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Existence of solutions for Two-Point Boundary Value Problem of Fractional Differential Equations at Resonance

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Dedication

I dedicate this modest work to my dear parents,
my God grant her a long life,
who helped me to face the difficulties
To all my sisters and brothers
To all my teachers for theire useful advice, their patience,their perseverance.

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Contents

Notation

The following notations are used throughout the work:
$\mathbb{R}$: set of real numbers.
$\mathbb{R}_+$: set of positive real numbers.
$\mathbb{R}_+^*$: set of strictly positive real numbres.
$\mathbb{N}$: set of natural numbers.
$\mathbb{C}$: set of complex numbers.
$C([a,b])$: the space of functions f continuous on $[a,b]$ at real values.
$\bar{C}^k$: the space of functions with values in R, K times differentiable in Ω.
(M,d) : metric space.
$d(.,.)$: distance application.
Ω : bounded open set.
$\overline{\Omega} = \Omega + \partial\Omega$: this is the closure of Ω.
$u'(t)$: the ordinary derivative with respact to t.
U an open set.
deg : Topological degree.
deg_B : Topological degree e of Brouwer.
$deg_L S$: Leray-Schauder topological degree.
$\bar{B}$: The closed unit ball.
max : Maximum.
$\Gamma(.)$: The Gamma function.
$\beta(.,.)$: The Beta function.
I_α^{a+} : The Riemann-Liouville fractional integral of order a.
$\langle , \rangle$ Scalar product
${}^R D^\alpha(f(t))$: Fractional derivation to theleft in the sense of Riemman-Liouville of order $\alpha > 0$
${}^c D^\alpha(f(t))$: Left fractional derivative in the Caputo sense of order $\alpha > 0$
$D^n = \dfrac{d^n}{dt^n}$: Ordinary derivative with respact to t of integer order n.

Im : Image of an application.
Ker : Noyau.
P, Q : Two continuous projections.
$\oplus$: The direct amount.
L : Fredholm's operator.
dom : Domain.
ind : Index.
dim : Dimension.
$codim$: Codimension.
$coker$: Conoyan.
Kp : The linear operator.
N : L-compact sur Ω.
$\|.\|_\infty = max|.|$.

Introduction

The objective of this works is to present solution existence resulte for a class of resonance boundary value problems for nonlinear implicit differential equations with fractional derivatives in the Caputo sense. A boundary value problem is said to bi in resonance if the corresponding linear homogeneous problem has a non-trivial solution. The technique used to present the results is based on Mawhin's degree of coincidence theory to examine the existence of solutions. Mawhin's theory allows the use of a topological degree approach to problems that can be written as an abstract operator equation of the form $Lx = Nx$, where L is a non-invertible linear operator and N is an operator nonlinear acting on a given Banach space. In 1972, Mawhin's devloped a method for solving this equation in this famous paper Topological and Boundary Degree Problems for Nonlinear Differential Equations [25], he assumed that L is a Fredholm operator of index zero. By Therefore, he developed a new theory of topological degree known as degree of coincidence for L, N: which is equally known as Mawhin's degree of coincidence theory.

Fractional differential equations have been studied extensively. It is caused both by the intensive development of the theory of fractional calculs itself and by the applications such as physics, chemistry, phenomena arising in engineering, economy, and sinece, see;for example ,[1-5].

Recently, more and more authors have paid their attentions to the boundary value problems of fractional differential equations. Moreover, there have been many works related to the existence of solutions for boundary value problems at resonance. It is considerable that there are many papers that have dealt with the solutions of multipoint boundary value problemsof fractional differential equations at resonance (see,e.g.,[12-16]). In [12] Bai and Zhang considered a three-point boundary value problem of fractional differential equations with non-linear growth given by

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f(t, u(t), D_{0^+}^{\alpha-1}u(t)), & 0 < t < 1, \\ u(0) = 0, \\ u(1) = \sigma u(\eta), \end{cases} \tag{1}$$

Where $1 < \alpha < 2, 0 < \eta, \sigma < 1 > 0, \sigma\eta^{\alpha-1} = 1$, $D_{0^+}^{\alpha}$ is Riemann-Liouville fractional derivative, and $f, g : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ are given functions.
In [13], Hu et al. have studied a two-point boundary value problem for fractional differential equation at resonance

$$\begin{cases} D_{0^+}^{\alpha}x(t) = f(t, x(t), x'(t)), & 0 \le t \le 1, \\ x(0) = 0, \\ x'(0) = x'(1), \end{cases} \tag{2}$$

Where $1 < \alpha \le 2$, $D_{0^+}^{\alpha}$ is Caputo fractional derivative, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ satisfies Caratheodory conditions.
As far as we know, there are few works on the existence of two-point boundary value problems of the fractional differential equations at resonance. Motivated by the works above, we discuss the existence and uniqueness of solutions for the following non-linear fractional differential equation:

$$\begin{cases} D_{0^+}^{\alpha}u(t) = f\big(t, u(t), D_{0^+}^{\alpha-1}u(t), D_{0^+}^{\alpha-2}u(t), ..., D_{0^+}^{\alpha-(N-1)}u(t)\big), \\ u(0) = D_{0^+}^{\alpha-2}u(0) = ... = D_{0^+}^{\alpha-(N-1)}u(0) = 0, \\ D_{0^+}^{\alpha-1}u(0) = D_{0^+}^{\alpha-1}u(1), \end{cases} \tag{3}$$

Where $0 < t < 1, N-1 < \alpha < N$, $D_{0^+}^{\alpha}$ is Riemann-Liouville fractional derivative, and $f : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is continuous function.
More precisely, we use the coincidence degree theorem due to Mawhin[22] The rest of this paper is organized as follows.
The two-point boundary value problem (3) happens to be at resonance in the sense that the associated linear homogeneous boundary value problem

$$\begin{cases} D_{0^+}^{\alpha}u(t) = 0 \\ u(0) = D_{0^+}^{\alpha-2}u(0) = ... = D_{0^+}^{\alpha-(N-1)}u(0) = 0, \\ D_{0^+}^{\alpha-1}u(0) = D_{0^+}^{\alpha-1}u(1), \end{cases} \tag{4}$$

Has $u(t) = c_1 t^{\alpha-1}$ as a nontrivial solution.

This dissertations is composed of four chapters:

In the first chapter we present the useful preliminary notions, which are used in the other chapters, we present a review of some fixed point theorems, in particular Banach's contraction principle, the nonlinear alternative of Leray-Schauder, and we also introduce an important notion on homotopy Also we discuss the concept of the topological degree and it's properties, we define two degrees the Brouwer degree in finite dimension then the Leray-Schauder degree in infinite dimension.

In the seconde chapter we start by presenting some special functions of the theory of fractional differential equations, and we define the fractional integral of Riemman-Liouville and Caputo and their properties as well as the relation and the difference between the two.

In the third chapter we define Mawhin's coincidence degree theorem, or at least construct it. We will need to identify an important class of operators:
Fredholm operators with index zero. These operators can be obtained from the projection. Finally, we prove the theorem.

In the fourth chapter we present a brief introduction to some notations and certain fundamental results involved in the reformulation of the problem as well as the main theorem, that of the existence of the solution obtained from Mawhin's.

In this dissertation, we are interested in the following two-point Boundary value problem of fractional differential equations at resonance.

Chapter 1

Reminders and fundamental concepts

The aim of this chapter is to study some fixed point theorems , we will start with the simplest and best known of them : Banach's fixed point theorem for contracting applications. we will then see Brouwer's fixed point theorem (valid in finite dimension) then Schauder's fixed point theorem (which is the "generalization" in infinite dimension). see([1] [14])

1.1 fixed point theorem

In this section we present some theories and characteristics of the banach fixed point, and also fixed point for the application is not a contraction on the whole metric space, with studying the principles of continuity.

1.1.1 Banach fixed point theorem

Banach's fixed point theorem (also known as the contracting map theorem) is a simple theorem to prove, which guarantees the existence of a unique fixed point for any contacting map, applies to complete spaces and which has many applications, these applications include theorems of the existence of solutoins for differential equations or integral equatoins and the study of the convergence of certain numerical methods.

Definition 1.1.1 ***[Metric spece]***

A metric space $(X;d)$*is a set* X *provided with a map* $d : \mathbb{X} \times \mathbb{X} \to \mathbb{R}$ *called distance or* *metric, having the following three properties :*

i) $\forall$ x ,y $\in$ X : $d(x, y) \geq 0$ and$d(x, y)$= 0 $\Longleftrightarrow$ x =y

ii) $\forall$ x,y $\in$ X : $d(x, y)$=$d(y, x)$ (*symmetry*)

iii) $\forall$ x,y $\in$ X :$d(x, z)$ $\leq$$d(x, y)$ + d(y,z)

Definition 1.1.2 ***[Full metric space]***
The metric space (x, d) is said to be complete if any Cauchy sequence in X converges in X.
Cauchy sequence:
We say that the sequence $(x_n)_n$ in the metric space (X,d) is Cauchy if

$$\forall \epsilon > 0, \exists N_\epsilon > 0 \text{ such that } n, m > N_\epsilon \Rightarrow d(x_n, x_m) < \epsilon.$$

We then write

$$d(x_n, x_m) \rightarrow 0, \text{ when } n, m \rightarrow +\infty$$

Definition 1.1.3 ***[Fixed point]***
Let $F : \mathbb{X} \rightarrow \mathbb{X}$ be an application .we call fixed point $x \in X$ such that $F(x) = x$

Definition 1.1.4 ***[The Lipschitzian application]***
Let (x, d) be a complet metric space and the map $F : \mathbb{R} \rightarrow \mathbb{R}$,we say that F is a lipschitzian map if there exists a positive constant $k \geq 0$ such that we have , for any pair of elements x, y of X , the inequality :

$$d(F(x), F(y)) \leq K(d(x, y)), \forall x, y \in M \tag{1.1}$$

if $K \geq 1$ the application F is called not expansive .
if $K < 1$ the applicatin F is called contraction .

Theorem 1.1.1 ((fixed point of banach (1922))) ***[1]***
Let(x, d) be a complet metric space and let $F : \mathbb{X} \rightarrow \mathbb{X}$ be a contracting map of contraction constant K , then :

(a) F admits a unique fixed point $\alpha \in X$

(b) For all $x \in X$, $\alpha = \lim_{n\to\infty} F^n(x)$ or $F^0(x)$ and $F^n(x) = F(F^{n-1}(x))$

(c) The speed of convergence can be estimated by :

$$d(F^n, \alpha) \leq (K^n(1 - K)^{-1})d(x, F(x)) \quad n \geq 1 \tag{1.2}$$

Proof.

(1) **The uniqueness of the fixed point :**
We assume that there is $x, y \in X$ with $x = F(x)$; $y = F(y)$
$d(x, y) = d(F(x), F(y)) \leq kd(x, y) 0 < k < 1.$
Then the last inequality is correctly in the case
$d(x, y) = 0 \Rightarrow x = y$
Then $\exists! x \in X$ such that $F(x) = x$

(2) **The existence of the fixed point :**
We assume that $F^n(x)$ is a cauchy sequence for $n \in N$.
Or
$d(F_n(x), F_{n+1}(x)) \leq kd(F^{n-1}(x), F^n(x)) \leq ..... \leq k^n d(x, F(x))$
If $m > n$ where $n \in 0, 1, ...,$ we obtain

$$\begin{aligned} d(F_n(x), F_m(x)) &\leq d(F^n(x), F^{n+1}(x)) + d(F^{n+1}(x), F^{n+2}(x)) + ... + d(F^{m-1}(x), F^m(x)) \\ &\leq k^n d(x, F(x)) + k^{n+1} d(x, F(x)) + k^{m-1} d(x, F(x)) \\ &\leq k^n d(x, F(x))[1 + k + k^2 + ...] \\ &\leq \frac{k^n}{1-k} d(x, F(x)). \end{aligned}$$

For $m > n$; $n \in 0, 1, ...$, we have

$$d(x_n, x_m) \leq \frac{k^n}{1-k} d(x_0, x_1) \tag{1.3}$$

Then F^n is a cauchy sequence in the complete space X subsequently then there exists $\alpha \in X$ with

$$\lim_{n \to +\infty} F^n = \alpha$$

Moreover by the continuity of F

$$\alpha = \lim_{n \to \infty} (F^{n+1}(x)) = \lim_{n \to \infty} (F(F^n(x))) = F(\lim_{n \to \infty} (F^n(x))) = F(\alpha)$$

So α is a fixed point of F .
Finally, $m \to \infty$ in $(1, 3)$, we obtain

$$d(F^n(x), x) \leq \frac{k^n}{1-k} d(x, F(x)).$$

■

Example 1.1.1
Consider the application $T : \mathbb{R} \to \mathbb{R}$ *defined by* $T(x) = \frac{x}{2} + \frac{1}{2}$ *,then* T *a contraction with* $0 < k = \frac{1}{2} < 1$ *, and admits as fixed point* $x = 1$ *moreover* $\lim\{T^n(x)\}_{n-1}^{\infty} = 1$

Remark 1.1.1 *The conditions of this theorem are necessary, consider the following examples*

Example 1.1.2 ***(Closing condition)***

(1) *If X is not stable by $F : F(x = \sqrt{x^{2+1}})$ on $X = [0, 1]$, we have a X is closed in R therefore it is complete (because at is complete) ,*
Moreover $F'(x) = \frac{x}{\sqrt{x^{2+1}}} < 1$,which implies that $\sup_{x\in[0,1]} F'(x) < 1$, therefore F is contracting on $[0, 1]$,
But F does not admit a fixed point because $F([0, 1]) = [1, \sqrt{2}] \subset [0, 1]$,$i, e$ X not stable by F

(2) *$F :]0, 1] \to]0, 1], F(x) = \frac{x}{2}$, is contracting and satisfies $F(]0, 1]) \subset]0, 1]$ but does not admit a fixed point, The problem is that $]0, 1]$ is not closed $\lim_{x_n} = 0$ is not contained in $]0, 1]$*

Example 1.1.3 ***(Contraction condition)***

(1) *$F(x) = \sqrt{x^2 + 1}$ on $X = [0, +\infty[$, We have $F([0, +\infty[) \subset ([0, +\infty[)$ and X is a closed in R, Then X is complete, But F does not have a fixed point because $\sup_{x\in X} F'(x) = 1$ i, e F is not contracting*

(2) *$F : \mathbb{R} \to \mathbb{R}, F(x) = x + \dfrac{1}{1 + e^x}$ checks $F(x) - F(y) < x - y$ for all $x \neq y$,but does not admit a fixed point , The problem is that T is not contracting, and for all $x_0 \in R$ we obtain $x_n \to +\infty$*

1.1.2 Fixed point theorems for the application is not a contraction on the whole metric space

Let (X, d) be a complete metric space, functions defined only on a subset of X will not necessarily have a fixed point ,Additional conditions will be necessary ,to ensure this

Theorem 1.1.2
Let K be a closed set in X and $F : \mathbb{K} \to \mathbb{X}$ a k-contraction, suppose that there exists $x_0 \in K$ and $r > 0$ such that

$$\overline{B(x_0, r)} \subset K \text{ and } d(x_0, F(x_0)) < (1 - k)r.$$

Then F has a single fixed point in $B(x_0, r)$
In certain applications,there are cases where F is lipschitz without being a contraction, while a certain power of F is a contraction see(1).In this case we have the following theorem

Theorem 1.1.3 *Let (X, d) be a complete metric space, $F : \mathbb{X} \to \mathbb{X}$ a lipschitzian map (not necessarily a contraction)*

$$d(T^n(x), T^m(y)) \leq kd(x, y), x, y \in M.$$

For a certain $m > n$ or $0 < k < 1$, then T admits a unique fixed $x^* \in M$.

Proof.

