UNIVERSITY OF KASDI MERBAH - OUARGLA
FACULTY OF MATHEMATICS AND MATTER SCIENCES
DEPARTMENT OF PHYSICS

N° d'ordre:
N° de série:

DOCTORAL THESIS

Speciality: Theoritical Physics

By :

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Quantum mechanics applications with a new type of extended uncertainty principle: Path integral approach

Thesis defended on 23rd June, 2024 before the jury composed of :

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Dedications

To my beloved mother, who meant so much to me and continues to mean so much. Though she is no longer with us, her memory remains deeply dear to me

To my beloved father

To my dear husband

To all my brothers and sisters

To all my family and friends

To every educator who has shaped my academic path

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Acknowledgements

First of all, praise be to Allah, who enabled me to successfully complete this thesis.

I would like to express my sincere gratitude to my thesis advisor, Prof. ***Chohra Thouria****, for her support and guidance throughout my journey to complete this doctoral project. Additionally, I extend my acknowledgements to Prof.* ***Boudjedaa Tahar*** *for his tremendous help and advice along this journey.*

I extend my acknowledgements and gratitude to the discussion committee members, each by name, starting with the committee president, Prof. ***Bentouila Omar****. My thanks also extend to Prof.* ***Douis Said****, Prof.* ***Khodja Lamin****, Dr.* ***Ferkous Noureddine****, and Dr.* ***Korichi Zineb****, each of whom served as discussants.*

I would like to express my gratitude to everyone who has guided and supported me in my career over the past years, whether near or far.

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Abstract

In the context of the non-relativistic Snyder-de Sitter model, this research aims to investigate the relativistic oscillators and non-relativistic quantum mechanics in momentum space using the path integral formalism. This examination is demonstrated in three main parts. In the first part, we formulated the D-dimensional momentum space path integral transition amplitude for both harmonic oscillators and free particles. Through the application of quantum correction rules, we derived the exact energy spectrum and the normalized radial momentum space eigenfunctions, while also investigating limiting cases for small parameter values. In the second part, we examined the relativistic Green function employing the same algebraic model, under the influence of a homogeneous electric field, for a Dirac oscillator particle with mass m and charge e. This analysis is followed by computing the propagator function and determining the associated spectral energies. Additionally, we examine the thermodynamic properties of an electron gas at high temperatures across four deformation parameter sets, analyzing their impact and deducing limiting cases for small parameter values. In the last part, we applied the path integral formalism to the Green function equation of a $(1+2)-$dimensional Dirac oscillator under a homogeneous magnetic field, utilizing the modified Snyder algebra by *S. Mignemi* (2012). Across radial coordinates transformation, we computed the Green function and electron propagator, extracting exact bound states and their associated spectral energies. Our investigation unveiled that when $m\bar{\omega} \rightarrow m\omega_c/2$ and $c \rightarrow V_F$, the behavior of the Dirac oscillator system under a uniform magnetic field within the Snyder-de Sitter algebra mirrors the dynamics of the monolayer Graphene problem within the same algebraic framework. Moreover, we derived the thermodynamic properties of the electron gas at elevated temperatures across four deformation parameter cases, assessing their influence and deducing limiting behaviors for small parameters.

Keywords: Propagator, Green function, Dirac oscillator equation, Snyder-de Sitter model, Homogeneous electric field, Homogeneous magnetic field, thermos dynamic properties.

Résumé

Dans le contexte du modèle de Snyder-de Sitter non relativiste, cette recherche vise à étudier les oscillateurs relativistes et la mécanique quantique non relativiste dans l'espace des impulsions en utilisant le formalisme du l'intégrale de chemin. Cette étude est présentée en trois parties principales. Dans la première partie, nous avons formulé l'amplitude de transition du l'intégrale de chemin de l'espace des impulsions D-dimensionnelles pour les oscillateurs harmoniques et les particules libres. En appliquant les règles de correction quantique, nous avons dérivé le spectre d'énergie exact et les fonctions propres normalisées de l'espace des impulsions radiales, tout en examinant les cas limites pour de petites valeurs de paramètres. Dans la deuxième partie, nous avons examiné la fonction de Green relativiste en utilisant le même modèle algébrique, sous l'influence d'un champ électrique homogène, pour une particule oscillateur de Dirac avec une masse m et une charge e. Cette analyse est suivie par le calcul de la fonction de propagateur et la détermination des énergies spectrales associées. De plus, nous examinons les propriétés thermodynamiques d'un gaz d'électrons à haute température à travers quatre ensembles de paramètres de déformation, en analysant leur impact et en déduisant des cas limites pour de petits paramètres. Dans la dernière partie, nous avons appliqué le formalisme du l'intégrale de chemin à l'équation de la fonction de Green d'un oscillateur de Dirac (1+2) dimensionnel sous un champ magnétique homogène, en utilisant l'algèbre de Snyder modifiée par *S. Mignemi* (2012). À travers la transformation des coordonnées radiales, nous avons calculé la fonction de Green et le propagateur d'électrons, extrayant les états liés exacts et leurs énergies spectrales associées. Notre étude a révélé que lorsque $m\bar{\omega} \to m\omega_c/2$ et $c \to V_F$, le comportement du système d'oscillateur de Dirac sous un champ magnétique uniforme dans l'algèbre de Snyder-de Sitter reflète la dynamique du problème du Graphène monocouche dans le même cadre algébrique. De plus, nous avons dérivé les propriétés thermodynamiques du gaz d'électrons à des températures élevées à travers quatre cas de paramètres de déformation, évaluant leur influence et déduisant des comportements limites pour de petits paramètres.

Mots clés: Propagateur, Fonction de Green, Équation de l'oscillateur de Dirac, Modèle de Snyder-de Sitter, Champ électrique homogène, Champ magnétique homogène, Propriétés thermodynamiques.

ملخـــــص

في سياق نموذج سنايدر (Snyder) غير النسبي، يهدف هذا البحث إلى دراسة المذبذبات النسبية وميكانيكا الكم غير النسبية في فضاء الزخم باستخدام تكامل المسار لفاينمان، وقد تمت هذه الدراسة في ثلاثة أجزاء رئيسية. في الجزء الأول، قمنا بصياغة سعة الانتقال لتكامل مسار الزخم الفضائي ذو البعد D لكل من المذبذبات التوافقية والجسيمات الحرة. من خلال تطبيق قواعد التصحيح الكمي، قمنا باشتقاق طيف الطاقة الدقيق والوظائف الذاتية لمساحة الزخم الشعاعي الطبيعي، مع التحقيق أيضًا في الحالات المحدودة لقيم المعلمات الصغيرة. في الجزء الثاني، قمنا بدراسة الدالة النسبية جرين (Green) باستخدام نفس النموذج الجبري، تحت تأثير مجال كهربائي متجانس، لجسيم مذبذب ديراك ذو الكتلة m والشحنة e. ويتبع هذا التحليل حساب منتشر فاينمان وتحديد الطاقات الطيفية المرتبطة بها. بالإضافة إلى ذلك، قمنا بفحص الخصائص الديناميكية الحرارية لغاز الإلكترون عند درجات حرارة عالية عبر أربع مجموعات من معلمات التشوه، وتحليل تأثيرها واستنباط الحالات المحددة لقيم المعلمات الصغيرة. في الجزء الأخير، قمنا بتطبيق تكامل المسار لفاينمان على معادلة دالة جرين (Green) لمذبذب ديراك(Dirac) ثنائي الأبعاد (2+1) تحت مجال مغناطيسي متجانس، وذلك باستخدام جبر سنايدر (Snyder) المعدّل بواسطة *(2012)* S. Mignemi. عن طريق تحويل الإحداثيات الشعاعية، قمنا بحساب دالة جرين (Green) ومنتشر الإلكترون، مع استخراج الحالات المحددة الدقيقة والطاقات الطيفية المرتبطة بها. وقد كشف تحقيقنا أنه عندما يكون $m\overline{\omega} \rightarrow m\omega_c/2$ و $c \rightarrow V_F$ ، فإن سلوك نظام مذبذب ديراك (Dirac) تحت مجال مغناطيسي موحد داخل جبر سنايدر-دي سيتر(Snyder de-Sitter) يعكس ديناميكيات مشكلة الجرافين (Graphene) أحادي الطبقة ضمن نفس الإطار الجبري. علاوة على ذلك، قمنا باشتقاق الخصائص الديناميكية الحرارية لغاز الإلكترون عند درجات حرارة مرتفعة عبر أربع حالات لمعلمات التشوه، وتقييم تأثيرها واستنباط السلوكيات المقيدة للمعلمات الصغيرة.

الكلمات المفتاحية: المنتشر، دالة Green، معادلة مذبذب ديراك (Dirac) ، نقطة تمييز، نموذج سنايدر-دي سيتر(Snyder de-Sitter)، مجال كهربائي متجانس، مجال مغناطيسي متجانس، خصائص ديناميكية حرارية.

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Chapter 1

General Introduction

In the late 19^{th} century, the main focus of physics revolved around classical Newtonian mechanics to explain the dynamics of material bodies, along with Maxwell's electromagnetic theory to described radiation in the form of electromagnetic waves. Thermodynamics was also effectively utilized to study interactions between matter and radiation. Building upon these accomplishments, physicists believed they had attained a comprehensive understanding of nature, grounded in the principle of determinism. However, as the 20^{th} century dawned, classical physics faced major challenges and complexities, especially in light of its inability to explain phenomena at the microscopic level. This shortcoming has become clear with the emergence of modern techniques for examining atomic and subatomic structures, such as studying atomic and molecular structures and the interactions of light with them. Phenomena like the emission of radiation by a black body, the occurrence of the photoelectric effect, and the stability of atoms have played crucial roles in motivating the exploration of quantum mechanics. In 1900, *Planck* presented a precise explanation of black body radiation (the concept of the quantum of energy), which subsequently initiated a cascade of new discoveries, leading to solutions for some of the most significant problems of that era. These developments include Compton's confirmation in 1923 of the photoelectric effect and the scattering of X-ray photons, *Bohr*'s demonstration in 1913 of atomic stability in his model of the hydrogen atom, and *Einstein*'s solution in 1905 to the photoelectric effect problem. which have the characteristics of particles with momentum $h\nu/c$ where ν is the frequency of the X-rays. Up until 1925, all these discoveries were collectively referred to as "*old quantum mechanics*". Only when *Heisenberg*, *Dirac*, and *Schrödinger* established the accurate mathematical framework governing these microphysical phenomena, which is rooted in the principle of probability. Historically, three independent formulations of quantum mechanics have emerged. The first, known as matrix mechanics or relies "matrix algebra", was developed by *Heisenberg-Dirac* in 1925. *Schrödinger* introduced wave mechanics as the second theory, which is an extension of *de Broglie*'s hypothesis. This formulation, which is more straightforward compared to matrix mechanics, explains the behaviour of tiny particles through a wave equation called the Schrödinger equation, instead of the eigenvalue matrix employed by *Heisenberg*. In 1948, *Feynman* introduced his well-known path integral representation of the kernel of the Schrödinger equation in his renowned paper published in Reviews of Modern Physics. This method merged the concepts of probability and determinism. As is well known, the transition from classical to quantitative description primarily depends on the Hamiltonian function, On the other hand, the quantum mechanics formulation essentially ignored the Lagrangian formulation. In 1933, *Paul. A. M. Dirac* was the first to highlight the potential importance of the Lagrangian in quantum mechanics in a paper (referred to as Paper Dirac). Building on *Dirac*'s insights, *Feynman* developed what he called the Lagrangian formulation of quantum mechanics. *Feynman* initiated his exploration of the classical action $S(x(t))$ and linked it to the principles of quantum mechanics probabilities. In this connection, he illustrated that the probability amplitude of a particle following a specific path or trajectory $x(t)$ is linked to the exponential factor $\exp\left[\frac{i}{\hbar}S\left(x\left(t\right)\right)\right]$. The path integral method provides an effective and adaptable framework for studying various physical systems and phenomena. This approach has become an essential tool in theoretical physics, encompassing quantum mechanics, quantum field theory, statistical physics, condensed matter physics, cosmology, and black hole physics.

After *Feynman* successfully established his approach based on *Heisenberg*'s uncertainty principle in standard quantum mechanics $\triangle x_i \triangle p_j \geq \frac{\hbar}{2}\delta_{ij}$ [1–3], the practical applications of this approach continued in both relativistic and non-relativistic cases [4–9]. Subsequently, this approach experienced a broadening of its applications to systems that stem from the generalisation of Heisenberg's principle. For instance, in the domain of quantum gravity [10, 11], the behaviour of systems deformed quadratic algebra is used to model the dynamics of systems with variable masses in semiconductor heterostructures [12], the description of the low energy excitations of Graphene, and the Fermi velocity. This results in the momentum commutator being proportional to pseudo-spin [13] ... etc.

Our research attempts to cover the recent developments in the field of path integrals within these kind of deformed algebras, especially the context of quantum gravity and string theory

[14–17]. In 1947, *Snyder* [18] presented his model to address the divergences that arise in Quantum Field Theory (QFT) when discretizing spacetime. Snyder's model can be understood as a form of Doubly Special Relativity (DSR) that includes an additional universal constant alongside c, the speed of light in a vacuum

$$\begin{aligned}\left[\hat{J}_{\mu\nu},\hat{X}_{\sigma}\right] &= i\hbar\left(\eta_{\mu\sigma}\hat{X}_{\nu}-\eta_{\nu\sigma}\hat{X}_{\mu}\right),\left[\hat{J}_{\mu\nu},\hat{P}_{\sigma}\right]=i\hbar\left(\eta_{\mu\sigma}\hat{P}_{\nu}-\eta_{\nu\sigma}\hat{P}_{\mu}\right),\\ \left[\hat{X}_{\mu},\hat{P}_{\nu}\right] &= i\hbar\left(\eta_{\mu\nu}+\beta\hat{P}_{\mu}\hat{P}_{\nu}\right),\left[\hat{X}_{\mu},\hat{X}_{\nu}\right]=i\hbar\beta\hat{J}_{\mu\nu};\quad \left[\hat{P}_{\mu},\hat{P}_{\nu}\right]=0.\end{aligned} \tag{1.1}$$

With $\eta_{\mu\nu}=(-1,1,1,1)$. The coupling constant denoted by β is approximately equal to the Planck length and has dimensions$[\beta]=[momentum]^{-2}$. The operators $\hat{J}_{\mu\nu}=\hat{X}_{\mu}\hat{P}_{\nu}-\hat{P}_{\nu}\hat{X}_{\mu}$ serve as the generators that maintain the Lorentz symmetry. In addition, the generalisation of this model to spacetimes with uniform curvature involves the introduction of a novel fundamental constant that is directly proportional to the cosmological constant. This modified model is characterised by three unchanging scales: the velocity of light in a vacuum c, a mass β, and a length α. It is known as Triply Special Relativity (TSR) or the Snyder de Sitter (SdS) model [20–24].

$$\begin{aligned}\left[\hat{J}_{\mu\nu},\hat{X}_{\sigma}\right] &= i\hbar\left(\eta_{\mu\sigma}\hat{X}_{\nu}-\eta_{\nu\sigma}\hat{X}_{\mu}\right),\left[\hat{J}_{\mu\nu},\hat{P}_{\sigma}\right]=i\hbar\left(\eta_{\mu\sigma}\hat{P}_{\nu}-\eta_{\nu\sigma}\hat{P}_{\mu}\right),\\ \left[\hat{X}_{\mu},\hat{P}_{\nu}\right] &= i\hbar\left(\eta_{\mu\nu}+\alpha\hat{X}_{\mu}\hat{X}_{\nu}+\beta\hat{P}_{\mu}\hat{P}_{\nu}+\sqrt{\alpha\beta}\left(\hat{P}_{\mu}\hat{X}_{\nu}+\hat{X}_{\nu}\hat{P}_{\mu}-\hat{J}_{\mu\nu}\right)\right),\\ \left[\hat{X}_{\mu},\hat{X}_{\nu}\right] &= i\hbar\beta\hat{J}_{\mu\nu};\quad \left[\hat{P}_{\mu},\hat{P}_{\nu}\right]=i\hbar\alpha\hat{J}_{\mu\nu}.\end{aligned} \tag{1.2}$$

To the best of our knowledge, only a limited studies have used the path integral approach to investigate the characteristics of SdS space in relativistic and nonrelativistic quantum mechanical systems.

The primary objective of this thesis is to employ the path integral approach within the framework of Snyder model. It aims to explore the behavior of both relativistic and non-relativistic particles with spin $1/2$ travelling in a homogeneous magnetic and electric field within momentum space representations.

Structured into four chapters, the thesis begins with an introduction and ends with a conclusion. In chapter two, we provide an overview of the path integral approach in quantum mechanical systems without deformation. Chapter three, formulates the D-dimensional momentum space path integral transition amplitude for the harmonic oscillator and free particle, deriving exact expressions for energy spectrum and relative wave functions. Chapter four, focusing on the employment of the path integral approach to derive the relativistic Green function for a (1+1)-Dirac oscillator system under a uniform electric field within the SdS model. It computes the propagator function, relevant spectral energies, and thermodynamic characteristics of a single electron at high temperatures. In Chapter five, we analyze the behavior of the (1+2)-dimensional Dirac oscillator system in a uniform magnetic field within the SdS algebra model, demonstrating its similarity to monolayer Graphene dynamics. It derives precise bound states, energy eigenvalues, and plots thermodynamic functions for the system. The final chapter presents a summary of findings and overall conclusions drawn from the study.

Chapter 2

Concept of Path Integral Formalism in Standard Quantum Mechanics

2.1 Historic backgrounds

In the general introduction to this thesis, we have outlined the historical evolution of the mathematical frameworks governing quantum physics. These frameworks are primarily based on two fundamentally distinct formulations: *Schrödinger*'s differential equation and *Heisenberg*'s matrix mechanics, introduced between 1925 and 1926. Quantum mechanics builds upon the Hamiltonian formulation of classical mechanics, where the quantum Hamiltonian operator $\hat{H}(\hat{x},\hat{p})$ is derived from the classical Hamiltonian $H(x,p)$ by simply replacing $p \to \hat{p} = -i\hbar\partial/\partial x$. This concept may be illustrated schematically as follows: [3]: While the

	Classical Mechanics	Quantum Mechanics
1. Variables:	x, p (c-numbers), $\{x,p\} = 1$	$\hat{x}$, $\hat{p}$ (operators), $[\hat{x},\hat{p}] = 1$
2. Hamiltonian:	$H(x, p)$	$\hat{H}(\hat{x}, \hat{p})$
3. Dynamical Law:	$df(x,p)/dt = \{f,H\}$	a) Heisenberg Eq : $i\hbar\frac{df}{dt} = [\hat{f},\hat{H}]$
	H. J. equation ?	b) Schrödinger Eq : $i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$
		$\hat{H} = H(x, -i\hbar\partial/\partial x)$
4: Lagrangian:	$L(x, \dot{x})$	?

Figure 2.1: Classical description versus quantum description

formulation of quantum mechanics is not directly based on the Lagrangian function (see Table 2.1), implying the impossibility of precisely measuring a quantum particle's position and momentum simultaneously, it was from this conceptual framework that the third mathematical formulation of quantum mechanics emerged, devised by *Richard Feynman* in 1948. On the other hand, the probability of an event is represented by the square of the modulus of the amplitude. In other words, for an event such as a particle leaving point (a) at time (t_a) and arriving at point (b) at time (t_b) with $(t_b \geq t_a)$, the event's probability is expressed as the square of the amplitude. The probability amplitude associated with this event is denoted as $K(a,b)$, the probability of event is given by [25]

$$P(b,a) = |K(a,b)|^2 . \tag{2.1}$$

Physically, from slit experiment, in quantum mechanics, there are multiple ways for an event to happen. The probability amplitude of the event is the sum of the probability amplitudes corresponding to each way.

$$K(a,b) = \sum_i K_i(a,b) . \tag{2.2}$$

Moreover, if the particle travels from point (a) to (b) via point (c) at time (t_c), where $(t_a \leq t_c \leq t_b)$, the probability amplitude of transitioning from (a) to (c) and then from (c) to (b) (Intermediate Principle) is defined as

$$K(a,b) = K(a,c)\,K(c,b) . \tag{2.3}$$

Unlike classical particle, which follow specific paths (classical paths) from point (a) to point (b), quantum particle traverse all possible paths between these two points. Therefore, according to the principle of superposition, the probability amplitude of such an event can be expressed as

$$K(a,b) = \sum_{\substack{\text{All the path} \\ \text{possible}}} \Phi(x(t)) . \tag{2.4}$$

Where $\Phi(x(t))$ is the probability amplitude of path $x(t)$ linking (a) and (b).

Let us postulate, additionally, that the contribution of each path acquires a phase proportional to the action $S(x(t))$, corresponding to the path $x(t)$

$$\Phi(x(t)) = C \exp\left[\frac{i}{\hbar} S(x(t))\right] , \tag{2.5}$$

where

$$S(x(t)) = \int_{t_a}^{t_b} L(x,\dot{x},t)\,dt . \tag{2.6}$$

Here, C represents the normalization constant of the amplitude, appropriately chosen. However, the selection of $\Phi(x(t))$ and $S(x(t))$ is not arbitrary. It results from an analogy between quantum systems and their classical correspondents, particularly when $\hbar \to 0$. It is easy to see that $S(x(t))$ defined by equation (2.6) becomes the classical action and the probability amplitude takes the classical form $K(a,b) = C\exp\left[\frac{i}{\hbar}S_{cl}\right]$ where $S_{cl} = S(\bar{x})$ is the classic action,with $(\bar{x})$ denoting the classical path of the particle.

To sum over all paths, we initially divide the time interval $(t_b - t_a)$ into N intervals, where $t_j - t_{j-1} = \varepsilon$, $t_b - t_a = N\varepsilon$. Consequently, each path can be discretely defined by a sequence of points$(x_0 = a, x_1 .... x_{N-1}, x_N = b)$, which, as $\varepsilon \to 0$, converges to the continuous path $x(t)$. Thus, the amplitude of the path $x(t)$ becomes a function of this sequence of points and is denoted as $\phi(x_0, ..., x_N)$. In the limit as $\varepsilon \to 0$, it essentially depends on the continuous path $x(t)$.

Now, applying the principle given by the equation (2.3), it follows that

$$\phi(x_1, ..., x_n) = \prod_{j=1}^{N} \phi(x_j, x_{j-1}), \tag{2.7}$$

where $\phi(x_j, x_{j-1})$ represents the probability amplitude of the part of the path bounded by the points x_j and x_{j-1}, which we will determine subsequently. The sum over all paths is then obtained by integrating over all points $(x_0, \ x_1, ... x_{N-1}, \ x_N)$, with the initial and final points (a) and (b) being fixed. This yields the following rough expression for the amplitude:

$$K(a,b) \simeq \int \prod_{j=1}^{N-1} dx_j \prod_{j=1}^{N} \phi(x_j, x_{j-1}). \tag{2.8}$$

By taking the limit $\varepsilon \to 0$ of this expression, we include all points along the paths, resulting in the correct expression:

$$K(x_b, t_b; x_a, t_a) = \lim_{\varepsilon \to 0} \int \prod_{j=1}^{N-1} dx_j \prod_{j=1}^{N} \phi(x_j, x_{j-1}), \tag{2.9}$$

where

$$\phi(x_j, x_{j-1}) = \frac{1}{A} \exp\left[\frac{i}{\hbar} S(x_j, x_{j-1})\right], \tag{2.10}$$

(A) represents the amplitude normalization constant, and S denotes the action between the instants (t_j) and (t_{j-1}). Furthermore, we require S to be a classical action.

$$S\left(x_j, x_{j-1}\right) = \int_{t_{j-1}}^{t_j} L(x, \dot{x}, t)\, dt, \tag{2.11}$$

and finally we obtain

$$K(x_b, t_b; x_a, t_a) = \lim_{\varepsilon \to 0} \int \prod_{j=1}^{N-1} dx_j \prod_{j=1}^{N} \left[\frac{1}{A} \exp\left[\frac{i}{\hbar} S\left(x_j, x_{j-1}\right)\right]\right]. \tag{2.12}$$

So the Eq. (2.12) is written as

$$K(x_b, t_b; x_a, t_a) = \int_{(x_a, t_a)}^{(x_b, t_b)} D[x(t)] \exp\left[\frac{i}{\hbar} \int_{t_a}^{t_b} L(x, \dot{x}, t)\, dt\right]. \tag{2.13}$$

The expression for S defined by equation (2.11) is challenging to compute for arbitrary ε. However, an approximation can be made on $S\left(x_j, x_{j-1}\right)$ by retaining only the first order in ε. This approximation arises from the fact that errors of order higher than ε. i.e $\varepsilon^{1+\eta>1}$ in $S\left(x_j, x_{j-1}\right)$, will not contribute significantly to the calculation of expression (2.12). These errors accumulate into an error ε that vanishes as $\varepsilon \to 0$. In the case of a quadratic Lagrangian in $(\dot{x})$, a good approximation of $S\left(x_j, x_{j-1}\right)$ is given by

$$S\left(x_j, x_{j-1}\right) = \varepsilon \mathcal{L}\left(\frac{x_j + x_{j-1}}{2}, \frac{x_j - x_{j-1}}{\varepsilon}, \frac{t_j + t_{j-1}}{2}\right), \tag{2.14}$$

here, we already observe the emergence of that we term mid-point principle.

For a Lagrangian independent of time and not containing a linear term in $(\dot{x})$, expression (4.14) can be simplified to

$$S\left(x_j, x_{j-1}\right) = \varepsilon \mathcal{L}\left(x_j, \frac{x_j - x_{j-1}}{\varepsilon}\right). \tag{2.15}$$

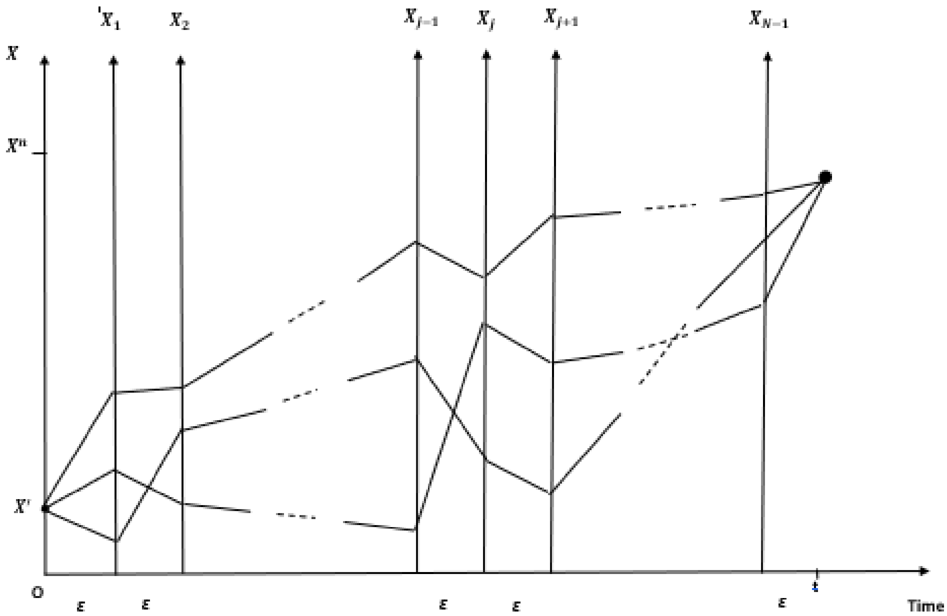


Figure 2.2: In between kicks by potential the system moves a very short time

For instance, given a particle subjected to the scalar potential $V(x)$ we have:

$$S\left(x_j,x_{j-1}\right)=\frac{m}{2}\left(\frac{\left(x_j-x_{j-1}\right)^2}{\varepsilon}\right)-\varepsilon V\left(x_j\right). \tag{2.16}$$

Note that from Eq. (2.9) an interesting property of the amplitude follows:

$$K\left(x_b,t_b;x_a,t_a\right)=\int\prod_{j=1}^{N-1}dx_c K\left(x_b,t_b;x_c,t_c\right)K\left(x_c,t_c;x_a,t_a\right), \tag{2.17}$$

with $t_a<t_c<t_b$, indicating that events occur sequentially in time. Let us select an instant (t_k) from the subdivision $t_0,t_1...,t_k...t_N$, positioned between (t_0) and (t_N) but sufficiently distant from them so that the durations t_k-t_0 and t_N-t_k are measurable. In other words, $t_0....,t_k....,t_{k+l}$, where (k) and (l) both tend to infinity. Then, we write:

$$\begin{aligned}K\left(x_b,t_b;x_a,t_a\right) &= \lim_{\substack{\varepsilon\to0\\(k,l\to\infty)}}\int\prod_{j=1}^{k-1}dx_j\prod_{j=1}^{k}\phi\left(x_j,x_{j-1}\right)dx_k\\ &\quad \prod_{j=k+1}^{k+l-1}dx_j\prod_{j=k+1}^{k+l}\phi\left(x_j,x_{j-1}\right),\end{aligned} \tag{2.18}$$

or

$$K\left(x_b,t_b;x_a,t_a\right)=\int dx_c K\left(x_b,t_b;x_c,t_c\right)K\left(x_c,t_c;x_a,t_a\right), \tag{2.19}$$

where $t_k = t_c$, this relation remains valid only for systems having an action which verifies the locality relation

$$S(b,a) = S(b,c) + S(c,b). \tag{2.20}$$

2.2 Wave Function

Previously, we established that $K(x_b,t_b;x_a,t_b)$ represents the probability amplitude for the particle to transition from (x_a) at time (t_a) to (x_b) at time (t_b), where (x_a,t_a) and (x_a,t_a) denote, respectively, the past and future states of the event. Disregarding its past (x_a,t_a), given the knowledge that it previously existed somewhere, this probability amplitude can then signify the likelihood of presence at point (x_b) at time (t_b), commonly referred to as the particle's wave function.

$$K(x_b,t_b,.,.) = \psi(x_b,t_b). \tag{2.21}$$

The dots signify that the past is irrelevant. With this definition, it becomes evident that this wave function satisfies the following integral equation:

$$\psi(x,t) = \int_{-\infty}^{+\infty} K(x,t;x_0,t_0)\,\psi(x_0,t_0)\,dx_0, \tag{2.22}$$

where $K(x,t;x_0,t_0)$ is the amplitude of probability of going from (x_0,t_0) to (x,t) often called propagator of the particle, $K(x,t,.,.) = \psi(x,t)$ and $K(x_0,t_0,.,.) = \psi(x_0,t_0)$, the dots designate a past tense before (x_0,t_0). Equation (2.22) implies that given the wave function at time (t_0) it is possible to determine its future at time $(t \succ t_0)$, all its past prior to (t_0) enclosed in the initial wave function $\psi(x_0,t_0)$. Taking the limit as $(t \to t_0)$ in equation (2.22) it comes

$$\psi(x,t) = \int_{-\infty}^{+\infty} \lim_{t\to t_0} K(x,t;x_0,t_0)\,\psi(x_0,t_0)\,dx_0, \tag{2.23}$$

which shows that $K(x,t;x_0,t_0)$ satisfies the following property called "normalization condition".

$$\lim_{t\to t_0} K(x,t;x_0,t_0) = \delta(x-x_0). \tag{2.24}$$

2.3 Schrödinger Equation

We will examine the case of a particle subject to the scalar potential $V(x)$ using expression (2.24) for $S(x_j, x_{j-1})$. Conversely, for the case of a vector potential, the incorporation of the midpoint principle becomes essential. Considering a duration $t - t_0 = \varepsilon \to 0$, equation (2.24) represents the propagator expression, which we can express as

$$\psi(x, t+\varepsilon) = \int \frac{1}{A} \exp\left[\frac{i}{\hbar}\left[\frac{m}{2}\frac{(x-x_0)^2}{\varepsilon} - \varepsilon V(x)\right]\right] \psi(x_0, t_0)\, dx_0, \tag{2.25}$$

setting $x - x_0 = \eta$, $t_0 = t$, becomes

$$\psi(x, t+\varepsilon) = \int d\eta \frac{1}{A} \exp\left[\frac{i}{\hbar}\left[\frac{m}{2}\frac{\eta^2}{\varepsilon}\right] \exp\left[\frac{-i\varepsilon V(x)}{\hbar}\right]\right] \psi(x-\eta, t). \tag{2.26}$$

The first exponential varies quickly, perhaps jump to the neighborhood of $\eta \sim 0$ so that η is of the order of $\sqrt{\varepsilon}$. Knowing that the other actors are slitally variable with η (continuity of the wave function $\psi(x(t))$), it follows that the only paths that contribute to the path integral are for which one $\triangle x_j \simeq \sqrt{\varepsilon}$ which shows the Brownian character of quantum motion (velocity discontinuity $\frac{\triangle x_j}{\varepsilon}$). Let us expand $\psi(x-\eta, t)$ in a Taylor series to order 2 in η therefore to order (i) in ε then let us integrate on η and let us expand $\psi(x, t+\varepsilon)$ and $\exp\left[\frac{i\varepsilon V(x)}{\hbar}\right]$ to order i in ε, we easily deduce the Schrödinger equation for $\psi(x,t)$, after having identified (A) to $A = \sqrt{\frac{2\pi i\hbar\varepsilon}{m}}$

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi, \tag{2.27}$$

or $\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(x)$ and $\hat{p} = -i\hbar\frac{\partial}{\partial x}$.

A generalization to the time-dependent potential $V(x,t)$ is possible, giving a result analogous to equation (2.27). Except perhaps we should use discretization $\frac{t_j + t_{j-1}}{2}$ for the time axis.

Let us now show that the propagator can be considered as a Green function of the Schrödinger equation. It was defined for (t), as a wave function at the point $(t_b \succ t_a)$ then we can say that it satisfied the Schrödinger equation (2.27).

$$\left[i\hbar\frac{\partial}{\partial t_b} - H_b\right] K = 0 \qquad \text{pour } t_b \succ t_a.$$

Furthermore, let us impose the condition $K(x_b,t_b;x_a,t_a)=0$, $t_b \prec t_a$ which expresses the fact that the propagator does not propagate the wave functions towards the past, which is consistent with a non-relativistic theory. It is then appropriate to pose:

$$\bar{K}(x_b,t_b;x_a,t_a)=\theta(t_b-t_a)K(x_b,t_b;x_a,t_a), \tag{2.28}$$

and easily shows, using the property (2.24) that $\bar{K}$ satisfies the following Schrödinger equation

$$\left(i\hbar\frac{\partial}{\partial t_b}-H\right)\bar{K}=i\hbar\delta(t_b-t_a)\,\delta(x_b-x_a). \tag{2.29}$$

For a time-independent Hamiltonian, the solution wave functions of the Schrödinger equation have the simple form $\exp\left(-\frac{itE}{\hbar}\right)\phi(x)$, where the $\phi(x)$ satisfy the eigenvalue equation $H\phi=E\phi$ and thus constitute a closed orthogonal system for the Hilbert space "space of wave functions". Let us then expand the wave function on this basis and compare the expression obtained with equation (2.24), we will obtain the following property

$$K(x_b,t_b;x_a,t_a)=\left\{\begin{array}{ll}\sum\!\!\!\!\int\phi(x_b)\,\phi^*(x_a)\exp\left[-\frac{iE}{\hbar}(t_b-t_a)\right] & \text{pour }\ t_b\succ t_a\\ 0 & \text{pour }\ t_b\prec t_a\end{array}\right\}, \tag{2.30}$$

which expresses the development of the propagator as a wave function, the symbol $\sum\!\!\!\!\int$ designates a summation over the discrete and continuous states. The developments carried out so far are valid for one-dimensional systems.

2.4 Path Integral Formalism in Phase Space (Trotter's Formula)

Follows the well-known canonical steps of Trotter's formula, and in non-relativistic quantum mechanics, the construction of the phase space path integral representation of the transition amplitude for standard quantum systems is:

$$K(x_b,t_b;x_a;t_a)=\left\langle x_b\,|\,\hat{U}(t_b,t_a)\,|\,x_a\right\rangle,$$

with $\hat{U}(t_b,t_a) = \exp\left(-\frac{iT}{\hbar}\hat{H}(\hat{x},\hat{p})\right)$ is the evolution operator and $T = (t_b - t_a)$. Formally, let's divide the time interval $[t_b,t_a]$ in $(N+1)$ intervals equal to $\varepsilon = \frac{T}{N+1}$, and we note that we can write this expenential as

$$K(x_b,t_b;x_a;t_a) = \left\langle x_b \left| \left\{ \exp\left[\frac{-i\varepsilon}{\hbar}\hat{H}(\hat{x},\hat{p})\right]\right\}^{N+1} \right| x_a \right\rangle. \tag{2.31}$$

Now let's insert the following closure relation based on the coordinates $\int |x_j\rangle\langle x_j| dx_j = 1$ by N times between even $\exp\left[\frac{-i\varepsilon}{\hbar}\hat{H}(\hat{x},\hat{p})\right]$, with the standard form of the Hamiltonian ($\hat{H}(\hat{x},\hat{p}) = \frac{\hat{p}^2}{2m} + V(\hat{x})$). Then Eq. (2.31) becomes as

$$K(x_b,t_b;x_a;t_a) = \lim_{N\to\infty}\prod_{j=1}^{N}\left[\int dx_j\right]\prod_{j=1}^{N+1} K\left(x_j,t_j;x_{j-1};t_{j-1}\right). \tag{2.32}$$

Where the infinitesimal transition amplitude is defined by

$$K\left(x_j,t_j;x_{j-1};t_{j-1}\right) = \left\langle x_j \left| \exp\left[\frac{-i\varepsilon}{\hbar}\hat{H}(\hat{x},\hat{p})\right] \right| x_{j-1} \right\rangle. \tag{2.33}$$

As we know, since $\varepsilon \ll 1$ when $N \gg 1$, following Trotter formula we have

$$\exp\left\{-\frac{i\varepsilon}{\hbar}\left[\frac{\hat{p}^2}{2m} + V(\hat{x})\right]\right\} = \exp\left[-\frac{i\varepsilon}{\hbar}\frac{\hat{p}^2}{2m}\right]\exp\left[-\frac{i\varepsilon}{\hbar}V(\hat{x})\right]. \tag{2.34}$$

For a second time, let's insert the closure relation based on the momentum $\int |p_j\rangle\langle p_j| dp_j = 1$, where we record each p_j as a pulse corresponding to each of the time periods, and we find obtain

$$K\left(x_j,t_j;x_{j-1};t_{j-1}\right) = \int dp_j \langle x_j| \exp\left[-\frac{i\varepsilon}{\hbar}\frac{\hat{p}^2}{2m}\right]|p_j\rangle\langle p_j| \exp\left(-\frac{i}{\hbar}V(\hat{x})\varepsilon\right)|x_{j-1}\rangle.$$

Furthermore, we have the actions of the following operators

$$\hat{p}|p_j\rangle = p_j|p_j\rangle, \text{ and } \hat{x}|x_{j-1}\rangle = x_{j-1}|x_{j-1}\rangle, \tag{2.35}$$

more general

$$e^{-\frac{i\varepsilon}{\hbar}\frac{\hat{p}^2}{2m}}\left|p_j\right\rangle = e^{-\frac{i\varepsilon}{\hbar}\frac{p_j^2}{2m}}\left|p_j\right\rangle, \text{ and } e^{-\frac{i\varepsilon}{\hbar}V(\hat{x})}\left|x_{j-1}\right\rangle = e^{-\frac{i\varepsilon}{\hbar}V(x_{j-1})}\left|x_{j-1}\right\rangle. \tag{2.36}$$

The propagator can be expressed like this:

$$K\left(x_j,t_j;x_{j-1};t_{j-1}\right) = \int dp_j e^{-\frac{i\varepsilon}{\hbar}\frac{p_j^2}{2m}} e^{-\frac{i\varepsilon}{\hbar}V(x_{j-1})}\left\langle x_j\left|p_j\right\rangle\left\langle p_j\right|x_{j-1}\right\rangle. \tag{2.37}$$

As we know in standard quantum mechanics $\langle x_j | p_j\rangle$ represents the plane wave

$$\langle x_j | p_j\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}p_j x_j}, \ \langle p_j | x_{j-1}\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{-\frac{i}{\hbar}p_j x_{j-1}}. \tag{2.38}$$

We will then write

$$K\left(x_j,t_j;x_{j-1};t_{j-1}\right) = \int dp_j e^{-\frac{i\varepsilon}{\hbar}\frac{p_j^2}{2m}} e^{-\frac{i\varepsilon}{\hbar}V(x_{j-1})} e^{\frac{i}{\hbar}p_j\left(x_j - x_{j-1}\right)}. \tag{2.39}$$

Substituting the equality of Eq. (2.39) into Eq. (2.32) The path integral representation of the transition amplitude for a particle in the potential $V(x)$ is expressed by

$$K(x_b,t_b;x_a;t_a) = \lim_{N\to\infty}\int\prod_{j=1}^{N}dx_j\prod_{j=1}^{N+1}\int\frac{dp_j}{2\pi\hbar}\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{j=1}^{N}\left[p_j\left(\frac{x_j - x_{j-1}}{\varepsilon}\right) - \frac{p_j^2}{2m} - V(x_{j-1})\right]\right\}. \tag{2.40}$$

Note that the integrations with respect to p_j are Gaussian and can be readily performed

$$\int\frac{dp_j}{2\pi\hbar}\exp\left[-\frac{i}{\hbar}\frac{p_j^2}{2m}\varepsilon + \frac{i}{\hbar}p_j\left(x_j - x_{j-1}\right)\right] = \sqrt{\frac{m}{2i\pi\hbar\varepsilon}}\exp\left[\frac{i}{\hbar}\frac{m}{2\varepsilon}\left(x_j - x_{j-1}\right)^2\right], \tag{2.41}$$

then

$$K(x_b,t_b;x_a;t_a) = \lim_{N\to\infty}\prod_{j=1}^{N}\left[\int dx_j\right]\left(\frac{m}{2i\pi\hbar\varepsilon}\right)^{(N+1)/2}\exp\left[\frac{i}{\hbar}\sum_{j=1}^{N+1}\left(\frac{m}{2\varepsilon}\left(x_j - x_{j-1}\right)^2 - \varepsilon V(x_{j-1})\right)\right]. \tag{2.42}$$

Note that at the limit $N \to \infty$ where the time interval ε tends to zero, the exponential in the integral seen as Rienmman integral is proportional to the classical Lagrangian of the system

$$\sum_{j=1}^{N+1}\left(\frac{m}{2\varepsilon}(x_j - x_{j-1})^2 - V(x_{j-1})\varepsilon\right) = \sum_{j=1}^{N+1}\varepsilon\left[\left(\frac{m}{2}\left(\frac{x_j - x_{j-1}}{\varepsilon}\right)^2 - V(x_{j-1})\right)\right]$$
$$= \int_{ta}^{t_b}\left(\frac{m}{2}\dot{x}_j^2 - V(x_j)\right)dt = \int_{ta}^{t_b} L(x_j,\dot{x}_j,t_j)\,dt, \tag{2.43}$$

where $\dot{x}_j = \lim_{\varepsilon\to o}\frac{x_j - x_{j-1}}{\varepsilon}$ is the speed of the particle at time t_j. Finally, we have the path integral formulation for the propagator $K(x_b,t_b;x_a;t_a)$ given by Feynman and which we will write in the following continuous form:

$$K(x_b,t_b;x_a;t_a) = \int_{(x_a,t_a)}^{(x_b,t_b)} D[x(t)]\exp\left[\frac{i}{\hbar}S[x(t)]\right], \tag{2.44}$$

$S[x(t)]$ is called classical action

$$S[x(t)] = \int_{ta}^{t_b} L(x_j,\dot{x}_j,t_j)\,dt. \tag{2.45}$$

The Feynman measure is denoted by

$$D[x(t)] = \lim_{N\to\infty}\prod_{j=1}^{N}[dx_j]\left(\frac{m}{2i\pi\hbar\varepsilon}\right)^{\frac{N+1}{2}}. \tag{2.46}$$

2.5 Green Function

In non-relativistic quantum mechanics the propagation takes place towards the future (non-relativistic causality) we then define the propagation by the Green function in following time

$$G(t_b,t_a) = \Theta(t_b - t_a)\exp\left(-\frac{i}{\hbar}(t_b - t_a)\hat{H}\right), \tag{2.47}$$

where $\Theta(t_b - t_a)$ is the Heaviside function ensures this causality.

