - (f, w_t + \rho(v - w_t)) \\ &= J(U) + \rho \boldsymbol{a}(U; W-U) + \frac{\rho^2}{2} \boldsymbol{a}(W-U; W-U) - \rho (f, v - w_t) \end{aligned}$$

$\Longrightarrow \rho \boldsymbol{a}(U;W-U)+\frac{\rho^2}{2}\boldsymbol{a}(W-U;W-U)-\rho\,(f,v-w_t)\geq 0$,
Since the inequality is true for all $\rho \in (0,1)$, we can divide by ρ, tending $\rho \to 0$, we find

$$\boldsymbol{a}(U;W-U) \geq (f, v-w)$$
$$\boldsymbol{a}((\theta_t,w_t);(\eta-\theta,v-w)) \geq (f,v-w_t)$$

Hence, the varitional inequality reads

$$\begin{cases} \text{Find } (\theta_t,w_t)\in K \\ \boldsymbol{a}((\theta_t,w_t);(\eta-\theta_t,v-w_t)) \geq (f,v-w_t), \quad \forall(\eta,v)\in K \end{cases} \tag{2.41}$$

where

$$\boldsymbol{a}((\theta,w);(\eta,v)) = \int_0^L \theta'\eta'\;dx + t^{-2}\int_0^L (w'-\theta)(v'-\eta)\;dx$$

2.2.2 Existence and uniqueness

Consider the problem

$$\begin{cases} \text{Find } (\theta,w)\in K \\ \boldsymbol{a}((\theta,w);(\eta-\theta,v-w)) \geq L((\eta-\theta,v-w)),\;\; \forall(\eta,v)\in K \end{cases} \tag{2.42}$$

where

$$\boldsymbol{a}((\theta,w);(\eta,v)) = a((\theta,w);(\eta,v)) + t^{-2}b((\theta,w);(\eta,v)), \quad L((\eta,v)) = (f,v)$$
$$a((\theta,w);(\eta,v)) = a(\theta,\eta) = \int_0^L \theta'\eta'\;dx, \qquad b((\theta,w);(\eta,v)) = \int_0^L (w'-\theta)(v'-\eta)\;dx$$

Propostion 2.4

The set K is convex closed in V.

Proof:

$$K = \{(\eta,v)\in V; v\geq\psi\}$$

(i) To prove that K is convex, it is necessary and sufficient to show that:

$$\begin{aligned}(\eta_1,v_1),(\eta_2,v_2)\in K &\Longrightarrow \rho(\eta_1,v_1)+(1-\rho)(\eta_2,v_2)\in K, && \forall\rho\in(0,1)\\ &\Longleftrightarrow (\rho\eta_1+(1-\rho)\eta_2,\rho v_1+(1-\rho)v_2)\in K, && \forall\rho\in(0,1)\\ &\Longleftrightarrow \rho v_1+(1-\rho)v_2\geq\psi, && \forall\rho\in(0,1)\end{aligned}$$

Indeed, let $\rho\in(0,1)$,

$$\begin{cases}(\eta_1,v_1)\in K\\(\eta_2,v_2)\in K\end{cases} \Longrightarrow \begin{cases}v_1\geq\psi\\v_2\geq\psi\end{cases} \Longrightarrow \begin{cases}\rho v_1\geq\rho\psi\\(1-\rho)v_2\geq(1-\rho)\psi\end{cases}$$

by the sum of the last two inequalities, we get

$$\rho v_1+(1-\rho)v_2 \geq (\rho+1-\rho)\psi = \psi$$

(ii) to prove that K is closed, we have to show that:

$$(\eta_n, v_n) \in K \text{ s.t. } \|(\eta_n, v_n) - (\eta, v)\|_V \to 0 \text{ as } n \mapsto \infty \implies (\eta, v) \in K$$

Indeed,

$$\begin{aligned} (\eta_n, v_n) \in K &\implies \psi \leq v_n = v_n - v + v \leq \|v_n - v\|_1 + v \quad \big) n \to \infty \\ &\implies v \geq \psi \\ &\implies (\eta, v) \in K \end{aligned}$$

■

Theorem 2.6

The problem (2.42) admits a unique solution (θ, w).

Proof: We apply Stampacchia theorem,

- K is convex closed set of a Hilbert space V.
- $\boldsymbol{a}$ is bilinear continuous coercive form (proved in the previous section).
- L is linear continuous form.

Therefore, the problem (2.42) admits a unique solution (θ, w). ■

2.2.3 Complementarity system

Theorem 2.7

The solution (θ, w) satisfies the complementarity system:

$$\begin{cases} -\theta'' - t^{-2}(w' - \theta) = 0, & \text{in } (0, L) \\ -t^{-2}(w' - \theta)' \geq f, & \text{in } (0, L) \\ w - \psi \geq 0, & \text{in } (0, L) \\ (-t^{-2}(w' - \theta)' - f)(w - \psi) = 0, & \text{in } (0, L) \\ \theta(0) = \theta(L) = 0 \\ w(0) = w(L) = 0 \end{cases} \tag{2.43}$$

Proof: We have $(\theta, w) \in K$, so $(\theta, w) \in V$.

1. We take $(\eta, v) = (\theta \pm \xi, w)$ in (2.42), with $\xi \in \mathcal{C}_0^\infty(0, L)$, we get

$$\int_0^L \theta'(\pm\xi)' dx - t^{-2} \int_0^L (w' - \theta)(\pm\xi) dx \geq 0, \qquad \forall \xi \in \mathcal{C}_0^\infty(0, L)$$

$$\int_0^L \theta'\xi' dx - t^{-2} \int_0^L (w' - \theta)(\xi) dx = 0, \qquad \forall \xi \in \mathcal{C}_0^\infty(0, L)$$

$$-\int_0^L \theta''\xi dx - t^{-2} \int_0^L (w' - \theta)(\xi) dx = 0, \qquad \forall \xi \in \mathcal{C}_0^\infty(0, L)$$

$$\int_0^L \big(-\theta'' - t^{-2}(w' - \theta)\big)(\xi) dx = 0, \qquad \forall \xi \in \mathcal{C}_0^\infty(0, L)$$

$$-\theta'' - t^{-2}(w' - \theta) = 0, \quad \text{in } (0, L).$$

2. Now, we take $(\eta, v) = (\theta, w + \varphi)$ in (2.42), with $\varphi \in \mathcal{C}_0^\infty(0, L), \varphi \geq 0$, we get

$$
\begin{aligned}
t^{-2}\int_0^L (w' - \theta)(\varphi')dx &\geq (f, \varphi) && \forall \varphi \in \mathcal{C}_0^\infty(0, L)\\
-t^{-2}\int_0^L (w' - \theta)'(\varphi)dx &\geq (f, \varphi), && \forall \varphi \in \mathcal{C}_0^\infty(0, L)\\
\int_0^L \big(-t^{-2}(w' - \theta)' - f\big)(\varphi)dx &\geq 0, && \forall \varphi \in \mathcal{C}_0^\infty(0, L)\\
-t^{-2}(w' - \theta)' \geq f, \quad \text{in } (0, L).
\end{aligned}
$$

3. Since $(\theta, w) \in K$; $w \geq \psi$ i.e., $w - \psi \geq 0$.
4. We suppose that $(0, L) = N \cup C$ such that

$$
\begin{aligned}
N(\theta, w) &= \{x \in (0, L); w(x) > \psi(x)\}\\
C(\theta, w) &= \{x \in (0, L); w(x) = \psi(x)\}
\end{aligned}
$$

(i) If $x \in C$, then $w - \psi = 0$, so $(-t^{-2}(w' - \theta)' - f)(w - \psi) = 0$ in C.

(ii) If $x \in N$, we take $(\eta, v) = (\theta, w \pm \varepsilon\varphi)$ in (2.42); $\varphi \in \mathcal{C}_0^\infty(0, L), \varphi \geq 0$, we obtain

$$t^{-2}\int_0^L (w' - \theta)(\pm\varepsilon\varphi')dx \geq (f, \pm\varepsilon\varphi), \qquad \forall \varphi \in \mathcal{C}_0^\infty(0, L)$$

$$\Longrightarrow t^{-2}\int_0^L (w' - \theta)(\varphi')dx = (f, \varphi), \qquad \forall \varphi \in \mathcal{C}_0^\infty(0, L)$$

$$\Longrightarrow -t^{-2}\int_0^L (w' - \theta)'(\varphi)dx = (f, \varphi), \qquad \forall \varphi \in \mathcal{C}_0^\infty(0, L)$$

$$\Longrightarrow \int_0^L \big(-t^{-2}(w' - \theta)' - f\big)(\varphi)dx = 0, \qquad \forall \varphi \in \mathcal{C}_0^\infty(0, L)$$

$$\Longrightarrow -t^{-2}(w' - \theta)' - f = 0.$$

So $(-t^{-2}(w' - \theta)' - f)(w - \psi) = 0$, in N.

$\therefore (-t^{-2}(w' - \theta)' - f)(w - \psi) = 0, \quad \text{in } (0, L)$. ∎

2.2.4 Regularity

Theorem 2.8

If $f \in L^2(0, L)$, then $(\theta, w) \in (H^2(0, L))^2$.

Proof: The first equation in the system (2.43) and the fact that θ and w are a priori in $H^1(0, L)$ imply that

$$-\theta'' + t^{-2}\theta \in L^2(0, L)$$

then by the elliptic regularity we have $\theta \in H^2(0, L)$.

Now the complementarity system imply that

$$f \leq -t^{-2}(w' - \theta)' \leq f\chi_{\{w>\psi\}}$$

hence, $w'' - \theta' \in L^2(0, L)$, this implies that $w'' \in L^2(0, L)$, again by the elliptic regularity $w \in H^2(0, L)$.

■

2.2.5 Asymptotic Analysis

$$\begin{cases} \text{Find } (\theta_t, w_t) \in K \\ \boldsymbol{a}((\theta_t, w_t); (\eta - \theta_t, v - w_t)) \geq L((\eta - \theta_t, v - w_t)), \ \ \forall (\eta, v) \in K \end{cases} \tag{2.44}$$

where

$$\boldsymbol{a}((\theta, w); (\eta, v)) = a((\theta, w); (\eta, v)) + t^{-2} b((\theta, w); (\eta, v)), \quad L((\eta, v)) = (f, v)$$

$$a((\theta, w); (\eta, v)) = a(\theta, \eta) = \int_0^L \theta' \eta' \ dx, \qquad b((\theta, w); (\eta, v)) = \int_0^L (w' - \theta)(v' - \eta) \ dx$$

Propostion 2.5

The solution (θ_t, w_t) of (2.17) converges strongly as t goes to 0 to the solution of the following problem:

$$\begin{cases} \text{Find } (\theta_0, w_0) \in K_0 = K \cap \ker B \quad \text{such that} \\ a(\theta_0, \eta - \theta_0) \geq (f, v - w_0), \quad \forall (\eta, v) \in K_0 \end{cases} \tag{2.45}$$

Proof: The proof is divided into several steps:

(i) $\exists C > 0$, $\|(\theta_t, w_t)\|_V < C$, $\forall t > 0$.

(ii) $\exists (\theta_0, w_0) \in K_0$, $\exists ((\theta_{t'}, w_{t'})) \subset ((\theta_t, w_t))$ s.t. $(\theta_{t'}, w_{t'}) \rightharpoonup (\theta_0, w_0)$ as $t' \to 0$.

(iii) (θ_0, w_0) is solution of the problem (2.45).

(iv) $\|(\theta_{t'}, w_{t'}) - (\theta_0, w_0)\|_V \to 0$, as $t' \to 0$.

(v) $\|(\theta_t, w_t) - (\theta_0, w_0)\|_V \to 0$, as $t \to 0$.

Proof of (i):

Let (η_0, v_0) a fixed element of K_0. Then

$$\boldsymbol{a}((\theta_t, w_t); (\eta_0 - \theta_t, v_0 - w_t)) \geq L((\eta_0 - \theta_t, v_0 - w_t))$$

Thus

$$\boldsymbol{a}((\theta_t - \eta_0, w_t - v_0); (\eta_0 - \theta_t, v_0 - w_t)) + \boldsymbol{a}((\eta_0, v_0); (\eta_0 - \theta_t, v_0 - w_t)) \geq L((\eta_0 - \theta_t, v_0 - w_t))$$

Hence

$$\begin{aligned}\boldsymbol{a}((\theta_t - \eta_0, w_t - v_0); (\theta_t - \eta_0, w_t - v_0)) &\leq L((\theta_t - \eta_0, w_t - v_0)) - \boldsymbol{a}((\eta_0, v_0); (\theta_t - \eta_0, w_t - v_0)) \\ &\leq L((\theta_t - \eta_0, w_t - v_0)) - a((\eta_0, v_0); (\theta_t - \eta_0, w_t - v_0))\end{aligned}$$

The coercivity of the form $\boldsymbol{a}$, the continuity of the forms a and L yield

$$\frac{1}{3}\|(\theta_t, w_t) - (\eta_0, v_0)\|_V^2 \leq C_1\|(\theta_t, w_t) - (\eta_0, v_0)\|_V + C_2\|(\eta_0, v_0)\|_V\|(\theta_t, w_t) - (\eta_0, v_0)\|_V$$

Which leads to

$$\|(\theta_t, w_t) - (\eta_0, v_0)\|_V \leq C. \tag{2.46}$$

Proof of (ii):

(i) and V is Hilbert space i.e., V is a reflexive space $\Longrightarrow$ $\exists(\theta_0, w_0) \in V$, $\exists((\theta_{t'}, w_{t'})) \subset ((\theta_t, w_t))$ s.t. $(\theta_{t'}, w_{t'}) \rightharpoonup (\theta_0, w_0)$ in V, as $t' \to 0$. It remains to show that $(\theta_0, w_0) \in K_0$.

We have

$$(\theta_{t'}, w_{t'}) \rightharpoonup (\theta_0, w_0) \quad \text{in } V$$

So

$$\begin{aligned}(\theta_{t'}, w_{t'}) &\rightharpoonup (\theta_0, w_0) \quad \text{in } (L^2(0,L))^2 \\ (\theta'_{t'}, w'_{t'}) &\rightharpoonup (\theta'_0, w'_0) \quad \text{in } (L^2(0,L))^2\end{aligned}$$

Then

$$w'_{t'} - \theta_{t'} \rightharpoonup w'_0 - \theta_0 \quad \text{in } (L^2(0,L))^2$$

It is easy to prove if $(\theta_{t'}, w_{t'}) \in K$, then $(\theta_0, w_0) \in K$.