As T^p is a contraction, it follows from theorem (1.1.2) that T^p has a unique fixed point, so $x^* = T^p x^*$
So

$$T^p(T(x^*)) = T(T^p(x^*)) = T(x^*)$$

then,$T(x^*)$ is a fixed point of T^P
But T^p admits a unique fixed point, hence $T(x^*) = x^*$. So T a unique fixed point (x^*), and it is unique because evrey fixed point of T is also fixed point of T^p. ■

Example 1.1.4

Knows M a metric space given by $M = C[a;b]$, is a Banach space with respect to the norm $||u|| = \max_{t\in[a,b]} |u(t)|, u \in M$.
We define $T : \mathbb{M} \to \mathbb{M}$ by:

$$Tu(t) = \int_a^t u(s)ds$$

We show that $||T(u) - T(v)|| \leq (b-a)||u-v||$,we have

$$\begin{aligned} ||T(u) - T(v)|| &= \max_{t\in[a,b]} |\int_a^t u(s)ds - \int_a^t v(s)ds| \\ &\leq \max_{t\in[a,b]} \int_a^t |u(s) - v(s)|ds \end{aligned}$$

According to the increase,we obtain

$$\begin{aligned} ||T(u) - T(v)|| &\leq \int_a^t ds||u(s) - v(s)|| \\ &\leq (t-a)||u(s) - v(s)||, \qquad \forall t \in [a,b] \\ &\leq (b-a)||u(s) - v(s)||. \end{aligned}$$

therefore $(b-a)$ is the best Lipschitz constant for T.
On the other hand, we have :

$$T^2(u)(t) = \int_a^t (\int_a^s u(\mu)d\mu)ds = \int_a^t (t-s)u(s)d(s).$$

And by induction

$$T^m u(t) = \frac{1}{(m-1)!}\int_a^t (t-s)^{m-1}u(s)ds.$$

Since then

$$\begin{aligned}
||T^m u(t) - T^m v(t)|| &= \max_{t\in[a,b]} |T^m u(t) - T^m v(t)| \\
&= \max_{t\in[a,b]} \left| \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} u(s) ds - \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} v(s) ds \right| \\
&= \max_{t\in[a,b]} \left| \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} (u(s) - v(s)) ds \right| \\
&\leq \max_{t\in[a,b]} \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} |(u(s) - v(s))| ds \\
&\leq \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} ds ||(u(s) - v(s))|| \\
&\leq \frac{-1}{(m-1)! \times m} \left[(t-s)^m\right]_a^t ||(u(s) - v(s))|| \\
&\leq \frac{1}{m!} (t-a)^m ||(u(s) - v(s))|| \qquad \forall t \in [a,b] \\
&\leq \frac{1}{m!} (b-a)^m ||(u(s) - v(s))||.
\end{aligned}$$

And so T^m would be a contraction if $\dfrac{(b-a)^m}{m!} \leq 1$.

1.1.3 Continuation principles

Anouther way to obtain the existence of fixed point for undefined map over the whole space is obtained via a continuation process.This one consists of deforming our application into another simpler one for which we know the existence of a fixed point. It goes without saying that this deformation known as will have to meet certain conditions.

Definition 1.1.5 ***(Homotopic applications)***

Let X and Y be their topological spaces. Thier continuous applications $f, g : \mathbb{X} \to \mathbb{Y}$ are called homotopic when there is a continuous application

$$H : \mathbb{X} \times [0,1] \to \mathbb{Y},$$

sush that: $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

Remark 1.1.2

In other words, there exists a family of applications of X in Y, to know $x \to H(x,t)$ for $0 \leq t \leq 1$,which starts from to arrive at g, and varies continuously. we note $f \simeq g$.

Example 1.1.5

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be the constant map $f(x) = 0$, and $g : \mathbb{R}^n \to \mathbb{R}^n$ the map $g(x) = x$. Let us show that f and g are homotopic. It suffices to take:

$$H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$$

such as

$$H(x,t) = (1-t)f(x) + tg(x).$$

We have

$$H(x,0) = (1-0) \times 0 + 0 \times x = 0,$$

And

$$H(x,0) = (1-1) \times 0 + 1 \times x = x.$$

Then $H(x,t) = tx$ *and* $H(x,1) = g(x)$ *and* $H(x,0) = f(x)$

Example 1.1.6

Let $X = Y = \mathbb{R}^n - \{0\}$, *we consider this time,* $p(x) = \dfrac{x}{||x||}$ *and* $q(x) = x$. *We see that* p *and* q *are homotopic by taking:*

$$H : (\mathbb{R}^n - \{0\}) \times [0,1] \to \mathbb{R}^n - \{0\}$$

Such that: $H(x,t) = (1-t)q(x) + tp(x)$, *we have*

$$H(x,0) = (1-0) \times x + 0 \times \frac{x}{||x||} = x,$$

And

$$H(x,1) = (1-1) \times x + 1 \times \frac{x}{||x||} = \frac{x}{||x||}.$$

Then $H(x,t) = (1-t)x + t\frac{x}{||x||}$ *and* $H(x,0) = q(x)$ *and* $H(x,1) = p(x)$.

Definition 1.1.6 ***(Homotopy equivalence)***

Let $f : \mathbb{X} \to \mathbb{Y}$ *be a continuous map. We say that* f *is a homotopy equivalence when there exists* $g : \mathbb{Y} \to \mathbb{X}$ *such that* $\begin{cases} g \circ f = id_x, \quad \text{and} \quad f \circ g = id_y \end{cases}$ *We say that* X *and* Y *are of the same type of homotopy, and we write* $X \simeq Y$.

Example 1.1.7

Let $X = \mathbb{R}^n - \{0\}$ *and* $Y = S^{n-1}$, *we then take* $f : \mathbb{X} \to \mathbb{Y}$ *defined by* $f(x) == x/||x||$, *and* $g : \mathbb{Y} \to \mathbb{X}$ *inclusion. So* $f \circ g = id_y$, *and example (1.1.6) shows that* $g \circ f \simeq id_x$. *so* $\mathbb{R}^n - \{0\}$ *has the same type of homotopy as the sphere* S^{n-1}.

Definition 1.1.7 ***(The homotopy properties)***

Let (X, d) *be a complete metric space, and* U *be an open subset of* X.
We consider $F : \overline{U} \to \mathbb{X}$ *and* $G : \overline{U} \to \mathbb{X}$ *two contractions, we say that* F *and* G *are homotopic if there exists* $H : \overline{U} \times [0,1] \to \mathbb{X}$ *verifying the following properties:*

(a) $H(.,0) = G$ *and* $H(.,1) = F$.

(b) $H(x,t) \neq x$ *for all* $x \in \partial U$ *and* $t \in [0,1]$.

(c) There exists $\alpha \in [0,1]$ *such that* $d\big(H(x,t), H(y,t)\big) \leq \alpha d(x,y)$ *for all* $x, y \in \overline{U}$, *and* $t \in [0,1]$.

(d) There exists $M \geq 0$ such that $d\big(H(x,t), H(x,s)\big) \leq M|t-s|$ for all $x \in \overline{U}$, and $t, s \in [0,1]$.

Theorem 1.1.4

Let $F : \overline{U} \to \mathbb{X}$ and $G : \overline{U} \to \mathbb{X}$ be two homotopically contractive applications and G be a fixed point in U. Then, F admits a fixed point in U

Proof.

We se the set $Q = \{(\lambda \in [0,1], x = H(x, \lambda))\}$ for certain $x \in U$ and H is a homotopy between F and G a described in definition (1.1.5). Note that Q is not empty since G has a fixed point and $0 \in Q$.

We show that Q is both open and closed in $[0,1]$ so show $Q = [0,1]$. Therefore F a fixed point.

(i) **Let us show that Q is a closed set in [0,1]:**
Let $\{\lambda_n\}_{n \in N}$ be a sequence in Q such that $\lim_{n\to\infty} \lambda_n = \lambda$, then we must show that $\lambda \in Q$.
Like $\lambda_n \in Q$ for $n = 1, ..., 2$ there exists $x_n \in U$ where $x_n = H(x_n, \lambda_n)$.
We have for $n, m \in \{1, 2...\}$

$$\begin{aligned} d(x_n, x_m) &= d\big(H(x_n, \lambda_n), H(x_m, \lambda_n)\big) \\ &\leq d\big(H(x_n, \lambda_n), H(x_m, \lambda_n)\big) + d\big(H(x_n, \lambda_m), H(x_m, \lambda_m)\big) \\ &\leq M|\lambda_n - \lambda_m| + \partial d(x_n, x_m). \end{aligned}$$

So

$$d(x_n, x_m) \leq \frac{M}{1-\alpha}|\lambda_n - \lambda_m|$$

Therefor $\{x_n\}$ is a Cauchy sequence of X (because $\{\lambda_n\}$ is also) and, since X is complete, there exists $x \in \overline{U}$ such that $\lim_{n\to\infty} x_n = x$.
By the continuity of H

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} H(x_n, \lambda_n) = H(x, \lambda)$$

So$\lambda \in Q$ and Q is closed in $[0,1]$.

(ii) **Let us show that Q is an open in [0,1]:**
Let $\lambda_0 \in Q$, then there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$.
Since, by hypothesis $x_0 \in U$, we can find $r > 0$ such that the open ball $B(x_0, r) = \{x \in X : (x, x_0) < r\} \subseteq U$.
Let us choose $\epsilon > 0$ such that $\epsilon \leq \frac{(1-\alpha)r}{M}$ where $r \leq dist(x_0, \partial U)$, and

$$dist(x_0, \partial U) = inf\, \{(x_0, x) : x \in \partial U\}.$$

Let us fix $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. So for to $x_0 \in \overline{B(x_0, r)}$

$$\begin{aligned} d(x_0, H(x, \lambda)) &\leq d(H(x_0, \lambda_0) H(x, \lambda_0)) + d(H(x, \lambda_0), H(x, \lambda)) \\ &\leq \alpha d(x_0, x) + M|\lambda, \lambda_0| \\ &\leq \alpha r + (1-\alpha) r = r \end{aligned}$$

Then for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ fixed

$$H(.,\lambda) : \overline{B(x_0, r)} \to \overline{B(x_0, r)}.$$

By theorem (1.1.1),(1.1.2),we deduce that $H(.,\lambda)$ a fixed point in U .
Then $\lambda \in Q$ for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. And therefore Q is open in $[0, 1]$ so $Q = [0, 1]$.

■ From the previous theorem, we deduce the following result:

Theorem 1.1.5 ***(non-linear alternative of Leray-Schauder)***
Let $U \subset E$ be an open set of a banach space E such that $0 \in U$, and let $F : \overline{U} \to E$ be a contraction such that $F(\overline{U})$ is bounded.
Then one of following two statements is verified:

(a) F a fixed point in $(\overline{U})$.

(b) There exists $\lambda \in (0, 1)$ and $x \in \alpha U$ such that $x = \lambda F(x)$.

Proof.
Suppose that (b) is not verified and that F does not have a fixed point on αU i.e $x \neq \lambda F(x)$ for all $x \in \alpha U$ and $\lambda \in [0, 1]$.
Let $H : \overline{U} \times [0, 1] \to E$ be given by $H(x, \lambda) = \lambda F(x)$, and let G be the zero map $(G(x) = 0)$. Note that G is a fixed point in U (namely $(G(0) = 0)$) and that F and G are two homotopically contractive maps. By theorem (1.1.4) F also has a fixed point and therefore statement (a) is verified. ■

1.2 Topological degree

In this section, we give a brief overview of the notion of topological degree whether in finite or infinite dimension. The degree, $\deg(f, \Omega, y)$ of F in Ω with respect to y gives information on the number of solutions of the equation $f(x) = y$ in an open set $\Omega \subset X$, where :$f : \mathbb{X} \to \mathbb{X}$ is continuous, $y \notin f(\partial\Omega)$ and X is a metric topological space most of the time.

1.2.1 Brouwer topological degree

Consider an open bounded Ω of R^n border $\partial\Omega$ and closing $\overline{\Omega}$.
$\overline{C}^k(\Omega, R^n)$ will designate the space of functions with value in R^n, K times differentiable in Ω which are continuous on $\overline{\Omega}$. This space will be equipped with its usual topology.

Definition 1.2.1 ***(Jacobian)***
Let $x_0 \in \Omega$, if f differentiable in x_0, we denote by $J_f(x_0) = \det f'(x_0)$ the jacobian from f in x_0.

Definition 1.2.2 ***(The critical point)***

Let f be a function of class C^1 on Ω. Let us denote by $J_f(x_0)$. Otherwise, x_0 is called a regular point.
We put $S_f(\Omega)$ the set of critical points.
That's to say :

$$S_f(\Omega) = \{x \in \Omega, J_f(x) = 0\}.$$

Definition 1.2.3 ***(Regular value)***

Consider y An element in R^n is said to be a regular value of f if $f^{-1}(y) \cap S_f\Omega = \emptyset$. Otherwise, y is said to be a singular value.

Definition 1.2.4 ***Topological degree***

Let $f \in \overline{C}^1(\Omega, R^n)$ and $y \in R^n$ $f(\partial\Omega)$ be a regular value of f. We call topological degree of f in Ω with respect to y the integer

$$\deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} Sgn \quad J_f(x).$$

Where $SgnJf(x)$ represents the sign of $J_f(x)$, defined by:

$$sgn(t) = \begin{cases} 1 & \text{if } t > 0, \\ -1 & \text{if } t < 0. \end{cases}$$

With the addition of these two notes

1) *if $f^{-1}(y) = \emptyset, deg(f, \Omega, y) = 0$.*

2) *$f^{-1}(y)$ contains a finite number of elements.*

Remark 1.2.1

In the case where $f^{-1}(y) \cap S_f(\Omega) \neq 0$, we move on to the following lemma

Lemma 1.2.1

Consider a function $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$.Then the set $f(S_f)$ of critical values of f has zero measure.

Proof.

It suffices to consider the case where $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$ *and* Ω is a cube of side a. For $k \geq 1$ integer, we divide the cube into k^N cubes C_i of side $\frac{a}{k}$ (with $1 \leq i \leq k$.)If i is such that $S \cap C_i \neq \varnothing$, let $x \in S \cap C_i$ *and* $y \in C_i$. As $J_f(x) = 0$, this means that $f'(x)(\mathbb{R}^n)$ is contained in a hyperplane of $\mathbb{R}^n$, let H_x.
By teaching by ϵ the uniform continuity module of f' on Ω, we have:

$$\begin{aligned} |f(y) - f(x) - f'(x)(y - x)| &\leq |y - x|\epsilon(|y - x|), \\ &\leq \frac{a\sqrt{N}}{k}\epsilon(|y - x|). \end{aligned}$$

So

$$d\big(f(y), f(x) + H_x\big) \leq \frac{a\sqrt{N}}{k}\epsilon\big(\frac{a\sqrt{N}}{k}\big).$$

Furthermore, by setting $L = \max_{x\in\Omega} ||f'||$, the finite increment theorem allows describe:

$$\begin{aligned} |f(y) - f(x)| &\leq |y - x| sup f'(x + t(y - x)) \\ &\leq L\frac{a\sqrt{N}}{k} \end{aligned}$$

Which ultimately implies that if $C_i \cap S \neq \varnothing$, then $f(C_i)$ is contained in a tile of thickness $2\epsilon(a\sqrt{N}/k)a\sqrt{N}/k$ and whose base has sides of length $2La\sqrt{N}/k$.
We deduce the estimate:

$$mes(f(c_i)) \leq 2\epsilon(\frac{a\sqrt{N}}{k})\frac{a\sqrt{N}}{k}(2L\frac{a\sqrt{N}}{k})^{N-1},$$

and, since there are k^N cubes C_i, we deduce that for all $k \geq 1$ we have

$$mes(f(c_i)) \leq 2^N L^{N-1}(a\sqrt{N})^N \epsilon(a\sqrt{N}/k).$$

By making $k \longrightarrow +\infty$ tend, we see that $mes\big(f(S)\big) = 0$.
We will now see that we can extend the notion of degree to the case where the function f is only continuous. ■

Definition 1.2.5
Let $\Omega \subset R^n$ be a bounded open, $f \in C(\overline{\Omega}, \mathbb{R}^n)$ and $y \in \mathbb{R}^n$ such that $y \notin f(\partial\Omega)$. We define the topological degree of f in Ω compared to y by

$$deg(f, \Omega, y) = [\lim_{n\to\infty} deg(f_n, \Omega, y)].$$

Where $\{f_n\}_{n\in N^*}$ is a sequence of function $C^1(\overline{\Omega}, \mathbb{R}^n)$ which converges uniformly to f in $\overline{\Omega}$.