The matrix element between states $|x_a\rangle$ and $|x_b\rangle$ is then written as

$$\begin{aligned} G(x_b,x_a;t_b,t_a) &= \langle x_b| G(t_b,t_a) |x_a\rangle = \Theta(t_b-t_a) \langle x_b| \exp\left(-\frac{i}{\hbar}(t_b-t_a)\hat{H}\right) |x_a\rangle \\ &= \Theta(t_b-t_a) K(x_b,t_b;x_a,t_a). \end{aligned} \tag{2.48}$$

In this case, $G(x_b,x_a;t_b,t_a)$ is a solution of the equation

$$\left[i\hbar\frac{\partial}{\partial t_b}+\hat{H}(x_b)\right] G(x_b,x_a;t_b,t_a) = -i\hbar\delta(x_b-x_a)\,\delta(t_b-t_a). \tag{2.49}$$

By introducing the Fourier transform of this Green function in time, we obtain the Green function in energy defined by

$$G(x_b,x_a;E) = \frac{1}{i\hbar}\int dT e^{\frac{i}{\hbar}E(t_b-t_a)} G(x_b,x_a;t_b,t_a). \tag{2.50}$$

For $\hat{H}$ independent of time, this Green function satisfies

$$\left(E-\hat{H}\right) G(x_b,x_a;E) = \delta(x_b-x_a), \tag{2.51}$$

$G(x_b,x_a;E)$ is the matrix element of an operator $\hat{G}(E)$

$$G(x_b,x_a;E) = \langle x_b| \hat{G}(E) |x_a\rangle, \tag{2.52}$$

and where formally we will write

$$\hat{G}(E) = \frac{1}{\left(E-\hat{H}\right)}, \tag{2.53}$$

and as

$$K(x_b,t_b;x_a,t_a) = \sum_n e^{\frac{i}{\hbar}E(t_b-t_a)} \varphi_n(x_b)\,\varphi_n^*(x_a), \tag{2.54}$$

then this energy-dependent Green function will be written as

$$\hat{G}(E) = \sum_n \frac{\varphi_n(x_b)\,\varphi_n^*(x_a)}{\left(E-\hat{H}\right)}. \tag{2.55}$$

2.6 The Path Integral in D-dimensional Polar Coordinates

The Feynman propagator of a particle of mass m moving in $D-$dimensional Euclidean space in a scalar potential $V(\mathbf{x})$ is given by

$$K(\mathbf{x}_b, \mathbf{x}_a; T) = \int D[\mathbf{x}(t)] \exp\left[\frac{i}{\hbar}\int_0^T \left(\frac{m}{2}\left(\vec{\dot{x}}^2\right) - V(\mathbf{x})\right) dt\right].$$

In discrete form, it is defined by:

$$K(\mathbf{x}_b, \mathbf{x}_a; T) = \lim_{N\to\infty}\left(\frac{m}{2i\pi\hbar\varepsilon}\right)^{(N+1)\frac{d}{2}} \int \prod_{j=1}^{N} d\mathbf{x}_j \exp\left[\frac{i}{\hbar}\sum_{j=1}^{N+1} S(j, j-1)\right], \tag{2.56}$$

with

$$S(j, j-1) = \frac{m}{2\varepsilon}\triangle\mathbf{x}_j^2 - \varepsilon V(\mathbf{x}_j), \tag{2.57}$$

where $\varepsilon = t_j - t_{j-1}$ and $\triangle u_j = u_j - u_{j-1}$, $(u = x_1, ..., x_d)$ are respectively the elementary time interval and the interval position. Let's go to polar coordinates:

$$x_i = r\prod_{k=1}^{i-1} \sin\theta_k \cos\theta_j \quad, j = 1, 2, ..., d-1, x_d = r\prod_{k=1}^{d-1} \sin\theta_k \sin\phi, \tag{2.58}$$

for convenience, we have set $\theta_0 = \pi/2$, $\theta_{d-1} = \phi$ and

$$0 \le \theta_k \le \pi; \ (k = 1, 2..., d-2), \quad 0 \le \phi \le 2\pi.$$

$$r = \left(\sum_{k=1}^{d} x_k^2\right)^{1/2}. \tag{2.59}$$

Following the usual polar decomposition, the propagator will be written in the polar representation as

$$K(r_b, r_a; T) = \lim_{N\to\infty}\left(\frac{m}{2i\pi\hbar\varepsilon}\right)^{\frac{N+1}{2}d} \int \prod_{j=1}^{N} r_j^{d-1} dr_j d\Omega_j \exp\left[\frac{i}{\hbar}\sum_{j=1}^{N+1} S(j, j-1)\right], \tag{2.60}$$

with

$$d\Omega_j = \prod_{\nu=1}^{d-1}\left(\sin\theta_j^{\nu}\right)^{d-1-\nu} d\theta_j^{\nu}, \tag{2.61}$$

which can also be put as follows

$$S(j,j-1)=\frac{m}{2\varepsilon}\left[r_j^2+r_{j-1}^2-2r_jr_{j-1}\cos\psi_{j,j-1}\right]-\varepsilon V\left(\bar{r}_j\right). \tag{2.62}$$

with

$$\cos\psi_{j,j-1}=\sum_{\nu=0}^{d-2}\cos\theta_{\nu+1}^{(j)}\cos\theta_{\nu+1}^{(j-1)}\prod_{\mu=0}^{\nu}\sin\theta_{\mu}^{(j)}\sin\theta_{\mu}^{(j-1)}+\prod_{\nu=1}^{d--1}\sin\theta_{\nu}^{(j)}\sin\theta_{\nu}^{(j-1)}. \tag{2.63}$$

We separate the radial part from the angular part by utilizing an alternate formula.

$$e^{z\cos\psi}=\left(\frac{z}{2}\right)^{-\nu}\Gamma(\nu)\sum_{l=0}^{\infty}(l+\nu)\ I_{l+\nu}(z)\ C_l^{\nu}(\cos\psi), \tag{2.64}$$

where $C_l^{\nu}$are Gegenbauer polynomals. In our case $\nu=(d-2)/2$. for $d=2$. On the other hand for $d=3$, $\nu=1/2$, $C_l^{1/2}(\cos\psi)=P_l(\cos\psi)$ (Legender polynomal) and Eq.(2.64) reduces to another familiar formula

$$e^{z\cos\psi}=\sqrt{\left(\frac{\pi}{2z}\right)}\sum_{l=0}^{\infty}(2l+1)\ I_{l+1/2}(z)\ P_l(\cos\psi). \tag{2.65}$$

Now if $\psi_{j,j-1}$ is the angle between two D-dimensional unit vectors $\vec{\Omega}^{(j-1)}$ and $\vec{\Omega}^{(j)}$ the following addition theorem applies

$$\sum_{\mu=1}^{M}S_l^{\mu}\left(\vec{\Omega}^{(j-1)}\right)S_l^{\mu}\left(\vec{\Omega}^{(j)}\right)=\frac{\Gamma(d/2)}{2\pi^{d/2}}\frac{(2l+d-2)}{(d-2)}C_l^{(d-2)/2}\left(\cos\psi_{j,j-1}\right), \tag{2.66}$$

where $S_l^{\mu}\left(\vec{\Omega}\right)$ are the real hyperspherical harmonic of degree 1 associated with unit vector $\vec{\Omega}, l=0,1,2,....,\infty$ while $\mu=1,2,.....,M$, with

$$M=\frac{(2l+d-2)(l+d-3)}{l!(d-2)!}. \tag{2.67}$$

The function $S_l^{\mu}\left(\vec{\Omega}\right)$ satisfy theorthonormality condition

$$\int d\Omega S_l^{\mu}(\Omega)\,S_{l'}^{\mu'}(\Omega)=\delta_{ll'},\delta_{\mu\mu'}. \tag{2.68}$$

For $d=3$, the formula (2.66) reduces to

$$\sum_{m=l}^{l} Y_{lm}^{*}\left(\overrightarrow{\Omega}^{(j-1)}\right) Y_{lm}\left(\overrightarrow{\Omega}^{(j)}\right) = \frac{2l+1}{4\pi} P_1\left(\cos\psi_{j,j-1}\right), \tag{2.69}$$

where $Y_{lm}\left(\overrightarrow{\Omega}\right)$ are the usual spherical harmonic. Using Eq. (2.64) and Eq. (2.66) in the path integral and performing the angular integrations with the help of the relation (2.68), we obtain

$$K^{(d)} = \frac{\Gamma(d/2)}{2\pi^{d/2}} \sum_{l=0}^{\infty} \frac{(2l+d-2)}{(d-2)} C_l^{(d-2)/2}\left(\cos\psi_{0,N}\right) K_l^{(d)}\left(r'',r';T\right), \tag{2.70}$$

for which the path integral defines the radial propagator.

$$K_l = \left(r'r''\right)^{(d-1)/2} \lim_{N\to\infty}\left(\frac{m}{i\hbar\varepsilon}\right)^{N/2} \int_0^{\infty} \prod_{j=1}^{N-1} \int dr_j \prod_{j=1}^{N} R_{j;j-1}, \tag{2.71}$$

where

$$R_{j;j-1} = \left(\frac{mr_j r_{j-1}}{i\hbar\varepsilon}\right)^{1/2} I_{l+(d-2)/2}\left(\frac{mr_j r_{j-1}}{i\hbar\varepsilon}\right) \exp\left\{\frac{i}{\hbar}\left[\frac{m}{2\varepsilon}\left(r_j^2 + r_{j-1}^2\right) - \varepsilon V(r_j)\right]\right\}. \tag{2.72}$$

The integrations over r_j can be readily performed according to the analytical expression for the potential function.

Chapter 3

Path Integral Approach to The D-dimensional Quantum Mechanics of The Non-Relativistic Snyder-de Sitter Model

3.1 Introduction

In physics, the theory of deformation often arises when considering systems in which the usual algebraic rules, such as commutativity, are not obeyed, through the introduction of parameters. For example, in quantum mechanics, operators representing physical observables like position and momentum may not commute with each other, leading to noncommutative algebraic structures. As it allows for a more general description of physical systems, it has applications in various areas of physics, including quantum field theory, string theory, and condensed matter physics. Over the past decades, noncommutativity in spacetime has garnered increasing interest. Historically, the Snyder model was the first attempt to study quantum spacetime, introducing a minimum measurable length [18]. This concept of fundamental length was predicted across all approaches related to quantum gravity on a Planck scale, leading to appearance of the Generalized Uncertainty Principle (GUP). Additionally, the Extended Uncertainty Principle (EUP) naturally arose from the preservation of local momentum symmetry or derived alternatively from the (anti)-de Sitter spacetime geometry.

In the latter context, the SdS algebra is defined as a nonlinear extension of the Poincaré algebra and also represents a generalization of the proposed minimal length uncertainty relation [10, 11]. It is the first proposal of quantum spacetime relying on an algebra constructed by spacetime coordinates and Lorentz generators, thus warranting special attention for its impact on quantum systems. In recent times, the study of quantum mechanical models within the SdS framework has gained significant attention. This model is regarded as an example of Doubly Special Relativity (DSR)[20], featuring a fundamental constant β. Depending on the positive or negative values of parameter β, it is also referred to as the Snyder or anti-Snyder model, respectively.

In light of this, the SdS model was created by extending the Snyder algebra to a curved de Sitter background, characterized by a positively curved spacetime that corresponds to the accelerated expansion of the universe. It can also be considered an example of Triply Special Relativity (TSR) or Yang's model [20]. In addition to the speed of light, the resulting model includes two more essential deformation parameters: the Planck energy and the Sitter ray, which are related to the cosmological constant. Its linked algebra is defined by the following commutation relationship [26–32],

$$\begin{aligned} \left[x_i, p_j\right] &= i\hbar\left(\delta_{ij} + \alpha x_i x_j + \beta p_i p_j + \sqrt{\alpha\beta}\left(p_i x_j + x_j p_i\right)\right), \\ \left[x_i, x_j\right] &= i\hbar\beta\varepsilon_{ijk}L_k \text{ and } \left[p_i, p_j\right] = i\hbar\alpha\varepsilon_{ijk}L_k, \end{aligned} \tag{3.1}$$

where L_k are components of angular momentum operator.

As a consequence, this algebra has captured particular interest in research and extensive studies of deformed physical models. In this regard, notable examples include the classical and quantum mechanics of a free particle and the harmonic oscillator [27], the two-dimensional relativistic Bosonic oscillator equation moving in a uniform magnetic field [33], the three-dimensional Dirac oscillator [32], and the exact solutions of the $(1+1)$-dimensional relativistic Klein-Gordon and Dirac equations with linear vector and scalar potentials [34].

Furthermore, the introduction of these deformed algebras into the path integral framework is crucial because the diffusion amplitudes in the ultraviolet regime are naturally regularised by this deformation. Additionally, it provides some insights into the regularization and renormalization of perturbed quantum field theory and statistical partition function, where the deformation parameters being the cut off of the theory. Thanks to the path integral approach, some problems have found solutions with a single deformation parameter. We mention some examples: the one-dimensional propagator for the DO [35], the one dimension relativistic spinning particle with vector and scalar linear potentials [36] the two dimensions relativistic DO [37], the one dimension harmonic oscillator [38], the Coulomb potential [39], the Klein Gordon particle [40], the D-dimensional harmonic oscillator [41] and the kernel for a free particle by [42]. Nevertheless, despite its successful outcomes, the Feynman approach still requires refinement as a quantification tool. This is particularly evident in cases involving deformation or constraints, where one does not know a priori how to select the discretization procedure. For instance, it has been found that the use of the mid-point prescription technique is privileged to be consistent with the direct method in the context of quantification with constraints, also known as the Faddeev-Senjanovic formulation [43–47]. Similarly, when studying the dynamics of quantum particles represented by deformed algebras using the path integral and only one deformation parameter, the discretization problem still exists [35–37]. In the following, the discretization problem will reappear, but this time in the presence of two deformation parameters. The primary aim of this chapter is to establish a path integral formulation for the SdS algebra in D-dimensional momentum space with two deformation parameters and investigate both the free particle and harmonic oscillator cases. As previously mentioned, for the discretization problem, the difficulty is identifying the most suitable quantum fluctuations associated with it. To transform the action and measure to the usual ones defined in the standard path integral. We employ the general form of the δ-point discretization method. It is worth noting that the overall correction depends on this δ-point discretization, and in this case, choosing the mid-point the discretization does not yield satisfactory and consistent results for computing the quantum corrections, as seen in the standard case [3]. In Section 2, we provide a concise overview of the quantum mechanics associated with the deformed SdS model. Section 3 outlines the construction of the path integral formalism within this deformed algebra framework. Subsequently, we compute the transition amplitude for both the free particle and the Harmonic oscillator in D-dimensional momentum space. To achieve this, we employ spherical coordinate transformation and relative angular decomposition [48], facilitating the conversion of the radial part to that of the Pöschel–Teller potential. This enables us to derive exact expressions for the energy spectrum and relative wave functions of the problem. Finally, in Section 4, we summarize our findings.

3.2 Quantum Mechanics with Generalized Snyder Model

According to [27], the generalized Snyder model's Heisenberg commutation relation in one dimension is as follows:

$$[\hat{x},\hat{p}] = i\hbar\left(1+\alpha^2\hat{x}^2+\beta^2\hat{p}^2+\alpha\beta\left(\hat{x}\hat{p}+\hat{p}\hat{x}\right)\right), \tag{3.2}$$

α and β denote a small positive parameters of deformation. The following Heisenberg uncertainty relation can be obtained directly from the above equation

$$\Delta x\Delta p \geq \frac{\hbar}{2}\frac{\left|1+\alpha^2\left(\Delta x\right)^2+\beta^2\left(\Delta p\right)^2\right|}{1+\hbar\alpha\beta}. \tag{3.3}$$

The above-mentioned relation (3.3) results in a non zero minimum length in position and momentum uncertainties

$$\Delta x_{\min} = \frac{\hbar\beta}{\sqrt{1+2\hbar\alpha\beta}}, \quad \Delta p_{\min} = \frac{\hbar\alpha}{\sqrt{1+2\hbar\alpha\beta}}. \tag{3.4}$$

According to *Mignemi* in [27] and *Stetsko* in [32], in D-dimensions, the connection between this deformed algebra and the Snyder algebra has been established, from which the representation of these position and momentum operators follows the SdS Heisenberg algebra, where

$$\hat{x}_i = \bar{x}_i + \tfrac{\beta}{\alpha}\lambda\,\bar{p}_i, \qquad \hat{p}_i = (1-\lambda)\,\bar{p}_i - \tfrac{\alpha}{\beta}\bar{x}_i, \tag{3.5}$$

to obtain the Hamiltonian symmetric, we can select a free parameter λ in each case. The following commutation relations are satisfied by the pair of operators $(\bar{x}_i,\bar{p}_i)$ [27].

$$\left[\bar{x}_i,\bar{p}_j\right] = i\hbar\left(1+\beta^2\bar{p}_i^2\right), \quad \left[\bar{x}_i,\bar{x}_j\right] = \beta^2\left(\bar{x}_i\bar{p}_j-\bar{x}_j\bar{p}_i\right), \quad \left[\bar{p}_i,\bar{p}_j\right] = 0. \tag{3.6}$$

Hence, it is feasible to redefine the expressions for these position and momentum coordinate operators to match to the Snyder-Heisenberg brackets (3.6) using auxiliary operators $\hat{X}$ and $\hat{P}$, which adhere to the standard commutation relations, as described in [27].

$$\hat{x}_i = \sqrt{1-\beta^2\hat{\mathbf{P}}^2}\hat{X}_i + \frac{\lambda\beta}{\alpha}\frac{\hat{P}_i}{\sqrt{1-\beta^2\hat{\mathbf{P}}_i^2}},\ \hat{p}_i = -\frac{\alpha}{\beta}\sqrt{1-\beta^2\hat{\mathbf{P}}^2}\hat{X}_i + (1-\lambda)\frac{\hat{P}_i}{\sqrt{1-\beta^2\hat{\mathbf{P}}^2}}. \tag{3.7}$$

Within the framework of the anti-SdS (aSdS) algebra model, all real values of P are permissible. However, if α^2, $\beta^2 > 0$, the permissible range of $\hat{P}$ is limited by $P^2 < 1/\beta^2$. The operators of $\hat{x}$ and $\hat{p}$ are symmetric only within the subspace $L^2(R^2, d\mathbf{P}/\sqrt{1-\beta^2\mathbf{P}^2})$, where the scalar product is defined as follows

$$\langle\psi \mid \phi\rangle = \int_{-1/\beta}^{1/\beta} \frac{d\mathbf{P}}{\sqrt{1-\beta^2\mathbf{P}^2}}\psi^*(P)\phi(P), \tag{3.8}$$

the wave function satisfies the periodic boundary conditions, with $\psi(-1/\beta) = \psi(1/\beta)$. This results in the following closure relation

$$\int_{-1/\beta}^{1/\beta} \frac{d\mathbf{P}}{\sqrt{1-\beta^2\mathbf{P}^2}} |\mathbf{P}\rangle\langle\mathbf{P}| = 1. \tag{3.9}$$

We define, following the approach of [10, 11], the projection relation for the free case as follow

$$\left\langle P|P^{'}\right\rangle = \left(\frac{1-\beta^2\mathbf{P}'^2}{1-\beta^2\mathbf{P}^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta^2\mathbf{P}^2}\,\delta\left(\mathbf{P}-\mathbf{P}'\right), \text{and } \gamma = i(1-\lambda)/2\hbar\alpha\beta, \tag{3.10}$$

on the other hand, regarding the harmonic oscillator potential, we have

$$\left\langle P|P^{'}\right\rangle = \left(\frac{1-\beta^2\mathbf{P}'^2}{1-\beta^2\mathbf{P}^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta^2\mathbf{P}^2}\,\delta\left(\mathbf{P}-\mathbf{P}'\right), \text{and } \gamma = i\lambda/2\hbar\alpha\beta. \tag{3.11}$$

When $\alpha^2, \beta^2 < 0$, it results in maximum momentum but no minimum positional uncertainty, this is known as the SdS algebra representation. In this instance, we alter the integration limits across the space in the equation above.

3.3 The Schrödinger Picture of the NR Snyder de-Sitter Model

As evident from this chapter, there are only a few cases where exact solvability is achievable. Next, we will proceed to calculating the solutions dynamics of a free particle and the harmonic oscillator potential, both in one and D dimensions.

3.3.1 Free Particle in 1D

Initially, we consider the Schrödinger equation for a free particle in one dimension for unit mass, as follows

$$\frac{\partial^2\psi}{\partial P^2}-\left(\beta-\frac{2i}{\hbar\alpha}\right)\frac{\beta P}{1-\beta^2P^2}\frac{d\psi}{dP}-\frac{\beta^2}{\hbar^2\alpha^2}\left[\frac{P^2-i\hbar\alpha/\beta}{(1-\beta^2P^2)^2}-\frac{2E}{1-\beta^2P^2}\right]\psi=0. \tag{3.12}$$

For SdS case, there are solutions of (3.12) that vanish at $P=\pm1/\beta$. These solutions have the following form:

$$\psi=const\times\left(1-\beta^2P^2\right)^{\frac{i}{2\hbar\alpha\beta}}\cos\left[\frac{\sqrt{2E}}{\hbar\alpha}\arcsin\beta P\right], \tag{3.13}$$

assuming an odd integer n in $E=\frac{\hbar^2\alpha^2n^2}{2}$, the values of $\triangle x$ in these solutions are finite. Additionally, for an SdS, the energy is not quantized, rather, the momentum eigenfunctions provide the pertinent solutions.

3.3.2 Harmonic Oscillator in 1D

We now consider the one-dimensional quantum harmonic oscillator, where the Hamiltonian is given below:

$$\hat{H}=\frac{\hat{p}^2}{2m}+\frac{m\omega_0^2\hat{x}^2}{2}, \tag{3.14}$$

to simplify the computations, we exploit the flexibility to choose the coefficient λ in the representation 3.6 such that the cross terms $\hat{\mathfrak{P}}\hat{\mathfrak{X}}+\hat{\mathfrak{X}}\hat{\mathfrak{P}}$ in the Hamiltonian vanish, setting

$$\lambda=\frac{\alpha^2}{\beta^2\omega_0^2+\alpha^2}, \tag{3.15}$$

given this selection, the Schrödinger equation takes the following form:

$$\frac{1}{2}\frac{\beta^2\omega_0^2}{\beta^2\omega_0^2+\alpha^2}\left[\hat{\mathcal{P}}^2+\frac{\left(\beta^2\omega_0^2+\alpha^2\right)^2}{\beta^4\omega_0^2}\hat{\mathcal{X}}^2\right]\psi=E\psi. \tag{3.16}$$

Using the realization (3.6) of the operators, (3.16) can be written as

$$\frac{\partial^2\psi}{\partial P^2}-\frac{\beta^2 P}{1-\beta^2P^2}\frac{d\psi}{dP}-\frac{1}{\hbar^2\omega^2}\left[\frac{P^2}{(1-\beta^2P^2)^2}-\frac{2\mathcal{E}}{1-\beta^2P^2}\right]\psi=0, \tag{3.17}$$

where $\omega=\left(1+\frac{\alpha^2}{\beta^2\omega_{2_0}}\right)\omega_0$ and $\mathcal{E}=\left(1+\frac{\alpha^2}{\beta^2\omega_{2_0}}\right)E$.

Specifically, when $\left(\alpha^2,\beta^2\right)\succ 0$, the equation takes a form similar to that of the flat Snyder model, albeit with distinct coefficients, and can be solved through the same methodology, from (3.17) we derive the standard Schrödinger equation for a potential by definig a variable $\bar{P}=\arcsin\beta P$ is defined

$$V=\frac{1}{\omega^2}\tan^2\bar{P}. \tag{3.18}$$

To obtain the explicit solution for (3.17), it is more practical to define the variable $z=(1+\beta P)/2$, which allows the equation to be expressed in the hypergeometric form

$$\frac{\partial^2\psi}{\partial z^2}+\frac{z-1/2}{z(z-1)}\frac{d\psi}{dz}-\left[\frac{\mu\,(z-1/2)^2}{z^2\,(z-1)^2}+\frac{\varepsilon}{z\,(z-1)}\right]\psi=0, \tag{3.19}$$

with

$$\mu=\frac{\omega_0^2}{\hbar^2\left(\beta^2\omega_0^2+\alpha^2\right)^2}\quad,\quad\varepsilon=\frac{2E}{\hbar^2\left(\beta^2\omega_0^2+\alpha^2\right)}. \tag{3.20}$$

Subsequently, the hypergeometric function $\mathbf{F}(a,b,c;z)$ can yield the solution through standard methods,

$$\psi=const\times\left(1-\beta^2P^2\right)^{(1+\sqrt{1+4\mu}/4)}\mathbf{F}(a,b,c;\frac{1+\beta P}{2}), \tag{3.21}$$

where

$$a=\frac{1}{2}\left(1+\sqrt{1+4\mu}\right)-\sqrt{\mu+\varepsilon},b=\frac{1}{2}\left(1+\sqrt{1+4\mu}\right)+\sqrt{\mu+\varepsilon},c=1+\frac{1}{2}\sqrt{1+4\mu}. \tag{3.22}$$

It is necessary for ψ to disappear at $P = \pm 1/\beta$, i.e.at $z = 0, 1$. This happens in the cases where either $a = -n$ or $b = -n$,

$$\varepsilon = \left(n + \frac{1}{2}\right)\left(1 + \sqrt{1+4\mu}\right) + n^2. \tag{3.23}$$

Additionally, Gegenbauer polynomials C_n^α can be used to express the solution as follow

$$\psi = const \times \left(1 - \beta^2 P^2\right)^{\alpha/2} C_n^\alpha\left(\beta P\right), \tag{3.24}$$

with $\alpha = \frac{1}{2}\left(1 + \sqrt{1+4\mu}\right)$. The energy spectrum is followed from (3.23),

$$E = \left(n + \frac{1}{2}\right)\hbar\omega_0\sqrt{1 + \frac{\hbar^2\left(\beta^2\omega_0^2 + \alpha^2\right)^2}{4\omega_0^2}} + \left(n^2 + n + \frac{1}{2}\right)\frac{\hbar^2\left(\beta^2\omega_0^2 + \alpha^2\right)}{2}. \tag{3.25}$$

This shows corrections of order $\hbar\left(\beta^2\omega_0^2 + \alpha^2/\omega_0\right)$ regarding the standard case and a duality for $\beta^2\omega_0^2 \leftrightarrow \alpha^2/\omega_0$.

The results for a flat Snyder space are recovered in the limit $\lambda \to 0$, while the energy spectrum on a 3-sphere is obtained in the case of $\beta \to 0$,

$$E = \left(n + \frac{1}{2}\right)\hbar\omega_0\sqrt{1 + \frac{\hbar^2\alpha^4}{4\omega_0^2}} + \left(n^2 + n + \frac{1}{2}\right)\frac{\hbar\alpha^2}{2}, \tag{3.26}$$

wherein ω_0 is independent of the energy shift with respect to the standard oscillator at first order.

When $\alpha^2, \beta^2 < 0$, the calculation can be done the same way. The energy spectrum is obtained by analytically continuing (3.25) for negative values of α^2 and β^2,In this case, the energy becomes negative for large n. Therefore, an upper bound on the permitted values of n must be applied in order to guarantee the positivity of energy.

3.3.3 The Free Particle in 3D

In the P_i-space, the representation (3.6) is adapted to three-dimensional spherical coordinates. First, we provide a definition for the operators

$$\hat{\mathcal{P}}_r = \mathcal{P}_r := \sqrt{\mathcal{P}_i^2} = \sqrt{\frac{P_i^2}{1-\beta^2 P_i^2}}, \quad \hat{\mathcal{X}}_r = \sqrt{1-\beta^2 P_r^2}\left(i\frac{\partial}{\partial P_r} + \frac{1}{P_r}\right), \tag{3.27}$$

with $[\hat{\mathcal{X}}_r, \hat{\mathcal{P}}_r] = i$. Following the rules of ordinary quantum mechanics, it follows that

$$\hat{\mathcal{P}}_i^2 = \hat{\mathcal{P}}_r^2 \ , \ \hat{\mathcal{X}}_i^2 = \hat{\mathcal{X}}_r^2 + \frac{\hat{L}^2}{\mathcal{P}_r^2} \ , \ \hat{\mathcal{X}}_i \hat{\mathcal{P}}_i + \hat{\mathcal{P}}_i \hat{\mathcal{X}}_i = \hat{\mathcal{X}}_r \hat{\mathcal{P}}_r + \hat{\mathcal{P}}_r \hat{\mathcal{X}}_r, \tag{3.28}$$

where $\hat{L}^2$ is the square of the angular momentum operator. Afterwards, the square of the momentum and position operators (3.6) can be expressed in terms of the radial operators like

$$\hat{x}_i^2 = \left(\hat{\mathcal{X}}_r + \frac{\beta}{\alpha}\lambda \hat{\mathcal{P}}_r\right)^2 + \frac{\hat{L}^2}{\mathcal{P}_r^2} \ , \ \hat{p}_i^2 = \left((1-\lambda)\hat{\mathcal{P}}_r - \frac{\alpha}{\beta}\hat{\mathcal{X}}_r\right)^2 + \frac{\alpha^{é}}{\beta^2}\frac{\hat{L}^2}{\mathcal{P}_r^2}, \tag{3.29}$$

and

$$\psi\left(P_r, P_\theta, P_\phi\right) = \sum_{l,m} \psi_{rlm}(P_r) Y_{lm}\left(P_\theta, P_\phi\right), \tag{3.30}$$

and only the radial functions need to be investigated in detail. Then, we will delete the (lm) indices in the radial functions.

In the space of the radial functions, the scalar product can be expressed as

$$(\psi_r, \phi_r) = \int_0^{1/\beta} \frac{P_r^2 dP_r}{\sqrt{1-\beta^2 p_r^2}} \psi_r^*(P_r)\, \phi_r(P_r). \tag{3.31}$$

The spectrum of the radial momentum and position operators is similar to that of the equivalent one-dimensional operators, with the exception that P_r can only take positive values. As a result, rather than delving deeper into it, we will move on to our examination of the Schrödinger equation. The framework utilized in this section additionally enables the prompt to discriminate single out the states that minimise the uncertainty relations between the position coordinates in various orientations.

$$\triangle x_i \triangle x_j \geq \left| \frac{\beta^2}{2} \langle \hat{J}_{ij} \rangle \right|, \tag{3.32}$$

it is also easy to extend the previous discussion to include the anti-Snyder model. Moreover, vanishing angular momentum states are those that minimise these uncertainty relations, and the momentum components are subject to the same considerations.

We will examine at the Schrödinger equation for a free particle in three dimensions. The radial part of the equation can also be expressed by choosing the gauge $\lambda = 0$, using the representation in equations (3.27), (3.29), and expanding the Eq. (3.30) in spherical harmonics as demonstrated below:

$$\frac{d^2\psi_r}{dP_r^2} - \frac{\left(3\beta^2 - 2i\frac{\beta}{\alpha}\right)P_r^2 - 2}{P_r(1-\beta^2 P_r^2)} \frac{d\psi_r}{dP_r} - \frac{\beta^2}{\alpha^2}\left[\frac{(1-4i\alpha\beta)P_r^2 + 3i\frac{\alpha}{\beta}}{(1-\beta^2P_r^2)^2} + \frac{l(l+1)\alpha^2}{\beta^2 P_r^2} - \frac{2E}{1-\beta^2P_r^2}\right]\psi_r = 0. \tag{3.33}$$

Defining now a function $u(P_r)$ such that $\psi_r = \left(1-\beta^2P_r^2\right)^{i/2\alpha\beta} u$, Eq. (3.33) simplifies to

$$\frac{d^2u}{dP_r^2} + \left(\frac{2}{P_r} - \frac{\beta^2 P_r}{(1-\beta^2P_r^2)}\right)\frac{du}{dP_r} - \left[\frac{l(l+1)}{P_r^2} - \frac{2\beta^2 E}{\alpha^2(1-\beta^2P_r^2)}\right]u = 0. \tag{3.34}$$

Eventually, the equation assumes the form of a hypergeometric differential equation following a change in the variable $z = \beta^2 P_r^2$,

$$\frac{d^2u}{dz^2} + \left(\frac{3-4z}{2z(1-z)}\right)\frac{du}{dz} - \frac{1}{4}\left[\frac{l(l+1)}{z^2} - \frac{\varepsilon}{z(1-z)}\right]u = 0, \tag{3.35}$$

where $\varepsilon = 2E/\alpha^2$, with the solution

$$u(P_r) = const \times \sqrt{1-\beta^2P_r^2}\,(\beta P_r)^l\, \mathbf{F}\left(a,b,c;\beta^2P_r^2\right), \tag{3.36}$$

F represents a hypergeometric function of parameters

$$a = 1 + \frac{l}{2} + \frac{\sqrt{1+\varepsilon+l(l+1)}}{2}, \quad b = 1 + \frac{l}{2} - \frac{\sqrt{1+\varepsilon+l(l+1)}}{2}, \quad c = l + \frac{3}{2}. \tag{3.37}$$

The solution of Eq. (3.33) is therefore

$$\psi_r = \left(1-\beta^2 P_r^2\right)^{\frac{1}{2}(1+i/\alpha\beta)} (\beta P_r)^l \, \mathbf{F}\left(a,b,c;\beta^2 P_r^2\right). \tag{3.38}$$

It is necessary to set the boundary conditions so that ψ_r vanishes at $P_r = 1/\beta$, i.e. at $z = 1$, when $b = -n$, with an integer n, this happens, leading to

$$E = \alpha^2 \left(2n^2 + 4n + 2nl + \frac{3}{2}l + \frac{3}{2}\right). \tag{3.39}$$

Thus, the energy's eigenvalues are quantized in the SdS case. Naturally, this is a result of the coordinate P_r having a finite range. The radial wave function can be written as

$$\psi_r = const \times \left(1-\beta^2 P_r^2\right)^{\frac{1}{2}(1+i/\alpha\beta)} (\beta P_r)^l \, \mathbf{P}_n^{\left(l+\frac{1}{2},\frac{1}{2}\right)} \left(1-2\beta^2 P_r^2\right), \tag{3.40}$$

with $P_n^{(\mu,\nu)}$a Jacobi polynomial. The solution (3.40) for the spherical wave with $l = 0$ has a simple form. By utilising the characteristics of the hypergeometric functions, one easily obtains

$$\psi_{r_0} = \left(1-\beta^2 P_r^2\right)^{i/2\alpha\beta} \frac{\sin\left[\sqrt{1+\varepsilon}\arcsin\beta P_r\right]}{\sqrt{1+\varepsilon}\beta P_r}. \tag{3.41}$$

That in the limit $\alpha \to 0, \beta \to 0$ coincides with the standard quantum-mechanical solution $\psi_{r_0} = \sin(\sqrt{2E}P_r)/(\sqrt{2E}P_r)$.

Assuming negative values of α^2 and β^2, the Schrödinger equation in the aSdS case is the analytical continuation of (3.33), requiring regularity for $P_r \to \infty$, the solution reads

$$\psi_r = const \times \frac{\left(1-\beta^2 P_r^2\right)^{i/2\alpha\beta}}{\left(\beta^2 P_r^2\right)^{\alpha}} \mathbf{F}\left(a, a-c+1, a-b+1; \frac{1}{\beta^2 P_r^2}\right), \tag{3.42}$$

and makes no indication of energy quantization.

3.3.4 The Harmonic Oscillator in 3D

One can follow the same steps as for the free particle to formulate the Schrödinger equation for a three-dimensional harmonic oscillator. Nevertheless, in this case, it is advantageous to choose λ to eliminate the mixed terms from the equation, as indicated by equation (3.15). After making this choice, one gets

$$\frac{1}{2}\frac{\beta^2\omega_0^2}{\beta^2\omega_0^2+\alpha^2}\left[\hat{\mathcal{P}}_i^2+\frac{\left(\beta^2\omega_0^2+\alpha^2\right)^2}{\beta^4\omega_0^2}\hat{\mathcal{X}}_i^2\right]\psi=E\psi. \tag{3.43}$$

Following certain algebraic modifications, the radial wave function equation expands into spherical harmonics, as in (3.30).

$$\frac{d^2\psi_r}{dP_r^2}+\left(\frac{2}{P_r}-\frac{\beta^2P_r}{1-\beta^2P_r^2}\right)\frac{d\psi_r}{dP_r}-\left[\frac{l(l+1)}{P_r^2}+\frac{1}{\omega^2}\frac{P_r^2}{(1-\beta^2P_r^2)^2}-\left(\frac{2\mathcal{E}}{\omega^2}-\beta^2\right)\frac{1}{1-\beta^2P_r^2}\right]\psi_r=0, \tag{3.44}$$

where $\omega=\left(1+\frac{\alpha}{\beta^2\omega_0^2}\right)\omega_0$ and $\mathcal{E}=\left(1+\frac{\alpha}{\beta^2\omega_0^2}\right)E$.
Defining a new variable $z=\beta^2P_r^2$, (3.44) is possible to express as a hypergeometric equation

$$\frac{d^2\psi_r}{dz^2}+\frac{3-4z}{2z(1-z)}\frac{d\psi_r}{dz}-\frac{1}{4}\left[\frac{l(l+1)}{z^2}+\frac{u}{(1-z)^2}-\frac{\varepsilon}{z(1-z)}\right]\psi_r=0, \tag{3.45}$$

with

$$u=\frac{1}{\beta^4\omega^2},\quad \varepsilon=\frac{2\mathcal{E}}{\beta^2\omega^2}=-1.$$

Note that (3.45) differs from the free particle equation (3.35) only for the term proportional to μ. Eq. (3.45) can be solved as

$$\psi_r=const\times(\beta P_r)^l\left(1-\beta^2P_r^2\right)^{(1+\sqrt{1+4\mu})/4}\mathbf{F}\left(a,b,c;\beta^2P_r^2\right), \tag{3.46}$$

with

$$a = \frac{1}{2}\left(l+\frac{3}{2}+\frac{\sqrt{1+4\mu}}{2}+\sqrt{1+\mu+\varepsilon+l\left(l+1\right)}\right), \tag{3.47}$$

$$b = \frac{1}{2}\left(l+\frac{3}{2}+\frac{\sqrt{1+4\mu}}{2}-\sqrt{1+\mu+\varepsilon+l\left(l+1\right)}\right), \tag{3.48}$$

$$c = l+\frac{3}{2}. \tag{3.49}$$

It is necessary for ψ to disappear at $P_r = 1/\beta$, i.e. at $z=1$. This happens when $b=-n$, and after that

$$\varepsilon = \left(2n+l+\frac{3}{2}\right)\sqrt{1+4u}+4n^2+4nl+6n+2l+\frac{3}{2}, \tag{3.50}$$

and

$$\psi_r = const \times (\beta P_r)^l \left(1-\beta^2 P_r^2\right)^{(1+\sqrt{1+4\mu})/4} \mathbf{P}_n^{\left(l+\frac{1}{2},\sqrt{1+4\mu}\right)}\left(1-2\beta^2 P_r^2\right),$$

with $P_n^{(\mu,\nu)}$ a Jacobi polynomial. From (3.50) it follows that

$$E = \left(2n+l+\frac{3}{2}\right)\omega_0\sqrt{1+\frac{\left(\beta^2\omega_0^2+\alpha^2\right)^2}{4\omega_0^2}}+\left(2n^2+3n+2nl+l+\frac{5}{4}\right)\left(\beta^2\omega_0^2+\alpha^2\right). \tag{3.51}$$

An alternative way to express the preceding expression is in terms of a new quantum number, $N=2n+l$, which is commonly introduced for the three-dimensional oscillator

$$E = \left(N+\frac{3}{2}\right)\omega_0\sqrt{1+\frac{\left(\beta^2\omega_0^2+\alpha^2\right)^2}{4\omega_0^2}}+\left(N^2+3N-l\left(l+1\right)+\frac{5}{2}\right)\frac{\left(\beta^2\omega_0^2+\alpha^2\right)}{2}. \tag{3.52}$$

Similarly to the one-dimensional, the harmonic oscillator's spectrum experiences corrections of order $(\beta^2\omega_0+\alpha^2/\omega_0)$ in ordinary quantum mechanics. The result for flat Snyder space is recovered in the limit $\alpha\to 0$, while the oscillator's spectrum on a 3-sphere can be derived in the case of $\beta\to 0$. Here, ω_0 has no bearing on the energy shift at first order with respect to the standard oscillator. By analytically continuing the energy spectrum of the SdS oscillator to negative α^2 and β^2, the energy spectrum of the 3-dimensional aSdS harmonic oscillator is obtained, and is therefore still governed by equation (3.51). To guarantee the positivity of the energy, some conditions on the quantum numbers need to be placed, just like in the one-dimensional case.

3.4 Path Integral Formalism in D-dimensional Momentum Space

In this section, we develop the path integral in momentum representation space in the D-dimensional for the non-relativistic propagator of a free particle and the harmonic oscillator, taking into account the presence of a nonzero minimum uncertainty in both momentum and position. The evolution operator ($\hat{U}(t_b,t_a)$) provides the standard formalism of path integration, which can be expressed as follows:

$$K(\mathbf{P}_b,t_b,\mathbf{P}_a,t_a)=\langle\mathbf{P}_b|\hat{U}(t_b,t_a)|\mathbf{P}_a\rangle$$

$$=\lim_{N\longrightarrow\infty}\prod_{j=1}^{N}\int_{-1/\beta}^{1/\beta}\frac{d\mathbf{P}_j}{\sqrt{1-\beta^2\mathbf{P}_j^2}}\prod_{j=1}^{N+1}\left\langle\mathbf{P}_j\right|e^{-\frac{i\varepsilon}{\hbar}\hat{\mathcal{H}}}\left|\mathbf{P}_{j-1}\right\rangle, \tag{3.53}$$

the standard form of the Hamiltonian defined on the D-dimensional sphere with the symmetry $SO(D+1)$ is represented by $\hat{\mathcal{H}}$, meaning that it is invariant under this group.

