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Theme

Asymptotic behaviour of solutions of some thermoelastic plate problems

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Before the jury composed of:

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Dedication

Praise be too Allah without whom no effort would be accomplished and no goal no be attained
To my dear parents, who always support
and
encourage me to achieve my ambition.
To my loving brothers and sisters
who were always there for me.

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Grate thanks go also to my best friend's and my colleagues in Math department.
I give special thanks to everyone in my life who helped me in my academic pursuit.

ABSTRACT

In this work, we care to study the Cauchy problem for thermoelastic plate systems, our interest was about the study done (in 2023) by Chen and Liu: "A not on asymptotic profiles for the thermoelastic plate system" proce. of the AMS 151(10): 4317 4329 2023. They consider the Cauchy problem for thermoelastic plate systems with Newton's Law of cooling.

Firstly, we define some important notations, and we recall some mathematical concepts that will be used throughout this dissertation. Then, we verified the results obtained in the article which is about the asymptotic behaviors of solutions, this was done by using the reduction, Pointwise estimates and auxiliary functions in the Fourier spaces, where we checked the growth ($n \leq 4$) or decay ($n \geq 5$) estimates and the asymptotic profiles of solutions for large time at the end.

Keywords:
thermoelastic plate system, Newton's law of cooling, Cauchy problem, growth estimate, decay estimate, Fourier spaces, asymptotic profile.

الملخص

في هذا العمل نهتم بدراسة مسألة كوشي لجملة معادلات الصفائح ذات المرونة الحرارية. اهتمامنا كان حول الدراسة التي قام بها المؤلفان Liu و Chen في المقال:

“A not on asymptotic profiles for the thermoelastic plate system” proce of AMS,2023

الذي تناول جملة معادلات الصفائح ذات المرونة الحرارية مع وجود قانون نيوتن للتبريد.

في البداية قدمنا بعض المفاهيم والمتباينات والمبرهنات المستخدمة في هذا المقال, ثم قمنا بالتحقق من النتائج المتحصل عليها في المقال حول السلوك التقاربي للحل وذلك باستخدام طريقة التخفيض، التقديرات النقطية والدوال المساعدة في فضاء فوربيه. حيث تحققنا من:

تزايد الحل من اجل $n \geq 4$

والتناقص من اجل $n \leq 5$

ثم في الأخير النسخة التقاربية للحل في زمن كبير.

الكلمات_المفتاحية_ جملة معادلات الصفائح ذات المرونة الحرارية، قانون نيوتن للتبريد، مسألة كوشي، تقدير التزايد، تقدير التناقص، فضاء فوربيه النسخة التقاربية.

Résumé

Dans ce travail nous soucions étudier le problème de Cauchy pour les systèmes de plaques thermoélastiques, notreL'intérêt était sur l'étude réalisée (en 2023) par Chen et Liu : " A not on AsymptotiqueProfils pour le système de plaques thermoélastiques" procédé de l'AMS 151(10): 4317 4329 2023. Ils considèrent le problème de Cauchy pour les systèmes thermoélastiques avec la loi de Newton de refroidissement. Tout d'abord, nous définissons quelques notations importantes, et nous rappelons quelques notions mathématiques. concepts qui seront utilisés tout au long de cette thèse. Ensuite, nous avons rerifié les résultats obtenus dans l'article qui concerne les comportements asymptotiques des solutions, cela a été fait en utilisant la réduction, les estimations ponctuelles et les fonctions auxiliaires dans les espaces de Fourier,où nous avons vérifié les estimations de croissance (n $\leq$4) ou de décroissance (n$\geq$ 5), et le asymptotique profils de solutions à la fin.

Mots clés: le séstéme de plaques thermolastique, la loi de Newton de refroidissement, probléme de Cauchy, estimations de croissance, estimations de décroissance, les espaces de Fourier, profil asymptotique

Contents

Notation

- we define the following zones of the Fourier space:
  $\mathcal{Z}_{int}(\varepsilon_0) := \{|\xi| \leq \varepsilon_0 \ll 1\}$,
  $\mathcal{Z}_{bdd}(\varepsilon_0, N_0) := \{\varepsilon_0 \leq |\xi| \leq N_0\}$,
  $\mathcal{Z}_{ext}(N_0) := \{|\xi| \geq N_0 \gg 1\}$.

  the cut-off functions χ_{int}, χ_{bdd}, $\chi_{ext} \in \mathcal{C}^\infty$ own supports in their corresponding zones $\mathcal{Z}_{int}(\varepsilon_0)$, $\mathcal{Z}_{bdd}(\varepsilon_0/2, 2N_0)$ and $\mathcal{Z}_{ext}(N_0)$, respectively.

  such that $\chi_{bdd}(\xi) = 1 - \chi_{int}(\xi) - \chi_{ext}(\xi)$ for all $\xi \in \mathbb{R}^n$
- $f \lesssim g$ means that there exists a positive constant C fulfilling $f \leq Cg$, which may be changed in different lines, analogously, for $f \gtrsim g$.
- the asymptotic behaviors $f \simeq g$ holds if and only if $g \lesssim f \lesssim g$.
- $\langle ., . \rangle$ by the inner product in Euclidean space.
- $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$ is the Japanese bracket.
- the weighted L^1 space :

$$L^{1,1} := \left\{ f \in L^1 : \|f\|_{L^{1,1}} := \int_{\mathbb{R}^n} (1 + |x|)|f(x)|dx < \infty \right\}.$$

so that $\|f\|_{L^1} \leq \|f\|_{L^{1,1}}$

- a summable function f are denoted by

$$P_f := \int_{\mathbb{R}^n} f(x)dx \ , \ M_f := \int_{\mathbb{R}^n} xf(x)dx$$

the time-independent functions

$$\mathcal{D}_n(t) := \begin{cases} t^{1-\frac{n}{4}} & \text{if } n \leq 3, \\ \sqrt{\ln t} & \text{if } n = 4, \\ t^{\frac{1}{2}-\frac{n}{8}} & \text{if } n \geq 5, \end{cases} \ \textit{and} \ \mathcal{B}_n(t) := \begin{cases} t^{\frac{1}{4}} & \text{if } n = 1, \\ \sqrt{\ln t} & \text{if } n = 2, \\ t^{\frac{1}{4}-\frac{n}{8}} & \text{if } n \geq 3, \end{cases} \tag{1}$$

Introduction

Problem modeling

Thin plates equation modeling

Thin plate theory has numerous practical applications in engineering, (e.g. airport runways, raft foundations, and road pavements). In the last century, various mathematical models have been developed to describe the motions of thin plates under different cases, these models for instance, Mindlin-Timoshenko models, Von Karman equations, and thermoelastic plate systems

Let us consider a homogeneous, elastic, and thermally isotropic plate subjected to temperature distribution. By combining the second law of thermodynamics for irreversible processes, the monographs [14, 15] modeled the well-known thermoelastic plate system with Fourier's law of heat conduction and Newton's law of cooling, namely,

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta\theta = 0, \\ \theta_t - \Delta\theta + \sigma\theta - \Delta u_t = 0, \end{cases} \tag{2}$$

Here, the scalar unknowns $u = u(t, x)$ and $\theta = \theta(t, x)$ denote, respectively, the vertical displacement and the temperature (relative to some reference temperature), the non-negative constant σ in the temperature equation is related to the heat transfer coefficient.For the heuristic derivations of the thermoelastic plate system (2), we mention [13]

Fourier's law of heat conduction

The classical thermoelasticity is concerned with the effect of heat on the deformation of an elastic solid and with the inverse effect of deformation on the thermal state of the solid. In the classical linear model for heat propagation, the heat flux is governed by Fourier's law of heat conduction, which states that the heat flux is proportional to the gradient of temperature. i.e

$$q(x,t) = -\delta \nabla \theta(x,t),$$

where x stands for the material point,t is the time, θ is the temperature (difference to a fixed constant reference temperature), q is the heat flux vector and δ is the coefficient of thermal conductivity.

Newton's law of cooling

Newton's law of cooling states that the rate of heat loss of a body is directly proportional to the temperature difference between the body and its surroundings. This means that the hotter on object is, the faster it will cool down. The law is often expressed mathematically as :

$$T(t) = T_{env} + (T_0 - T_{env})e^{-kt}$$

where $T(t)$ is the temperature of the object at given timet, T_{env} is the temperature of the environment, T_0 is initial temperature of the object, and k is the cooling constant.

Over the past thirty years, there has been extensive research on the thermoelastic plate system (2) from various communities, including partial differential equation (PDEs), controllability, inverse problems, and dynamical systems (in [4, 8, 9, 15, 24, 26]).

Indeed, most of the recent studies investigated the thermoelastic plate system (4) with $\sigma = 0$, for example the corresponding Cauchy problem for (4) with $\sigma = 0$ has been thoroughly investigated in [4, 23, 25], where sharp decay proprieties of an energy term $(u_t, \Delta u, \theta)$ have been discovered. Among these result, in the work [4] the authors discovered the critical dimension $n = 4$ for the vertical displacement, which determines different large-time behaviors, namely:

optimal growth for$n \leq 3$,
bounded behavior for $n = 4$,
decay estimates for $n \geq 5$
the optimality of these results is ensured by the same behaviors for both upper and lower bounds. Additionally, they founded the large-time asymptotic profile $\psi = \psi(t, x)$ by

$$\begin{aligned}\psi(t,x) :=& F^{-1}_{\xi\to x}\left(\frac{1}{|\xi|^2}\left(e^{-a_0|\xi|^2t} - \cos(a_2|\xi|^2t)e^{-a_0|\xi|^2t}\right)\right)P\Psi_0 \\ &+ F^{-1}_{\xi\to x}\left(\frac{\sin(a_2|\xi|^2t)}{a_2|\xi|^2}e^{-a_1|\xi|^2t}\right)\end{aligned} \tag{3}$$

with $\Psi_0 := 2a_1u_1+\theta_0$ and $\Psi_1 := (a_0^2+a_2^2-a_1^2)u_1+(a_0-a_1)\theta_0$, where the positive constants a_0, a_1, a_2 are defined in the second statement of Proposition 3.1.
Nevertheless, the Cauchy problem for thermoelastic plate system (4) with $\sigma > 0$ so far has not been explored yet. As for other works on the model (2) with $\sigma > 0$, for the exact controllability problem see [12, 13] , and for the exponential stability of semigroups [13, 14].

For the study of asymptotic behavior of solutions of some thermoelastic plate problem depending on the work of WENHUI CHEN and YAN LIU in [3] , we consider the corresponding Cauchy problem for the thermoelastic plate system (4), namely

$$\begin{cases} u_{tt} + \Delta^2 u + \Delta\theta = 0, \\ \theta_t - \Delta\theta + \sigma\theta - \Delta u_t = 0, \\ (u, u_t, \theta)(0, x) = (u_0, u_1, \theta_0)(x). \end{cases} \tag{4}$$

with the constant $\sigma > 0$, and (t,x)$\in \mathbb{R}_+ \times \mathbb{R}^n$ for any n≥ 1.
Our main focus is by employing WKB method(an initialism for Wentzal-Karmers-Brillouin) and Fourier analysis, the influence of the lower-order term $+\sigma\theta$ on the temperature equation (4) from Newton's law of cooling on large-time asymptotic behaviors of the vertical displacement. we obtain optimal growth estimates for n$\leq$4 and decay estimates for n$\geq$5 for the vertical displacement in the L^2 framework.
We will consider the L^2 norm of solutions because it may contribute to study global (in time) existence and asymptotic profiles of solutions to some corresponding nonlinear problems by Duhamel's principle and fixed-point theorem in suitable evolution spaces (see, for example [5, 9, 22]).

Among them, as we will state in Remark 1.1 and Table 1, the growth rates when $n \leq 4$ are the same as those for the pure plate model, but the growth rate when n=4 and decay rates n$\geq$5 are weakened by the lower-order term $+\sigma\theta$ comparing with the result in [4]. In addition, the dominant Fourier multiplier of asymptotic profiles has been greatly changed into

$$F^{-1}_{\xi\to x}\Big(\frac{\sin(|\xi|^2 t)}{|\xi|^2}e^{-\frac{1}{2\sigma}|\xi|^4 t}\Big),$$

which causes the weakened effect of decay rates when n$\leq$4. Indeed, the effect of the lower-order term $+\sigma\theta$ in the temperature (parabolic) equation will propagate throughout $(\Delta\theta, -\Delta u - t)^T$ to the plate model. It leads to slower decay rates for n$\leq$ 4 in comparison with the classical model with $\sigma = 0$.

This work is divided into three chapters:

- In chapter one, we define some important notations, and we recall some mathematical concepts that will be used throughout this memory .
- In chapter two, we study the asymptotic behaviors of solutions by reduction procedure, Pointwise estimates and auxiliary functions in the Fourier spaces .
- In chapter three we find the optimal estimates for the vertical displacement in the L^2 norm, and the asymptotic profile of solution.

Chapter 1

Preliminaries

In this chapter, we recall some notations and review some mathematical concepts that we will use later .