Now, we have

$$\boldsymbol{a}((\theta_{t'}, w_{t'}); (\eta_0 - \theta_{t'}, v_0 - w_{t'})) \geq L(\eta_0 - \theta_{t'}, v_0 - w_{t'}) \tag{2.47}$$

i.e.,

$$a((\theta_{t'}, w_{t'}); (\eta_0 - \theta_{t'}, v_0 - w_{t'})) + t'^{-2}b((\theta_{t'}, w_{t'}); (\eta_0 - \theta_{t'}, v_0 - w_{t'})) \geq L((\eta_0 - \theta_{t'}, v_0 - w_{t'}))$$

so

$$a((\theta_{t'}, w_{t'}); (\eta_0 - \theta_{t'}, v_0 - w_{t'})) - t'^{-2}b((\theta_{t'}, w_{t'}); (\theta_{t'}, w_{t'})) \geq L((\eta_0 - \theta_{t'}, v_0 - w_{t'}))$$

i.e.,

$$t'^{-2}b((\theta_{t'}, w_{t'}); (\theta_{t'}, w_{t'})) \leq -a((\theta_{t'}, w_{t'}); (\theta_{t'} - \eta_0, w_{t'} - v_0 -)) + L((\theta_{t'} - \eta_0, w_{t'} - v_0))$$

Using (2.46), the continuity of the forms a and L, we deduce

$$b((\theta_{t'}, w_{t'}); (\theta_{t'}, w_{t'})) \leq Ct'^2$$

In other hand,

$$\begin{aligned} b((\theta_0, w_0); (\theta_0, w_0)) &= \|w_0' - \theta_0\|_0^2 \\ &\leq \liminf_{t'\to 0} \left(\|w_{t'}' - \theta_{t'}\|_0^2\right) \\ &= \liminf_{t'\to 0} \left(b((\theta_{t'}, w_{t'}); (\theta_{t'}, w_{t'}))\right) \\ &\leq \liminf_{t'\to 0} \left(Ct'^2\right) \\ &= 0 \end{aligned}$$

Thus $b((\theta_0, w_0); (\theta_0, w_0)) = 0 \implies B(\theta_0, w_0) = 0$. Hence, $(\theta_0, w_0) \in \ker B$.

Therefore $(\theta_0, w_0) \in K_0$.

Proof of (iii):

Take $(\eta, v) \in K_0$ in (2.47), i.e., $v' = \eta$, then we deduce

$$a(\theta_{t'}, \eta - \theta_{t'}) \geq (f, v - w_{t'})$$

Hence

$$a(\theta_0, \eta - \theta_0) \geq (f, v - w_0), \quad \forall (\eta, v) \in K_0,$$

Proof of (iv):

From the coercivity of the form $\boldsymbol{a}$

$$\begin{aligned} \frac{1}{3}\|(\theta_{t'}, w_{t'}) - (\theta_0, w_0)\|_V^2 &\leq \boldsymbol{a}((\theta_{t'} - \theta_0, w_{t'} - w_0); (\theta_{t'} - \theta_0, w_{t'} - w_0)) \\ &= \boldsymbol{a}((\theta_{t'}, w_{t'}); (\theta_{t'} - \theta_0, w_{t'} - w_0)) - \boldsymbol{a}((\theta_0, w_0); (\theta_{t'} - \theta_0, w_{t'} - w_0)) \\ &\leq (f, w_{t'} - w_0) - \boldsymbol{a}((\theta_0, w_0); (\theta_{t'} - \theta_0, w_{t'} - w_0)) \\ &= (f, w_{t'} - w_0) - a(\theta_0, \theta_{t'} - \theta_0) \to 0, \quad \text{as} \quad t' \to 0 \end{aligned}$$

Thus

$$\|(\theta_{t'}, w_{t'}) - (\theta_0, w_0)\|_V \to 0, \quad \text{as} \quad t' \to 0$$

Proof of (v):

We can easily prove that the problem (2.45) admits a unique solution (θ_0, w_0), hence $\|(\theta_t, w_t) - (\theta_0, w_0)\|_V \to 0$, as $t \to 0$.

■

Propostion 2.6

The solution w_0 verify the the Euler-Bernoulli variational formulation

$$\begin{cases} \text{Find } w_0 \in \widetilde{K} = \left\{v \in H_0^2(0,L); v \geq \psi\right\}, \\ \displaystyle\int_0^L w_0''(v'' - w_0'')dx \geq \int_0^L f(v - w_0)dx, \quad \forall v \in \widetilde{K} \end{cases} \tag{2.48}$$

Proof: Taking $\theta_0 = w_0'$ and $\eta = v'$ in (2.45), we get

$$\int_0^L w_0''(v'' - w_0'')dx \geq \int_0^L f(v - w_0)\ dx$$

Since $(\theta_0, w_0) \in (H_0^1(0,L))^2$, $w_0 \in H^2(0,L)$ and $w_0' = \theta_0$, we deduce that $w_0 \in H_0^2(0,L)$. ■

Propostion 2.7

The solution w_0 verify the Euler-Bernoulli problem

$$\begin{cases} w_0^{(4)} = f \\ w_0 \geq \psi \\ w_0(0) = w_0'(0) = 0 \\ w_0(L) = w_0'(L) = 0 \end{cases} \tag{2.49}$$

Proof: We have form the system (2.43)

$$(-t^{-2}(w' - \theta)' - f)(w - \psi) = 0,$$

when $w - \psi \neq 0$ i.e., $w - \psi > 0$, we get

$$-t^{-2}(w' - \theta)' = f$$

We shall develop asymptotic expansions with respect to t for θ and w.
The expansions take the form

$$\theta = \theta_0 + t\theta_1 + t^2\theta_2 + t^3\theta_3 + ...$$
$$w = w_0 + tw_1 + t^2w_2 + t^3w_3 + ...$$

Inserting the expansion of θ and w in

$$-\theta'' - t^{-2}(w' - \theta) = 0$$
$$-t^{-2}(w' - \theta)' = f$$

and equating like powers of t gives the equations:

$$(\text{For } i = 0) \begin{cases} \theta_0 - w_0' = 0, \\ \theta_0' - w_0'' = 0, \end{cases}$$

$$(\text{For } i = 1) \begin{cases} \theta_1 - w_1' = 0, \\ \theta_1' - w_1'' = 0, \end{cases}$$

$$(\text{For } i = 2) \begin{cases} \theta_2 - w_2' = \theta_0'', \\ \theta_2' - w_2'' = f, \end{cases}$$

$$(\text{For } i = 3) \begin{cases} \theta_3 - w_3' = \theta_1'', \\ \theta_3' - w_3'' = 0, \end{cases}$$

$$\vdots$$

which can be written

$$(\text{For } i = 0, 1, ...) \begin{cases} \theta_i - w_i' = \theta_{i-2}'', \\ \theta_i' - w_i'' = \delta_{i2} f, \end{cases}$$

In particular , for $i = 2$

$$\begin{cases} \theta_2 - w_2' = \theta_0'', \\ \theta_2' - w_2'' = f, \end{cases}$$

it means

$$\begin{cases} \theta_2 - w_2' = \theta_0'', \\ (\theta_2 - w_2')' = f, \end{cases}$$

so

$$\theta_0^{(3)} = f \implies w_0^{(4)} = f$$

As $w \geq \psi$, we infer $w_0 \geq \psi$. Since $(\theta_0, w_0) \in (H_0^1(0, L))^2$, $w_0 \in (H^2(0, L))$ and $w_0' = \theta_0$, we deduce that $w'(0) = w'(L) = 0$. ■

2.2.6 Locking phenomena

Let assume that the interval $[0, L]$ is divided into sub-intervals of length h and that $(\eta_h, v_h) \in K_h = K \cap (V_h^1 \times V_h^1)$, we consider the discrete problem

$$\begin{cases} \text{Find } (\theta_t^h, w_t^h) \in K_h \text{ such that} \\ a(\theta_t^h, \eta_h - \theta_t^h) + t^{-2} b((\theta_t^h, w_t^h), (\eta_h - \theta_t^h, v_h -, w_t^h)) \geq (f, v_h - w_t^h), \quad \forall (\eta_h, v_h) \in K_h \end{cases} \tag{2.50}$$

and

$$B(\eta_h, v_h) = v_h' - \eta_h$$

Suppose there exist $(\theta_0, w_0) \in \ker B$ with

$$d := (f, w_0) > 0, \quad a\left((\theta_0, w_0), (\theta_0, w_0)\right) = a\left((\theta_0, \theta_0)\right) \leq (f, w_0) \tag{2.51}$$

In particular, the energy of the minimal solution satisfies

$$J\left((\theta_t, w_t)\right) \leq J\left((\theta_0, w_0)\right) = \frac{1}{2} a\left((\theta_0, \theta_0)\right) - (f, w_0) \leq -\frac{1}{2} d \tag{2.52}$$

with a bound that is independent of t. Thus,

$$\frac{1}{2} d \leq -J\left((\theta_t, w_t)\right) \leq (f, w_t) \leq \|f\|_0 \|(\theta_t, w_t)\|_V$$

and so for all $t > 0$

$$\frac{1}{2} d \leq \|f\|_0 \|(\theta_t, w_t)\|_V \iff \|(\theta_t, w_t)\|_V \geq \frac{d}{2} \|f\|_0^{-1} > 0 \tag{2.53}$$

Now , note that the inverse inequality claims that:

$$\|\phi_h\|_1 \leq C h^{-1} \|\phi_h\|_0, \quad \phi_h \in \mathbb{P}_k$$

so

$$\int_{x_i}^{x_i+h} (\alpha x + \beta)^2 \, dx \geq C h^2 \int_{x_i}^{x_i+h} \alpha^2 \, dx$$

whenever $\alpha, \beta \in \mathbb{R}$. From this inequality it follows that on each subinterval of the partition

$$\int_{x_i}^{x_i+h} (v_h' - \eta_h)^2 \, dx \geq C h^2 \int_{x_i}^{x_i+h} (\eta_h')^2 \, dx$$

After summing over all subintervals we get

$$\|v_h' - \eta_h\|_0 \geq C h \|\eta_h\|_1 \tag{2.54}$$

The triangle inequality and Poincaré's inequality yield

$$\|v_h\|_1 \leq \|v_h' - \eta_h\|_0 + \|\eta_h\|_0 \leq \|v_h' - \eta_h\|_0 + \|\eta_h\|_1 \tag{2.55}$$

then,

$$h\|v_h\|_1 \leq \|v_h' - \eta_h\|_0 + h\|\eta_h\|_1 \leq C \|v_h' - \eta_h\|_0 \tag{2.56}$$

So, by (2.54) we get

$$\|B(\eta_h, v_h)\|_0 = \|v_h' - \eta_h\|_0 \geq C\ h \left(\|v_h\|_1 + \|\eta_h\|_1\right) \tag{2.57}$$

Hence

$$\|B(\eta_h, v_h)\|_0 \geq C\ h \|(\eta_h, v_h)\|_V \tag{2.58}$$

We have also (cf. part(ii) of the proof of Proposition 2.5)

$$\|B(\theta_t^h, w_t^h)\|_V^2 \leq Ct^2. \tag{2.59}$$

by using (2.58), we deduce

$$\|(\theta_t^h, w_t^h)\|_V \leq C\frac{t}{h} \tag{2.60}$$

For h fixed and t tends to 0 ,

$$\lim_{t\to 0} \|(\theta_t^h, w_t^h)\|_V = 0$$

however,

$$\lim_{t\to 0} \|(\theta_t, w_t)\|_V \neq 0$$

As a result , we come to the conclusion that there is a locking phenomena.

2.3 Standard finite elemtent approximation

Let $k \geq 1$ and let $\mathcal{I}_h$ be a partition of $[0, L]$. An obvious discretization of (2.10) results from employing Galerkin's method with the subspace V_h^k of V.

2.3.1 Without obstacle

$$\begin{cases} \text{Find } (\theta_h, w_h) \in V_h^k \text{ such that} \\ a(\theta_h, \eta_h) + t^{-2} b((\theta_h, w_h), (\eta_h, v_h)) = (f, v_h), \quad \forall (\eta_h, v_h) \in V_h^k \end{cases} \tag{2.61}$$

It is easy to prove that for fixed t, θ_h and w_h converge to θ and w in both L^2 and H^1 at the approximation theoretic optimal rate with respect to h. Such convergence however is not uniform with respect to the thickness t. The following theorem gives the rates of convergence which hold both for constant t and uniform in t.

Theorem 2.9

Let $f \in H^{k-1}(0, L)$. Then there exists a constant C independent of $t \in (0, 1]$, and the mesh size h such that

$$\|\delta\theta\|_0 \leq \begin{cases} C \min(t^{-2}h^2, 1)\|f\|_0 & \text{if} \quad k = 1 \\ C \min(t^{-1}h^{k+1}, h^k)\|f\|_{k-1} & \text{if} \quad k > 1 \end{cases}$$

$$\|(\delta\theta)'\|_0 \leq C \min(t^{-1}h^k, h^{k-1})\|f\|_{k-1}$$

$$\|\delta w\|_0 \leq \begin{cases} C \min(t^{-2}h^2, 1)\|f\|_0 & \text{if} \quad k = 1 \\ C \min(t^{-1}h^{k+1}, h^k)\|f\|_{k-1} & \text{if} \quad k > 1 \end{cases}$$

$$\|(\delta w)'\|_0 \leq C \min(t^{-1}h^k, h^{k-1})\|f\|_{k-1}$$

where

$$\delta\theta = \theta - \theta_h, \delta w = w - w_h$$

Proof: See [1], p.409. ■

2.3.2 With obstacle

Let $K_h \subset V_h^k$.

$$\begin{cases} \text{Find } (\theta_h, w_h) \in K_h \text{ such that } \forall (\eta_h, v_h) \in K_h \\ a(\theta_h, \eta_h - \theta_h) + t^{-2} b((\theta_h, w_h), (\eta_h - \theta_h, v_h -, w_h)) \geq (f, v_h - w_h) \end{cases} \tag{2.62}$$

Even in the presence of the obstacle, the convergence of (θ_h, w_h) to (θ, w) is not uniform with respect to the parameter t.