Theorem 1.2.1 ***(Some important properties of the Brouwer topological degree)[7]***
Let $\Omega \in \mathbb{R}^n$ be a bounded open, and let

$$A(\Omega) = \{f \in C(\overline{\Omega}, \mathbb{R}^n) : y \notin f(\partial\Omega)\}.$$

The map $deg(f, \Omega, y) : A(\Omega) \to \mathbb{Z}$ satisfies the following properties:

(1) ***(Normalization)*** *$deg(I, \Omega, y) = 1$ if $y \in \Omega$ and $deg(I, \Omega, y) = 0$ if $y \in \mathbb{R}^n\Omega$ where I designates the identity map on $\overline{\Omega}$*

(2) ***(Solvency)*** *if $deg(f, \Omega, y) \neq 0$, then $f(x) = y$ admits at least one solution in Ω.*

(3) ***(Invariance by homotopy)*** *For all:$h : [0, 1]\times\Omega \to \mathbb{R}^n$ and all $y : [0, 1] \to \mathbb{R}^n$ continuous such that $y(t) \notin h(t, \partial\Omega)$ for all $t \in [0, 1]$, $deg(h(t, .), \Omega, y(t))$ is independent oft.*

*(4) **(Additivity)** Suppose that Ω_1 and Ω_2 are two disjoint and open subset of Ω and $y \notin f(\overline{\Omega}(\Omega_1 \cup \Omega_2))$. then*

$$deg(f, \Omega, y) = deg(f, \Omega_1, y) + deg(f, \Omega_2, y).$$

(5) $deg(f, \Omega, y)$ is constant on any related components of $\mathbb{R}^n f(\partial\Omega)$.

(6) $deg(f, \Omega, y) = deg(f - y, \Omega, 0)$.

*(7) Let $g : \overline{\Omega} \to F_m$be a continuous map where F_m is a subspace of $\mathbb{R}^n, dimF_m = m, 1 \leq m \leq n$. Suppose that y is such that $y \notin (I - g)\partial\Omega$.
Then*

$$deg(f, \Omega, y) = deg(I - g)_{\overline{\Omega} \cap F_m}, \Omega \cap F_m, y$$

Remark 1.2.2

In order to demonstrate the existence of solutions of nonlinear equations in $\mathbb{R}_n$, property (2) of the theorem above is often supplemented by the property of invariance by homotopy of the degree. The main interest of this notion lies in the fact that if two maps are homotopic, they have the same degree.

Example 1.2.1

Let $\Omega(-1; 1)$ and consider

$$h : (t; x) \in [0, 1] \times \Omega \to h(t, x) = (1 - t)x + txe^x$$

It is clear that this map satisfies:

1) h is continuous on $[0; 1] \times \overline{\Omega}$

2) $h(0; x) = x$ and $h(1; x) = xe^x$

*3) For all $T \in [0; 1]$, the function $h(t; x)$ does not cancel out at $\{-1, 1\}$, So if $f(x) = xe^x$.
Then*

$$deg\big(f, (-1, 1), 0\big) = deg\big(I, (-1, 1), 0\big) = 1$$

1.2.2 Leray-Schauder topological degree

Let X be a standardized vector space of infinite dimension, $\Omega \subset X$ an open and bounded set, $f : \overline{\Omega} \to \mathbb{X}$ an contunuous function and $y \in X$ such that $y \notin f(\partial\Omega)$. In the previous section, we saw that in finite dimension, $C(\overline{\Omega}, X)$ is a suitable class of function for which there exists a unique degree function,the degree of Brouwer, satisfying the properties $1, 2and3$ of the theorem. Unfortunately, in infinite dimension, $C(\overline{\Omega}, X)$ is not. Indeed, an example from leray shows that it is necessary to restrict the class of functions for which there is existence and uniqueness of a Leray-Schauder degree function, to a set strictly contained in $C(\overline{\Omega}, X)$.

Definition 1.2.6

Let x be a Banach space and Ω closed part of X. If $T : \Omega \to X$ is a continuous operator, we say that T is compact if for any bounded part B of $\Omega, T(B)$ is relatively compact in X.

Remark 1.2.3

Note in particular that if T is compact, then T is bounded on the bounded parts of X.

Definition 1.2.7

Let X be a Banach space and Ω be a part of X. We say that the map $T : \Omega \to X$ is of finite rank if $dim(Im(T)) < \infty$, in other words if $Im(T)$ is a subspace of finite dimension of X.

Lemma 1.2.2

Let X be a Banach space, $\Omega \subset X$. a bounded open and $T : \overline{\Omega} \to X$ be a compact map, then for all $\epsilon > 0$, there exists a space of finite dimension denoted F and a continuous map $T_\epsilon : \overline{\Omega} \to F$ Such that:

$$||T_\epsilon x - Tx|| < \epsilon \forall x \in \overline{\Omega}.$$

Definition 1.2.8

*Let X be a Banach space, $\Omega \subset X$. a bounded open and $T : \overline{\Omega} \to X$ a compact map. Now suppose $0 \notin (I - T)(\partial\Omega)$. There exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$, the Brouwer degree $deg(I - T_\epsilon, \Omega \cap F_\epsilon, 0)$ is well defined as in **lemma** (1.2.1). Therefore, we define the Leray-Schauder degree as:*

$$deg(I - T\Omega, 0) = deg(I - T_\epsilon, \Omega \cap F_\epsilon, 0)$$

Remark 1.2.4

This definition depends only on T and on Ω. If $Y \in X$ is such that $y \notin (I - T)(\partial\Omega)$, the degree of $I - T$ in Ω with respect to y is defined as

$$deg(I - T\Omega, y) = deg(I - T - y, \Omega, 0).$$

Theorem 1.2.2 ***(Some important properties of the Leray-Schauder topological degree)***

*Let X be a Banach space and $A = \{(I - T, \Omega, 0), \Omega$ **A bounded open of** X, $T : \overline{\Omega} \to X$ compact , $0 \notin (I - T)(\partial\Omega)\}$, then, there exists a single map $deg(f, \Omega, y) : A \to \mathbb{Z}$ called the Leray-Schauder topological degree such that:*

(1) ***(Normality)*** *If $0 \in \Omega$, then $deg(I, \Omega, 0) = 1$.*

(2) ***(Solvency)*** *If $deg(I - T, \Omega, 0) \neq 0$, then exists $x \in \Omega$ such that $(I - T)x = 0$.*

(3) ***(Invariance by homotopy)*** *Let $H : [0, 1] \times \overline{\Omega}$ a compact homotopy, such that $0 \notin (I - H(t, .))(\partial\Omega)$.*
Then

$$deg\big(I - H(t, .), \Omega, 0\big) \quad \text{does not depend on} \quad t \in [0, 1].$$

(4) ***(Additivity)*** *Let Ω_1 and Ω_2 be two open disjoint subsets of Ω and*

$$0 \notin (I-T)(\bar{\Omega})\backslash(\Omega_1 \cup \Omega_2).$$

Then

$$deg(I-T,\Omega,0) = deg(I-T,\Omega_1,0) + deg(I-T,\Omega_2,0).$$

The Leray-Schuader degree retains all the basic properties of the Brouwer degree.
Finally and as a cnsequence of this notion of degree we are going to prove some topological fixed point theorems in particular the nonlinear alternative of Leray-Schuader

1.3 Topological fixed point theorem

Theorem 1.3.1 ***(Brouwer)***
Let B be the closed unit ball of $\mathbb{R}^n$ and $f : \bar{B} \to \bar{B}$ continues. Then f be a fixed point: there exists $x \in \bar{B}$ such that $f(x) = x$.

Proof.
We will show the existence of the solution of $f(x) = x$ on $\bar{B}$:

(i) If there is $x \in \partial\Omega$, then there is noting to prove.

(ii) Otherwise consider the continuous map $h(t,x) = x - tf(x)$.
Then

$$h(0,x) = x - 0 * f(x) = x,$$

and

$$h(1,x) = x - 1 * f(x) = x - f(x).$$

If we assume that $h(t,x_0) = 0$ as $x_0 \in \partial B$, then we obtain $x_0 = tf(x_0)$ which implies as $0 \geq t \geq 1$, that $f(x_0) \in \partial B$, contradiction, As is an admissible homotopy between $I - f$ and I.
Therefore

$$deg(I-f,\Omega,0) = deg(I,\Omega,0) = 1.$$

In conclusion $\exists x \in B$ such that $x - tf(x) = 0$ i.e$f(x) = x$.

■

Theorem 1.3.2 ***(Schauder)***
Let B be the closed unit ball of a Banach E and $f = \bar{B} \to \bar{B}$ compact. Then f an fixed point $\exists x \in \bar{B}$ such that $f(x) = x$.

Proof.
Let $h(t,x) = tf(x)$ be a compact function on $[0,1] \times \bar{B}$.
If for $t \in [0,1]$ and a $x \in \partial B$, we have $x - h(t,x) = 0$, then $tf(x) = x$; as $|x| = 1$ and $|f(x)| \leq 1$, this imposes $t = 1$ and $x = f(x)$ therefore a fixed point on ∂B situation which we

have excluded.
We can therefore apply the properites of normalization and invariance by homotopy of the given degree

$$deg(I-f,B,0) = deg(I,B,0) = 1$$

Since $h(0,.) = 0$ and $h(1,0) = f$ therefore the existence of a fixed point. ■

Theorem 1.3.3 ***(Non-linear alternative of Leray-Schauder)[4]***
Let $\Omega \subset X$ be a bounded open subset of a Banach space X such that $0 \in \Omega$, and let $T : \bar{\Omega} \to X$ be a compact operator. Then one of the following two statements is verified:

(1) T a fixed point in Ω.

(2) there exists $\lambda > 1$ and $x \in \partial\Omega$ such that $Tx = \lambda x$.

Proof.
If (2) is true then we have nothing to prove. Otherwise we define the homotopy

$$H(t,x) = tTx \forall t \in [0,1].$$

Thus defined $H(t,x)$ is compact $H(0,x) = 0$ and $H(1,x) = Tx$.
Suppose that $H(t,x_0) = x_0$ for a certain $t \in [0,1]$ and $x_0 \in \partial\Omega$. Then we have $tTx_0 = x_{0.}$
If $t = 0$ or $t = 1$ we have (1); Otherwise $Tx_0 = \dfrac{1}{t}x_0$ for a certain $t \in (0,1)$, And then we have (2). Otherwise, we have

$$deg(I-T,\Omega,0) = deg(I,\Omega,0) = 1.$$

And then T has a fixed point in Ω. ■

Theorem 1.3.4 ***(Brouwer)***
Let M be a convex , compact and non-empty part of a finite dimension standardized space $(X, ||.||)$ and let $A : M \to M$ be a continuous map, then A admits a fixed point.

Theorem 1.3.5 ***(Schauder)***
Let M be a bounded , closed,convex and non-empty part of a Banach space X and let $A : M \to M$ be a compact map, then A admits a fixed point.

Theorem 1.3.6 ***(Ascoli-Arzela)***
Consider $X = C([a,b])$ equipped with the standard $||u|| = \max_{a\leq t\leq b}|u(t)|$, with $-\infty < a < b < +\infty$.
If M is a subset of X such that:

(i) M is bounded, there exists a constant $r > 0$ such that

$$||u|| \leq r \quad \forall u \in M.$$

(ii) M is equicontinuous,then

$$\forall \epsilon > 0, \exists \delta > 0 \ st \ , |t_1 - t_2| < \delta \ and \ \forall u \in M \Rightarrow |u(t_1 - ut_2)| \leq \epsilon.$$

Then, M ***is relatively compact****.*

Chapter 2

Fractional

In this chapter,
we first present their important functions in the theory of fractional calculation, the Gamma function, the Beta function.Next,we will present the definition of fractional integral and study the fractional derivatives of Riemann Liouville and Caputo as well as their properties. See ([4]-[22])

2.1 Useful specific functions

In this section, we present the definition and some properties of the Gamma Euler function and the Beta function which is linked to this function.

2.1.1 Euler Gamma Functions[3]

One of the basic tools of fractional calculus is the Gamma function which extends the factorial function to positive real numbres (and even to complex numbers with positive real parts).

Definition 2.1.1

Let $x \in \mathbb{R}_*^+$, the function Gamma is defined by:

$$\Gamma(x) = \int_0^{+\infty} e^{-t} t^{x-1} dt. \tag{2.1}$$

This integral converges for all positive real numbres.

Example 2.1.1
Let's calculate $\Gamma(1)$ *and* $\Gamma(\frac{1}{2})$

$$\Gamma(1) = \int_0^{+\infty} t^{1-1}e^{-t}dt = 1$$

$$\Gamma(\frac{1}{2}) = \int_0^{+\infty} t^{-(\frac{1}{2})}e^{-t}dt$$

Posonst $t = x^2; dt = 2xdx$. *So*

$$\Gamma(\frac{1}{2}) = 2\int_0^{+\infty} e^{-x^2}dx$$

To calculate this integral let us put

$$A = \int_0^{+\infty} e^{-x^2}dx$$

Let's take

$$\begin{aligned} A^2 &= \int_0^{+\infty} e^{-y^2}dy \int_0^{+\infty} e^{-x^2}dx, \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-(x^2+y^2)}dxdy. \end{aligned}$$

The calculation is simple to carry out if we carry out the polar coordinates

$$\begin{aligned} A^2 &= \int_0^{\pi/2}\int_0^{+\infty} re^{-r^2}drd0 \\ &= \pi/4 \\ A &= \frac{\sqrt{\pi}}{2}. \end{aligned}$$

Then

$$\Gamma(\frac{1}{2}) = \sqrt{\pi}$$

Proposition 2.1.1 *For all* $x \in \mathbb{R}_*^+, t > 0, n \in \mathbb{N}$, *we have*

(1) $\Gamma(x+1) = x\Gamma(x)$.

(2) $\Gamma(n+1) = (n)!$.

(3) $\Gamma(0) = \infty$.

(4) $\Gamma(n+\frac{1}{2}) = \dfrac{(2n)!\sqrt{(\pi)}}{4^n n!}$.

Proof.

(1) Let us represent $\Gamma(x+1)$ by the Euler integral and integrate by parts:

$$\begin{aligned}\Gamma(x+1) &= \int_0^{+\infty} t^{(x+1)-1}e^{-t}dt \\ &= \int_0^{+\infty} t^{x}e^{-t}dt \\ &= \left[-t^{x}e^{-t}\right]_0^{\infty} + x\int_0^{+\infty} t(x-1)e^{-t}dt \\ \Gamma(x+1) &= x\Gamma(x)\end{aligned}$$

Hence the so-called recurrence relation.

(2) Euler's Gamma function generalizes the factorial becuase

$$\Gamma(x+1) = (n)!.$$

Indeed, $\Gamma(1) = 1$; and using the property (1) we obtain:

$$\begin{aligned}\Gamma(2) &= 1.\Gamma(1) = 1! \\ \Gamma(3) &= 2.\Gamma(2) = 2.1 = 2! \\ \Gamma(4) &= 3.\Gamma(3) = 3.2 = 3! \\ \Gamma(n+1) &= n.\Gamma(n) = n.\Gamma(n-1)! = n!\end{aligned}$$

(3) OF (1) we have

$$\begin{aligned}\Gamma(x) &= \frac{\Gamma(x+1)}{x} \\ \lim_{x\to 0}\Gamma(x) &= \lim_{x\to 0}\frac{\Gamma(x+1)}{x}, \\ \lim_{x\to 0}\Gamma(x) &= \infty.\end{aligned}$$

(4) We will prove the formula Gamma$(n+\frac{1}{2}) = \frac{(2n)!\sqrt{(\pi)}}{4^n n!}$, by recurrence for $n \in \mathbb{N}$. for $n = 0$; we have $\Gamma(n+\frac{1}{2}) = \sqrt{\pi}$

Suppose that the formula is verified for $(n-1)$ and consider n. That is to say that

$$\begin{aligned}\Gamma(n+\frac{1}{2}) &= \big(n-\frac{1}{2}\big)\Gamma(n-\frac{1}{2}), \\ &= (n+\frac{1}{2})\frac{2(n-1)!\sqrt{\pi}}{4^{n-1}(n-1)!}, \\ &= (\frac{2n-1}{2})\frac{(2n-2)!\sqrt{\pi}}{4^{n-1}(n-1)!}, \\ &= \frac{2n}{2n}\frac{(2n-1)}{2}\frac{(2n-2)!\sqrt{\pi}}{4^{n-1}(n-1)!}, \\ \Gamma(n+\frac{1}{2}) &= \frac{(2n)!\sqrt{\pi}}{4n}.\end{aligned}$$

So the formula is verified for n.

■

2.1.2 Beta Function [4](or the Bessel function of the second kind)

It is not the basic functions of fractional calculation. This function plays an important role when combined with the Gamma function.