- $u_t = \dfrac{\partial u}{\partial t}$, $u_{tt} = \dfrac{\partial^2 u}{\partial t^2}$, $u_{ttt} = \dfrac{\partial^3 u}{\partial t^3}$
- $\nabla u = \left(\dfrac{\partial u}{\partial x_1}, ..., \dfrac{\partial u}{\partial x_n}\right)$
- $\Delta u = \sum_{i=1}^{n} \dfrac{\partial^2 u}{\partial x_i^2}$
- $\widehat{u+v} = \hat{u} + \hat{v}$
- $\widehat{u * v} = \hat{u} * \hat{v}$
- $\widehat{bu} = b\hat{u}$
- $\widehat{\nabla u(x,t)} = |\xi|\hat{u}(\xi, t)$
- $\widehat{\Delta u(x,t)} = -|\xi|^2\hat{u}(\xi, t)$

1.1 Functional spaces

1.1.1 Lebesgue spaces

Definition 1 *Let Ω be a domain in $\mathbb{R}^n (n \in \mathbb{N})$ for $1 \leq P < \infty$, the Lebesgue space $L^p(\Omega)$ is defined by:*

$$L^p(\Omega) = \Big\{ u : \Omega \longrightarrow \mathbb{R},\ u \text{ is measurable and } \textstyle\int_\Omega |u(x)|^p dx < \infty \Big\},$$

with the norm

$$\|u\|_p = \big(\textstyle\int_\Omega |u(x)|^p dx \big)^{\frac{1}{p}} \ .$$

In addition, we define $L^\infty(\Omega)$by:

$L^\infty(\Omega) = \{u : \Omega \longrightarrow \mathbb{R},\ u$ is measurable and $\exists\ c > 0$ such that $|u(x)| \leq c$ a.e on $\Omega\}$,

equipped with the norm

$\|u\|_\infty = ess \sup |u(x)| = \inf\{c : |u(x)| \leq c \ a.e \ on \ \Omega\}$.

1.1.2 Fourier spaces

Definition 2 *Let $u \in L^1(\mathbb{R}^n)$, we define its Fourier transform*

$$\hat{u}(\xi) = \textstyle\int_{\mathbb{R}^n} u(x) e^{-ix.\xi} dx,\ \forall \xi \in \mathbb{R}^n,$$

and its inverse Fourier transform

$$u(x) = \frac{1}{(2\pi)^n} \textstyle\int_{\mathbb{R}^n} \hat{u}(\xi) e^{ix.\xi} d\xi$$

Theorem 3 ***(Plancherel's theorem)*** *Assume that $u \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Then $\hat{u} \in L^2(\mathbb{R}^n)$ and*

$$\|\hat{u}\|_2 = \|u\|_2,$$

1.2 Some inequalities

Theorem 4 *(**Hausdorff-Young inequality**) For every $u \in L^p(\mathbb{R}^n)$ we have the estimate*

$$\|\hat{u}\|_{p'} \leq \|u\|_p, \tag{1.1}$$

whenever $1 \leq p \leq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$.

Theorem 5 *(**Holder's inequality**) Let $1 \leq p \leq \infty$. if $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, then $uv \in L^1(\Omega)$ and*

$$\|uv\|_1 \leq \|u\|_p \|v\|_{p'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.
By taking $p = p' = 2$, we have the Cauchy-Schwarz inequality .

Theorem 6 *(**Minkowski's inequality**) Let $p \in [1, \infty]$, f,$g \in L^p(\Omega)$. Then, $f + g \in L^p(\Omega)$ and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p$$

Chapter 2

ASYMPTOTIC BEHAVIORS OF SOLUTION IN THE FOURIER SPACE

2.0.1 Reduction procedure

To begin with this section, strongly motivated by the recent work[3], we employ the reduction procedure with respect to the vertical displacement. Due to

$$(\partial_t - \Delta + \sigma I)\theta = \Delta u_t$$

equipping the identity operator I, we act this diffusion operator (4) to arrive at the third-order (in time) evolution equation as follows:

$$\begin{aligned}
&(\partial_t - \Delta + \sigma I)(u_{tt} + \Delta^2 u + \Delta\theta) = 0\\
&u_{ttt} + \Delta^2 ut + \Delta\theta_t - \Delta u_{tt} - \Delta\Delta^2 u - \Delta\Delta\theta + \sigma u_{tt} + \sigma\Delta^2 u + \sigma\Delta\theta = 0\\
&u_{ttt} + \Delta^2 ut + \Delta\theta_t + (\sigma - \Delta)u_{tt} + (\sigma - \Delta)\Delta^2 u - \Delta^2\theta + \sigma\Delta\theta = 0
\end{aligned}$$

by the second equation (4) we get

$$\begin{aligned}
\theta_t - \Delta\theta + \sigma\theta &= \Delta u_t\\
\Delta\theta_t - \Delta^2\theta + \sigma\Delta\theta &= \Delta^2 u_t
\end{aligned}$$

$$\begin{aligned}
&u_{ttt} + (\sigma - \Delta)u_{tt} + \Delta^2 ut + \Delta\theta_t - \Delta^2\theta + \sigma\Delta\theta + (\sigma - \Delta)\Delta^2 u = 0\\
&u_{ttt} + (\sigma - \Delta)u_{tt} + \Delta^2 ut + \Delta^2 ut + (\sigma - \Delta)\Delta^2 u = 0
\end{aligned}$$

$$\begin{cases} u_{ttt} + (\sigma I - \Delta)u_{tt} + 2\Delta^2 u_t + (\sigma - \Delta)\Delta^2 u = 0\\ (u, u_t, u_{tt})(0, x) = (u_0, u_1, u_2)(x), \end{cases} \tag{2.1}$$

with $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$, where the third data is determined by $u_2 := -\Delta^2 u_0 - \Delta\theta_0$.
An application of the partial Fourier transform to the model (4) ,with respect to spatial variables yields

$$\begin{cases} \hat{u}_{ttt} + (\sigma + |\xi|^2)\hat{u}_{tt} + 2|\xi|^4\hat{u}_t + (\sigma + |\xi|^2)|\xi|^4\hat{u} = 0\\ (\hat{u}, \hat{u}_t, \hat{u}_{tt})(0, \xi) = (\hat{u}_0, \hat{u}_1, \hat{u}_2)(\xi), \end{cases} \tag{2.2}$$

with $(t, \xi) \in \mathbb{R}_+ \times \mathbb{R}^n$. Note that $\hat{u}_2 = -|\xi|^4\hat{u}_0 + |\xi|^2\hat{\theta}_0$

Its characteristic equation is given by

$$\lambda^3 + (\sigma + |\xi|^2)\lambda^2 + 2|\xi|^4\lambda + (\sigma + |\xi|^2)|\xi|^4 = 0 \tag{2.3}$$

we have the cubic equation $x^3 + ax^2 + bx + c$, here discriminant given by

$$\Delta_{dis} = a^2b^2 + 18abc - 4b^3 - 4a^3c - 27c^2$$

so, with
$a = (\sigma + |\xi|^2)$
$b = 2|\xi|^4$
c=$(\sigma + |\xi|^2)|\xi|^4$

This cubic owns the strictly negative discriminant

$$\begin{aligned}
\Delta_{dis} = {} & [(\sigma + |\xi|^2)]^2[2|\xi|^4]^2 + 18[(\sigma + |\xi|^2)][2|\xi|^4][(\sigma + |\xi|^2)|\xi|^4] - 4[2|\xi|^4]^3 \\
& - 4[(\sigma + |\xi|^2)]^3[(\sigma + |\xi|^2)|\xi|^4] - 27[(\sigma + |\xi|^2)|\xi|^4]^2 \\
& = (\sigma + |\xi|^2)^2 4|\xi|^8 + 18(\sigma + |\xi|^2)2|\xi|^4(\sigma + |\xi|^2)|\xi|^4 - 4(8|\xi|^{12}) - 4(\sigma + |\xi|^2)^4|\xi|^4 - 27[(\sigma + |\xi|^2)|\xi| \\
& = 4(\sigma + |\xi|^2)^2|\xi|^8 + 36(\sigma + |\xi|^2)|\xi|^8 - 32|\xi|^{12} - 4(\sigma + |\xi|^2)^4|\xi|^4 - 27[(\sigma + |\xi|^2)|\xi|^8] \\
& = (4 + 36 - 27)(\sigma + |\xi|^2)^2|\xi|^8 - 32|\xi|^{12} - 4(\sigma + |\xi|^2)^4|\xi|^4 \\
& = -4(\sigma + |\xi|^2)^4|\xi|^4 - 32|\xi|^{12} + 13(\sigma + |\xi|^2)^2|\xi|^8 \\
& = -4(\sigma + |\xi|^2)^4|\xi|^4 - 32|\xi|^{12} + 16\sqrt{2}(\sigma + |\xi|^2)^2|\xi|^8 - 16\sqrt{2}(\sigma + |\xi|^2)^2|\xi|^8 + 13(\sigma + |\xi|^2)^2|\xi|^8 \\
& = -4[(\sigma + |\xi|^2)^2|\xi|^2 - 2\sqrt{2}|\xi|^6]^2 - (16\sqrt{2} - 13)(\sigma + |\xi|^2)^2|\xi|^8) < 0.
\end{aligned}$$

For this reason, the last cubic (2.3) has one real root λ_1 and two complex conjugate roots $\lambda_{2/3} = \lambda_R \pm i\lambda_I$ carrying $\lambda_R \in \mathbb{R}$.
Different from the homogeneous characteristic equation of the model with $\sigma = 0$ in [4],

$$\lambda^3 + |\xi|^2\lambda^2 + 2|\xi|^4\lambda + |\xi|^6 = 0$$

her discriminant given by

$$\Delta_{dis} = -4(|\xi|^6 - 2\sqrt{2}|\xi|^6)^2 - (16\sqrt{2} - 13)|\xi|^{12} < 0.$$

has one real root and two complex (non-real) conjugate roots as follows
$\lambda_1 = -a_0|\xi|^2 := -\dfrac{1 + \alpha_-}{3}|\xi|^2,$
$\lambda_{2,3} = -a_1|\xi|^2 \pm ia_2|\xi|^2 := -\dfrac{2 - \alpha_-}{6}|\xi|^2 \pm i\dfrac{\sqrt{3}\alpha_+}{6}|\xi|^2,$

with $\alpha_{\pm} := \sqrt[3]{\frac{1}{2}(3\sqrt{69}+11)} \pm \sqrt[3]{\frac{1}{2}(3\sqrt{69}-11)}$

for λ_1 ,

$$\begin{aligned}
&\lambda_1^3 + |\xi|^2\lambda_1^2 + 2|\xi|^4\lambda_1 + |\xi|^6 \\
&= \big(-\frac{1+\alpha_-}{3}|\xi|^2\big)^3 + |\xi|^2\big(-\frac{1+\alpha_-}{3}|\xi|^2\big)^2 + 2|\xi|^4\big(-\frac{1+\alpha_-}{3}|\xi|^2\big) + |\xi|^6 \\
&= \frac{11-\alpha_-^3-15\alpha_-}{27}|\xi|^6
\end{aligned}$$

$$\alpha_- = \sqrt[3]{\frac{1}{2}(3\sqrt{69}+11)} - \sqrt[3]{\frac{1}{2}(3\sqrt{69}-11)} = (a-b)$$

$$\begin{aligned}
&11-\alpha_-^3-15\alpha_- \\
&= 11-(a-b)^3-15(a-b) \\
&= 11-(a^3-b^3+3ab(a-b))-15(a-b) \\
&= 11-(\frac{1}{2}(3\sqrt{69}+11)-\frac{1}{2}(3\sqrt{69}-11))+3ab(a-b)-15(a-b) \\
&= 3ab(a-b)-15(a-b) \\
&= (a-b)(3ab-15)
\end{aligned}$$

$(a-b)(3ab-15)=0$ si $ab=5$

$$ab = (\sqrt[3]{\frac{1}{2}(3\sqrt{69}+11)})(\sqrt[3]{\frac{1}{2}(3\sqrt{69}-11)}) = 2.618795544 \times 1.909274671 = 5$$

According to the size of frequencies $|\xi|$, one may separate the discussion into three zones. To be specific, we employ Taylor-like asymptotic expansions for $\xi \in \mathcal{Z}_{int}(\varepsilon_0) \cup \mathcal{Z}_{ext}(N_0)$, and a contradiction argument combined with continuity of the roots for $\xi \in \mathcal{Z}_{bdd}(\varepsilon_0, N_0)$. By proceeding with lengthy but straightforward computations, we state the next behaviors for the roots.

Proposition 7 *The characteristic roots $\lambda_j = \lambda_j(|\xi|)$ with j=1,2,3 to the cubic equation* (2.3) *can be expanded by the next forms.*

- *Concerning $\xi \in Z_{int}(\varepsilon_0)$, three characteristic roots behave as*

$$\lambda_1 = -\sigma + \frac{\sigma}{2-3\sigma}|\xi|^2 + O(|\xi|^4),$$
$$\lambda_{2/3} = \pm i|\xi|^2 - \frac{1}{2\sigma}|\xi|^4 + \big(\frac{1}{2\sigma^2} \pm \frac{(2\sigma+1)i}{8\sigma^2}\big)|\xi|^6 + O(|\xi|^8),$$

namely, $\lambda_R = -\frac{1}{2\sigma}|\xi|^4 + \frac{1}{2\sigma^2}|\xi|^6 + O(|\xi|^8)$ *and* $\lambda_i = |\xi|^2 + \frac{2\sigma+1}{8\sigma^2}|\xi|^6 + O(|\xi|^8)$,

- *Concerning* $\xi \in \mathcal{Z}_{ext}(N_0)$*, three characteristic roots behave as*

$$\lambda_1 = -a_0|\xi|^2 + O(1),$$
$$\lambda_{2/3} = -a_1|\xi|^2 \pm ia_2|\xi|^2 + O(1),$$

where $a_0 := \frac{1+\alpha_-}{3} \approx 0.57$, $a_1 := \frac{2-\alpha_-}{6} \approx 0.22$ *and* $a_2 := \frac{\sqrt{3}\alpha_+}{6} \approx 1.31$ *carrying the constants*

$$\alpha_\pm := \sqrt[3]{\tfrac{1}{2}(3\sqrt{69}+11)} \pm \sqrt[3]{\tfrac{1}{3}(3\sqrt{69}-11)},$$

namely, $\lambda_R = -a_1|\xi|^2 + O(1)$ *and* $\lambda_I = -a_2|\xi|^2 + O(1)$.

- *Concerning* $\xi \in \mathcal{Z}_{bdd}(\varepsilon_0, N_0)$*, three characteristic roots fulfill* $R\lambda_j < 0$ *for all* $j = 1, 2, 3$.