3.4.1 The Free Particle

For this algebra, the new Hamiltonian in the free case is expressed as

$$\hat{\mathcal{H}}=\frac{1}{2m}\left[\hat{p}_i^2+\alpha^2\hat{L}_i^2\right]. \tag{3.54}$$

We construct the corresponding transition amplitude for this Hamiltonian (3.54), in $D-$dimensions, by formulate the propagator corresponding to the old Hamiltonian $\hat{H}=\hat{p}_i^2/2m$, and then integrating it with energy values for $\alpha^2\hat{L}_i^2/2m$. And the other energy term are obtained from the spectral decomposition of the radial transition amplitude. This means that the operator $\hat{H}=\hat{p}_i^2/2m$ can be written like this

$$\hat{H}=\frac{1}{2m}\left[\frac{\mathbf{P}^2(1-i\hbar\alpha\beta(1-D))-i\hbar D\frac{\alpha}{\beta}}{1-\beta^2\mathbf{P}^2}+\left(\alpha^2\hbar^2-2i\hbar\frac{\alpha}{\beta}\right)P_i\frac{\partial}{\partial P_i}-\hbar^2\frac{\alpha^2}{\beta^2}\left(1-\beta^2\mathbf{P}^2\right)\frac{\partial^2}{\partial P_i^2}\right]. \tag{3.55}$$

The latter is achieved by substituting the operator $\hat{p}_i$ with expressions involving auxiliary operators $\hat{X}_i$ and $\hat{P}_i$ obeying the canonical commutation relations (in this free case we select $\lambda = 0$). After doing simple calculations, we get the following outcome:

$$
\begin{aligned}
&K(\mathbf{P}_b, t_b, \mathbf{P}_a, t_a) \\
&= \lim_{N\longrightarrow\infty} \prod_{j=1}^{N} \int_{-1/\beta}^{1/\beta} \frac{d\mathbf{P}_j}{\sqrt{1-\beta^2\mathbf{P}_j^2}} \prod_{j=1}^{N+1} \left[\int \frac{d\mathbf{q}_j}{(2\pi\hbar)^D} \left(\frac{1-\beta^2\mathbf{P}_{j-1}^2}{1-\beta^2\mathbf{P}_j^2} \right)^{\frac{\gamma}{2}} \sqrt{1-\beta^2\mathbf{P}_j^2} \right] \\
&\times \exp\left[\sum_{j=1}^{N+1} \frac{i\varepsilon}{\hbar} \left\{ -\frac{\mathbf{q}_j \Delta\mathbf{P}_j}{\varepsilon} - \frac{\alpha^2}{2m\beta^2}\left(1-\beta^2\mathbf{P}_j^2\right)\mathbf{q}_j^2 - i\hbar \frac{2\alpha^2(2\gamma-3+2i/\alpha\beta\hbar)}{2m} \mathbf{q}_j \mathbf{P}_j \right.\right. \\
&+ \frac{\hbar^2\alpha^2}{2m\beta^2} \frac{(\gamma-1)\beta^2\left(D+\beta^2\mathbf{P}_j^2(\gamma+(1-D))\right)}{\left(1-\beta^2\mathbf{P}_j^2\right)} - \frac{1}{2m} \frac{\mathbf{P}_j^2(1-i\hbar\alpha\beta(1-D))-i\hbar D\frac{\alpha}{\beta}}{1-\beta^2\mathbf{P}_j^2} \\
&\left.\left. - \frac{\hbar^2\beta^2\left(\alpha^2-2i\frac{\alpha}{\beta\hbar}\right)(\gamma-1)}{2m} \frac{\mathbf{P}_j^2}{\left(1-\beta^2\mathbf{P}_j^2\right)} \right] \right\}. \qquad (3.56)
\end{aligned}
$$

After executing the multiple Gaussian integrations over $\mathbf{q}_j$, The equation above can be simplified to the following form

$$
\begin{aligned}
&K(\mathbf{P}_b, t_b, \mathbf{P}_a, t_a) = \\
&\lim_{N\longrightarrow\infty} \prod_{j=1}^{N} \int_{-1/\beta}^{1/\beta} \frac{d\mathbf{P}_j}{\sqrt{1-\beta^2\mathbf{P}_j^2}} \prod_{j=1}^{N+1} \left[\left(\frac{1-\beta^2\mathbf{P}_{j-1}^2}{1-\beta^2\mathbf{P}_j^2} \right)^{\frac{\gamma}{2}} \left[\sqrt{\frac{m}{2\pi i\hbar\varepsilon\alpha^2/\beta^2}} \right]^D \left[1-\beta^2\mathbf{P}_j^2\right]^{\frac{1-D}{2}} \right] \\
&\times \exp\left\{ \frac{i\varepsilon}{\hbar} \sum_{j=1}^{N+1} \left[\frac{m\beta^2\left(\Delta\mathbf{P}_j\right)^2}{2\alpha^2\varepsilon^2\left(1-\beta^2\mathbf{P}_j^2\right)} + \frac{i\hbar\beta^2}{\varepsilon}\left(\gamma - \frac{3}{2} + i/\alpha\beta\hbar\right) \frac{\mathbf{P}_j\Delta\mathbf{P}_j}{1-\beta^2\mathbf{P}_j^2} \right.\right. \\
&- \frac{\hbar^2\alpha^2\beta^2\left(\gamma-\frac{3}{2}+i/\alpha\beta\hbar\right)^2}{2m} \frac{\mathbf{P}_j^2}{\left(1-\beta^2\mathbf{P}_j^2\right)} + \frac{\hbar^2\alpha^2}{2m\beta^2} \frac{(\gamma-1)\beta^2\left(D+\beta^2\mathbf{P}_j^2(\gamma+(1-D))\right)}{\left(1-\beta^2\mathbf{P}_j^2\right)} \\
&\left.\left. - \frac{1}{2m} \frac{\mathbf{P}_j^2(1-i\hbar\alpha\beta(1-D))-i\hbar D\frac{\alpha}{\beta}}{1-\beta^2\mathbf{P}_j^2} - \frac{\hbar^2\beta^2\left(\alpha^2-2i\frac{\alpha}{\beta\hbar}\right)(\gamma-1)}{2m} \frac{\mathbf{P}_j^2}{\left(1-\beta^2\mathbf{P}_j^2\right)} \right] \right\}. \qquad (3.57)
\end{aligned}
$$

Moreover, all terms associated with γ will be nullified by the term terms associated with $\left(\frac{1-\beta^2\mathbf{P}_{j-1}^2}{1-\beta^2\mathbf{P}_j^2}\right)^{\frac{\gamma}{2}}$, will be nullified by the term

$$\frac{1}{2}\ln\left(\frac{1-\beta^2\mathbf{P}_j^2}{1-\beta^2\mathbf{P}_{j-1}^2}\right) = -\frac{\beta^2\mathbf{P}_j\Delta\mathbf{P}_j}{1-\beta^2\mathbf{P}_j^2} + \frac{\beta^2}{2}\left[\frac{-\left(1-\beta^2\mathbf{P}_j^2\right)\left(\Delta\mathbf{P}_j\right)^2 - 2\beta^4\sum_i P_j^2\left(\Delta P_j\right)^2}{\left(1-\beta^2\mathbf{P}_j^2\right)^2}\right], \tag{3.58}$$

then, next making up this term $\left(\frac{-i}{\alpha\beta\hbar}\frac{\beta^2\mathbf{P}_j\Delta\mathbf{P}_j}{1-\beta^2\mathbf{P}_j^2}\right)$ by the above equivalence (3.58). Then Eq. (3.57) can be written as

$$\begin{aligned} K\left(\mathbf{P}_b,t_b,\mathbf{P}_a,t_a\right) = & \lim_{N\longrightarrow\infty}\prod_{j=1}^{N}\int_{-1/\beta}^{1/\beta}\frac{d\mathbf{P}_j}{\sqrt{1-\beta^2\mathbf{P}_j^2}} \\ & \times\prod_{j=1}^{N+1}\left[\left(\frac{1-\beta^2\mathbf{P}_{j-1}^2}{1-\beta^2\mathbf{P}_j^2}\right)^{\frac{\gamma}{2}}\left[\sqrt{\frac{m}{2\pi i\hbar\varepsilon\alpha^2/\beta^2}}\right]^D\left[1-\beta^2\mathbf{P}_j^2\right]^{\frac{1-D}{2}}\right] \\ & \times\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{j=1}^{N+1}\left[\frac{m\beta^2\left(\Delta\mathbf{P}_j\right)^2}{2\alpha^2\varepsilon^2\left(1-\beta^2\mathbf{P}_j^2\right)} + \frac{i\hbar\beta^2}{\varepsilon}\left(\gamma-\frac{3}{2}+i/\alpha\beta\hbar\right)\frac{\mathbf{P}_j\Delta\mathbf{P}_j}{1-\beta^2\mathbf{P}_j^2}\right.\right. \\ & -\frac{\hbar^2\alpha^2\beta^2\left(\gamma-\frac{3}{2}+i/\alpha\beta\hbar\right)^2}{2m}\frac{\mathbf{P}_j^2}{\left(1-\beta^2\mathbf{P}_j^2\right)} + \frac{\hbar^2\alpha^2}{2m\beta^2}\frac{(\gamma-1)\beta^2\left(D+\beta^2\mathbf{P}_j^2(\gamma+(1-D))\right)}{\left(1-\beta^2\mathbf{P}_j^2\right)} \\ & \left.\left.-\frac{1}{2m}\frac{\mathbf{P}_j^2\left(1-i\hbar\alpha\beta\left(1-D\right)\right)-i\hbar D\frac{\alpha}{\beta}}{1-\beta^2\mathbf{P}_j^2} - \frac{\hbar^2\beta^2\left(\alpha^2-2i\frac{\alpha}{\beta\hbar}\right)(\gamma-1)}{2m}\frac{\mathbf{P}_j^2}{\left(1-\beta^2\mathbf{P}_j^2\right)}\right]\right\}. \end{aligned} \tag{3.59}$$

Next, we present spherical coordinates for momentum variables P in the D-dimension, which are defined by

$$\begin{aligned} P_{\Omega_1} &= P\cos\phi_1 \\ P_{\Omega_2} &= P\sin\phi_1\cos\phi_2 \\ &\vdots \\ P_{\Omega_{D-1}} &= P\sin\phi_1\ldots\sin\phi_{D-2}\cos\phi_{D-1} \\ P_{\Omega_D} &= P\sin\phi_1\ldots\sin\phi_{D-2}\sin\phi_{D-1}, \end{aligned} \tag{3.60}$$

with $P^2 = \sum_{\nu=1}^{D} (P^{\nu})^2$. and $0 \le \phi_\nu \le \pi$ $(\nu = 1,...D-2)$, $0 \le \phi_{D-1} \le 2\pi$. This results in a transformation over the measure term, as shown below

$$\prod_{j=1}^{N} \int \frac{d\mathbf{P}_j}{\sqrt{1-\beta^2 \mathbf{P}_j^2}} = \prod_{j=1}^{N} \int \frac{P_j^{D-1} dP_j}{\sqrt{1-\beta^2 P_j^2}} dP_{\Omega_j}, \quad dP_{\Omega_j} = \prod_{k}^{D-1} \left(\sin \phi^k\right)^{D-1-k} d\phi_k, \tag{3.61}$$

where dP_{Ω_j} is the $(D-1)$-dimensional surface element on the unit sphere, and ΔP_{Ω_n} is the relative angle between D -dimensional vectors $\mathbf{p}_j$ and $\mathbf{p}_{j-1}$. As we known in polar coordinates, we can conclude the correction of $\left(\Delta P_{\Omega_j}\right)^2$ from the kinetic energy term like

$$\exp\left[\frac{i}{\hbar} \frac{m\beta^2 \left(\Delta \mathbf{P}_j\right)^2}{2\alpha^2 \varepsilon \left(1-\beta^2 \mathbf{P}_j^2\right)}\right] = \exp\left[\frac{i}{\hbar} \frac{m\beta^2}{2\alpha^2 \varepsilon} \frac{\left(P_j^2 + P_{j-1}^2 - 2P_j P_{j-1} \cos \Delta P_{\Omega_j}\right)}{1-\beta^2 P_j^2}\right]. \tag{3.62}$$

We must simplify this term $\left(-\frac{3}{2}\frac{\beta^2(\mathbf{P}_j \Delta \mathbf{P}_j)}{(1+\beta \mathbf{p}_j^2)}\right)$ in order to execute the path integration over the angular variables as follows

$$\frac{3}{2}\frac{\beta^2(\mathbf{P}_j \Delta \mathbf{P}_j)}{(1+\beta \mathbf{p}_j^2)} = \left[\frac{3}{2}\frac{\beta^2 P_j \Delta P_j}{(1-\beta^2 P_j^2)} + \frac{3}{2}\frac{\beta^2 P_j P_{j-1}(1-\cos\Omega_j)}{(1-\beta^2 P_j^2)}\right], \tag{3.63}$$

and

$$2\left(1-\cos\Omega_j\right) = \sum_{k=1}^{D-1} \left(\prod_{l=1}^{k-1} \sin^2 \bar{\phi}_l^{(j)}\right) \left(\Delta \phi_k^{(j)}\right)^2 \approx \frac{i\varepsilon}{\hbar} \frac{\alpha^2 \hbar^2 (D-1)}{m\beta^2} \frac{\left(1-\beta^2 P_j^2\right)}{P_j P_{j-1}}. \tag{3.64}$$

As a result, in the first order of ε, Eq. (3.63) takes the following form

$$\exp\left(\frac{3}{2}\frac{\beta^2(\mathbf{P}_j \Delta \mathbf{P}_j)}{(1+\beta \mathbf{p}_j^2)}\right) = \exp\left[\frac{3}{2}\frac{\beta^2 P_j \Delta P_j}{(1-\beta^2 P_j^2)} + \frac{i\varepsilon}{\hbar}\frac{3}{2}\frac{\alpha^2 \hbar^2 (D-1)}{2m}\right]. \tag{3.65}$$

To execute the P_{Ω_j}-integrals. And according to [3], it is helpful to expand (3.62) into a factorised series

$$e^{h\cos\Delta P_{\Omega_j}} = (h/2)^{-\frac{D-2}{2}} \Gamma\left(\tfrac{D-2}{2}\right) \sum_{l_j=0}^{+\infty} \left(l_j + (D/2) - 1\right) I_{l_j+(D/2)-1}(h)$$
$$\times C_{l_j}^{(D/2)-1}\left(\cos \Delta P_{\Omega_j}\right), \tag{3.66}$$

where $C_l^{\nu}(x)$ are Gegenbauer polynomials, and it checks the following addition theorem

$$\sum_{m=1}^{M_{l_j}} S_{l_j}^m\left(P_{\Omega_{j-1}}\right) S_{l_j}^m\left(P_{\Omega_j}\right) = \frac{(2l_j+D-2)\Gamma(D/2)}{(2\pi)^{D/2}(D-2)} C_{l_j}^{(D/2)-1}\left(\cos\Delta P_{\Omega_j}\right), \tag{3.67}$$

where $M_l = \frac{(2l+D-2)(l+D-3)!}{l!(D-2)!}$ and $S_l^m(P_\Omega)$ are the real hyperspherical harmonics of degree l associated with unit vector P_Ω. While $I_l(h)$ are related to the modified Bessel functions $\tilde{I}_l(h)$

$$I_l(h) = e^h (2\pi h)^{-1/2} \tilde{I}_l(h). \tag{3.68}$$

This leads to the following D-dimensional time-sliced path integral

$$\begin{aligned} K(\mathbf{P}_b,\mathbf{P}_a,T) &= \frac{-i}{(P_bP_a)^{(D-1)/2}} \\ &\times \lim_{N\to\infty}\left(\frac{1-\beta^2P_a^2}{1-\beta^2P_b^2}\right)^{-i/2\alpha\beta\hbar} e^{\frac{iT}{\hbar}\frac{3\alpha^2\hbar^2(D-1)}{4m}} \prod_{j=1}^{N}\left(\int_0^\infty \frac{dP_j}{\sqrt{1-\beta^2P_j^2}}\int dP_{\Omega_j}\right) \\ &\times \prod_{j=1}^{N+1}\left[\sqrt{\frac{m}{2\pi i\hbar\varepsilon\alpha^2/\beta^2}}\sum_{l_j=0}^{+\infty}\sum_{m_j=1}^{M_{l_j}} S_{l_j}^{m_j}\left(P_{\Omega_{j-1}}\right)S_{l_j}^{m_j}\left(P_{\Omega_j}\right)\tilde{I}_{l_j+\frac{D}{2}-1}\left(-\frac{i}{\hbar}\frac{m\beta^2P_jP_{j-1}}{\alpha^2\varepsilon^2\left(1-\beta^2P_j^2\right)}\right)\right] \\ &\times \exp\left\{\frac{i\varepsilon}{\hbar}\sum_{j=1}^{N+1}\left[\frac{m\beta^2(\Delta P_j)^2}{2\alpha^2\varepsilon^2\left(1-\beta^2P_j^2\right)} - \frac{3i\hbar}{2\varepsilon}\frac{\beta^2P_j\Delta P_j}{1-\beta^2P_j^2} - \frac{5\hbar^2\alpha^2}{8m}\frac{\beta^2P_j^2}{\left(1-\beta^2P_j^2\right)} - \frac{\hbar^2\alpha^2\left(D+(1-D)\beta^2P_j^2\right)}{2m\left(1-\beta^2P_j^2\right)}\right]\right\}. \end{aligned} \tag{3.69}$$

Now, the N symbols δ of Kronecker can be obtained by performing the N-integrations over the P_{Ω_j}-variables.

$$\int dP_\Omega S_l^m\left(P_{\Omega_j}\right)S_{l'}^{m'}\left(P_{\Omega_j}\right) = \delta_{l;l'}\delta_{m,m'}. \tag{3.70}$$

By using these, all angular integrations can be eliminated, allowing the amplitude of the time evolution to follow as an expansion.

$$K(\mathbf{p}_b,p_{0b},\mathbf{p}_a,p_{0a}) = \sum_{l=0}^{+\infty}\sum_{m=1}^{M_l}\frac{1}{(P_bP_a)^{\frac{D-1}{2}}}K_l(P_b,P_a,T)S_l^m\left(P_{\Omega_b}\right)S_l^m\left(P_{\Omega_a}\right), \tag{3.71}$$

where the $K_l(P_b,P_a,T)$ is obviously given by the radial path integral

$$K_l\left(P_b,P_a,T\right)=(-i)\lim_{N\to\infty}\left(\frac{1-\beta^2P_a^2}{1-\beta^2P_b^2}\right)^{-i/2\alpha\beta\hbar}\exp\left[\frac{iT}{\hbar}\frac{3\alpha^2\hbar^2(D-1)}{4m}\right]$$
$$\times\prod_{j=1}^{N}\left(\int_0^\infty\frac{\beta dP_j}{\sqrt{1-\beta^2P_j^2}}\right)\prod_{j=1}^{N+1}\left[\frac{m}{2\pi i\hbar\varepsilon\alpha^2}\right]^{1/2}\tilde{I}_{l+\frac{D}{2}-1}\left(-\frac{i}{\hbar}\frac{m\beta^2P_jP_{j-1}}{\alpha^2\varepsilon\left(1-\beta^2P_j^2\right)}\right)$$
$$\times\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{j=1}^{N+1}\left[\frac{m\beta^2\left(\Delta P_j^2\right)}{2\alpha^2\varepsilon^2\left(1-\beta^2P_j^2\right)}-\frac{3i\hbar}{2\varepsilon}\frac{\beta^2P_j\Delta P_j}{1-\beta^2P_j^2}-\frac{5\hbar^2\alpha^2}{8m}\frac{\beta^2P_j^2}{\left(1-\beta^2P_j^2\right)}-\frac{\hbar^2\alpha^2\left(D+(1-D)\beta^2P_j^2\right)}{2m\left(1-\beta^2P_j^2\right)}\right]\right\}.$$
(3.72)

In the continuum limit $\varepsilon\to0$, the asymptotic expression for the modified Bessel function

$$\tilde{I}_l(z)_{z\to\infty}=\exp\left(-\frac{l^2-1/4}{2z}\right),\tag{3.73}$$

it allows the radial function $K_l\left(P_b,P_a,T\right)$ to be obtained as the following

$$K_l\left(P_b,P_a,T\right)=$$
$$(-i)\left(\frac{1-\beta^2P_a^2}{1-\beta^2P_b^2}\right)^{-i/2\alpha\beta\hbar}\exp\left[\frac{iT}{\hbar}\frac{3\alpha^2\hbar^2\left(D-1\right)}{4m}\right]\lim_{N\to\infty}\prod_{j=1}^{N}\left(\int_0^\infty\frac{\beta dP_j}{\sqrt{1-\beta^2P_j^2}}\right)$$
$$\times\prod_{j=1}^{N+1}\left[\sqrt{\frac{m}{2\pi i\hbar\varepsilon\alpha^2}}\right]\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{j=1}^{N+1}\left[\frac{m\beta^2\left(\Delta P_j\right)^2}{2\alpha^2\varepsilon^2\left(1-\beta^2P_j^2\right)}-\frac{3i\hbar}{2\varepsilon}\frac{\beta^2P_j\Delta P_j}{1-\beta^2P_j^2}\right.\right.$$
$$\left.\left.-\frac{5\hbar^2\alpha^2}{8m}\frac{\beta^2P_j^2}{\left(1-\beta^2P_j^2\right)}-\frac{\hbar^2\alpha^2\left(D+(1-D)\beta^2P_j^2\right)}{2m\left(1-\beta^2P_j^2\right)}-\hbar^2\frac{\alpha^2\left[\left(l+\frac{D}{2}-1\right)^2-1/4\right]}{2m\beta^2}\frac{\left(1-\beta^2P_j^2\right)}{P_jP_{j-1}}\right]\right\}.\tag{3.74}$$

The propagator (3.74) takes on a more complex shape as a result of the term measure. To simplify, we will employ the point transformation method (see, Ref. [48]), in which the $\delta-$point discretization interval is

$$P^{(\delta)}=\delta P+(1-\delta)P_{-1}.\tag{3.75}$$

Based on [35, 37], three quantum corrections are extracted by the expression (3.74) as follows: the term measure $\left(dP_j/\sqrt{1-\beta^2P_j^2}\right)$, the kinetic energy term, and the second term in action (3.74). Therefore, we use the $\delta-$point discretization interval to expand all of these terms. Then, we adopt the coordinate transformation $P=g(x)$, to return the standard kinetic term $\left(m\left(\Delta x\right)^2/2\alpha^2\varepsilon\right)$. The selection of g is based on the following condition:

$((\partial g(x)/\partial x) = \sqrt{1-\beta^2P^2})$. Next, we use the $\delta-$point discretization interval to develop the measure terms and the kinetic energy [48]. Assuming the $\delta-$point discretization, we can determine the correction total C_T as

$$C_T = \frac{i\hbar\alpha^2\varepsilon}{2m}\left[\frac{5}{4}\tan^2(\beta x_j) - \left(2\delta^2-\delta-1\right)\cos^{-2}(\beta x_j)\right], \tag{3.76}$$

then this new propagator expression is obtained.

$$K_l(P_b,P_a,T) = (-i)\left(\frac{\cos^2(\beta x_a)}{\cos^2(\beta x_b)}\right)^{-i/2\alpha\beta\hbar} e^{\frac{i(t_b-t_a)}{\hbar}\left[\frac{\hbar^2\alpha^2}{2m}\left[\left(l+\frac{D}{2}-1\right)^2-1/4\right]+\frac{\hbar^2\alpha^2(D-1)}{4m}\right]}$$
$$\times\lim_{N\to\infty}\prod_{j=1}^{N}\left(\int_0^\infty dx_j\right)\prod_{j=1}^{N+1}\sqrt{\frac{m}{2\pi i\hbar\varepsilon\alpha^2}}\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N+1}\left[\frac{m\left(\Delta x_j\right)^2}{2\alpha^2\varepsilon}-\frac{\hbar^2\alpha^2\varepsilon}{2m}\frac{\left[\left(l+\frac{D}{2}-1\right)^2-1/4\right]}{\sin^2\left(\beta x_j\right)}\right]\right\}. \tag{3.77}$$

As stated in reference [49], the transition amplitude pertaining to the Pöschel–Teller potential leads to the following result. Thus, the ultimate expression for the radial transition amplitude in the context of the D-dimensional free particle under SdS space is provided as:

$$\begin{aligned} K_l(x_b,x_a,T) &= (-i)\left(\frac{\cos^2(\beta x_a)}{\cos^2(\beta x_b)}\right)^{-i/2\alpha\beta\hbar}\sum_{n=0}^{\infty} e^{\frac{iT}{\hbar}\left[\frac{\hbar^2\alpha^2}{2m}\left[\left(l+\frac{D}{2}-1\right)^2-1/4+\frac{(D-1)}{2}\right]\right]} \\ &\quad\times e^{-\frac{iT}{\hbar}\frac{\alpha^2\hbar^2}{2m}(k+\lambda+2n)^2}\Phi_n^*(x_a)\Phi_n(x_b), \end{aligned} \tag{3.78}$$

where

$$\begin{aligned} \Phi_n(x) &= \left[\frac{2n(k+\lambda+2n)\Gamma(k+\lambda+n)}{\Gamma\left(k+\frac{1}{2}+n\right)\Gamma\left(\lambda+\frac{1}{2}+n\right)}\right]^{1/2}\sin^\kappa(x)\cos^\lambda(x) \\ &\quad\times P_n^{(\kappa-1/2,\lambda-1/2)}\left(2\sin^2(x)-1\right), \end{aligned} \tag{3.79}$$

when the following parameters have been employed:

$$\kappa = 1, \quad \lambda = \pm\left(l+\frac{D}{2}-1\right)+\frac{1}{2}, \tag{3.80}$$

and $P_n^{(\kappa-1/2,\lambda-1/2)}$ denotes Jacobi polynomials. As a result, the energy eigenvalues E_n have the following expression:

$$E_n = \frac{\alpha^2\hbar^2}{2m}\left[\left(l+2n+\frac{D+1}{2}\right)^2 - \frac{(D-1)}{2}\right]. \tag{3.81}$$

For $D=3$, we obtain the exact result reported in [27].

3.4.2 Harmonic Oscillator

Here, the D-dimensional harmonic oscillator's Hamiltonian is as follows

$$\hat{\mathcal{H}} = \frac{1}{2m}\left(\hat{p}_i^2+\alpha^2\hat{L}_i^2\right)+\frac{m\omega_0^2}{2}\left(x_i^2+\beta^2\hat{L}_i^2\right). \tag{3.82}$$

This Hamiltonian is defined on the D-dimensional sphere having the symmetry $SO(D+1)$ meaning it remains invariant under this group. For this Hamiltonian (3.82), we take the same steps as in the free case to obtain the path integral, namely

$$\begin{aligned} K(\mathbf{P}_b,t_b,\mathbf{P}_a,t_a) &= \langle \mathbf{P}_b|\hat{U}(t_b,t_a)|\mathbf{P}_a\rangle \\ &= \lim_{N\longrightarrow\infty}\prod_{j=1}^{N}\int_{-1/\beta}^{1/\beta}\frac{d\mathbf{P}_j}{\sqrt{1-\beta^2\mathbf{P}_j^2}}\prod_{j=1}^{N+1}\left\langle \mathbf{P}_j\right| e^{-\frac{i\varepsilon}{\hbar}\hat{H}}\left|\mathbf{P}_{j-1}\right\rangle, \end{aligned} \tag{3.83}$$

where the harmonic oscillator standard Hamiltonian is provided by

$$\hat{H} = \frac{\omega_0}{2m\omega}\left[\frac{P_i^2}{1-\beta^2P_i^2}+\hbar^2m^2\omega^2\beta^2P_i\frac{\partial}{\partial P_i}-\hbar^2m^2\omega^2\left(1-\beta^2P_i^2\right)\frac{\partial^2}{\partial P_i^2}\right]. \tag{3.84}$$

Following the definition of the delta Dirac function in equation (3.11), we obtain the phase space path integral expression for the kernel by applying the Hamiltonian operator to it

$$\begin{aligned}
&K\left(\mathbf{P}_b,t_b,\mathbf{P}_a,t_a\right)= \\
&\lim_{N\longrightarrow\infty}\prod_{j=1}^{N}\left[\int_{-1/\beta}^{1/\beta}\frac{d\mathbf{P}_j}{\sqrt{1-\beta^2\mathbf{P}_j^2}}\int\frac{d\mathbf{q}_j}{(2\pi\hbar)^D}\right]\prod_{j=1}^{N+1}\left[\left(\frac{1-\beta^2\mathbf{P}_{j-1}^2}{1-\beta^2\mathbf{P}_j^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta^2\mathbf{P}_j^2}\right] \\
&\times\exp\left\{\sum_{j=1}^{N+1}\frac{i\varepsilon}{\hbar}\left[\frac{\mathbf{q}_j\Delta\mathbf{P}_j}{\varepsilon}-\frac{m\omega\omega_0}{2}\left[\left(1-\beta^2\mathbf{P}_j^2\right)\mathbf{q}_j^2+i\hbar\beta^2\left(2\gamma-3\right)\mathbf{q}_j\mathbf{P}_j\right]\right.\right. \\
&\left.\left.+\frac{\hbar^2m\omega\omega_0}{2}\frac{(\gamma-1)^2\beta^4P_j^2+(\gamma-1)\left(\beta^2D\left(1-\beta^2P_j^2\right)\right)}{\left(1-\beta^2\mathbf{P}_j^2\right)}-\frac{\omega_0}{2m\omega}\frac{P_j^2}{1-\beta^2P_j^2}\right]\right\}. \qquad (3.85)
\end{aligned}$$

After executing the multiple Gaussian integrations over $\mathbf{q}_j$, The equation above can be simplified to the following form

$$\begin{aligned}
&K\left(\mathbf{P}_b,t_b,\mathbf{P}_a,t_a\right)=\lim_{N\longrightarrow\infty}\prod_{j=1}^{N}\left(\int_{-1/\beta}^{1/\beta}\frac{d\mathbf{P}_j}{\sqrt{1-\beta^2\mathbf{P}_j^2}}\right) \\
&\times\prod_{j=1}^{N+1}\left[\left(\frac{1-\beta^2\mathbf{P}_{j-1}^2}{1-\beta^2\mathbf{P}_j^2}\right)^{\frac{\gamma}{2}}\left[\sqrt{\frac{1}{2\pi i\hbar\varepsilon m\omega\omega_0}}\right]^D\left[1-\beta^2\mathbf{P}_j^2\right]^{\frac{1-D}{2}}\right] \\
&\times\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\left(\Delta\mathbf{P}_j\right)^2}{2m\omega\omega_0\varepsilon^2\left(1-\beta^2\mathbf{P}_j^2\right)}+\frac{i\hbar\beta^2}{\varepsilon}\left(\gamma-3/2\right)\frac{\mathbf{P}_j\Delta\mathbf{P}_j}{1-\beta^2\mathbf{P}_j^2}-\frac{\hbar^2\beta^4m\omega\omega_0\left(\gamma-\frac{3}{2}\right)^2}{2}\frac{\mathbf{P}_j^2}{\left(1-\beta^2\mathbf{P}_j^2\right)}\right.\right. \\
&\left.\left.+\frac{\hbar^2m\omega\omega_0}{2}\frac{(\gamma-1)^2\beta^4P_i^2+(\gamma-1)\beta^2D\left(1-\beta^2\mathbf{P}_j^2\right)}{\left(1-\beta^2\mathbf{P}_j^2\right)}-\frac{\omega_0}{2m\omega}\frac{P_j^2}{1-\beta^2P_j^2}\right]\right\}, \qquad (3.86)
\end{aligned}$$

moreover, all terms associated with γ will be nullified by the term terms associated with $\left(\frac{1-\beta^2\mathbf{P}_{j-1}^2}{1-\beta^2\mathbf{P}_j^2}\right)^{\frac{\gamma}{2}}$ will be nullified by the term

$$\begin{aligned}
&K\left(\mathbf{P}_b,t_b,\mathbf{P}_a,t_a\right)= \\
&\lim_{N\longrightarrow\infty}\prod_{j=1}^{N}\left[\int_{-1/\beta}^{1/\beta}\frac{d\mathbf{P}_j}{\sqrt{1-\beta^2\mathbf{P}_j^2}}\right]\prod_{j=1}^{N+1}\left[\left[\sqrt{\frac{1}{2\pi i\hbar\varepsilon m\omega\omega_0}}\right]^D\left[1-\beta^2\mathbf{P}_j^2\right]^{\frac{1-D}{2}}\right] \\
&\times\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\left(\Delta\mathbf{P}_j\right)^2}{2m\omega\omega_0\varepsilon^2\left(1-\beta^2\mathbf{P}_j^2\right)}-\frac{3i\hbar}{2\varepsilon}\frac{\beta^2P_j\Delta P_j}{1-\beta^2P_j^2}-\frac{9\hbar^2\beta^4m\omega\omega_0}{8}\frac{P_j^2}{\left(1-\beta^2\mathbf{P}_j^2\right)}\right.\right. \\
&\left.\left.-\frac{3(D-1)}{4}m\omega\omega_0\beta^2\hbar^2+\frac{\hbar^2m\omega\omega_0\beta^2D}{2}-\frac{\omega_0}{2m\omega}\frac{P_j^2}{1-\beta^2P_j^2}\right]\right\}, \qquad (3.87)
\end{aligned}$$

through this path integral, the radial propagator $K_l(P_b, P_a, T)$ associated with this kernel (3.87) is obtained

$$K_l(P_b,P_a,T) = (-i)\lim_{N\to\infty} e^{-\frac{iTm\omega\omega_0\beta^2}{\hbar}\left(\frac{3(D-1)}{4}-\frac{D}{2}\right)}\prod_{j=1}^{N}\left(\int_0^\infty \frac{dP_j}{\sqrt{1-\beta^2P_j^2}}\right)$$
$$\times\prod_{j=1}^{N+1}\sqrt{\frac{1}{2\pi i\hbar\varepsilon m\omega\omega_0}}\exp\left\{\frac{i\varepsilon}{\hbar}\sum_{j=1}^{N+1}\left[\frac{(\Delta P_j)^2}{2m\omega\omega_0\varepsilon^2\left(1-\beta^2P_j^2\right)}-\frac{3i\hbar}{2\varepsilon}\frac{\beta^2P_j\Delta P_j}{1-\beta^2P_j^2}-\frac{\omega_0}{2m\omega}\frac{P_j^2}{1-\beta^2P_j^2}\right.\right.$$
$$\left.\left.+\frac{m\omega\omega_0\hbar^2}{2}\frac{\beta^4P_j^2(-9/4)}{\left(1-\beta^2P_j^2\right)}-\frac{m\omega\omega_0\hbar^2\left[(l+(D/2)-1)^2-1/4\right]}{2}\frac{\left(1-\beta^2P_j^2\right)}{P_jP_{j-1}}\right]\right\}. \tag{3.88}$$

Where C_T, the corresponding total quantum correction, is provided as

$$C_T = \frac{i}{\hbar}\frac{m\omega\omega_0\hbar^2\beta^2\varepsilon}{2}\left[\frac{5}{4}\tan^2(x_j)-\left(2\delta^2-\delta-1\right)\cos^{-2}(x_j)\right]. \tag{3.89}$$

In this case, the problem is transformed into the relative the Pöschel–Teller radial propagator, by employing the coordinate transformation $\beta P = \sin(x)$.

$$K_l(x_b,x_a,T) = (-i)\lim_{N\to\infty} e^{\left[-\frac{iTm\omega\omega_0\beta^2\hbar^2}{2\hbar}\left[\frac{D}{2}-\frac{5}{2}-\left[(l+(D/2)-1)^2-1/4\right]\right]+\frac{i}{\hbar}\frac{\omega_0T}{2m\omega\beta^2}\right]}$$
$$\times\prod_{j=1}^{N}\left(\int_0^\infty dx_j\right)\prod_{j=1}^{N+1}\left[\sqrt{\frac{1}{2\pi i\hbar\varepsilon m\omega\omega_0\beta^2}}\right]\exp\left\{\frac{i}{\hbar}\sum_{j=1}^{N+1}\left[\frac{(\Delta x_j)^2}{2m\omega\omega_0\beta^2\varepsilon}\right.\right.$$
$$\left.\left.-\frac{m\omega\omega_0\beta^2\varepsilon}{2}\left(\frac{1/m^2\omega^2\beta^4}{\cos^2(x_j)}\right)-\frac{m\omega\omega_0\beta^2\hbar^2\varepsilon\left[(l+(D/2)-1)^2-1/4\right]}{2}\frac{1}{\sin^2(x_j)}\right]\right\}, \tag{3.90}$$

furthermore, [49] provides the solution to the latter path integral

$$K_l(P_b,P_a,T) = (-i)\lim_{N\to\infty} e^{\left[-\frac{iTm\omega\omega_0\beta^2\hbar^2}{2\hbar}\left[\frac{D}{2}-\frac{5}{2}-\left[(l+(D/2)-1)^2-1/4\right]\right]+\frac{i}{\hbar}\frac{\omega_0T}{2m\omega\beta^2}\right]}$$
$$\times\sum_{n=0}^{\infty}\exp\left[\frac{-im\omega\omega_0\beta^2}{2\hbar}(k+\lambda+2n)^2T\right]\Phi_n^*(P_a)\,\Phi_n(P_b), \tag{3.91}$$

with

$$\Phi_n(x) = \left[\frac{2n(k+\lambda+2n)\Gamma(k+\lambda+n)}{\Gamma\left(k+\frac{1}{2}+n\right)\Gamma\left(\lambda+\frac{1}{2}+n\right)}\right]^{1/2}\sin^{\kappa}(x)\cos^{\lambda}(x)\,P_n^{(\kappa-1/2,\lambda-1/2)}\left(2\sin^2(x)-1\right), \tag{3.92}$$

where $P_n^{(\kappa-1/2,\lambda-1/2)}$ indicates Jacobi polynomials, and κ, λ parameters are defined as

$$\lambda=\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{4}{m^2\omega^2\beta^4}},\ \kappa=\pm\left(l+\frac{D}{2}-1\right)+\frac{1}{2}. \tag{3.93}$$

Consequently, the energy eigenvalues E_n can be expressed as follows:

$$\begin{aligned} E_n &= \left[\frac{m\omega\omega_0\beta^2}{2}\left[\left(2n+l+\frac{D}{2}\right)^2-\left(\frac{D}{2}-\frac{5}{2}\right)\right]\right. \\ &\left.+\omega_0\left(2n+l+\frac{D}{2}\right)\sqrt{1+\frac{m^2\omega^2\beta^4}{4}}\right]. \end{aligned} \tag{3.94}$$

For $D=3$ we find the exact result obtained in [27].

3.5 Conclusion

In this section, we study the path integral of a free particle and an isotropic harmonic oscillator in the momentum space representation in D-dimensional within the framework of the SdS algebra with two fundamental deformation parameters. To perform this path integral, we simplify the problem to a purely radial one using the D-dimensional spherical coordinates for momentum variables. Next, we apply the coordinate transformation method with a $\delta-$point discretization interval to convert the problem to that of a particle in a symmetric Pöschel–Teller potential. Notably, this approach is consistent with the choice made in the single-parameter cases, indicating that the discretization is dependent on the $\delta-$point discretization in a similar way [35, 37]. Through radial spectral decomposition of the transition amplitude, we determine the momentum space wave functions and energy levels. The energy levels exhibit a dependence on $(2n+l)^2$, just like the energy levels of a particle trapped in a potential well.

Chapter 4

Thermal Properties of a One-Dimensional Dirac Oscillator in a Homogeneous Electric Field with Generalized Snyder Model: Path Integral Treatment

4.1 Introduction

Snyder's 1947 work [18, 19], which proposed the Heisenberg generalisation principle in quantum field theory to solve the divergence problem, has been of great interest to the field of quantum physics. Such as dynamics based on variable masses in semiconductor heterostructures, as expressed by the generalised displacement operator [50], the behaviour of an impurity atom with 3He in a Bose liquid, as examined in [51], and the description of low-energy excitations in Graphene in conjunction with Fermi velocity through the application of the generalised Heisenberg algebra, which involves determining the momentum commutator that is proportional to pseudo-spin [13]. Furthermore, It also plays a fundamental role in non-commutative geometries [53], string theory [52], black hole physics [54], and quantum gravity [55]. According to the concept of the (GUP), these theories require the existence of a minimum length on the order of the Planck mass ($m_P = \sqrt{\hbar c/G}$), $\sqrt{\beta} \sim 10^8\,\mathrm{kg}^{-1}$ (i.e., $\beta \sim (m_P)^{-2}$), or the presence a minimum momentum on the order of the square root of the cosmological constant,$\sqrt{\alpha} \sim 10^{-24}\,\mathrm{cm}^{-1}$, as in the context of the (EUP) [56]. The combination of these results in the (SdS) model, or (TSR) [27]. TSR relates three invariant scales: the cosmological constant Λ [27] , the Snyder parameter β, and the speed of light in vacuum c. Through different methods, these theories have provided solutions to a number of quantum mechanical problems [57–64]. However, the Feynman path integral formalism is a mathematical framework derived from ideas regarding classical trajectories that is used to explain quantum mechanics. Choosing the discretization interval period is necessary in order to use this mathematical technique. For an exact result curved spaces, choosing the midpoint as the discretization schema provides in the context of usual Heisenberg commutation relations, for details see the reference [65]. But when we generalise the Heisenberg principle, this choice quickly becomes problematic as we see in the cases of non-zero minimal momentum [66, 67] and non-zero minimal length [35, 37]. Moreover, the path integral approach in $D-$dimensional quantum mechanics has been developed by the authors of [68], who took into account the coexistence of minimal position and momentum uncertainty.

In this chapter, we extend our study to the relativistic case focusing, in particular, on the 1D-DO in a uniform electric field. After that, the difference in the midpoint discretization interval within the aSdS-model is then confirmed. According to [27], the alteration of the commutation relation between the position and momentum operators in one dimension is articulated as follows:

$$\left[\hat{X},\hat{P}\right]=\imath\hbar\left(1+\beta\hat{P}^{2}+\alpha\hat{X}^{2}+\sqrt{\alpha\beta}\left(\hat{X}\hat{P}+\hat{P}\hat{X}\right)\right). \tag{4.1}$$

Eq. (4.1) yields the generalised uncertainty relation shown below if we put ($\langle X\rangle=\langle P\rangle=0$).

$$\left(\Delta X\right)\left(\Delta P\right)\geq\frac{\hbar}{2}\frac{\left(1+\alpha\left(\Delta X\right)^{2}+\beta\left(\Delta P\right)^{2}\right)}{1+\hbar\sqrt{\alpha\beta}}. \tag{4.2}$$

Thus, modifying this deformed algebra yields minimal uncertainty in momentum and position.

$$\left(\Delta X\right)_{\min}=\frac{\hbar\sqrt{\beta}}{1+2\hbar\sqrt{\alpha\beta}},\quad\left(\Delta P\right)_{\min}=\frac{\hbar\sqrt{\alpha}}{1+2\hbar\sqrt{\alpha\beta}}. \tag{4.3}$$

In momentum representation, the position operator $\hat{X}$ and momentum operator $\hat{P}$ that adhere to the algebra (4.1) can be expressed as follows:

$$\hat{X}=\hat{\mathfrak{X}}+\sqrt{\frac{\beta}{\alpha}}\kappa\hat{\mathfrak{P}},\quad\hat{P}=-\sqrt{\frac{\alpha}{\beta}}\hat{\mathfrak{X}}+\left(1-\kappa\right)\hat{\mathfrak{P}}, \tag{4.4}$$

small positive parameters (α,β) are used here. Moreover, κ is a free parameter that ensuring the Hamiltonian's hermiticity (in all scenarios). According to [69], the operators $(\hat{X},\hat{P})$ satisfy the commutation relation:

$$\left[\hat{\mathfrak{X}},\hat{\mathfrak{P}}\right]=\imath\hbar\left(1+\beta\hat{\mathfrak{P}}^{2}\right). \tag{4.5}$$

Alternatively, by utilizing the auxiliary operators $\hat{x}$ and $\hat{p}$, these position and momentum coordinate operators can be written in a way that satisfies the Snyder-Heisenberg commutation relation $\hat{p}$, obeying standard commutation relation (i.e., $[\hat{x},\hat{p}]=\imath\hbar$), defined by the following relations

$$\hat{\mathfrak{X}}=\sqrt{1-\beta\hat{p}^{2}}\hat{x},\quad \hat{\mathfrak{P}}=\frac{\hat{p}}{\sqrt{1-\beta\hat{p}^{2}}}. \tag{4.6}$$

In the aSdS-model case, all real values of p are acceptable, but if $\alpha,\beta>0$, the permitted range of values of p is limited by $p^2<1/\beta$ in the SdS-model. Additionally, the operators of $\hat{X}$ and $\hat{P}$ for the SdS-model are only symmetric in the subspace $L^2(R,dp/\sqrt{1-\beta p^2})$, where the scalar product has the following definition

$$\langle\psi\mid\phi\rangle=\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp}{\sqrt{1-\beta p^{2}}}\psi^{*}\left(p\right)\phi\left(p\right), \tag{4.7}$$

periodic boundary conditions are satisfied by the wave function, $\psi\left(-1/\sqrt{\beta}\right)=\psi\left(1/\sqrt{\beta}\right)$, resulting to the following closure relation:

$$\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp}{\sqrt{1-\beta p^{2}}}\left|p\right\rangle\left\langle p\right|=I. \tag{4.8}$$

Notably, the (α,β) parameters in the aSdS-model are negative. Thus, we modify the integration limits in Eq. (5.9) across all spaces. In addition, as described in [27], the corresponding formal eigenvectors coincide with those of the $\hat{X}-$position operator.

$$\langle p|x\rangle_{\alpha,\beta}=\frac{1}{\sqrt{2\pi\hbar}}\left(1-\beta p^{2}\right)^{-\frac{\gamma}{2}}\exp\left(-\frac{\imath x}{\hbar}\frac{\arcsin\sqrt{\beta}p}{\sqrt{\beta}}\right),\ \gamma=\imath\kappa/\hbar\sqrt{\alpha\beta}. \tag{4.9}$$

Afterwards, the closure relation for the maximally localised states is applied to Eq. (4.9), and we utilise the properties of the delta function $\delta f(x)=\sum_{i}\delta\left(x-x_i\right)/\left|f'(x_i)\right|$, where x_i are the roots of $f(x)$ [68]. At last, we have:

$$\langle p_j|p_{j-1}\rangle_{\alpha,\beta}=\int\frac{dx_j}{2\pi\hbar}\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta p_j^2}\exp\left(-\frac{\imath x_j}{\hbar}\left(p_j-p_{j-1}\right)\right). \tag{4.10}$$

It is appropriate to apply the delta functions (4.10) on the subspace $L^2(R,dp/\sqrt{1-\beta p^2})$, when both α and β (i.e., SdS or aSdS) signs are taken. Moreover, regaining the standard projection relation $\langle p|p'\rangle_{(\alpha,\beta)\to 0}=\delta\left(p-p'\right)$ occurs when both α and β are equal to zero. On the other hand, the time p_0-component is expressed as follows since no deformation is applied to it

$$\langle p_0|p_0'\rangle=\delta\left(p_0-p_0'\right)=\int\frac{dt}{2\pi\hbar}e^{-\frac{\imath}{\hbar}t\left(p_0-p_0'\right)}. \tag{4.11}$$

Consequently, the elements matrix of the operators $\hat{\mathfrak{X}}$ and $\hat{\mathfrak{X}}^2$ are provided, respectively, by

$$\left\langle p_j\left|\hat{\mathfrak{X}}\right|p_{j-1}\right\rangle_{\alpha,\beta}=\langle p_j|p_{j-1}\rangle_{\alpha,\beta}\left[(\gamma-1)\frac{\imath\hbar\beta p}{\sqrt{1-\beta p_j^2}}+\sqrt{1-\beta p^2}x_j\right], \tag{4.12}$$

$$\left\langle p_j\left|\hat{\mathfrak{X}}^2\right|p_{j-1}\right\rangle_{\alpha,\beta}$$
$$=\langle p_j|p_{j-1}\rangle_{\alpha,\beta}\left[-\gamma(\gamma-1)\frac{\hbar^2\beta^2p_j^2}{1-\beta p_j^2}-\hbar^2\beta\left(\gamma-1\right)+\left(1-\beta p_j^2\right)x_j^2+2\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)p_jx_j\right]. \tag{4.13}$$

We introduce in section 2 the formulation of the path integral in one-dimensional momentum space of the SdS model for DO particles exposed to a uniform electric field, without the need for Grassmann variables, as demonstrated in [35, 70]. This method, which was previously used in [71, 72], entails executing the path integration over the components of the Green function matrix for 1D-DO particles exposed to the uniform electric field using the SdS model. In section 3, we use the coordinate transformation method to obtain the local kinetic part, which leads to the propagator of Rosen-Morse type I and II [49]. The precise propagator and the associated energy eigenvalues are inferred. In section 4, we assess the thermodynamic properties of this system and offer a detailed physical analysis of the associated plotted graphs.