Theorem 2.10

Let $f \in H^{k-1}(0, L)$. Then there exists a constant C independent of $t \in (0, 1]$, and the mesh size h such that

$$\|\delta\theta\|_0 \leq \begin{cases} C \min(t^{-2}h^2, 1)\|f\|_0 & \text{if} \quad k = 1 \\ C \min(t^{-1}h^{k+1}, h^k)\|f\|_{k-1} & \text{if} \quad k > 1 \end{cases}$$

$$\|(\delta\theta)'\|_0 \leq C \min(t^{-1}h^k, h^{k-1})\|f\|_{k-1}$$

$$\|\delta w\|_0 \leq \begin{cases} C \min(t^{-2}h^2, 1)\|f\|_0 & \text{if} \quad k = 1 \\ C \min(t^{-1}h^{k+1}, h^k)\|f\|_{k-1} & \text{if} \quad k > 1 \end{cases}$$

$$\|(\delta w)'\|_0 \leq C \min(t^{-1}h^k, h^{k-1})\|f\|_{k-1}$$

where

$$\delta\theta = \theta - \theta_h, \delta w = w - w_h$$

CHAPTER 3

MIXED FORMULATIONS

3.1 A mixed formulation in the presence of the obstacle

Theorem 3.1

Let $\boldsymbol{\mu}$ be the space defined by,

$$\boldsymbol{\mu} := \{\mu \in H^{-1}(0,L), \langle \mu, \varphi \rangle \geq 0, \quad \forall \varphi \geq 0, \varphi \in C_0^\infty(0,L)\}$$

The mixed formulation reads:

$$\begin{cases} \text{Find } (\theta, w, \lambda) \in V \times \boldsymbol{\mu} \\ \boldsymbol{a}((\theta,w),(\eta,v)) + \boldsymbol{b}((\eta,v),\lambda) = F((\eta,v)), & \forall (\eta,v) \in V \\ \boldsymbol{b}((\theta,w),\mu-\lambda) \leq G(\mu-\lambda), & \forall \mu \in \boldsymbol{\mu} \end{cases} \tag{3.1}$$

where

$$\boldsymbol{b}((\eta,v),\mu) = -\langle \lambda, v \rangle, \quad F((\eta,v)) = (f,v), \quad G(\mu) = -\langle \mu, \psi \rangle$$

Proof: Let $\lambda = -t^{-2}(w'-\theta)' - f$ in the system (2.43), then the complementarity system reads:

$$\begin{cases} -\theta'' - t^{-2}(w'-\theta) = 0, & x \in (0,L) \\ -t^{-2}(w'-\theta)' - \lambda = f, & x \in (0,L) \\ \lambda \geq 0, & x \in (0,L) \\ w - \psi \geq 0, & x \in (0,L) \\ \lambda(w-\psi) = 0, & x \in (0,L) \\ \theta(0) = \theta(L) = 0 \\ w(0) = w(L) = 0 \end{cases} \tag{3.2}$$

Let $(\eta, v) \in V$ and $\mu \in \boldsymbol{\mu}$.

- We multiply the first two equations of (3.2) by η and v respectively, and integrate over $(0,L)$, we get

$$\begin{cases} -\int_0^L \theta''\eta \, dx - t^{-2}\int_0^L (w'-\theta)\eta \, dx = 0 \\ -t^{-2}\int_0^L (w'-\theta)'v \, dx - \langle \lambda, v \rangle = \int_0^L fv \, dx \end{cases}$$

  Using integration by parts,

$$\begin{cases} \int_0^L \theta'\eta' \, dx - t^{-2}\int_0^L (w'-\theta)\eta \, dx \, dx = 0 \\ t^{-2}\int_0^L (w'-\theta)v' \, dx - \langle \lambda, v \rangle = \int_0^L fv \, dx \end{cases}$$

  so

$$\int_0^L \theta'\eta' \, dx + t^{-2}\int_0^L (w' - \theta)(v' - \eta)\, dx - \langle \lambda, v\rangle = \int_0^L fv\, dx$$

i.e.,

$$\boldsymbol{a}((\theta, w), (\eta, v)) - \langle \lambda, v\rangle = (f, v), \quad \forall (\eta, v) \in V \tag{3.3}$$

- We have

$$\begin{cases} w - \psi \geq 0 \\ \lambda(w - \psi) = 0 \end{cases}$$

then

$$\begin{cases} \mu(w - \psi) \geq 0 \\ \lambda(w - \psi) = 0 \end{cases}$$

thus

$$\begin{aligned} &(\mu - \lambda)(w - \psi) \geq 0 \\ \Longrightarrow\ &\langle \mu - \lambda, w - \psi\rangle \geq 0 \\ \Longrightarrow\ &\langle \mu - \lambda, w\rangle \geq \langle \mu - \lambda, \psi\rangle \\ \Longrightarrow\ &-\langle \mu - \lambda, w\rangle \leq -\langle \mu - \lambda, \psi\rangle \end{aligned} \tag{3.4}$$

■

3.2 Finite element approximations

Let $\mathcal{I}_h$ be a discretization of $(0, L)$. We consider two finite-dimensional spaces V_h and Q_h such that $V_h \subset V$, $Q_h \subset Q = H^{-1}(0, L)$.
Moreover

$$\boldsymbol{\mu}_h := \{\mu_h \in Q_h; \quad \mu_h \geq 0\} \subset \boldsymbol{\mu}$$

The discret problem

$$\begin{cases} \text{Find } (\theta_h, w_h, \lambda_h) \in V_h \times \boldsymbol{\mu}_h \\ \boldsymbol{a}((\theta_h, w_h), (\eta_h, v_h)) + \boldsymbol{b}((\eta_h, v_h), \lambda_h) = F((\eta_h, v_h)), \quad \forall (\eta_h, v_h) \in V_h \\ \boldsymbol{b}((\theta_h, w_h), \mu_h - \lambda_h) \leq G(\mu_h - \lambda_h), \quad \forall \mu_h \in \boldsymbol{\mu}_h \end{cases} \tag{3.5}$$

The spaces V_h and Q_h must satisfy a compatibility condition for (3.5) be stable. This condition is known as *the discrete inf-sup condition*

$$\sup_{(\eta_h, v_h) \in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \gtrsim \|\mu_h\|_{-1} \quad \forall \mu_h \in Q_h \tag{3.6}$$

The problem (3.5) stable means regular system.

3.2.1 A pair leading to a regular system

$V_h = V_h^1 \times V_h^2$, $Q_h = V_h^0$.

Proof: First, we introduce the following discrete norm

$$\|\eta_h\|_{-1,h}^2 = \sum_{I_i \in \mathcal{I}_h} h_i^2 \|\eta_h\|_{0,I_i}^2, \quad \forall \eta_h \in Q_h$$

We define the function $b_i(x)$ by

$$b_i(x) = \begin{cases} (x - x_{i-1})(x_{i+1} - x) & \text{if } x \in I_i \\ 0 & \text{otherwise} \end{cases}$$

and $\mu_h = (-1)^i$ on $]x_i, x_{i+1}[$.

Then

$$\|b_i\|_{L^1} = \frac{h_i^3}{6}, \quad \|b_i'\|_{L^2}^2 = \frac{h_i^3}{3}, \quad \|\mu_h\|_{L^2}^2 = h_i.$$

We set $v_h|_{I_i} = b_i \mu_h$, so

$$\langle \mu_h, v_h \rangle = \langle \mu_h, \mu_h b_i \rangle = \sum_{I_i \in \mathcal{I}_h} \int_{I_i} \mu_h^2 b_i = \frac{h_i^3}{6} = \sum_{I_i \in \mathcal{I}_h} \frac{h_i^2}{6} \|\mu_h\|_{0,I_i}^2.$$

$$\|v_h\|_1^2 = \|v_h'\|_0^2 = \sum_{I_i \in \mathcal{I}_h} \|v_h'\|_{0,I_i}^2 = \sum_{I_i \in \mathcal{I}_h} \|\mu_h b_i'\|_{0,I_i}^2 = \sum_{I_i \in \mathcal{I}_h} \|b_i'\|_{0,I_i}^2 = \frac{h_i^3}{3} = \sum_{I_i \in \mathcal{I}_h} \frac{h_i^2}{3} \|\mu_h\|_{0,I_i}^2$$

$$\Longrightarrow \|v_h\|_1 = \left(\sum_{I_i \in \mathcal{I}_h} \frac{h_i^2}{3} \|\mu_h\|_{0,I_i}^2 \right)^{1/2}$$

Thus

$$\frac{\langle \mu_h, v_h \rangle}{\|v_h\|_1} = \frac{\sqrt{3}}{6} \|\mu_h\|_{-1,h}^2 \Longrightarrow \sup_{(\eta_h, v_h) \in V_h} \frac{\langle \mu_h, v_h \rangle}{\|v_h\|_1} \geq C_1 \|\mu_h\|_{-1,h} \tag{3.7}$$

In other hand

$$\|\mu_h\|_{-1} = \sup_{v \in H_0^1(0,L)} \frac{\langle \mu_h, v \rangle}{\|v\|_1} \Longrightarrow \langle \mu_h, v \rangle \geq C_2 \|\mu_h\|_{-1} \|v\|_1,$$

so, using Cauchy-Schwarz inequality, we have

$$\begin{aligned} \langle \mu_h, v_h \rangle &= \langle \mu_h, v_h - v \rangle + \langle \mu_h, v \rangle \\ &= \sum_{I_i \in \mathcal{I}_h} \int_{I_i} h_i(\mu_h) h_i^{-1} (v_h - v) + \langle \mu_h, v \rangle \\ &\geq - \left(\sum_{I_i \in \mathcal{I}_h} h_i^2 \|\mu_h\|_{0,I_i}^2 \right)^{1/2} \left(\sum_{I_i \in \mathcal{I}_h} \left(h_i^{-1} \|v_h - v\|_{0,I_i} \right)^2 \right)^{1/2} + \langle \mu_h, v \rangle \\ &\geq -C_3 \|\mu_h\|_{-1,h} \|v\|_1 + \langle \mu_h, v \rangle \end{aligned}$$

then

$$\begin{aligned}\frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} &\geq -C_3\|\mu_h\|_{-1,h} + \frac{\langle \mu_h, v\rangle}{\|v_h\|_1}\\ &\geq -C_3\|\mu_h\|_{-1,h} + C_2\|\mu_h\|_{-1}\end{aligned}$$

i.e.,

$$\sup_{(\eta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \geq -C_3\|\mu_h\|_{-1,h} + C_2\|\mu_h\|_{-1} \tag{3.8}$$

Using (3.7) and (3.8) and letting $k \in (0,1)$, we have

$$\begin{aligned}\sup_{(\eta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} &= (1-k)\sup_{(\eta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} + k\sup_{(\eta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1}\\ &\geq C_1(1-k)\|\mu_h\|_{-1,h} + k\left(C_2\|\mu_h\|_{-1} - C_3\|\mu_h\|_{-1,h}\right)\\ &= kC_2\|\mu_h\|_{-1} + \left((1-k)C_1 - C_3k\right)\|\mu_h\|_{-1,h}\end{aligned}$$

It is sufficient $(1-k)C_1 - C_3k > 0 \implies 0 < k < \dfrac{C_1}{C_1+C_3}$, to conclude

$$\sup_{(\eta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \gtrsim \|\mu_h\|_{-1}$$

■

3.2.2 A pair leading to a singular system

$V_h = V_h^1 \times V_h^1$, $Q_h = V_h^0$.

Proof: View the following problem

$$\begin{cases}\displaystyle\int_0^L \theta_h'\eta_h'\,dx + \int_0^L w_h v_h\,dx + \langle \mu_h, w_h\rangle = \langle f, v_h\rangle\\ \langle \lambda_h, v_h\rangle = 0\end{cases}$$

We take $\eta_h = v_h = 0$ and $\mu_h = (-1)^i$ on $]x_i, x_{i+1}[$.

$$\begin{aligned}\langle \mu_h, w_h\rangle &= \int_0^L \mu_h\varphi_i\,dx\\ &= \int_{x_{i-1}}^{x_{i+1}} \mu_h\varphi_i\,dx\\ &= \int_{x_{i-1}}^{x_i} (-1)\left(\frac{x-x_{i-1}}{h}\right)\,dx + \int_{x_i}^{x_{i+1}} (+1)\left(\frac{x_{i+1}-x}{h}\right)\\ &= 0\end{aligned}$$

we have $\langle \mu_h, w_h\rangle = 0$ although $\mu_h \neq 0$, which means the matrix extracted from $\langle \mu_h, w_h\rangle$ is not injective, thus the condition (3.6) is not verified. ■

Remark 5

Note that the inf-sup condition is not valid for low-order finite element space.

CHAPTER 4

STABILIZED FORMULATIONS

Let $\mathcal{H} = V \times Q$. Define the norm

$$\| (\eta_h, v_h, \mu_h) \|_{\mathcal{H}}^2 = \|\eta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1}^2$$

4.1 A compact formulation

We define the bilinear form $\mathcal{A} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ by:

$$\mathcal{A}((\eta, v, \mu), (\xi, z, \rho)) = \boldsymbol{a}((\eta, v), (\xi, z)) - \langle \mu, z \rangle - \langle \rho, v \rangle$$

and the linear form $\mathcal{L} : \mathcal{H} \to \mathbb{R}$ by:

$$\mathcal{L}((\xi, z, \rho)) = (f, z) - \langle \rho, \psi \rangle$$

Theorem 4.1

If $((\theta, w, \lambda)$ is the solution of the mixed problem, then it is a solution for the following compact problem:

$$\begin{cases} \text{Find } (\theta, w, \lambda) \in V \times \boldsymbol{\mu} \\ \mathcal{A}((\theta, w, \lambda), (\eta, v, \mu - \lambda)) \leq \mathcal{L}((\eta, v, \mu - \lambda)) \quad \forall (\eta, v, \mu) \in V \times \boldsymbol{\mu} \end{cases} \tag{4.1}$$

Proof: We have the problem (3.1)

$$\begin{cases} \text{Find } (\theta, w, \lambda) \in V \times \boldsymbol{\mu} \\ \boldsymbol{a}((\theta, w), (\eta, v)) - \langle \lambda, v \rangle = (f, v), & \forall (\eta, v) \in V \\ - \langle \mu - \lambda, w \rangle \leq - \langle mu - \lambda, \psi \rangle & \forall \mu \in \boldsymbol{\mu} \end{cases}$$

Summing the two expressions, we obtain

$$\boldsymbol{a}((\theta, w), (\eta, v)) - \langle \lambda, v \rangle - \langle \mu - \lambda, w \rangle \leq (f, v) - \langle \mu - \lambda, \psi \rangle$$

Thus

$$\mathcal{A}((\theta, w, \lambda), (\eta, v, \mu - \lambda)) \leq \mathcal{L}((\eta, v, \mu - \lambda))$$

■

For the well posedness of the compact problem, we need to prove the inf-sup condition for the form $\mathcal{A}$.

Propostion 4.1

For all $(\eta, v, \mu) \in V \times \boldsymbol{\mu}$, there exists $z \in H_0^1(0, L)$ such that:

$$\mathcal{A}\left((\eta, v, \mu); (\eta, z, -\mu)\right) \gtrsim \| (\eta, v, \mu) \|_{\mathcal{H}}^2 \tag{4.2}$$

$$\|z\|_1 \lesssim \|v\|_1 + \|\mu\|_{-1} \tag{4.3}$$

Proof : Let $\mu \in Q$.