Remark 8 *In the small frequencies portion of Proposition* (7)*, we not only obtain pairwise distinct characteristic roots with negative real parts, but also investigate some higher-order terms, i.e.* $|\xi|^2$*-terms in* $\lambda_{2,3}$ *when* $\xi \in \mathcal{Z}_{int}(\varepsilon_0)$*. These nonvanishing higher-order terms will contribute to further expansions of solutions in the Fourier space.*

2.0.2 Pointwise estimates and auxiliary functions in the Fourier space

Throughout this subsection, we consider $\xi \in \mathcal{Z}_{int}(\varepsilon_0) \cup \mathcal{Z}_{ext}(N_0)$ only because of exponential decay estimates for $\xi \in \mathcal{Z}_{bdd}(\varepsilon_0, N_0)$. According to the representation of solution to the third-order model (2.2) and the expression of the last data

$$\hat{u}_2 = -|\xi|^4 \hat{u}_0 + |\xi|^2 \hat{\theta}_0$$

we may arrive at

$$\begin{aligned}
\hat{u} =& \frac{(|\xi|^4 - \lambda_I^2 - \lambda_R^2)\hat{u}_0 + 2\lambda_R \hat{u}_1 - |\xi|^2 \hat{\theta}_0}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2} e^{\lambda_1 t} \\
&+ \frac{(2\lambda_R\lambda_1 - \lambda_1^2 - |\xi|^4)\hat{u}_0 - 2\lambda_R \hat{u}_1 + |\xi|^2 \hat{\theta}_0}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2} \cos(\lambda_I t) e^{\lambda_R t} \\
&+ \frac{[\lambda_1(\lambda_R\lambda_1 + \lambda_I^2 - \lambda_R^2) + |\xi|^4(\lambda_R - \lambda_1)]\hat{u}_0}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} \sin(\lambda_I t) e^{\lambda_R t} \\
&+ \frac{(\lambda_R^2 - \lambda_I^2 - \lambda_1^2)\hat{u}_1 - |\xi|^2(\lambda_R - \lambda_1)\hat{\theta}_0}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} \sin(\lambda_I t) e^{\lambda_R t}
\end{aligned}$$

whose idea should be traced back to [5].

To analyze asymptotic behaviors of solution, we firstly set $\hat{g}_j = \hat{g}_j(t, \xi)$ with $j = 1, 2$ such that

$$\begin{aligned}
\hat{g}_1 &:= \frac{-\lambda_1^2 \sin(\lambda_I t) e^{\lambda_R t}}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} \hat{u}_1 \\
\hat{g}_2 &:= \frac{-\lambda_1^2 \cos(\lambda_I t) e^{\lambda_R t}}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} \hat{u}_0 + \frac{|\xi|^2 \lambda_1 \sin(\lambda_I t) e^{\lambda_R t}}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} \hat{\theta}_0,
\end{aligned}$$

whose origins are extractions of leading terms of $\hat{u}$ for small frequencies. Applying the asymptotic expansions derived in Proposition (7), with

$$\lambda_1 = -\sigma + \frac{\sigma}{2 - 3\sigma}|\xi|^2 + O(|\xi|^4)$$

$$\begin{aligned}
\lambda_R &= -\frac{1}{2\sigma}|\xi|^4 - \frac{1}{2\sigma^2}|\xi|^2 + O(|\xi|^8) \\
\lambda_I &= |\xi|^2 + \frac{2\sigma + 1}{8\sigma^2}|\xi|^6 + O(|\xi|^8)
\end{aligned}$$

the next error estimate holds:

$$\begin{aligned}
&\hat{u} - \hat{g}_1 - \hat{g}_2 = \\
&\frac{(|\xi|^4 - \lambda_I^2 - \lambda_R^2)\hat{u}_0 + 2\lambda_R \hat{u}_1 - |\xi|^2 \hat{\theta}_0}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2} e^{\lambda_1 t} + \frac{(2\lambda_R\lambda_1 - \lambda_1^2 - |\xi|^4)\hat{u}_0 - 2\lambda_R \hat{u}_1 + |\xi|^2 \hat{\theta}_0}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2} \cos(\lambda_I t) e^{\lambda_R t} \\
&+ \left(\frac{[\lambda_1(\lambda_R\lambda_1 + \lambda_I^2 - \lambda_R^2) + |\xi|^4(\lambda_R - \lambda_1)]\hat{u}_0}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} + \frac{(\lambda_R^2 - \lambda_I^2 - \lambda_1^2)\hat{u}_1 - |\xi|^2(\lambda_R - \lambda_1)\hat{\theta}_0}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} \right) \sin(\lambda_I t) e^{\lambda_R t} \\
&- \frac{-\lambda_1^2 \sin(\lambda_I t) e^{\lambda_R t}}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} \hat{u}_1 - \frac{-\lambda_1^2 \cos(\lambda_I t) e^{\lambda_R t}}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} \hat{u}_0 - \frac{|\xi|^2 \lambda_1 \sin(\lambda_I t) e^{\lambda_R t}}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} \hat{\theta}_0
\end{aligned}$$

$$\hat{u}-\hat{g}_1-\hat{g}_2=$$
$$\frac{(|\xi|^4-\lambda_I^2-\lambda_R^2)\hat{u}_0+2\lambda_R\hat{u}_1-|\xi|^2\hat{\theta}_0}{2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2}e^{\lambda_1 t}+\frac{(2\lambda_R\lambda_1-|\xi|^4)\hat{u}_0-2\lambda_R\hat{u}_1+|\xi|^2\hat{\theta}_0}{2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2}\cos(\lambda_I t)e^{\lambda_R t}$$
$$+\Big(\frac{[\lambda_1(\lambda_R\lambda_1+\lambda_I^2-\lambda_R^2)+|\xi|^4(\lambda_R-\lambda_1)]\hat{u}_0}{\lambda_I(2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2)}+\frac{(\lambda_R^2-\lambda_I^2)\hat{u}_1-|\xi|^2\lambda_R\hat{\theta}_0}{\lambda_I(2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2)}\Big)\sin(\lambda_I t)e^{\lambda_R t}$$
$$\chi_{int}(\xi)|\hat{u}-\hat{g}_1-\hat{g}_2|$$
$$\leq\chi_{int}(\xi)\Big(\frac{(\frac{1}{2\sigma})^2|\xi|^6|\hat{u_0}|+(\frac{1}{\sigma}|\xi|^4+\frac{1}{\sigma^2}|\xi|^6)|\hat{u}_1|+|\xi|^2|\hat{\theta}_0|}{\sigma^2+\frac{2\sigma^2}{2+3\sigma}|\xi|^2}\Big)e^{(-\sigma+\frac{\sigma}{2-3\sigma}|\xi|^2)t}$$
$$+\chi_{int}(\xi)\Big(\frac{(\frac{1}{\sigma}|\xi|^6+\frac{1}{2+3\sigma}|\xi|^6)|\hat{u}_0|+(\frac{1}{\sigma}|\xi|^4+\frac{1}{\sigma^2}|\xi|^6)|\hat{u}_1|+|\xi|^2|\hat{\theta}|}{\sigma^2+\frac{2\sigma^2}{2+3\sigma}|\xi|^2}\Big)|\cos(|\xi|^2t)|e^{(-\frac{1}{2\sigma}|\xi|^4-\frac{1}{2\sigma^2}|\xi|^2)t}$$
$$+\chi_{int}(\xi)\Big(\frac{|\xi|^2(\frac{3\sigma}{2}|\xi|^2+\frac{2\sigma^2+\sigma+2}{4\sigma+6\sigma^2}|\xi|^4)|\hat{u}_0|}{|\xi|^2(\sigma^2+\frac{2\sigma^2}{2+3\sigma}|\xi|^2)}\Big)|\sin(|\xi|^2t)|e^{(-\frac{1}{2\sigma}|\xi|^4-\frac{1}{2\sigma^2}|\xi|^2)}$$
$$+\chi_{int}(\xi)\Big(\frac{|\xi|^2((\frac{1}{2\sigma})^2|\xi|^4+|\xi|^2)|\hat{u}_1|+|\xi|^2(\frac{1}{2\sigma}|\xi|^4+\frac{1}{2\sigma^2}|\xi|^6)|\hat{\theta}_0|}{|\xi|^2(\sigma^2+\frac{2\sigma^2}{2+3\sigma}|\xi|^2)}\Big)|\sin(|\xi|^2t)|e^{(-\frac{1}{2\sigma}|\xi|^4-\frac{1}{2\sigma^2}|\xi|^2)}$$

$$\lesssim\chi_{int}(\xi)\big(|\xi|^6|\hat{u}_0|+|\xi|^4|\hat{u}_1|+|\xi|^2|\hat{\theta}_0|\big)e^{-ct}+\chi_{int}(\xi)\big(|\xi|^6|\hat{u}_0|+|\xi|^4|\hat{u}_1|+|\xi|^2|\hat{\theta}_0|\big)|\cos(|\xi|^2t)|e^{-c|\xi|^2t}$$
$$+\chi_{int}(\xi)\big(|\xi|^2|\hat{u}_0|+|\xi|^2|\hat{u}_1|+|\xi|^4|\hat{\theta}_0|\big)|\sin(|\xi|^2t)|e^{-c|\xi|^2t}$$

$$\chi_{int}(\xi)|\hat{u}-\hat{g}_1-\hat{g}_2|$$
$$\lesssim\chi_{int}(\xi)\big(e^{-ct}+|\cos(|\xi|^2t)|e^{-c|\xi|^4t}|\big)\big(|\xi|^6|\hat{u}_0|+|\xi|^4|\hat{u}_1|+|\xi|^2|\hat{\theta}_0|\big)$$
$$+\chi_{int}(\xi)\big(|\xi|^2|\hat{u}_0|+|\xi|^2|\hat{u}_1|+|\xi|^4|\hat{\theta}_0|\big)|\sin(|\xi|^2t)|e^{-c|\xi|^4t}$$

$$\lesssim\chi_{int}(\xi)(e^{-ct}+|\cos(|\xi|^2t)|)\big(|\xi|^6|\hat{u_0}|+|\xi|^4|\hat{u_1}|+|\xi|^2|\hat{\theta}_0|\big)$$
$$+\chi_{int}(\xi)\sin(|\xi|^2t)e^{-c|\xi|^4t}\big(|\xi|^2|\hat{u_0}|+|\xi|^2|\hat{u_1}|+|\xi|^4|\hat{\theta}_0|\big)$$
$$\chi_{int}(\xi)|\hat{u}-\hat{g}_1-\hat{g}_2|\lesssim\chi_{int}(\xi)|\xi|^2e^{-c|\xi|^4t}(|\hat{u_0}|+|\hat{u_1}|+|\hat{\theta}_0|)\qquad(2.4)$$

Where we took advantages of $|\xi|^4-\lambda_I^2=O(|\xi|^8)$ as well as $2\lambda_R\lambda_1-|\xi|^4=O(|\xi|^8)$ for

$\xi \in \mathcal{Z}_{int}(\varepsilon)$.

Due to the estimates that

$$\begin{aligned}
\chi_{int}(\xi)|\hat{g}_1| =& \chi_{int}(\xi)\frac{\lambda_1^2|\sin(|\lambda_I t)|}{\lambda_I(2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2)}e^{\lambda_R t}|\hat{u}_1| \\
&\leq \chi_{int}(\xi)\frac{\sigma^2+\dfrac{2\sigma^2}{2+3\sigma}|\xi|^2|\sin(|\xi|^2 t)|}{|\xi|^2(\sigma^2+\dfrac{2\sigma^2}{2+3\sigma}|\xi|^2)}e^{(-\frac{1}{2\sigma}|\xi|^4-\frac{1}{2\sigma^2}|\xi|^6)t}|\hat{u}_1| \\
&\lesssim \chi_{int}(\xi)\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}e^{-c|\xi|^4 t}|\hat{u}_1|
\end{aligned}$$

and

$$\begin{aligned}
\chi_{int}(\xi)|\hat{g}_2| &= \chi_{int}(\xi)\big(\frac{\lambda_1^2|\cos(\lambda_1 t)|}{2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2}e^{\lambda_R t}|\hat{u}_0| + \frac{|\xi|^2\lambda_1|\sin(\lambda_I t)|}{\lambda_I(2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2)}e^{\lambda_R t}|\hat{\theta}_0|\big) \\
&= \chi_{int}(\xi)e^{\lambda_R t}\big(\frac{\lambda_1^2|\cos(\lambda_1 t)|}{2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2}|\hat{u}_0| + \frac{|\xi|^2\lambda_1|\sin(\lambda_I t)|}{\lambda_I(2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2)}|\hat{\theta}_0|\big) \\
&\leq \chi_{int}(\xi)e^{-\frac{1}{2\sigma}|\xi|^4-\frac{1}{2\sigma^2}|\xi|^6}\big(\frac{\sigma^2+\dfrac{2\sigma^2}{2+3\sigma}|\xi|^2|\cos(|\xi|^2 t)|}{\sigma^2+\dfrac{2\sigma^2}{2+3\sigma}|\xi|^2}|\hat{u}_0| + \frac{|\xi|^2(\sigma+\dfrac{\sigma}{2+3\sigma}|\xi|^2)|\sin(|\xi|^2 t)|}{|\xi|^2(\sigma^2+\dfrac{2\sigma^2}{2+3\sigma}|\xi|^2)}|\hat{\theta}_0|\big) \\
&\leq \chi_{int}(\xi)e^{-\frac{1}{2\sigma}|\xi|^4-\frac{1}{2\sigma^2}|\xi|^6}\big(\cos(|\xi|^2 t)|\hat{u}_0| + \frac{\sigma+\dfrac{\sigma}{2+3\sigma}|\xi|^2|\sin(|\xi|^2 t)|}{\sigma^2+\dfrac{2\sigma^2}{2+3\sigma}|\xi|^2}|\hat{\theta}_0|\big) \\
&\lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u}_0| + |\hat{\theta_0}|\big)
\end{aligned}$$

we have :