4.2 Path Integral Formulation in (anti) Snyder de Sitter

The Green function for a relativistic DO particle in one-dimensional space is defined as the inverse of the Dirac operator in the absence of electric field interaction.

$$\left(\gamma^0 \imath\hbar\partial_t - \hat{H}\right)\hat{S} = -I, \tag{4.14}$$

where $\hat{H}$ denotes the DO equation's Hamiltonian operator and is provided by

$$\hat{H} = c\gamma^1\left(\hat{P} - \imath m\omega\gamma^0\hat{X}\right) + mc^2, \tag{4.15}$$

where ω is the oscillator's classical frequency and mc^2 denotes the rest mass. For further detailed consideration, we select that the time component ($\hat{P}_0 = \imath\hbar\partial_0 = \imath\hbar\partial/\partial ct$, $\hat{X}_0 = \hat{x}^0 := ct$) is deformation-free, and that the momenta $\hat{P}$ and position $\hat{X}$ operators verify the Eq. (4.6). Accordingly, we can generalize the Green's function (4.14) equation for the $(1+1)-$dimensional DO in the presence of a uniform electric field $\mathcal{E}$ to include the following:

$$\left[\gamma^0\left(\imath\hbar\partial_t + e\mathcal{E}\hat{X}\right) - c\gamma^1\left(\hat{P} - \imath m\omega\gamma^0\hat{X}\right) - mc^2\right]\hat{S} = -I. \tag{4.16}$$

The $\gamma^\mu-$Dirac matrices in the $(1+1)$ dimension are represented by the Pauli matrices after the choice

$$\gamma^0 = \sigma_3,\ \gamma^1 = \imath\sigma_2,\ \gamma^2 = -\imath\sigma_1. \tag{4.17}$$

Following that, the solution to Eq. (4.15) is given as

$$\hat{S} = \left[\mathcal{O}_-^D\right]^{-1} = \left[\mathcal{O}_+^D\right]\left[\mathcal{O}_-^D\mathcal{O}_+^D\right]^{-1}, \tag{4.18}$$

the operator $\mathcal{O}_\pm^D$ is defined by

$$\mathcal{O}_\pm^D = \left[\gamma^0\left(\imath\hbar\partial_t + e\mathcal{E}\hat{X}\right) - c\gamma^1\left(\hat{P} - \imath m\omega\gamma^0\hat{X}\right) \pm mc^2\right]. \tag{4.19}$$

Using the Schwinger proper-time method [73], and noting that $\hat{S}=\left[\mathcal{O}_{+}^{D}\right]\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]^{-1}$, it is more practical to express the $\hat{S}$ Green's function in this way.

$$\hat{S}=\left[\mathcal{O}_{+}^{D}\right]\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]^{-1}=(\imath/\hbar)\left[\mathcal{O}_{+}^{D}\right]\int_{0}^{\infty}d\lambda\exp\left(\frac{\imath}{\hbar}\lambda\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]\right), \tag{4.20}$$

the parameter λ in the above equation indicates is a invariant parameter and is an even variable, with the $\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]$ operator acting as a Hamiltonian, and it can be expressed as follows

$$\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]=\left\{\left(\hat{P}_{0}+e\mathcal{E}\hat{X}\right)^{2}-c^{2}\hat{P}^{2}-c^{2}m^{2}\omega^{2}\hat{X}^{2}-m^{2}c^{4}-\left(ce\mathcal{E}\gamma^{0}\gamma^{1}-\imath m\omega c^{2}\gamma^{0}\right)\left[\hat{X},\hat{P}\right]\right\}. \tag{4.21}$$

Using the SdS algebra provided by Eq. (4.1), we obtain:

$$\begin{aligned}\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]=&\left\{\hat{P}_{0}^{2}-m^{2}c^{4}+2e\mathcal{E}\hat{P}_{0}\hat{X}-\varpi^{2}\hat{X}^{2}-c^{2}\hat{P}^{2}\right.\\&\left.-\imath\hbar\left(ec\mathcal{E}\gamma^{0}\gamma^{1}-\imath c^{2}m\omega\gamma^{0}\right)\left(1+\beta\hat{P}^{2}+\alpha\hat{X}^{2}+\sqrt{\alpha\beta}\left(\hat{X}\hat{P}+\hat{P}\hat{X}\right)\right)\right\},\end{aligned} \tag{4.22}$$

with $\varpi^{2}=\left(c^{2}m^{2}\omega^{2}-e^{2}\mathcal{E}^{2}\right)$.

Moreover, we need to express this Hamiltonian using position and momentum operators consistent with the flat Snyder model, characterized by the modified commutation relationship defined in Eq. (4.5) [27]. By substituting operators $\left(\hat{\mathfrak{X}},\hat{\mathfrak{P}}\right)$ into an expression $\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]$, the Eq. (4.22) becomes,

$$\begin{aligned}\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]=&\left\{\hat{P}_{0}^{2}-m^{2}c^{4}+2e\mathcal{E}\hat{P}_{0}\hat{\mathfrak{X}}+2e\mathcal{E}\hat{P}_{0}\kappa\sqrt{\tfrac{\beta}{\alpha}}\hat{\mathfrak{P}}+c^{2}\left(-\frac{\varpi^{2}}{c^{2}}\frac{\beta}{\alpha}\kappa^{2}-(1-\kappa)^{2}\right)\hat{\mathfrak{P}}^{2}\right.\\&+\left(c^{2}(1-\kappa)\sqrt{\tfrac{\alpha}{\beta}}-\kappa\varpi^{2}\sqrt{\tfrac{\beta}{\alpha}}\right)\left(\hat{\mathfrak{X}}\hat{\mathfrak{P}}+\hat{\mathfrak{P}}\hat{\mathfrak{X}}\right)\\&\left.-\left(\varpi^{2}+c^{2}\tfrac{\alpha}{\beta}\right)\hat{\mathfrak{X}}^{2}-\imath\hbar\left(ec\mathcal{E}\gamma^{0}\gamma^{1}-\imath c^{2}m\omega\gamma^{0}\right)\left(1+\beta\hat{\mathfrak{P}}^{2}\right)\right\}.\end{aligned} \tag{4.23}$$

In order for the previously mentioned term $\left(\hat{\mathfrak{X}}\hat{\mathfrak{P}}+\hat{\mathfrak{P}}\hat{\mathfrak{X}}\right)$ to terminate at zero, we impose a condition on κ,

$$\kappa=\left(1-\frac{\beta}{\alpha}\varpi^{2}/c^{2}\right)^{-1}. \tag{4.24}$$

As a result, the Hamiltonian operator becomes as

$$\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]=\left\{\hat{P}_0^2-m^2c^4+2e\mathcal{E}\hat{P}_0\hat{\mathcal{X}}+\frac{2e\mathcal{E}\hat{P}_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\hat{\mathcal{P}}-\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\hat{\mathcal{X}}^2\right.$$
$$\left.-\frac{\frac{\beta}{\alpha}\varpi^2}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\hat{\mathcal{P}}^2-\imath\hbar\left(ec\mathcal{E}\gamma^0\gamma^1-\imath c^2m\omega\gamma^0\right)\hat{F}\left(\hat{\mathcal{P}}\right)\right\}, \tag{4.25}$$

with

$$\hat{F}\left(\hat{\mathcal{P}}\right)=1+\beta\hat{\mathcal{P}}^2. \tag{4.26}$$

In momentum representation, the $\left[\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]$ serves as

$$\mathcal{G}\left(p_b,p_a,p_{0b},p_{0a}\right)=\left(\imath/\hbar\right)\int_0^{\infty}d\lambda\left\langle p_b,p_{0b}\left|\exp\left(-\frac{\imath}{\hbar}\lambda\left[-\mathcal{O}_{-}^{D}\mathcal{O}_{+}^{D}\right]\right)\right|p_a,p_{0a}\right\rangle. \tag{4.27}$$

Before moving further,in order to avoid computing their Feynman path integral expression for matrices, it is appropriate to introduce the following exponential matrix and simplify its form as follows

$$e^{\lambda\left(ec\mathcal{E}\gamma^0\gamma^1-\imath c^2m\omega\gamma^0\right)\hat{F}\left(\hat{\mathcal{P}}\right)}=\frac{1}{2}\sum_{s=\pm1}\left\{\mathbb{I}-\begin{pmatrix}\frac{scm\omega}{\varpi} & \imath s\frac{e\mathcal{E}}{c\varpi}\\ \imath s\frac{e\mathcal{E}}{c\varpi} & -s\frac{cm\omega}{\varpi}\end{pmatrix}\right\}e^{\imath s\lambda c\varpi\hat{F}\left(\hat{\mathcal{P}}\right)}, \tag{4.28}$$

subsequently, we execute the following equality [71]:

$$\cosh\left(\delta\right)=\frac{cm\omega}{\varpi},\ \sinh(\delta)=\frac{e\mathcal{E}}{c\varpi}, \tag{4.29}$$

after conducting a few computational calculations, we obtain:

$$e^{-\lambda\left(ec\mathcal{E}\gamma^0\gamma^1-\imath c^2m\omega\gamma^0\right)\hat{F}\left(\hat{\mathcal{P}}\right)}=\sum_{s=\pm1}\exp\left(-\frac{\delta}{2}\sigma_2\right)\mathbb{X}_s\mathbb{X}_s^{+}\exp\left(\frac{\delta}{2}\sigma_2\right)e^{\imath s\lambda c\varpi\hat{F}\left(\hat{\mathcal{P}}\right)}. \tag{4.30}$$

Here, $\mathbb{X}_s=\frac{1}{2}\begin{pmatrix}(1+s) & (1-s)\end{pmatrix}^T$ and $\mathbb{X}_s^{+}$ is the transpose of the vector $\mathbb{X}_s$, denoted as $\mathbb{X}_s^{+}=\mathbb{X}_s^{T}$.

Thus, the expression (4.27) can be presented as follows:

$$\mathcal{G}(p_b, p_a, p_{0b}, p_{0a}) = (\imath/\hbar) \sum_{s=\pm 1} \exp\left(-\frac{\delta}{2}\sigma_2\right) \mathbb{X}_s \mathbb{X}_s^+ \exp\left(\frac{\delta}{2}\sigma_2\right) \int_0^\infty d\lambda \left\langle p_b, p_{0b} \right| \exp\left(-\frac{\imath}{\hbar}\hat{\mathcal{H}}^{(s)}\right) \left| p_a, p_{0a} \right\rangle, \tag{4.31}$$

with

$$\hat{\mathcal{H}}^{(s)} = -\lambda \left\{ \hat{P}_0^2 - m^2c^4 + 2e\mathcal{E}\hat{P}_0\hat{\mathcal{X}} + \frac{2e\mathcal{E}\hat{P}_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\hat{\mathcal{P}} - \left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)\hat{\mathcal{X}}^2 \right.$$
$$\left. - \frac{\varpi^2\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\hat{\mathcal{P}}^2 + s\hbar c\varpi \hat{F}\left(\hat{\mathcal{P}}\right) \right\}. \tag{4.32}$$

Following this, we will use the path integral framework to construct the Green function, which decomposes the exponential $\exp(-\imath\hat{\mathcal{H}}^{(s)})$ into $(N+1)$ exponential $\exp(-\imath\varepsilon\hat{\mathcal{H}}^{(s)})$, with $\varepsilon = \tau_j - \tau_{j-1} = 1/(N+1)$. Then, we insert N times resolution identity (4.8) between each pair of infinitesimal operator $\exp(-\imath\varepsilon\hat{\mathcal{H}}^{(s)})$. We shall obtain

$$\mathcal{G}(p_b, p_a, p_{0b}, p_{0a}) = (\imath/\hbar) \sum_{s=\pm 1} \left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^+ e^{\frac{\delta}{2}\sigma_2}\right] \lim_{N\longrightarrow\infty} \int_0^\infty d\lambda \prod_{j=1}^{N} \left[\int_{-\infty}^{+\infty} dp_{0j} \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \frac{dp_j}{\sqrt{1-\beta p_j^2}}\right]$$
$$\times \prod_{j=1}^{N+1} \left\langle p_j, p_{0j} \right| \exp\left(-\frac{\imath\varepsilon}{\hbar}\hat{\mathcal{H}}^{(s)}\right) \left| p_{j-1}, p_{0j-1} \right\rangle_{\alpha,\beta}. \tag{4.33}$$

It is convenient to develop the exponential up to the first order of ε to facilitate the calculation. Thus, we write

$$\lim_{N\longrightarrow\infty,\varepsilon\to 0} \left\langle p_j, p_{0j} \right| e^{-\frac{\imath\varepsilon}{\hbar}\hat{\mathcal{H}}^{(s)}} \left| p_{j-1}, p_{0j-1} \right\rangle_{\alpha,\beta}$$
$$= \lim_{N\longrightarrow\infty,\varepsilon\to 0} \left[\left\langle p_j, p_{0j} \mid p_{j-1}, p_{0j-1} \right\rangle_{\alpha,\beta} - \frac{\imath\varepsilon}{\hbar} \left\langle p_j, p_{0j} \right| \hat{\mathcal{H}}^{(s)} \left| p_{j-1}, p_{0j-1} \right\rangle_{\alpha,\beta} \right]. \tag{4.34}$$

The Hamiltonian operator represented in the SdS framework on the projection relation $\left\langle p_j \mid p_{j-1} \right\rangle_{\alpha,\beta}$ given in Eq. (4.10) is then eliminated by replacing all of the operators ($\hat{\mathcal{X}}$,$\hat{\mathcal{P}}$, $\hat{\mathcal{X}}^2$, $\hat{\mathcal{P}}^2$). As a result, the expression $\mathcal{G}(p_b, p_a, p_{0b}, p_{0a})$ is converted into the following path integral in phase-space

$$\mathcal{G}\left(p_b, p_a, p_{0b}, p_{0a}\right) = (\imath/\hbar) \lim_{\substack{N\to\infty \\ \varepsilon\to 0}} \sum_{s=\pm 1} \left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right] \int_0^{\infty} d\lambda \prod_{j=1}^{N} \left[\int_{-\infty}^{+\infty} dp_{0j} \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \frac{dp_j}{\sqrt{1-\beta p_j^2}}\right]$$
$$\times \prod_{j=1}^{N+1} \left[\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}} \sqrt{1-\beta p_j^2} \int \frac{dx_j}{2\pi\hbar} \frac{dt_j}{2\pi\hbar} e^{\frac{\imath}{\hbar} t_j \Delta p_{0j}}\right] \exp\left\{\frac{\imath}{\hbar} \sum_{j=1}^{N+1} \left[-x_j \Delta p_j + \lambda\varepsilon\left(p_{0j}^2 - m^2c^4\right)\right.\right.$$
$$+\lambda\varepsilon\gamma(\gamma-1)\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2 p_j^2}{1-\beta p_j^2} + \lambda\varepsilon\hbar^2\beta\left(\gamma-1\right)\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)$$
$$-\lambda\varepsilon\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)x_j^2 - \lambda\varepsilon 2\imath\hbar\beta\left(\gamma - \frac{3}{2}\right)\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)p_j x_j$$
$$+\lambda\varepsilon 2e\mathcal{E}p_{0j}\sqrt{1-\beta p^2}x_j + \lambda\varepsilon\frac{2e\mathcal{E}p_{0j}\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}} + 2\lambda\varepsilon e\mathcal{E}p_{0j}\left(\gamma-1\right)\frac{\imath\hbar\beta p_j}{\sqrt{1-\beta p_j^2}}$$
$$\left.\left.-\lambda\varepsilon\frac{\varpi^2\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2} + \varepsilon s\hbar\lambda c\varpi\left(1 + \frac{\beta p_j^2}{1-\beta p_j^2}\right)\right]\right\}. \tag{4.35}$$

As is customary, we perform Gaussian integration over t_j and x_j, for this system, we determine the Lagrangian expression

$$\mathcal{G}\left(p_b,p_a,p_{0b},p_{0a}\right)=(\imath/\hbar)\lim_{\substack{N\to\infty\\ \varepsilon\to 0}}\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\int_0^{\infty}d\lambda\prod_{j=1}^{N}\left[\int_{-\infty}^{+\infty}dp_{0j}\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_j}{\sqrt{1-\beta p_j^2}}\right]$$
$$\times\prod_{j=1}^{N+1}\left[\delta\left(p_{0j}-p_{0j-1}\right)\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta p_j^2}\sqrt{\frac{1}{4\imath\pi\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)}}\right]$$
$$\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\Delta p_j^2}{4\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)}+\frac{\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)}{\left(1-\beta p_j^2\right)}p_j\Delta p_j+\lambda\varepsilon\left(p_{0j}^2-m^2c^4\right)\right.\right.$$
$$-\frac{\lambda\varepsilon\hbar^2\beta^2\left(\gamma-\frac{3}{2}\right)^2\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}{\left(1-\beta p_j^2\right)}p_j^2+\lambda\varepsilon\gamma(\gamma-1)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2p_j^2}{1-\beta p_j^2}$$
$$+\lambda\varepsilon\hbar^2\beta\left(\gamma-1\right)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)-2\imath\hbar\beta\lambda\varepsilon e\mathcal{E}p_{0j}\left(\gamma-\frac{3}{2}\right)\frac{p_j}{\sqrt{1-\beta p_j^2}}$$
$$-\frac{e\mathcal{E}p_{0j}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\Delta p_j}{\sqrt{1-\beta p_j^2}}+\frac{\lambda\varepsilon e^2\mathcal{E}^2p_{0j}^2}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}+\lambda\varepsilon\frac{2e\mathcal{E}p_{0j}\sqrt{\frac{\beta}{\alpha}}}{\left(1-\frac{\beta}{\alpha}\frac{e^2\mathcal{E}^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}}$$
$$\left.\left.+\lambda\varepsilon 2e\mathcal{E}p_{0j}\left(\gamma-1\right)\frac{\imath\hbar\beta p_j}{\sqrt{1-\beta p_j^2}}-\lambda\varepsilon\frac{\varpi^2\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2}+\varepsilon s\hbar\lambda c\varpi\left(1+\frac{\beta p_j^2}{1-\beta p_j^2}\right)\right]\right\}. \tag{4.36}$$

We execute the following equality to the first order of ε, in order to simplify the above expression,

$$\frac{e\mathcal{E}p_{0j}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\Delta p_j}{\sqrt{1-\beta p_j^2}}=\frac{e\mathcal{E}p_{0j}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\Delta\arcsin\left(\sqrt{\beta}p_j\right)}{\sqrt{\beta}}$$
$$+\frac{e\mathcal{E}p_{0j}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\left(\Delta p_j\right)^2}{2}\frac{\beta p_j}{\left(1-\beta p_j^2\right)^{3/2}}, \tag{4.37}$$

where

$$\left(\Delta p_j\right)^2\sim 2\imath\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right). \tag{4.38}$$

By substituting Eq. (4.38) in Eq. (4.37) and then into Eq. (4.36), can be written the below equation

$$\mathcal{G}\left(p_b,p_a,p_{0b},p_{0a}\right)=(\imath/\hbar)\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\delta\left(p_{0b}-p_{0a}\right)\int_0^{\infty}d\lambda$$
$$\times e^{\frac{\imath}{\hbar}\lambda\left[p_{0b}^2-m^2c^4+\lambda\varepsilon\hbar^2\beta(\gamma-1)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\right]}\times e^{-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_{0b}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\arcsin\left(\sqrt{\beta}p_b\right)-\arcsin\left(\sqrt{\beta}p_a\right)}{\sqrt{\beta}}}$$
$$\times\lim_{\substack{N\to\infty\\ \varepsilon\to0}}\prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_j}{\sqrt{1-\beta p_j^2}}\right]\prod_{j=1}^{N+1}\left[\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}\sqrt{\frac{1}{4\pi\imath\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}}\right]$$
$$\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\Delta p_j^2}{4\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)}+\frac{\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)}{\left(1-\beta p_j^2\right)}p_j\Delta p_j\right.\right.$$
$$-\lambda\varepsilon\left(\gamma-\frac{3}{2}\right)^2\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2p_j^2}{\left(1-\beta p_j^2\right)}+\lambda\varepsilon\gamma(\gamma-1)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2p_j^2}{1-\beta p_j^2}$$
$$-2\imath\hbar\beta\lambda\varepsilon\left(\gamma-\frac{3}{2}\right)\frac{e\mathcal{E}p_0p_j}{\sqrt{1-\beta p_j^2}}-\imath\hbar\lambda\varepsilon\frac{e\mathcal{E}p_0\beta p_j}{\sqrt{1-\beta p_j^2}}+2\imath\hbar\lambda\varepsilon\left(\gamma-1\right)\frac{e\mathcal{E}p_0\beta p_j}{\sqrt{1-\beta p_j^2}}$$
$$+\lambda\varepsilon\frac{2e\mathcal{E}p_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}}-\lambda\varepsilon\frac{\beta}{\alpha}\frac{\left(c^2m^2\omega^2-e^2\mathcal{E}^2\right)}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2}$$
$$\left.\left.+\lambda\varepsilon\frac{e^2\mathcal{E}^2p_0^2}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}+\lambda\varepsilon s\hbar c\varpi\left(1+\frac{\beta p_j^2}{1-\beta p_j^2}\right)\right]\right\},\tag{4.39}$$

and with some simplifications, we will find

$$\mathcal{G}(p_b,p_a,p_{0b},p_{0a})=(\imath/\hbar)\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\delta(p_{0b}-p_{0a})\int_0^{\infty}d\lambda\prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_j}{\sqrt{1-\beta p_j^2}}\right]$$
$$\times e^{-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\arcsin\left(\sqrt{\beta}p_b\right)-\arcsin\left(\sqrt{\beta}p_a\right)}{\sqrt{\beta}}}e^{\frac{\imath}{\hbar}\lambda\left(p_{0j}^2+\frac{e^2\mathcal{E}^2p_0^2}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}-m^2c^4+\hbar^2\beta(\gamma-1)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\right)}$$
$$\times\lim_{\substack{N\to\infty\\ \varepsilon\to0}}\prod_{j=1}^{N+1}\left[\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}\sqrt{\frac{1}{4\pi\imath\hbar\varepsilon\lambda\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}}\right]$$
$$\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\Delta p_j^2}{4\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)}+\frac{\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)}{1-\beta p_j^2}p_j\Delta p_j\right.\right.$$
$$-\lambda\varepsilon\hbar^2\beta^2\left(\gamma^2-3\gamma+\frac{9}{4}\right)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{p_j^2}{1-\beta p_j^2}$$
$$+\lambda\varepsilon\left(\gamma^2-\gamma\right)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2p_j^2}{1-\beta p_j^2}+\lambda\varepsilon\frac{2e\mathcal{E}p_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}}$$
$$\left.\left.-\lambda\varepsilon\frac{\left(c^2m^2\omega^2-e^2\mathcal{E}^2\right)\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2}+\varepsilon s\hbar\lambda c\varpi\left(1+\frac{\beta p_j^2}{1-\beta p_j^2}\right)\right]\right\}. \tag{4.40}$$

Moreover, all terms associated with (γ) in Eq. (4.40) will be nullified by the term $\left(\left(1-\beta p_{j-1}^2\right)/\left(1-\beta p_j^2\right)\right)^{\frac{\gamma}{2}}$ [68],

$$\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}=\exp\left[\left(-\frac{\gamma}{2}\Delta p_j\frac{-2\beta p_j}{\left(1-\beta p_j^2\right)}+\frac{\gamma}{2}\imath\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left[-2\beta-\frac{4\beta^2p_j^2}{1-\beta p_j^2}\right]\right)\right]. \tag{4.41}$$

When we substitute the aforementioned result (4.1) into Eq. (4.40), we get

$$\mathcal{G}(p_b,p_a,p_{0b},p_{0a})=(\imath/\hbar)\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\delta(p_{0b}-p_{0a})\int_0^{\infty}d\lambda$$
$$\times e^{-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\varpi^2+c^2\frac{\alpha}{\beta}}\frac{\arcsin\left(\sqrt{\beta}p_b\right)-\arcsin\left(\sqrt{\beta}p_a\right)}{\sqrt{\beta}}}e^{\frac{\imath}{\hbar}\lambda\left(\frac{\left(c^2m^2\omega^2+c^2\frac{\alpha}{\beta}\right)p_0^2}{\varpi^2+c^2\frac{\alpha}{\beta}}-m^2c^4-\hbar^2\beta\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\right)}\times\mathbb{K}(p_b,p_a,\lambda), \tag{4.42}$$

where the kernel propagator $\mathbb{K}(p_b, p_a, \lambda)$ is defined by the following path integral

$$\mathbb{K}(p_b, p_a, \lambda) = \lim_{N\to\infty} \prod_{j=1}^{N} \left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \frac{dp_j}{\sqrt{1-\beta p_j^2}} \right] \prod_{j=1}^{N+1} \left[\sqrt{\frac{1}{4\pi\imath\hbar\lambda\varepsilon\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)}} \right]$$
$$\times \exp\left\{ \frac{\imath}{\hbar} \sum_{j=1}^{N+1} \left[\frac{\left(\Delta p_j\right)^2}{4\lambda\varepsilon\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)} - \frac{3}{2}\frac{\imath\hbar\beta}{\left(1-\beta p_j^2\right)} p_j \Delta p_j \right.\right.$$
$$-\lambda\varepsilon\hbar^2\beta^2\frac{9}{4}\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)\frac{p_j^2}{\left(1-\beta p_j^2\right)} + \lambda\varepsilon\frac{2e\mathcal{E}p_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}}$$
$$\left.\left. -\lambda\varepsilon\frac{\varpi^2\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2} + \lambda\varepsilon s\hbar c\varpi\left(1+\frac{\beta p_j^2}{1-\beta p_j^2}\right)\right]\right\}. \tag{4.43}$$

Typically, in order to obtain the conventional form of the Feynman path integral for systems based on the principle of generalization, three quantum corrections must be applied: the measure term ($dp_j/\sqrt{1-\beta p_j^2}$), the action term ($\left(\Delta p_j\right)^2/2\varepsilon\left(1-\beta p_j^2\right)$), and the factor term ($p_j\Delta p_j/\left(1-\beta p_j^2\right)$. The quantum corrections from these three terms can be computed using a two-step process, as per [35, 37, 66, 67]. Initially, this Kernel is written at the $\eta-$point discretization interval ($p_j^{(\eta)} = \eta p_j + \left(1-\eta\right)p_{j-1}$). This avoids the use of the midpoint interval in the case of the presence of the SdS model [35, 37, 66, 67]. Throughout the second step, we need to use the momentum coordinate transformation method given by ($\sqrt{\beta}p = \sin\sqrt{\beta}\tilde{q}$) to obtain the usual kinetic term($\left(\Delta\tilde{q}_j\right)^2/2\varepsilon$). According to [35, 67], the formal treatment of the selection of the $\eta-$point discretization interval in the presence of the deformation coefficient has been addressed. Following simple calculations, we obtain the total quantum correction,

$$C_T = \imath\hbar\lambda\varepsilon\beta\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)\left[1 + \frac{9}{4}\tan^2\left(\sqrt{\beta}\tilde{q}\right)\right], \tag{4.44}$$

and this corresponds to fixing $\eta = \frac{1}{2}(1 \pm 1/\sqrt{2})$.

Substituting Eq. (4.44) in Eq. (4.43) and then into Eq. (4.42) we get:

$$\mathcal{G}(p_b,p_a,p_{0b},p_{0a})=(\imath/\hbar)\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\delta(p_{0b}-p_{0a})\int_0^{\infty}d\lambda$$
$$\times e^{-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\arcsin\left(\sqrt{\beta}p_b\right)-\arcsin\left(\sqrt{\beta}p_a\right)}{\sqrt{\beta}}}\, e^{\frac{\imath}{\hbar}\lambda\left(\frac{\left(\beta m^2\omega^2+\alpha\right)p_0^2}{\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)}-\frac{\varpi^2}{\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)}-m^2c^4\right)}\times\bar{K}(\tilde{q}_b,\tilde{q}_a,\lambda), \quad (4.45)$$

the propagator kernel $\bar{K}(\tilde{q}_b,\tilde{q}_a,\lambda)$ is precisely the path integral representation of the transition amplitude in relation to the Rosen–Morse of kind (I) potential [74]:

$$\bar{K}(\tilde{q}_b,\tilde{q}_a,\lambda)=\lim_{N\to\infty}\prod_{j=1}^{N}\left[\int d\tilde{q}_j\right]\prod_{j=1}^{N+1}\left[\sqrt{\frac{1}{4\pi\imath\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}}\right]$$
$$\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{(\Delta\tilde{q}_j)^2}{4\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}+\lambda\varepsilon\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)}\tan\left(\sqrt{\beta}\tilde{q}_j\right)\right.\right.$$
$$\left.\left.-\lambda\varepsilon\left(\frac{\varpi^2}{\alpha+\beta\frac{\varpi^2}{c^2}}-s\hbar c\varpi\right)\frac{1}{\cos^2\left(\sqrt{\beta}\tilde{q}_j\right)}\right]\right\}. \quad (4.46)$$

In this case, Eq. (4.46) transforms as follows when $(\alpha,\ \beta)$ are negative,

$$\bar{K}(q_b,q_a,\lambda)=\sqrt{\beta}\int D[q(t)]\exp\left\{\frac{\imath}{\hbar}\int_0^1\left[\frac{M}{2}\dot{q}^2(t)-A\tanh(q(t))+\frac{B}{\cosh^2(q(t))}\right]dt\right\}, \quad (4.47)$$

with $q(t)$, M, A and B defined by

$$q(t)=\sqrt{\beta}\tilde{q}(t),\quad M=\frac{1}{2\lambda c^2\bar{\theta}}\ \text{ and } A=\lambda\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\bar{\theta}}, B=\lambda\hbar c\varpi\left(\frac{\varpi}{\hbar c\bar{\theta}}+s\right), \quad (4.48)$$

here $\bar{\theta}=\left(\alpha+\beta\varpi^2/c^2\right)$. Following Ref. [3, 49], we can express $\bar{K}(q_b,q_a,\lambda)$,

$$\bar{K}(q_b,q_a,\lambda)=\sqrt{\beta}\sum_{n=0}^{\infty}\Psi_n(q_b)\Psi_n^{*}(q_a)\exp\left(-\frac{\imath}{\hbar}\lambda\bar{E}_n\right), \quad (4.49)$$

where

$$\Psi(q)=\left[\left(1-\frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)\,n!\Gamma(s-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]^{1/2}2^{n+(1-\bar{s})/2}$$
$$\times(1-\tanh q)^{\frac{\bar{s}}{2}-k_2-n}(1+\tanh q)^{k_2-\frac{1}{2}}P_n^{(\bar{s}-2k_2-2n,2k_2-1)}(\tanh q). \tag{4.50}$$

$P_n^{(\eta_1,\eta_2)}(z)$ signifies the Jacobi polynomial, and

$$\bar{E}_n=-\left[\frac{\hbar^2(\bar{s}-2n-1)^2}{8M}+\frac{2MA^2}{\hbar^2(\bar{s}-2n-1)^2}\right], \tag{4.51}$$

where

$$\bar{s}=\sqrt{1+8MB/\hbar^2},\;\; k_1=\frac{1}{2}(1+\bar{s}),\;\; k_2=\frac{1}{2}\left(1+\frac{1}{2}(\bar{s}-2n-1)-\frac{2MA}{\hbar(\bar{s}-2n-1)}\right). \tag{4.52}$$

After recompensing for every value (M, A, B, $\bar{s}$) in Eq. (4.51), we get

$$\bar{E}_n=-\lambda\hbar^2c^2\bar{\theta}\left[\left(\nu_s-n-\frac{1}{2}\right)^2+\frac{\alpha e^2\mathcal{E}^2p_0^2}{\hbar^4c^4\bar{\theta}^4\left(\nu_s-n-\frac{1}{2}\right)^2}\right], \tag{4.53}$$

with

$$\nu_s=\frac{\sqrt{m^2\omega^2-\frac{e^2\mathcal{E}^2}{c^2}}}{\hbar\bar{\theta}}+\frac{s}{2}. \tag{4.54}$$

Thus, the values ($\bar{s}$, k_2, $2k_2-1$ and $(\bar{s}-2k_2-2n)$) are transformed into the following formulas:

$$\bar{s}=2\nu_s,\;\; k_2=\frac{1}{2}\left(1+\frac{1}{2}(2\nu_s-2n-1)-\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\hbar c^2\bar{\theta}^2}\frac{1}{(2\nu_s-2n-1)}\right), \tag{4.55}$$

and

$$2k_2-1=\frac{1}{2}(2\nu_s-2n-1)-\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\hbar c^2\bar{\theta}^2}\frac{1}{(2\nu_s-2n-1)}=\eta_{n,s}^-, \tag{4.56}$$
$$\bar{s}-2k_2-2n=\frac{1}{2}(2\nu_s-2n-1)+\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\hbar c^2\bar{\theta}^2}\frac{1}{(2\nu_s-2n-1)}=\eta_{n,s}^+. \tag{4.57}$$

We can now write

$$\begin{aligned}
\mathcal{G}\left(p_b,p_a,p_{0b},p_{0a}\right) &= \left(\imath/\hbar\right)\delta\left(p_{0b}-p_{0a}\right)\sqrt{\beta}\sum_{s=\pm1}\sum_{n}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\int_0^{\infty}d\lambda \\
&\times\exp\left[\frac{\imath\lambda}{\hbar}\left(\left(\frac{\beta m^2\omega^2+\alpha}{\alpha+\beta\frac{\varpi^2}{c^2}}p_0^2-\frac{\varpi^2}{\alpha+\beta\frac{\varpi^2}{c^2}}-m^2c^4\right)\right.\right. \\
&\left.\left.+\hbar^2c^2\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)\left[\left(\nu_s-n-\frac{1}{2}\right)^2+\frac{\alpha e^2\mathcal{E}^2p_0^2}{\hbar^4c^4\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)^4\left(\nu_s-n-\frac{1}{2}\right)^2}\right]\right)\right] \\
&\times\frac{\left(\bar{s}-2k_2-2n\right)n!\Gamma\left(\bar{s}-n\right)}{\Gamma\left(\bar{s}+1-n-2k_2\right)\Gamma\left(2k_2+n\right)}2^{2n+(1-2\nu_s)}e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}\left(\sqrt{\beta}p_b\right)-\sinh^{-1}\left(\sqrt{\beta}p_a\right)}{\hbar\sqrt{\beta}}\right]} \\
&\times\left(1-\tanh q_b\right)^{\frac{\eta_{n,s}^{+}}{2}}\left(1+\tanh q_b\right)^{\frac{\eta_{n,s}^{-}}{2}}P_n^{\left(\eta_{n,s}^{+},\eta_{n,s}^{-}\right)}\left(\tanh q_b\right) \\
&\times\left(1-\tanh q_a\right)^{\frac{\eta_{n,s}^{+}}{2}}\left(1+\tanh q_a\right)^{\frac{\eta_{n,s}^{-}}{2}}P_n^{\left(\eta_{n,s}^{+},\eta_{n,s}^{-}\right)}\left(\tanh q_a\right). \qquad (4.58)
\end{aligned}$$

In the following section, we will compute the propagator within the framework of the aSdS model. Next, we will extract the energy levels and their mapping in specific special cases of deformation parameters.

4.3 Extracting Energy Levels for (1D-DO) in a Homogeneous Electric Field

For an accurate assessment of the propagator expression, it is convenient to integrate at the proper time λ and perform the Fourier transformation to Eq. (4.58). Following a straightforward calculation, we get

$$\begin{aligned}
\mathcal{G}\left(p_b,p_a,t_b,t_a\right) &= \sqrt{\beta}\sum_{s=\pm1}\sum_{n}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\int\frac{dp_0}{2\pi\hbar}\frac{e^{-\frac{\imath}{\hbar}p_0\left(t_b-t_a\right)}}{p_0^2-\left(E_{n,s}^{(\alpha,\beta)}\right)^2} \\
&\times\frac{\left(\bar{s}-2k_2-2n\right)n!\Gamma\left(\bar{s}-n\right)}{\Gamma\left(\bar{s}+1-n-2k_2\right)\Gamma\left(2k_2+n\right)}2^{2n+(1-\bar{s})}e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\hbar\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}\left(\sqrt{\beta}p_b\right)-\sinh^{-1}\left(\sqrt{\beta}p_a\right)}{\sqrt{\beta}}\right]} \\
&\times\left(1-\tanh q_b\right)^{\frac{\eta_{n,s}^{+}}{2}}\left(1+\tanh q_b\right)^{\frac{\eta_{n,s}^{-}}{2}}P_n^{\left(\eta_{n,s}^{+},\eta_{n,s}^{-}\right)}\left(\tanh q_b\right) \\
&\times\left(1-\tanh q_a\right)^{\frac{\eta_{n,s}^{+}}{2}}\left(1+\tanh q_a\right)^{\frac{\eta_{n,s}^{-}}{2}}P_n^{\left(\eta_{n,s}^{+},\eta_{n,s}^{-}\right)}\left(\tanh q_a\right), \qquad (4.59)
\end{aligned}$$

where

$$\left(E_{n,s}^{(\alpha,\beta)}\right)^2 = \frac{\bar{\theta}}{\beta m^2\omega^2 + \alpha\left(1 + \frac{e^2\mathcal{E}^2/\hbar^2c^2}{\bar{\theta}^2\left(\nu_s - n - \frac{1}{2}\right)^2}\right)}\left[m^2c^4 + \hbar c\varpi\left(2n+1-s\right) - \hbar^2c^2\bar{\theta}\left(n + \frac{1}{2} - \frac{s}{2}\right)^2\right]. \tag{4.60}$$

The energy spectrum is determined by integrating over the variable p_0. One way to accomplish this integration is to turn the problem into a complex integral along a specific choice of contour C. The residue theorem is employed for obtaining

$$\int_{-\infty}^{+\infty} f(p_0)\frac{dp_0}{2\pi\hbar}\frac{e^{-\frac{\imath}{\hbar}p_0(t_b-t_a)}}{p_0^2 - \left(E_{n,s}^{(\alpha,\beta)}\right)^2} = -\imath\sum_{\varepsilon=\pm1} f(\varepsilon E_{n,s}^{(\alpha,\beta)})\frac{e^{\frac{-\imath}{\hbar}\varepsilon E_{n,s,\varepsilon}^{(\alpha,\beta)}(t_b-t_a)}}{2E_{n,s}^{(\alpha,\beta)}}\Theta\left(\varepsilon\left(t_b-t_a\right)\right), \tag{4.61}$$

which has the poles

$$\varepsilon E_{n,s}^{(\alpha,\beta)} = E_{n,s,\varepsilon}^{(\alpha,\beta)} = \varepsilon\sqrt{\frac{\bar{\theta}}{\beta m^2\omega^2 + \alpha\left(1 + \frac{e^2\mathcal{E}^2}{\varpi^2 - 2\bar{\theta}\hbar c\varpi\left(n+\frac{1}{2}-\frac{s}{2}\right) + \bar{\theta}^2\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2}\right)}}$$
$$\times\left[m^2c^4 + \hbar c\varpi\left(2n+1-s\right) - \hbar^2c^2\bar{\theta}\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]^{\frac{1}{2}}, \tag{4.62}$$

here, Θ denote the Heaviside function. In Eq. (4.62), n is a quantum number, and the Dirac spinor's two components are described by the parameter $s = \pm 1$. Here, $\varepsilon = +1$ corresponds to positive energy states, while $\varepsilon = -1$ corresponds to negative energy states. A negatively (for $(e = -1)$) or positively (for $(e = +1)$ charged particle is described by the parameter $e = \mp 1$ where ω is the oscillator's angular frequency and $\mathcal{E}$ denotes the strength of the uniform electric field. For the DO in the context of the aSdS model, the corresponding spectral energy is as follows when the electric field $\mathcal{E}$ is set to zero:

$$E_{n,s}^{(\alpha,\beta)}\left(\mathcal{E}=0\right) = \pm\left[m^2c^4 + \hbar c^2 m\omega\left(2n+1-s\right) - \hbar^2c^2\left(\alpha+\beta m^2\omega^2\right)\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]^{\frac{1}{2}}. \tag{4.63}$$

Also, for $\omega = \mathcal{E} = 0$, the corresponding energy levels drop to

$$E_{n,s}^{(\alpha,\beta)}(\omega = \mathcal{E} = 0) = \pm\left[m^2c^4 - \hbar^2c^2\alpha\left(n + \frac{1}{2} - \frac{s}{2}\right)^2\right]^{\frac{1}{2}}. \tag{4.64}$$

As can be seen from Eq. (4.64), even in the absence of $\omega-$oscillation and $\mathcal{E}-$ electric fields, the energy levels that depend on n^2 remain continuous within the context of the aSdS model. In the presence of a uniform electric field, we were able to extract the interacting Dirac oscillator's spectral energy. Although the corresponding normalised eigenspinors are complex and long, for the purposes of this discussion, we will only be concerned with finding the Green function in momentum space. Thus, we can write the elements matrix of $S^{(\alpha,\beta)}(p_b, p_a, t_b, t_a)$ as follows using Eqs. (4.20) and (4.27)

$$S^{(\alpha,\beta)}(p_b, p_a, t_b, t_a) = \left[\mathcal{O}_+^D\right]_b \mathcal{G}(p_b, p_a, t_b, t_a), \tag{4.65}$$

in this case, Eq. (4.19) defines $\mathcal{O}_+^D$ and Eq. (4.59) precisely calculates $\mathcal{G}(p_b, p_a, t_b, t_a)$. Therefore, we obtain,

$$\begin{aligned} S^{(\alpha,\beta)}(p_b, p_a, t_b, t_a) = {} & -\imath\sqrt{\beta}\sum_{s=\pm 1}\sum_n \left[\gamma^0\left(\hat{p}_{0b} + e\mathcal{E}\hat{X}_b\right) - c\gamma^1\left(\hat{P}_b - \imath m\omega\gamma^0\hat{X}_b\right) + mc^2\right]\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^+ e^{\frac{\delta}{2}\sigma_2}\right] \\ & \times \sum_{\varepsilon=\pm 1}\frac{e^{-\frac{\imath}{\hbar}\varepsilon E_{n,s}^{(\alpha,\beta)}(t_b - t_a)}}{2E_{n,s}^{(\alpha,\beta)}}\Theta\left(\varepsilon\left(t_b - t_a\right)\right)\frac{\bar{\theta}\left[\left(1 - \frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{(\beta m^2\omega^2 + \alpha) + \frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s - n - \frac{1}{2}\right)^2}} \\ & \times 2^{2n+(1-\bar{s})} e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\hbar\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}\left(\sqrt{\beta}p_b\right) - \sinh^{-1}\left(\sqrt{\beta}p_a\right)}{\sqrt{\beta}}\right]} \\ & \times \left(1 - \tanh q_b\right)^{\frac{\eta_{n,s}^+}{2}}\left(1 + \tanh q_b\right)^{\frac{\eta_{n,s}^-}{2}} P_n^{\left(\eta_{n,s}^+, \eta_{n,s}^-\right)}\left(\tanh q_b\right) \\ & \times \left(1 - \tanh q_a\right)^{\frac{\eta_{n,s}^+}{2}}\left(1 + \tanh q_a\right)^{\frac{\eta_{n,s}^-}{2}} P_n^{\left(\eta_{n,s}^+, \eta_{n,s}^-\right)}\left(\tanh q_a\right). \end{aligned} \tag{4.66}$$

Afterward, writing the relations

$$\sum_{\varepsilon=\pm 1} f(\varepsilon)\Theta\left(\varepsilon\left(t_b - t_a\right)\right) = f(s)\Theta\left(s\left(t_b - t_a\right)\right) + f(-s)\Theta\left(-s\left(t_b - t_a\right)\right), \tag{4.67}$$

and

$$\gamma^3 e^{A\sigma_2} = \sigma_3 e^{A\sigma_2} = -e^{A\sigma_2}\sigma_3, \;\; \sigma_3\chi_s = s\chi_s, \tag{4.68}$$

$$\gamma^1 e^{A\sigma_2} = \imath\sigma_2 e^{A\sigma_2} = \imath e^{A\sigma_2}\sigma_2, \;\; \sigma_2\chi_s = \imath s\chi_{-s}, \tag{4.69}$$

$$\gamma^2 e^{A\sigma_2} = -\imath\sigma_1 e^{A\sigma_2} = \imath\sigma_1 e^{A\sigma_2}, \sigma_1\chi_s = \chi_{-s}. \tag{4.70}$$