We consider the following auxiliary problem

$$\begin{cases} \text{Find } \; q \in H_0^1(0,L) \\ a(q,p) + (q,p) = \langle \mu, p \rangle \,, \quad \forall p \in H_0^1(0,L) \end{cases} \tag{4.4}$$

It is obvious that this problem has a unique solution (Lax-Milgram theorem). Taking $p = q$, we get

$$\|q\|_1^2 \leq \langle \mu, q \rangle \implies \|q\|_1 \leq \|\mu\|_{-1}$$

Besides

$$\|\mu\|_{-1} = \sup_{p \in H_0^1(0,L)} \frac{\langle \mu, p \rangle}{\|p\|_1} = \sup_{p \in H_0^1(0,L)} \frac{a(q,p) + (q,p)}{\|p\|_1} \leq \|q\|_1$$

Thus

$$\|\mu\|_{-1} = \|q\|_1 \tag{4.5}$$

- We put $z = v - q$, using Young's, Cauchy–Schwarz inequalities

$$\begin{aligned}
\mathcal{A}((\eta, v, \mu); (\eta, z, -\mu)) &= \mathcal{A}((\eta, v, \mu); (\eta, v - q, -\mu)) \\
&= \boldsymbol{a}((\eta, v), (\eta, v - q)) - \langle \mu, v - q \rangle + \langle \mu, q \rangle \\
&= a(\eta, \eta) + t^{-2} b((\eta, v), (\eta, v - q)) + \langle \mu, q \rangle \\
&\geq a(\eta, \eta) + b((\eta, v), (\eta, v - q)) + \langle \mu, q \rangle \\
&= a(\eta, \eta) + b((\eta, v), (\eta, v)) - b((\eta, v), (0, q)) + \langle \mu, q \rangle \\
&= \|\eta'\|_0^2 + \|v' - \eta\|_0^2 - \int_0^L (v' - \eta) q' \; dx + \langle \mu, q \rangle \\
&\geq \|\eta'\|_0^2 + \|v' - \eta\|_0^2 - \|v' - \eta)\|_0^2 \|q'\|_0^2 + \langle \mu, q \rangle \\
&\geq \|\eta'\|_0^2 + \|v' - \eta\|_0^2 - \frac{1}{2}\|v' - \eta\|_0^2 - \frac{1}{2}\|q'\|_0^2 + \|q\|_1^2 \\
&= \|\eta'\|_0^2 + \frac{1}{2}\|v' - \eta\|_0^2 + \frac{1}{2}\|q\|_1^2 \\
&\geq \frac{1}{2}\left(\|\eta'\|_0^2 + \|v' - \eta\|_0^2\right) + \frac{1}{2}\|q\|_1^2
\end{aligned}$$

Utilizing (4.5) and Lemma 2.1

$$\mathcal{A}((\eta, v, \mu); (\eta, z, -\mu)) \geq \frac{1}{6}\left(\|\eta\|_1^2 + \|v\|_1^2\right) + \frac{1}{2}\|\mu\|_{-1}^2$$

$$\geq \frac{1}{6}\left(\|\eta\|_1^2 + \|v\|_1^2 + \|\mu\|_{-1}^2\right)$$

- Again, using (4.5)

$$\|z\|_1 = \|v - q\|_1 \leq \|v\|_1 + \|q\|_1 = \|v\|_1 + \|\mu\|_{-1}$$ ■

Propostion 4.2

$$\sup_{(\xi,z,\rho)\in V\times Q} \frac{\mathcal{A}((\eta, v, \mu), (\xi, z, \rho))}{\|(\xi, z, \rho)\|_{\mathcal{H}}} \gtrsim \|(\eta, v, \mu)\|_{\mathcal{H}} \tag{4.6}$$

Proof: We proved :

$$\mathcal{A}\left((\eta, v, \mu); (\eta, z, -\mu)\right) \gtrsim \|\,(\eta, v, \mu)\,\|_{\mathcal{H}}^2$$

i.e.,

$$\frac{\mathcal{A}\left((\eta, v, \mu); (\eta, z, -\mu)\right)}{\|\,(\eta, v, \mu)\,\|_{\mathcal{H}}} \gtrsim \|\,(\eta, v, \mu)\,\|_{\mathcal{H}}$$

Additionally

$$\|\,(\eta, z, -\mu)\,\|_{\mathcal{H}}^2 = \|\eta\|_1^2 + \|z\|_1^2 + \|\mu\|_{-1}^2 \lesssim \|\eta\|_1^2 + \|v\|_1^2 + \|\mu\|_{-1}^2 = \|\,(\eta, v, \mu)\,\|_{\mathcal{H}}^2$$

i.e.,

$$\frac{1}{\|\,(\eta, z, -\mu)\,\|_{\mathcal{H}}} \gtrsim \frac{1}{\|\,(\eta, v, \mu)\,\|_{\mathcal{H}}}$$

Then

$$\frac{\mathcal{A}\left((\eta, v, \mu); (\eta, z, -\mu)\right)}{\|\,(\eta, v, -\mu)\,\|_{\mathcal{H}}} \gtrsim \|\,(\eta, v, \mu)\,\|_{\mathcal{H}}$$

Thus

$$\sup_{(\xi,z,\rho)\in V\times Q} \frac{\mathcal{A}((\eta, v, \mu), (\xi, z, \rho))}{\|(\xi, z, \rho)\|_{\mathcal{H}}} \gtrsim \|(\eta, v, \mu)\|_{\mathcal{H}}$$ ■

4.2 Finite element approximations

4.2.1 Stable formulations: Inf-sup condition

We consider the following discret problem

$$\begin{cases} \text{Find } (\theta_h, w_h, \lambda_h) \in V_h \times \mu_h \\ \mathcal{A}((\theta_h, w_h, \lambda_h), (\eta_h, v_h, \mu_h - \lambda_h)) \leq \mathcal{L}((\eta_h, v_h, \mu_h - \lambda_h)) \quad \forall (\eta_h, v_h, \mu_h) \in V_h \times \mu_h \end{cases} \tag{4.7}$$

where V_h and μ_h are the same defined in the section (3.2).

Assume that V_h and Q_h are such that

$$\sup_{(\eta_h,v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \gtrsim \|\mu_h\|_{-1} \quad \forall \mu_h \in Q_h \tag{4.8}$$

Propostion 4.3

Let $W_h \subset H_0^1(0,L)$ a finite-dimensional space.

For all $(\eta_h, v_h, \mu_h) \in V_h \times \mu_h$, there exists $z_h \in W_h$ such that:

$$\mathcal{A}\left((\eta_h, v_h, \mu_h); (\eta_h, z_h, -\mu_h)\right) \gtrsim \|\,(\eta_h, v_h, \mu_h)\,\|_{\mathcal{H}}^2 \tag{4.9}$$

$$\|z_h\|_1 \lesssim \|v_h\|_1 + \|\mu_h\|_{-1} \tag{4.10}$$

Proof: Let $\mu_h \in Q_h$.
We observe the following auxiliary problem

$$\begin{cases} \text{Find } \ q_h \in W_h \\ a(q_h, p_h) + (q_h, p_h) = \langle \mu_h, p_h\rangle\,, \quad \forall p_h \in W_h \end{cases} \tag{4.11}$$

It is obvious that this problem has a unique solution (Lax-Milgram theorem). Taking $p_h = q_h$, we get

$$\|q_h\|_1^2 \le \langle \mu_h, q_h\rangle \implies \|q_h\|_1 \le \|\mu_h\|_{-1}$$

Besides, from the condition (4.8) we have

$$\|\mu_h\|_{-1} \lesssim \sup_{p_h\in W_h} \frac{\langle \mu_h, p_h\rangle}{\|p_h\|_1} = \sup_{p_h\in W_h} \frac{a(q_h,p_h) + (q_h,p_h)}{\|p_h\|_1} \le \|q_h\|_1$$

Thus

$$\exists C > 0, \quad C\|\mu_h\|_{-1} \le \|q_h\|_1 \le \|\mu_h\|_{-1} \tag{4.12}$$

- We put $z_h = v_h - q_h$, using the result (4.12) , Young's , Cauchy–Schwarz inequalities and Lemma (2.1)

$$
\begin{aligned}
\mathcal{A}((\eta_h, v_h, \mu_h); (\eta_h, z_h, -\mu_h)) &= \mathcal{A}((\eta_h, v_h, \mu_h); (\eta_h, v_h - q_h, -\mu_h)) \\
&= \boldsymbol{a}((\eta_h, v_h), (\eta_h, v_h - q_h)) - \langle \mu_h, v_h - q_h \rangle + \langle \mu_h, q_h \rangle \\
&= a(\eta_h, \eta_h) + t^{-2} b((\eta_h, v_h), (\eta_h, v_h - q_h)) + \langle \mu_h, q_h \rangle \\
&\geq a(\eta_h, \eta_h) + b((\eta_h, v_h), (\eta_h, v_h - q_h)) + \langle \mu_h, q_h \rangle \\
&= a(\eta_h, \eta_h) + b((\eta_h, v_h), (\eta_h, v_h)) - b((\eta_h, v_h), (0, q_h)) + \langle \mu_h, q_h \rangle \\
&= \|\eta_h'\|_0^2 + \|v_h' - \eta_h\|_0^2 - \int_0^L (v_h' - \eta_h) q_h' \, dx + \langle \mu_h, q_h \rangle \\
&\geq \|\eta_h'\|_0^2 + \|v_h' - \eta\|_0^2 - \|v_h' - \eta_h)\|_0^2 \|q_h'\|_0^2 + \langle \mu_h, q_h \rangle \\
&\geq \|\eta_h'\|_0^2 + \|v_h' - \eta\|_0^2 - \frac{1}{2}\|v_h' - \eta\|_0^2 - \frac{1}{2}\|q_h'\|_0^2 + \|q_h\|_1^2 \\
&= \|\eta_h'\|_0^2 + \frac{1}{2}\|v_h' - \eta\|_0^2 + \frac{1}{2}\|q_h\|_1^2 \\
&\geq \frac{1}{2}\left(\|\eta_h'\|_0^2 + \|v_h' - \eta\|_0^2\right) + \frac{1}{2}\|q_h\|_1^2 \\
&\geq \frac{1}{6}\left(\|\eta_h\|_1^2 + \|v_h\|_1^2\right) + \frac{1}{2}C^2\|\mu_h\|_{-1}^2 \\
&\geq \min\left(\frac{1}{6}, \frac{1}{2}C^2\right)\left(\|\eta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1}^2\right)
\end{aligned}
$$

- Again applying (4.12)

$$\|z_h\|_1 = \|v_h - q_h\|_1 \leq \|v_h\|_1 + \|q_h\|_1 \leq \|v_h\|_1 + \|\mu_h\|_{-1}$$

■

Propostion 4.4

$$\sup_{(\xi_h, z_h, \rho_h) \in V_h \times Q_h} \frac{\mathcal{A}((\eta_h, v_h, \mu_h), (\xi_h, z_h, \rho_h))}{\|(\xi_h, z_h, \rho_h)\|_{\mathcal{H}}} \gtrsim \|(\eta_h, v_h, \mu_h)\|_{\mathcal{H}} \qquad (4.13)$$

Proof: We proved :

$$\mathcal{A}\left((\eta_h, v_h, \mu_h); (\eta_h, z_h, -\mu_h)\right) \gtrsim \|\left(\eta_h, v_h, \mu_h\right)\|_{\mathcal{H}}^2$$

i.e.,

$$\frac{\mathcal{A}\left((\eta_h, v_h, \mu_h); (\eta_h, z_h, -\mu_h)\right)}{\|\, (\eta_h, v_h, \mu_h)\,\|_{\mathcal{H}}} \gtrsim \|\, (\eta_h, v_h, \mu_h)\,\|_{\mathcal{H}}$$

Additionally

$$\|\, (\eta_h, z_h, -\mu_h)\,\|^2_{\mathcal{H}} = \|\eta_h\|^2_1 + \|z_h\|^2_1 + \|\mu_h\|^2_{-1} \lesssim \|\eta_h\|^2_1 + \|v_h\|^2_1 + \|\mu_h\|^2_{-1} = \|\, (\eta_h, v_h, \mu_h)\,\|^2_{\mathcal{H}}$$

i.e.,

$$\frac{1}{\|\, (\eta_h, z_h, -\mu_h)\,\|_{\mathcal{H}}} \gtrsim \frac{1}{\|\, (\eta_h, v_h, \mu_h)\,\|_{\mathcal{H}}}$$

Then

$$\frac{\mathcal{A}\left((\eta_h, v_h, \mu_h); (\eta_h, z_h, -\mu_h)\right)}{\|\, (\eta_h, v_h, -\mu_h)\,\|_{\mathcal{H}}} \gtrsim \|\, (\eta_h, v_h, \mu_h)\,\|_{\mathcal{H}}$$

Thus

$$\sup_{(\xi_h, z_h, \rho_h) \in V_h \times Q_h} \frac{\mathcal{A}((\eta_h, v_h, \mu_h), (\xi_h, z_h, \rho_h))}{\|(\xi_h, z_h, \rho_h)\|_{\mathcal{H}}} \gtrsim \|(\eta_h, v_h, \mu_h)\|_{\mathcal{H}}$$

■

Both propositions were proven under the assumption (4.8), which is not always verified for any subspaces V_h, Q_h. Still there is some spaces that can hold it.

Lemma 4.1

The finite element spaces $V_h = V_h^1 \times V_h^2$, $Q_h = V_h^0$ satisfy the condition (4.8).

Proof: See the proof of (3.2.1). ■

Lemma 4.2

We can generalize the result of the previous lemma: $V_h = V_h^k \times V_h^{k+1}$, $Q_h = V_h^{k-2}$, $k \geq 1$

Lemma 4.3

The finite element spaces $V_h = V_h^1 \times V_h^1$, $Q_h = V_h^0$ can not satisfy the condition (4.8).