$|\hat{u}-\hat{g}_1| = |\hat{u}-\hat{g}_1-\hat{g}_2+\hat{g}_2|$

We use the triangle inequality to obtain

$$\begin{aligned}
|\hat{u}-\hat{g}_1| &\leq |\hat{u}-\hat{g}_1-\hat{g}_2| + |\hat{g}_2| \\
\chi_{int}(\xi)|\hat{u}-\hat{g}_1| &\leq \chi_{int}(\xi)|\hat{u}-\hat{g}_1-\hat{g}_2| + \chi_{int}(\xi)|\hat{g}_2| \\
&\lesssim \chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u_0}|+|\hat{u_1}|+|\hat{\theta}_0|) + \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u_0}|+|\hat{\theta_0}|\big) \\
&\lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}|\xi|^2|\hat{u_1}| + (1+|\xi|^2)(|\hat{u_0}|+|\hat{\theta}_0|)
\end{aligned}$$

Hence

$$\chi_{int}(\xi)|\hat{u}-\hat{g}_1| \lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u_0}| + |\xi|^2|\hat{u_1}| + |\hat{\theta_0}|\big) \tag{2.5}$$

And,
$|\hat{u}| = |\hat{u} - \hat{g_1} + \hat{g_1}|$
We use triangle inequality to obtain

$$\begin{aligned}
|\hat{u}| &\le |\hat{u} - \hat{g_1}| + |\hat{g_1}| \\
\chi_{int}(\xi)|\hat{u}| &\lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u}_0| + |\xi|^2\hat{u}_1| + |\hat{\theta_0}|\big) + \chi_{int}(\xi)\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}e^{-c|\xi|^4 t}|\hat{u}_1| \\
&\lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u}_0| + |\hat{\theta_0}|\big) + \chi_{int}(\xi)e^{-c|\xi|^4 t}(|\xi|^2 + \frac{|\sin(|\xi|^2 t)|}{|\xi|^2}|\hat{u}_1|) \\
&\lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u}_0| + |\hat{\theta_0}|\big) + \chi_{int}(\xi)\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}e^{-c|\xi|^4 t}|\hat{u}_1|
\end{aligned}$$

Hence

$$\chi_{int}(\xi)|\hat{u}| \lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u}_0| + \frac{|\sin(|\xi|^2 t)|}{|\xi|^2}|\hat{u}_1| + |\hat{\theta_0}|\big) \tag{2.6}$$

Let us introduce some approximations $\hat{J}_j = \hat{J}_j(t, |\xi|)$ with $j = 0, 1$ such that

$$\hat{J}_0 := \frac{\sin(|\xi|^2 t)|}{|\xi|^2}e^{-\frac{1}{2\sigma}|\xi|^4 t} \text{ and } \hat{J}_1 := \cos(|\xi|^2 t)e^{-\frac{1}{2\sigma}|\xi|^4 t}.$$

They are the Fourier transformations of higher-order diffusion-plates. Then, we are able to claim the refined estimates by subtracting some approximated functions.

Proposition 9 *Concerning $\xi \in \mathcal{Z}_{int}(\varepsilon_0)$, the following refined estimates hold:*

$$\chi_{int}(\xi)|\hat{g_1} - \hat{J}_0\hat{u}_1| \lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}|\hat{u}_1| \tag{2.7}$$

$$\chi_{int}(\xi)|\hat{g_1} - (\hat{J}_0 + \hat{H}_0 + \hat{H}_1)|\hat{u_1}| \lesssim \chi_{int}(\xi)|\xi|^2|e^{-c|\xi|^4 t}|\hat{u}_1|, \tag{2.8}$$

$$\chi_{int}(\xi)|\hat{g_2} - \hat{J}_1\hat{u}_0 - \sigma^{-1}|\xi|^2|\hat{J}_0\hat{\theta}_0| \lesssim \chi_{int}(\xi)|\xi|^2|e^{-c|\xi|^4 t}\big(|\hat{u_0}| + |\hat{\theta_0}||\big), \tag{2.9}$$

Where the auxiliary functions $\hat{H}_j := \hat{H}_j(t, \xi)$ with $j = 0, 1$ are defined by

$$\hat{H}_0 := \frac{2\sigma + 1}{8\sigma^2}|\xi|^4 t\cos(|\xi|^2 t)e^{-\frac{1}{2\sigma}|\xi|^4 t},$$

$$\hat{H}_1 := \frac{1}{2\sigma^2}|\xi|^4 t\sin(|\xi|^2 t)e^{-\frac{1}{2\sigma}|\xi|^4 t},$$

Proof. A direct subtraction associated with suitable decomposition implies $\hat{g}_1 - (\hat{J}_0 + \hat{H}_0 + \hat{H}_1)\hat{u}_1 = \sum_{j=0}^{3} \hat{E}_j \hat{u}_1$,
Where the error terms are

$$\hat{E}_0 := \frac{-\lambda_1^2 \sin(\lambda_I t) e^{\lambda_R t}}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} - \frac{1}{|\xi|^2} \sin(\lambda_I t)| e^{\lambda_R t}$$

$$\hat{E}_1 := \frac{1}{|\xi|^2}\big(\sin(\lambda_I t) - \sin(|\xi|^2 t) - \frac{2\sigma+1}{8\sigma^2}|\xi|^6 t \cos(|\xi|^2 t)\big) e^{\lambda_R t},$$

$$\hat{E}_2 := \frac{1}{|\xi|^2} \sin(|\xi|^2 t)\big(e^{\lambda_R t} - e^{-\frac{1}{2\sigma}|\xi|^4 t} - \frac{1}{2\sigma^2}|\xi|^6 t e^{-\frac{1}{2\sigma}|\xi|^4 t}\big)$$

$$\hat{E}_3 := \frac{2\sigma+1}{8\sigma^2}|\xi|^4 t \cos(|\xi|^2 t)\big(e^{\lambda_R t} - e^{-\frac{1}{2\sigma}|\xi|^4 t}\big)$$

For one thing, the asymptotic behavior stated in proposition(7) leads to

$$\begin{aligned}
\chi_{int}(\xi)|\hat{E}_0| &= \chi_{int}(\xi)\left|\frac{-\lambda_1^2 \sin(\lambda_I t) e^{\lambda_R t}}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} - \frac{\sin(|\xi|^2 t)|}{|\xi|^2} e^{\lambda_R t}\right| \\
&= \chi_{int}(\xi)\left|\frac{-\lambda_1^2|\xi|^2 + \lambda_1^2\lambda_I - \lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2)}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)|\xi|^2}\right| |\sin(\lambda_I t)| e^{\lambda_R t} \\
&= \chi_{int}(\xi)\left|\frac{\lambda_1^2(\lambda_I - |\xi|^2) - \lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2)}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)|\xi|^2}\right| \sin(\lambda_I t)| e^{\lambda_R t} \\
&= \chi_{int}(\xi)\left|\frac{\lambda_1^2(o(|\xi|^6)) - \lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2)}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)|\xi|^2}\right| |\sin(\lambda_I t)| e^{\lambda_R t} \\
&\lesssim \chi_{int}(\xi)\frac{\sigma^2 + \frac{2\sigma^2}{2+3\sigma}|\xi|^2 + \frac{\sigma^2}{(2+3\sigma)^2}|\xi|^4}{|\xi|^2(\sigma^2 + \frac{2\sigma^2}{2+3\sigma}|\xi|^2)|\xi|^2} |\sin(|\xi|^2 t)| e^{-\frac{1}{2\sigma}|\xi|^4 t + \frac{1}{2\sigma^2}|\xi|^6} \\
&\lesssim \chi_{int}(\xi)\left(\frac{1}{|\xi|^4} + \frac{\frac{\sigma^2}{(2+3\sigma)^2}|\xi|^4}{|\xi|^4(\sigma^2 + \frac{2\sigma^2}{2+3\sigma}|\xi|^2)} e^{-c|\xi|^4 t}\right) \\
&\lesssim \chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}
\end{aligned}$$

because of $\lambda_I - |\xi|^2 = o(|\xi|^6)$ for $\xi \in \mathcal{Z}_{int}(\varepsilon_0)$.
For another, with the help of Taylor's expansions as $|\xi| \ll 1$, we notice

$$\sin(\lambda_I t) = \sin(|\xi|^2 t) + \frac{2\sigma+1}{8\sigma^2}|\xi|^6 t \cos(|\xi|^2 t) + o(|\xi|^{12})t^2,$$
$$e^{\lambda_R t} = e^{-\frac{1}{2\sigma}|\xi|^4 t} + \frac{1}{2\sigma^2}|\xi|^6 t e^{-\frac{1}{2\sigma}|\xi|^4 t} + o(|\xi|^{12} t) t^2 e^{-\frac{1}{2\sigma}|\xi|^4 t}$$

As a consequence,

$$\chi_{int}(\xi)(\hat{E}_1+\hat{E}_2+\hat{E}_3) = \chi_{int}(\xi)(\frac{1}{|\xi|^2}e^{\lambda_R t}(o(|\xi|^{12})t^2)+\frac{1}{|\xi|^2}\sin(|\xi|^2t)(o(|\xi|^{12}t)t^2e^{-\frac{1}{2\sigma}|\xi|^4t}))$$
$$+\chi_{int}(\xi)(\frac{2\sigma+1}{8\sigma^2}|\xi|^4t\cos(|\xi|^2t)\big(\frac{1}{2\sigma^2}|\xi|^6te^{-\frac{1}{2\sigma}|\xi|^4t}+o(|\xi|^{12}t)t^2e^{-\frac{1}{2\sigma}|\xi|^4t}\big))$$
$$=\chi_{int}(\xi)(e^{\lambda_R t}(o(|\xi|^{10})t^2)+\sin(|\xi|^2t)(o(|\xi|^{10}t)t^2e^{-\frac{1}{2\sigma}|\xi|^4t}))$$
$$+\chi_{int}(\xi)(\frac{2\sigma+1}{8\sigma^2}|\xi|^4t\cos(|\xi|^2t)\big(\frac{1}{2\sigma^2}|\xi|^6te^{-\frac{1}{2\sigma}|\xi|^4t}+o(|\xi|^{12}t)t^2e^{-\frac{1}{2\sigma}|\xi|^4t}\big))$$

$$\chi_{int}(\xi)(|\hat{E}_1|+|\hat{E}_2|+|\hat{E}_3|)\leq\frac{2\sigma+1}{(8\sigma^2)(2\sigma^2)}|\xi|^{10}t^2|\cos(|\xi|^2t)|e^{-c|\xi|^4t}$$

$$\chi_{int}(\xi)(|\hat{E}_1|+|\hat{E}_2|+|\hat{E}_3|)\lesssim\chi_{int}(\xi)|\xi|^{10}t^2e^{-c|\xi|^4t}\lesssim\chi_{int}(\xi)|\xi|^2e^{-c|\xi|^4t}$$

Summarizing the last estimates, we conclude
$\hat{g}_1-(\hat{J}_0+\hat{H}_0+\hat{H}_1)\hat{u}_1=\sum_{j=0}^{3}\hat{E}_j\hat{u}_1$
$\chi_{int}(\xi)|\hat{g}_1-(\hat{J}_0+\hat{H}_0+\hat{H}_1)\hat{u}_1|\lesssim\chi_{int}(\xi)\sum_{j=0}^{3}|\hat{E}_j||\hat{u}_1|$
$\chi_{int}(\xi)|\hat{g}_1-(\hat{J}_0+\hat{H}_0+\hat{H}_1)\hat{u}_1|\lesssim\chi_{int}(\xi)|\xi|^2e^{-c|\xi|^4}t|\hat{u}_1|$,
which completed the proof of (2.8).