The propagator $S^{(\alpha,\beta)}(p_b,p_a)$ in the momentum space can then be expressed as follows

$$\begin{aligned}
S^{(\alpha,\beta)}(p_b,p_a) = -\imath\sqrt{\beta}\sum_{s=\pm1}\sum_{n}\Bigg\{ & \Bigg[\frac{e^{\frac{-\imath}{\hbar}E_{n,s}^{(s)}(t_b-t_a)}}{2E_{n,s}^{(s)}}\Theta\left(s\left(t_b-t_a\right)\right) \\
& \times e^{-\frac{\delta}{2}\sigma_2}\left[\left(-\left(E_{n,s}^{(s)}+se\mathcal{E}\hat{X}_b\right)+mc^2\right)\mathbb{X}_s\mathbb{X}_s^{+}-\imath sc\left(\hat{P}_b-\imath m\omega\hat{X}_b\right)\mathbb{X}_{-s}\mathbb{X}_s^{+}\right]e^{\frac{\delta}{2}\sigma_2} \\
& \times\bar{\theta}\frac{\left[\left(1-\frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{(\beta m^2\omega^2+\alpha)+\frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}}2^{2n+(1-\bar{s})} \\
& \times e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}E_{n,s}^{(s)}}{\left(c^2m^2\omega^2-e^2\mathcal{E}^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}\left(\sqrt{\beta}p_b\right)-\sinh^{-1}\left(\sqrt{\beta}p_a\right)}{\hbar\sqrt{\beta}}\right]} \\
& \times\left(1-\tanh q_b\right)^{\frac{\eta_{n,s}^{+}}{2}}\left(1+\tanh q_b\right)^{\frac{\eta_{n,s}^{-}}{2}}P_n^{\left(\eta_{n,s}^{+},\eta_{n,s}^{-}\right)}\left(\tanh q_b\right) \\
& \times\left(1-\tanh q_a\right)^{\frac{\eta_{n,s}^{+}}{2}}\left(1+\tanh q_a\right)^{\frac{\eta_{n,s}^{-}}{2}}P_n^{\left(\eta_{n,s}^{+},\eta_{n,s}^{-}\right)}\left(\tanh q_a\right) \\
& +\left[\frac{e^{\frac{-\imath}{\hbar}E_{n,s}^{(-s)}(t_b-t_a)}}{2E_{n,s}^{(-s)}}\Theta\left(-s\left(t_b-t_a\right)\right)\bar{\theta}\frac{\left[\left(1-\frac{4MA}{\hbar(\tilde{s}-2n-1)^2}\right)\frac{(\tilde{s}-2k_2-2n)n!\Gamma(\tilde{s}-n)}{\Gamma(\tilde{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{(\beta m^2\omega^2+\alpha)+\frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}}2^{2n+(1-\tilde{s})}\right. \\
& \times\left[e^{-\frac{\delta}{2}\sigma_2}\left[\left[\left(E_{n,s}^{(-s)}-se\mathcal{E}\hat{X}_b\right)+mc^2\right]\mathbb{X}_s\mathbb{X}_s^{+}-\imath sc\left(\hat{P}_b-\imath m\omega\gamma^0\hat{X}_b\right)\mathbb{X}_{-s}\mathbb{X}_s^{+}\right]e^{\frac{\delta}{2}\sigma_2}\right] \\
& \times e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}E_{n,s}^{(-s)}}{\left(c^2m^2\omega^2-e^2\mathcal{E}^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}\left(\sqrt{\beta}p_b\right)-\sinh^{-1}\left(\sqrt{\beta}p_a\right)}{\hbar\sqrt{\beta}}\right]} \\
& \times\left(1-\tanh q_b\right)^{\frac{\eta_{n,s}^{+}}{2}}\left(1+\tanh q_b\right)^{\frac{\eta_{n,s}^{-}}{2}}P_n^{\left(\eta_{n,s}^{+},\eta_{n,s}^{-}\right)}\left(\tanh q_b\right) \\
& \times\left(1-\tanh q_a\right)^{\frac{\eta_{n,s}^{+}}{2}}\left(1+\tanh q_a\right)^{\frac{\eta_{n,s}^{-}}{2}}P_n^{\left(\eta_{n,s}^{+},\eta_{n,s}^{-}\right)}\left(\tanh q_a\right)\Bigg\}.
\end{aligned} \tag{4.71}$$

The expression of the Heaviside function $\Theta\left(-s\left(t_b-t_a\right)\right)$ must be unified by substituting s to $(-s)$ for all terms multiplied by $\Theta\left(-s\left(t_b-t_a\right)\right)$. Furthermore, to unify the same energy, we make the following mapping

$$n \rightarrow n-s. \tag{4.72}$$

So, in the context of the aSdS model in the momentum space, the propagator $S^{(\alpha,\beta)}(p_b,p_a)$ of the $(1+1)$-dimensional DO subjected to an electric field becomes as

$$\begin{aligned}
S^{(\alpha,\beta)}(p_b,p_a,T) = & -\imath\sqrt{\beta}\sum_{s=\pm1}\sum_{n}\left\{\frac{e^{\frac{-\imath}{\hbar}E^{(s)}_{n,s}(t_b-t_a)}}{2E^{(s)}_{n,s}}\Theta\left(s\left(t_b-t_a\right)\right)\right.\\
&\times\left[\bar{\theta}\frac{\left[\left(1-\frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{(\beta m^2\omega^2+\alpha)+\frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}}2^{2n+(1-\bar{s})}\right.\\
&\times e^{-\frac{\delta}{2}\sigma_2}\left[\left(-\left(E^{(s)}_{n,s}+se\mathcal{E}\hat{X}_b\right)+mc^2\right)\mathbb{X}_s\mathbb{X}^+_s-\imath sc\left(\hat{P}_b-\imath m\omega\hat{X}_b\right)\mathbb{X}_{-s}\mathbb{X}^+_s\right]e^{\frac{\delta}{2}\sigma_2}\\
&\times e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}E^{(s)}_{n,s}}{\hbar\left(c^2m^2\omega^2-e^2\mathcal{E}^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}\left(\sqrt{\beta}p_b\right)-\sinh^{-1}\left(\sqrt{\beta}p_a\right)}{\sqrt{\beta}}\right]}\\
&\times\left(1-\tanh q_b\right)^{\frac{\eta^+_{n,s}}{2}}\left(1+\tanh q_b\right)^{\frac{\eta^-_{n,s}}{2}}P_n^{\left(\eta^+_{n,s},\eta^-_{n,s}\right)}\left(\tanh q_b\right)\\
&\times\left(1-\tanh q_a\right)^{\frac{\eta^+_{n,s}}{2}}\left(1+\tanh q_a\right)^{\frac{\eta^-_{n,s}}{2}}P_n^{\left(\eta^+_{n,s},\eta^-_{n,s}\right)}\left(\tanh q_a\right)\\
&+\left[\bar{\theta}\frac{\left[\left(1-\frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{(\beta m^2\omega^2+\alpha)+\frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}}2^{2n+(1-\bar{s})}\right.\\
&\times\left[e^{-\frac{\delta}{2}\sigma_2}\left[\left(\left(E^{(s)}_{n,s}+se\mathcal{E}\hat{X}_b\right)+mc^2\right)\mathbb{X}_{-s}\mathbb{X}^+_{-s}+\imath sc\left(\hat{P}_b-\imath m\omega\gamma^0\hat{X}_b\right)\mathbb{X}_s\mathbb{X}^+_{-s}\right]e^{\frac{\delta}{2}\sigma_2}\right]\\
&\times e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}E^{(s)}_{n,s}}{\hbar\left(c^2m^2\omega^2-e^2\mathcal{E}^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}\left(\sqrt{\beta}p_b\right)-\sinh^{-1}\left(\sqrt{\beta}p_a\right)}{\sqrt{\beta}}\right]}\\
&\times\left(1-\tanh q_b\right)^{\frac{\eta^+_{n,s}}{2}}\left(1+\tanh q_b\right)^{\frac{\eta^-_{n,s}}{2}}P_{n-s}^{\left(\eta^+_{n,s},\eta^-_{n,s}\right)}\left(\tanh q_b\right)\\
&\left.\times\left(1-\tanh q_a\right)^{\frac{\eta^+_{n,s}}{2}}\left(1+\tanh q_a\right)^{\frac{\eta^-_{n,s}}{2}}P_{n-s}^{\left(\eta^+_{n,s},\eta^-_{n,s}\right)}\left(\tanh q_a\right)\right\}. \qquad (4.73)
\end{aligned}$$

Additionally, by substituting $(-\alpha,-\beta)$ for α and β, one can construct the SdS space from the propagator's function and spectral energies, which are defined in Eqs. (4.73) and (4.62), respectively. Also, the Jacobi polynomial is replaced by Romanovski polynomials [75],

$$P_n^{\left(\eta^+_{n,s},\eta^-_{n,s}\right)}\left(\imath\tan q\right)\to R_n^{\left(\eta^+_{n,s},\eta^-_{n,s}\right)}\left(\tan q\right). \qquad (4.74)$$

Further, the energy levels are expressed in n^2 in both cases for the signal parameters α and β. Since the values of α and β are typically very small according to theory of deformation, we expand (4.62) to first order in α and β, and as a result, we find

$$E_{n,s}^{(\alpha,\beta)} = \pm\sqrt{\frac{c^2m^2\omega^2 - e^2\mathcal{E}^2}{c^2m^2\omega^2}}\left[m^2c^4 + \hbar c\sqrt{c^2m^2\omega^2 - e^2\mathcal{E}^2}\,(2n+1-s)\right] \mp \frac{\bar{\theta}}{2}\frac{\sqrt{\frac{c^2m^2\omega^2-e^2\mathcal{E}^2}{c^2m^2\omega^2}}\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2}{\left[m^2c^4 + \hbar c\sqrt{c^2m^2\omega^2 - e^2\mathcal{E}^2}\,(2n+1-s)\right]^{\frac{1}{2}}}. \tag{4.75}$$

The first term in this case denotes the Dirac oscillator's Landau levels in a homogeneous electric field without deformation, and the second term denotes the quantum gravity correction. Remarkably, the bounded eigenstates are absent at large electric field value then critical one $\frac{e\mathcal{E}}{c} > m\omega$ the bounded eigenstates are absent. Now, let us consider the following particular cases.

1- In limit case $\alpha \to 0$, the expression of Eq. (4.62) reduces to that of the flat Snyder model,

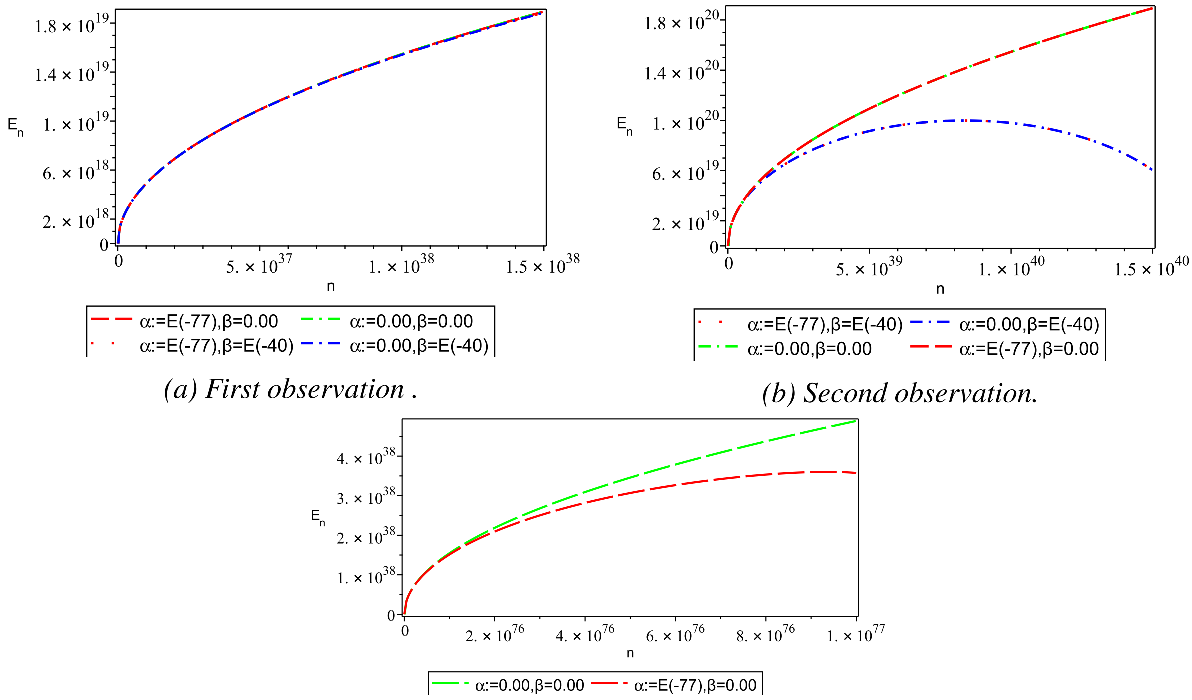
$$E_{n,s}^{(\alpha=0)} = \pm\frac{\varpi}{cm\omega}\left[m^2c^4 + \hbar c\varpi\,(2n+1-s) - \hbar^2\beta\varpi^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]^{\frac{1}{2}}. \tag{4.76}$$

1. In limit case $\beta \to 0$, one recovers the spectral energies for the Heisenberg algebra in an (anti-)de Sitter background [69],

$$E_{n,s}^{(\beta=0)} = \pm\left(1 + \frac{e^2\mathcal{E}^2}{\left(\varpi^2 - 2\bar{\theta}\hbar c\varpi\left(n+\frac{1}{2}-\frac{s}{2}\right) + \bar{\theta}^2\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right)}\right)^{-1/2} \times\left[m^2c^4 - \hbar^2c^2\alpha\left(n+\frac{1}{2}-\frac{s}{2}\right)^2 + \hbar c\varpi\,(2n+1-s)\right]^{\frac{1}{2}}. \tag{4.77}$$

To investigate the differences caused by the presence or absence of (aSdS) algebra, as well as to understand the impact of incorporating one parameter without the other on the energy levels, we plot the energy levels $E_{n,s=+1}^{(\alpha,\beta)}$ as a function of the quantum numbers n. we employ the natural unit system, setting $\hbar$, c to 1, which leads to dimensionless parameters. Furthermore, we specify the electron mass as $m = 0.5MeV$ ànd an electric field $\mathcal{E}$ of $0.2MeV^2$, $e = 0.303$, $\omega = 2MeV$. We use four different deformation parameter values for this: (i.e., ($\alpha = 10^{-77}MeV$, $\beta = 10^{-40}MeV$), ($\alpha = 10^{-77}MeV$, $\beta = 0.0MeV$), ($\alpha = 0.0MeV$, $\beta = 10^{-40}MeV$) and ($\alpha = 0.0$, $\beta = 0.0$)), as illustrated in Fig. 4.1. This later is broken down into three sub-figures (Fig.

4.1a, Fig. 4.1b, Fig. 4.1c).



(a) First observation .

(b) Second observation.

(c) Third observation.

Figure 4.1: $E_{n,\alpha,\beta}$ versus the quantum number n for different values of the deformation parameters.

We observe that all energy level cases in Fig. 4.1a apply when the quantum number principle n between 0 and 2×10^{38}. Meanwhile, Fig. 4.1b illustrates this separation, which happens, when $n=10^{38}$ and $n=10^{40}$. Curves $\beta\neq0$ no longer appear when $n>10^{41}$. On the other hand, in Figure 4.1c, the case ($\alpha=10^{-77}$, $\beta=0.0$) plot is shown at $\{n>10^{76}$ and vanishes at $n>10^{77}$. These data clearly show that the $\alpha-$ parameter has a greater effect than the $\beta-$ parameter.

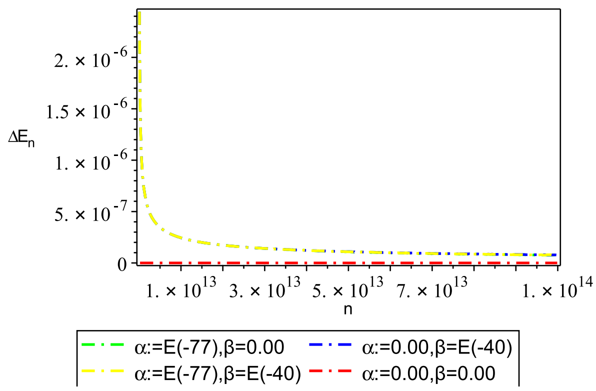


Figure 4.2: The energy spacing between adjacent levels as a function of n

As we also see in Fig. 4.2, the energy spacing between adjacent levels is constant, which indicates strong confinement.

Just as in the case of SdS algebra, we can plot all the energy level curves. For instance, we can see in Fig. 4.1c that the HUP algebra's energy spectrum curve is below the case of ($\alpha = 10^{-77}$, $\beta = 0.0$).

Additionally, the nonrelativistic energy level is determined by taking into account that a larger portion of the system's total energy is contained in the rest energy (mc^2) of the particle [76], which is, $E_{n,s}^{(\alpha,\beta)} = mc^2 + E_{n,s,\alpha,\beta}^{(NR)}$, where $mc^2 \gg E_{n,s,\alpha,\beta}^{(NR)}$ and $mc^2 \gg \sqrt{m^2\omega^2 - e^2\mathcal{E}^2/c^2}$. The energy spectrum of a nonrelativistic particle in the context of the Snyder (anti-)de Sitter model at first-order approximation can therefore be obtained by applying this prescription in Eq. (4.62),

$$E_{n,s,\alpha,\beta}^{(NR)} = \sqrt{\frac{\bar{\theta}}{\beta m^2\omega^2 + \alpha\left(1 + \frac{e^2\mathcal{E}^2}{\left(\varpi^2 - 2\bar{\theta}\hbar c\varpi\left(n+\frac{1}{2}-\frac{s}{2}\right) + \bar{\theta}^2\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right)}\right)}}$$
$$\times \left[\frac{\hbar}{2m}(\varpi/c)(2n+1-s) - \frac{\hbar^2}{2m}\bar{\theta}\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]. \quad (4.78)$$

In limit case $\alpha \to 0$, Eq. (4.78) becomes as,

$$E_{n,s,\alpha=0,\beta}^{(NR)} = \sqrt{\frac{m^2\omega^2 - e^2\mathcal{E}^2/c^2}{m^2\omega^2}}\left[\frac{\hbar}{2m}\sqrt{m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}}(2n+1-s)\right.$$
$$\left. - \frac{\hbar^2}{2m}\beta\left(m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}\right)\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]. \quad (4.79)$$

In limit case $\beta \to 0$, Eq. (4.78) transforms as,

$$E_{n,s,\alpha,\beta=0}^{(NR)} = \pm\left(1 + \frac{e^2\mathcal{E}^2}{\left(\varpi^2 - 2\alpha\hbar c\varpi\left(n+\frac{1}{2}-\frac{s}{2}\right) + \alpha^2\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right)}\right)^{-1/2}$$
$$\times \left[\frac{\hbar}{2m}(\varpi/c)(2n+1-s) - \frac{\hbar^2}{2m}\alpha\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]. \quad (4.80)$$

In accordance with Equation (4.78) and in the first order of (α, β), we can derive the energy spectrum for a spinless non-relativistic particle ($s = 0$) subject to a uniform electric field

$$E_{n,s,\alpha,\beta}^{(NR)}=\left[\hbar\left(\frac{m^2\omega^2-e^2\mathcal{E}^2/c^2}{m^2\omega}\right)\left(n+\frac{1}{2}\right)-\frac{\hbar^2}{2m}\bar{\theta}\sqrt{\frac{m^2\omega^2-e^2\mathcal{E}^2/c^2}{m^2\omega^2}}\left(n+\frac{1}{2}\right)^2\right]. \quad (4.81)$$

The first term of the equation (4.81) corresponds to the energy level of a spinless non-relativistic oscillator with frequency, interacting with a uniform electric field within conventional quantum mechanics (HUP). The second term signifies the relativistic correction within the framework of a modified Heisenberg algebra. Additionally, if we consider the limit as $\mathcal{E}\to 0$, Eq. (4.81) transforms to:

$$E_{n,s,\alpha,\beta}^{(NR)}=\left[\hbar\omega\left(n+\frac{1}{2}\right)-\frac{\hbar^2}{2m}\bar{\theta}\left(n+\frac{1}{2}\right)^2\right]. \quad (4.82)$$

The initial term in this case represents the energy level of a spinless non-relativistic oscillator with a frequency of ω particles in HUP, while the subsequent term represents the initial correction of deformation in the non-relativistic case.

4.4 Thermodynamic Functions

Within the context of the aSdS model, we will talk about the thermodynamic properties of the DO particle interacting with a homogenous electric field using modified algebra (4.1). Determining the appropriate partition function is necessary in order to arrive at these thermodynamic properties. And so we have:

$$Z=\sum_{n=0}^{\infty}e^{-\bar{\beta}E_n}, \quad (4.83)$$

where T represents the system's equilibrium temperature, k_B denotes the Boltzmann constant, so that $\bar{\beta}=1/(k_BT)$. To simplify, we chose the first order of (α,β) positive energy level for spin up $(s=+1)$, as provided by Eq. (4.91). So, the sum (4.83) reads,

$$Z(T,\alpha,\beta)=\sum_{n=0}^{\infty}\exp\left[-\bar{\beta}\sqrt{b+an}-\bar{\beta}\frac{\bar{\theta}}{2}\frac{(\varpi/c)^2}{m^2\omega^2}\frac{\hbar^2c^2n^2}{\sqrt{b+an}}\right], \quad (4.84)$$

with $a=2\frac{(\hbar c^2)(\varpi/c)^3}{m^2\omega^2}$, $b=\frac{(\varpi/c)^2m^2c^4}{m^2\omega^2}$.

At the first order of (α,β), the partition function (4.83) becomes

$$Z(T,\alpha,\beta)=Z^0(\bar\beta)+\bar\theta\Delta Z^{(1)}(\bar\beta), \tag{4.85}$$

where

$$Z^0(\bar\beta)=\sum_{n=0}^{\infty}e^{-\bar\beta\sqrt{b+an}} \text{ and } \Delta Z^{(1)}(\bar\beta)=-\bar\beta\frac{\hbar^2c^2}{2}\frac{(\varpi/c)^2}{m^2\omega^2}\sum_{n=0}^{\infty}\frac{n^2}{\sqrt{b+an}}e^{-\bar\beta\sqrt{b+an}}. \tag{4.86}$$

By applying the Euler-Maclaurin summation formula [77], we can assess the sums in (4.85).

$$\sum_{n=0}^{\infty}f(n)=\frac{1}{2}f(0)+\int_0^{\infty}f(x)dx-\sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0), \tag{4.87}$$

here, the Bernoulli numbers are denoted by B_{2p}, where $B_2=1/6$, $B_4=-1/30$, ..., and $f^{(2k-1)}(0)$ is the order $(2k-1)$ derivative at $x=0$, and its values are as follows:

$$f(0)=e^{-\bar\beta\frac{(\varpi/c)}{m\omega}mc^2},\quad f^{(1)}(0)=-\left(\hbar c^2\right)\bar\beta\frac{(\varpi/c)^2}{m\omega}\frac{e^{-\bar\beta\frac{(\varpi/c)}{m\omega}mc^2}}{mc^2}, \tag{4.88}$$

$$\begin{aligned}f^{(3)}(0)=&\left\{-\frac{\left(\hbar c^2\right)^3(\varpi/c)^6}{\left(mc^2\right)^3(m\omega)^3}\bar\beta^3-\frac{3\,(\hbar c)^3\varpi^5}{(cm\omega)^4\left(mc^2\right)^4}\bar\beta^2-\frac{3\left(\hbar c^2\right)^3(\varpi/c)^4}{m\omega\left(mc^2\right)^5}\bar\beta\right.\\&\left.+6\bar\theta\bar\beta^2\frac{\hbar^3c^3}{2}\frac{\varpi^3}{\left(c^2m^2\omega^2\right)\left(m^2c^4\right)}+6\bar\theta\bar\beta\frac{\hbar^3c^3}{2}\frac{\varpi^2}{2cm\omega\left(mc^2\right)^3}\right\}e^{-\bar\beta\frac{(\varpi/c)}{m\omega}mc^2}.\end{aligned} \tag{4.89}$$

Next, in Eq. (4.87), the integral over (x) has the form

$$\int_0^{\infty}f(x)dx=\left\{\frac{2\sqrt{b}}{a\bar\beta}+\frac{2}{a\bar\beta^2}-\frac{\bar\theta}{2}\frac{\hbar^2c^2(\varpi/c)^2}{m^2\omega^2}\left[\frac{16b}{a^3\bar\beta^2}+\frac{48\sqrt{b}}{a^3\bar\beta^3}+\frac{48}{a^3\bar\beta^4}\right]\right\}e^{-\bar\beta\sqrt{b}}. \tag{4.90}$$

As a result, the partition function is expressed as

$$\begin{aligned}Z(T,\alpha,\beta)=&\left\{\frac{1}{2}+\frac{(m\omega)\left(mc^2\right)}{\left(\hbar c^2\right)\left(\varpi^2/c^2\right)}\frac{1}{\bar\beta}+\frac{m^2\omega^2}{\left(\hbar c^2\right)\left(\varpi^2/c^2\right)^{3/2}}\frac{1}{\bar\beta^2}\right.\\&-\frac{\bar\theta}{c^2}\left[\frac{\left(m^2\omega^2\right)\left(m^2c^4\right)}{\left(\hbar c^2\right)(\varpi/c)^5\bar\beta^2}+\frac{3\left(m^3\omega^3\right)\left(mc^2\right)}{\left(\hbar c^2\right)(\varpi/c)^6\bar\beta^3}\right.\\&\left.\left.+\frac{3m^4\omega^4}{\left(\hbar c^2\right)(\varpi/c)^7\bar\beta^4}\right]\right\}e^{-\bar\beta\frac{(\varpi/c)}{m\omega}mc^2}-\sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0).\end{aligned} \tag{4.91}$$

To calculate this partition function, we are required to compute the sum outlined in the above expression. However, in our case, this computation can only be executed using numerical methods. Up to $k=2$, this sum can be represented as

$$\sum_{k=1} \frac{B_{2k}}{(2k-1)!} f^{(2k-1)}(0) = -\frac{\bar{\beta}}{6} \frac{\hbar c^2 \left(\varpi/c\right)^2}{m\omega} \frac{e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2}}{mc^2} - \frac{1}{180}\left[-\bar{\beta}\frac{3\left(\hbar c^2\right)^3\left(\varpi/c\right)^4}{m\omega\left(mc^2\right)^5} -\bar{\beta}^2\frac{3\left(\hbar c^2\right)^3\left(\varpi/c\right)^5}{c^2\left(m^4\omega^4\right)\left(m^2c^4\right)^2} + 6\bar{\theta}\bar{\beta}^2\frac{\left(\hbar c^2\right)^3}{2}\frac{\left(\varpi/c\right)^3}{c^2\left(m^2\omega^2\right)\left(m^2c^4\right)} - \bar{\beta}^3\frac{\left(\hbar c^2\right)^3\left(\varpi/c\right)^6}{\left(mc^2\right)^3\left(m\omega\right)^3} + 3\bar{\theta}\bar{\beta}\frac{\left(\hbar c^2\right)^3}{2}\frac{\left(\varpi/c\right)^2}{c^2\left(m\omega\right)\left(mc^2\right)^3}\right] e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2}. \tag{4.92}$$

When the high temperature ($\bar{\beta} \ll 1$), every term in the sum of Eq. (4.92) has a positive power in $\bar{\beta}$, which is significantly smaller than the other term in Eq. (4.91). Therefore, we can neglect the terms that have $\bar{\beta}^n$ and the terms that do not have $\bar{\beta}$. Furthermore, we expand the function $(e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2})$ to the orders of $\bar{\beta}$ in Eq. (4.91), and then, after making a few simplifications, we disregard all of the positive exponents of $\bar{\beta}$. Eq. (4.91) is given the following result:

$$Z(T,\alpha,\beta) \simeq \frac{m^2\omega^2}{\left(\hbar c^2\right)\left(\varpi/c\right)^3}\frac{1}{\bar{\beta}^2} - \frac{\bar{\theta}}{c^2}\left[\frac{3m^4\omega^4}{\left(\hbar c^2\right)\left(\varpi/c\right)^7}\frac{1}{\bar{\beta}^4} - \frac{\left(m^2\omega^2\right)\left(mc^2\right)^2}{2\left(\hbar c^2\right)\left(\varpi/c\right)^5}\frac{1}{\bar{\beta}^2}\right]. \tag{4.93}$$

We can see that the $\bar{\theta}-$deformation parameter is relatively small, so the partition function is reformulated as:

$$Z(T,\alpha,\beta) \simeq \frac{m^2\omega^2}{\left(\hbar c^2\right)\left(\varpi/c\right)^3}\left(k_BT\right)^2 e^{-\bar{\theta}\left[\frac{3m^2\omega^2}{(\varpi/c)^4}\left(k_BT\right)^2 - \frac{1}{2}\frac{m^2c^4}{(\varpi/c)^2}\right]}. \tag{4.94}$$

The term associated with $\bar{\theta}$ represents a contribution of the SdS algebra to the $Z-$function, and when $\bar{\theta}$ tends to zero, it gives the partition function for a DO with $(1+1)-$dimensions subject to a uniform electric field in the HUP algebra. We can now obtain all thermodynamic functions, including the $F-$Helmholtz free energy, the $\Xi-$mean energy, the $C-$heat capacity, and the $S-$entropy, with the aid of the partition function in Eq. (4.94).

$$F(T,\alpha,\beta) = -\left(k_BT\right)\ln(Z) = F_0\left(\bar{\beta}\right) + \bar{\theta}\Delta F^1\left(\bar{\beta}\right). \tag{4.95}$$

First, let us discuss the Helmholtz free energy for a (1+1)-dimensional DO with a homogenous electric field in HUP algebra, and with $F_0\left(\bar{\beta}\right)$.

$$F_0\left(\bar{\beta}\right) = -2\left(k_B T\right)\ln\left(\frac{m\omega(k_B T)}{\sqrt{\hbar c^2\left(m^2\omega^2 - \frac{e^2\varepsilon^2}{c^2}\right)^{3/2}}}\right), \tag{4.96}$$

where $\Delta F^1\left(\bar{\beta}\right)$ is the first-order correction for the SdS deformation

$$\Delta F^1\left(\bar{\beta}\right) = -\frac{1}{2k_B}\frac{m^2c^2}{\left(m^2\omega^2 - \frac{e^2\varepsilon^2}{c^2}\right)}\left(k_B T\right) + \frac{3m^2\omega^2}{c^2 k_B\left(m^2\omega^2 - \frac{e^2\varepsilon^2}{c^2}\right)^2}\left(k_B T\right)^3. \tag{4.97}$$

The following expression gives the relation of the mean energy and the partition function

$$\Xi(T,\alpha,\beta) = -\frac{\partial \ln(Z)}{\partial\bar{\beta}} = 2k_B T\exp\left(-3\bar{\theta}\frac{m^2\omega^2}{c^2\left(m^2\omega^2 - \frac{e^2\varepsilon^2}{c^2}\right)^2}\left(k_B T\right)^2\right), \tag{4.98}$$

then

$$\Xi(T,\alpha,\beta) = -\frac{\partial \ln(Z)}{\partial\bar{\beta}} = 2k_B T - 6\bar{\theta}\frac{m^2\omega^2}{c^2\left(m^2\omega^2 - \frac{e^2\varepsilon^2}{c^2}\right)^2}\left(k_B T\right)^3 \tag{4.99}$$

In HUP algebra, the usual case of mean energy is recovered when $\bar{\theta} \to 0$.

Regarding heat capacity, we have

$$C(T,\alpha,\beta) = \frac{\partial \Xi}{\partial T} = C_0\left(\bar{\beta}\right) + \bar{\theta}\Delta C^1\left(\bar{\beta}\right), \tag{4.100}$$

where, in the absence of aSdS algebra, $C_0\left(\bar{\beta}\right) = 2k_B$ is constant, and the first correction to the heat capacity, $\Delta C^1\left(\bar{\beta}\right)$ is dependent on T^2.

$$\Delta C^1\left(\bar{\beta}\right) = 18\frac{m^2\omega^2 k_B^3 T^2}{c^2\left(m^2\omega^2 - \frac{e^2\varepsilon^2}{c^2}\right)^2}. \tag{4.101}$$

At last, entropy is presented as

$$S(T,\alpha,\beta) = k_B\ln(Z) - k_B\bar{\beta}\frac{\partial \ln(Z)}{\partial\bar{\beta}} = S_0\left(\bar{\beta}\right) + \bar{\theta}\Delta S^1\left(\bar{\beta}\right). \tag{4.102}$$

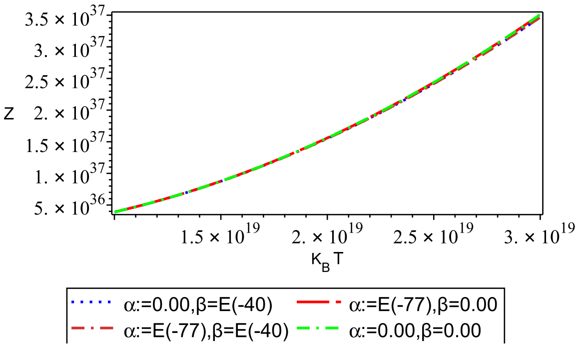
The entropy of the (1+1)-dimensional DO under the uniform electric field in HUP algebra is denoted by $S_0\left(\bar{\beta}\right)$ which can be expressed as follows:

$$S_0\left(\bar{\beta}\right) = 2k_B + 2k_B \ln\left(\frac{m\omega}{\sqrt{\hbar c^2\left(m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}\right)^{3/2}}}\left(k_B T\right)\right), \tag{4.103}$$

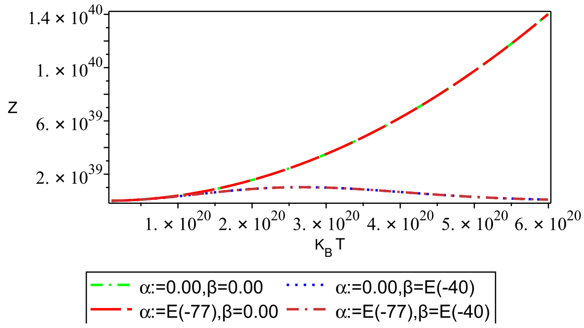
whereas, the term correction of entropy in the first order of (α, β) is denoted by $\Delta S^1\left(\bar{\beta}\right)$

$$\Delta S^1\left(\bar{\beta}\right) = k_B\left[\frac{m^2c^2}{2\left(m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}\right)} - \frac{9m^2\omega^2}{c^2\left(m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}\right)^2}\left(k_B T\right)^2\right]. \tag{4.104}$$

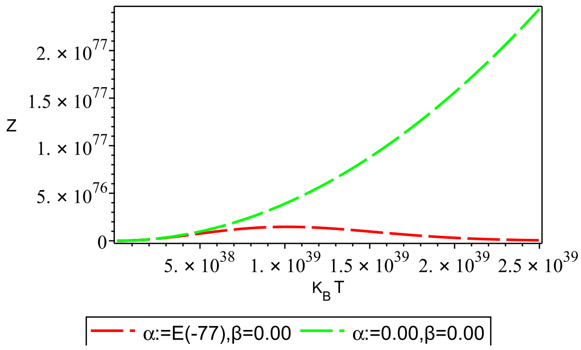
We illustrate the thermodynamic properties of our system under various deformation parameters in the following figures. For simplicity, we employ the natural unit system, where $\hbar$, c, and k_B are all set to 1, rendering all parameters dimensionless. This necessitates precise estimations of the relevant physical quantities. Therefore, in the high-temperature range, we have chosen an oscillator value of about $2MeV$, an electron mass of $m = 0.5MeV$, and an electric field $\mathcal{E}$ at $0.2MeV^2$. As a result, as functions of temperature ($(k_B T)$), the thermodynamic properties are depicted in figures (4.3), (4.4), (4.5), (4.6), and (4.7), four distinct deformation parameter values were used, namely, ($\alpha = 10^{-77}MeV$, $\beta = 10^{-35}MeV$), ($\alpha = 0.0MeV$, $\beta = 10^{-35}MeV$), ($\alpha = 10^{-77}MeV$, $\beta = 0.0MeV$) and ($\alpha = 0.0MeV$, $\beta = 0.0MeV$).



(a) First observation .



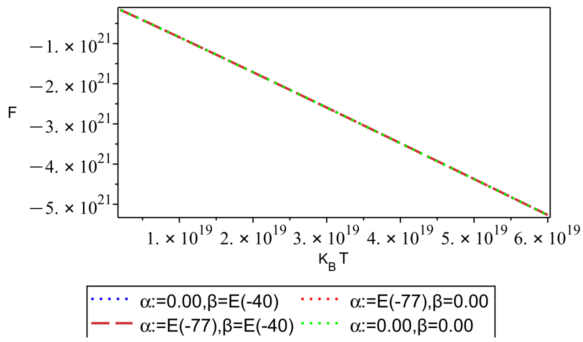
(b) Second observation .



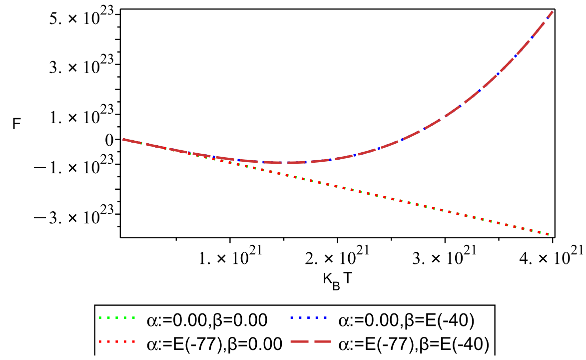
(c) Third observation.

Figure 4.3: Partition function for the DO with uniform electric field as a function of temperature k_BT for different values of the deformation parameters.

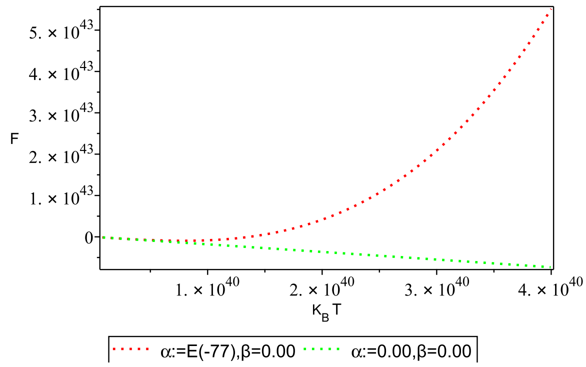
As evident from the sub-figure (4.3a), the aSdS algebra produces an increase in the partition function from $k_BT = 1\times10^{19}$ to approximately $k_BT \sim 2.5\times10^{19}MeV$. Then, after the temperature $k_BT \sim 10^{20}MeV.$, the curves (4.3b) corresponding to ($\alpha = 10^{-77}MeV$, $\beta = 10^{-35}MeV$) and ($\alpha = 0.0MeV$, $\beta = 10^{-35}MeV$) decrease to zero. Meanwhile, in Fig. (4.3c), the curves for ($\alpha = 10^{-77}MeV$, $\beta = 0.0MeV$) are close to zero when k_BT exceeds $10^{39}MeV$. In contrast, the other two curves closely align up to $k_BT \sim 5\times10^{38}MeV$.



(a) First observation .



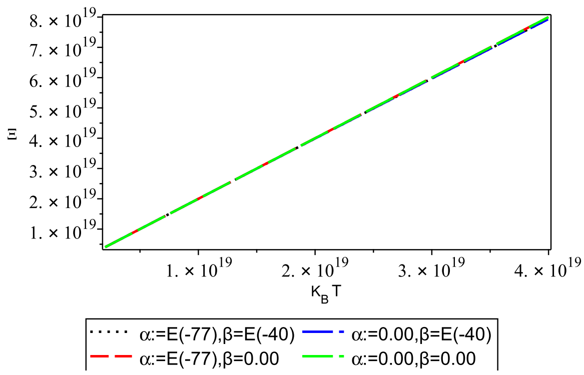
(b) Second observation .



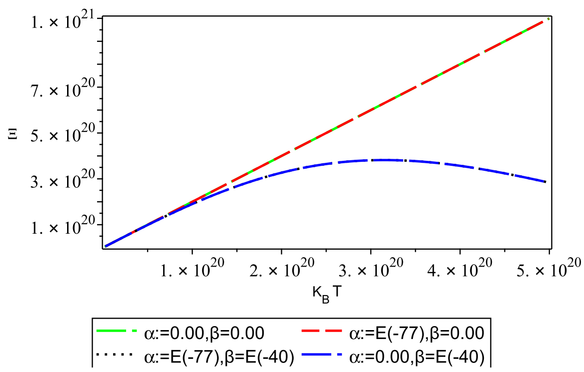
(c) Third observation.

Figure 4.4: The Helmholtz free energy function for the DO with uniform electric field as a function of temperature $k_B T$ for different values of the deformation parameters.

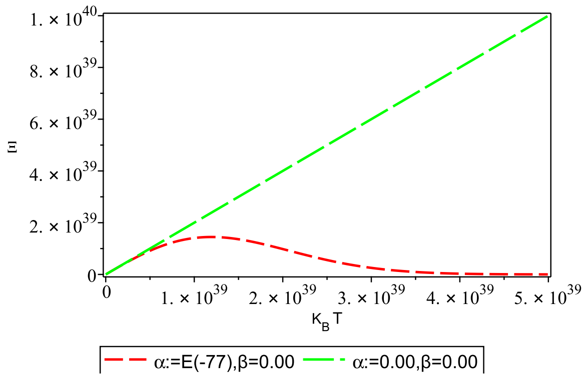
Figure (4.4) presents the Helmholtz free energy for the one-dimensional DO within the aSdS context in terms of $k_B T$. This representation shows that the aSdS algebra results in a decrease in the F-function, which varies from $k_B T = 1 \times 10^{19}$ to $k_B T \sim 6 \times 10^{19} MeV$ for each of the four deformation parameter cases in Fig. (4.4a). After $k_B T > 10^{39}$, the curves (4.4b) for (($\alpha = 10^{-77} MeV$, $\beta = 10^{-35} MeV$) and ($\alpha = 0.0 MeV$, $\beta = 10^{-35} MeV$)) disappear when $\beta \neq 0$. In the meantime, Fig. (4.4c) shows that the case represented by ($\alpha = 10^{-77} MeV$, $\beta = 0.0 MeV$) has an effect up to temperature $k_B T > 10^{21} MeV$.



(a) First observation .



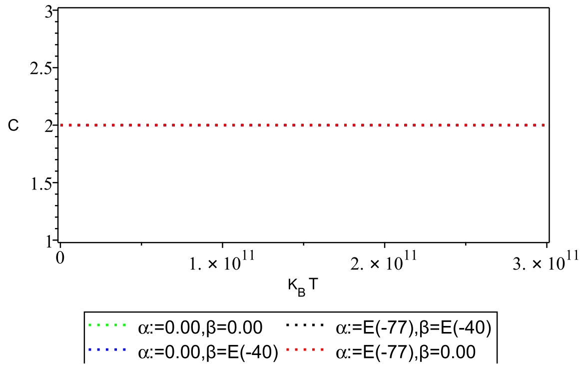
(b) Second observation .



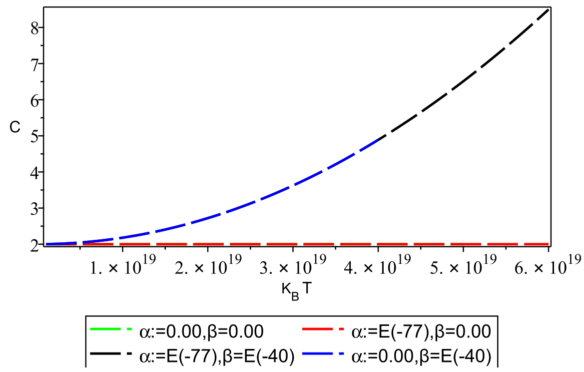
(c) Third observation.

Figure 4.5: The mean energy function for the DO with uniform electric field as a function of temperature k_BT for different values of the deformation parameters.

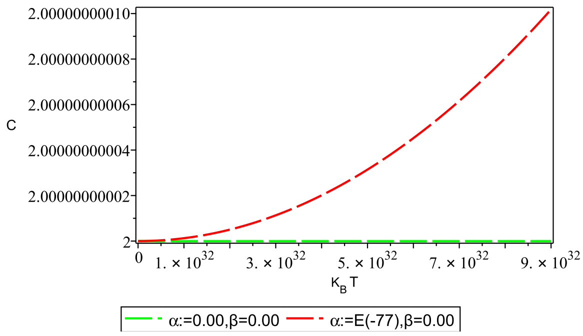
Moreover, as Figure (4.5a) illustrates, the mean energy in the aSdS model increases with temperature. In Fig. (4.5b) it is demonstrated that in the cases ($\alpha = 10^{-77}MeV$, $\beta = 10^{-35}MeV$) and ($\alpha = 0.0MeV$, $\beta = 10^{-35}MeV$), the curves decline to zero after reaching the temperature $k_BT \sim 10^{21}MeV$. But in this case, ($\alpha = 10^{-77}MeV$, $\beta = 0.0MeV$), the curve (4.5c) goes down to zero when k_BT surpasses $5\times10^{39}MeV$.



(a) First observation .



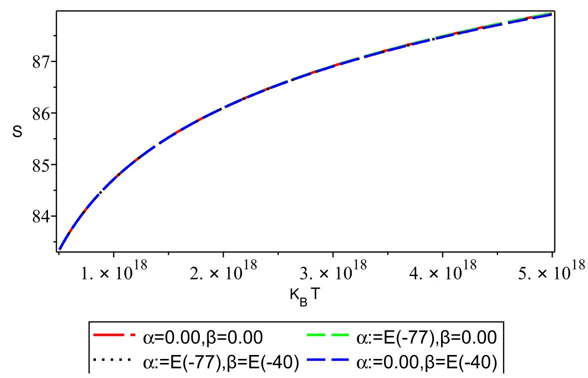
(b) Second observation .



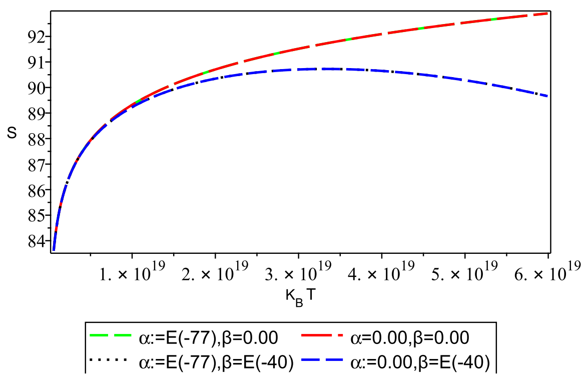
(c) Third observation.