Definition 2.1.2

The Beta function id defined by:

$$\beta(p,q) = \int_0^1 t^{p-1}(1-t)^{y-1}dt \quad \forall x,y>0 \tag{2.2}$$

Theorem 2.1.1

The Beta function is connected with the Gamma function by the following relationship:

$$\beta(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}, \quad \forall x,y>0$$

Proof.

Let $D = [0,+1[\times[0,+1[$, we have

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^{+\infty} e^{-x}x^{p-1}dx \int_0^{+\infty} e^{-y}y^{q-1}dy \\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-(x+y)}x^{p-1}y^{q-1}dxdy\end{aligned}$$

We pose $y = u - x; dy = du$

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^{+\infty}\int_0^{u} e^{-u}x^{p-1}(u-x)^{q-1}dxdy \\ &= \int_0^{+\infty} e^{-u}\int_0^{u} x^{p-1}(u-x)^{q-1}dxdy\end{aligned}$$

We pose $x = tu; dx = udt$

$$\begin{aligned}\Gamma(p)\Gamma(q) &= \int_0^{+\infty} e^{-u}\int_0^{1} t^{p-1}(u)^{p-1}(1-t)u^q dxdy \\ \Gamma(p)\Gamma(q) &= \Gamma(x+y)\beta(x,y)\end{aligned}$$

Therefore

$$\beta(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

■

Example 2.1.2

Let's calculate $\beta(\frac{1}{2},\frac{1}{2})$.
According to the example of $\Gamma(\frac{1}{2}) = \sqrt{\pi}$,

$$\beta(\frac{1}{2},\frac{1}{2}) = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)} = \pi.$$

Proposition 2.1.2

(1) $B(p,q) = B(q,p)$.

(2) $B(p,q) = B(p+1,q) + B(p,q+1)$.

(3) $B(p,q+1) = \frac{p}{q}B(p+1,q) = \frac{p}{p+q}B(p,q)$.

(4) $B(p,q) = \int_0^{+\infty} \frac{t^{p-1}}{(1+t)^{p+q}}dt = 2\int_0^{\pi/2}(\sin t)^{2p-1}(\cos t)^{2q-t}dt$.

2.2 The integral and the Fractional derivative

This section contains definitions and some properties of integrals and fractional derivatives of the Riemann-Liouville and Caputo type.

2.2.1 The Fractional integral in the sense of Riemann-Liouville[16][17]

Let $f : [a,b[\to \mathbb{R}$ be a continuous or integral function and: A primitive of f is given by the expression

$$I^{(1)}f(x) = \int_a^x f(t)dt.$$

For a second antideriative and according to Fubini's theorem we will have:

$$\begin{aligned}
I^{(2)}f(x) &= \int_a^x I^{(1)}f(u)du \\
&= \int_a^x \int_a^u \big(f(y)dy\big)du \\
&= \int_a^t f(u)du \int_a^x dt \\
&= \int_a^x (x-t)f(t)dt.
\end{aligned}$$

And for a third primitive we will have:

$$\begin{aligned}
I^3 f(x) &= \int_a^x dx_1 \int_a^{x1} dx_2 \int_a^{x2} f(x_3)dx_3 \\
&= \int_a^x dx_1 \int_a^{x1} (x_1 - x_2)f(x_2)dx_2 \\
&= \frac{1}{(3-1)!}\int_a^x (x-t)^{(3-1)}f(t)dt \\
&= \frac{1}{(2)!}\int_a^x (x-T)^{(2)}f(t)dt \\
&= \frac{1}{2}\int_a^x (x-T)^{(2)}f(t)dt.
\end{aligned}$$

In the general case for any integral n and by induction we have the Cauchy formula:

$$I^n f(x) = \int_a^x dx_1 \int_a^{x_1} dx_2 ... \int_a^{x_{(n-1)}} f(x_n)dx_n, \tag{2.3}$$

$$= \frac{1}{(n-1)!} \int_a^x (x-t)^{n-1} f(t)dt. \tag{2.4}$$

For any integer n.
Since the generalization of the factorial by the function Gamma: $(n-1)! = \Gamma(n)$, Riemann realized that the second member of (2.4) could have a meaning even when $n$taking a non-integer value,it was natural to define fractional integration as follows:

Definition 2.2.1

Let$f(x) \in C[a,b]$, the Riemann-Liouville fractional integral $I^{\alpha}_{a^+} f(x)$ (left) and $I^{\beta}_{a^+} f(x)$ (right) of order $\alpha > 0$ are defined by:

$$I^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt.$$
$$I^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_b^x (x-t)^{\alpha-1} f(t)dt.$$

Proposition 2.2.1

(i) $I^0_{a^+} f(x) = f(x)$.

(ii) the integral operator I^{α}_0 *is linear.*

Theorem 2.2.1

For $f \in C[a,b]$ Riemann-Liouville fractional integral owns the property of semigroup:

$$I^{\alpha}_{a^+}[I^{\beta}_{a^+} f(x)] = I^{(\alpha+\beta)}_{a^+} f(x), \quad \alpha, \beta > 0.$$

Lemma 2.2.1

The relation

$$I^{\alpha}_{a+} I^{\beta}_{a+} f(x) = I^{\alpha+\beta}_{a+} f(x) \tag{2.5}$$

is valid in following cases $\beta > 0, \alpha + \beta > 0$ and $f(x) \in L_1(a,b)$.

Now let us recall some notations about the coincidence degree continuation theorem.

1- Let Y and Z be real Banach spaces.

2- let $L : dom(L) \subset Y \to Z$ be a Fredholm map of index zero.

3- and let $P : Y \to Y, Q : Z \to Z$ be continuous projectors such that

$$\{Ker(L) = Im(P)Im(L) = Ker(Q)\}and\{Y = Ker(L) \oplus Ker(P)Z = Im(L) \oplus Im(Q)\}$$

. It follows that $L|_{dom(L)} \cap Ker(P) : dom(L) \cap Ker(P) \to Im(L)$ is invertible.
We denote the inverse of this map by K_P.

4- If Ω is an open bounded subset of Y, the map N will be called L-compact on $\overline{\Omega}$ if:
(i)$QN(\overline{\Omega})$ is bounded.
(ii) $K_{P,Q}N = K_P(I - Q)N : \overline{\Omega} \to Y$ is compact.

Proof.

From the definition we find:

$$\begin{aligned}
I^{\alpha}_{a^+}[I^{\beta}_{a^+}f(x)] &= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{1}{(x-t)^{1-\alpha}} I^{\beta}_{a^+}f(y)dy \\
&= \frac{1}{\Gamma(\alpha)}\int_a^x \frac{1}{(x-y)^{1-\alpha}} \frac{1}{\Gamma(\beta)}\int_a^y \frac{f(t)}{(y-t)^{1-\beta}}dtdy \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x \frac{1}{(x-y)^{1-\alpha}}\int_a^y \frac{f(t)}{(y-t)^{1-\beta}}dtdy.
\end{aligned}$$

According to Fubini's theorem we have:

$$I^{\alpha}_{a^+}[I^{\beta}_{a^+}f(x)] = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(t)dt\int_t^x (x-y)^{\alpha-1}(y-t)^{\beta-1}dy.$$

By changing the variable $y = t + (x-t)s$,then $dy = (x-t)ds$. We obtain

$$\begin{aligned}
\int_t^x (x-y)^{\alpha-1}(y-t)^{\beta-1}dy &= \int_0^1 \Big[[x-(t+(x-t)s)]^{\alpha-1}\,[(t+(x-t)s)-t]^{\beta-1} \Big](x-t)ds \\
&= \int_0^1 \Big[\big((x-t)-(x-t)s\big)^{\alpha-1}\big((x-t)s\big)^{\beta-1} \Big](x-t)ds \\
&= \int_0^1 (1-s)^{\alpha-1}(x-t)^{\alpha-1}(x-t)^{\beta-1}s^{\beta-1}(x-t)ds \\
&= (x-t)^{\alpha+\beta-1}\int_0^1 (1-s)^{\alpha-1}s^{\beta-1}ds \\
&= (x-t)^{\alpha+\beta-1}\beta(\alpha,\beta) \\
&= (x-t)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.
\end{aligned}$$

From where

$$\begin{aligned}
I^{\alpha}_{a^+}[I^{\beta}_{a^+}f(x)] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x f(t)dt(x-t)^{\alpha+\beta-1}\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \\
I^{\alpha}_{a^+}[I^{\beta}_{a^+}f(x)] &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^x (x-t)^{\alpha+\beta-1}f(t)dt \\
&= I^{\alpha+\beta}_{a^+}f(x).
\end{aligned}$$

Such that $\beta(\alpha,\beta)$ is the Beta function. ∎

Lemma 2.2.2

The fractional integral operator I_a^α with $\alpha > 0$ is bounded in $L^p([a,b]), 0 \leq p \leq +\infty$

$$||I_a^\alpha||_p \leq \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}||f||_p.$$

2.2.2 Fractional derivatives in the Riemann-Liouville

There are several definitions of fractional derivatives. In this part we will present the Riemann-Liouville derivative, which is the most used.

Definition 2.2.2 *Let f be an integrable function on $[a;b[$ then fractional derivative of order $\alpha($ with $n-1 \leq \alpha < n; n \neq 0)$ in the sense of Riemann-Liouville defined by:*

$$\begin{aligned} {}^R D_a^\alpha &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (x-t)^{n-\alpha-1} f(s) ds \\ &= \frac{d^n}{dt^n}(I^{n-\alpha} f(t)) \end{aligned}$$

Or $n = [\alpha] + 1$.

In particular *for $\alpha = 0$ and for $\alpha = n$ we have:*

$$\begin{aligned} {}_{a^+}^{R} D^0 f(t) &= f(t) \\ {}_{a^+}^{R} D^n f(t) &= f^n(t) \end{aligned}$$

Example 2.2.1

1. ***The non-integer derivative of a constant function in the sense of Riemann-Liouville***

   IN general the non-integer derivative of a constant function in the sense of Riemann-Liouville is neither zero nor constant but we have

$$\begin{aligned} {}^R D_{a^+}^\alpha C &= \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t C(t-\tau)^{n-\alpha-1} \\ &= \frac{C}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t (x-t)^{n-\alpha-1} \\ &= \frac{C}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} [\frac{(t-\tau)^{n-\alpha}}{n-\alpha}]_a^t \\ &= \frac{C}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} [\frac{(t-\alpha)^{n-\alpha}}{n-\alpha}] \\ &= \frac{C}{\Gamma(n-\alpha)} \frac{\Gamma(n-\alpha)}{1-\alpha}(t-a)^{-\alpha} \\ {}^R D_{a^+}^\alpha C &= \frac{C}{\Gamma(1-\alpha)}(t-a)^{-\alpha} \end{aligned}$$

2. ***The derivative of $f(t) = (t-a)^\alpha$ in the Riemann-Liouville sense.***
Let α non-integre and $0 \leq n-1 < \alpha < n$ and $\alpha > -1$ then we have:

$$^{R}D^{\alpha}_{a^+}(t-a)^\alpha = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_a^t C(t-\tau)^{n-\alpha-1}(\tau-a)^\beta d\tau.$$

By changing variable $\tau = a + s(t-a)$ we have $d\tau = (t-a)ds$ and then:

$$\begin{aligned}
^{R}D^{\alpha}_{a^+}(t-a)^\alpha &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}(t-a)^{n-\alpha+\beta}\int_a^t (1-s)^{n-\alpha-1}s^\beta ds \\
&= \frac{\Gamma(n-\alpha+\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}\beta(n-\alpha,\beta+1) \\
&= \frac{\Gamma(n-\alpha+\beta+1)\Gamma(n-\alpha)\Gamma(\beta+1)}{\Gamma(n-\alpha)\Gamma(\beta-\alpha+1)\Gamma(n-\alpha+\beta+1)}(t-a)^{\beta-\alpha} \\
^{R}D^{\alpha}_{a^+}(t-a)^\alpha &= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(t-a)^{\beta-\alpha}.
\end{aligned}$$

For example $^{R}D^{0.5}_{0}t^{0.5} = \frac{\Gamma(1.5)}{\Gamma(1)} = \Gamma(1.5)$.

Proposition 2.2.2 (20)

1) ***Composition with the fractional integral***
-The fractional derivation operator in the sense of Riemann-Liouville is a left inverse of the fractional integration operator,

$$^{R}D^{\alpha}(I^\alpha f(t)) = f(t),$$

in general we have

$$^{R}D^{\alpha}(I^\beta f(t)) =^{R} D^{\alpha-\beta}f(t)$$

and if $\alpha - \beta < 0,^{R} D^{\alpha-\beta}f(t) = I^{\beta-\alpha}f(t)$
-In general, fractional derivation and integration do not commute.

$$^{R}D^{-\alpha}(^{R}_{a}D^{\beta}_{t}f(t)) =^{R} D^{\beta-\alpha}f(t) - \sum_{k=1}^{m}[^{R}D^{\beta-k}_{t}f(t)]_{t=a}\frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}$$

with $m-1 \leq \beta < m$

2) ***Composition with integre derivatives***
The fractional derivation and the conventional derivation (of integer order) only commute if: $f^k(a) = 0$ for all $k = 0, 1, 2..., n-1$.

$$\frac{d^n}{dt^n}(^{R}D^{\alpha}f(t)) =^{R} D^{n+\alpha}f(t)$$

but

$$^{R}D^{p}(\frac{d^n}{dt^n}f(t)) =^{R} D^{n+p}f(t) - \sum_{k=1}^{m}\frac{f^{(k)}(a)(t-a)^{k-p-n}}{\Gamma(k-p-n+1)}$$

3) ***Composition with fractional derivatives***
Let $n-1 \leq \alpha < n$ and $m-1 \leq \beta < m$, then

$$
{}^{R}D^{\alpha}({}^{R}D_{t}^{\alpha} f(t)) = {}^{R} D^{\alpha+\beta} f(t) - \sum_{k=1}^{m} \left[{}^{R}D^{\beta-k} f(t)\right]_{t=a} \frac{(t-a)^{-\alpha-k}}{\Gamma(q-k+1)}
$$

$$
{}^{R}D^{\alpha}({}^{R}D_{t}^{\beta} f(t)) = {}^{R} D^{\alpha+\beta} f(t) - \sum_{k=1}^{m} \left[{}^{R}D^{\beta-k} f(t)\right]_{t=a} \frac{(t-a)^{-\alpha-k}}{\Gamma(-\beta-k+1)}
$$

subsequently two fractional differentiation operators ${}^{R}D^{\alpha}$ and ${}^{R}D^{\beta}(\alpha=\beta)$, only commute if and

$$
\left[{}^{R}D^{\beta-k} f(t)\right]_{t=a} = 0
$$

for all $k = 1, 2, ..., n$, and $\left[{}^{R}D^{\beta-k} f(t)\right]_{t=a}$ for all $k = 1, 2, ..., m$.

2.2.3 Fractional derivatives in the sense of Caputo

The partial derivation in the sense of Riemman-Liouville played an active role in the development of microcomputing in pure and applied mathematics in late 1960 ans and required a revision which led many authors, includig Caputo,to find a new definition of fractional derivation due to problems applied to optical flexibility and mechanics.

Definition 2.2.3 (19)
Let f be an integrabel function on $[a; b[$ then fractional derivative of order $\alpha > 0($ with $n-1 < \alpha < n, n \in \mathbb{N}^{*}$ in the sense of Caputo defined by:

$$
\begin{aligned}
{}^{c}D_{a^{+}}^{\alpha} f(t) &= \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} (x-t)^{n-\alpha-1} f^{n}(t) dt \\
&= I^{n-\alpha} \frac{d^{n}}{dt^{n}} (f(t))
\end{aligned}
$$

with $n = [\alpha+1], [\alpha]$ denotes the integer part.

Remark 2.2.1
A fractional derivative in the sense of Riemann-Liouville of order $]n-1, n[$ is obtained by an map of the fractional integration operator of order $n-\alpha$ followed by a classical derivation of order n, while the derivative fractional in the sense of Caputo is the result of the permutation of these two operations.