Additionally, by the use of the triangle inequality associated and with

$$\hat{H}_0+\hat{H}_1=\frac{2\sigma+1}{8\sigma^2}|\xi|^4t\cos(|\xi|^2t)e^{-\frac{1}{2\sigma}|\xi|^4t}+\frac{1}{2\sigma^2}|\xi|^4t\sin(|\xi|^2t)e^{-\frac{1}{2\sigma}|\xi|^4t}$$
$$\chi_{int}(\xi)\big(|\hat{H}_0|+|\hat{H}_1|\big)\leq\chi_{int}(\xi)\big(\frac{2\sigma+1}{8\sigma^2}|\xi|^4t|\cos(|\xi|^2t)|e^{-\frac{1}{2\sigma}|\xi|^4t}+\frac{1}{2\sigma^2}|\xi|^4t|\sin(|\xi|^2t)|e^{-\frac{1}{2\sigma}|\xi|^4t}\big)$$
$$\chi_{int}(\xi)\big(|\hat{H}_0|+|\hat{H}_1|\big)\leq\chi_{int}(\xi)\big(te^{-c|\xi|^4t}+te^{-c|\xi|^4t}\big)$$
$$\chi_{int}(\xi)\big(|\hat{H}_0|+|\hat{H}_1|\big)\lesssim\chi_{int}(\xi)te^{-c|\xi|^4}$$
$$\chi_{int}(\xi)\big(|\hat{H}_0|+|\hat{H}_1|\big)\lesssim\chi_{int}(\xi)e^{-c|\xi|^4}$$

we have
$|\hat{g}_1-\hat{J}_0\hat{u}_1|=|\hat{g}_1-\hat{J}_0\hat{u}_1+(\hat{H}_0+\hat{H}_1)\hat{u}_1-(\hat{H}_0+\hat{H}_1)\hat{u}_1|$
$|\hat{g}_1-\hat{J}_0\hat{u}_1|\leq|\hat{g}_1-\hat{J}_0\hat{u}_1+(\hat{H}_0+\hat{H}_1)\hat{u}_1|+|(\hat{H}_0+\hat{H}_1)\hat{u}_1|$
$\chi_{int}(\xi)|\hat{g}_1-\hat{J}_0\hat{u}_1|\leq\chi_{int}(\xi)|\hat{g}_1-(\hat{J}_0+\hat{H}_0+\hat{H}_1)\hat{u}_1|+\chi_{int}(\xi)(|\hat{H}_0|+|\hat{H}_1|)|\hat{u}_1|$
$\chi_{int}(\xi)|\hat{g}_1-\hat{J}_0\hat{u}_1|\lesssim\chi_{int}(\xi)|\xi|^2e^{-c|\xi|^4t}|\hat{u}_1|+\chi_{int}(\xi)e^{-c|\xi|^4}|\hat{u}_1|$
$\chi_{int}(\xi)|\hat{g}_1-\hat{J}_0\hat{u}_1|\lesssim\chi_{int}(\xi)e^{-c|\xi|^4}|\hat{u}_1|(|\xi|^2+1)$
$\chi_{int}(\xi)|\hat{g}_1-\hat{J}_0\hat{u}_1|\lesssim\chi_{int}(\xi)e^{-c|\xi|^4}|\hat{u}_1|$

complete proof of (2.7)

By a similar approach to the above, we may prove (2.9) easily

$$
\begin{aligned}
\hat{g}_2 - \hat{J}_1\hat{u}_0 - \sigma^{-1}|\xi|^2\hat{J}_0\hat{\theta}_0 &= \frac{-\lambda_1^2\cos(\lambda_I t)e^{\lambda_R t}}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2}\hat{u}_0 + \frac{|\xi|^2\lambda_1\sin(\lambda_I t)e^{\lambda_R t}}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)}\hat{\theta}_0 \\
&\quad - \cos(|\xi|^2 t)e^{-\frac{1}{2\sigma}|\xi|^4 t}\hat{u}_0 - \sigma^{-1}|\xi|^2\frac{\sin(|\xi|^2 t)|}{|\xi|^2}e^{-\frac{1}{2\sigma}|\xi|^4 t}\hat{\theta}_0 \\
&= \Big(\frac{-\lambda_1^2\cos(\lambda_I t)e^{\lambda_R t}}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2} - \cos(|\xi|^2 t)e^{-\frac{1}{2\sigma}|\xi|^4 t}\Big)\hat{u}_0 \\
&\quad + \Big(\frac{|\xi|^2\lambda_1\sin(\lambda_I t)e^{\lambda_R t}}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} - \sigma^{-1}|\xi|^2\frac{\sin(|\xi|^2 t)|}{|\xi|^2}e^{-\frac{1}{2\sigma}|\xi|^4 t}\Big)\hat{\theta}_0 \\
&= \Big(\frac{-\lambda_1^2}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2} - 1\Big)\cos(\lambda_I t)e^{\lambda_R t}\hat{u}_0 \\
&\quad + \Big(\frac{\lambda_1}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} - \sigma^{-1}\Big)\sin(\lambda_I t)e^{\lambda_R t}\hat{\theta}_0 \\
&= \Big(\frac{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2}{2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2}\Big)\cos(\lambda_I t)e^{\lambda_R t}\hat{u}_0 \\
&\quad + \Big(\frac{\lambda_1}{\lambda_I(2\lambda_R\lambda_1 - \lambda_I^2 - \lambda_R^2 - \lambda_1^2)} - \sigma^{-1}\Big)\sin(\lambda_I t)e^{\lambda_R t}\hat{\theta}_0
\end{aligned}
$$

$$
\begin{aligned}
\chi_{int}(\xi)|\hat{g}_2 - \hat{J}_1\hat{u}_0 - \sigma^{-1}|\xi|^2\hat{J}_0\hat{\theta}_0| &\le \chi_{int}(\xi)e^{-c|\xi|^4 t}\Bigg(\frac{\frac{1}{\sigma(2-3\sigma)}|\xi|^6}{\sigma^2 + \frac{2\sigma^2}{2-3\sigma}|\xi|^2}|\cos(|\xi|^2 t)||\hat{u}_0| \\
&\quad + \Big(\frac{\sigma + \frac{\sigma}{2-3\sigma}|\xi|^2 + \sigma^{-1}|\xi|^2(\sigma^2 + \frac{2\sigma^2}{2-3\sigma}|\xi|^2)}{|\xi|^2(\sigma^2 + \frac{2\sigma^2}{2-3\sigma}|\xi|^2)}\Big)|\sin(|\xi|^2 t)||\hat{\theta}_0|\Bigg) \\
&\lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\Big(\frac{\frac{1}{\sigma(2-3\sigma)}|\xi|^2}{\sigma^2 + \frac{2\sigma^2}{2-3\sigma}|\xi|^2}|\cos(|\xi|^2 t)||\hat{u}_0| \\
&\quad + \Big(\frac{\sigma + \frac{\sigma}{2-3\sigma}|\xi|^2 + \sigma|\xi|^2 + \frac{2\sigma}{2-3\sigma}|\xi|^4}{|\xi|^2(\sigma^2 + \frac{2\sigma^2}{2-3\sigma}|\xi|^2)}\Big)|\sin(|\xi|^2 t)||\hat{\theta}_0|\Big)
\end{aligned}
$$

$$\chi_{int}(\xi)|\hat{g}_2-\hat{J}_1\hat{u}_0-\sigma^{-1}|\xi|^2\hat{J}_0\hat{\theta}_0 \lesssim \chi_{int}(\xi)e^{-c|\xi|^4t}\Big(\frac{\frac{1}{\sigma(2-3\sigma)}}{\sigma^2+\frac{2\sigma^2}{2-3\sigma}|\xi|^2}|\xi|^2|\cos(|\xi|^2t)||\hat{u}_0|$$
$$+\frac{\frac{2\sigma}{2-3\sigma}|\xi|^2}{|\xi|^2(\sigma^2+\frac{2\sigma^2}{2-3\sigma}|\xi|^2)}|\xi|^2|\sin(|\xi|^2t)||\hat{\theta}_0|\Big)$$
$$\lesssim \chi_{int}(\xi)e^{-c|\xi|^4t}(|\xi|^2|\cos(|\xi|^2t)||\hat{u}_0|+|\xi|^2|\sin(|\xi|^2t)||\hat{\theta_0}|)$$
$$\lesssim \chi_{int}(\xi)|\xi|^2e^{-c|\xi|^4t}\big(|\hat{u}_0|+|\hat{\theta}_0|\big)$$

completed the proof of (2.9). ■

Eventually, we propose pointwise estimates under bounded and large frequencies such that

$$\hat{u}=\frac{(|\xi|^4-\lambda_I^2-\lambda_R^2)\hat{u}_0+2\lambda_R\hat{u}_1-|\xi|^2\hat{\theta}_0}{2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2}e^{\lambda_1t}$$
$$+\frac{(2\lambda_R\lambda_1-\lambda_1^2-|\xi|^4)\hat{u}_0-2\lambda_R\hat{u}_1+|\xi|^2\hat{\theta}_0}{2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2}\cos(\lambda_It)e^{\lambda_Rt}$$
$$+\frac{[\lambda_1(\lambda_R\lambda_1+\lambda_I^2-\lambda_R^2)+|\xi|^4(\lambda_R-\lambda_1)]\hat{u}_0}{\lambda_I(2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2)}\sin(\lambda_It)e^{\lambda_Rt}$$
$$+\frac{(\lambda_R^2-\lambda_I^2-\lambda_1^2)\hat{u}_1-|\xi|^2(\lambda_R-\lambda_1)\hat{\theta}_0}{\lambda_I(2\lambda_R\lambda_1-\lambda_I^2-\lambda_R^2-\lambda_1^2)}\sin(\lambda_It)e^{\lambda_Rt}$$

$$\chi_{ext}(\xi)\hat{u}\leq\chi_{ext}(\xi)\Bigg(\Big(\frac{(|\xi|^4+a_2^2|\xi|^4+a_1^2|\xi|^4)\hat{u}_0+2a_1|\xi|^2\hat{u}_1+|\xi|^2\hat{\theta}_0}{2a_0a_1|\xi|^4+a_2^2|\xi|^4+a_1^2|\xi|^4+a_0^2|\xi|^4}e^{-a_0|\xi|^2t}\Big)$$
$$+\Big(\frac{(2a_0a_1|\xi|^4+a_0^2|\xi|^4+|\xi|^4)|\hat{u}_0|+2a_1|\xi|^2|\hat{u}_1|+|\xi|^2|\hat{\theta}_0|}{2a_0a_1|\xi|^4+a_2^2|\xi|^4+a_1^2|\xi|^4+a_0^2|\xi|^4}|\cos(-a_2|\xi|^2t)|e^{-a_1|\xi|^2t}\Big)$$
$$+\Big(\frac{[-a_0|\xi|^2(a_0a_1|\xi|^4+a_2^2|\xi|^4+a_1^2|\xi|^4)+|\xi|^4(a_1|\xi|^2+a_0|\xi|^2)]|\hat{u}_0|}{-a_2|\xi|^2(2a_0a_1|\xi|^4+a_2^2|\xi|^4+a_1^2|\xi|^4+a_0^2|\xi|^4)}|\sin(-a_2|\xi|^2t)|e^{-a_1|\xi|^2t}\Big)$$
$$+\Big(\frac{(a_1^2|\xi|^4+a_2^2|\xi|^4+a_0^2|\xi|^4)|\hat{u}_1|+|\xi|^2(a_1|\xi|^2+a_0|\xi|^2)|\hat{\theta}_0|}{-a_2|\xi|^2(2a_0a_1|\xi|^4+a_2^2|\xi|^4+a_1^2|\xi|^4+a_0^2|\xi|^4)}|\sin(-a_2|\xi|^2t)|e^{-a_1|\xi|^2t}\Big)\Bigg)$$

$$
\begin{aligned}
&\lesssim \chi_{ext}(\xi)\big(e^{-c|\xi|^2 t} + |\cos(|\xi|^2 t)|e^{-c|\xi|^2 t}\big)\big(|\hat{u}_0| + \frac{1}{|\xi|^2}|\hat{u}_1| + \frac{1}{|\xi|^2}|\hat{\theta}_0|\big)\\
&+ \chi_{ext}(\xi)|\sin(|\xi|^2 t)|e^{-c|\xi|^2 t}\big(|\hat{u}_0| + \frac{1}{|\xi|^2}|\hat{u}_1| + \frac{1}{|\xi|^2}|\hat{\theta}_0|\big)\\
&\lesssim \chi_{ext}(\xi)e^{-ct}\big(|\hat{u}_0| + \frac{1}{|\xi|^2}(|\hat{u}_1| + |\hat{\theta}_0|)\big)\\
&\lesssim \chi_{ext}(\xi)e^{-ct}\big(|\hat{u}_0| + |\xi|^{-2}(|\hat{u}_1| + |\hat{\theta}_0|)\big)
\end{aligned}
$$

$$
\big(\chi_{bdd}(\xi) + \chi_{ext}(\xi)\big)|\hat{u}| \lesssim \big(\chi_{bdd}(\xi) + \chi_{ext}(\xi)\big)e^{-ct}\big(|\hat{u}_0| + \langle\xi\rangle^{-2}|\hat{u}_1| + \langle\xi\rangle^{-2}|\hat{\theta_0}|\big) \tag{2.10}
$$

whose proof is standard basing on Proposition (7).On may see the second estimate in [4].They will not influence on large-time behaviors of solutions since exponential decays.

Chapter 3

ASYMPTOTIC BEHAVIORS OF SOLUTION

3.1 The vertical displacement estimates

Our first result contributes to optimal estimates for the vertical displacement in the L^2 norm . Especially, the solution grows to infinite polynomially (lower-dimension n=1,2,3) or logarithmically (critical-dimension n=4) as t$\rightarrow \infty$.