Figure 4.6: The heat capacity function for the DO with uniform electric field as a function of temperature k_BT for different values of the deformation parameters.

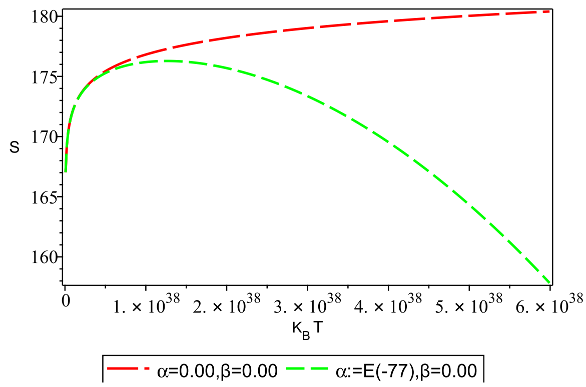
In addition, the heat capacity in Fig. (4.6a) is a constant $C = 2k_B$ when $k_BT < 10^{11}$. When $k_BT > 10^{19}MeV$, the cases ($\alpha = 10^{-77}MeV$, $\beta = 10^{-35}MeV$) and ($\alpha = 0.0MeV$, $\beta = 10^{-35}MeV$) show an increase with temperature, as shown in Fig. (4.6b). Figure (4.6c) depicts the capacity increase for the case ($\alpha = 10^{-77}MeV$, $\beta = 0.0MeV$) with rising temperature at $k_BT > 10^{32}$.



(a) First observation .



(b) Second observation .



(c) Third observation.

Figure 4.7: The entropy function for the DO with uniform electric field as a function of temperature $k_B T$ for different values of the deformation parameters.

Lastly, we plot the effect of aSdS on the entropy function in three graphs in Fig. (4.7a). As per Fig. (4.7b), at temperature $k_B T > 10^{19}$, the aSdS reduces the values of entropy with temperature for the cases ($\alpha = 10^{-77} MeV$, $\beta = 10^{-35} MeV$) and ($\alpha = 0.0 MeV$, $\beta = 10^{-35} MeV$)). On the other hand, the entropy function for the case ($\alpha = 10^{-77} MeV$, $\beta = 0.0 MeV$) in Fig. (4.7c) decreases as the temperature rises, with $k_B T > 10^{38}$.

As we mentioned earlier, it has been observed that the $\alpha-$parameter in the aSdS algebra affects energy eigenvalues more strongly than the $\beta-$parameter, the same holds true for thermodynamic functions. Likewise, simply by substituting (α and β) with ($-\alpha$,$-\beta$), we can deduce thermodynamic properties and suitable curves for the SdS model case. Ultimately, our findings coincide precisely with those reported in Ref. [78] when the aSdS parameters $\alpha = \beta = 0$ and the electric field $\mathcal{E} \to 0$.

4.5 Conclusion

This chapter outlines the formulation of the 1D-DO within the framework of the Snyder (anti-)de Sitter model, along with its exposure to a uniform electric field, all represented in momentum space. We derive the precise causal Green function and its corresponding propagator using the coordinate transformation technique, from which we extract the relevant energy values. The Green function and its corresponding propagator are expressed in terms of Jacobi polynomials when (α, β) are negative, and in terms of Romonovski polynomials when (α, β) are positive, in both cases for the sign deformation parameters. Moreover, we have shown that within the framework of Snyder (anti)-de Sitter space, the energy depends on n^2 and remains continuous even in the absence of oscillation and electric fields.Furthermore, we have constructed the non-relativistic energy level in the context of the aSdS algebra and derived limit cases for deformation parameters, taking into account both spin and non-spin situations.
The thermodynamic quantities of our system, including the partition function Z, the Helmholtz free energy F, the mean energy Ξ, the entropy S, and the heat capacity C, have all been determined in the first order of (α,β) at high temperatures using the Euler-MacLaurin formula. And we have demonstrated the importance of the $\alpha-$deformation parameter over the $\beta-$parameter by plotting the EUP terms of thermodynamic functions at temperature $k_B T$. Nevertheless, current experimental techniques are unable to identify these effects.

Chapter 5

Exact Green's Function for 2D Dirac Oscillator in Constant Magnetic Field within Snyder model, and its Thermal Properties

5.1 Introduction

The Dirac oscillator (DO) model combines harmonic oscillator (HO) elements with the Dirac equation to describe a relativistic quantum mechanical system. It describes the behavior of a relativistic particle with spin one half in the presence of a HO potential type, derived by transforming the momentum vector ($\mathbf{p} \to \mathbf{p} - im\omega\gamma^0\mathbf{x}$), where γ^0 is the Dirac matrix. Because of its tight link with several physical phenomena in quantum physics, many different versions of this physical system have been described. The original study by Ito et al. [79] was later developed upon by Moshinsky and Szczepaniak in [80]. When the non-relativistic limit is considered, the behaviour of the quantum HO may be restored, however, a spin-orbit coupling factor emerges in this limit as well. References such as [81–86] provide several examples from various branches of physics. Moreover, following the appearance of deformation theories grounded in Heisenberg's generalization principle [10, 27, 69], many researchers have promptly sought to explore its impact on relativistic oscillators. The Green's function technique is used in referencesfthermodynamic functions for this system. [35, 37] to present the DO model with a minimum length in one and two dimensions. In addition to determining the high-temperature thermodynamic properties of the DO in one dimension, see Ref. [62]. Furthermore, anti-de Sitter commutation relations result in the appearance of minimal uncertainty. Ref. [87] describes the DO in one dimension using the position space representation, and analyses the properties thermodynamic functions for relativistic harmonic oscillators in high temperatures. Later, within the extended uncertainty principle framework, Benzair et al. [88, 89] calculated the energy spectrum of the DO using the path integral formulation in one and two dimensions, respectively. Further, the study of thermodynamic properties for relativistic oscillator particles within deformed algebra is proved by the citations [34, 62, 89–93]. Referring to [94] as well, where the authors examine the relativistic spinning massless particle in the presence of a constant magnetic field within the Graphene layer. The behaviour of the DO in the Som–Raychaudhuri space-time was also examined by de Montigny et al. [95], with particular attention to the impact of the vorticity parameter and frequency. Then, as discussed in Ref. [96], this study was generalized to the DKP oscillator case for a zero spin field under cosmic-string background space-time, which is characterized by a stationary cylindrical symmetric metric.

Despite extensive discussions, only a limited number of studies have explored the DO using the path integral formulation. These applications are grounded in 3-model deformed algebras: the first one, known as GUP [10, 97], is based on DSR theories and confirms the presence of a minimum measurable length. Moreover, the second requires the existence of a minimum measurable momentum, which calls for an EUP to be created in place of the HUP [69, 98, 99]. On the other hand, the third is created by fusing GUP and EUP, which is prominent from a DSR model on a anti de-Sitter background. This results in the SdS model, also known as TSR [27, 94].

The following algebraic relationship is followed by the operators for position $\hat{X}_\mu$, momentum $\hat{P}_\mu$, momentum $\hat{P}_\mu$, and Lorentz generator $\hat{J}_{\mu\nu}$ to construct of the SdS model algebra.

$$\begin{aligned}
&\left[\hat{J}_{\mu\nu},\hat{X}_\sigma\right]=i\hbar\left(\eta_{\mu\sigma}\hat{X}_\nu-\eta_{\nu\sigma}\hat{X}_\mu\right), \quad \left[\hat{J}_{\mu\nu},\hat{P}_\sigma\right]=i\hbar\left(\eta_{\mu\sigma}\hat{P}_\nu-\eta_{\nu\sigma}\hat{P}_\mu\right),\\
&\left[\hat{X}_\mu,\hat{P}_\nu\right]=i\hbar\left(\eta_{\mu\nu}+\alpha\hat{X}_\mu\hat{X}_\nu+\beta\hat{P}_\mu\hat{P}_\nu+\sqrt{\alpha\beta}\left(\hat{P}_\mu\hat{X}_\nu+\hat{X}_\nu\hat{P}_\mu-\hat{J}_{\mu\nu}\right)\right),\\
&\left[\hat{X}_\mu,\hat{X}_\nu\right]=i\hbar\beta\hat{J}_{\mu\nu}; \qquad \left[\hat{P}_\mu,\hat{P}_\nu\right]=i\hbar\alpha\hat{J}_{\mu\nu}.
\end{aligned} \tag{5.1}$$

In this context, $\hat{J}_{\mu\nu}=\hat{X}_\mu\hat{P}_\nu-\hat{X}_\nu\hat{P}_\mu$ denotes the Lorentz symmetry generators and $\eta_{\mu\nu}=diag(1,-1,-1,-1)$ is the flat Minkowski space-time metric. The coupling constants, denoted as α and β, have inverse mass and inverse length, respectively. When the limit as $\alpha \to 0$, the above commutations relations (5.1) give to the Snyder model in flat space [69]. Moreover, the algebra (5.1) becomes the de Sitter algebra when $\beta \to 0$, and this parameter has an effect corresponding to the $\Lambda = -3\alpha$ cosmological constant [69, 98, 99].

In this chapter, our objective is to rigorously formulate the path integral approach in momentum space representation for the (2+1)-dimensional 2D-DO subjected to a uniform magnetic field and the Snyder model (SdS). Additionally, as outlined in [27], it furnishes the deformed Heisenberg algebra in the three-dimensional case within the non-relativistic SdS model.

$$\left[\hat{X}_i,\hat{P}_j\right] = \imath\hbar\left(\delta_{ij} + \alpha\hat{X}_i\hat{X}_j + \beta\hat{P}_i\hat{P}_j + \sqrt{\alpha\beta}\left(\hat{P}_i\hat{X}_j + \hat{X}_j\hat{P}_i\right)\right),$$
$$\left[\hat{X}_i,\hat{X}_j\right] = \imath\hbar\beta\hat{J}_{ij}, \quad \left[\hat{P}_i,\hat{P}_j\right] = \imath\hbar\alpha\hat{J}_{ij}. \tag{5.2}$$

Where $\hat{J}_{ij} = \hat{X}_i\hat{P}_j - \hat{X}_j\hat{P}_i$. In the limits $\alpha \to 0$, $\beta \to 0$ and $((\alpha,\beta) \to 0)$, the Snyder model in flat space is recovered to the de Sitter algebra and the undeformed Heisenberg algebra, respectively [69]. Given these commutation relations, it becomes vital to investigate the transformation that links this deformed algebra with the Snyder algebra. *Mignemi* citeMignemi2one first presented this transformation, and it is described as,

$$\hat{X}_i = \hat{\mathfrak{X}}_i + \sqrt{\frac{\beta}{\alpha}}\kappa\hat{\mathfrak{P}}_i = \imath\hbar\sqrt{1-\beta\mathbf{p}^2}\frac{\partial}{\partial p_i} + \sqrt{\frac{\beta}{\alpha}}\kappa\frac{p_i}{\sqrt{1-\beta\mathbf{p}^2}}, \tag{5.3}$$
$$\hat{P}_i = -\sqrt{\frac{\alpha}{\beta}}\hat{\mathfrak{X}}_i + (1-\kappa)\hat{\mathfrak{P}}_i = -\imath\hbar\sqrt{\frac{\alpha}{\beta}}\sqrt{1-\beta\mathbf{p}^2}\frac{\partial}{\partial p_i} + (1-\kappa)\frac{p_i}{\sqrt{1-\beta\mathbf{p}^2}}. \tag{5.4}$$

The index $(i = 1,2)$ stand the components vector of the position $\hat{X}_i := (\hat{X},\hat{Y})$or $\hat{P}_i := \left(\hat{P}_X,\hat{P}_Y\right)$ momentum operators. Here, κ is a free parameter that can be selected in each case. And in order to guarantee the symmetry of the Hamiltonian and that $(\hat{\mathfrak{X}}_i,\hat{\mathfrak{P}}_i)$ satisfies the below deformed Heisenberg commutation relations [69],

$$\left[\hat{\mathfrak{X}}_i,\hat{\mathfrak{P}}_j\right] = \imath\hbar\left(\delta_{ij} + \beta\hat{\mathfrak{P}}_i\hat{\mathfrak{P}}_j\right), \quad \left[\hat{\mathfrak{X}}_i,\hat{\mathfrak{X}}_j\right] = \beta\left(\hat{\mathfrak{X}}_i\hat{\mathfrak{P}}_j - \hat{\mathfrak{X}}_j\hat{\mathfrak{P}}_i\right), \quad \left[\hat{\mathfrak{P}}_i,\hat{\mathfrak{P}}_j\right] = 0. \tag{5.5}$$

Thus, the position $\hat{\mathfrak{X}}_i$ and momentum $\hat{\mathfrak{P}}_i$ operators of the Snyder Heisenberg brackets (5.5) can therefore be expressed in terms of auxiliary operators $\hat{x}_i = \imath\hbar\partial/\partial p_i$ and $\hat{p}_i = p_i$, which maintain the following relations:

$$\hat{\mathfrak{X}}_i = \sqrt{1-\beta\mathbf{p}^2}\hat{x}_i,\ \hat{\mathfrak{P}}_i = \frac{\hat{p}_i}{\sqrt{1-\beta\mathbf{p}^2}}. \tag{5.6}$$

It's crucial to stress that for positive values of α and β the momentum operator p_i is constrained within the interval $(-1/\sqrt{\beta})$ to $(1/\sqrt{\beta})$. Specifically, when $\langle P_i\rangle$ and $\langle X_i\rangle$ are both equal to zero, the uncertainty relation appears in the following form:

$$(\Delta X)_i(\Delta P)_j \geq \frac{\hbar}{2}\left(\delta_{ij}+\alpha(\Delta X)_i(\Delta X)_j+\beta(\Delta P)_i(\Delta P)_j+\sqrt{\alpha\beta}\left((\Delta P)_i(\Delta X)_j+(\Delta X)_i(\Delta P)_j\right)\right). \tag{5.7}$$

It is important to note that the concept of minimal uncertainties fails to apply in the cases where (α and β)< 0 (i.e., aSdS), meaning that all real values of p_i are allowed. Prior to delving into the specifics, which are covered in the section that follows, Notably, the scalar product has changed. It seems that the symmetry of the operators of $\hat{X}_i$ and $\hat{P}_i$ appears to be limited to the subspace $L^2(R^2,d\mathbf{p}/\sqrt{1-\beta\mathbf{p}^2})$), we use the following form shown in [27],

$$\langle\psi\mid\phi\rangle = \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{d\mathbf{p}}{\sqrt{1-\beta\mathbf{p}^2}}\psi^*(p)\phi(p), \tag{5.8}$$

these wave functions satisfy the periodic boundary conditions $\psi(-1/\sqrt{\beta}) = \psi(1/\sqrt{\beta})$, and accordingly, the modified closure relation is provided by [68], where,

$$\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{d\mathbf{p}}{\sqrt{1-\beta\mathbf{p}^2}}|\mathbf{p}\rangle\langle\mathbf{p}| = I. \tag{5.9}$$

Thus, we derive the following expression by applying the closure relation to the maximally localised states [68]:

$$\left\langle\mathbf{p}|\mathbf{p}'\right\rangle_{\alpha,\beta} = \left(\frac{1-\beta\mathbf{p}'^2}{1-\beta\mathbf{p}^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta\mathbf{p}^2}\delta^2(\mathbf{p}-\mathbf{p}'), \text{and } \gamma = i\kappa/\hbar\sqrt{\alpha\beta}. \tag{5.10}$$

It is noteworthy to observe that for ($\alpha<0$, $\beta<0$), in the above equation, we adjust the limits of integration to encompass the entire space. Furthermore, in the case that α and β are both equal to zero, we recover the standard projection relation, denoted by $\left\langle\mathbf{p}|\mathbf{p}'\right\rangle_{(\alpha,\beta)\to 0} = \delta^2(\mathbf{p}-\mathbf{p}')$.

Recalling that the time-momentum relationship in the time component is not deformed is essential.

$$\left\langle p_0 | p_0' \right\rangle = \delta\left(p_0 - p_0'\right) = \int \frac{dt}{2\pi\hbar} e^{-\frac{\imath}{\hbar} t\left(p_0 - p_0'\right)}. \tag{5.11}$$

In turn, the operators matrix elements $\mathcal{P}_i^2$, $\left(\hat{\mathcal{P}}_1\hat{\mathcal{X}}_2 - \hat{\mathcal{P}}_2\hat{\mathcal{X}}_1\right)$ and $\mathcal{X}_i^2$ are given, respectively, as follows,

$$\mathcal{P}_i^2 \left\langle \mathbf{p} \mid \mathbf{p}' \right\rangle_{\alpha,\beta} = \left\langle \mathbf{p} \mid \mathbf{p}' \right\rangle_{\alpha,\beta} \left[\frac{p_i^2}{1 - \beta p_i^2}\right], \tag{5.12}$$

and

$$\left(\hat{\mathcal{P}}_1\hat{\mathcal{X}}_2 - \hat{\mathcal{P}}_2\hat{\mathcal{X}}_1\right) \left\langle \mathbf{p} \mid \mathbf{p}' \right\rangle_{\alpha,\beta} = \left\langle \mathbf{p} \mid \mathbf{p}' \right\rangle_{\alpha,\beta} \left(p_y x - p_x y\right), \tag{5.13}$$

then

$$\begin{aligned} \hat{\mathcal{X}}_i^2 \left\langle \mathbf{p} \mid \mathbf{p}' \right\rangle_{\alpha,\beta} = \left\langle \mathbf{p} \mid \mathbf{p}' \right\rangle_{\alpha,\beta} \Bigg[& -\gamma(\gamma-1)\frac{\hbar^2\beta^2 p_i^2}{1-\beta p_i^2} - \hbar^2 2\beta\left(\gamma-1\right) \\ & -2\imath\hbar\beta\left(\gamma - \frac{3}{2}\right)\left(p_x x + p_y y\right) + \left(1 - \beta p_i^2\right) x_i^2 \Bigg]. \end{aligned} \tag{5.14}$$

This chapter is structured in six sections. section 2, our focus lies on formulating the path integral for particles have spin 1/2 within the context of Snyder model space-time. It is significant to note that, as proved in [35, 70], this formulation is achieved here without the need of Grassmann variables. This method relies on calculating the path integral on the Green function's elements matrix. A similar technique has been applied in previous studies [72, 89]. We perform to separate the radial part from the angular part, in section 3 by applying the polar coordinate transformation. The process of separating variables leads to the derivation of the Pöschel–Teller radial propagator [37, 49]. In contrast, for section 4, we have derived The exact solutions of the bound states and the appropriate energy eigenvalues. As section 5, we illustrates, the behaviour of the DO system in the presence of a uniform magnetic field, within the SdS algebra closely resembles the dynamics of the monolayer Graphene problem, assuming the following equality $m\bar{\omega} \to m\omega_c/2$ and $c \to V_F$. In section 6, we examined and discussed the special cases that result from these studies. We conclude section 7 by testing and plotting the thermodynamic functions for this system.

5.2 Path Integral analysis in Snyder-de Sitter space for (2+1)-Dirac oscillator

Now, we proceed to study the Green function $\hat{S}$ for the (2+1)-dimensional DO in momentum space representation, within the context of SdS space, and in the presence of a uniform magnetic field ($\mathbf{B} = \mathcal{B}\mathbf{k}$), which is given by the following equation [5],

$$\left(\hat{H} - \imath\hbar\partial_t\right)\hat{S} = I. \tag{5.15}$$

In this case, the unit matrix is I. The Hamiltonian expression for the DO without electromagnetic interaction is defined by [5],

$$\hat{H} = c\alpha.\left(\overrightarrow{\hat{P}} - \imath m\omega\beta\overrightarrow{\hat{X}}\right) + \beta mc^2, \tag{5.16}$$

Eq. (5.6) is used to verify the momenta $\overrightarrow{\hat{P}}-$ and position $\overrightarrow{\hat{X}}-$ operators in this case. Where the parameters denote m, c and ω to the mass of the particle, the speed of light, and the angular frequency of the oscillator, respectively. For each α and β , the σ_i-Pauli matrices serve as:

$$\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{5.17}$$

Following the application of minimal electromagnetic coupling in Eq. (5.16) [5], Eq. (5.15) minimises:

$$\left[\gamma^0\hat{P}_0 - \gamma.(\overrightarrow{\hat{P}} - \frac{e}{c}\mathbf{A}) + \imath m\omega\gamma^0\gamma.\overrightarrow{\hat{X}} - mc\right]\hat{S} = -\mathbb{I}. \tag{5.18}$$

For a relativistic particle, the parameter $e = \mp|e|$ describes a particle with positive charge ($e = |e|$) or negative charge ($e = -|e|$). Moreover, the Pauli matrices in two dimensions represent the $\gamma^\mu-$Dirac matrices

$$\gamma^0 = \sigma_3, \ \gamma^1 = \imath\sigma_2, \ \gamma^2 = -\imath\sigma_1. \tag{5.19}$$

Observe that the time component is not deforming.($\hat{P}_0 = \imath\hbar\partial_0 = \imath\hbar\partial/\partial ct$, $\hat{X}_0 = \hat{x}^0 := ct$). The two components of the vector potential ($\mathbf{A} = \frac{\mathcal{B}}{2c}\left(-\hat{X}_2, \hat{X}_1\right)$) are the potential of a uniform magnetic field $\mathcal{B}$. Eq. (5.18) can therefore be expressed as stated in [5],

$$\left[\gamma^0 \imath\hbar\partial/\partial t - c\gamma^1\left(\hat{P}_1 + m\bar{\omega}\hat{X}_2\right) - c\gamma^2\left(\hat{P}_2 - m\bar{\omega}\hat{X}_1\right) - mc^2\right]\hat{S} = -\mathbb{I}, \tag{5.20}$$

here, $\bar{\omega}$ denotes to $\bar{\omega} = \omega \mp \omega_c/2$, with $\omega_c = \frac{|e|\mathcal{B}}{mc}$ is the cyclotron frequency. The formal solution of Eq. (5.20) is

$$\hat{S} = -\left[\mathcal{O}_-^D\right]^{-1} = -\mathcal{O}_+^D\left[\mathcal{O}_-^D\mathcal{O}_+^D\right]^{-1}, \tag{5.21}$$

where $\mathcal{O}_\pm^D$ is defined by

$$\mathcal{O}_\pm^D = \left[\gamma^0 \imath\hbar\partial/\partial t - c\gamma^1\left(\hat{P}_1 + m\bar{\omega}\hat{X}_2\right) - c\gamma^2\left(\hat{P}_2 - m\bar{\omega}\hat{X}_1\right) \pm mc^2\right]. \tag{5.22}$$

Using the Schwinger proper-time method [73], and after mentioning that $\hat{S} = -\left[\mathcal{O}_+^D\right]\left[\mathcal{O}_-^D\mathcal{O}_+^D\right]^{-1}$, the Green's matrix operator $\hat{S}$ can be conveniently expressed as

$$\hat{S} = \left[\mathcal{O}_+^D\right]\hat{\mathcal{G}}, \tag{5.23}$$

with

$$\hat{\mathcal{G}} = \frac{\imath}{\hbar}\int_0^\infty d\lambda \exp\left(-\frac{\imath}{\hbar}\lambda\hat{\mathcal{H}}\right), \tag{5.24}$$

where an even variable is denoted by λ. The formula $\hat{\mathcal{H}}-$operator is represented below:

$$\begin{aligned}\hat{\mathcal{H}} = -&\left[\gamma^0 \imath\hbar\partial/\partial t - c\gamma^1\left(\hat{P}_1 + m\bar{\omega}\hat{X}_2\right) - c\gamma^2\left(\hat{P}_2 - m\bar{\omega}\hat{X}_1\right) - mc^2\right] \\ &\times\left[\gamma^0 \imath\hbar\partial/\partial t - c\gamma^1\left(\hat{P}_1 + m\bar{\omega}\hat{X}_2\right) - c\gamma^2\left(\hat{P}_2 - m\bar{\omega}\hat{X}_1\right) + mc^2\right].\end{aligned} \tag{5.25}$$

After the equation (5.25) is simplified, we obtain:

$$\begin{aligned}\hat{\mathcal{H}} = -\Big[&-\hbar^2\partial_t^2 - m^2c^4 - c^2\left(\hat{P}_1^2 + \hat{P}_2^2\right) - c^2\left(m\bar{\omega}\right)^2\left(\hat{X}_1^2 + \hat{X}_2^2\right) \\ &-c^2 m\bar{\omega}\left[\left(\hat{X}_2\hat{P}_1 + \hat{P}_1\hat{X}_2\right) - \left(\hat{X}_1\hat{P}_2 + \hat{P}_2\hat{X}_1\right)\right] \\ &+c^2\gamma^1\gamma^2\left\{\left[\hat{P}_1, \hat{P}_2\right] + \left(m\bar{\omega}\right)^2\left[\hat{X}_1, \hat{X}_2\right] + m\bar{\omega}\left[\hat{X}_2, \hat{P}_2\right] + m\bar{\omega}\left[\hat{X}_1, \hat{P}_1\right]\right\}\Big].\end{aligned} \tag{5.26}$$

To express this Hamiltonian, we must use the position and momentum operators, which achieve the deformed quantum algebra described by Snyder and are predicated on the adapted commutation relation provided in the previous section (see, Eq. (5.5)) [27]. By inserting the operators ($\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_j\right)$) into the Hamiltonian expression $\hat{\mathcal{H}}-$, we find:

$$\hat{\mathcal{H}} = -\left[-\hbar^2\partial_t^2 - m^2c^4 - c^2\left((m\bar{\omega})^2 + \frac{\alpha}{\beta}\right)\left(\hat{\mathcal{X}}_1^2 + \hat{\mathcal{X}}_2^2\right) - 2c^2 m\bar{\omega}\left(\hat{\mathcal{X}}_2\hat{\mathcal{P}}_1 - \hat{\mathcal{X}}_1\hat{\mathcal{P}}_2\right)\right.$$
$$\left. - c^2\left((1-\kappa)^2 + \kappa^2\frac{\beta}{\alpha}(m\bar{\omega})^2\right)\left(\hat{\mathcal{P}}_1^2 + \hat{\mathcal{P}}_2^2\right) - \imath\hbar\gamma^1\gamma^2\hat{F}(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i)\right], \tag{5.27}$$

with

$$\hat{F}(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i) = c^2\beta\left(\frac{\alpha}{\beta} + (m\bar{\omega})^2\right)\left(\hat{\mathcal{X}}_2\hat{\mathcal{P}}_1 - \hat{\mathcal{X}}_1\hat{\mathcal{P}}_2\right) - c^2 m\bar{\omega}\left(2 + \beta\left(\hat{\mathcal{P}}_1^2 + \hat{\mathcal{P}}_2^2\right)\right). \tag{5.28}$$

Based on the given value of κ, the equation above indicates that the term $(\hat{\mathcal{P}}_i\hat{\mathcal{X}}_i + \hat{\mathcal{P}}_i\hat{\mathcal{X}}_i)$ will be evidently absent.

$$\kappa = \frac{1}{1 + \frac{\beta}{\alpha}(m\bar{\omega})^2}. \tag{5.29}$$

When $\hat{\mathcal{G}}$ is represented in momentum space, the corresponding element matrix is

$$\mathcal{G}(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a}) = -\frac{\imath}{\hbar}\int_0^\infty d\lambda \left\langle \mathbf{p}_b, p_{0b}\left|\exp\left(-\frac{\imath}{\hbar}\lambda\hat{\mathcal{H}}\right)\right|\mathbf{p}_a, p_{0a}\right\rangle. \tag{5.30}$$

Before going into building the Green function $\mathcal{G}(\mathbf{p}_b, \mathbf{p}_a, p_{0b}, p_{0a})$ using the path integral formulation. First, we need to remove the Pauli matrices, as they are not in line with Feynman's formulation, by constructing the following exponential matrix. Then, we can simplify it as follows

$$\exp\left(\lambda\gamma^1\gamma^2\hat{F}\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i\right)\right) = \cos\left(\lambda\hat{F}\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i\right)\right) + \gamma^1\gamma^2\sin\left(\lambda\hat{F}\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i\right)\right), \tag{5.31}$$

given that $(\gamma^1\gamma^2)^2 = -1$, considering the properties of Dirac matrices, Eq. (5.31) can be expressed:

$$\exp\left(\lambda\gamma^1\gamma^2\hat{F}\left(\hat{\mathcal{X}}_i,\hat{\mathcal{P}}_i\right)\right) = \frac{1}{2}\sum_{s=\pm1}\left[1+\imath s\gamma^1\gamma^2\right]\exp\left(-\frac{\imath}{\hbar}s\lambda\hbar\hat{F}\left(\hat{\mathcal{X}}_i,\hat{\mathcal{P}}_i\right)\right). \tag{5.32}$$

As a result, Eq. (5.33) may be expressed as follows:

$$\mathcal{G}\left(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}\right) = \frac{\imath}{2\hbar}\sum_{s=\pm1}\left[1+\imath s\gamma^1\gamma^2\right]\int_0^\infty d\lambda\left\langle\mathbf{p}_b,p_{0b}\right|\exp\left(-\frac{\imath}{\hbar}\hat{\mathcal{H}}^{(s)}\right)\left|\mathbf{p}_a,p_{0a}\right\rangle, \tag{5.33}$$

with

$$\begin{aligned}\hat{\mathcal{H}}^s = -\lambda\Bigg[&-\hbar^2\partial_t^2 - c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(\hat{\mathcal{X}}_1^2+\hat{\mathcal{X}}_2^2\right) - \frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta}+(m\bar{\omega})^2}\left(\hat{\mathcal{P}}_1^2+\hat{\mathcal{P}}_2^2\right)\\ &-2c^2m\bar{\omega}\left(\hat{\mathcal{P}}_1\hat{\mathcal{X}}_2-\hat{\mathcal{P}}_2\hat{\mathcal{X}}_1\right) - s\hbar\hat{F}\left(\hat{\mathcal{X}}_i,\hat{\mathcal{P}}_i\right) - m^2c^4\Bigg].\end{aligned} \tag{5.34}$$

We break down the exponential $\exp(-\imath\lambda\hat{\mathcal{H}}^{(s)})$ for the kernel of (5.33) into $(N+1)$ exponential $\exp(-\imath\varepsilon\hat{\mathcal{H}}^{(s)})$, with $\varepsilon = \tau_j - \tau_{j-1} = 1/(N+1)$. Next, between every pair of infinitesimal operator $\exp(-\imath\varepsilon\hat{\mathcal{H}}^{(s)})$ we insert N resolutions of identities (5.9). In fact, we have [68],

$$\mathcal{G}\left(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}\right) = \frac{\imath}{2\hbar}\sum_{s=\pm1}\left[1+\imath s\gamma^1\gamma^2\right]\lim_{N\longrightarrow\infty}\int_0^\infty d\lambda\prod_{j=1}^{N}\int\frac{dp_{0j}d\mathbf{p}_j}{\sqrt{1-\beta\mathbf{p}_j^2}}\prod_{j=1}^{N+1}\left\langle\mathbf{p}_j,p_{0j}\right|e^{-\frac{\imath\varepsilon}{\hbar}\hat{\mathcal{H}}^{(s)}}\left|\mathbf{p}_{j-1},p_{0j-1}\right\rangle_{\alpha,\beta}. \tag{5.35}$$

To proceed further, the exponential can be developed to the first order of ε. Consequently, we obtain

$$\begin{aligned}&\lim_{N\longrightarrow\infty,\varepsilon\to0}\left\langle\mathbf{p}_j,p_{0j}\right|e^{-\frac{\imath\varepsilon}{\hbar}\hat{\mathcal{H}}^{(s)}}\left|\mathbf{p}_{j-1},p_{0j-1}\right\rangle_{\alpha,\beta}\\ &=\lim_{N\longrightarrow\infty,\varepsilon\to0}\left[\left\langle\mathbf{p}_j,p_{0j}\mid\mathbf{p}_{j-1},p_{0j-1}\right\rangle_{\alpha,\beta} - \frac{\imath\varepsilon}{\hbar}\left\langle\mathbf{p}_j,p_{0j}\right|\hat{\mathcal{H}}^{(s)}\left|\mathbf{p}_{j-1},p_{0j-1}\right\rangle_{\alpha,\beta}\right].\end{aligned} \tag{5.36}$$

Next, we add each and every operators ($\mathcal{X}_i^2$, $\mathcal{P}_i^2$, $\hat{\mathcal{P}}_1\hat{\mathcal{X}}_2-\hat{\mathcal{P}}_2\hat{\mathcal{X}}_1$) to the projection relationship $\left\langle\mathbf{p}_j\mid(.)\mid\mathbf{p}_{j-1}\right\rangle_{\alpha,\beta}$. In order to remove the Hamiltonian operator in the SdS framework, the expression $\mathcal{G}\left(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}\right)$ is converted into the following path integral in phase space:

$$\mathcal{G}\left(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}\right)=-\frac{\imath}{2\hbar}\lim_{\substack{N\to\infty\\ \varepsilon\to 0}}\sum_{s=\pm 1}\left[1+\imath s\gamma^1\gamma^2\right]\int_0^\infty d\lambda\prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_{0j}d\mathbf{p}_j}{\sqrt{1-\beta\mathbf{p}_j^2}}\right]$$
$$\times\prod_{j=1}^{N+1}\left[\left(\frac{1-\beta\mathbf{p}_{j-1}^2}{1-\beta\mathbf{p}_j^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta\mathbf{p}_j^2}\int\frac{dx_j^\mu}{(2\pi\hbar)^3}\right]\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[-x_j^\mu\Delta p_{j_\mu}+\lambda\varepsilon\left[p_{0j}^2-m^2c^4\right.\right.\right.$$
$$-c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(-\gamma(\gamma-1)\frac{\hbar^2\beta^2\mathbf{p}_j^2}{1-\beta\mathbf{p}_j^2}-2\beta\hbar^2(\gamma-1)\right.$$
$$\left.+2\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)\left(x_jp_{x_j}+y_jp_{y_j}\right)+\left(1-\beta\mathbf{p}_j^2\right)\left(x_j^2+y_j^2\right)\right)+s\hbar c^2m\bar{\omega}\left(2+\frac{\beta\mathbf{p}_j^2}{1-\beta\mathbf{p}_j^2}\right)$$
$$\left.\left.\left.-\frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta}+(m\bar{\omega})^2}\frac{\mathbf{p}_j^2}{1-\beta\mathbf{p}_j^2}+c^2\left(2m\bar{\omega}+s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right)\left(p_{y_j}x_j-p_{x_j}y_j\right)\right]\right]\right\}. \tag{5.37}$$

It is convenient to eliminate all terms multiply by the $\gamma-$parameter by applying the term $\left(\left(1-\beta\mathbf{p}_{j-1}^2\right)/\left(1-\beta\mathbf{p}_j^2\right)\right)^{\frac{\gamma}{2}}$. With the following analysis, this can be made clear [68],

$$\ln\left(\frac{1-\beta\mathbf{p}_{j-1}^2}{1-\beta\mathbf{p}_j^2}\right)^{\gamma/2}=-\frac{\gamma}{2}\ln\left(\frac{1-\beta\mathbf{p}_j^2}{1-\beta\mathbf{p}_{j-1}^2}\right)$$
$$=\beta\gamma\frac{\left(p_{xj}\Delta p_{xj}+p_{yj}\Delta p_{yj}\right)}{1-\beta\mathbf{p}_j^2}-\frac{2\imath\varepsilon}{\hbar}\gamma\beta c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)$$
$$-\frac{\imath\varepsilon}{\hbar}2\beta^2\gamma c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{\mathbf{p}_j^2}{1-\beta\mathbf{p}_j^2}. \tag{5.38}$$

Furthermore, after performing the multiple Gaussian integrations over (x,y,t), the Lagrangian path integral representation will be obtained by converting the Green function $\mathcal{G}\left(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}\right)$ to the following form:

$$\begin{aligned}
\mathcal{G}(\mathbf{p}_b,\mathbf{p}_a,p_{0b},p_{0a}) &= -\frac{\imath}{2\hbar}\lim_{N\longrightarrow\infty}\sum_{s=\pm1}\left[1+\imath s\gamma^1\gamma^2\right]\delta\left(p_{0b}-p_{0a}\right)\int_0^\infty d\lambda\prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{d\mathbf{p}_j}{\sqrt{1-\beta\mathbf{p}_j^2}}\right] \\
&\times\prod_{j=1}^{N+1}\left[\sqrt{1-\beta\mathbf{p}_j^2}\left(\frac{1}{(2\pi\hbar)}\sqrt{\frac{\pi}{\frac{\imath}{\hbar}\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta\mathbf{p}_j^2\right)}}\right)^2\right] \\
&\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\left(\Delta\mathbf{p}_j\right)^2}{4\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta\mathbf{p}_j^2\right)}+\lambda\varepsilon\left(p_0^2-m^2c^4\right)\right.\right. \\
&-\frac{9}{4}\hbar^2\beta^2c^2\lambda\varepsilon\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{\mathbf{p}_j^2}{1-\beta\mathbf{p}_j^2}-\frac{3}{2}\imath\hbar\beta\frac{\left(p_{y_j}\Delta p_{y_j}+p_{x_j}\Delta p_{x_j}\right)}{\left(1-\beta\mathbf{p}_j^2\right)} \\
&+\lambda\varepsilon\frac{c^2\left(2m\bar{\omega}+s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right)^2}{4\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)}\frac{\mathbf{p}_j^2}{1-\beta\mathbf{p}_j^2}-2\beta c^2\hbar^2\lambda\varepsilon\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right) \\
&+\frac{2m\bar{\omega}+s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)}{2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)}\frac{\left(p_{x_j}\Delta p_{y_j}-p_{y_j}\Delta p_{x_j}\right)}{\left(1-\beta\mathbf{p}_j^2\right)} \\
&\left.\left.-\lambda\varepsilon\frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta}+(m\bar{\omega})^2}\frac{\mathbf{p}_j^2}{1-\beta\mathbf{p}_j^2}+s\hbar c^2m\bar{\omega}\lambda\varepsilon\left(2+\frac{\beta\mathbf{p}_j^2}{1-\beta\mathbf{p}_j^2}\right)\right]\right\}. \qquad (5.39)
\end{aligned}$$

In the following section, using spherical two-dimensional coordinates, we were able to successfully complete the calculation. Given the established importance of symmetries in preserving the physical quantities of this system, we need to determine the best way to account for them.

5.3 Green Function Analysis In Polar Coordinates

Firstly, let us start by using relative polar coordinates (p_ρ, p_θ) in order to simplify the path integrals above (5.39), where in the 2-dimensional spherical coordinates of the momentum variables $\mathbf{p}$ are given by

$$p_x = p_\rho\cos\left(p_\theta\right),\ p_y = p_\rho\sin\left(p_\theta\right), \qquad (5.40)$$

with $0 < p_\theta < \pi$ and $\mathbf{p}^2 = p_x^2 + p_y^2$. This leads to the transformation of the measure term, kinetic term, and the action terms [68],

$$\prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \frac{d\mathbf{p}_j}{\sqrt{1-\beta \mathbf{p}_j^2}}\right] = \prod_{j=1}^{N}\left[\int \frac{p_{\rho_j} dp_{\rho_j}}{\sqrt{1-\beta p_{\rho_j}^2}} dp_{\theta_j}\right]. \tag{5.41}$$

$$(\Delta \mathbf{p}_j)^2 = p_{\rho_j}^2 + p_{\rho_{j-1}}^2 - 2p_{\rho_j}p_{\rho_{j-1}}\cos\left(\Delta p_{\theta_j}\right). \tag{5.42}$$

$$\mathbf{p}_j \Delta \mathbf{p}_j = p_{\rho_j}\Delta p_{\rho_j} + p_{\rho_j}p_{\rho_{j-1}}\left(\frac{1}{2}\left(\Delta p_{\theta_j}\right)^2 + \ldots\right). \tag{5.43}$$

$$p_{x_j}\Delta p_{y_j} - p_{y_j}\Delta p_{x_j} = p_{\rho_j}p_{\rho_{j-1}}\sin\left(\Delta p_{\theta_j}\right), \tag{5.44}$$

the kinetic energy term is used to calculate the correction $\left(\Delta p_{\theta_j}\right)^2$, which is equal to [68],

$$\left(\Delta p_{\theta_j}\right)^2 \sim 2\imath\hbar\lambda\varepsilon c^2\left(\frac{\alpha}{\beta} + (m\bar{\omega})^2\right)\frac{1-\beta p_{\rho_j}^2}{p_{\rho_j}p_{\rho_{j-1}}}. \tag{5.45}$$

After incorporating that into the Eq. (5.39), the Green function becomes:

$$\begin{aligned}
\mathcal{G}\left(p_{\rho_b}, p_{\rho_a}, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right) &= -\frac{\imath}{2\hbar}\lim_{N\longrightarrow\infty}\sum_{s=\pm1}\left[1+\imath s\gamma^1\gamma^2\right]\delta\left(p_{0b}-p_{0a}\right) \\
&\times\int_0^\infty d\lambda \prod_{j=1}^{N}\left[\int \frac{p_{\rho_j}dp_{\rho_j}}{\sqrt{1-\beta p_{\rho_j}^2}} dp_{\theta_j}\right]\prod_{j=1}^{N+1}\left[\frac{\sqrt{1-\beta p_{\rho_j}^2}}{4\pi\imath\hbar\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}\right] \\
&\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{p_{\rho_j}^2+p_{\rho_{j-1}}^2-2p_{\rho_j}p_{\rho_{j-1}}\cos\left(\Delta p_{\theta_j}\right)}{4\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)} - \imath\hbar\frac{3}{2}\frac{\beta p_{\rho_j}\Delta p_{\rho_j}}{\left(1-\beta p_{\rho_j}^2\right)} + \lambda\varepsilon\left(p_0^2 - m^2c^4\right)\right.\right. \\
&-\frac{9}{4}\hbar^2\beta^2\lambda\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{p_{\rho_j}^2}{\left(1-\beta p_{\rho_j}^2\right)} - 2\beta\hbar^2\lambda\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right) \\
&+\frac{3}{2}\beta\hbar^2c^2\lambda\varepsilon\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right) + \lambda\varepsilon\frac{c^2\left(2m\bar{\omega}+s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right)^2 p_{\rho_j}^2}{4\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)} \\
&+\frac{2m\bar{\omega}+s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)}{2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)}\frac{p_{\rho_j}p_{\rho_{j-1}}\sin\left(\Delta p_{\theta_j}\right)}{\left(1-\beta p_{\rho_j}^2\right)} + s\hbar c^2 m\bar{\omega}\lambda\varepsilon\left(2+\frac{\beta p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\right) \\
&\left.\left.-\lambda\varepsilon\frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta}+(m\bar{\omega})^2}\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\right]\right\}.
\end{aligned} \tag{5.46}$$

The third term in the kinetic energy, along with the final term in the procedure, indicates the possibility of an angle shift according to the following relation

$$p_{\theta_j} \rightarrow p_{\theta_j} + \tau_j \lambda c^2 \left(2m\bar{\omega} + s\hbar\beta\left(\frac{\alpha}{\beta} + (m\bar{\omega})^2\right)\right), \tag{5.47}$$

The τ represents time in physics. The path integral over the angle p_{θ_j} will then be calculated using the well-known relation [3]

$$\exp\left(a\cos p_\theta\right) = \sum_{\ell=-\infty}^{+\infty} I_\ell\left(a\right)\exp\left(\imath\ell p_\theta\right), \tag{5.48}$$

where the modified Bessel functions are denoted by $I_\ell\left(a\right)$. Following simple calculation, Eq. (5.46) can be expressed as

$$\begin{aligned}
\mathcal{G}\left(p_{\rho_b}, p_{\rho_a}, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right) &= -\frac{\imath}{2\hbar}\lim_{N\longrightarrow\infty}\sum_{s=\pm1}\left[1+\imath s\gamma^1\gamma^2\right]\delta\left(p_{0b}-p_{0a}\right)\int_0^\infty d\lambda \prod_{j=1}^N\left[\int\frac{p_{\rho_j}dp_{\rho_j}}{\sqrt{1-\beta p_{\rho_j}^2}}\right]\\
&\times\prod_{j=1}^{N+1}\left[\left(\frac{\sqrt{1-\beta p_{\rho_j}^2}}{4\pi\imath\hbar\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}\right)\sum_{\ell_j=-\infty}^{+\infty} I_{\ell_j}\left(-\frac{\imath}{\hbar}\frac{p_{\rho_j}p_{\rho_{j-1}}}{2\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}\right)\right]\\
&\times\prod_{j=1}^N\left[\int dp_{\theta_j}\right]\prod_{j=1}^{N+1}\left[e^{\imath\ell_j\left(\Delta p_{\theta_j}+\lambda\varepsilon c^2\left(2m\bar{\omega}+s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right)\right)}\right]\\
\times\exp\Bigg\{\frac{\imath}{\hbar}\Bigg[&\frac{p_{\rho_j}^2+p_{\rho_{j-1}}^2}{4\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)} + \lambda\varepsilon\left(p_0^2-m^2c^4\right) - \frac{9}{4}\lambda\varepsilon\hbar^2c^2\beta^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\\
&-\imath\hbar\frac{3}{2}\frac{\beta p_{\rho_j}\Delta p_{\rho_j}}{1-\beta p_j^2} + \frac{3}{2}\lambda\varepsilon\hbar^2c^2\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right) - 2\lambda\varepsilon\hbar^2c^2\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\\
&-\lambda\varepsilon\frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta}+(m\bar{\omega})^2}\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2} + \lambda\varepsilon c^2 s\hbar m\bar{\omega}\left(2+\frac{\beta p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\right)\Bigg]\Bigg\},
\end{aligned} \tag{5.49}$$

it is now possible to perform the $N-$integrations over the $p_{\theta_j}-$variables, yielding the N symbols of Kronecker [3],

$$\prod_{j=1}^N\left[\int_0^{2\pi}dp_{\theta_j}\right]\prod_{j=1}^{N+1}\left[e^{\imath\ell_j\Delta p_{\theta_j}}\right] = \prod_{j=1}^N\left(2\pi\delta_{\ell_j,\ell_{j+1}}\right)e^{\imath\ell_{N+1}p_{\theta_{N+1}}-\imath\ell_1 p_{\theta_0}}. \tag{5.50}$$

With the exception of ℓ, these symbols can remove all summations. The radial time evolution amplitudes are defined as follows in relation to the azimuthal ℓ-quantum numbers [3]:

$$\mathcal{G}\left(p_{\rho_b},p_{\rho_a},p_{\theta_b},p_{\theta_a};p_{0b},p_{0a}\right)=\frac{1}{2\pi}\frac{1}{\sqrt{p_{\rho_b}p_{\rho_a}}}\sum_{\ell=-\infty}^{+\infty}e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\mathcal{G}_\ell\left(p_{\rho_b},p_{\rho_a};p_{0b},p_{0a}\right), \quad (5.51)$$

and

$$\begin{aligned}\mathcal{G}_\ell\left(p_{\rho_b},p_{\rho_a};p_{0b},p_{0a}\right)=&-\frac{\imath}{2\hbar}\delta\left(p_{b0}-p_{a0}\right)\sum_{s=\pm1}\left[1+\imath s\gamma^1\gamma^2\right]\lim_{N\longrightarrow\infty}\int_0^\infty d\lambda\, e^{\frac{\imath}{\hbar}\lambda\left(p_0^2-m^2c^4\right)}\\&\times\prod_{j=1}^{N}\int_{-1/\beta}^{1/\beta}\left[\frac{dp_{\rho_j}}{\sqrt{1-\beta p_{\rho_j}^2}}\right]\prod_{j=1}^{N+1}\left[4\pi\imath\hbar\lambda\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right]^{-1/2}\\&\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\left(\Delta p_{\rho_j}\right)^2}{4\lambda\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}-\imath\hbar\frac{3}{2}\frac{\beta p_{\rho_j}\Delta p_{\rho_j}}{\left(1-\beta p_{\rho_j}^2\right)}\right.\right.\\&+\lambda\varepsilon\left(-\frac{9}{4}\hbar^2\beta^2c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}-\frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta}+(m\bar{\omega})^2}\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\right.\\&+s\hbar c^2m\bar{\omega}\frac{\beta p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}-\hbar^2c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{\left(\ell^2-1/4\right)\left(1-\beta p_{\rho_j}^2\right)}{p_{\rho_j}p_{\rho_{j-1}}}\\&\left.\left.\left.-\frac{c^2\hbar^2\beta}{2}\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)+2c^2s\hbar m\bar{\omega}+\hbar\ell c^2\left[2m\bar{\omega}+s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right]\right)\right]\right\}. \quad (5.52)\end{aligned}$$

This is accomplished while considering the following relationship

$$I_\ell(z)=e^z(2\pi z)^{-1/2}\tilde{I}_\ell(z). \quad (5.53)$$

The modified Bessel functions $\tilde{I}_\ell(z)=\exp\left(-\frac{\ell^2-1/4}{2z}\right)$ exhibit asymptotic equality as $|z|\to\infty$, with $|\arg z|<0$ [3].