Example 2.2.2

1) ***The derivative of a constant function in the Caputo sense***
The derivative of a constant function in the Caputo sense is zero

$$
{}^{c}D^{\alpha} C = 0
$$

*2) **The derivative of** $f(t) = (t-a)^\alpha$ **in the sense of Caputo***
Let P *be an integer and* $0 \leq n-1 < p < n$ *with* $\alpha > n-1$, *then we have*

$$f^n(\tau) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-n+1)}(\tau-a)^{\alpha-n}.$$

From where

$$^cD^p(t-a)^\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)}\int_a^t (t-\tau)^{n-p-1}(\tau-a)^{\alpha-n}d\tau.$$

Carrying out the change of variable $\tau = a + s(t-a)$ *we obtain*

$$\begin{aligned}
^cD^p(t-a)^\alpha &= \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)}\int_a^t (t-\tau)^{n-p-1}(\tau-a)^{\alpha-n}d\tau \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)}(t-a)^{\alpha-p}\int_0^1 (1-s)^{n-p-1}s^{\alpha-n}ds \\
&= \frac{\Gamma(\alpha+1)\beta(n-p,\alpha-n+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)}(t-a)^{\alpha-p} \\
&= \frac{\Gamma(\alpha+1)\Gamma(n-p)\Gamma(\alpha-n+1)}{\Gamma(n-p)\Gamma(\alpha-n+1)\Gamma(\alpha-p+1)}(t-a)^{\alpha-p} \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-p+1)}(t-a)^{\alpha-p},
\end{aligned}$$

2.3 Relationship between the Riemann-Liouville derivative and that of Caputo:[15]

Theorem 2.3.1
Let $\alpha \geq 0, n = [\alpha]+1$, *if* f *has* $n-1$ *derivative in* a *and si* $^RD_a^\alpha$ *exsits then:*

$$^cD_a^\alpha f(x) =^R D_a^\alpha[f(x) - \sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k].$$

For almost all $x \in [a,b]$.

Proof.

According to the definition we have:

$$
\begin{aligned}
{}^{R}D_a^{\alpha}[f(x)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k] &={}^{R} D^n I_a^{n-\alpha}[f(x)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k]\\
&=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x (x-t)^{n-\alpha-1}[f(t)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k]dt\\
&=\frac{-1}{\Gamma(n-\alpha+1)}[(x-t)^{n-\alpha}(f(t)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k)]_a^x\\
&+\frac{1}{\Gamma(n-\alpha+1)}\int_a^x (x-t)^{n-\alpha}[D(f(t)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k)]dt\\
&=I_a^{n-\alpha+1}D[(f(x)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k)]
\end{aligned}
$$

Same way for n time then:

$$
I_a^{n-\alpha}D[(f(x)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k)]=I_a^{n-\alpha}I_a^n D^n[(f(x)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k)].
$$

Or $\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k$ is a polynomial of degree $n-1$ then:

$$
\begin{aligned}
I_a^{n-p}D[(f(x)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k)] &= I_a^{n-p}I_a^n D^n f(x)\\
&= D^n I_a^{n-p}[(f(x)-\sum_{k=0}^{n-1}\frac{f^k(a)}{k!}(x-a)^k)]\\
&= D^n I_a^n I_a^{n-p}D^n f(x)\\
&= I_a^{n-p}D^n f(x)\\
&={}^{c} D_a^p f(x)
\end{aligned}
$$

■

Remark 2.3.1

From the relation we notice that the derivation in the sense of Caputo of a function f is a fractional derivation of remainder in the Taylor expansion of f.

2.4 General properties of fractional derivatives

2.4.1 Linearity

Fractional derivation is a linear operator

$$D^\alpha(\lambda f(t) + \mu g(t)) = \lambda D^\alpha f(t) + \mu D^\alpha g(t).$$

2.4.2 Leibniz's rule

For n integer we have

$$\frac{d^n}{dt^n}(f(t)g(t)) = f^{(k)}(t)g^{n-k}(t).$$

Generalizing this formula gives us $D^\alpha(f(t)g(t)) = f^{(k)}(t)D^{\alpha-k}g(t) + R_n^\alpha(t)$. When $n > \alpha + 1$ and

$$R_n^\alpha(t) = \frac{1}{\Gamma(-\alpha)}\int_a^t (t-\tau)^{-\alpha-1}g(\tau)d(\tau)\int_\tau^t f^{(n+1)}(\xi)(\tau-\xi)^n d\xi.$$

With

$$\lim R_n^\alpha(t) = 0.$$

If f and g are continuous in $[a,t]$ and all their derivatives the formula becomes:

$$D^p(f(t)g(t)) = f^{(k)}(t)D^{\alpha-k}g(t).$$

2.5 Fundamental lemmas

Lemma 2.5.1 (16) *[21]*
Let $\alpha > 0$ then the differential equation

$$^cD_{0^+}^\alpha f(t) = 0.$$

Admits solutions

$$f(t) = c_0 + c_1 t + c_2 t^2 ... + c_{n-1}t^{n-1}, c_i \in \mathbb{R}, i = 0, 1, 2, ..., n-1, n = [\alpha] + 1.$$

Proof.
Soppose that

$$^cD_{0^+}^\alpha f(t) = 0.$$

According to the definition of the fractional derivative in the sense of Caputo we obtain

$$I^{n-\alpha}(\frac{d}{dt})^n f(t) = 0$$

That's to say
$$\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)n-\alpha-1(\frac{d}{dt})^n f(s)ds = 0.$$
Since $\frac{1}{\Gamma(n-\alpha)} \neq 0$, we have
$$\int_0^t (t-s)n-\alpha-1(\frac{d}{dt})^n f(s)ds = 0,$$
and consequently
$$I^{n-\alpha-1} * f^n(s) = 0.$$

■

Lemma 2.5.2

Let $\alpha > 0$, *then*
$$I_{0+}^{\alpha}\ {}^{c}D_{0+}^{\alpha} f(t) = f(t) + c_0 + c_1 t + c_2 t^2 + ... + c_{n-1}t^{n-1}.$$
for $c_i \in \mathbb{R}, i = 0, 1, ..., n-1, n = [\alpha] + 1$.

Proof.

We have by the definition of the Caputo fractional derivative
$${}^{c}D_{0+}^{\alpha} f(t) = I_{0+}^{n-\alpha} f^n(t).$$
We apply the fractional integral operator to both sides of the equality
$$\begin{aligned}
I_{0+}^{\alpha}\ {}^{c}D_{0+}^{\alpha} f(t) &= I_{0+}^{\alpha} I_{0+}^{n-\alpha} f^n(t)\\
&= I_{0+}^{n}\ {}^{R}D_{0+}^{n} f(t)\\
&= f(t) - \sum_{j=1}^{n} \frac{t^{n-j}}{\Gamma(n-j+1)} \lim_{t\to 0}\left(\frac{d}{dt}\right)^{n-j} I^{n-n} f(t)\\
&= f(t) - \sum_{j=1}^{n} \frac{t^{n-j}}{\Gamma(n-j+1)} \left(\left(\frac{d}{dt}\right)^{n-j} f(t)\right)(0)\\
&= f(t) - \sum_{j=1}^{n} \frac{t^{n-j}}{\Gamma(n-j+1)} f^{(n-j)}(0).
\end{aligned}$$
by changing the variable $k = n - j$ we obtain:
$$\begin{aligned}
I_{0+}^{\alpha}\ {}^{c}D^{\alpha} f(t) &= f(t) - \sum_{k=0}^{n-1} \frac{f^k(0)t^k}{k!}\\
&= f(t) - \sum_{k=0}^{n-1} \frac{C_k^1 t^k}{k!}\\
&= f(t) + \sum_{k=0}^{n-1} \frac{C_k t^k}{k!}
\end{aligned}$$

■

Chapter 3

Mawhin's continuation theorem

In this chapter,
we present Mawhin's continuation theorem. But before giving a more detailed presentation of the theorem, let us briefly review the state of the art on Fredholm operators as well as the main notions relating to them. On the other hand, as well as see later, these operators can be obtained from the projections, so we devote another section to it.See ([23]-[32])

3.1 Topological supplement

Let E and F be two closed subspaces of a $\mathbb{R}$ normalized vector space X. We say that E is a topological supplement of F if

$$X = F \oplus E.$$

3.2 Projection

Let X be a vector space. We say that a linear operator $P : \mathbb{X} \to \mathbb{X}$ is a projection if $P(P(x)) = P(x), \forall x \in X(i - e$ if $P^2 = P)$.

Proposition 3.2.1

Let X be a vector space. A linear operator $P : X \to X$ is a projection and only if $(I - P)$ is a projection. Moreover, if the space X is normalized, then P is continues if and only if $(I - P)$ is continuous.

Proof.

1) **We show that**: P projection $\Leftrightarrow$ $(I-P)$ projection.
   $(\Rightarrow)P$ a projection, then:

$$\begin{aligned}(I-p)^2 &= (I-P)\left[(I-P)(x)\right]x \\ &= (I-P)\left[x-P(x)\right] \\ &= I(x-P(x))-P(x-P(x)) \\ &= x-P(x)-P(x)+P^2(x) \\ &= x-P(x) \\ &= (I-P)(x).\end{aligned}$$

   $(\Leftarrow)(I-P)$ is a projection, then:

$$I-(I-P)=I-I+P+P.$$

   is also a projection.

2) **We shaw that**: P is continuous $\Leftrightarrow$ $(I-P)$ is continuous.
   $(\Rightarrow)P$ is continuous, then $(I-P)$ continuous
   $(\Leftarrow)(I-P)$ is continuous, then P is continuous.
   For the topological framework, as identity is a countinuous map and that the sum.

■

Proposition 3.2.2
If P is a projection in X then:

$$\begin{cases} Ker(P)=Im(I-P), \\ Im(P)=Ker(I-P). \end{cases}$$

Proof.

1) **We show that** $Ker(P)+Im(P)$.
   (i) $Ker(P)\subset Im(I-P)$
   If $x\in Ker(P)\Rightarrow P(x)=0$ We replace P by $(I-P)$

$$\begin{aligned}(I-P)(x)=x-P(x)&=x-0=x \\ &\Rightarrow x\in Im(I-P) \\ &\Rightarrow Ker(P)\subset(I-P).\end{aligned}$$

   (ii) $Ker(P)\supset Im(I-P)$
   If $x\in Im(I-P)$, we define the map:

$$\begin{aligned}P\big((I-P)(x)\big) &= P(x)-P^2(x)=P(x)-P(x)=0 \\ &\Rightarrow x\in Ker(P) \\ &= Ker(P)\supset Im(I-P).\end{aligned}$$

   So

$$Ker(P)=Im(I-P).$$

2) **We show that** $Im(P) = Ker(I-P)$.
(i) $Im(P) \subset Ker(I-P)$
If $x \in Im(P) \Rightarrow P(x) = x$, we replace P by $(I-P)$. Then

$$\begin{aligned}(I-P)(x) &= xP(x)x - x = 0\\ &\Rightarrow x \in Ker(I-P)\\ &\Rightarrow Im(P) \subset Ker(I-P).\end{aligned}$$

(ii) $Im(P) \supset Ker(I-P)$
If $x \in Ker(I-P)(x)$

$$\begin{aligned}(I-P)(x) = 0 &\Leftrightarrow x - P(x) = 0\\ &\Leftrightarrow x = P(x)\\ &\Rightarrow Im(P) \supset Ker(I-P).\end{aligned}$$

So

$$Im(P) = Ker(I-P).$$

■

Definition 3.2.1 ***(A Hausdorff space)***
A topological space X is separated (or hausdorff) if

$$\forall x \neq y \in X, \exists x U_x, y \in U_y \text{ open such that } U_x \cup U_y = \emptyset$$

Corollary 3.2.1
Any continuous projection in a Hausdorff space is closed image. In particular,the continuous projections of Banach spaces are closed images.

Theorem 3.2.1
If P is a continuous projectin into a topological Hausdorff vector space X,then X is the direct sum of $Im(P)$ and $Ker(P), (i.e. X = Im(P) \oplus Ker(P))$.

Proof.
By the preceding corollary, $Im(P)$ and $Ker(P)$ are closed in X, where

$$\begin{aligned}Ker(P) &= \{x \in X, P(x) = 0\},\\ Im(P) &= \{x \in X, P(x) = x\}.\end{aligned}$$

We put $x = P(x) + (I-P)(x)$
1-i) $P(x) \in Im(P)$ because $P(P(x)) = P^2(x) = P(x)$.
ii) $(I-P)(x) \in Ker(P)$ because $P((I-P)(x)) = P(x)P^2(x)P(x) - P(x) = 0$.
Then

$$X = Im(P) + Ker(P).$$

2-i) $P(x) \in Im(P) = Ker(I-P) \Rightarrow P(x) = (I-P)P(x) = P(x) - P^2(x) = P(x) - P(x) = 0$
ii) $(I-P)(x) \in Ker(P) \Rightarrow (I-P)(x) = 0$.
Then $x \in Im(P) \supset Ker(P) + \{0\}$
According to (1) and (2)

$$X = Im(P) \oplus Ker(P).$$

■

3.3 Dimensional and codimensional subspace finished

Lemma 3.3.1 ***(Projection onto a finite-dimensional subspace)***

If E is finite-dimensional vector suspace of a standardized space X then there exists a continuous projection P on X such that $Im(P) = E$.

Proof.

We choose a base $e_1, ..., e_n$ of E, and we designate by $e_j^*, j = 1, ..., n$ the linear forms coordinated on E, i,e

$$e_j^*(e_i) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Using the Hahn-Banach theorem we can extend these linear forms on E into continuous linear forms $x_1^*, ..., x_n^*$ on X. We obtain that the map P
defined by

$$\forall x \in X, P(x) = \sum_{j=1}^{n} x_j^*(x) e_j.$$

Is a continuous projection of X on E that answers the problem. ∎

Corollary 3.3.1

If E is a finite-dimensional vector subspace of a normalized space X, there exists a closed vector subspace $Y \subset X$ such that $X = E \oplus Y$

Definition 3.3.1 ***(Codimension of a vector subspace)****.*

If the quotient space X/Y is of finite dimension, we say that the closed vector subspace $Y \subset X$ is of finite codimension in X that we write

$$codim(Y) = dim(X/Y)$$

Lemma 3.3.2

Let E be a normalized vector space, M and N two closed vector subspace of E such that $M \cup N = \{0\}$. If $dim(M) = codim(N) < \infty$, then $E = M \oplus N$.

Proof.

Let $\pi = \pi_{|N}$ the canonical surjection of E in E/N. as $M \cap N = \{0\}$, the π_M is injective. From where

$$dim(\pi(M)) = dim(M) = codim(N) = dim(E/N).$$

Thus $\pi(M) = E/N = \pi(E)$. Now if any $x \in E$, then $\pi(x) \in \pi(E) = \pi(M)$. Thus there exists x_M such that $\pi(x) = \pi(x_M)$. From where $\pi(x - x_M) \in Ker\pi = N$. we therefore deduce that $x \in M + N$. This proves that $E = M + N$ and so finally $E = M \oplus N$. ∎

3.4 Fredholm operator

Definition 3.4.1

Let X and Y be two normalized vector $\mathbb{R}$-spaces, we say that a linear map $L : dom(L) \subset X \to Y$ is of Fredholm if it satisfies the following conditions:

1 $Ker(L) = L^{-1}(\{0\})$ *is of finite dimension.*

2 $Im(L) = L(dom(L))$ *is closed and of finite codimension.*

Recall that the codimension of $Im(L)$ is the dimension of $coKer(L) = dim(Y/Im(L))$.

Definition 3.4.2 ***(Index)***

If L is a Fredholm operator, then its index is the integer

$$ind(L) = dim(Ker(L)) - codim(Im(L)).$$

Example 3.4.1

1. *If X and Y are of finite dimension, then every linear map $L : X \to Y$ is of Fredholm with*
$$\begin{aligned} ind(L) &= dim(Ker(L)) - codim(Im(L)) \\ &= dim(Ker(L)) - dimcoKer(L) \\ &= dim(Ker(L)) - (dim(Y) - dim(Im(L))) \\ &= dim(Ker(L)) + dim(Im(L)) - dim(Y) \\ &= dim(X) - dim(Y). \end{aligned}$$

2. *the identity $I : X \to X$ is a Fredholm operator withindex* 0.
$$\begin{aligned} ind(L) &= dim(Ker(I)) - codim(Im(I)) \\ &= dim(Ker(I)) - dimcoKer(I) \\ &= dim(Ker(I)) - dim\left(\frac{X}{Im(I)}\right) \\ &= dim\{0\} - dim\{0\} = 0. \end{aligned}$$

3. *If X and Y are Banach spaces and $L : X \to Y$ is a bijective linear map then L is a Frefholm operator with index* 0. *Indeed, it follows from the bijective that $Ker(L) = \{0\}_X$, whose dimension is zero, and $Im(L) = Y$ and is codimension is zero, so*
$$\begin{aligned} ind(L) &= dim(Ker(L)) - codim(Im(L)) \\ &= dim(Ker(L)) - dim\left(\frac{Y}{Im(Y)}\right) \\ &= dim\{0\}_X - dim\{0\} = 0. \end{aligned}$$

Theorem 3.4.1

If L is a Fredholm operator, k is a compact linear map, then $L+K$ is of Fredholm and

$$ind(L+K) = ind(L).$$

In particular, any disturbance of the identity is a Fredholm operator of index 0.