Proof: See the proof of (3.2.2). ■

4.2.2 A priori error estimation

Theorem 4.2

$$\|\theta-\theta_h\|_1+\|w-w_h\|_1+\|\lambda-\lambda_h\|_{-1}\lesssim \inf_{(\eta_h,v_h)\in V_h}(\|\theta-\eta_h\|_1+\|w-v_h\|_1) + \inf_{\mu_h\in\Lambda_h}(\|\lambda-\mu_h\|_{-1}+\sqrt{\langle w-\psi,\mu_h\rangle}) \quad (4.14)$$

Proof: We have from (4.9)

$$\|\theta_h-\eta_h\|_1^2+\|w_h-v_h\|_1^2+\|\lambda_h-\mu_h\|_1^2\lesssim \mathcal{A}((\theta_h-\eta_h,w_h-v_h,\lambda_h-\mu_h);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))$$

$$=\mathcal{A}((\theta_h,w_h,\lambda_h);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))-\mathcal{A}((\eta_h,v_h,\mu_h);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))$$

$$-\mathcal{A}((\theta,w,\lambda);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))+\mathcal{A}((\theta,w,\lambda);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))$$

$$=\mathcal{A}((\theta_h,w_h,\lambda_h);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))-\mathcal{A}((\theta,w,\lambda);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))$$

$$+\mathcal{A}((\theta-\eta_h,w-v_h,\lambda-\mu_h);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))$$

$$\le\mathcal{L}((\theta_h-\eta_h,z_h,\mu_h-\lambda_h))-\mathcal{A}((\theta,w,\lambda);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))$$

$$+\mathcal{A}((\theta-\eta_h,w-v_h,\lambda-\mu_h);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))$$

$$=(f,z_h)-\langle\mu_h-\lambda_h,\psi\rangle-\int_0^L\theta'(\theta_h-\eta_h)'\,dx-t^{-2}\int_0^L(w'-\theta)(z_h'-(\theta_h-\eta_h))\,dx+\langle\lambda,z_h\rangle$$

$$+\langle\mu_h-\lambda_h,w\rangle+\mathcal{A}((\theta-\eta_h,w-v_h,\lambda-\mu_h);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))$$

$$=(f,z_h)-\langle\mu_h-\lambda_h,\psi\rangle+\int_0^L\theta''(\theta_h-\eta_h)\,dx+t^{-2}\int_0^L(w'-\theta)'(z_h)\,dx+t^{-2}\int_0^L(w'-\theta)(\theta_h-\eta_h)\,dx$$

$$+\langle\lambda,z_h\rangle+\langle\mu_h-\lambda_h,w\rangle+\mathcal{A}((\theta-\eta_h,w-v_h,\lambda-\mu_h);(\theta_h-\eta_h,z_h,\mu_h-\lambda_h))$$

Since, from (3.2),

$$\theta''+t^{-2}(w'-\theta)=0$$
$$t^{-2}(w'-\theta)'+\lambda+f=0$$

Then

$$\|\theta_h - \eta_h\|_1^2 + \|w_h - v_h\|_1^2 + \|\lambda_h - \mu_h\|_1^2 \lesssim \mathcal{A}((\theta - \eta_h, w - v_h, \lambda - \mu_h); (\theta_h - \eta_h, z_h, \mu_h - \lambda_h))$$
$$+ \langle \mu_h - \lambda_h, w - \psi \rangle$$

We have $w - \psi \geq 0, \quad \lambda_h \geq 0$, so

$$\|\theta_h - \eta_h\|_1^2 + \|w_h - v_h\|_1^2 + \|\lambda_h - \mu_h\|_1^2$$

$$\lesssim \mathcal{A}((\theta - \eta_h, w - v_h, \lambda - \mu_h); (\theta_h - \eta_h, z_h, \mu_h - \lambda_h)) + \langle \mu_h, w - \psi \rangle$$

$$\lesssim (\|\theta - \eta_h\|_1 + \|w - v_h\|_1 + \|\lambda - \mu_h\|_{-1}) (\|\theta_h - \eta_h\|_1 + \|z_h\|_1 + \|\lambda_h - \mu_h\|_{-1}) + \langle \mu_h, w - \psi \rangle$$

$$\lesssim \|\theta - \eta_h\|_1^2 + \|w - v_h\|_1^2 + \|\lambda - \mu_h\|_{-1}^2 + \|\theta_h - \eta_h\|_1^2 + \|z_h\|_1^2 + \|\lambda_h - \mu_h\|_{-1}^2 + \langle \mu_h, w - \psi \rangle$$

Because $\|z_h\|_1 \lesssim \|\theta_h - \eta_h\|_1 + \|\lambda_h - \mu_h\|_{-1}$, we obtain

$$\|\theta_h - \eta_h\|_1^2 + \|w_h - v_h\|_1^2 + \|\lambda_h - \mu_h\|_{-1}^2 \lesssim \|\theta - \eta_h\|_1^2 + \|w - v_h\|_1^2 + \|\lambda - \mu_h\|_{-1}^2 + \langle \mu_h, w - \psi \rangle$$

This implies that

$$\|\theta_h - \eta_h\|_1 + \|w_h - v_h\|_1 + \|\lambda_h - \mu_h\|_{-1} \lesssim \|\theta - \eta_h\|_1 + \|w - v_h\|_1 + \|\lambda - \mu_h\|_{-1} + \sqrt{\langle \mu_h, w - \psi \rangle}$$

On the other hand

$$\|\theta - \theta_h\|_1 + \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1} \leq \|\theta - \eta_h\|_1 + \|w - v_h\|_1 + \|\lambda - \mu_h\|_{-1} + \|\theta_h - \eta_h\|_1$$
$$+ \|w_h - v_h\|_1 + \|\lambda_h - \mu_h\|_{-1}$$

Thus

$$\|\theta - \theta_h\|_1 + \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1} \lesssim \|\theta - \eta_h\|_1 + \|w - v_h\|_1 + \|\lambda - \mu_h\|_{-1} + \sqrt{\langle \mu_h, w - \psi \rangle}$$

Hence

$$\|\theta - \theta_h\|_1 + \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1} \lesssim \inf_{(\eta_h, v_h) \in V_h} (\|\theta - \eta_h\|_1 + \|w - v_h\|_1)$$
$$+ \inf_{\mu_h \in \Lambda_h} (\|\lambda - \mu_h\|_{-1} + \sqrt{\langle \mu_h, w - \psi, \rangle}) \qquad \blacksquare$$

4.2.3 A stabilized formulation

It is not difficult to show that the $V_h = V_h^1 \times V_h^1$, $Q_h = V_h^0$ leads to a singular system for the compact formulation, even it is conforming approximation. In this section we will propose a stabilized formulation that will be well posed for all conforming approximations.

Let $\alpha > 0$, we introduce the following bilinear and linear forms:

$$\begin{aligned}\mathcal{S}_h((\sigma,\tau,\rho);(\eta,v,\mu)) = h^2t^4 \int_0^L & \Big[\big(\sigma'' + t^{-2}(\tau' - \sigma)\big)\big(\eta'' + t^{-2}(v' - \eta)\big) \\ & + \big(t^{-2}(\tau'' - \sigma') + \rho\big)\big(t^{-2}(v'' - \eta') + \mu\big)\Big]\, dx \\ \mathscr{L}_h((\eta,v,\mu)) = -h^2t^4 \int_0^L & f\big(t^{-2}(v'' - \eta') + \mu\big)\, dx\end{aligned}$$

and then we define, the bilinear form $\mathcal{A}_h$ and the linear form $\mathcal{L}_h$

$$\begin{aligned}\mathcal{A}_h((\sigma,\tau,\rho);(\eta,v,\mu)) &= \mathcal{A}((\sigma,\tau,\rho);(\eta,v,\mu)) - \alpha\mathcal{S}_h((\sigma,\tau,\rho);(\eta,v,\mu)) \\ \mathcal{L}_h((\eta,v,\mu)) &= \mathcal{L}((\eta,v,\mu)) - \alpha\mathscr{L}_h((\eta,v,\mu))\end{aligned}$$

Remark 6

Note that

$$\mathcal{S}_h((\theta,w,\lambda);(\eta,v,\mu)) = \mathscr{L}_h((\eta,v,\mu)), \quad \forall(\eta,v,\mu) \in V_h \times \mu_h$$

Consider the discret problem

$$\begin{cases} \text{Find } (\theta_h,w_h,\lambda_h) \in V_h \times \mu_h \\ \mathcal{A}_h((\theta_h,w_h,\lambda_h),(\eta_h,v_h,\mu_h - \lambda_h)) \le \mathcal{L}_h((\eta_h,v_h,\mu_h - \lambda_h)),\ \forall(\eta_h,v_h,\mu_h) \in V_h \times \mu_h \end{cases} \tag{4.15}$$

Lemma 4.4

There exists a positive constant C_I such that for any $\phi_h \in V_h$

$$C_I \sum_{I_i \in \mathcal{I}_h} h_i^2 \|\phi_h'\|_{0,I_i}^2 \le \|\phi_h\|_0^2 \tag{4.16}$$

$$C_I \sum_{I_i \in \mathcal{I}_h} h_i^2 \|\phi_h''\|_{0,I_i}^2 \le \|\phi_h'\|_0^2 \tag{4.17}$$

Proof: To prove (4.16), note that

$$\|\phi_h\|_0 = \sum_{I_i \in \mathcal{I}_h} h_i \|\phi_h\|_{0,I_i}$$

We have from the inequality (1.3)

$$\|\phi_h'\|_{0,I_i} \le C h_i^{-1} \|\phi_h\|_{0,I_i}$$

i.e.,

$$C^{-1} h_i \|\phi_h'\|_{0,I_i} \le \|\phi_h\|_{0,I_i}$$

Then

$$C_I \sum_{I_i \in \mathcal{I}_h} h_i^2 \|\phi_h'\|_{0,I_i}^2 \le \sum_{I_i \in \mathcal{I}_h} \|\phi_h\|_{0,I_i}^2$$

Hence

$$C_I \sum_{I_i \in \mathcal{I}_h} h_i^2 \|\phi_h'\|_{0,I_i}^2 \le \|\phi_h\|_0^2$$

Same proof for (4.17). ∎

Propostion 4.5

Suppose that $0 < \alpha < \min\left(\frac{t^{-2}}{2h^2 + C_I^{-1}}, \frac{C_I}{2t^4}\right)$. Then for any $(\eta_h, v_h, \mu_h) \in V_h \times \mu_h$, there exits $(u_h, p_h) \in V_h$ such that

$$\mathcal{A}_h((\eta_h, v_h, \mu_h); (u_h, p_h, -\mu_h)) \gtrsim \|\eta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1}^2 \tag{4.18}$$

$$\|u_h\|_1 + \|p_h\|_1 \lesssim \|\eta_h\| + \|v_h\|_1 + \|\mu_h\|_{-1} \tag{4.19}$$

Proof: The proof is divided into several steps:

Step I : Using (4.17) and Young's inequalities

$$\mathcal{A}_h((\eta_h, v_h, \mu_h); (\eta_h, v_h, -\mu_h)) = \mathcal{A}((\eta_h, v_h, \mu_h); (\eta_h, v_h, -\mu_h)) - \alpha \mathcal{S}_h((\eta_h, v_h, \mu_h); (\eta_h, v_h, -\mu_h))$$

$$= \boldsymbol{a}((\eta_h, v_h); ((\eta_h, v_h)) - \alpha \mathcal{S}_h((\eta_h, v_h, \mu_h); (\eta_h, v_h, -\mu_h))$$

$$= \|\eta_h'\|_0^2 + t^{-2}\|v_h' - \eta_h\|_0^2 - \alpha h^2 \left(t^4 \int_0^L \left(\eta_h'' + t^{-2}(v_h' - \eta_h)\right)^2 \, dx + \int_0^L (v_h'' - \eta_h')^2 \, dx - t^4 \int_0^L \mu_h^2 \, dx \right)$$

$$\ge \|\eta_h'\|_0^2 + t^{-2}\|v_h' - \eta_h\|_0^2 - \alpha t^4 h^2 \int_0^L \left(\eta_h'' + t^{-2}(v_h' - \eta_h)\right)^2 \, dx - \alpha C_I^{-1} \int_0^L (v_h' - \eta_h)^2 \, dx + \alpha t^4 h^2 \int_0^L \mu_h^2$$

$$\ge \|\eta_h'\|_0^2 + t^{-2}\|v_h' - \eta_h\|_0^2 - 2\alpha t^4 h^2 \|\eta_h''\|_0^2 - 2\alpha h^2 \|v_h' - \eta_h\|_0^2 - \alpha C_I^{-1} \|v_h' - \eta_h\|_0^2 + \alpha t^4 h^2 \|\mu_h\|_0^2$$

$$\ge \|\eta_h'\|_0^2 + t^{-2}\|v_h' - \eta_h\|_0^2 - 2\alpha C_I^{-1} t^4 \|\eta_h'\|_0^2 - 2\alpha h^2 \|v_h' - \eta_h\|_0^2 - \alpha C_I^{-1} \|v_h' - \eta_h\|_0^2 + \alpha t^4 h^2 \|\mu_h\|_0^2$$

$$\ge (1 - 2\alpha t^4 C_I^{-1}) \|\eta_h'\|_0^2 + (t^{-2} - 2\alpha h^2 - \alpha C_I^{-1}) \|v_h' - \eta_h\|_0^2 + \alpha t^4 h^2 \|\mu_h\|_0^2$$

Because $0 < \alpha < \min\left(\frac{t^{-2}}{2h^2+C_I^{-1}}, \frac{C_I}{2t^4}\right)$, we have

$$1 - 2\alpha t^4 C_I^{-1} > 0 \quad , \quad t^{-2} - 2\alpha h^2 - \alpha C_I^{-1} > 0$$

Hence

$$\mathcal{A}_h((\eta_h, v_h, \mu_h); (\eta_h, v_h, -\mu_h)) \geq C_3(\|\eta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1,h}^2) \tag{4.20}$$

where $C_3 = \min(\alpha t^4, 1 - 2\alpha t^4 C_I^{-1}, t^{-2} - 2\alpha h^2 - \alpha C_I^{-1})$.