Theorem 10 *Let us consider the thermoelastic plat system* (4) *with* $\sigma > 0$ *and carrying initial datum* $(u_0, u_1, \theta_0) \in (L^2 \cap L^1)^3$. *Then, its vertical displacement fulfills the following optimal estimates:*

$$D_n(t)|P_{u_1}| \lesssim \|u(t,.)\|_{L^2} \lesssim D_n(t)\|(u_0, u_1, \theta_0)\|_{(L^2 \cap L^1)^3} \tag{3.1}$$

for t $\gg$*1, where the time-dependent coefficient* $D_n(t)$ *was defined in* (1). *Namely, if* $|P_{u_1}| \neq 0$ *then the optimal estimates* $\|u(t,.)\|_{L^2} \simeq D_n(t)$ *hold for any* $n \geq 1$ *as t* $\gg$*1.*

Proof. let us firstly recall the optimal estimates proposed in [9,propositions 3.1-3.3] as follows:

$$\left\| \frac{\sin(|\xi|^2 t)}{|\xi|^2} e^{-c|\xi|^4 t} \right\|_{L^2} \simeq D_n(t) \text{ and } \left\| |\xi|^k e^{-c|\xi|^4 t} \right\|_{L^2} \simeq t^{-\frac{k}{4}-\frac{n}{8}} \tag{3.2}$$

for any $k \in \mathbb{N}_0$ and $t \gg 1$, where $\mathcal{D}_n(t)$ was introduced in (1). Then, by applying Holder's inequality, Hausdorff-Young inequality and Plancherel's theorem we can obtain form (2.6) and that (2.10)

from (2.6) we obtain

$$\chi_{int}(\xi)|\hat{u}| \lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u}_0| + \frac{|\sin(|\xi|^2 t)|}{|\xi|^2}|\hat{u}_1| + |\hat{\theta_0}|\big)$$

$$\chi_{int}(\xi)|\hat{u}| \lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u}_0| + |\hat{\theta_0}|\big) + \chi_{int}(\xi)e^{-c|\xi|^4 t}\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}|\hat{u}_1|$$

and, from (2.10) we obtain

$$\big(\chi_{bdd}(\xi) + \chi_{ext}(\xi)\big)|\hat{u}| \lesssim \big(\chi_{bdd}(\xi) + \chi_{ext}(\xi)\big)e^{-ct}\big(|\hat{u}_0| + \langle\xi\rangle^{-2}|\hat{u}_1| + \langle\xi\rangle^{-2}|\hat{\theta_0}|\big)$$

$$\big(1 - \chi_{int}(\xi)\big)|\hat{u}| \lesssim \big(1 - \chi_{int}(\xi)\big)e^{-ct}\big(|\hat{u}_0| + \langle\xi\rangle^{-2}|\hat{u}_1| + \langle\xi\rangle^{-2}|\hat{\theta_0}|\big)$$

$$|\hat{u}| - \chi_{int}(\xi)|\hat{u}| \lesssim e^{-ct}\big(|\hat{u}_0| + \langle\xi\rangle^{-2}|\hat{u}_1| + \langle\xi\rangle^{-2}|\hat{\theta_0}|\big) - \chi_{int}(\xi)e^{-ct}\big(|\hat{u}_0| + \langle\xi\rangle^{-2}|\hat{u}_1| + \langle\xi\rangle^{-2}|\hat{\theta_0}|\big)$$

$$|\hat{u}| - \chi_{int}(\xi)|\hat{u}| \lesssim e^{-ct}\big(|\hat{u}_0| + \langle\xi\rangle^{-2}|\hat{u}_1| + \langle\xi\rangle^{-2}|\hat{\theta_0}|\big)$$

So, by summation

$$\chi_{int}(\xi)|\hat{u}| + |\hat{u}| - \chi_{int}(\xi)|\hat{u}| \lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u}_0| + |\hat{\theta_0}|\big) + \chi_{int}(\xi)e^{-c|\xi|^4 t}\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}|\hat{u}_1| + e^{-ct}\big(|\hat{u}_0| + \langle\xi\rangle^{-2}|\hat{u}_1| + \langle\xi\rangle^{-2}|\hat{\theta_0}|\big)$$

$$|\hat{u}| \lesssim \chi_{int}(\xi)e^{-c|\xi|^4 t}\big(|\hat{u}_0| + |\hat{\theta_0}|\big) + \chi_{int}(\xi)e^{-c|\xi|^4 t}\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}|\hat{u}_1| + e^{-ct}\big(|\hat{u}_0| + \langle\xi\rangle^{-2}|\hat{u}_1| + \langle\xi\rangle^{-2}|\hat{\theta_0}|\big)$$

$$\begin{aligned}\int |\hat{u}|^2 \mathrm{d}\xi &\lesssim \int \big|\chi_{int}(\xi)e^{-c|\xi|^4 t}\big(\hat{u}_0 + \hat{\theta}_0\big)\big|^2 \mathrm{d}\xi + \int \big|\chi_{int}(\xi)e^{-c|\xi|^4 t}\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}\hat{u}_1\big|^2 \mathrm{d}\xi \\ &+ e^{-ct}\int \big|\big(\hat{u}_0 + \hat{u}_1 + \hat{\theta}_0\big)\big|^2 \mathrm{d}\xi \\ &\lesssim \int \big|\chi_{int}(\xi)e^{-c|\xi|^4 t}\big|^2\big|\big(\hat{u}_0 + \hat{\theta}_0\big)\big|^2 \mathrm{d}\xi + \int \big|\chi_{int}(\xi)e^{-c|\xi|^4 t}\big|^2\big|\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}\hat{u}_1\big|^2 \mathrm{d}\xi \\ &+ e^{-ct}\int \big|\big(\hat{u}_0 + \hat{u}_1 + \hat{\theta}_0\big)\big|^2 \mathrm{d}\xi\end{aligned}$$

by Holder's inequality and the Hausdorff-young inequality

$$\begin{aligned}\|u\|_{L^2}^2 &\lesssim \|\chi_{int}(\xi)e^{-c|\xi|^4 t}\|_{L^2}\big(\|u_0\|_{L^1} + \|\theta_0\|_{L^1}\big)\| + \|\chi_{int}(\xi)\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}e^{-c|\xi|^4 t}\|_{L^2}\|u_1\|_{L^1} \\ &+ e^{-ct}\big(\|u_0\|_{L^2} + \|u_1\|_{L^2} + \|\theta_0\|_{L^2}\big)\end{aligned}$$

$$\begin{aligned}\|u\|_{L^2} &\lesssim \|\chi_{int}(\xi)e^{-c|\xi|^4 t}\|_{L^2}\big(\|u_0\|_{L^1} + \|\theta_0\|_{L^1}\big) \\ &+ \|\chi_{int}(\xi)\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}e^{-c|\xi|^4 t}\|_{L^2}\|u_1|\|_{L^1} \\ &+ e^{-ct}\big(\|u_0\|_{L^2} + \|u_1\|_{L^2} + \|\theta_0\|_{L^2}\big) \\ &\lesssim \|\chi_{int}(\xi)e^{-c|\xi|^4 t}\|_{L^2}\big(\|u_0\|_{L^1} + \|\theta_0\|_{L^1}\big) + e^{-ct}\big(\|u_0\|_{L^2} + \|\theta_0\|_{L^2}\big) \\ &+ \|\chi_{int}(\xi)\frac{|\sin(|\xi|^2 t)|}{|\xi|^2}e^{-c|\xi|^4 t}\|_{L^2}\|u_1\|_{L^1} + e^{-ct}\big(\|u_1\|_{L^2}\big)\end{aligned}$$

$$\begin{aligned}&\lesssim \|\chi_{int}(\xi)e^{-c|\xi|^4t}\|_{L^2}\big(\|u_0\|_{L^1}+\|\theta_0\|_{L^1}\big)+e^{-ct}\big(\|u_0\|_{L^2}+\|\theta_0\|_{L^2}\big)\\&+\|\chi_{int}(\xi)\frac{|\sin(|\xi|^2t)|}{|\xi|^2}e^{-c|\xi|^4t}\|_{L^2}\|u_1\|_{L^1}+e^{-ct}\big(\|u_1\|_{L^2}\big)\end{aligned}$$

$$\lesssim t^{-\frac{n}{8}}\big(\|u_0\|_{L^2\cap L^1}+\|\theta_0\|_{L^2\cap L^1}\big)+\mathcal{D}_n(t)\|u_1\|_{L^2\cap L^1}$$

for large-time $t \gg 1$, which gets the upper bound estimate.

For the lower one, from (2.5) and (2.7) we actually know

$$\chi_{int}(\xi)|\hat{u}-\hat{g_1}| \lesssim \chi_{int}(\xi)e^{-c|\xi|^4t}\big(|\hat{u}_0|+|\xi|^2|\hat{u}_1|+|\hat{\theta_0}|\big)$$

$$\chi_{int}(\xi)|\hat{g_1}-\hat{J}_0\hat{u}_1| \lesssim \chi_{int}(\xi)e^{-c|\xi|^4t}|\hat{u}_1|$$

we use the triangle inequality to obtain

$$\big|\chi_{int}(\xi)(\hat{u}-\hat{g_1})+\chi_{int}(\xi)(\hat{g_1}-\hat{J}_0\hat{u}_1)\big| \le \big|\chi_{int}(\xi)(\hat{u}-\hat{g_1})\big|+\big|\chi_{int}(\xi)(\hat{g_1}-\hat{J}_0\hat{u}_1)\big|$$

$$\chi_{int}(\xi)|\hat{u}-\hat{g_1}+\hat{g_1}-\hat{J}_0\hat{u}_1| \le \big|\chi_{int}(\xi)(\hat{u}-\hat{g_1})\big|+\big|\chi_{int}(\xi)(\hat{g_1}-\hat{J}_0\hat{u}_1)\big|$$

$$\begin{aligned}\chi_{int}(\xi)|\hat{u}-\hat{J}_0\hat{u}_1|+(1-\chi_{int}(\xi))(|\hat{u}-\hat{J}_0\hat{u}_1|) \le&\big|\chi_{int}(\xi)(\hat{u}-\hat{g_1})\big|+\big|\chi_{int}(\xi)(\hat{g_1}-\hat{J}_0\hat{u}_1)\big|\\&+(1-\chi_{int}(\xi))(|\hat{u}-\hat{J}_0\hat{u}_1|)\end{aligned}$$

$$|\hat{u}-\hat{J}_0\hat{u}_1| \le \big|\chi_{int}(\xi)(\hat{u}-\hat{g_1})\big|+\big|\chi_{int}(\xi)(\hat{g_1}-\hat{J}_0\hat{u}_1)\big|+(1-\chi_{int}(\xi))(|\hat{u}|)+(1-\chi_{int}(\xi))(|\hat{J}_0\hat{u}_1|)$$

$$\begin{aligned}\int_{\mathbb{R}^n}|\hat{u}-\hat{J}_0\hat{u}_1|^2\mathrm{d}\xi \le &\int_{\mathbb{R}^n}\big|\chi_{int}(\xi)(\hat{u}-\hat{g_1})\big|^2\mathrm{d}\xi+\int_{\mathbb{R}^n}\big|\chi_{int}(\xi)(\hat{g_1}-\hat{J}_0\hat{u}_1)\big|^2\mathrm{d}\xi\\&+\int_{\mathbb{R}^n}|(1-\chi_{int}(\xi))\hat{u}|^2\mathrm{d}\xi+\int_{\mathbb{R}^n}|(1-\chi_{int}(\xi))(\hat{J}_0\hat{u}_1)|^2\mathrm{d}\xi\end{aligned}$$

$$\begin{aligned}\|\hat{u}-\hat{J}_0\hat{u}_1\|_{L^2}^2 \le&\|\chi_{int}(\xi)(\hat{u}-\hat{g_1})\|_{L^2}^2+\|\chi_{int}(\xi)(\hat{g_1}-\hat{J}_0\hat{u}_1)\|_{L^2}^2\\&+\|(1-\chi_{int}(\xi))\hat{u}\|_{L^2}^2+\|(1-\chi_{int}(\xi))(\hat{J}_0\hat{u}_1)\|_{L^2}^2\end{aligned}$$

$$\begin{aligned}\|\hat{u}-\hat{J}_0\hat{u}_1\|_{L^2} \le&\|\chi_{int}(\xi)(\hat{u}-\hat{g_1})\|_{L^2}+\|\chi_{int}(\xi)(\hat{g_1}-\hat{J}_0\hat{u}_1)\|_{L^2}\\&+\|(1-\chi_{int}(\xi))\hat{u}\|_{L^2}+\|(1-\chi_{int}(\xi))(\hat{J}_0\hat{u}_1)\|_{L^2}\end{aligned}$$

by Plancherel's theorem

$$\begin{aligned}\|u(t,.)-J_0(t,|D|)u_1(.)\|_{L^2} &\lesssim \big\|\chi_{int}(\xi)(\hat{u}(t,\xi)-\hat{g}_1(t,\xi))\|_{L^2}+\|\chi_{int}(\xi)(\hat{g}_1(t,\xi)-\hat{J}_0(t,|\xi|)\hat{u}_1(\xi))\|_{L^2}\\ &\quad+\|(1-\chi_{int}(\xi))\hat{u}(t,\xi)\|_{L^2}+\|(1-\chi_{int}(\xi))\hat{J}_0(t,|\xi|)\hat{u}_1(\xi)\|_{L^2}\end{aligned}$$

$$\begin{aligned}\|u(t,.)-J_0(t,|D|)u_1(.)\|^2_{L^2} &\lesssim \int_{\mathbb{R}^n}\big|\chi_{int}(\xi)(\hat{u}-\hat{g}_1)\big|^2\mathrm{d}\xi+\int_{\mathbb{R}^n}\big|\chi_{int}(\xi)(\hat{g}_1-\hat{J}_0\hat{u}_1)\big|^2\mathrm{d}\xi\\ &\quad+\int_{\mathbb{R}^n}\big|(1-\chi_{int}(\xi))\hat{u}\big|^2\mathrm{d}\xi+\int_{\mathbb{R}^n}\big|(1-\chi_{int}(\xi))\hat{J}_0(t,|\xi|)\hat{u}_1(\xi)\big|^2\mathrm{d}\xi\end{aligned}$$

$$\begin{aligned}&\|u(t,.)-J_0(t,|D|)u_1(.)\|^2_{L^2}\\ &\lesssim \int_{\mathbb{R}^n}\big|\chi_{int}(\xi)e^{-c|\xi|^4t}\big(|\hat{u}_0|+|\xi|^2|\hat{u}_1|+|\hat{\theta_0}|\big)\big|^2\mathrm{d}\xi+\int_{\mathbb{R}^n}\big|\chi_{int}(\xi)e^{-c|\xi|^4t}\hat{u}_1\big|^2\mathrm{d}\xi\\ &+\int_{\mathbb{R}^n}\big|(1-\chi_{int}(\xi))e^{-c|\xi|^4t}\big(|\hat{u}_0|+\frac{\sin(|\xi|^2t)}{|\xi|^2}|\hat{u}_1|+|\hat{\theta}_0|\big)\big|^2\mathrm{d}\xi\\ &+\int_{\mathbb{R}^n}\big|(1-\chi_{int}(\xi))\frac{\sin(|\xi|^2t)}{|\xi|^2}e^{-c|\xi|^4t}\hat{u}_1(\xi)\big|^2\mathrm{d}\xi\\ &\lesssim \int_{\mathbb{R}^n}\big|\chi_{int}(\xi)e^{-c|\xi|^4t}\big|^2\big|\big(|\hat{u}_0|+|\xi|^2|\hat{u}_1|+|\hat{\theta_0}|\big)\big|^2\mathrm{d}\xi+\int_{\mathbb{R}^n}\big|\chi_{int}(\xi)e^{-c|\xi|^4t}\big|^2\big|\hat{u}_1\big|^2\mathrm{d}\xi\\ &+\int_{\mathbb{R}^n}\big|(1-\chi_{int}(\xi))e^{-c|\xi|^4t}\big|^2\big|\big(|\hat{u}_0|+\frac{\sin(|\xi|^2t)}{|\xi|^2}|\hat{u}_1|+|\hat{\theta}_0|\big)\big|^2\mathrm{d}\xi\\ &+\int_{\mathbb{R}^n}\big|(1-\chi_{int}(\xi))\big|^2\big|\frac{\sin(|\xi|^2t)}{|\xi|^2}e^{-c|\xi|^4t}\hat{u}_1(\xi)\big|^2\mathrm{d}\xi\\ &\lesssim t^{\frac{-n}{8}}\|\big(u_0,u_1,\theta_0\big)\|_{(L^2\cup L^1)}\end{aligned}$$

$$\|u(t,.)-J_0(t,|D|)u_1(.)\|_{L^2}\lesssim t^{\frac{-n}{8}}\|\big(u_0,u_1,\theta_0\big)\|_{(L^2\cup L^1)}$$

for $t\gg 1$. Here, (3.2) has been used.