Proposition 3.4.1

If L is a Fredholm operator with zero index, then L is surjective if and only if L is injective.

Proof.

If L is surjective, then $Im(L) = Y + \{0\}$ and consequently, $dim\{0\} = dim(Ker(L)) = 0$, so $Ker(L) = \{0\}$, hence L is injective.
In everything that follows (unless otherwise stated) $L : om(L) \subset X \to Y$ denotes a Fredholm operator of index 0. If L is Fredholm, then according to the above, there exists two continuous projections, $P : X \to X$ and $Q : Y \to Y$ such that

$$Im(P) = Ker(L), \quad Ker(Q) = Im(L).$$

We pose

$$X_1 = Im(I - P) = Ker(P) \text{ and } Y_1 = Im(Q)$$

then we had to write

$$X = Ker(L) \oplus X_1; \quad Y = Im(L) \oplus Y_1,$$

Consider an isomorphism

$$J : Ker(L) \to Im(Q).$$

whose existence is ensured by the fact that $dimKer(L) = dimIm(Q) = n$.
Let's notice

$$dom(L) = Ker(L) \oplus (dom(L) \cap X_1).$$

And that the restrictions of L to $dom(L) \cap X_1$ is a isomorphism on $Im(L)$, let us denote by L_p this restriction, that is to say $Lp : dom(L) \cap X_1 \to Im(L)$, then ■

Lemma 3.4.1

L_p is an algebraic isomorphism.

Proof.

1- **Let us show that L_p is injective:**
Let $x \in Ker(L_p) \subset Ker(L) = Im(P)$, then there exists a $y \in Dom(P)$ such that $x = P_y$
As P is a projection, we obtain

$$x = P_y = P_Y^2 = P(P_y) = Px = 0.$$

Therefore, $x = 0$ and therefore $Ker(L_p) = \{0\}$, which means the injection of L^p

2- **the surjection of L^p:**
Since P is a projection, then we can write the vector space X as a direct sum:

$$X = Ker(P) \oplus Im(P) = Ker(P) \oplus Ker(L).$$

Take $z \in Im(L)$, so there exists $x \in dom(L) \subset X$ such that $L_x = z$.
As $X = Ker(P) \oplus Ker(L)$, then there exists two unique elements

$$\Big\{ e \in Ker(P) \text{ and } f \in ker(L) \text{ such that } x = e + f.$$

We have

$$z = Lx = L(e + f) = Le + Lf = Le + 0 = Le,$$

thus $e \in dom(L)$.
Finally, we obtain $e \in dom(L), e \in Ker(P)$ and $L_p e = z$, from where L_p is indeed surjective.

Let $K_p : Im(L) \subset Y \to dom(L) \cap Ker(P)$ is bijectif, defined by $K_p := L_p^{-1}$, that $PK_p = 0$, and that it satisfies the properties. ■

Lemma 3.4.2

1 *On $Im(L)$, we have $LK_p = I$.*

2 *On $dom(L)$, we have $K_p L = (I - P)$.*

Proof.

(1) Take $x \in Im(L)$, then $LK_p x = L(K_p(x)) = L_p(K_p(x)) = Ix$.

(2) As $Im(P)(L) = Ker(L)$, then $LP = 0$, and by consequently $K_p L = K_p L(I - P)$.

So showing (2), amounts to verifying that $K_p L(I - P) = K_p L_p(I - P)$.
If we have $Im(I - P) \subseteq dom(L_p) = dom(L) \cap Ker(P)$, then the result follows.
Let us take $x \in dom(L)$ As $P(x) \in Ker(L) \subset dom(L)$ and $dom(L)$ is a vector subspace of X, we have

$$(x - Px) \in dom(L)$$

Since

$$P(x - Px) = P(x) - P^2(x) = P(x) - P(x) = 0.$$

Then $(x - Px) \in Ker(P)$ and by consequently $(x - Px) \in dom(L) \cup Ker(P)$.
Erom here obtains $Im(I - P) \subset dom(L) \cap Ker(P)$. Hence using (1):

$$K_p L(I - P) = K_p L_p(I - P).$$

ensues.
Now define the operator $K_{P,Q} : Y \to X$, then $K_{P,Q} = L_p^{-1}(I - Q)$. ■

Lemma 3.4.3

The operator $L + JP : dom(L) \to Y$ is an isomorphism and

$$(L + JP)^{-1} = K_{P,Q} + J^{-1}Q.$$

In particular

$$(L + JP)^{-1}x = J^{-1}x \text{ for all } x \in Im(Q).$$

Proof.

For the injectivity of $L + JP$, let $x \in dom(L)$ such that

$$(L + JP)x = 0 \tag{3.1}$$

From this equality we deduce that

$$Lx \in Im(L) \cap Im(J) = Ker(Q) \cap Im(Q) = \{0\},$$

hence $x \in Ker(L)$. Consequently, $Px = x$ and taking into account (3.1) where $0 < t < 1$, $Jx = 0$, consequently $x = 0$. For the surjectivity of $L + JP, y \in Y$.
Let us affirm that

$$x = (K_{P,Q} + J^{-1}Q)y,$$

is a solution of

$$(L + JP)x = y.$$

Indeed, like $J^{-1}Q \in Ker(L)$, it follows that

$$\begin{aligned} Lx &= LK_{P,Q} \\ &= LL_p^{-1}(I - Q)y \\ &= (I - Q)y. \end{aligned}$$

As $K_{P,Q}y \in dom(L) \cap Ker(P)$ it follows that

$$JPx = JJ^{-1}Qy = Qy,$$

Consequently

$$(L + JP)x = (I - Q) = Qy,$$

and

$$(L + JP)^{-1} = K_{P,Q} + J^{-1}Q.$$

■

Lemma 3.4.4

If $N : \triangle \subset X \to X$ is a map, the problem

$$x \in dom(L) \cap \triangle, Lx = Nx.$$

And equivalent to the fixed point problem

$$x \in \triangle, x = Px + J^{-1}QNx + K_{P,Q}Nx.$$

Proof.

We have

$$\begin{aligned}
&[x \in dom(L) \cap \triangle, Lx = Nx] \\
\Leftrightarrow &[x \in dom(L) \cap \triangle, (L+JP)x = (N+JP)x] \\
\Leftrightarrow &\left[x \in \triangle, x = (L+JP)^{-1}(N+JP)x\right].
\end{aligned}$$

On the other part, using lemma (3.4.3):

$$\begin{aligned}
(L+JP)^{-1}(N+JP) &= (K_{P,Q} + J^{-1}Q)(N+JP) \\
&= K_{P,Q}N + K_{P,Q}JP + J^{-1}QN + J^{-1}QJP.
\end{aligned}$$

Since $Im(J) = Im(Q) = Ker(I-Q)$, it follows that

$$K_{P,Q}JP = L_p^{-1}(I-Q)JP = 0.$$

Since $Q_{|Im(Q)} = I_{|Im(Q)}$ and $Im(J)Im(Q)$, we deduce that

$$J^{-1}QJP = J^{-1}JP = P.$$

Therefore, $(L+JP)^{-1}(N+JP) = P + J^{-1}QN + K_{P,Q}N$.
Let X, Y be be two Banach spaces and $L : dom(L) \subset X \to Y$ be a Fredholm operator with index 0. ■

Definition 3.4.3 ***(The L-compact map)***

Let Ω be a bounded open subset of X such that $dom(L) \cap \neq \emptyset$, the map $N : X \to Y$ is called L-compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_{P,Q}N : \overline{\Omega} \to X$ is compact.
As a consequence of lemmas (3.5) and (3.6) we have the following definition:

Definition 3.4.4 ***(The degree of Mawhin)***

If the operator L and N satisfied the owners mentioned above, then the degree of coincidence of L and N on Ω is defined by

$$deg[(L,N),\Omega] = deg_{LS}(I - M, \Omega, 0).$$

M well designate the quantity given by:

$$M(P,J,Q) = P + J^{-1}QN + K_{P,Q}N.$$

3.5 Proof of Mawhin's theorem

Theorem 3.5.1

Let L be a Fredholm operator with index zero, and N is L-compact on $\overline{\Omega}$. Suppose the following conditions are satisfied:

(i) $Lx \neq Nx$ for all $[(x,\lambda) \in [(dom(L) \backslash Ker(L)) \cap \partial\Omega] \times]0,1[]$

(ii) $QNx \neq 0$ for all $x \in Ker(L) \cap \partial\Omega$

(iii) $deg_B\big(J^{-1}QN|_{Ker(L)}, \Omega \cap Ker(L), 0\big) \neq 0$ *where* $Q : Y \to Y$ *is the projection defined by above with* $Im(L) = Ker(Q)$.

Then the equation $Lx = Nx$ *admits at least one solution in* $dom(L) \cap \overline{\Omega}$.

Proof.

For $\lambda \in [0, 1]$, consider the family of problems

$$x \in dom(L) \cap \overline{\Omega}, Lx = \lambda Nx + (1 - \lambda)QNx \tag{3.2}$$

Let $M : [0, 1] \times \overline{\Omega} \to Y$ be a homotopy defined by

$$M(\lambda, x) = Px + J^{-1}QNx + K_{P,Q}Nx$$

By virtue of lemma(3.4.4), problem(3.2) is equivalent to a fixed point problem $x \in \overline{\Omega}$ and

$$\begin{aligned} x &= Px + J^{-1}Q(\lambda N + (1 - \lambda)QN)x + K_{P,Q}(\lambda N + (1 - \lambda)QN)x \\ &= Px + J^{-1}QNx + (1 - \lambda)J^{-1}QNx + K_{P,Q}Nx + (1 - \lambda)K_{P,Q}QNx \\ &= M(\lambda, x) \end{aligned}$$

So, this last equation is equivalent to a fixed point problem

$$x \in \overline{\Omega}, \quad x = M(\lambda, x) \tag{3.3}$$

If there exists a $x \in \partial\Omega$ such that $Lx = Nx$, then we are done. Now suppose that

$$Lx \neq Nx \text{ for all } x \in dom(L) \cap \Omega, \tag{3.4}$$

And on the other part

$$Lx \neq \lambda Nx + (1 - \lambda)QNx \tag{3.5}$$

For all $(\lambda, x) \in]0, 1[\times(dom(L) \cap \Omega)$. if

$$Lx = Nx + (1 - \lambda)QNx.$$

For all $(\lambda, x) \in]0, 1[\times(dom(L) \cap \Omega)$, we obtain by the map of Q to the two members of the previous equality

$$QNx = 0, \quad Lx = Nx.$$

The first of these equalities and condition (ii) imply that $x \neq Ker(L) \cap \partial\Omega, i.ex \in \partial\Omega \cap dom(L)\backslash Ker(L)$ and therefore the second equality contradicts (i). Using (ii) again, it follows that

$$Lx \neq QNx \quad \text{for all} \quad x \in dom(L) \cap \partial\Omega. \tag{3.6}$$

By virtue of (3.4),(3.5) and (3.6) we deduce that

$$x \neq M(\lambda, x) \quad \text{for all} \quad (\lambda, x) \in [0, 1] \times \partial\Omega. \tag{3.7}$$

It is easy to verify that $M(\lambda, x)$ is compact because N L-compact on $\overline{\Omega}$, hence using the homotopy invariance property of the Leray-Schauder degree, we obtain

$$degLS(I - M(0,.), \Omega, 0) = degLS(I - M(1,.), \Omega, 0). \tag{3.8}$$

On other part we have

$$deg_{LS}(I - M(0,.), \Omega, 0) = deg_{LS}(I - (P + J^{-1}QN), \Omega, 0). \tag{3.9}$$

But the rank of $P + J^{-1}QN$ is continuous in $Ker(L)$, hence using the property of reduction of the degree of Leary-Schauder and the fact that $P|_{Ker(L)} = I|_{Ker(L)}$, (because $Ker(L) = Im[(P)] = Ker(I - P)$) we obtain

$$deg_{LS}(I - (P + J^{-1}QN), \Omega, 0) = deg_B(I - (P + J^{-1}QN), \Omega \cap Ker(L), 0) \tag{3.10}$$

$$= deg_B(J^{-1}QN, \Omega \cap Ker(L), 0). \tag{3.11}$$

By virtu of (3.9),(3.10) and (3.11), it follows that $deg_{LS}(I - M(1,.), \Omega, 0) \neq 0$, and hence the existence property of the Leray-Schuader degree implies the existence of an $x \in \overline{\Omega}$ such that

$$x = M(1, x) \quad i, e \quad x \in dom(L) \cap \Omega, \quad Lx = Nx$$

■

Chapter 4

Theorem application

4.1 Introduction

In this section we present a brief introduction to some notations and certain fundamental results involved in the reformulation of the problem as well as the main theorem, that of the existence of the solution obtained from Mawhin's.

4.2 Main Results

In this section, we will prove the existence results for (3). We use the Banach space $E = C[0,1]$ with the norm $||u||_\infty = \max_{0\leq t\leq 1} |u(t)|$. For $\alpha > 0, N = [\alpha] + 1$, we define a linear space

$$X = \{u|u, D_{0+}^{\alpha-1}u \in E, i = 1, 2, ..., N-1\}. \tag{4.1}$$

By means of the functional analysis theory, we can prove that X is a Banach space with the norm

$$||u||_X = ||D_{0+}^{\alpha-1}u||_\infty + ... + ||D_{0+}^{\alpha-(N-1)}u||_\infty + ||u||_\infty.$$

Define L to be the linear operator from $dom(L) \cap X$ to E with:

$$dom(L) = \Big\{u \in X | D_{0+}^{\alpha}u(t) \in E, u(0) = D_{0+}^{\alpha-2}u(0) = ... = D_{0+}^{\alpha-(N-1)}u(0) = 0, D_{0+}^{\alpha-1}u(0) = D_{0+}^{\alpha-1}u(1)\Big\}$$

and

$$Lu = D^{\alpha}_{0+}u, \qquad u \in dom(L). \tag{4.2}$$

We define $N : X \to E$ by

$$Nu(t) = f\big(t, u(t), D^{\alpha-1}_{0^+}u(t), D^{\alpha-2}_{0^+}u(t), ..., D^{\alpha-(N-1)}_{0^+}u(t)\big). \tag{4.3}$$

Then the problem (3) can be writen by $Lu = Nu$.

Lemma 4.2.1

The mapping $L : dom(L) \subset E$ is a Fredholm operator of index zero.

Proof.

It is clear that

$$Ker(L) = \big\{c_1 t^{\alpha-1}\big\} \cong \mathbb{R}^1. \tag{4.4}$$

Let $x \in Im(L)$, so there exists a function $u \in dom(L)$ which satisfies $Lu = x$. By(4.9) and Lemma(4.3.1), we have

$$u(t) = I^{\alpha}_{0+}x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + ... + c_N t t^{\alpha-N}. \tag{4.5}$$

By $u(0) = D^{\alpha-2}_{0^+}u(0) = ... = D^{\alpha-(N-1)}_{0^+}u(0) = 0$, we can obtain $c_2 = ... = c_N = 0$. Hence

$$u(t) = I^{\alpha}_{0+}x(t) + c_1 t^{\alpha-1} \tag{4.6}$$

Then, we have

$$\begin{aligned} D^{\alpha-1}_{0^+}u(t) &= D^{\alpha-1}_{0^+}\Bigg(I^{\alpha}_{0+}x(t) + c_1 t^{\alpha-1}\Bigg) \\ &= D^{\alpha-1}_{0^+} I^{\alpha}_{0+}x(t) + c_1 \frac{\Gamma(\alpha)}{\Gamma(1)} \\ &= \int_0^t x(s)ds + c_1\Gamma(\alpha). \end{aligned} \tag{4.7}$$

Taking into account $D^{\alpha-1}_{0^+}u(0) = D^{\alpha-1}_{0^+}u(1)$, we obtain

$$\int_0^1 x(s)ds = 0. \tag{4.8}$$

On the other hand, suppose x satisfy $\int_0^1 x(s)ds = 0$. Let $u(t) = I^{\alpha}_{0+}x(t)$, we can easily prove $u(t) \in dom(L)$.