Step II : We have from (3.8)

$$\sup_{(\eta_h, v_h)\in V_h} \frac{\langle \mu_h, v_h\rangle}{\|v_h\|_1} \geq C_1\|\mu_h\|_{-1} - C_1\|\mu_h\|_{-1,h}$$

we deduce that $\forall \mu_h \in Q_h, \quad \exists(\zeta_h, q_h) \in V_h$ such that

$$\|\mu_h\|_{-1} = \|(\zeta_h, q_h)\|_V \tag{4.21}$$

$$\langle \mu_h, q_h\rangle \geq C_1\|\mu_h\|_{-1}^2 - C_2\|\mu_h\|_{-1}\|\mu_h\|_{-1,h} \tag{4.22}$$

Step III :

$$\begin{aligned}
\mathcal{A}_h((\eta_h, v_h, \mu_h); (\zeta_h, q_h, 0)) &= \mathcal{A}((\eta_h, v_h, \mu_h); (\zeta_h, q_h, 0)) - \alpha\mathcal{S}_h((\eta_h, v_h, \mu_h); (\zeta_h, q_h, 0)) \\
&= \boldsymbol{a}((\eta_h, v_h); (\zeta_h, q_h)) - < \mu_h, q_h > - \alpha\mathcal{S}_h((\eta_h, v_h, \mu_h); (\zeta_h, q_h, 0)) \\
&= \int_0^L \eta_h'\zeta_h'\, dx + t^{-2}\int_0^L (v_h' - \eta_h)(q_h' - \zeta_h)\, dx - < \mu_h, q_h > \\
&- \alpha\int_0^L \Big(h^2t^4\eta_h''\zeta_h'' + h^2t^2\left(\eta_h''(q_h' - \zeta_h) + \zeta_h''(v_h' - \eta_h)\right)\Big)\, dx \\
&- \alpha\int_0^L \Big(h^2(v_h' - \eta_h)(q_h' - \zeta_h) + h^2(v_h'' - \eta_h')(q_h'' - \zeta_h')\Big)\, dx \\
&- \alpha\int_0^L h^2t^2\mu_h(q_h'' - \mu_h')\, dx
\end{aligned}$$

Using (4.22) and Cauchy-Schwarz inequalities

$$
\begin{aligned}
\mathcal{A}_h((\eta_h, v_h, \mu_h); (\zeta_h, q_h, 0)) \geq & - \|\eta_h'\|_0 \|\zeta_h'\|_0 - t^{-2}\|v_h' - \eta_h\|_0 \|q_h' - \zeta_h\|_0 + C_1\|\mu_h\|_{-1}^2 - C_2\|\mu_h\|_{-1}\|\mu_h\|_{-1,h} \\
& - \alpha t^4 (h\|\eta_h''\|_0)(h\|\zeta_h''\|_0) - \alpha t^2 \big(h\|\eta_h''\|_0 \|h(q_h' - \zeta_h)\|_0 + h\|\zeta_h''\|_0 \|h(v_h' - \eta_h)\|_0\big) \\
& - \alpha\big((h\|v_h' - \eta_h\|_0)(h\|q_h' - \zeta_h\|_0) + (h\|v_h'' - \eta_h'\|_0)(h\|q_h'' - \zeta_h'\|_0)\big) \\
& - \alpha t^2 (h\|\mu_h\|_0)(h\|q_h'' - \mu_h'\|_0)
\end{aligned}
$$

Using (4.17), (4.21) and Young's inequalities

$$
\begin{aligned}
\mathcal{A}_h((\eta_h, v_h, \mu_h); (\zeta_h, q_h, 0)) \geq & - (1 + \alpha t^4 C_I^{-1})\|\eta_h'\|_0 \|\zeta_h'\|_0 + C_1\|\mu_h\|_{-1}^2 - C_2\|\mu_h\|_{-1}\|\mu_h\|_{-1,h} \\
& - \alpha t^2 \sqrt{C_I^{-1}} \big(\|\eta_h'\|_0 \|h(q_h' - \zeta_h)\|_0 + \|\zeta_h'\|_0 \|h(v_h' - \eta_h)\|_0\big) \\
& - \alpha\big((h\|v_h' - \eta_h\|_0)(h\|q_h' - \zeta_h\|_0)\big) - \alpha C_I^{-1}(\|v_h' - \eta_h\|_0)(\|q_h' - \zeta_h\|_0) \\
& - \alpha t^2 \sqrt{C_I^{-1}} (h\|\mu_h\|_0)(\|q_h' - \mu_h\|_0) - t^{-2}\|v_h' - \eta_h\|_0 \|q_h' - \zeta_h\|_0 \\
& \geq (C_1 - \frac{\epsilon}{2}C_2)\|\mu_h\|_{-1}^2 - \frac{\epsilon}{2}\left(\frac{\alpha t^4}{C_I} + 1\right)\|\zeta_h'\|_0^2 - \frac{1}{2\epsilon}\left(\frac{\alpha t^4}{C_I} + 1\right)\|\eta_h'\|_0^2 \\
& - \frac{\alpha t^2}{\sqrt{C_I}}\left(\frac{\epsilon}{2}\big(\|\zeta_h'\|_0^2 + h^2\|q_h' - \zeta_h\|_0^2\big) + \frac{1}{2\epsilon}\big(\|\eta_h'\|_0^2 + h^2\|v_h' - \eta_h\|_0^2\big)\right) \\
& - (\alpha h^2 + \alpha C_I^{-1} + t^{-2})\left(\frac{\epsilon}{2}\|q_h' - \zeta_h\|_0^2 + \frac{1}{2\epsilon}\|v_h' - \eta_h\|_0^2\right) \\
& - \frac{\alpha t^2}{\sqrt{C_I}}\left(\frac{\epsilon}{2}\|q_h' - \zeta_h\|_0^2 + \frac{1}{2\epsilon}h^2\|\mu_h\|_0^2\right) - \frac{C_2}{2\epsilon}\|\mu_h\|_{-1,h}^2 \\
& \geq C_4\|\mu_h\|_{-1}^2 - C_5(\|\eta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1,h}^2)
\end{aligned}
$$

We choose ϵ small enough so that $C_4 > 0$.

With the same steps we get

$$
\mathcal{A}_h((\eta_h, v_h, \mu_h); (\delta\zeta_h, \delta q_h, 0)) \geq C_4\|\mu_h\|_{-1}^2 - \delta C_5(\|\eta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1,h}^2) \tag{4.23}
$$

Step IV : From (4.20) , (4.23) we got

$$\begin{cases} \mathcal{A}_h((\eta_h, v_h, \mu_h); (\eta_h, v_h, -\mu_h)) \geq C_3(\|\eta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1,h}^2) \\ \mathcal{A}_h((\eta_h, v_h, \mu_h); (\delta\zeta_h, \delta q_h, 0)) \geq C_4\|\mu_h\|_{-1}^2 - \delta C_5(\|\eta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1,h}^2) \end{cases}$$

So

$$\mathcal{A}_h((\eta_h, v_h, \mu_h); (\eta_h+\delta\zeta_h, v_h+\delta q_h, -\mu_h)) \geq C_4\|\mu_h\|_{-1}^2+(C_3-\delta C_5)(\|\eta_h\|_1^2+\|v_h\|_1^2+\|\mu_h\|_{-1,h}^2)$$

We choose $\delta > 0$ such that $C_3 - \delta C_5 > 0$ i.e., $0 < \delta < \frac{C_3}{C_5}$

Hence there exist $u_h = \eta_h + \delta\zeta_h$, $p_h = v_h + \delta q_h$ such that

$$\mathcal{A}_h((\eta_h, v_h, \mu_h); (u_h, p_h, -\mu_h)) \gtrsim \|\eta_h\|_1^2 + \|v_h\|_1^2 + \|\mu_h\|_{-1}^2$$

Step V : Using (4.21), we obtain

$$\begin{aligned} \|u_h\|_1 + \|p_h\|_1 &= \|\eta_h + \delta\zeta_h\|_1 + \|v_h + \delta q_h\|_1 \\ &\leq \|\eta_h\|_1 + \|v_h\|_1 + \delta(\|\zeta_h\|_1 + \|q_h\|_1) \\ &\lesssim \|\eta_h\| + \|v_h\|_1 + \|\mu_h\|_{-1} \end{aligned}$$

■

Propostion 4.6

$$\sup_{(u_h,p_h,\xi_h)\in V_h\times Q_h} \frac{\mathcal{A}_h((\eta_h, v_h, \mu_h), ((u_h, p_h, \xi_h))}{\|((u_h, p_h, \xi_h)\|_{\mathcal{H}}} \gtrsim \|(\eta_h, v_h, \mu_h)\|_{\mathcal{H}} \tag{4.24}$$

Proof: We proved:

$$\mathcal{A}_h((\eta_h, v_h, \mu_h); (u_h, p_h, -\mu_h)) \gtrsim \| (\eta_h, v_h, \mu_h) \|_{\mathcal{H}}^2$$

i.e.,

$$\frac{\mathcal{A}_h ((\eta_h, v_h, \mu_h); (u_h, p_h, -\mu_h))}{\| (\eta_h, v_h, \mu_h) \|_{\mathcal{H}}} \gtrsim \| (\eta_h, v_h, \mu_h) \|_{\mathcal{H}}$$

Additionally

$$\| (u_h, p_h, -\mu_h) \|_{\mathcal{H}}^2 = \|u_h\|_1^2+\|p_h\|_1^2+\|\mu_h\|_{-1}^2 \lesssim \|\eta_h\|_1^2+\|v_h\|_1^2+\|\mu_h\|_{-1}^2 = \| (\eta_h, v_h, \mu_h) \|_{\mathcal{H}}^2$$

i.e.,

$$\frac{1}{\| (u_h, p_h, -\mu_h) \|_{\mathcal{H}}} \gtrsim \frac{1}{\| (\eta_h, v_h, \mu_h) \|_{\mathcal{H}}}$$

Then

$$\frac{\mathcal{A}_h ((\eta_h, v_h, \mu_h); (u_h, p_h, -\mu_h))}{\| (u_h, p_h, -\mu_h) \|_{\mathcal{H}}} \gtrsim \| (\eta_h, v_h, \mu_h) \|_{\mathcal{H}}$$

Thus

$$\sup_{(u_h,p_h,\xi_h)\in V_h\times Q_h} \frac{\mathcal{A}_h((\eta_h, v_h, \mu_h), (u_h, p_h, \xi_h))}{\|(u_h, p_h, \xi_h)\|_{\mathcal{H}}} \gtrsim \|(\eta_h, v_h, \mu_h)\|_{\mathcal{H}}$$

■

4.2.4 Residual a posteriori error estimation

First of all, we define the local indicators:

$$\eta_i^{(1)} = h_i \|t^{-2}(w_h'' - \theta_h') + f + \lambda_h\|_{0,I_i}$$
$$\eta_i^{(2)} = \|\theta_h'' + t^{-2}(w_h' - \theta_h)\|_{0,I_i}$$

We also define the global indicator

$$\eta^2 = \sum_{I_i \in \mathcal{I}_h} \left(\left(\eta_i^{(1)}\right)^2 + \left(\eta_i^{(2)}\right)^2 \right) \tag{4.25}$$

$$S_m = \|\psi - w_h\|_1 + \sqrt{\langle \lambda_h, (\psi - w_h)_+ \rangle} \tag{4.26}$$

Lemma 4.5

$$\mathcal{A}((\theta - \theta_h, w - w_h, \lambda - \lambda_h); (\theta - \theta_h, z, \lambda_h - \lambda)) \gtrsim \|\theta - \theta_h\|_1^2 + \|w - w_h\|_1^2 + \|\lambda - \lambda_h\|_{-1}^2$$
$$\|z\|_1 \lesssim \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1}$$

Proof: In (4.2), (4.3), we take $(\eta, v, \mu) = (\theta - \theta_h, w - w_h, \lambda - \lambda_h)$. ■

Lemma 4.6

Let z_h the Lagrange interpolation of z.

$$0 \leq \mathcal{L}_h((0, -z_h, 0)) - \mathcal{A}_h((\theta_h, w_h, \lambda_h); (0, -z_h, 0)). \tag{4.27}$$

Proof: In the discret problem (4.15), we take $(\eta, v, \mu) = (0, -z_h, \lambda_h)$. ■

Propostion 4.7

$$\|\theta - \theta_h\|_1^2 + \|w - w_h\|_1^2 + \|\lambda - \lambda_h\|_{-1}^2$$

$$\lesssim \mathcal{L}((\theta - \theta_h, z - z_h, \lambda_h - \lambda)) - \mathcal{A}((\theta_h, w_h, \lambda_h); (\theta - \theta_h, z - z_h, \lambda_h - \lambda))$$

$$+ \alpha h^2 t^2 \int_0^L \Big(\big(- f - t^{-2}(w_h'' - \theta_h') - \lambda_h \big) \big(z_h'' \big) + \big(- \theta_h'' - t^{-2}(w_h' - \theta_h) \big) \big(z_h' \big) \Big) \, dx$$

Proof: From Lemma (4.5) and adding (4.27)

$$\|\theta-\theta_h\|_1^2+\|w-w_h\|_1^2+\|\lambda-\lambda_h\|_{-1}^2 \lesssim \mathcal{A}((\theta-\theta_h,w-w_h,\lambda-\lambda_h);(\theta-\theta_h,z,\lambda_h-\lambda))$$

$$=\mathcal{A}((\theta,w,\lambda);(\theta-\theta_h,z,\lambda_h-\lambda))-\mathcal{A}((\theta_h,w_h,\lambda_h);(\theta-\theta_h,z,\lambda_h-\lambda))$$

$$\leq \mathcal{A}((\theta,w,\lambda);(\theta-\theta_h,z,\lambda_h-\lambda))-\mathcal{A}((\theta_h,w_h,\lambda_h);(\theta-\theta_h,z,\lambda_h-\lambda))+\mathcal{L}_h((0,-z_h,0))$$

$$-\mathcal{A}_h((\theta_h,w_h,\lambda_h);(0,-z_h,0))$$

$$\leq \mathcal{A}((\theta,w,\lambda);(\theta-\theta_h,z,\lambda_h-\lambda))-\underline{\mathcal{A}((\theta_h,w_h,\lambda_h);(\theta-\theta_h,z,\lambda_h-\lambda))}+\underline{\underline{\mathcal{L}((0,-z_h,0))}}$$

$$-\underline{\mathcal{A}((\theta_h,w_h,\lambda_h);(0,-z_h,0))}-\alpha\mathscr{L}_h((0,-z_h,0))+\alpha\mathcal{S}_h((\theta_h,w_h,\lambda_h);(0,-z_h,0))$$

$$+\underline{\underline{\mathcal{L}((\theta-\theta_h,z,\lambda-\lambda_h))}}-\mathcal{L}((\theta-\theta_h,z,\lambda-\lambda_h))$$

$$=\mathcal{L}((\theta-\theta_h,z-z_h,\lambda_h-\lambda))-\mathcal{A}((\theta_h,w_h,\lambda_h);(\theta-\theta_h,z-z_h,\lambda_h-\lambda))-\alpha\mathscr{L}_h((0,-z_h,0))$$

$$+\alpha\mathcal{S}_h((\theta_h,w_h,\lambda_h);(0,-z_h,0))\underbrace{+\mathcal{A}((\theta,w,\lambda);(\theta-\theta_h,z,\lambda_h-\lambda))-\mathcal{L}((\theta-\theta_h,z,\lambda-\lambda_h))}_{\text{negative }(-)\text{ from(4.1)}}$$

$$\leq \mathcal{L}((\theta-\theta_h,z-z_h,\lambda_h-\lambda))-\mathcal{A}((\theta_h,w_h,\lambda_h);(\theta-\theta_h,z-z_h,\lambda_h-\lambda))$$

$$+\alpha h^2t^2\int_0^L\Big(\big(-f-t^{-2}(w_h''-\theta_h')-\lambda_h\big)\big(z_h''\big)+\big(-\theta_h''-t^{-2}(w_h'-\theta_h)\big)\big(z_h'\big)\Big)\,dx$$

■

Lemma 4.7

$$\langle\lambda_h-\lambda,w_h-\psi\rangle\leq\frac{\|\lambda-\lambda_h\|_{-1}^2}{2}+S_m^2 \tag{4.28}$$