For another thing, the decomposition

$$\begin{aligned}&J_0(t,|D|)u_1(x)-J_0(t,x)P_{u_1}\\ &\quad=\int_{|y|\le t^{\alpha_0}}(J_0(t,x-y)-J_0(t,x))u_1(y)\mathrm{d}y\\ &\quad+\int_{|y|\ge t^{\alpha_0}}J_0(t,x-y)u_1(y)\mathrm{d}y-J_0(t,x)\int_{|y|\ge t^{\alpha_0}}u_1(y)\mathrm{d}y\end{aligned}$$

with a sufficiently small constant $\alpha_0 > 0$, as well as Taylor's expansion

$$|J_0(t,x-y) - J_0(t,x)| \lesssim |y||\nabla J_0(t,x-\gamma_0 y)| \tag{3.3}$$

with a constant $\gamma_0 \in (0,1)$, shows

$$\begin{aligned}|J_0(t,|D|)u_1(.) - J_0(t,.)P_{u_1}| &\le \int_{|y|\le t^{\alpha_0}} |(J_0(t,x-y) - J_0(t,x))u_1(y)dy| \\ &+ \int_{|y|\ge t^{\alpha_0}} |J_0(t,x-y)u_1(y)|dy + |J_0(t,x)| \int_{|y|\ge t^{\alpha_0}} |u_1(y)|dy \\ &\lesssim \int_{|y|\le t^{\alpha_0}} |y||\nabla J_0(t,x-\gamma_0 y)||u_1(y)|\mathrm{d}y + \int_{|y|\ge t^{\alpha_0}} |J_0(t,x-y)u_1(y)|\mathrm{d}y \\ &+ |J_0(t,x)| \int_{|y|\ge t^{\alpha_0}} |u_1(y)|\mathrm{d}y\end{aligned}$$

$$\begin{aligned}\int_{\mathbb{R}^n} \left|J_0(t,|D|)u_1(.) - J_0(t,.)P_{u_1}\right|^2 \mathrm{d}x &\lesssim \int_{\mathbb{R}^n} \left|\int_{|y|\le t^{\alpha_0}} |y||\nabla J_0(t,x-\gamma_0 y)u_1(y)|\mathrm{d}y\right|^2 \mathrm{d}x \\ &+ \int_{\mathbb{R}^n} \left|\int_{|y|\ge t^{\alpha_0}} \left|J_0(t,x-y)u_1(y)\right|\mathrm{d}y\right|^2 \mathrm{d}x \\ &+ \int_{\mathbb{R}^n} \left|J_0(t,x)\int_{|y|\ge t^{\alpha_0}} |u_1(y)|dy\right|^2 \mathrm{d}x\end{aligned}$$

$$\begin{aligned}&\lesssim \int_{\mathbb{R}^n} \left|\int_{|y|\le t^{\alpha_0}} |y||\nabla J_0(t,x-\gamma_0 y)|\mathrm{d}y\right|^2 \mathrm{d}x \int_{|y|\le t^{\alpha_0}} |u_1(y)|\mathrm{d}y \\ &+ \int_{\mathbb{R}^n} \left|\int_{|y|\ge t^{\alpha_0}} \left|J_0(t,x-y)\right|\mathrm{d}y\right|^2 \mathrm{d}x \int_{|y|\ge t^{\alpha_0}} |u_1(y)|\mathrm{d}y \\ &+ \int_{\mathbb{R}^n} \left|J_0(t,x)\right|^2 \mathrm{d}x \int_{|y|\ge t^{\alpha_0}} |u_1(y)|\mathrm{d}y\end{aligned}$$

$$\begin{aligned}\|J_0(t,|D|)u_1(x) - J_0(t,x)P_{u_1}\|_{L^2}^2 &\lesssim t^{\alpha_0}\|\nabla J_0(t,x-\gamma_0 y)\|_{L^2}\|u_1\|_{L^1} + \|J_0(t,x-y)\|_{L^2}\|u_1\|_{L^1} \\ &+ \|J_0(t,x)\|_{L^2} \int_{|y|\ge t^{\alpha_0}} |u_1(y)|\mathrm{d}y \\ &\lesssim t^{\alpha_0}\|\hat{\nabla} J_0(t,|\xi|)\|_{L^2}\|u_1\|_{L^1} + \|\hat{J}_0(t,|\xi|)\|_{L^2}\|u_1\|_{L^1} \\ &+ \|\hat{J}_0(t,|\xi|)\|_{L^2} \int_{|y|\ge t^{\alpha_0}} |u_1(y)|\mathrm{d}y\end{aligned}$$

$$\|J_0(t,|D|)u_1(x) - J_0(t,x)P_{u_1}\|_{L^2} \lesssim t^{\alpha_0}\left\|\frac{\sin(|\xi|^2 t)}{|\xi|}e^{-c|\xi|^4 t}\right\|_{L^2}\|u_1\|_{L^1} + \|\hat{J}_0(t,|\xi|)\|_{L^2}\int_{|y|\geq t^{\alpha_0}}|u_1(y)|\mathrm{d}y$$
$$\lesssim t^{\alpha_0}t^{-\frac{1}{4}}D_n(t)\|u_1\|_{L^1} + \|\hat{J}_0(t,|\xi|)\|_{L^2}$$
$$\lesssim t^{\alpha_0-\frac{1}{4}}\mathcal{D}_n(t)\|u_1\|_{L^1} + o(D_n(t)) \tag{3.4}$$

for t»1, where we considered

$$\left\|\frac{\sin(|\xi|^2 t)}{|\xi|}\mathrm{e}^{-c|\xi|^4 t}\right\|_{L^2} = \left(\int_{\mathbb{R}^n}\left||\xi|\frac{\sin(|\xi|^2 t)}{|\xi|^2}\mathrm{e}^{-\frac{c}{2}|\xi|^4 t}\mathrm{e}^{-\frac{c}{2}|\xi|^4 t}\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}$$
$$\lesssim \left(\int_{\mathbb{R}^n}\left|\text{ess sup}\big||\xi|\mathrm{e}^{-\frac{c}{2}|\xi|^4 t}\big|\frac{\sin(|\xi|^2 t)}{|\xi|^2}\mathrm{e}^{-\frac{c}{2}|\xi|^4 t}\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}$$
$$\lesssim \text{ess sup}\big||\xi|\mathrm{e}^{-\frac{c}{2}|\xi|^4 t}\big|\left(\int_{\mathbb{R}^n}\left|\frac{\sin(|\xi|^2 t)}{|\xi|^2}\mathrm{e}^{-\frac{c}{2}|\xi|^4 t}\right|^2\mathrm{d}\xi\right)^{\frac{1}{2}}$$
$$\lesssim \left\||\xi|\mathrm{e}^{-\frac{c}{2}|\xi|^4 t}\right\|_{L^\infty}\left\|\frac{\sin(|\xi|^2 t)}{|\xi|^2}\mathrm{e}^{-\frac{c}{2}|\xi|^4 t}\right\|_{L^2}$$
$$\lesssim t^{-\frac{1}{4}}D_n(t),$$

and the fact that $u_1 \in L^1$associated with

$$\lim_{t\to\infty}\int_{|y|\geq t^{\alpha_0}}|u_1(y)|dy = 0.$$

In conclusion, an application of Minkowski's inequality immediately implies

$$\|\hat{J}_0(t,|\xi|)|P_{u_1}| - u(t,.) + u(t,.)\|_{L^2} \leq \|\hat{J}_0(t,|\xi|)|P_{u_1}| - u(t,.)\|_{L^2} + \|u(t,.)\|_{L^2}$$
$$\|u(t,.)\|_{L^2} \gtrsim \|\hat{J}_0(t,|\xi|)\|_{L^2}|P_{u_1}| - \|u(t,.) - J_0(t,.)P_{u_1}\|_{L^2}$$
$$\|u(t,.)\|_{L^2} \gtrsim \|\hat{J}_0(t,|\xi|)\|_{L^2}|P_{u_1}| - \|u(t,.) - J_0(t,|D|)u_1(.) + J_0(t,|D|)u_1(.) - J_0(t,.)P_{u_1}\|_{L^2}$$
$$\gtrsim \left\|\frac{\sin(|\xi|^2 t)}{|\xi|^2}\mathrm{e}^{-c|\xi|^4 t}\right\|_{L^2}|P_{u_1}| - \|u(t,.) - J_0(t,|D|)u_1(.)\|_{L^2} - \|J_0(t,|D|)u_1(.) - J_0(t,.)P_{u_1}\|_{L^2}$$
$$\gtrsim \mathcal{D}_n(t)|P_{u_1}| - \|u(t,.) - J_0(t,|D|)u_1(.)\|_{L^2} - \|J_0(t,|D|)u_1(.) - J_0(t,.)P_{u_1}\|_{L^2}$$
$$\gtrsim \mathcal{D}_n(t)|P_{u_1}| - t^{-\frac{n}{8}}\|(u_0,u_1,\theta_0)\|_{(L^2\cap L^1)^3} - t^{\alpha_0-\frac{1}{4}}\mathcal{D}_n(t)\|u_1\|_{L^1} - o(\mathcal{D}_n(t))$$
$$\gtrsim \mathcal{D}_n(t)|P_{u_1}|$$

for $t \gg 1$ by taking sufficiently small. So, our proof of **Theorem 10** is complete ■

Remark 11 *Let us recall the large-time behavior of the pure plate model*

$$\begin{cases} w_{tt} + \Delta^2 w = 0, \\ (w, w_t)(0, x) = (w_0, w_1)(x), \end{cases} \tag{3.5}$$

with $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$. *Where in [11] the author derived the optimal growth estimates*

$$\|w(t, .)\|^2_{L^2} \simeq \begin{cases} t^{2-\frac{n}{2}} \text{ if } n \leq 3, \\ \ln t \text{ if } n = 4, \end{cases} \tag{3.6}$$

for $t \gg 1$, *where initial datum are taken in* $L^2 \cap L^1$.

For another, Wenhui Chen and Ryo Ikehata in [4] found the derived optimal estimates for the classical thermoelastic plat system (or the model (4) *with* $\sigma = 0$*) as follows:*

$$\begin{cases} v_{tt} + \Delta^2 v + \Delta\theta = 0, \\ \theta_t - \Delta\theta - \Delta v_t = 0, \\ (v, v_t, \theta)(0, x) = (v_0, v_1, \theta_0)(x). \end{cases} \tag{3.7}$$

To be specific, for all dimensions $n \geq 1$, *they obtained*

$$\|v(t, .)\|^2_{L^2} \simeq t^{2-\frac{n}{2}} \tag{3.8}$$

for $t \gg 1$.

In Comparison with optimal estimates (3.5) *in Theorem 2.1,* (3.6) *in [11],* (3.8) *in [4], and the table*

Table 3.1: Influence from the plate model, Fourier's law and the lower-order term

Dimensions(dim.)	n≥3 (Lower-dim)	n=4 (critical-dim)	n≤5(Higher-dim.)
pure plate	$t^{2-\frac{n}{2}}$	log t	-
Thermoelastic plates with σ=0	$t^{2-\frac{n}{2}}$	1	$t^{2-\frac{n}{2}}$
Thermoelastic plates with $\sigma > 0$	$t^{2-\frac{n}{2}}$	log t	$t^{1-\frac{n}{4}}$

we claim the following points :

- *In the lower-dimensions* n=1, 2, 3*, the growth rates of the models*(4) *and* (3.7) *are the same as those for the pure plate model* (3.5)*. This implies that the plate equation plays the decisive role in the general thermoelastic plat system* (4) *for any* $\sigma \geq 0$.