Thus, we conclude that

$$Im(L) = \left\{x : \int_0^1 x(s)ds = 0\right\}. \tag{4.9}$$

Consider the linear operators $Q : E \to E$ defined by

$$Qx(t) = \int_0^1 x(s)ds. \tag{4.10}$$

Take $x(t) \in E$, then:

$$\begin{aligned} Q(Qx(t)) &= Q\bigg(\int_0^1 x(s)ds\bigg) \\ &= \int_0^1 \bigg(\int_0^1 x(t)dt\bigg)ds \\ &= \int_0^1 x(s)ds = Qx(t). \end{aligned} \tag{4.11}$$

We can see $Q^2 = Q$.
For $x(t) \in E$ in the type $x(t) = x(t) - Qx(t) + Qx(t)$, obviously, $x(t) - Qx(t) \in Ker(Q) = Im(L)$ and $Qx(t) \in Im(Q)$. That is to say

$$E = Im(L) + Im(Q).$$

If $u \in Im(L) \cap Im(Q)$, we have $u = c_1$, then $\int_0^1 c_1 ds = 0$.
As a result $c_1 = 0$, and we get

$$E = Im(L) \oplus Im(Q).$$

Note that and $L = dimKer(L) - codimIm(L) = 0$. Then L is a Fredholm mapping of index zero.
We can define the operators $P : X \to X$, where

$$Pu = \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0) t^{\alpha-1}. \tag{4.12}$$

For $u \in X$,

$$\begin{aligned} P(Pu) &= P\bigg(\frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0) t^{\alpha-1}\bigg) \\ &= \frac{1}{\Gamma(\alpha)} D_{0^+}^{\alpha-1} u(0) t^{\alpha-1} \\ &= Pu. \end{aligned} \tag{4.13}$$

So we have $P^2 = P$. Note that

$$Ker(P) = \left\{u : D_{0^+}^{\alpha-1} u(0) = 0\right\}. \tag{4.14}$$

Since $u = u - Pu + Pu$, it is easy to say that $u - Pu \in Ker(P)$ and $Pu \in ker(L)$. So we have:

$$X = Ker(P) + Ker(L).$$

If $u \in Ker(L) \cap Ker(P)$, then $u = c_1 t^{\alpha-1}$. We can derive $c_1 = 0$ from $D_{0^+}^{\alpha-1} c_1 t_{t=0}^{\alpha-1} = 0$. Then

$$X = Ker(L) \oplus Ker(P). \tag{4.15}$$

For $u \in X$,

$$\begin{aligned} ||Pu||_X &= \frac{1}{\Gamma(\alpha)}\big|D_{0^+}^{\alpha-1}u(0)\big|.\big|\big|t^{\alpha-1}\big|\big|_X \\ &= \frac{1}{\Gamma(\alpha)}\big|D_{0^+}^{\alpha-1}u(0)\big|.\,\big[\big|\big|t^{\alpha-1}\big|\big|_\infty + \big|\big|D_{0^+}^{\alpha-1}t^{\alpha-1}\big|\big|_\infty + ... + \big|\big|D_{0^+}^{\alpha-1}t^{\alpha-(N-1)}\big|\big|_\infty\big] \\ &= \left(\sum_{i=1}^{N-1}\frac{1}{\Gamma(i)}\frac{1}{\Gamma(\alpha)}\right)\big|D_{0^+}^{\alpha-1}u(0)\big| \\ &= a\big|D_{0^+}^{\alpha-1}u(0)\big|. \end{aligned} \tag{4.16}$$

where $a = 1/\Gamma(\alpha) + \sum_{i=1}^{N-1}(1/\Gamma(i))$.
We define $K_P : Im(L) \to dom(L) \cap Ker(P)$ by $K_Px = I_{0+}^\alpha x = x$. For $x \in Im(L)$, we have:

$$LK_Px = LI_{0+}^\alpha x = D_{0+}^\alpha I_{0+}^\alpha x = x. \tag{4.17}$$

For $u \in dom(L) \cap Ker(P)$, we have

$$D_{0^+}^{\alpha-1}u(0) = 0.$$

And for $u \in dom(L)$, the coefficients $c_1, ..., c_N$ in the expressions

$$u = I_{0+}^\alpha D_{0+}^\alpha u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + ... + c_Nt^{\alpha-N} \tag{4.18}$$

Are all equal to zero. Thus, we obtain

$$K_PLu = I_{0+}^\alpha D_{0+}^\alpha u = u. \tag{4.19}$$

This shows that $K_P = \big(L_{dom(L)\cap Ker(P)}\big)^{-1}$. Again for each $x \in Im(L)$,

$$\begin{aligned} \big|\big|K_Px\big|\big|_X &= \big|\big|I_{0+}^\alpha x\big|\big|_X \\ &= \big|\big|I_{0+}^\alpha x\big|\big|_\infty + \big|\big|D_{0^+}^{\alpha-1}I_{0+}^\alpha x\big|\big|_\infty + ... + \big|\big|D_{0^+}^{\alpha-(N-1)}I_{0+}^\alpha x\big|\big|_\infty \\ &\le \left(\sum_{i=1}^{N-1}\frac{1}{\Gamma(i+1)} + \frac{1}{\Gamma(\alpha+1)}\right)||x||_\infty \\ &= b||x||_\infty, \end{aligned} \tag{4.20}$$

where $b = 1/\Gamma(\alpha+1) + \sum_{i=1}^{N-1}(1/\Gamma(i+1))$. ■

Lemma 4.2.2
Assume $\Omega \subset Y$ an open bounded subset such that $dom(L) \cap Y \neq \emptyset$; then map N is L-compact on $\overline{\Omega}$.

Proof.
By the continuity of f, we can get that $QN(\overline{\Omega})$ and $K_P(I-Q)N(\overline{\Omega})$ are bounded. So, in view of the Arzela-Ascoli theorem, we need only to prove that $K_P(I-Q)N(\overline{\Omega})$ is equicontinuous.

From the continuity of f, there exists a constant $r > 0$, such that $|(I-Q)N(u(t))| < r$, for all $u \in \overline{\Omega}, \quad t \in [0,1]$.
For $0 \leq t_1 \leq t_2 \leq 1, \quad u \in \Omega$, we have

$$\begin{aligned} |K_{P,Q}Nu(t_2) - K_{P,Q}Nu(t_1)| &= \frac{1}{\Gamma(\alpha)}\bigg| \int_0^{t_2} (t_2-s)^{\alpha-1}(I-Q)N(u(s))ds \\ &\quad - \int_0^{t_1} (t_1-s)^{\alpha-1}(I-Q)N(u(s))ds \bigg| \\ &\leq \frac{r}{\Gamma(\alpha)} \int_0^{t_2} \left[(t_1-s)^{\alpha-1} - (t_2-s)^{\alpha-1}\right] ds + \frac{r}{\Gamma(\alpha)} \int_{t_1}^{t_2} (t_2-s)^{\alpha-1} ds \\ &= \frac{r}{\Gamma(\alpha+1)}(t_2^\alpha - t_1^\alpha). \end{aligned} \tag{4.21}$$

Furthermore, we have

$$\begin{aligned} |D_{0+}^{\alpha-1}K_{P,Q}Nu(t_2) - D_{0+}^{\alpha-1}K_{P,Q}Nu(t_1)| &= \frac{1}{\Gamma(i)}\bigg| \int_0^{t_2} (t_2-s)^{i-1}(I-Q)N(u(s))ds \\ &\quad - \int_0^{t_1} (t_1-s)^{i-1}(I-Q)N(u(s))ds \bigg| \\ &\leq \frac{r}{\Gamma(i)} \int_0^{t_2} \left[(t_1-s)^{i-1} - (t_2-s)^{i-1}\right] ds \\ &\quad + \frac{r}{\Gamma(i)} \int_{t_1}^{t_2} (t_2-s)^{i-1} ds \\ &= \frac{r}{\Gamma(i+1)}(t_2^i - t_1^i). \end{aligned} \tag{4.22}$$

where $i = 1, 2, ..., N-1$. Since t^α and t^i are uniformly continuous on $[0,1]$, we can get that $K_P(I-Q)N : \overline{\Omega} \to Y$ is compact. The proof is completed.
To obtain our main, we need the following conditions.
(H_1) There exists $\varphi, \psi_i \in L^1, i = 1, N$, such that for all $u \in \mathbb{R}^2, t \in [0,1]$,

$$|f(t, x_1, x_2, ..., x_N)| \leq \varphi + \psi_1|x_1| + \psi_2|x_2| + ... + \psi_N|x_N|. \tag{4.23}$$

(H_2) There exists a constant $A > 0$ such that for every $y \in \mathbb{R}$, if $|x_2| > A$ for all $t \in [0,1]$, then

$$f(t, x_1, x_2, ..., x_N) \neq 0. \tag{4.24}$$

(H_3) There exists a constant $D > 0$ such that, for each $c_i, i = 1, 2$ satisfying $\min\{|c_1|, |c_2|\} > D$. We have either at least one of the following:

$$c_1 N\big(c_1 t^{\alpha-1}\big) > 0 \tag{4.25}$$

or

$$c_1 N\big(c_1 t^{\alpha-1}\big) < 0 \tag{4.26}$$

(H_4) $\sum_{i=2}^N \rho_i < 1$, where $\rho_{i+1} = (a+b)||\psi_i||_1, i = 1, 2, ..., N$. ∎

Lemma 4.2.3

$\Omega_1 = \{u \in dom(L)\backslash Ker(L) | Lu = \lambda Nu \in [0,1]\}$*is bounded.*

Proof.

For $u \in \Omega_1, \lambda \neq 0$ and $Lu = \lambda Nu$. By (), $Lu = \lambda Nu \in Im(L) = Ker(Q)$ that is;

$$\lambda \int_0^1 f\big(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), ..., D_{0+}^{\alpha-(N-1)}u(t)\big)dt = 0. \tag{4.27}$$

By the integral mean value theorem, there exits a constant $t_0 \in [0,1]$ such that

$$f\big(t_0, u(t_0), D_{0+}^{\alpha-1}u(t_0), D_{0+}^{\alpha-2}u(t_0), ..., D_{0+}^{\alpha-(N-1)}u(t_0)\big) = 0. \tag{4.28}$$

From (H_2), we can get $|D_{0+}^{\alpha-1}u(t_0)| \leq A$.
Again for $u \in \Omega_1, (I-P)u \in dom(L)\backslash Ker(L)$ and $LPu = 0$.
From (27), we have

$$||(I-P)u||_X = \big|\big|K_P L(I-P)u\big|\big|_X = \big|\big|K_P Lu\big|\big|_X \leq b||Nu||_\infty. \tag{4.29}$$

Now by Lemma (4.2.2)

$$\begin{aligned} \big|D_{0+}^{\alpha-1}u(0)\big| &\leq \big|D_{0+}^{\alpha-1}u(t_0)\big| + \left|\int_0^{t_0} D_{0+}^{\alpha}u(s)ds\right| \\ &\leq \big|D_{0+}^{\alpha-1}u(t_0)\big| + |t_0| \max_{0\leq t\leq t_0} \big|D_{0+}^{\alpha}u(t)\big| \\ &\leq \big|D_{0+}^{\alpha-1}u(t_0)\big| + ||D_{0+}^{\alpha}u(t)||_\infty \\ &\leq A + ||Lu||_\infty = A + ||Nu||_\infty. \end{aligned} \tag{4.30}$$

That is,

$$\big|D_{0+}^{\alpha-1}u(0)\big| \leq A + ||Nu||_\infty. \tag{4.31}$$

From (23) and (36), we have

$$\begin{aligned} ||u||_X = ||Pu + (I-P)u||_X &\leq ||Pu||_X + ||(I-P)u||_X \\ &\leq a\big|D_{0+}^{\alpha-1}u(0)\big| + b||Nu||_\infty. \end{aligned} \tag{4.32}$$

Furthermore, it follows from (38) and (H_1) that

$$\begin{aligned} ||u||_X &\leq \big(a\big|D_{0+}^{\alpha-1}u(0)\big| + b||Nu||_\infty\big) \\ &\leq a(A + ||Nu||_\infty) + b||Nu||_\infty = aA + (a+b)||Nu||_\infty \\ &\leq aA + (a+b) \times \left|\left|f\big(t, u(t), D_{0+}^{\alpha-1}u(t), D_{0+}^{\alpha-2}u(t), ..., D_{0+}^{\alpha-(N-1)}u(t)\big)\right|\right|_\infty \\ &\leq aA + (a+b)\big(||\varphi||_1 + ||\psi||_1||u||_\infty + ||\psi_2||_1||D_{0+}^{\alpha-1}u||_\infty + ... + ||\psi_N||_1||D_{0+}^{\alpha-((N-1))}u||_\infty\big) \\ &= aA + (a+b)||\varphi||_1 + \rho_2||u||_\infty + \rho_3||D_{0+}^{\alpha-1}u||_\infty + \rho_4||D_{0+}^{\alpha-2}u||_\infty + ... + \rho_{N+1}||D_{0+}^{\alpha-(N-1)}u||_\infty. \end{aligned} \tag{4.33}$$

By the definition $||u||_X$ and (H_4), it is easy to see that $||D_{0+}^{\alpha-1}u||_\infty, ..., ||D_{0+}^{\alpha-(N-1)}u||_\infty$ and $||u||_\infty$ are bounded. So Ω_1 is bounded. ■

Lemma 4.2.4

$\Omega_2 = \{u \in Ker(L) : Nu \in Im(L)\}$ is bounded.

Proof.

Let $u \in Ker(L)$, so we have $u = c_1 t^{\alpha-1}, c_1 \in \mathbb{R}$. For $Nu \in Im(L) = Ker(L)$,

$$\int_0^1 f\big(t, c_1 t^{\alpha-1}, c_1\Gamma(\alpha), ..., \frac{\Gamma(\alpha)}{\Gamma(N-1)} c_1 t^{N-1}\big)dt = 0. \tag{4.34}$$

By the integral mean value theorem, there exits a constant $t_1 \in [0,1]$ such that

$$f\big(t, c_1 t^{\alpha-1}, c_1\Gamma(\alpha), ..., \frac{\Gamma(\alpha)}{\Gamma(N-1)} c_1 t^{N-1}\big) = 0. \tag{4.35}$$

From (H_2), it follows that $|c_1| \leq A/\Gamma(\alpha)$. Hence, Ω_2 is bounded. ■

Lemma 4.2.5

$\Omega_3 = \{u \in Ker(L) : \lambda Nu + (1-\lambda)QNu = 0, \lambda \in [0,1]\}$ is bounded.

Proof.

Let $u \in Ker(L)$, so we have $u = c_1 t^{\alpha-1}, c_1 \in \mathbb{R}$. If $\lambda = 0$, then $|c_1| \leq D$. If $\lambda = 1$, we have $c_1 = 0$.
If $\lambda \neq 0$ and $\lambda \neq 1$, then

$$\lambda c_1 t^{\alpha-1} + (1-\lambda)QNu = 0. \tag{4.36}$$

It follows that

$$\lambda c_1 t^{\beta-1} + (1-\lambda) \times \int_0^1 f\big(t, c_1 t^{\alpha-1}, c_1\Gamma(\alpha), ..., \frac{\Gamma(\alpha)}{\Gamma(N-1)} c_1 t^{N-1}\big)dt = 0. \tag{4.37}$$

Then we get

$$\lambda c_1^2 t^{\alpha-1} + (1-\lambda) \int_0^1 c_1 f\big(t, c_1 t^{\alpha-1}, ..., c_1\Gamma(\alpha)\big)dt = 0. \tag{4.38}$$

which, together with (H_3), implies $|c_1| \leq D$. Here, Ω_3 is bounded. ■

Remark 4.2.1

If the other parts of (H_3) hold,
then the set $\Omega_3' = \{u \in Ker(L) : -\lambda u + (1-\lambda)QNu = (0,0), \lambda \in [0,1]\}$ is bounded.