Proof: If we take $\eta_h=v_h=0$ in the discret problem (4.7) we get

$$-\langle\mu_h-\lambda_h,w_h\rangle\leq-\langle\mu_h-\lambda_h,\psi\rangle \tag{4.29}$$

and for the choice $\mu_h=0$, $\mu_h=2\lambda_h$ in (4.29), we find that

$$\langle\lambda_h,w_h-\psi\rangle=0$$

Then,

$$\begin{aligned}\langle\lambda_h-\lambda,w_h-\psi\rangle&=\langle\lambda,\psi-w_h\rangle\\&\leq\langle\lambda,(\psi-w_h)_+\rangle\\&=\langle\lambda-\lambda_h,(\psi-w_h)_+\rangle+\langle\lambda_h,(\psi-w_h)_+\rangle\\&\leq\|\lambda-\lambda_h\|_{-1}\|(\psi-w_h)_+\|_1+\langle\lambda_h,(\psi-w_h)_+\rangle\end{aligned}$$

Hence, by Young's inequality

$$\begin{aligned}\langle \lambda_h - \lambda, w_h - \psi \rangle &\le \frac{\|\lambda - \lambda_h\|_{-1}^2}{2} + \frac{\|(\psi - w_h)_+\|_1^2}{2} + \langle \lambda_h, (\psi - w_h)_+ \rangle \\ &\le \frac{\|\lambda - \lambda_h\|_{-1}^2}{2} + S_m^2\end{aligned}$$

■

Theorem 4.3 (*Indicator reliability*)

$$\|\theta - \theta_h\|_1 + \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1} \lesssim \eta + S_m \tag{4.30}$$

Proof: From Proposition (4.7)

$$\|\theta - \theta_h\|_1^2 + \|w - w_h\|_1^2 + \|\lambda - \lambda_h\|_{-1}^2$$

$$\lesssim \mathcal{L}((\theta - \theta_h, z - z_h, \lambda_h - \lambda)) - \mathcal{A}((\theta_h, w_h, \lambda_h); (\theta - \theta_h, z - z_h, \lambda_h - \lambda))$$

$$+ \alpha h^2 t^2 \int_0^L \Big(\big(- f - t^{-2}(w_h'' - \theta_h') - \lambda_h \big) \big(z_h'' \big) + \big(- \theta_h'' - t^{-2}(w_h' - \theta_h) \big) \big(z_h' \big) \Big) \, dx$$

$$= (f, z - z_h) - \langle \lambda_h - \lambda, \psi \rangle - a(\theta_h, \theta - \theta_h) - t^{-2} b((\theta_h, w_h); (\theta - \theta_h, z - z_h)) + \langle \lambda_h, z - z_h \rangle$$

$$+ \langle \lambda_h - \lambda, w_h \rangle + \alpha h^2 t^2 \int_0^L \Big(\big(- f - t^{-2}(w_h'' - \theta_h') - \lambda_h \big) \big(z_h'' \big) + \big(- \theta_h'' - t^{-2}(w_h' - \theta_h) \big) \big(z_h' \big) \Big) \, dx$$

$$\lesssim (f, z - z_h) - \int_0^L \theta_h'(\theta - \theta_h)' \, dx - t^{-2} \int_0^L (w_h' - \theta_h)(z - z_h)' \, dx + t^{-2} \int_0^L (w_h' - \theta_h)(\theta - \theta_h) \, dx$$

$$+ \langle \lambda_h, z - z_h \rangle + \alpha t^2 \int_0^L \Big(h \big(- f - t^{-2}(w_h'' - \theta_h') - \lambda_h \big) h \big(z_h'' \big) \Big) \, dx$$

$$+ \alpha t^2 \int_0^L \Big(\big(- \theta_h'' - t^{-2}(w_h' - \theta_h) \big) h \big(z_h' \big) \Big) \, dx + \langle \lambda_h - \lambda, w_h - \psi \rangle$$

Using (4.16), (4.17), Cauchy-Schwarz and Young's inequalities

$$\|\theta-\theta_h\|_1^2+\|w-w_h\|_1^2+\|\lambda-\lambda_h\|_{-1}^2 \lesssim \sum_{I_i\in\mathcal{I}_h}(f,z-z_h)_{0,I_i}-\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\theta_h'(\theta-\theta_h)'dx$$

$$-t^{-2}\sum_{I_i\in\mathcal{I}_h}\int_{I_i}(w_h'-\theta_h)(z-z_h)'dx+t^{-2}\sum_{I_i\in\mathcal{I}_h}\int_{I_i}(w_h'-\theta_h)(\theta-\theta_h)dx+\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\lambda_h(z-z_h)dx$$

$$+\alpha t^2\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\Big(h_i\big(-f-t^{-2}(w_h''-\theta_h')-\lambda_h\big)h_i\big(z_h''\big)\Big)dx$$

$$+\alpha t^2\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\Big(\big(-\theta_h''-t^{-2}(w_h'-\theta_h)\big)h_i\big(z_h'\big)\Big)\,dx+\langle\lambda_h-\lambda,w_h-\psi\rangle$$

$$=\sum_{I_i\in\mathcal{I}_h}\int_{I_i}h_i\big(t^{-2}(w_h''-\theta_h')+f+\lambda_h\big)h_i^{-1}\big(z-z_h\big)dx+\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\big(\theta_h''+t^{-2}(w_h'-\theta_h)\big)\big(\theta-\theta_h\big)dx$$

$$+\alpha t^2\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\Big(h_i\big(-f-t^{-2}(w_h''-\theta_h')-\lambda_h\big)h_i\big(z_h''\big)\Big)dx$$

$$+\alpha t^2\sum_{I_i\in\mathcal{I}_h}\int_{I_i}\Big(\big(-\theta_h''-t^{-2}(w_h'-\theta_h)\big)h_i\big(z_h'\big)\Big)\,dx+\langle\lambda_h-\lambda,w_h-\psi\rangle$$

$$\le\Big(\sum_{I_i\in\mathcal{I}_h}h_i^2\|t^{-2}(w_h''-\theta_h')+f+\lambda_h\|_{0,I_i}^2\Big)^{1/2}\Big(\sum_{I_i\in\mathcal{I}_h}\big(h_i^{-1}\|z-z_h\|_{0,I_i}\big)^2\Big)^{1/2}$$

$$+\Big(\sum_{I_i\in\mathcal{I}_h}\|\theta_h''+t^{-2}(w_h'-\theta_h)\|_{0,I_i}^2\Big)^{1/2}\Big(\sum_{I_i\in\mathcal{I}_h}\|\theta-\theta_h\|_{0,I_i}^2\Big)^{1/2}$$

$$+\alpha\Big(\sum_{I_i\in\mathcal{I}_h}h_i^2\|t^{-2}(w_h''-\theta_h')+f+\lambda_h\|_{0,I_i}^2\Big)^{1/2}\Big(\sum_{I_i\in\mathcal{I}_h}h_i^2\|z_h''\|_{0,I_i}^2\Big)^{1/2}$$

$$+\alpha\Big(\sum_{I_i\in\mathcal{I}_h}\|\theta_h''+t^{-2}(w_h'-\theta_h)\|_{0,I_i}^2\Big)^{1/2}\Big(\sum_{I_i\in\mathcal{I}_h}h_i^2\|z_h'\|_{0,I_i}^2\Big)^{1/2}+\langle\lambda_h-\lambda,w_h-\psi\rangle$$

$$\lesssim\frac12\sum_{I_i\in\mathcal{I}_h}\Big(\eta_i^{(1)}\Big)^2+\frac12\|z\|_1^2+\frac12\sum_{I_i\in\mathcal{I}_h}\Big(\eta_i^{(2)}\Big)^2+\frac12\|\theta-\theta_h\|_1^2+\frac{\alpha}{2}\sum_{I_i\in\mathcal{I}_h}\Big(\eta_i^{(1)}\Big)^2+\frac{\alpha}{2}\|z_h\|_1^2$$

$$+\frac{\alpha}{2}\sum_{I_i\in\mathcal{I}_h}\Big(\eta_i^{(2)}\Big)^2+\frac{\alpha}{2}\|z_h\|_0^2+\langle\lambda_h-\lambda,w_h-\psi\rangle$$

Since

$$\|z_h\|_1 \lesssim \|z\|_1$$
$$\|z_h\|_0 \lesssim \|z_h\|_1 \lesssim \|z\|_1$$

Then

$$\|\theta - \theta_h\|_1^2 + \|w - w_h\|_1^2 + \|\lambda - \lambda_h\|_{-1}^2 \lesssim \eta^2 + \|z\|_1^2 + \langle \lambda_h - \lambda, w_h - \psi \rangle$$

$$\lesssim \eta^2 + \|w - w_h\|_1^2 + \|\lambda - \lambda_h\|_{-1}^2 + \frac{\|\lambda - \lambda_h\|_{-1}^2}{2} + S_m^2$$

So

$$\|\theta - \theta_h\|_1^2 + \|w - w_h\|_1^2 + \|\lambda - \lambda_h\|_{-1}^2 \lesssim \eta^2 + S_m^2$$

Hence

$$\|\theta - \theta_h\|_1 + \|w - w_h\|_1 + \|\lambda - \lambda_h\|_{-1} \lesssim \eta + S_m$$

■

CHAPTER 5

NUMERICAL SIMULATIONS

5.1 Numerical tests for the mixed formulation

In this section, we check the results proven in the chapter 3 using the Uzawa method.

We have the discrete mixed variational problem

$$\begin{cases} \text{Find } (\theta_h, w_h, \lambda_h) \in V_h \times \boldsymbol{\mu}_h \\ \boldsymbol{a}((\theta_h, w_h), (\eta_h, v_h)) + \boldsymbol{b}((\eta_h, v_h), \lambda_h) = F(\eta_h, v_h), & \forall (\eta_h, v_h) \in V_h \\ \boldsymbol{b}((\theta_h, w_h), \mu_h - \lambda_h) \leq G(\mu_h - \lambda_h), & \forall \mu_h \in \boldsymbol{\mu}_h \end{cases} \tag{5.1}$$

The variational form of Uzawa method

$$\begin{cases} \boldsymbol{a}((\theta_h^k, w_h^k); (\eta_h, v_h)) = F(\eta_h, v_h) - \boldsymbol{b}((\eta_h, v_h), \lambda_k), \quad \forall (\eta_h, v_h) \in V_h \\ (\widetilde{\lambda}_h^{k+1}, \mu_h) = (\lambda_h^k, \mu_h) + \alpha \boldsymbol{b}((\theta_h^k, w_h^k), \mu_h) - \alpha G(\mu_h), \quad \forall \mu_h \in \boldsymbol{\mu}_h \\ \lambda_h^{k+1} = \mathcal{P}_{\mu_h}(\widetilde{\lambda}_h^{k+1}) \end{cases} \tag{5.2}$$

Now the algorithm of this method

Algorithm 1 Uzawa method

1. Give some initial value λ_h^0
2. $k = 0$
3. Repeat:
4. Compute (θ_h^k, w_h^k) from the equation
$$\boldsymbol{a}((\theta_h^k, w_h^k); (\eta_h, v_h)) = F(\eta_h, v_h) - \boldsymbol{b}((\eta_h, v_h), \lambda_k)$$
5. Compute $\widetilde{\lambda}_h^{k+1}$ from the equation
$$(\widetilde{\lambda}_h^{k+1}, \mu_h) = (\lambda_h^k, \mu_h) + \alpha \boldsymbol{b}((\theta_h^k, w_h^k), \mu_h) - \alpha G(\mu_h)$$
6. Take λ_h^{k+1} as the projection of $\widetilde{\lambda}_h^{k+1}$ on $\boldsymbol{\mu}_h$:
$$\lambda_h^{k+1} = \mathcal{P}_{\mu_h}(\widetilde{\lambda}_h^{k+1})$$
7. $k = k + 1$
8. Until $\|\lambda_h^{k+1} - \lambda_h^k\| \leq \varepsilon$

We consider the domain $\Omega = (0, 1)$, the obstacle defined by $\psi(x) = 4x^3 - 9x^2 + 6x - 1$, the force $f = 0$.

Previously, we have proven that Timoshenko problem converges to the Euler-Bernoulli problem when t tends to 0. For the previous data, we can give the exact solution of the Euler-Bernoulli problem

$$w(x) = \begin{cases} -4x^3 + 3x^2 & \text{if } x < \frac{1}{2} \\ 4x^3 - 9x^2 + 6x - 1 & \text{if } x \geq \frac{1}{2} \end{cases}$$

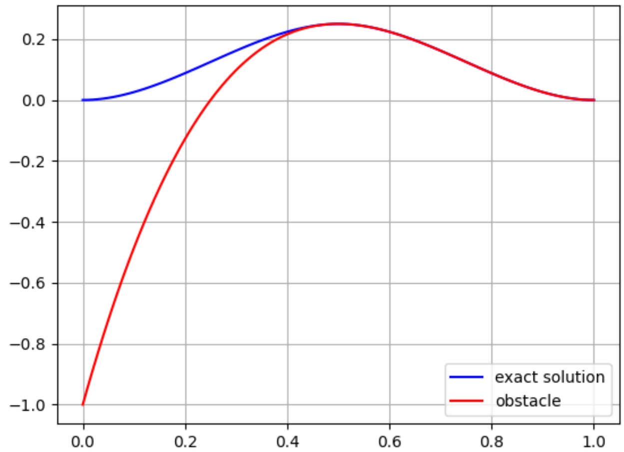


Figure 5.1: The solution $w(x)$ and the obstacle $\psi(x)$

Now, we test different finite element spaces to verify which pair leads to a stable system. In other words, locking free.

We took $\alpha = 0.001$ and $t = 10^{-5}$.