This propagator's expression (5.52) takes on a more complex form because it contains the term measure. Using the point transformation method (refer to Ref. [35]) to simplify this, the Υ-point discretization interval is defined as

$$p_{\rho_j}^{(\delta)}=\Upsilon p_{\rho_j}+(1-\Upsilon)p_{\rho_{j-1}}. \quad (5.54)$$

Based on Refs. [35, 37], this usually gives rise to three quantum corrections: a momentum measure term $(dp_{\rho_j}/\sqrt{1-\beta p_{\rho_j}^2})$, a kinetic energy term, and the last term in action (5.52). We extend these corrections using the discretization interval of $p_{\rho_j}^{(\Upsilon)}$-points in order to analyse these corrections even more. Subsequently, we employ a coordinate transformation to restore the conventional kinetic term $(\frac{(\Delta x_j)^2}{4\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)})$. The function $f(x)$ is defined by the following condition:

$$df(x)/dx=\sqrt{1-\beta p_{\rho}^2}\Rightarrow\sqrt{\beta}p_{\rho}=\sin x. \tag{5.55}$$

Subsequently, we utilize discretization intervals, referred to as Υ-points, to express the kinetic energy and measurement terms. This facilitates the determination of the overall correction, denoted by the symbol C_T.

$$C_T=\imath\hbar\varepsilon c^2\left(\alpha+\beta(m\bar{\omega})^2\right)\left[\frac{5}{4}\tan^2 x-\left(2\Upsilon^2-\Upsilon-1\right)\frac{1}{\cos^2 x}\right]. \tag{5.56}$$

With the predefined Υ-values (i.e., $\Upsilon=0,1/2$) given in Refs.[35, 37], C_T assumes the following form:

$$C_T=\imath\hbar\varepsilon\beta c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left[1+\frac{9}{4}\tan^2 x\right]. \tag{5.57}$$

In doing so, the radial propagator $\mathcal{G}\ell(xb,x_a;p_{0b},p_{0a})$ is converted into the following expression:

$$\begin{aligned}\mathcal{G}_{\ell}(x_b,x_a;p_{0b},p_{0a})&=-\frac{\imath}{2\hbar}\delta(p_{b0}-p_{a0})\sum_{s=\pm1}\left[1+\imath s\gamma^1\gamma^2\right]\\ \times\int_0^{\infty}d\lambda\exp\Bigg\{&\frac{\imath\lambda}{\hbar}\Bigg[p_0^2-m^2c^4+\frac{1}{2}\hbar^2c^2\left(\alpha+\beta(m\bar{\omega})^2\right)+\hbar\ell c^2\left[2m\bar{\omega}+s\hbar\left(\alpha+\beta(m\bar{\omega})^2\right)\right]\\ +2c^2s\hbar m\bar{\omega}+\hbar^2c^2&\left(\alpha+\beta(m\bar{\omega})^2\right)\left(\ell^2-1/4\right)-c^2\left[s\hbar m\bar{\omega}-\frac{(m\bar{\omega})^2}{\left(\alpha+\beta(m\bar{\omega})^2\right)}\right]\Bigg]\Bigg\}\mathcal{K}_{\ell}(x_b,x_a,\lambda).\end{aligned} \tag{5.58}$$

As shown in reference [49], the kernel radial propagator $\mathcal{K}_{\ell}(x_b,x_a,\lambda)$ equates precisely to the path integral of a particle subjected to the Pöschel–Teller potential (PTP),

$$\mathcal{K}_{\ell}(x_b,x_a,\lambda)=\lim_{N\longrightarrow\infty}\prod_{j=1}^{N}\left[\int dx_j\right]\prod_{j=1}^{N+1}\left[4\pi\imath\hbar\varepsilon\beta c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right]^{-1/2}$$
$$\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{(\Delta x_j)^2}{4\varepsilon c^2\left(\alpha+\beta(m\bar{\omega})^2\right)}-\varepsilon\hbar^2c^2\left(\alpha+\beta(m\bar{\omega})^2\right)\left[\frac{\frac{(m\bar{\omega})^2}{\alpha+\beta(m\bar{\omega})^2}-s\hbar m\bar{\omega}}{\hbar^2\left(\alpha+\beta(m\bar{\omega})^2\right)}\frac{1}{\cos^2 x}+\frac{\ell^2-1/4}{\sin^2 x}\right]\right]\right\}. \tag{5.59}$$

According to *Grosch* [49], the transition amplitude with respect to the PTP results is defined by:

$$K=\lim_{N\longrightarrow\infty}\prod_{j=1}^{N}\int dx_j\prod_{j=1}^{N+1}\left(\sqrt{\frac{M}{2\pi\imath\hbar\varepsilon}}\right)\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{M}{2\varepsilon}(\Delta x_j)^2-\varepsilon\frac{\hbar^2}{2M}\left[\frac{(\nu^2-1/4)}{\cos^2 x}+\frac{(\delta^2-1/4)}{\sin^2 x}\right]\right]\right\}$$
$$=\sum_n\Phi_n(x_b)\Phi_n^*(x_a)\exp\left[-\frac{\imath}{\hbar}\left(\frac{\hbar^2}{2M}(\delta+\nu+2n+1)^2\right)\right], \tag{5.60}$$

and

$$\Phi_n^{(\delta,\nu)}(x)=\left[2(\delta+\nu+2n+1)\frac{n!\Gamma(\delta+\nu+n+1)}{\Gamma(\delta+n+1)\Gamma(\nu+n+1)}\right]^{1/2}(\sin x)^{\delta+1/2}(\cos x)^{\nu+1/2}P_n^{(\delta,\nu)}(\cos 2x). \tag{5.61}$$

In contrast, M, δ, and ν, respectively, can be identified as:

$$M:=\frac{1}{2c^2\left(\alpha+\beta(m\bar{\omega})^2\right)},\quad \delta=|\ell|, \tag{5.62}$$

and

$$\nu_s=\pm\left(\frac{1}{2}-s\frac{m\bar{\omega}/\hbar}{\alpha+\beta(m\bar{\omega})^2}\right). \tag{5.63}$$

Following the condition of the generalized uncertainty principle, as outlined in the introduction, we will adopt the following values:

$$\nu_+=-\frac{1}{2}+\frac{m\bar{\omega}}{\hbar\left(\alpha+\beta(m\bar{\omega})^2\right)},\quad \nu_-=\frac{1}{2}+\frac{m\bar{\omega}}{\hbar\left(\alpha+\beta(m\bar{\omega})^2\right)}, \tag{5.64}$$

negative values, on the other hand, are discarded, resulting in the following outcome

$$\nu_{-s} = \nu_s + s, \text{ with } \nu_s = -\frac{s}{2} + \frac{m\bar{\omega}}{\hbar\bar{\theta}}, \tag{5.65}$$

here, $\bar{\theta} = \alpha + \beta\,(m\bar{\omega})^2$. Subsequently, in accordance with the $\gamma^\mu-$ properties outlined below, the expression $\frac{1}{2}\sum_{s=\pm1}\left[1+\imath s\gamma^1\gamma^2\right] = \sum_{s=\pm1}\chi_s\chi_s^+$ holds, where $\chi_s^+ = \frac{1}{2}\begin{pmatrix} 1+s & 1-s \end{pmatrix}$. Equation (5.52) is then transformed into the following form

$$\begin{aligned}\mathcal{G}_\ell\left(x_b, x_a; p_{0b}, p_{0a}\right) = {} & \frac{\imath}{\hbar}\delta\left(p_{0b}-p_{0a}\right)\sum_n\sum_{s=\pm1}\chi_s\chi_s^+\int d\lambda\, e^{\frac{\imath\lambda}{\hbar}\left(p_0^2-\omega_{s,\ell,n}^2\right)} \\ & \times\left[2\left(|\ell|+\nu_s+2n+1\right)\frac{n!\Gamma\left(|\ell|+\nu_s+n+1\right)}{\Gamma\left(|\ell|+n+1\right)\Gamma\left(\nu_s+n+1\right)}\right] \\ & \times\left(\sin x_b\right)^{|\ell|+1/2}\left(\cos x_b\right)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}\left(\cos 2x_b\right) \\ & \times\left(\sin x_a\right)^{|\ell|+1/2}\left(\cos x_a\right)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}\left(\cos 2x_a\right),\end{aligned} \tag{5.66}$$

with

$$\omega_{s,\ell,n}^2 = m^2c^4 + \hbar^2c^2\bar{\theta}\left[2n+1-s+|\ell|-\ell\right]\left[|\ell|+\ell+2\nu_s+2n+1+s\right]. \tag{5.67}$$

To accurately assess the propagator expression, we shall apply the Fourier transformation of (5.66) with regard to the p_{0b} and p_{0a} variables. At this point, integrating over λ results in the equation,

$$\begin{aligned}\mathcal{G}\left(p_{\rho_b}, p_{\rho_a}, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right) = {} & \sum_n\sum_{s=\pm1}\sum_{\ell=-\infty}^{+\infty}\frac{e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}}{2\pi}\chi_s\chi_s^+\int\frac{dp_0}{2\pi\hbar}\frac{e^{-\frac{\imath}{\hbar}p_0\left(t_b-t_a\right)}}{p_0^2-\omega_{s,\ell,n}^2} \\ & \times\left[2\left(|\ell|+\nu_s+2n+1\right)\frac{n!\Gamma\left(|\ell|+\nu_s+n+1\right)}{\Gamma\left(|\ell|+n+1\right)\Gamma\left(\nu_s+n+1\right)}\right] \\ & \times\left(\sin x_b\right)^{|\ell|}\left(\cos x_b\right)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}\left(\cos 2x_b\right) \\ & \times\left(\sin x_a\right)^{|\ell|}\left(\cos x_a\right)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}\left(\cos 2x_a\right).\end{aligned} \tag{5.68}$$

Now, let us apply the theorem of residue at the pole p_0 which allows us to express

$$\int \frac{dp_0}{2\pi\hbar}\frac{e^{-\frac{\imath}{\hbar}p_0(t_b-t_a)}}{p_0^2-\omega_{n,s,\ell}^2}=-\imath\sum_{\varepsilon=\pm1}\frac{e^{-\frac{\imath}{\hbar}\varepsilon E_{n,s,\ell}(t_b-t_a)}}{2E_{n,s,\ell}}\Theta\left(\varepsilon\left(t_b-t_a\right)\right), \tag{5.69}$$

which has the poles

$$E_{n,s,\ell}=\pm\sqrt{m^2c^4+4\hbar^2c^2\bar{\theta}\left[n+\frac{1}{2}-\frac{s}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right]\left[\frac{|\ell|}{2}+\frac{\ell}{2}+\nu_s+n+\frac{1}{2}+\frac{s}{2}\right]}. \tag{5.70}$$

Moreover, it confirms the identity that follows for an arbitrary function

$$\sum_{s=\pm1}\sum_{\varepsilon=\pm1}f(\varepsilon)\Theta\left(\varepsilon\left(t_b-t_a\right)\right)=\sum_{s=\pm1}f(s)\Theta\left(s\left(t_b-t_a\right)\right)+f(-s)\Theta\left(-s\left(t_b-t_a\right)\right), \tag{5.71}$$

we refer to $\Theta(x)$ as the Heaviside function. As a consequence, the Green function is expressed as:

$$\begin{aligned}\mathcal{G}\left(x_b,x_a,p_{\theta_b},p_{\theta_a};p_{0b},p_{0a}\right)=\imath\sum_n\sum_{s=\pm1}\sum_{\ell=-\infty}^{+\infty}\frac{e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}}{2\pi}\chi_s\chi_s^+\left\{\left[\frac{e^{-\frac{\imath}{\hbar}sE_{n,s}(t_b-t_a)}}{2E_{n,s}}\Theta\left(s\left(t_b-t_a\right)\right)\right]\right.\\ \times\left[2\left(|\ell|+\nu_s+2n+1\right)\frac{n!\Gamma\left(|\ell|+\nu_s+n+1\right)}{\Gamma\left(|\ell|+n+1\right)\Gamma\left(\nu_s+n+1\right)}\right]\\ \times\left(\sin x_b\right)^{|\ell|}\left(\cos x_b\right)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}\left(\cos 2x_b\right)\left(\sin x_a\right)^{|\ell|}\left(\cos x_a\right)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}\left(\cos 2x_a\right)\\ +\left[\frac{e^{\imath sE_{n,s}(t_b-t_a)}}{2E_{n,s}}\Theta\left(-s\left(t_b-t_a\right)\right)\right]\left[2\left(|\ell|+\nu_{-s}+2n+1\right)\frac{n!\Gamma\left(|\ell|+\nu_{-s}+n+1\right)}{\Gamma\left(|\ell|+n+1\right)\Gamma\left(\nu_{-s}+n+1\right)}\right]\\ \left.\times\left(\sin x_b\right)^{|\ell|}\left(\cos x_b\right)^{\nu_{-s}+1/2}P_n^{(|\ell|,\nu_{-s})}\left(\cos 2x_b\right)\left(\sin x_a\right)^{|\ell|}\left(\cos x_a\right)^{\nu_{-s}+1/2}P_n^{(|\ell|,\nu_{-s})}\left(\cos 2x_a\right)\right\}.\end{aligned} \tag{5.72}$$

Furthermore, we apply the transformation $(s\to -s)$ to the terms multiplied by $\Theta\left(-s\left(t_b-t_a\right)\right)$ to unify the energy expression between the terms $\Theta\left(s\left(t_b-t_a\right)\right)$ and $\Theta\left(-s\left(t_b-t_a\right)\right)$. This results in

$$n\to n-s,\quad |\ell|\to|\ell|+s,\quad \nu_{-s}=\nu_s+s. \tag{5.73}$$

Thus, the Green function is represented by:

$$\mathcal{G}\left(x_b, x_a, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right) = \imath \sum_n \sum_{s=\pm 1} \sum_{\ell=-\infty}^{+\infty} \frac{e^{\imath \ell\left(p_{\theta_b} - p_{\theta_a}\right)}}{2\pi} \frac{e^{-\frac{\imath}{\hbar} s E_{n,s}(t_b - t_a)}}{2E_{n,s}} \Theta\left(s\left(t_b - t_a\right)\right)$$
$$\times \left\{ \left[2\left(|\ell| + \nu_s + 2n + 1\right) \frac{n!\Gamma\left(|\ell| + \nu_s + n + 1\right)}{\Gamma\left(|\ell| + n + 1\right)\Gamma\left(\nu_s + n + 1\right)}\right] \mathbf{F}_n^{(|\ell|, \nu_s)}\left(x_b\right) \mathbf{F}_n^{(|\ell|, \nu_s)}\left(x_a\right) \chi_s \chi_s^+ \right.$$
$$\left. + \left[2\left(|\ell| + \nu_s + 2n + 1\right) \frac{(n-s)!\Gamma\left(|\ell| + \nu_s + s + n + 1\right)}{\Gamma\left(|\ell| + n + 1\right)\Gamma\left(\nu_s + n + 1\right)}\right] \mathbf{F}_{n-s}^{(|\ell|+s, \nu_s+s)}\left(x_b\right) \mathbf{F}_{n-s}^{(|\ell|+s, \nu_s+s)}\left(x_a\right) \chi_{-s} \chi_{-s}^+ \right\}, \tag{5.74}$$

where

$$\mathbf{F}_n^{(|\ell|, \nu_s)}(x) = (u)^{|\ell|} (\upsilon)^{\nu_s + 1/2} P_n^{(|\ell|, \nu_s)}\left(1 - 2u^2\right). \tag{5.75}$$

$$\mathbf{F}_{n-s}^{(|\ell|+s, \nu_s+s)}(x) = (u)^{|\ell|+s} (\upsilon)^{\nu_s + s + 1/2} P_{n-s}^{(|\ell|, \nu_s+s)}\left(1 - 2u^2\right), \tag{5.76}$$

so that $u = \sin x$, $\upsilon = \cos x$.

In the subsequent section, we leverage the symmetrical properties of the propagator to derive a precise solution for our problem, enabling us to compute the normalized wave functions and their associated energy spectra.

5.4 Spinor Eigenstates and Energy Levels

In order to obtain an accurate assessment of the Green function $S\left(p_b, p_a\right)$ expression, we utilise the operator $\left[\mathcal{O}_+^D\right]_b$ on the function (5.72).Using the relationships that are provided, we can apply the operator $\left[\mathcal{O}_+^D\right]_b$ to $\chi_s \chi_s^+$, represented as follows:

$$\left[\mathcal{O}_+^D\right]_b \chi_s \chi_s^+ = \left[\chi_s \chi_s^+ \left(s \imath \hbar \partial_{tb} + mc^2\right) + \chi_{-s} \chi_s^+ \left\{\left(s\hat{P}_{1b} + \imath \hat{P}_{2b}\right) + m\bar{\omega}\left(s\hat{X}_{2b} - \imath \hat{X}_{1b}\right)\right\}\right]. \tag{5.77}$$

Expressed as follows in polar coordinates:

$$\begin{aligned}&\left[\mathcal{O}_{+}^{D}\right]_{b}\chi_{s}\chi_{s}^{+}=\chi_{s}\chi_{s}^{+}\left(s\imath\hbar\partial_{tb}+mc^{2}\right)\\&+\chi_{-s}\chi_{s}^{+}\left\{\begin{array}{c}\left[s\hbar\sqrt{\frac{\alpha}{\beta}}\sqrt{1-\beta p_{b}^{2}}e^{\imath sp_{\theta_{b}}}\left[-\imath\frac{\partial}{\partial p_{b}}+\frac{s}{p_{\rho_{b}}}\frac{\partial}{\partial p_{\theta_{b}}}\right]+s\frac{(1-\kappa)pe^{\imath sp_{\theta_{b}}}}{\sqrt{1-\beta p_{b}^{2}}}\right]\\+m\bar{\omega}\left[\imath\hbar\sqrt{1-\beta p_{b}^{2}}e^{\imath sp_{\theta_{b}}}\left[-\imath\frac{\partial}{\partial p_{b}}+\frac{s}{p_{b}}\frac{\partial}{\partial p_{\theta_{b}}}\right]-\imath\frac{\kappa\sqrt{\frac{\beta}{\alpha}}pe^{\imath sp_{\theta_{b}}}}{\sqrt{1-\beta p_{b}^{2}}}\right]\end{array}\right\}.\end{aligned}\tag{5.78}$$

Finally, we obtain the spectral decomposition of the Green function $S(p_b, p_a)$, that is shown below:

$$\begin{aligned}S(\mathbf{p}_b,\mathbf{p}_a;t_b,t_a)&=\frac{\imath}{\sqrt{2\pi}}\sum_{s=\pm1}\sum_{n}\frac{e^{-\frac{\imath}{\hbar}sE_{n,s}(t_b-t_a)}}{2E_{n,s}}\Theta\left(s\left(t_b-t_a\right)\right)\\&\times\left\{\left[2\left(|\ell|+\nu_s+2n+1\right)\frac{n!\Gamma\left(|\ell|+\nu_s+n+1\right)}{\Gamma\left(|\ell|+n+1\right)\Gamma\left(\nu_s+n+1\right)}\right]\mathbf{F}_n^{(\ell,\nu_s)}\left(x_a\right)\right.\\&\times\left[\begin{array}{c}\chi_s\chi_s^{+}\left(E_{n,s,\ell}+mc^2\right)+e^{\imath s\theta_b}\chi_{-s}\chi_s^{+}\\\times\left\{\begin{array}{c}\left[s\hbar\sqrt{\frac{\alpha}{\beta}}\sqrt{1-\beta p_{\rho_b}^2}\left[-\imath\frac{\partial}{\partial p_{\rho_b}}+\frac{s}{p_{\rho_b}}\frac{\partial}{\partial p_{\theta_b}}\right]+\frac{\frac{\beta}{\alpha}(m\bar{\omega})^2}{1+\frac{\beta}{\alpha}(m\bar{\omega})^2}\frac{sp_{\rho_b}}{\sqrt{1-\beta p_{\rho_b}^2}}\right]\\+\imath m\bar{\omega}\left[\hbar\sqrt{1-\beta p_{\rho_b}^2}\left[-\imath\frac{\partial}{\partial p_{\rho_b}}+\frac{s}{p_{\rho_b}}\frac{\partial}{\partial p_{\theta_b}}\right]-\frac{\sqrt{\frac{\beta}{\alpha}}}{1+\frac{\beta}{\alpha}(m\bar{\omega})^2}\frac{p_{\rho_b}}{\sqrt{1-\beta p_{\rho_b}^2}}\right]\end{array}\right\}\end{array}\right]e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\mathbf{F}_n^{(\ell,\nu_s)}\left(x_b\right)\\&+\left[2\left(|\ell|+\nu_s+2n+1\right)\frac{(n-s)!\Gamma\left(|\ell|+\nu_s+s+n+1\right)}{\Gamma\left(|\ell|+n+1\right)\Gamma\left(\nu_s+n+1\right)}\right]\mathbf{F}_{n-s}^{(\ell+s,\nu_s+s)}\left(x_a\right)\\&\times\left[\begin{array}{c}\chi_{-s}\chi_{-s}^{+}\left(-E_{n,s,\ell}+mc^2\right)+e^{-\imath sp_{\theta_b}}\chi_s\chi_{-s}^{+}\\\times\left\{\begin{array}{c}\left[-s\hbar\sqrt{\frac{\alpha}{\beta}}\sqrt{1-\beta p_{\rho_b}^2}\left(-\imath\frac{\partial}{\partial p_{\rho_b}}-\frac{s}{p_{\rho_b}}\frac{\partial}{\partial p_{\theta_b}}\right)-s\frac{\frac{\beta}{\alpha}(m\bar{\omega})^2}{1+\frac{\beta}{\alpha}(m\bar{\omega})^2}\frac{p_{\rho_b}}{\sqrt{1-\beta p_{\rho_b}^2}}\right]\\+m\bar{\omega}\left[\hbar\sqrt{1-\beta p_{\rho_b}^2}\left(\frac{\partial}{\partial p_{\rho_b}}-\imath\frac{s}{p_{\rho_b}}\frac{\partial}{\partial p_{\theta_b}}\right)-\imath\frac{\sqrt{\frac{\beta}{\alpha}}}{1+\frac{\beta}{\alpha}(m\bar{\omega})^2}\frac{p_{\rho_b}}{\sqrt{1-\beta p_{\rho_b}^2}}\right]\end{array}\right\}\end{array}\right]\\&\left.\times e^{\imath(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}\mathbf{F}_{n-s}^{(\ell+s,\beta_s+s)}\left(x_b\right)\right\}.\end{aligned}\tag{5.79}$$

As demonstrated below, we used a simple calculation to express the Green function

$$
\begin{aligned}
S(\mathbf{p}_b,\mathbf{p}_a;t_b,t_a) &= \frac{\iota\beta}{\sqrt{2\pi}}\sum_{s=\pm1}\sum_n \frac{e^{-\frac{\iota}{\hbar}sE_{n,s}(t_b-t_a)}}{2E_{n,s}}\Theta\left(s\left(t_b-t_a\right)\right)\\
&\times\left\{\left[2\left(|\ell|+\nu_s+2n+1\right)\frac{n!\Gamma\left(|\ell|+\nu_s+n+1\right)}{\Gamma\left(|\ell|+n+1\right)\Gamma\left(\nu_s+n+1\right)}\right]\right.\\
\times&\left[\begin{array}{c}
\chi_s\chi_s^+\left(E_{n,s,\ell}+mc^2\right)e^{\iota\ell\left(p_{\theta_b}-p_{\theta_a}\right)}+e^{\iota sp_{\theta_b}}e^{\iota\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_{-s}\chi_s^+\frac{\hbar\sqrt{\beta}}{\sqrt{1-\eta_b^2}}\\
\times\left\{\begin{array}{c}
\iota\sqrt{\frac{\alpha}{\beta}}\left\{(1-s)\left[(|\ell|-\nu_s)+(|\ell|+\nu_s)\eta_b\right]P_n^{(\ell,\nu_s)}(\eta_b)+2s\left(1-\eta_b^2\right)\frac{dP_n^{(\ell,\nu_s)}(\eta_b)}{d\eta_b}\right\}\\
+m\bar{\omega}\left\{(1-s)\left[(|\ell|-\nu_s)+(|\ell|+\nu_s)\eta_b\right]P_n^{(\ell,\nu_s)}(\eta_b)-2\left(1-\eta_b^2\right)\frac{dP_n^{(\ell,\nu_s)}(\eta_b)}{d\eta_b}\right\}
\end{array}\right\}
\end{array}\right]\\
&\times\left(\sin x_b\right)^{|\ell|}\left(\cos x_b\right)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}\left(\cos 2x_b\right)\left(\sin x_a\right)^{|\ell|}\left(\cos x_a\right)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}\left(\cos 2x_a\right)\\
&+\left[2\left(|\ell|+\nu_s+2n+1\right)\frac{(n-s)!\Gamma\left(|\ell|+\nu_s+s+n+1\right)}{\Gamma\left(|\ell|+n+1\right)\Gamma\left(\nu_s+n+1\right)}\right]\\
\times&\left[\begin{array}{c}
-\chi_{-s}\chi_{-s}^+\left(E_{n,s,\ell}-mc^2\right)e^{\iota(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}+e^{-\iota sp_{\theta_b}}e^{\iota(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_s\chi_{-s}^+\frac{\hbar\sqrt{\beta}}{\sqrt{\left(1-\eta_b^2\right)}}\\
\times\left\{\begin{array}{c}
\iota\sqrt{\frac{\alpha}{\beta}}\left\{\begin{array}{c}(1+s)\left[(|\ell|+1)-(\nu_s+1)+((|\ell|+1)+(\nu_s+1))\eta_b\right]P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)\\ -2s\left(1-\eta_b^2\right)\frac{d}{d\eta_b}P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)\end{array}\right\}\\
+m\bar{\omega}\left\{\begin{array}{c}(1+s)\left[(|\ell|+1)-(\nu_s+1)+((|\ell|+1)+(\nu_s+1))\eta_b\right]P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)\\ -2\left(1-\eta_b^2\right)\frac{d}{d\eta_b}P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)\end{array}\right\}
\end{array}\right\}
\end{array}\right]\\
&\left.\times\left(\sin x_b\right)^{|\ell|+s}\left(\cos x_b\right)^{\nu_s+s+1/2}P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)\left(\sin x_a\right)^{|\ell|+s}\left(\cos x_a\right)^{\nu_s+s+1/2}P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_a)\right\}.
\end{aligned}
\tag{5.80}
$$

Based on the characteristics of Jacobi's polynomials, as detailed in [100], we find,

$$
\frac{dP_n^{(|\ell|,\nu_s)}(\eta)}{d\eta}=\frac{1}{2}\frac{\Gamma\left(n+|\ell|+\nu_s+2\right)}{\Gamma\left(n+|\ell|+\nu_s+1\right)}P_{n-1}^{(|\ell|+1,\nu_s+1)}(\eta), \tag{5.81}
$$

and

$$
\begin{aligned}
(1-\eta)^{\alpha_1}(1+\eta)^{\beta_1}\frac{d}{d\eta}P_n^{(\alpha_1,\beta_1)}(\eta) &= -2(n+1)(1-\eta)^{\alpha_1-1}(1+\eta)^{\beta_1-1}P_{n+1}^{(\alpha_1-1,\beta_1-1)}(\eta)\\
&+\left(\alpha(1-\eta)^{\alpha_1-1}(1+\eta)^{\beta_1}-\beta_1(1-\eta)^{\alpha_1}(1+\eta)^{\beta_1-1}\right)P_n^{(\alpha_1,\beta_1)}.
\end{aligned}
\tag{5.82}
$$

So, the following expression can be used to reformulate the Green's function:

$$
\begin{aligned}
S\left(\mathbf{p}_b,\mathbf{p}_a;t_b,t_a\right) &= \frac{\imath\beta}{\sqrt{2\pi}}\sum_{s=\pm1}\sum_{n}\frac{e^{-\frac{\imath}{\hbar}sE_{n,s}(t_b-t_a)}}{2E_{n,s}}\Theta\left(s\left(t_b-t_a\right)\right)\\
&\times\Bigg\{\Bigg[\left[\frac{2\left(E_{n,s,\ell}+mc^2\right)n!(|\ell|+\nu_s+2n+1)\Gamma(|\ell|+\nu_s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}\right]\mathbf{F}_n^{(|\ell|,\nu_s)}\left(\eta_b\right)\mathbf{F}_n^{(\ell,\nu_s)}\left(\eta_a\right)e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_s\chi_s^{+}\\
&\quad+\left[\frac{2\hbar\sqrt{\beta}\left(\imath\sqrt{\frac{\alpha}{\beta}}-sm\bar{\omega}\right)\left(n-\frac{s}{2}+\frac{1}{2}\right)!}{n!}\frac{n!2(|\ell|+\nu_s+2n+1)\Gamma(|\ell|+\nu_s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}\frac{\Gamma\left(n+|\ell|+\nu_s+\frac{s}{2}+\frac{1}{2}+1\right)}{\Gamma(n+|\ell|+\nu_s+1)}\right]\\
&\quad\times\mathbf{F}_{n-s}^{(\ell+s,\nu_s+s)}\left(\eta_b\right)\mathbf{F}_n^{(\ell,\nu_s)}\left(\eta_a\right)e^{\imath s p_{\theta_b}}e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_{-s}\chi_s^{+}\Bigg]\\
&+\Bigg[-\frac{2\left(E_{n,s,\ell}-mc^2\right)(n-s)!(|\ell|+\nu_s+2n+1)\Gamma(|\ell|+\nu_s+s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}\mathbf{F}_{n-s}^{(\ell+s,\nu_s+s)}\left(\eta_b\right)\mathbf{F}_{n-s}^{(\ell+s,\nu_s+s)}\left(\eta_a\right)e^{\imath(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_{-s}\chi_{-s}^{+}\\
&\quad+\left[\frac{2s\hbar\sqrt{\beta}\left(\imath\sqrt{\frac{\alpha}{\beta}}+sm\bar{\omega}\right)\left(n-\frac{s}{2}+\frac{1}{2}\right)!}{(n-s)!}\frac{2(n-s)!(|\ell|+\nu_s+2n+1)\Gamma(|\ell|+\nu_s+s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}\frac{\Gamma(n+|\ell|+\nu_{-s}+1)}{\Gamma\left(n+|\ell|+\nu_{-s}-\frac{s}{2}+\frac{1}{2}\right)}\right]\\
&\quad\times\mathbf{F}_n^{(\ell,\nu_s)}\left(\eta_b\right)\mathbf{F}_{n-s}^{(\ell+s,\nu_s+s)}\left(\eta_a\right)e^{-\imath s p_{\theta_b}}e^{\imath(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_s\chi_{-s}^{+}\Bigg]\Bigg\}.
\end{aligned}
\tag{5.83}
$$

It can be expressed in compressed form by taking advantage of the propagator's symmetry properties:

$$
S^c\left(\mathbf{p}_b,\mathbf{p}_a;t_b,t_a\right)=\imath\sum_{n=0}^{\infty}\sum_{\ell}\sum_{s=\pm1}\left[s\Phi_{n,\ell}^{s}\left(p_{\rho_b},p_{\theta_b};t_b\right)\left(\Phi_{n,\ell}^{s}\left(p_{\rho_a},p_{\theta_a};t_a\right)\right)^{\dagger}\sigma_3 e^{-\frac{\imath}{\hbar}sE_{n,s,\ell}(t_b-t_a)}\Theta\left(s\left(t_b-t_a\right)\right)\right].
\tag{5.84}
$$

It is possible to deduce that our system's normalised eigenspinors are as follows :

$$
\begin{aligned}
\Phi_{n,\ell}^{s}\left(p_\rho,p_\theta\right) &= \frac{\sqrt{\beta}}{\sqrt{2E_{n,s,\ell}}}\Bigg\{\sqrt{2\left(E_{n,s,\ell}+mc^2\right)\frac{n!(\ell+\nu_s+2n+1)\Gamma(\ell+\nu_s+n+1)}{\Gamma(\ell+n+1)\Gamma(\nu_s+n+1)}}\mathbf{F}_n^{(\ell,\nu_s)}\left(\eta\right)e^{\imath\ell p_\theta}\chi_s\\
&\quad+\sqrt{-2\left(E_{n,s,\ell}-mc^2\right)\frac{(n-s)!(\ell+\nu_s+2n+1)\Gamma(\ell+\nu_s+s+n+1)}{\Gamma(\ell+n+1)\Gamma(\nu_s+n+1)}}\mathbf{F}_{n-s}^{(\ell+s,\nu_s+s)}\left(\eta\right)e^{-\imath(\ell+s)p_\theta}\chi_{-s}\Bigg\},
\end{aligned}
\tag{5.85}
$$

here, $\eta=\cos 2x=1-2\beta p_\rho^2$ serves to revert us back to the original variables.

It is worth noting that for $\omega=0$, we can substitute $m\bar{\omega}$ with $(e\mathcal{B}/2c)$ and $\bar{\theta}$ with $\left(\alpha+\beta\left(e\mathcal{B}/2c\right)^2\right)$ in Eq. (5.70). However, the spectral energies remain as the pole expressions given in Eq. (5.70). As $\mathcal{B}$ approaches 0, we get the following outcome:

$$
E_{s,\ell,n}^{\pm}=\pm\sqrt{m^2c^4+4\hbar^2c^2\alpha\left[n+\frac{1}{2}-\frac{s}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right]\left[n+\frac{1}{2}+\frac{|\ell|}{2}+\frac{\ell}{2}\right]}.
\tag{5.86}
$$

This result highlights the fact that, within the framework of the aSdS model, the energy levels remains depends on n^2 even when magnetic field $\mathcal{B}$ and $\omega-$oscillation are absent. Given that the values of α and β are typically very small in deformation theory, we can perform a first-order expansion of Eq. (5.70) with respect to α and β. This leads us to the following outcome:

$$E_{s,\ell,n} = E^0_{s,\ell,n} + \bar{\theta}\Delta E^1_{s,\ell,n}, \tag{5.87}$$

the first term in this equation represents the Landau levels of a (2+1)-dimensional DO,

$$E^0_{s,\ell,n} = \pm\sqrt{m^2c^4 + 4\hbar c^2 m\bar{\omega}\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]}, \tag{5.88}$$

and the quantum gravity correction adjustment is the second term,

$$\Delta E^1_{s,\ell,n} = 2\hbar^2c^2 \frac{\left(n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right)\left(n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right)}{\sqrt{m^2c^4 + 4\hbar c^2 m\bar{\omega}\left(n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right)}}. \tag{5.89}$$

1- In limit case as $\alpha \to 0$, the expression of the flat Snyder model can be obtained by reducing Eq. (5.70).

$$E_{s,\ell,n} = E^0_{s,\ell,n} + \beta\Delta E^{\alpha=0}_{s,\ell,n}, \tag{5.90}$$

with

$$\Delta E^{\alpha=0}_{s,\ell,n} = 2\hbar^2c^2\,(m\bar{\omega})^2 \frac{\left(n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right)\left(n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right)}{\sqrt{m^2c^4 + 4\hbar c^2 m\bar{\omega}\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]}}. \tag{5.91}$$

1. In limit case $\beta \to 0$, in a anti de-Sitter background, the spectral energies of the Heisenberg algebra are recovered [21],

$$E_{s,\ell,n} = E^0_{s,\ell,n} + \alpha\Delta E^{\beta=0}_{s,\ell,n}, \tag{5.92}$$

with

$$\Delta E^{\beta=0}_{s,\ell,n} = 2\hbar^2 c^2 \frac{\left(n+\frac{1}{2}-\frac{s}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right)\left(n+\frac{1}{2}+\frac{|\ell|}{2}+\frac{\ell}{2}\right)}{\sqrt{m^2c^4+4\hbar c^2 m\bar{\omega}\left[n+\frac{1}{2}-\frac{s}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right]}}. \tag{5.93}$$

Using natural units ($\hbar = c = 1$), we calculated the conventional energy eigenvalues of the DO and the corrections introduced within in the context of SdS model for a single electron. This calculation is carried out using Eqs. (5.88), (5.89), (5.91) and (5.93), with $\alpha = 10^{-70}$ and $\beta = 10^{-40}$, $m = 0.5MeV$, and $m\omega = 1MeV^2$, while considering the case $s = +1$. Thus, table (5.1) displays the specific energy spectrum values corresponding to various combinations of n and ℓ. It is worth mentioning that the ground energy values in table (5.1) remain unaltered due to the SdS model.

Table 5.1: The energy eigenvalues, both ordinary and corrected, of the 2D-DO in the presence of a homogeneous magnetic field (in MeV) for a single electron at various values of n with $s = +1$

state n	ℓ	$E^0_{s,\ell,n}$	$\Delta E^1_{s,\ell,n} \times (10^{-70} + 0.9 \times 10^{-43})$	$\Delta E^{\alpha=0}_{s,\ell,n} \times (10^{-70})$	$\Delta E^{\beta=0}_{s,\ell,n} \times (10^{-40})$
0	0	0.510999	0	0	0
1	-1	0.709591	8.45557	0.007763	8.45557
	0	0.61832	4.851854	0.004454	4.851854
	1	0.61832	8.086424	0.007424	8.086424
2	-2	0.863667	23.157084	0.02126	23.157084
	-1	0.790392	18.977918	0.017423	18.977918
	0	0.709591	14.092617	0.012938	14.092617
	1	0.709591	19.729664	0.018114	19.729664
	2	0.709591	25.36671	0.023289	25.36671
3	-3	0.994143	42.2474	0.038787	42.24745
	-2	0.931193	37.586202	0.034508	37.586202
	-1	0.863667	32.419918	0.029764	32.419918
	0	0.790392	26.569085	0.024393	26.569085
	1	0.790392	34.160252	0.031362	34.160252
	2	0.790392	41.751419	0.038332	41.751419
	3	0.790392	49.342586	0.045301	49.342586

Hence, it can be noted that the energy level spacing produces a stable case in the subsequent Fig. (5.1)

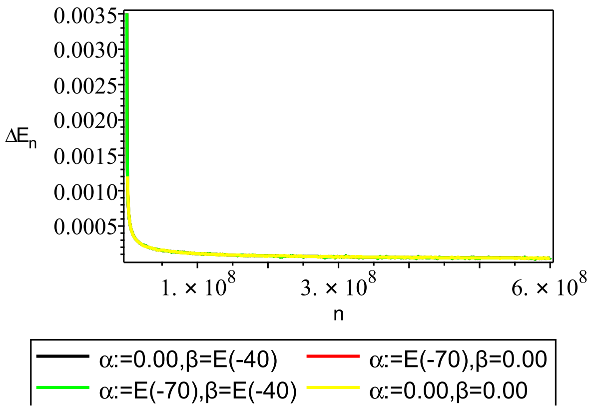


Figure 5.1: The energy spacing between neighboring levels as a function of n and $\ell = 0$ for $s = 1$.

Similar results were obtained for the (2D-DO) in the context of the EUP model algebra [89]. Further, comparable outcomes were obtained for the 1-dimensional DO in anti-de Sitter space [87] and in scenarios involving minimal lengths [38]. That is, the energy level spacing of the 2D-DO is obviously zero in the absence of the SdS algebra. The energy level spacing for the 2D-DO is zero. In conventional space, energy levels tend to converge to continuous states for large values of n, whereas the deformation coefficient preserves the separation of energy levels.