Theorem 4.2.1

Suppose $(H_1) - (H_4)$ hold, then the problem (3) has at least one solution in Y.

Proof.

Let Ω be a bounded open set of Y such that $\bigcup_{i=1}^{3} \overline{\Omega}_i \subset \Omega$. It fillows from Lemmas (4.3.3), (4.3.4) and (4.3.5), N is L-compact on Ω. By Lemma(4.3.3),(4.3.4), and, (4.3.5) we get the following:

(1) $Lu \neq Nu$, for every $u \in [(dom(L) \backslash Ker(L)) \cap \partial\Omega] \times (0,1)$.

(2) $Nu \neq Im(L)$ for every $u \in Ker(L) \cap \partial\Omega$.

(3) Let $H(u, \lambda) = \lambda Iu + (1-\lambda)JQNu$, where I is the identical operator. Via the homotopy property of degree, we obtain that:

$$\begin{aligned} deg(JQN|_{Ker(L)}, \Omega \cap Ker(L), 0) &= deg(H(.,0), \Omega \cap Ker(L), 0) \\ &= deg(H(.,1), \Omega \cap Ker(L), 0) \\ &= deg(I, \Omega \cap Ker(L), 0) = 1 \neq 0. \end{aligned} \tag{4.39}$$

Applying Theorem 1 ,we conclude that $Lu = Nu$ has at least one solution in $dom(L) \cap \overline{\Omega}$. Under the stronger conditions imposed on f, we can prove the uniqueness of solutions to the (3) studied above. ■

Theorem 4.2.2

Suppose the conditions (H_1) in the theorem are replaced by the following conditions.
$(H_1)'$ There exist positive constants $a_i, i = 0, 1, ..., N-1$, such that for all $(x_1, x_2, ..., x_N)$, $(y_1, y_2, ..., y_N) \in \mathbb{R}^N$, one has

$$|f(t, x_1, x_2, ..., x_N) - f(t, y_1, y_2, ..., y_N)| \leq a_0|x_1 - y_1| + ... + a_{N-1}|x_N - y_N|. \tag{4.40}$$

$(H_1)''$ Ther exist constants $l_i, i = 1, 2, ..., N-1$, such that for all $(x_1, x_2, ..., x_N), (y_1, y_2, ..., y_N) \in \mathbb{R}^N$, one has

$$|f(t, x_1, x_2, ..., x_N) - f(t, y_1, y_2, ..., y_N)| \leq -l_0|x_1 - y_1| + l_1|x_2 - y_2| - l_2|x_3 - y_3| - ... - l_{N-1}|x_N - y_N| \tag{4.41}$$

Then the BVP(3) has a uniqe solution, provided that

$$\frac{al_0}{l_1} + aa_0 + a_0c + \sum_{i=2}^{N-1} \frac{al_i}{l_1} + (a+c)\sum_{i=1}^{N-1} a_i < 1. \tag{4.42}$$

Proof.

Let $y_i = 0, i = 1, 2, ..., N$, and $\varphi_1 = |f(t, 0, ..., 0)|$, then the condition (H_1) is satisfied. According to Theorem(4.3.1) , BVP(3) has at least one solution. Suppose $u_i \in Y, i = 1, 2$ are two solutions of (3) then;

$$D^{\alpha}_{0+}u_i(t) = f\big(t, u_i(t), D^{\alpha-1}_{0+}u_i(t), D^{\alpha-2}_{0+}u_i(t), ..., D^{\alpha-(N-1)}_{0+}u_i(t)\big), i = 1, 2. \tag{4.43}$$

Note that $u = u_1 - u_2$, so u satisfy the equation

$$D^{\alpha}_{0+}u = f\big(t, u_1, D^{\alpha-1}_{0+}u_1, ..., D^{\alpha-(N-1)}_{0+}u_1\big) - f\big(t, u_2, D^{\alpha-1}_{0+}u_2, ..., D^{\alpha-(N-1)}_{0+}u_2\big). \tag{4.44}$$

According to $Im(L) = Ker(Q)$, we have

$$\int_0^1 f\big(t, u_1, D^{\alpha-1}_{0+}u_1, ..., D^{\alpha-(N-1)}_{0+}u_1\big) - f\big(t, u_2, D^{\alpha-1}_{0+}u_2, ..., D^{\alpha-(N-1)}_{0+}u_2\big)dt = 0. \tag{4.45}$$

By the integral mean value theorem, there exists $\eta \in [0,1]$, such that

$$f\big(\eta, u_1(\eta), D_{0^+}^{\alpha-1}u_1(\eta), ..., D_{0^+}^{\alpha-(N-1)}u_1(\eta)\big) - f\big(\eta, u_2(\eta), D_{0^+}^{\alpha-1}u_2(\eta), ..., D_{0^+}^{\alpha-(N-1)}u_2(\eta)\big) = 0. \tag{4.46}$$

By $(H_1)''$, we have

$$\begin{aligned} 0 &= \big|f\big(\eta, u_1(\eta), D_{0^+}^{\alpha-1}u_1(\eta), ..., D_{0^+}^{\alpha-(N-1)}u_1(\eta)\big) - f\big(\eta, u_2(\eta), D_{0^+}^{\alpha-1}u_2(\eta), ..., D_{0^+}^{\alpha-(N-1)}u_2(\eta)\big)\big| \\ &\geq -l_0\big|u(\eta)\big| + l_1\big|D_{0^+}^{\alpha-1}u(\eta)\big| - l_2\big|D_{0^+}^{\alpha-2}u(\eta)\big| - ... - l_{N-1}\big|D_{0^+}^{\alpha-(N-1)}u(\eta)\big| \end{aligned} \tag{4.47}$$

We can have

$$\begin{aligned} \big|D_{0^+}^{\alpha-1}u(\eta)\big| &\leq \frac{l_0}{l_1}\big|u(\eta)\big| + \frac{l_2}{l_1}\big|\big|D_{0^+}^{\alpha-2}u(\eta)\big| + ... + \frac{l_{(N-1)}}{l_1}\big|D_{0^+}^{\alpha-(N-1)}u(\eta)\big| \\ &\leq \frac{l_0}{l_1}||u||_\infty + \sum_{i=2}^{N-1}\frac{l_i}{l_1}\big|\big|D_{0^+}^{\alpha-i}u\big|\big|_\infty. \end{aligned} \tag{4.48}$$

Thus, we can obtain

$$\begin{aligned} \big|D_{0^+}^{\alpha-1}u(0)\big| &\leq \big|D_{0^+}^{\alpha-1}u(\eta)\big| + \left|\int_0^\eta D_{0+}^{\alpha}u(s)ds\right| \\ &\leq \big|D_{0^+}^{\alpha-1}u(\eta)\big| + |\eta|\max_{0\leq t\leq\eta}\big|D_{0+}^{\alpha}u(t)\big| \\ &\leq \frac{l_0}{l_1}||u||_\infty + \sum_{i=2}^{N-1}\frac{l_i}{l_1}\big|\big|D_{0^+}^{\alpha-i}u\big|\big|_\infty + \big|\big|D_{0^+}^{\alpha}u(t)\big|\big|_\infty \\ &\leq \frac{l_0}{l_1}||u||_\infty + \sum_{i=2}^{N-1}\frac{l_i}{l_1}\big|\big|D_{0^+}^{\alpha-i}u\big|\big|_\infty + ||Lu||_\infty. \end{aligned} \tag{4.49}$$

According to (23),(36), and (56), we have

$$\begin{aligned} ||u||_X &= ||Pu + (I-P)u||_X \leq ||Pu||_X + ||(I-P)u||_X \\ &= \frac{al_0}{l_1}||u||_X + \sum_{i=2}^{N-1}\frac{al_i}{l_1}\big|\big|D_{0^+}^{\alpha-i}u\big|\big|_\infty + a||Lu||_\infty + c||Lu||_\infty. \\ &\leq (a+c)\bigg(a_0||u||_\infty + \sum_{i=1}^{N-1}a_i\big|\big|D_{0^+}^{\beta-i}u\big|\big|_\infty\bigg). \end{aligned} \tag{4.50}$$

From the definition of $||u||_\infty$ and the assumption (49), we have $||u|| = 0$, so that $u_1 = u_2$. ■

4.3 Example

Let us consider the following boundary value problems:

$$\begin{cases} D_{0^+}^{2,5}u(t) = \frac{t}{5} + \frac{1}{9}D_{0^+}^{1,5}u(t) + \sin^2\big(D_{0^+}^{0,5}u(t)\big) + \arctan u(t), \quad 0 < t < 1, \\ u(0) = D_{0^+}^{0,5}u(0) = 0, \\ D_{0^+}^{1,5}u(0) = D_{0^+}^{1,5}u(1). \end{cases} \tag{4.51}$$

Corresponding to the problem(3), we have that $\alpha = 2.5$ and

$$f(t, x, y, z) = \frac{t}{5} + \arctan x + \frac{1}{9}y + \sin^2(z). \tag{4.52}$$

Moreover,

$$|f(t, x, y, z)| \leq \frac{t}{5} + \frac{\pi}{2} + \frac{1}{9}|y| + 1. \tag{4.53}$$

We can get that the condition (H_1) holds, that is $\varphi = (12 + 5\pi)/10, \psi_1 = \psi_3 = 0$, and $\psi_2 = 1/9$.
Taking $A = 25, D = 19$, we can calculate that $(H_2) - (H_4)$ hold.
Hence, by Theorem(4.3.1) , we obtain that (58) has at least one solution.

Conclusion

Our main goal in this dissertation, is to apply the Mawhin's degree of coincidence for the study of the existence of solution of a boundary value problem for a nonlinear fractional order differential equation with fractional derivative at sense of Caputo on a bounded interval, with boundary conditions.
Using Mawhin's coincidence theorem, we present solution existence results for this boundary value problem. We obtain a results on the existence of solution for the mentioned fractional boundary value problem. This results extends those obtained for ordinary differential equations of integer order. We give an example to illustrate our main results.

Bibliography

[1] V. IS. Banach, On operations in abstract sets and their applications to integral equations, Fundamenta Math., 3 (1922),pp. 133-181.

[2] R. P. Agarwal, M. Meehan , D. OiRegan, Fixed Point Theory and applications, Cambridge Tracts in Mathematics, Cambridge University Press, 141,2001.

[3] O.Kavian. Introduction to the theory of critical points and applications to elliptic problems. Flight. 13. Mathematics Applications. Springer-Verlag, 1993.

[4] Brahim Tellab (2018). Solving Fractional Differential Equations. Doctoral thesis. University of the Mentori Constantinel Brothers.

[5] K. Deimling, Nonlinear Fractional Analysis, Springer, Berlin, 1985.

[6] O. Kavian. Introduction to the theory of critical points and applications to elliptic problems. Flight. 13. Mathematics and Applications. Springer-Verlag,1993.

[7] J. R. Graef, B. Yang , Positive solutions of a third order eigenvalue problem, Dynam. Systems Appl. 15 (2006),97-110.

[8] D. OiRegan, Y. Je Cho and Y.Q. Chen, Topological Degree Theory and Applications, Series in Mathematical Analysis and Applications, vol. 10, Chapman and Hall/CRC,(2006).

[9] R. P. Agarwal, D. OiRegan and D. R. Sahu, Fixed points theory for lipschitzian- type mapping with applications, Vol 6. Cambridge university Press Springer, 2000.

[10] R. P. Agarwal, M. Meehan and D. OiRegan, Fixed point theory and applications, Cambridge Tracts in Mathematics, Vol. 141, Cambridge University Press 2001.

[11] S. Banach, On operations in abstract sets and their application, Fund. Math. 3(1922).

[12] E. Hairer and G. Wanner, Analysis by its history, Springer Verlag, Bererlin, 1997.

[13] R. P. Agarwal, D. OiRegan, and D. R. Sahu, Fixed point theory for lipschitzian- type mapping with applications, Vol 6. Cambridge university Press Springer, 2000.

[14] A. Sirma, S. Sevgin, A Note on coincidence Degree Theory, Int. J. Math. Math. Sc, Volume 2012Article Id 370946, 18 pages.

[15] Tidjani Menacer (2014). Synchronization of Chaotic Dynamical Systems with Fractional Derivatives. Doctoral thesis. Mentouri-Constantinel University.

[16] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematics Studies 204, Ed van Mill, Amsterdam, 2006.

[17] A. Ouhab, Fractional calculation. Mathematics Laboratory, University of Sidi Bel Abbes B. P. 89, 22000 Sidi Bel Abbes, Algeria.

[18] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, Theory and Applications of Fractional Differential Equations, vol. 204 of North-Holland Mathematics Studies, Elsevier, Amsterdam, The Netherlands, 2006.

[19] I. Podlubny, Fractional differential equations, Mathematics in Science and Engineering,vol,198, Academic Press, New York/Londin/Toronto,1999.

[20] S. Q. Zhang, The existence of a positive solution for a nonlinear Fractional differential equation, J. Math. Anal. Appl. 252(2000) , 804-812.

[21] Bai, Z., Lu, H. : Positive solutions for boundary value problem of nonlinear fractional differential equation. J. Math. Comput. 218, 1719-1725(2011).

[22] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Fraction al Differential Equations, John Wiley Sons, 1993.

[23] I. Podlubny, Fractional Differential Equation, Academic Press, San Diego, Calif, USA 1999.

[24] X. Su,"Boundary value problem for a coupled system of nonlinear fractional differential equations," Applied MathematicsLetters, vol. 22, no. 1,pp.64-69,2009.

[25] Z. Hu, W. Liu, and T. Chen, "Two-point boundary value problems for fractionl differential equations at resonance," Bulletin of the Malaysian Mathematical Sciences Society 2, vol.36, no.747 -755,2013.

[26] Z. Hu, W. Liu, and T. Chen, "Existence of solutions for a coupled system of fractional differential equations at resonance ," Boundary Value Problems, vol. 2012, article 98,13 pages, 2012.

[27] N. Xu, W. Liu, and L. Xiao, "The existence of solutions for nonlinear fractional multipoint boundary value problems at resonance," Boundary Value Problems, vol. 2012, article 65,10 pages, 2012.

[28] Z. Bai and Y. Zhang, "Solvability of fractional three-point boundary value problems with nonlinear growth," Applied Mathematics and Computation, vol. 218, no5, pp.1719-1725,2011.

[29] N. Xu, W. Liu, and L. Xiao, "The existence of solutions for nonlinear fractional multipoint boundary value problems at resonance," Boundary Value Problems, vol. 2012, article 65,10 pages, 2012.

[30] W. Jiang, "The existence of solutions to boundary value problems of fractional differential equations at resonance," Nonlinear Analysis: Theory, Methods Applications, vol.74, no. 5,pp. 1987-1994,2011.

[31] J. Mawhin,"Topological degree and boundary value problems for nonlinear differential equations," in Topological Methods for Ordinary Differential Equations, vol. 1537 of Lecture Notes in Mathematics,pp.74-142,Springer,Berlin,Germany,1993.

[32] Z. Bai and Y. Zhang, "Solvability of fractional three-point boundary value problems with nonlinear growth," Applied Mathematics and Computation, vol. 218, no5, pp.1719-1725,2011.

ملخص

في هذه المذكرة قمنا بتحديد نتائج وجود مسالة القيمة الحدية ذات النقطتين للمعادلات التفاضلية الكسرية عند الرنين عن طريق نظرية درجة التطابق .
وبلاضافة على ذلك تم الحصول على نتيجة بشان تفرد الحل. نعطي مثالا لتوضيح نتائجنا.
الكلمات المفتاحية: مشكلة الحدية ذات النقطتين الرنين نظرية درجة التطابق.

Resumé

Dans cette mémoire, nous établissons les résultats d'existence pour un probléme aux limite á deux points d'équations différentielles fractionnaires en résonance au moyen de la théorie du degré de coïncidence. De plus, un résultat sur leunicitéde la solution est obtenu. Nous donnons un exemple pour illustrer nos résultats.
***Mots-clés**: probléme aux limite á deux points, résonance, théorie du degré de coïncidence.*

Abstract

In this dissertation, we establish the existence results for two-point boundary value problem of fractional differential equations at resonance by means of the cioncidence degree theory. Furthermore, a results on the uniqueness of solution is obtained. We give an example to demonstrate our results.
***Keywords**: two-point boundary value problem, resonance, the cioncidence degree theory.*