Example 1 :

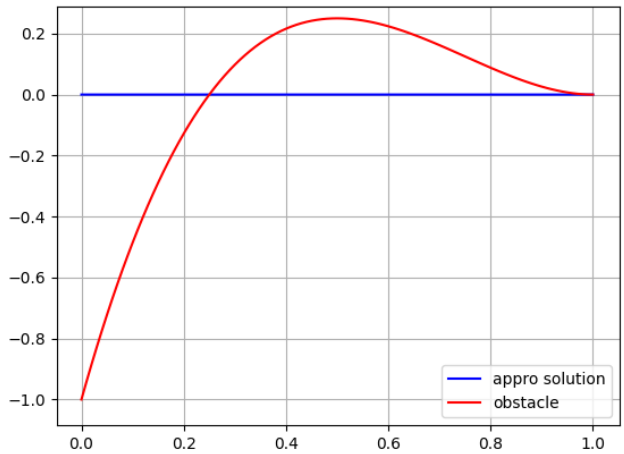


Figure 5.2: $V_h = V_h^1 \times V_h^1, Q_h = V_h^0$

Example 2 :

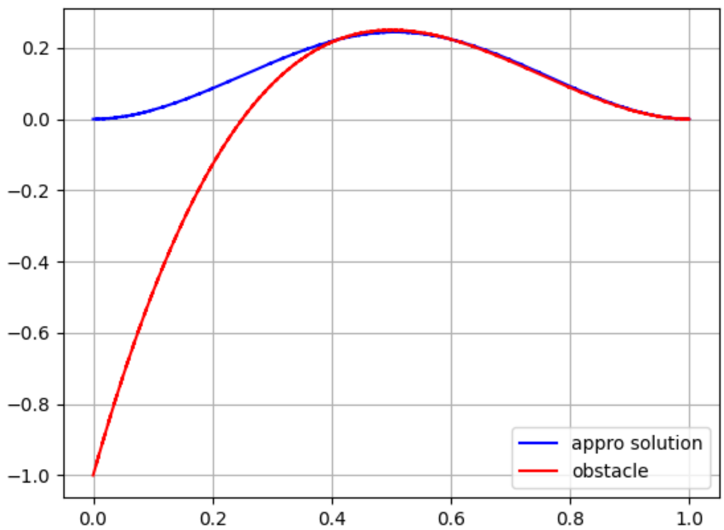


Figure 5.3: $V_h = V_h^2 \times V_h^1, Q_h = V_h^0$

Example 3 :

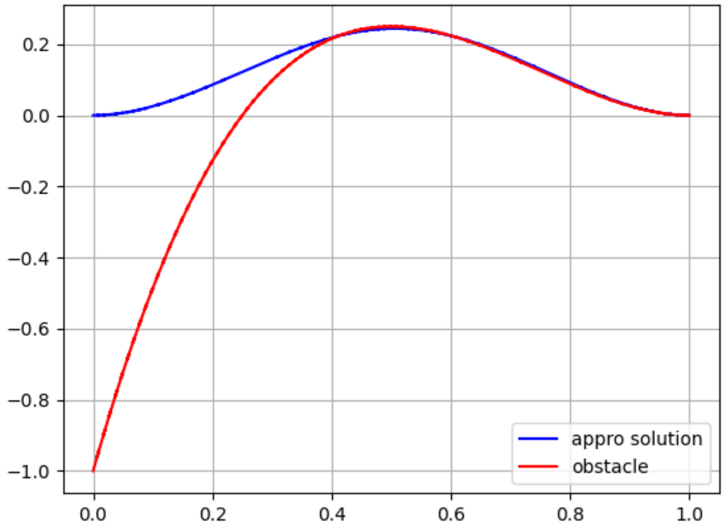


Figure 5.4: $V_h = V_h^3 \times V_h^2, Q_h = V_h^0$

Example 4 :

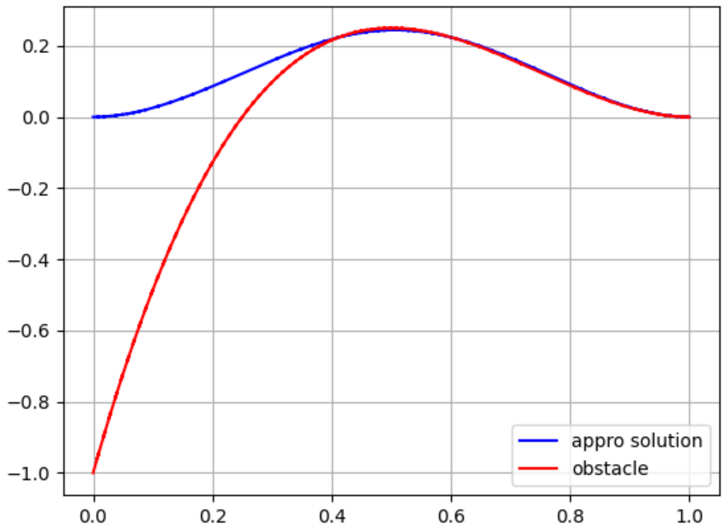


Figure 5.5: $V_h = V_h^3 \times V_h^1, Q_h = V_h^0$

5.2 Numerical tests for the stabilized formulation

In this section, we will prove numerically that the stabilized method leads to a stable system even with low-order finite elements using the Newton method.

Our idea is replacing the constrained minimization problem with an unconstrained minimization problem by adding a non linear term.

We have the discrete stabilized formulation

$$\begin{cases} \text{Find } (\theta_h, w_h, \lambda_h) \in V_h \times \mu_h \\ \mathcal{A}_h((\theta_h, w_h, \lambda_h), (\eta_h, v_h, \mu_h - \lambda_h)) \leq \mathcal{L}_h(\eta_h, v_h, \mu_h - \lambda_h),\ \forall (\eta_h, v_h, \mu_h) \in V_h \times \mu_h \end{cases} \tag{5.3}$$

With choice of test functions, (5.3) equivalent to the following mixed formulation

$$\begin{cases} \boldsymbol{a}_h((\theta_h, w_h); (\eta_h, v_h)) - \boldsymbol{b}_h((\eta_h, v_h), \lambda_h) = \ell_h(\eta_h, v_h) \\ \boldsymbol{b}_h((\theta_h, w_h), \mu_h - \lambda_h) + \langle \lambda_h, \mu_h - \lambda_h \rangle_h \geq g_h(\mu_h - \lambda_h) \end{cases} \tag{5.4}$$

where

$$\begin{cases} \boldsymbol{a}_h((\theta_h, w_h); (\eta_h, v_h)) = \boldsymbol{a}((\theta_h, w_h); (\eta_h, v_h)) - \alpha h^2 t^4 \displaystyle\int_0^L \Big[\big(\theta_h'' + t^{-2}(w_h' - \theta_h)\big)\big(\eta_h'' + t^{-2}(v_h' - \eta_h)\big) \\ \qquad + t^{-4}\big(w_h'' - \theta_h'\big)\big(v_h'' - \eta_h'\big)\Big]\, dx \\ \boldsymbol{b}_h((\eta_h, v_h), \mu_h) = \big\langle \mu_h, v_h + \alpha h^2 t^2 (v_h'' - \eta_h') \big\rangle \\ \ell_h(\eta_h, v_h) = (f, v_h) + \alpha h^2 t^2 \displaystyle\int_0^L f(v_h'' - \eta_h') \\ \langle \lambda_h, \mu_h - \lambda_h \rangle_h = \alpha h^2 t^4 \langle \lambda_h, \mu_h - \lambda_h \rangle \\ g_h(\mu_h - \lambda_h) = \big\langle \mu_h - \lambda_h, \psi - \alpha h^2 t^4 f \big\rangle \end{cases}$$

We can reformulate the problem (5.4) as a minimisation problem of the following nonlinear functional:

$$\begin{aligned} J_h(\eta_h, v_h) := &\frac{1}{2}\boldsymbol{a}_h((\eta_h, v_h); (\eta_h, v_h)) - \ell_h(\eta_h, v_h) \\ &+ \frac{1}{3\alpha h^2 t^4} \int_0^L \big(v_h - \psi + \alpha h^2 t^2 f + \alpha h^2 t^2 (v_h'' - \eta_h')\big)_-^3 \end{aligned} \tag{5.5}$$

$$F((\theta_h, w_h), (\eta_h, v_h)) = \langle J_h'(\theta_h, w_h), (\eta_h, v_h)\rangle = \boldsymbol{a}_h((\theta_h, w_h); (\eta_h, v_h)) - \ell_h(\eta_h, v_h) + \frac{1}{\alpha h^2 t^4}\int_0^L \big(w_h - \psi + \alpha h^2 t^2 f + \alpha h^2 t^2 (w_h'' - \theta_h')\big)_-^2 \big(v_h + \alpha h^2 t^2 (v_h'' - \eta_h')\big) \quad (5.6)$$

$$\langle F'((\theta_h, w_h); (\eta_h, v_h)), (d_{\theta_h}, d_{w_h})\rangle = \boldsymbol{a}_h((d_{\theta_h}, d_{w_h}); (\eta_h, v_h)) + \frac{2}{\alpha h^2 t^4}\int_0^L \big(w_h - \psi + \alpha h^2 t^2 f + \alpha h^2 t^2 (w_h'' - \theta_h')\big)_- \big(d_{w_h} + \alpha h^2 t^2 (d_{w_h}'' - d_{\theta_h}')\big)\big(v_h + \alpha h^2 t^2 (v_h'' - \eta_h')\big) \quad (5.7)$$

The variational form of Newton method

$$\begin{cases} \Big\langle F'((\theta_h^k, w_h^k); (\eta_h, v_h)), (d_{\theta_h^k}, d_{w_h^k})\Big\rangle = -F((\theta_h^k, w_h^k), (\eta_h, v_h)) \quad \forall (\eta_h, v_h) \in V_h \\ (\theta_h^{k+1}, w_h^{k+1}) = (\theta_h^k, w_h^k) + (d_{\theta_h^k}, d_{w_h^k}) \end{cases} \quad (5.8)$$

Now the algorithm of this method

Algorithm 2 stabilized method

1. Give some initiale value (θ_0, w_0)
2. $k = 0$
3. Repeat:
4. Compute $(d_{\theta_h^k}, d_{w_h^k})$ from the equation

$$\Big\langle F'((\theta_h^k, w_h^k); (\eta_h, v_h)), (d_{\theta_h^k}, d_{w_h^k})\Big\rangle = -F((\theta_h^k, w_h^k), (\eta_h, v_h))$$

5. Compute $(\theta_h^{k+1}, w_h^{k+1})$ from the equation

$$(\theta_h^{k+1}, w_h^{k+1}) = (\theta_h^k, w_h^k) + (d_{\theta_h^k}, d_{w_h^k})$$

7. $k = k + 1$
8. Until $\|d_\theta\|_\infty + \|d_w\|_\infty \leq \varepsilon$

To make sure that the method converges, we give as initial guess $(\theta_0, w_0) = (w', w)$ where w is the exact solution of Euler-Bernoulli problem. We program this method using the same data and parameters, $V_h = V_h^1 \times V_h^1$, $Q_h = V_h^0$ that led to a singular system.

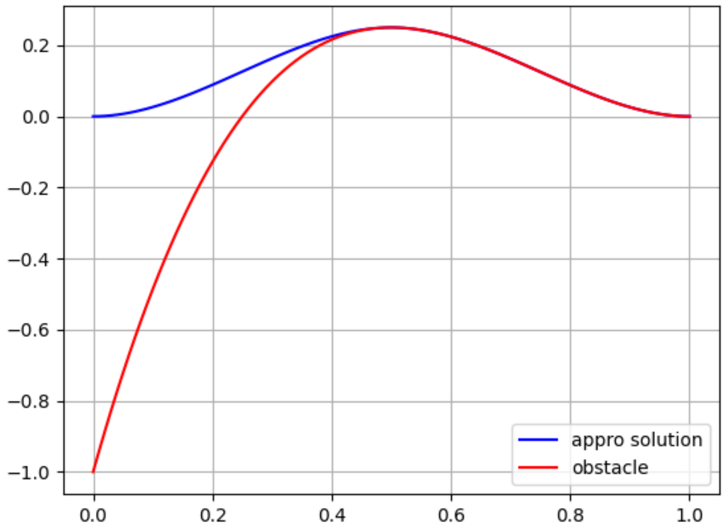


Figure 5.6: $V_h = V_h^1 \times V_h^1, Q_h = V_h^0$

Iteration	$\|d_\theta\|_\infty + \|d_w\|_\infty$
3	0.00156692
5	0.000292946
7	7.13867e-05
9	1.78467e-05
11	4.46167e-06
15	2.78841e-07
19	1.92356e-08
23	1.39771e-09
24	8.84545e-10

Table 5.1: The error estimate

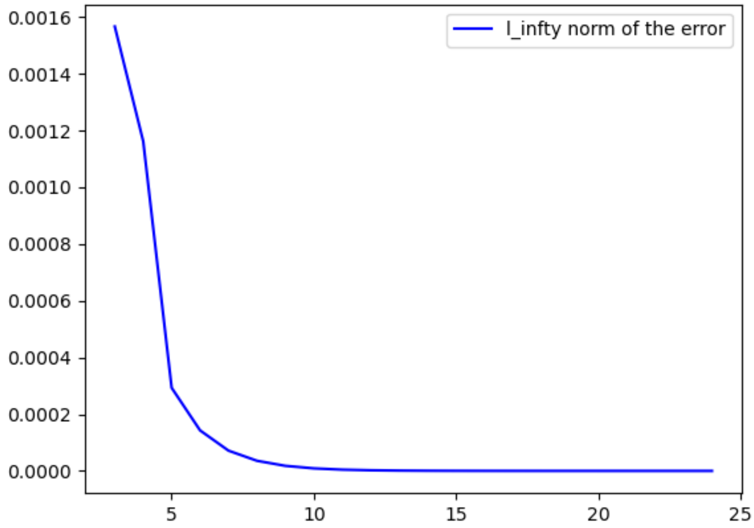


Figure 5.7: the error curve for the stabilized formulation $V_h = V_h^1 \times V_h^1, Q_h = V_h^0$

Now, we study the convergence of Newton method: each time, we modify the value of α, then we report the value of the error $\|\theta - \theta_h\|_0 + \|w - w_h\|_0$ in the following table

α	$\|\theta - \theta_h\|_0 + \|w - w_h\|_0$
0.001	0.00203667
0.01	0.00233247
0.1	0.00423994
1	0.00218611
1.5	0.000647212
2	0.00191294
2.5	0.00101347

Table 5.2: The error estimate.

CONCLUSION

In this work, we studied the Timoshenko obstacle problem, which is a significant challenge in numerical simulations due to the presence of a small parameter t. The locking phenomena arises for very small values of this parameter implies the necessity of the employing of specialized methods to prevent these effects. We used a stabilized method to address this issue, which involves defining a discrete stabilized formulation and verifying that the **inf-sup** condition is satisfied for almost all finite element spaces. This condition ensures the stability of the method. To further validate the efficacy of this method, we applied the Newton method for the non linear problem.

As perspectives or extensions of the present work, we believe that we can consider the following problems:

- The optimality of the the indicator η.
- The Reissner-Mindlin model.
- The Naghdi's shell model.

ABSTRACT

We are interested in the finite element approximation of the Timoshenko obstacle problem. This problem depends on a small parameter that causes the locking phenomenon. We used the Lagrange multiplier method with mixed and stabilized finite element methods. In both methods, we studied the mathematical and numerical analysis to see the difference between the two methods and determine which method is better in preventing the locking effect.

Keywords: Timoshenko model, locking phenomena,inf-sup condition, mixed formulation, stabilized formulation.

Résumé

Nous sommes intéressés à l'approximation par éléments finis du problème de la poutre de Timoshenko avec obstacle. Ce problème dépend d'un petit paramètre qui provoque le phénomène de verrouillage. Nous avons utilisé la méthode du multiplicateur de Lagrange avec des méthodes d'éléments finis mixtes et stabilisées. Dans les deux méthodes, nous avons étudié l'analyse mathématique et numérique pour voir la différence entre les deux méthodes et déterminer quelle méthode est la meilleure pour éviter l'effet de verrouillage.

Mots clés: modèle de Timoshenko, phénomène de verrouillage, condition d'inf-sup, formulation mixte, formulation stabilisée.

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