- *In the critical-dimension* n= 4*, the lower-order term* $+\sigma\theta$ *subdues the estimates in the classical thermoelastic plates* (3.7) *so that the plate equation has the decisive influence again with the same growth rate* $\log t$ *as the one in the plate model* (3.5)*.*

- *In the higher-dimensions* $n \geq 5$*, the lower-order term* $+\sigma\theta$ *weakens the decay rates from* $t^{2-\frac{n}{2}}$ *to* $t^{1-\frac{n}{4}}$*. This weakened effect is originated from the lower-order term propagates via the coupling* $(\delta\theta, -\delta u_t)^T$*. This effect is quit different from the evolution equations with the mass term, e.g. heat equations with mass*[8, *chapter*12.2]*, Klein-Gordon equation* [8, *chapter*11.3.4]*, damped waves with mass*[21]*, and strongly damped waves with mass*[5]*. To the best of authors' knowledge, it seems to be the first example that addition of lower-order term (sometimes we call it mass term) causes weakened dissipative properties.*

Remark 12 *In our result for optimal estimates, we just require additional* L^1 *regularity for initial datum rather than* $L^{1,1}$ *regularity in [4, 11]. Indeed, the weighted* L^1 *assumption for the plate model [4, 11] can be relaxed by* L^1 *assumption by following our approach.*

3.2 The asymptotic profile of solution

Before stating the result concerning asymptotic profiles, let us take

$$\begin{aligned} \varphi(t,x) :=& J_0(t,x)P_{u_1} + \langle \nabla J_0(t,x), M_{u_1}\rangle + H(t,x)P_{u_1} \\ & J_1(t,x)P_{u_0} - \sigma^{-1}\delta J_0(t,x)P_{\theta_0}, \end{aligned} \tag{3.9}$$

where some functions originated from higher-order diffusion-plates are chosen by
$J_0(t,x) := F^{-1}_{\xi\to x}\Big(\dfrac{\sin(|\xi|^2 t)}{|\xi|^2} \mathrm{e}^{\frac{-1}{2\sigma}|\xi|^4 t}\Big)$, $J_1(t,x) := F^{-1}_{\xi\to x}\Big(\cos(|\xi|^2 t)\mathrm{e}^{\frac{-1}{2\sigma}|\xi|^4 t}\Big)$,
as well as

$$H(t,x) := \frac{t}{8\sigma^2} F^{-1}_{\xi\to x}\big[\big((2\sigma+1)\cos(|\xi|^2 t) + 4\sin(|\xi|^2 t)\big)|\xi|^4 \mathrm{e}^{\frac{-1}{2\sigma}|\xi|^4 t}\big].$$

We now may show the asymptotic profile of solution, where the time-dependent coefficient of optimal estimates (3.1) has been improved as t$\gg$ 1 when we subtract the profile $\varphi = \varphi(t,x)$. Particularly, in the higher-dimensional case n$\geq$ 3, we claim $u(t,.) \to \varphi(t,.)$ as $t \to \infty$ in the L^2 framework.

Theorem 13 *Let us consider the thermoelastic plate system* (4) *with* $\sigma > 0$ *and carrying initial datum* $(u_0, u_1, \theta_0) \in (L^2 \cap L^{1,1})^3$. *Then, its vertical displacement fulfills the following refined estimates:*

$$\|u(t,.) - \varphi(t,.)\|_{L^2} = o(B_n(t))$$

for t$\gg$ 1, where the time-dependent coefficient $B_n(t)$ *was defined in* (1).

Proof. Recalling the profile defined in (3.9), we indeed my find the next decomposition:

$$u(t,x) - \varphi(t,x) = \varepsilon_0(t,x) + \varepsilon_1(t,x) + \varepsilon_2(t,x)$$

with the components

$\varepsilon_0(t,x) := u(t,x) - (J_0(t,|D|) + H(t,|D|))u_1(x) - J_1(t,|D|)u_0(x) + \sigma^{-1}\Delta J_0(t,|D|)\theta_0(x),$

$\varepsilon_1(t,x) := J_0(t,|D|)u_1(x) - J_0(t,x)P_{u_1} - \langle \nabla J_0(t,x), M_{u_1}\rangle,$

$\varepsilon_2(t,x) := (H(t,|D|)u_1(x) - H(t,x)P_{u_1}) + (J_1(t,|D|)u_0(x) - J_1(t,x)P_{u_0})$

$-\sigma^{-1}(\Delta J_0(t,|D|)\theta_0(x) - \Delta J_0(t,x)P_{\theta_0})$

First for all, we employ the derived estimates (2.4),(2.8) as well as (2.9) to get

$\chi_{int}(\xi)|\hat{u} - \hat{g}_1 - \hat{g}_2| \lesssim \chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)$

$\chi_{int}(\xi)|\hat{g}_1 - (\hat{J}_0 + \hat{H}_0 + \hat{H}_1)|\hat{u_1}| \lesssim \chi_{int}(\xi)|\xi|^2|e^{-c|\xi|^4 t}|\hat{u}_1|,$

$\chi_{int}(\xi)|\hat{g}_2 - \hat{J}_1\hat{u}_0 - \sigma^{-1}|\xi|^2|\hat{J}_0\hat{\theta}_0| \lesssim \chi_{int}(\xi)|\xi|^2|e^{-c|\xi|^4 t}\big(|\hat{u_0}| + |\hat{\theta}_0|\big),$

so,by summation

$$\chi_{int}(\xi)|\hat{u} - \hat{g}_1 - \hat{g}_2 + \hat{g}_1 - (\hat{J}_0 + \hat{H}_0 + \hat{H}_1)\hat{u_1} + \hat{g}_2 - \hat{J}_1\hat{u}_0 - \sigma^{-1}|\xi|^2\hat{J}_0\hat{\theta}_0|$$
$$\lesssim \chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)$$

$$\chi_{int}(\xi)|\hat{u} - (\hat{J}_0 + \hat{H}_0 + \hat{H}_1)\hat{u_1} - \hat{J}_1\hat{u}_0 - \sigma^{-1}|\xi|^2|\hat{J}_0\hat{\theta}_0| \lesssim \chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)$$

$\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)| \lesssim \chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)$
$\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)| + (1-\chi_{int}(\xi)|\hat{\varepsilon}_0(t,x))| \lesssim \chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)$
$+ (1-\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)|)$
$\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)| - \chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)|) + |\hat{\varepsilon}_0(t,\xi)| \lesssim \chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)$
$+ (1-\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)|)$
$|\hat{\varepsilon}_0(t,\xi)| \lesssim \chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|) + (1-\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)|)$
$\int |\hat{\varepsilon}_0(t,\xi)|^2 \mathrm{d}\xi \lesssim \int |\chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)|^2 \mathrm{d}\xi + \int |(1-\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)|)|^2 \mathrm{d}\xi$
$\|\hat{\varepsilon}_0(t,\xi)\|_{L^2}^2 \lesssim \|\chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)\|_{L^2}^2 + \|(1-\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)|)\|_{L^2}^2$
$\|\hat{\varepsilon}_0(t,\xi)\|_{L^2} \lesssim \|\chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)\|_{L^2} + \|(1-\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)|)\|_{L^2}$
by Plancherel's theorem

$$\begin{aligned}\|\varepsilon_0(t,x)\|_{L^2} &\lesssim \|\chi_{int}(\xi)|\xi|^2 e^{-c|\xi|^4 t}(|\hat{u}_0| + |\hat{u}_1| + |\hat{\theta}_0|)\|_{L^2} + \|(1-\chi_{int}(\xi)|\hat{\varepsilon}_0(t,\xi)|)\|_{L^2} \\ &\lesssim t^{-\frac{1}{2}-\frac{n}{8}}\|(u_0,u_1,\theta_0)\|_{(L^2\cap L^1)^3}\end{aligned}$$

for $t \gg 1$.

To treat the second error term, we re-formulate it by

$$\begin{aligned}\varepsilon_1(t,x) = &\int_{|y|\le t^{\alpha_1}} \big(J_0(t,x-y) - J_0(t,x) - \langle y, \nabla J_0(t,x)\rangle\big)u_1(y)\mathrm{d}y \\ &+ \int_{|y|\ge t^{\alpha_1}} \big(J_0(t,x-y) - J_0(t,x)\big)u_1(y)\mathrm{d}y \\ &+ \int_{|y|\ge t^{\alpha_1}} \langle -y, \nabla J_0(t,x)\rangle u_1(y)\mathrm{d}y,\end{aligned}$$

equipping a sufficiently small constant $\alpha_1 > 0$.

Again from (3.3) and

$$|J_0(t,x-y) - J_0(t,x) - \langle y, \nabla J_0(t,x)\rangle| \lesssim |y|^2|\Delta J_0(t,x-\gamma_1 y)|$$

with $\gamma_1 \in (0,1)$, on derives

$$
\begin{aligned}
|\varepsilon_1(t,x)| \leq & \int_{|y|\leq t^{\alpha_1}} |(J_0(t,x-y) - J_0(t,x) - \langle y, \nabla J_0(t,x)\rangle)u_1(y)|\mathrm{d}y \\
& + \int_{|y|\geq t^{\alpha_1}} |(J_0(t,x-y) - J_0(t,x))u_1(y)|\mathrm{d}y + \int_{|y|\geq t^{\alpha_1}} |\langle -y, \nabla J_0(t,x)\rangle u_1(y)|\mathrm{d}y, \\
\lesssim & \int_{|y|\leq t^{\alpha_1}} |y|^2|\Delta J_0(t,x-\gamma_1 y)u_1(y)|\mathrm{d}y + \int_{|y|\geq t^{\alpha_1}} |y||\nabla J_0(t,x-\gamma_0 y)||u_1(y)|\mathrm{d}y \\
& + \int_{|y|\geq t^{\alpha_1}} |\langle -y, \nabla J_0(t,x)\rangle u_1(y)|\mathrm{d}y
\end{aligned}
$$

$$
\begin{aligned}
\Big(\int_{\mathbb{R}^n} |\varepsilon_1(t,x)|^2\mathrm{d}x\Big)^{\frac{1}{2}} \lesssim & \Big(\int_{\mathbb{R}^n} \Big|\int_{|y|\leq t^{\alpha_1}} |y|^2|\Delta J_0(t,x-\gamma_1 y)u_1(y)|\mathrm{d}y\Big|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\
& + \Big(\int_{\mathbb{R}^n} \Big|\int_{|y|\geq t^{\alpha_1}} |y||\nabla J_0(t,x-\gamma_0 y)||u_1(y)|\mathrm{d}y\Big|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\
& + \Big(\int_{\mathbb{R}^n} \Big|\int_{|y|\geq t^{\alpha_1}} |\langle -y, \nabla J_0(t,x)\rangle u_1(y)|\mathrm{d}y\Big|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\
\|\varepsilon_1(t,.)\|_{L^2} \lesssim & \, t^{2\alpha_1}\|\Delta J_0(t,x-\gamma_1 y)\|_{L^2}\|u_1\|_{L^1} + \|\nabla J_0(t,x-\gamma_0 y)\|_{L^2}\int_{|y|\geq t^{\alpha_1}} |y||u_1(y)|\mathrm{d}y \\
& + \|\nabla J_0(t,x)\|_{L^2}\int_{|y|\geq t^{\alpha_1}} |y||u_1(y)|\mathrm{d}y \\
\lesssim & \, t^{2\alpha_1}\|\hat{\Delta} J_0(t,|\xi|)\|_{L^2}\|u_1\|_{L^1} + \|\hat{\nabla} J_0(t,|\xi|)\|_{L^2}\int_{|y|\geq t^{\alpha_1}} |y||u_1(y)|\mathrm{d}y \\
& + \|\hat{\nabla} J_0(t,|\xi|)\|_{L^2}\int_{|y|\geq t^{\alpha_1}} |y||u_1(y)|\mathrm{d}y \\
\lesssim & \, t^{2\alpha_1}\||\xi|^2\hat{J}_0(t,|\xi|)\|_{L^2}\|u_1\|_{L^1} + \||\xi|\hat{J}_0(t,|\xi|)\|_{L^2}\int_{|y|\geq t^{\alpha_1}} |y||u_1(y)|\mathrm{d}y \\
\|\varepsilon_1(t,.)\|_{L^2} \lesssim & \, t^{2\alpha_1-\frac{n}{8}}\|u_1\|_{L^1} + o\big(\mathcal{B}_n(t)\big)
\end{aligned}
$$

for $t \gg 1$, where we used [10] and the weighted assumption $u_1 \in L^{1,1}$.
Afterward, similarly to (3.4), we may calculate

$$
\begin{aligned}
\varepsilon_2(t,x) :=& (H(t,|D|)u_1(x) - H(t,x)P_{u_1}) + (J_1(t,|D|)u_0(x) - J_1(t,x)P_{u_0}) \\
& - \sigma^{-1}(\Delta J_0(t,|D|)\theta_0(x) - \Delta J_0(t,x)P_{\theta_0})
\end{aligned}
$$

$$\|\varepsilon_2(t,.)\|_{L^2} = o(t^{-\frac{n}{8}})$$

for $t \gg 1$ also, which can be guaranteed by the hypothesis $(u_0, u_1, \theta_0) \in L^{1,1}$.Summing up all obtained estimates from previous statements, we assert that

$$\|u(t,.) - \varphi(t,.)\|_{L^2} \lesssim \|\varepsilon_0(t,.)\|_{L^2} + \|\varepsilon_1(t,.)\|_{L^2} + \|\varepsilon_2(t,.)\|_{L^2} = o(B_n(t))$$

for large-time $t \gg 1$, and our proof is complete now. ∎

Remark 14 *The lower-order term $+\sigma\theta$ enables the asymptotic profiles of the thermoelastic plate system* (4) *to change$\psi = \psi(t,x)$ in* (3) *of the model with $\sigma = 0$ into $\varphi = \varphi(t,x)$ in* (3.9) *of the model with $\sigma > 0$. The crucial difference is the power of exponential function in the Fourier multipliers.*

Remark 15 *Physically, the three-dimensional thermoelastic plate system is more important than other cases because of its applications in practice. If we simply consider $\varphi_{sim}(t,x) = J_0(t,x)P_{u_1}$ to be the asymptotic profile, then it is not difficult to prove $\|u(t,.) - \varphi_{sim}(t,.)\|_{L^2} = o(D_n(t))$ as $t \gg 1$, which does not decay in $\mathbb{R}^3$ as $t \to \infty$. For this purpose, we construct the higher-order profile $\varphi(t,x)$ instead of $\varphi_{sim}(t,x)$.*

CONCLUTION

This dissertation studied the Cauchy problem for thermoelastic plate systems associated with Newton's law of cooling, by applying WKB method and Fourier analysis where, optimal growth for ($n \leq 4$) or decay for ($n \geq 5$) estimates for the vertical displacement and asymptotic profiles of solutions for large-time.

The additional lower-order term $+\sigma\theta$ in the temperature equation weakens the decay rates of the vertical displacement. This lower-order term leads to anew leading term, which differs from the classical thermoelastic plate model.

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