5.5 Dirac Fermions in Graphene Layers

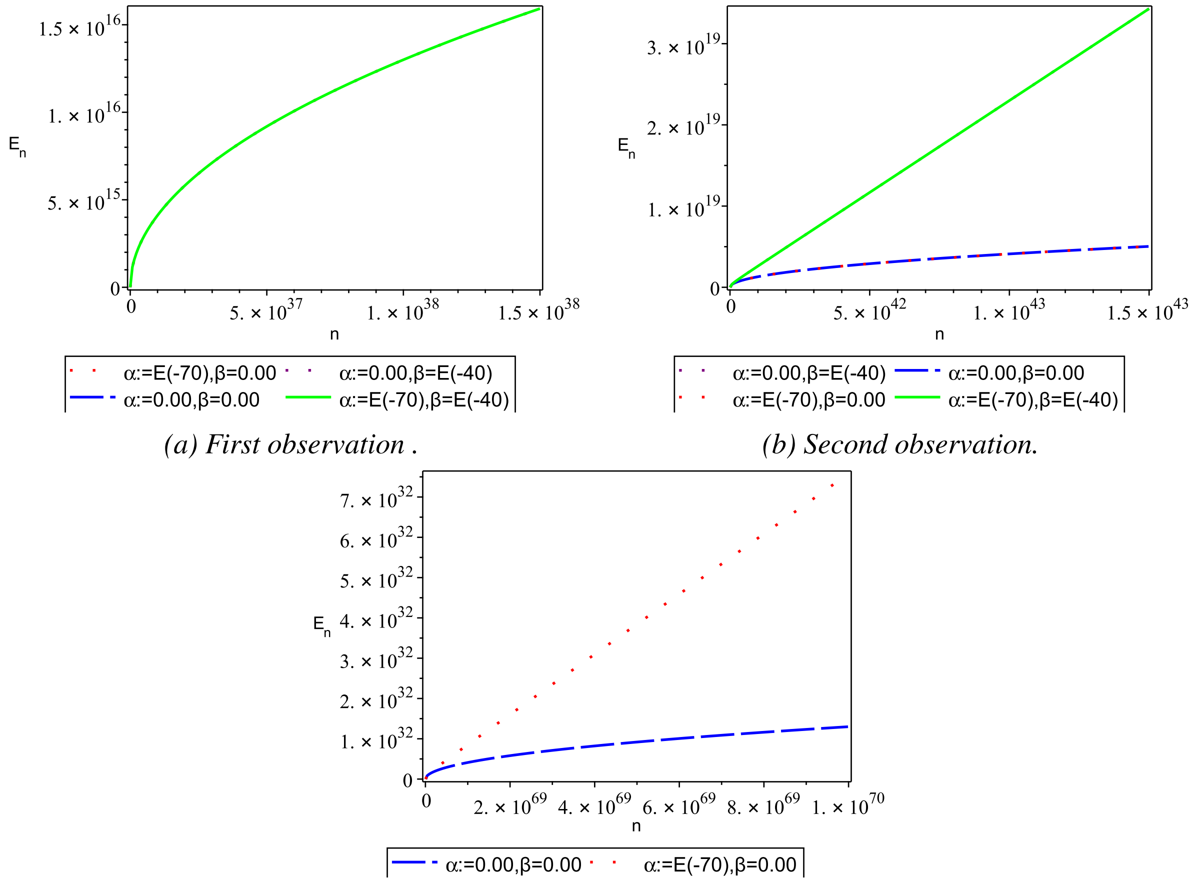
In this scenario, massless Dirac fermions are confined within a Graphene layer configured for the SdS mode and subjected to an external uniform magnetic field. We obtain the energy and wave function expressions by substituting $m\bar{\omega} \to m\omega_c/2$ and $c \to V_F$ into Eqs. (5.70) and (5.85). Consequently, we obtain the resulting energy spectra and corresponding eigenspinors:

$$E_{s,\ell,n} = \pm 2\hbar V_F \sqrt{\alpha + \beta\left(\frac{e\mathcal{B}}{2c}\right)^2} \sqrt{\left(n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right)\left(n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2} + \frac{\frac{e\mathcal{B}}{2\hbar c}}{\alpha + \beta\left(\frac{e\mathcal{B}}{2c}\right)^2}\right)}, \tag{5.94}$$

and

$$\Phi^s_{n,\ell}(p_\rho, p_\theta) = \sqrt{\beta}\left\{\sqrt{\frac{n!(\ell+\nu_s+2n+1)\Gamma(\ell+\nu_s+n+1)}{\Gamma(\ell+n+1)\Gamma(\nu_s+n+1)}}\mathbf{F}_n^{(\ell,\nu_s)}(\eta)\, e^{\imath\ell p_\theta}\chi_s \right.$$
$$\left. + \sqrt{\frac{(n-s)!(\ell+\nu_s+2n+1)\Gamma(\ell+\nu_s+s+n+1)}{\Gamma(\ell+n+1)\Gamma(\nu_s+n+1)}}\mathbf{F}_{n-s}^{(\ell+s,\nu_s+s)}(\eta)\, e^{-\imath(\ell+s)p_\theta}\chi_{-s}\right\}. \tag{5.95}$$

Our findings are consistent alongside those in Graphene within the Snyder model, as documented in Ref. [94]. It is noteworthy to emphasize that the wave function expressions were not calculated by these authors, so an exact solution was not provided. Nevertheless, we can use Eqs. (5.94) to produce plots that show the energy states of a single electron for $\alpha = 10^{-70}$, and $\beta = 10^{-40}$, $V_F = 0.00373$, while considering the case $s = +1$.



(a) First observation . *(b) Second observation.*

(c) Third observation.

Figure 5.2: $E_{n,\alpha,\beta}$−Energy levels versus the quantum number n for different values of the deformation parameters.

As we can see, all cases of the energy level curves in Fig. 5.2a are identical when the quantum number principle n lies between 0 and 1.5×10^{38}. After $n > 10^{41}$, the two cases' curves (($\alpha \neq 0$, $\beta \neq 0$) and ($\alpha = 0$, $\beta \neq 0$)) diverge from those of the cases (($\alpha \neq 0$, $\beta = 0$) and ($\alpha = 0$, $\beta = 0$)) as displayed in Fig. 5.2b. In contrast, the plot for t he state ($\alpha \neq 0$, $\beta = 0$) plot in Fig. 5.2c is separated from the state ($\alpha = 0$, $\beta = 0$) when the quantum number $n > 10^{69}$.

5.6 Non-Relativistic Approximation

The energy levels in the non-relativistic regime for the 2D-DO under a uniform magnetic field and within the anti de-Sitter space system are obtained by taking the limit as $mc^2 \to \infty$. Employing a second-order Taylor expansion of the equation (5.70), this yields the following outcome:

$$E_{s,\ell,n} = mc^2 + \hbar\bar{\omega}\left[2n+1-s+|\ell|-\ell\right] + 2\bar{\theta}\left(\hbar^2/m\right)\left[n+\frac{1}{2}-\frac{s}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right]\left[n+\frac{1}{2}+\frac{|\ell|}{2}+\frac{\ell}{2}\right], \tag{5.96}$$

the first term mc^2 represents the particle's rest energy, where the second and third terms represent the energy of the non-relativistic 2D HO with frequency $\bar{\omega}$ and the correction within the framework of Snyder model. In the non-relativistic limit, the normalized wave functions with spin 1/2 are provided by

$$\Phi^{s}_{n,\ell}\left(p_\rho, p_\theta\right) = \sqrt{\beta}e^{\iota \ell p_\theta}\sqrt{\frac{n!(\ell+\nu_s+2n+1)\Gamma(\ell+\nu_s+n+1)}{\Gamma(\ell+n+1)\Gamma(\nu_s+n+1)}}\left(\sqrt{\beta}p_\rho\right)^{|\ell|}\left(1-\beta p_\rho^2\right)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}\left(1-2\beta p_\rho^2\right)\chi_s, \tag{5.97}$$

where we have used the following limits:

$$\lim_{m\to\infty}\frac{E_{n,s,\ell}+mc^2}{E_{n,s,\ell}} = 2, \quad \lim_{m\to\infty}\frac{E_{n,s,\ell}-mc^2}{E_{n,s,\ell}} = 0. \tag{5.98}$$

As $\alpha \to 0$, Eq. (5.96) simplifies to,

$$E_{s,\ell,n} = mc^2 + \hbar\bar{\omega}\left[2n+1-s+|\ell|-\ell\right] + 2\beta\left(m\bar{\omega}\right)^2\left(\hbar^2/m\right)\left[n+\frac{1}{2}-\frac{s}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right]\left[n+\frac{1}{2}+\frac{|\ell|}{2}+\frac{\ell}{2}\right]. \tag{5.99}$$

As $\beta \to 0$, Eq. (5.96) transforms into,

$$E_{s,\ell,n} = mc^2 + \hbar\bar{\omega}\left[2n+1-s+|\ell|-\ell\right] + 2\alpha\left(\hbar^2/m\right)\left[n+\frac{1}{2}-\frac{s}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right]\left[n+\frac{1}{2}+\frac{|\ell|}{2}+\frac{\ell}{2}\right]. \tag{5.100}$$

The energy spectrum of a spinless nonrelativistic particle ($s=0$) in the presence of a uniform magnetic field can be found using Eq. (5.96) and in the first order of (α,β),

$$E_{n,s=0}^{(NR)}=\hbar\bar{\omega}\left(2n+1+|\ell|-\ell\right)+2\bar{\theta}\left(\hbar^{2}/m\right)\left[n+\frac{1}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right]\left[n+\frac{1}{2}+\frac{|\ell|}{2}+\frac{\ell}{2}\right]. \quad (5.101)$$

When a spinless non-relativistic oscillator with a frequency of ω interacts with a uniform magnetic field in usual quantum mechanics HUP, the first and second terms in Eq. (5.101) denote, respectively, the energy level and the relativistic correction, pertaining to the modification of the Heisenberg algebra. Moreover, in the limit as $\mathcal{B}$ approaches 0, Eq. (5.101) changes to

$$E_{n,s=0}^{(NR)}=\hbar\omega\left[2n+1+|\ell|-\ell\right]+2\left(\alpha+\beta(m\omega)^{2}\right)\left(\hbar^{2}/m\right)\left[n+\frac{1}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right]\left[n+\frac{1}{2}+\frac{|\ell|}{2}+\frac{\ell}{2}\right]. \quad (5.102)$$

This case involves two terms: The first represents the energy level for a spinless non-relativistic oscillator of frequency ω particle within the HUP, while the second corresponds to the initial deformation correction in the non-relativistic scenario.

5.7 Deformationless Scenario

We address the two limits to obtain the ordinary case:

1- Limit $\alpha\to0,\beta\neq0$:

To derive the conventional wave functions for the 2D-DO under a uniform magnetic field, we set $\alpha=0$, resulting in Eq. (5.85) becoming:

$$\Phi_{n,\ell}^{s}\left(p_{\rho},p_{\theta}\right)=\frac{\sqrt{\beta}}{\sqrt{2E_{n,s,\ell}^{\alpha=0}}}\left\{\sqrt{2\left(E_{n,s,\ell}^{\alpha=0}+mc^{2}\right)\frac{n!\left(\ell+v_{s}^{\beta}+2n+1\right)\Gamma\left(\ell+v_{s}^{\beta}+n+1\right)}{\Gamma(\ell+n+1)\Gamma\left(v_{s}^{\beta}+n+1\right)}}\mathbf{F}_{n}^{\left(\ell,v_{s}^{\beta}\right)}(\eta)\,e^{\imath\ell p_{\theta}}\chi_{s}\right.$$
$$\left.+\sqrt{-2\left(E_{n,s,\ell}^{\alpha=0}-mc^{2}\right)\frac{(n-s)!\left(\ell+v_{s}^{\beta}+2n+1\right)\Gamma\left(\ell+v_{s}^{\beta}+s+n+1\right)}{\Gamma(\ell+n+1)\Gamma\left(v_{s}^{\beta}+n+1\right)}}\mathbf{F}_{n-s}^{\left(\ell+s,v_{s}^{\beta}+s\right)}(\eta)\,e^{-\imath(\ell+s)p_{\theta}}\chi_{-s}\right\}, \quad (5.103)$$

in Snyder space, the energy spectrum for the 2D-DO is denoted by $E_{n,\ell,s}^{(\alpha=0)}$

$$E_{s,\ell,n}^{\alpha=0,\pm}=\pm\sqrt{m^2c^4+4\hbar^2c^2\beta\left(m\bar{\omega}\right)^2\left[n+\frac{1}{2}-\frac{s}{2}+\frac{|\ell|}{2}-\frac{\ell}{2}\right]\left[n+\frac{1}{2}+\frac{|\ell|}{2}+\frac{\ell}{2}+\frac{m\bar{\omega}/\hbar}{\beta\left(m\bar{\omega}\right)^2}\right]}. \tag{5.104}$$

While ν_s^β and $\mathbf{F}_n^{\left(\ell,\nu_s^\beta\right)}(\eta)$ are provided by, respectively,

$$\nu_s^\beta=-\frac{s}{2}+\frac{1}{\beta\hbar m\bar{\omega}},\quad \mathbf{F}_n^{\left(\ell,\nu_s^\beta\right)}(\eta)=(u)^{|\ell|}(\upsilon)^{\nu_s^\beta+1/2}P_n^{\left(|\ell|,\nu_s^\beta\right)}\left(1-2u^2\right). \tag{5.105}$$

2- Limit $\beta\to 0,\alpha\to 0$:

To return to the standard case, we set $\beta\to 0$ and $\alpha\to 0$ (i.e., $\bar{\theta}\to 0$) in the momentum space representation of the standard DO and derive the spinorial wave functions. So, we can write ν_s^β in Eq. (5.105) as follows:

$$\nu_s^\beta=\frac{1}{\beta\hbar m\bar{\omega}}. \tag{5.106}$$

Indeed, according to Ref. [100] we get

$$L_n^\mu(x)=\lim_{\nu_s\to\infty}P_n^{(\mu,\nu_s)}(1-\frac{2x}{\nu_s}),\quad \lim_{\bar{\mu}\to+\infty}\frac{\Gamma(\bar{\mu}+\mu)}{\Gamma(\bar{\mu})}e^{-\mu\ln(\bar{\mu})}=1, \tag{5.107}$$

with $x=\frac{p_\rho^2}{\hbar m\bar{\omega}}$ and $\bar{\mu}=\nu_s+n+1$, noting that (to $\mathcal{O}(\beta)$)

$$\lim_{\alpha,\beta\to 0}\left(1-\beta p_\rho^2\right)^{\frac{m\bar{\omega}/\hbar}{2\left(\beta(m\bar{\omega})^2\right)}}=e^{-\frac{p_\rho^2}{2\hbar m\bar{\omega}}}. \tag{5.108}$$

$L_k^\gamma(x)$ is the formula for Laguerre polynomials. Consequently, the spinorial wave functions become in the limit $\bar{\theta}\to 0$ as follows:

$$\begin{aligned}\lim_{\alpha\to 0,\beta\to 0}\Psi_{n,\ell,s}(p_\rho,p_\theta)&=(-1)^n e^{\imath\ell p_\theta}\sqrt{\frac{m\bar{\omega}n!}{2\pi\Gamma(n+\ell+1)}}\sqrt{\frac{E_{n,\ell,s}^{(\bar{\theta}=0)}+m}{E_{n,\ell,s}^{(\bar{\theta}=0)}}}\left(p_\rho/\sqrt{m\bar{\omega}}\right)^\ell e^{-\frac{p_\rho^2}{2m\bar{\omega}\hbar}}L_n^\ell(\frac{p_\rho^2}{m\bar{\omega}\hbar})\chi_s\\&-(-1)^n\imath e^{\imath(\ell+s)p_\theta}\sqrt{\frac{m\bar{\omega}(n-s)!}{2\pi\Gamma(n+\ell+1)}}\sqrt{\frac{E_{n,\ell,s}^{(\bar{\theta}=0)}-m}{E_{n,\ell,s}^{(\bar{\theta}=0)}}}\left(p_\rho/\sqrt{m\bar{\omega}}\right)^{\ell+s}e^{-\frac{p_\rho^2}{2m\bar{\omega}\hbar}}L_{n-s}^{\ell+s}(\frac{p_\rho^2}{m\bar{\omega}\hbar})\chi_{-s},\end{aligned} \tag{5.109}$$

where $E_{n,\ell,s}^{(\bar{\theta}=0)}$ is the usual energy spectrum for the 2D-DO (see Refs. [82, 101, 102])

$$E_{n,\ell,s}^{(\bar{\theta}=0)} = \pm\sqrt{m^2 + 2m\bar{\omega}\left(2n+1-s+|\ell|-\ell\right)}. \tag{5.110}$$

Hence, it is noteworthy that *Chetouani et al.* [5] addressed the identical issue utilizing the path integral approach without deformation parameters, resulting in the same formula for the conventional energy levels of the 2D-DO, as illustrated in Eq. (5.110). Conversely, the spinorial eigenfunction is revealed to be the Fourier transform of the eigenfunction previously derived in [5], as depicted in Eq. (5.109).

5.8 Thermodynamic Functions

We delve into the thermodynamic characteristics of a lone electron interacting with the DO within the altered algebra described by Eq. (5.2). Initially, we need to ascertain the partition function for this particular system in order to calculate these attributes. It is represented by the equation below:

$$Z = \sum_{n=0}^{\infty} e^{-\bar{\beta} E_n}. \tag{5.111}$$

In this case, the $\bar{\beta} = 1/(k_B T)$ appears. The Boltzmann constant is denoted by k_B, and the system temperature is indicated by T. Here, Eq. (5.87) determines the energy levels E_n. In our research, we focus specifically on: the positive energy levels because the summation in Eq. (5.111) diverges for negative energies. We also take into account $s = +1$ and $\ell = 0$,

$$Z = \sum_{n=0}^{\infty} \exp\left[-\bar{\beta}\sqrt{b+an} - \bar{\theta}\bar{\beta}\frac{2\hbar^2 c^2\left(n^2+\frac{n}{2}\right)}{\sqrt{b+an}}\right]. \tag{5.112}$$

The partition function expression that we obtain in the first-order approximation with respect to $\bar{\theta}$ is as follows:

$$Z(T,\alpha) = Z^0(\bar{\beta}) + \bar{\theta}\Delta Z^{(1)}(\bar{\beta}), \tag{5.113}$$

where

$$Z^0(\bar{\beta}) = \sum_{n=0}^{\infty} e^{-\bar{\beta}\sqrt{b+an}}, \quad \Delta Z^{(1)}(\bar{\beta}) = -2\hbar^2\bar{\beta}c^2\sum_{n=0}^{\infty}\frac{\left(n^2+n/2\right)}{\sqrt{b+an}}e^{-\bar{\beta}\sqrt{b+an}}, \tag{5.114}$$

with $a = 4\hbar m\bar{\omega}c^2, b = m^2c^4$. By utilizing the Euler-Maclaurin summation formula, we are able to assess the sums in (5.113), we assert

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)dx - \sum_{k=1} \frac{B_{2k}}{(2k-1)!} f^{(2k-1)}(0), \tag{5.115}$$

where B_{2p} are the Bernoulli numbers, $B_2 = 1/6$, $B_4 = -1/30$, ..., and $f^{(2k-1)}(0)$ is the derivative of order $(2k-1)$ at $x = 0$

$$f(0) = e^{-\bar{\beta}mc^2}, \quad f^{(1)}(0) = -2\bar{\beta}\hbar\bar{\omega}e^{-\bar{\beta}mc^2} - 2\bar{\theta}\bar{\beta}\hbar^2c^2\frac{e^{-\bar{\beta}mc^2}}{2}, \tag{5.116}$$

$$f^{(3)}(0) = -8\bar{\beta}\left(\hbar\bar{\omega}\right)^3 \frac{\left(\bar{\beta}mc^2\right)^2 + 3\bar{\beta}mc^2 + 3}{m^2c^4} e^{-\bar{\beta}mc^2} \tag{5.117}$$

$$+2\bar{\theta}\bar{\beta}\hbar^2c^2 \frac{6\hbar\bar{\omega}\left(2mc^2\left(\bar{\beta}mc^2+1\right) - \hbar\bar{\omega}\left(m^2c^4\bar{\beta}^2 + 3\bar{\beta}mc^2 + 3\right)\right)}{\left(mc^2\right)^3} e^{-\bar{\beta}mc^2}. \tag{5.118}$$

Eq. (5.113) provides the integral over x. It is determined by

$$\int_0^{\infty} f(x)dx = \frac{e^{-\bar{\beta}mc^2}\left(\bar{\beta}mc^2+1\right)}{2\bar{\beta}^2\hbar\bar{\omega}mc^2} - 2\bar{\theta}\hbar^2c^2\bar{\beta}\left[\frac{3e^{-\bar{\beta}mc^2}}{4\left(\hbar\bar{\omega}mc^2\right)^3\bar{\beta}^5} + \frac{3e^{-\bar{\beta}mc^2}}{4\left(\hbar\bar{\omega}\right)^3m^2c^4\bar{\beta}^4}\right.$$
$$\left. + \frac{e^{-\bar{\beta}mc^2}}{2\left(\hbar\bar{\omega}\right)^3mc^2\bar{\beta}^3} + \frac{e^{-\bar{\beta}mc^2}}{8\left(\hbar\bar{\omega}mc^2\right)^2\bar{\beta}^3} + \frac{e^{-\bar{\beta}mc^2}}{8\left(\hbar\bar{\omega}\right)^2mc^2\bar{\beta}^2}\right]. \tag{5.119}$$

As a result, the partition function has the following expression:

$$Z(T,\alpha) = \frac{1}{2}e^{-\bar{\beta}mc^2} + \frac{1+\bar{\beta}mc^2}{2\bar{\beta}^2\hbar\bar{\omega}mc^2}e^{-\bar{\beta}mc^2} - 2\bar{\theta}\hbar^2c^2\bar{\beta}\left[\frac{3}{4\left(\hbar\bar{\omega}mc^2\right)^3\bar{\beta}^5} + \frac{3}{4\left(\hbar\bar{\omega}\right)^3m^2c^4\bar{\beta}^4}\right.$$
$$\left. + \frac{1}{2\left(\hbar\bar{\omega}\right)^3mc^2\bar{\beta}^3} + \frac{1}{8\left(\hbar\bar{\omega}mc^2\right)^2\bar{\beta}^3} + \frac{1}{8\left(\hbar\bar{\omega}\right)^2mc^2\bar{\beta}^2}\right]e^{-\bar{\beta}mc^2} - \sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0). \tag{5.120}$$

To compute the partition function, we need to evaluate the sum presented in equation (5.120). In our specific scenario, this computation can only be accomplished using numerical methods.

Until $k=2$, it is possible to express the sum as:

$$\sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0)=-\frac{\bar{\beta}}{6}\left(2\hbar\bar{\omega}+\bar{\theta}\hbar^2c^2\right)e^{-\bar{\beta}mc^2}-\frac{1}{180}\left[-8\bar{\beta}\left(\hbar\bar{\omega}\right)^3\frac{\left(\bar{\beta}mc^2\right)^2+3\bar{\beta}mc^2+3}{m^2c^4}\right.$$
$$\left.+2\bar{\theta}\bar{\beta}\hbar^2c^2\frac{6\hbar\bar{\omega}\left(2mc^2\left(\bar{\beta}mc^2+1\right)-\hbar\bar{\omega}\left(m^2c^4\bar{\beta}^2+3\bar{\beta}mc^2+3\right)\right)}{\left(mc^2\right)^3}\right]e^{-\bar{\beta}mc^2}. \tag{5.121}$$

It should be noted here that at high temperatures ($\bar{\beta}\ll 1$), all terms in the sum in Eq. (5.121) have positive powers of $\bar{\beta}$, these terms are notably smaller than the remaining term in Eq. (5.120). Therefore, we will eliminate the terms that include $\bar{\beta}^n$ and the ones that do not contain $\bar{\beta}$, resulting in the following:

$$Z(T,\alpha,\beta)\simeq\frac{\left(k_BT\right)^2}{2\hbar\bar{\omega}mc^2}-\bar{\theta}\left[\frac{3\hbar^2c^2\left(k_BT\right)^4}{2\left(\hbar\bar{\omega}mc^2\right)^3}+\frac{\hbar^2c^2\left(k_BT\right)^2}{4\left(\hbar\bar{\omega}\right)^3mc^2}+\frac{\hbar^2c^2\left(k_BT\right)^2}{4\left(\hbar\bar{\omega}\right)^2\left(mc^2\right)^2}\right]. \tag{5.122}$$

According to standard quantum physics, the partition function's first term reflects the traditional 2D-DO. The terms indicate the effects of spatial deformation caused by the existence of the SdS model. Using the partition function, we may generate a variety of thermodynamic functions. For instance, the Helmholtz free energy of the 2D-DO in a uniform magnetic field at high temperatures may be represented as follows:

$$F(T,\bar{\theta})=-\left(k_BT\right)\ln(Z)$$
$$=-2T\ln\left(\frac{k_BT}{\sqrt{2\hbar\bar{\omega}mc^2}}\right)+\bar{\theta}\left[\frac{3\hbar^2c^2\left(k_BT\right)^3}{\left(\hbar\bar{\omega}\right)^2\left(mc^2\right)^2}+\frac{\hbar^2c^2T}{2\left(\hbar\bar{\omega}\right)^2}+\frac{\hbar^2c^2T}{2\left(\hbar\bar{\omega}\right)\left(mc^2\right)}\right]. \tag{5.123}$$

One can define the relationship between the partition function and the mean energy as follows:

$$\Xi(T,\alpha,\beta)=-\frac{\partial\ln(Z)}{\partial\bar{\beta}}=2k_BT+6\bar{\theta}\frac{\hbar^2c^2\left(k_BT\right)^3}{\left(\hbar\bar{\omega}\right)^2\left(mc^2\right)^2}. \tag{5.124}$$

As $\bar{\theta}\to 0$, we recover the standard mean energy associated with the HUP algebra.

In terms of the heat capacity, we find:

$$C(T,\alpha,\beta)=\frac{\partial\Xi}{\partial T}=2k_B+18\bar{\theta}\frac{\hbar^2c^2\left(k_B\right)^3T^2}{\left(\hbar\bar{\omega}\right)^2\left(mc^2\right)^2}. \tag{5.125}$$

As $\bar{\theta} \to 0$, which corresponds to the absence of the SdS algebra, the heat capacity remains constant, specifically $C = 2k_B$. Still, it is clear that the heat capacity shows exhibits temperature-dependent variations due to the modification brought by the standard Heisenberg algebra when SdS algebra is present. Finally, entropy can be expressed as:

$$S(T,\alpha,\beta) = k_B \ln(Z) - k_B \bar{\beta} \frac{\partial \ln(Z)}{\partial \bar{\beta}} = S_0\left(\bar{\beta}\right) + \bar{\theta} \Delta S^1\left(\bar{\beta}\right). \tag{5.126}$$

For the 2D-DO in the HUP algebra, the entropy is denoted by $S_0\left(\bar{\beta}\right)$. The following expression provides it:

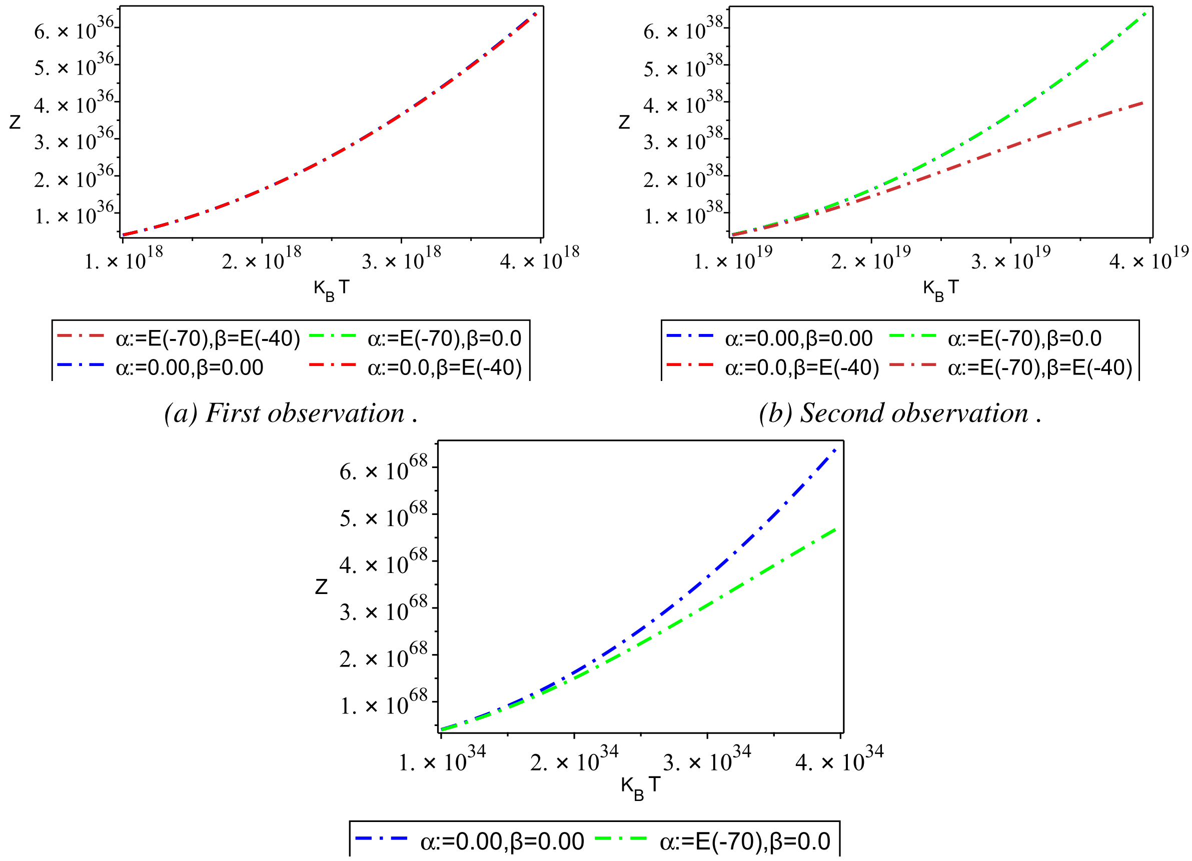
$$S_0\left(\bar{\beta}\right) = 2k_B + 2k_B \ln\left(\frac{k_B T}{\sqrt{2\hbar\bar{\omega} mc^2}}\right). \tag{5.127}$$

Meanwhile, the entropy expression is written at the first-order of (α,β) is indicated by $\Delta S^1\left(\bar{\beta}\right)$, it can be written as follows:

$$\Delta S^1\left(\bar{\beta}\right) = -k_B \left[\frac{9\hbar^2 c^2 (k_B T)^2}{(\hbar\bar{\omega})^2 (mc^2)^2} + \frac{\hbar^2 c^2}{2(\hbar\bar{\omega})^2} + \frac{\hbar^2 c^2}{2(\hbar\bar{\omega})(mc^2)}\right]. \tag{5.128}$$

Let's present, in the following figures, a comparative analysis of our system's thermodynamic properties under different deformation parameters, in the following figures. To make this presentation easier, we use the natural unit system, setting $\hbar$, c, and k_B to 1, which makes all parameters dimensionless. To guarantee accuracy, we have selected particular values for the relevant physical quantities, such as an oscillator parameter in the high-temperature regime of about $2MeV$, an electron mass of $m = 0.5MeV$, and a magnetic field $\mathcal{B}$ of $0.2MeV^2$.

Thus, we display the thermodynamic properties as functions of temperature $(k_B T)$ in figures (5.3), (5.4), (5.5), (5.6) and (5.7). All the figures depict the behavior of these properties for four distinct sets of deformation parameters, namely, ($\alpha = 10^{-70}$, $\beta = 10^{-40}$), ($\alpha = 0.0$, $\beta = 10^{-40}$), ($\alpha = 10^{-70}$, $\beta = 0.0$) and ($\alpha = 0.0$, $\beta = 0.0$).

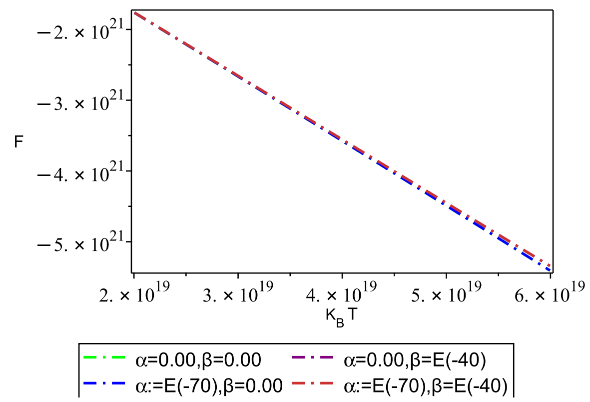


(a) First observation . *(b) Second observation .*

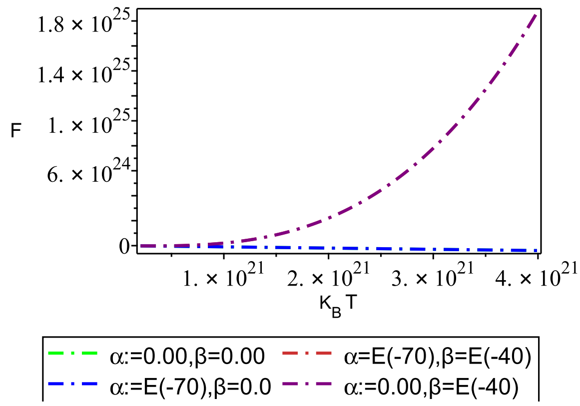
(c) Third observation.

Figure 5.3: The $Z(T, \alpha, \beta)$ Partition function for the 2D-DO in homogeneous magnetic field as a function of temperature $k_B T$ for various values of the deformation parameters.

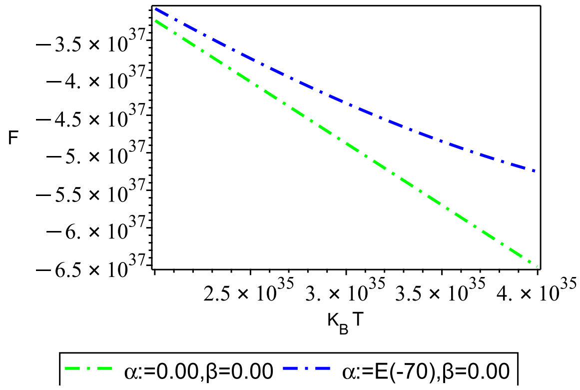
Specifically, Fig. (5.3a) shows that the SdS algebra causes the partition function to surge from $k_B T = 1 \times 10^{19}$ to about $k_B T \sim 2.5 \times 10^{19} MeV$. Following this, the curves (5.3b) corresponding to ($\alpha \neq 0$, $\beta \neq 0$) and ($\alpha = 0.0$, $\beta \neq 0$) decrease to zero at the temperature $k_B T \sim 10^{20} MeV$. Hence, Fig. (5.3c) shows that the curve for ($\alpha \neq 0$, $\beta = 0$) collapses to zero when $k_B T$ exceeds $10^{35} MeV$. The other two curves, however, closely align up to $k_B T \sim 5 \times 10^{38} MeV$.



(a) First observation .



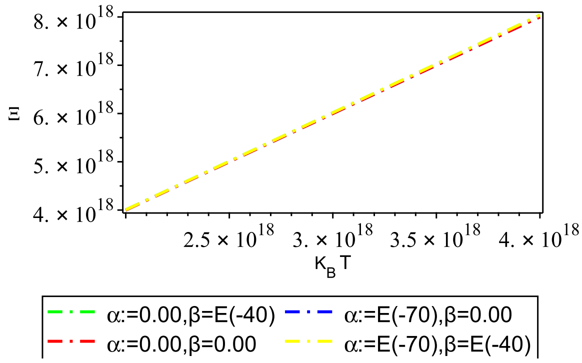
(b) Second observation .



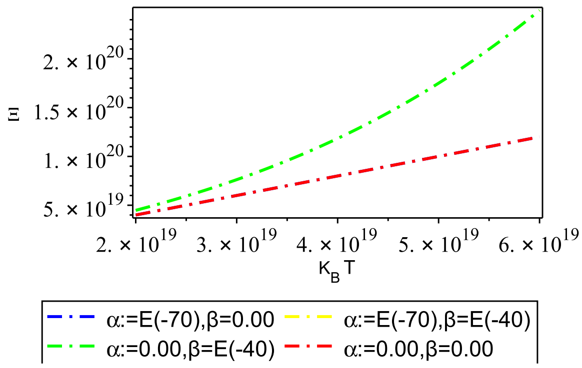
(c) Third observation.

Figure 5.4: The $F(T,\bar{\theta})$ Helmholtz free energy function for the 2D-DO in homogeneous magnetic field as a function of temperature k_BT for various values of the deformation parameters.

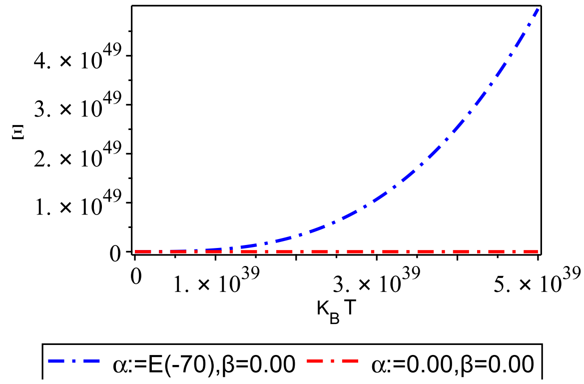
The Helmholtz free energy for 2D-DO in the SdS setting is shown in Figure (5.4) as a function of k_BT. This representation shows that the SdS algebra results in a decrease in the $F-$function, which proceeds from $k_BT = 1\times 10^{19}$ to $k_BT \sim 6\times 10^{19} MeV$ for each of the four deformation parameter cases in Fig. (5.4a). Beyond $k_BT > 10^{39}$, the curves (5.4b) for both $((\alpha \neq 0,\ \beta \neq 0)$ and $(\alpha = 0,\ \beta \neq 0))$ vanish when $\beta \neq 0$. Meanwhile, the case characterized by $((\alpha \neq 0,\ \beta = 0))$ has an effect up to temperature $k_BT > 10^{21} MeV$ in Fig. (5.4c).



(a) First observation .



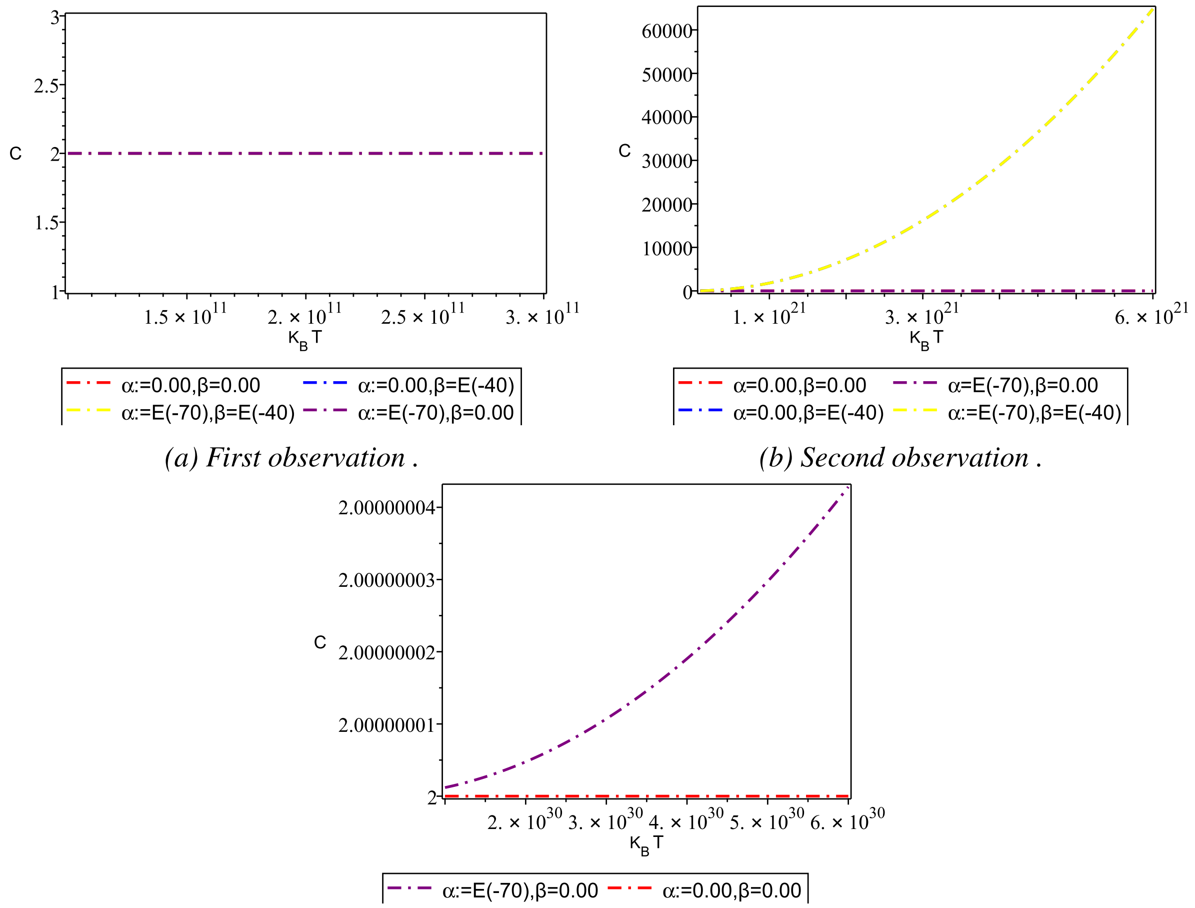
(b) Second observation .



(c) Third observation.

Figure 5.5: The $\Xi(T,\alpha,\beta)$ mean energy function for the 2D-DO in homogeneous magnetic field as a function of temperature k_BT for various values of the deformation parameters.

Moreover, the SdS model shows an increase in mean energy with temperature, as figure (5.5a) illustrates. while the curves for the cases ($\alpha \neq 0$, $\beta \neq 0$) and ($\alpha = 0$, $\beta \neq 0$) decrease to zero upon reaching the temperature $k_BT \sim 10^{21}MeV$, as illustrated in Fig. (5.5b). Nevertheless, in the case where $\alpha \neq 0$, $\beta = 0$, the curve (5.5c) approaches zero at the point where k_BT exceeds $5 \times 10^{39}MeV$.



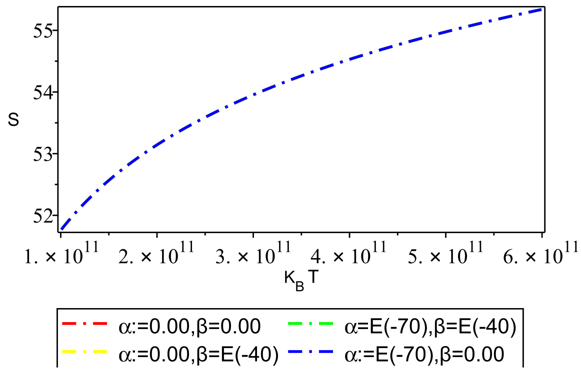
(a) First observation .

(b) Second observation .

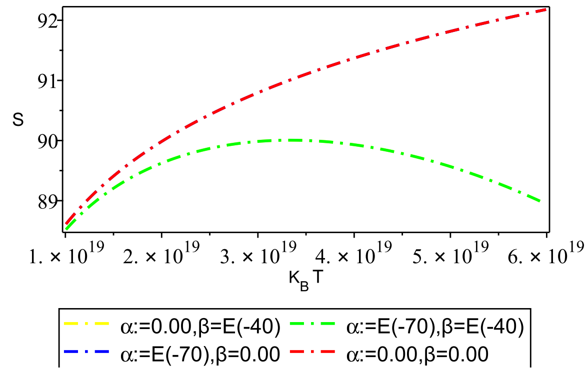
(c) Third observation.

Figure 5.6: The $C(T, \alpha, \beta)$ heat capacity function for the 2D-DO in homogeneous magnetic field as a function of temperature k_BT for various values of the deformation parameters.

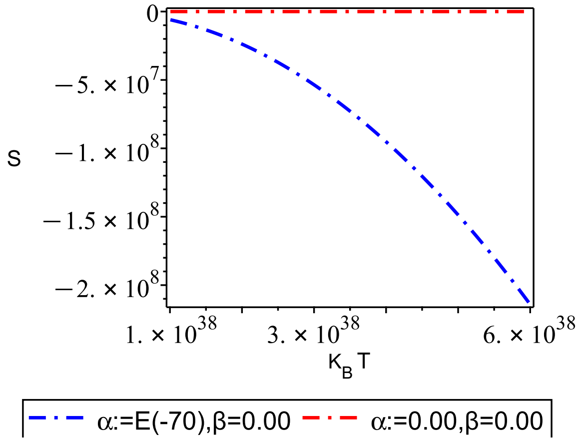
Additionally, for $k_BT < 10^{11}$, the heat capacity in Fig. (5.6a) is constant, $C = 2k_B$. Next, as $k_BT > 10^{19} MeV$, the cases ($\alpha \neq 0$, $\beta \neq 0$) and ($\alpha = 0$, $\beta \neq 0$)) increase with rising temperature, as shown in Fig. (5.6b). The capacity increases for the case ($\alpha \neq 0$, $\beta = 0$) as the temperature rises to $k_BT > 10^{32}$, as shown in Fig. (5.6c).



(a) First observation .



(b) Second observation .



(c) Third observation.

Figure 5.7: The $S(T,\alpha,\beta)$ entropy function for the 2D-DO in homogeneous magnetic field as a function of temperature k_BT for various values of the deformation parameters.

Lastly, we plot the SdS model's effect on the entropy function in three different images in Fig. (5.7a). As illustrated in Fig. (5.7b), at temperature $k_BT > 10^{19}$, the aSdS reduces the entropy values with temperature for the cases (($\alpha \neq 0$, $\beta \neq 0$) and ($\alpha = 0$, $\beta \neq 0$)). In contrast, the entropy function of the case ($\alpha \neq 0$, $\beta = 0$) decreases in Fig. (5.7c) when the temperature $k_BT > 10^{38}$. The above figures show that SdS algebra has a greater impact on thermodynamic functions when the $\alpha-$parameter is present compared to the $\beta-$parameter. Similarly, we can determine the thermodynamic functions and suitable all curves for the aSdS model scenario by simply replacing (α and β) with ($-\alpha$,$-\beta$). Finally, when $\alpha = \beta = 0$ and the magnetic field approaches zero, ($\mathcal{B} \to 0$), our findings are quite accurate. Ref. [103] examines the thermal properties of a three-dimensional DO in the framework of standard Heisenberg uncertainty principle. We can obtain the thermodynamic functions for massless Dirac fermions in a Graphene layer within the Snyder model by considering the limits $m\bar{\omega} \to m\omega_c/2$ and $c \to V_F$ (see Ref. [94]).

5.9 Conclusion

This chapter has examined the behavior of the 2D-DO under a uniform magnetic field using the momentum space representation within the framework of the SdS Heisenberg principle. In the beginning, we presented a new model for the Green function that works with the generalized SdS algebra. Subsequently, we straightforward integrate over even trajectories, leading to the precise calculation of the Green's function in polar coordinates. Transitioning to polar coordinates has made it possible to delineate the energy spectrum and identify the corresponding wave functions, streamlining the analysis of the system's quantum states. The SdS model has shown that energy levels exhibit a dependency on both quantum numbers n and ℓ, regardless of the absence of oscillations and magnetic fields. This effect leads to the emergence of phenomena such as harmonic oscillation, anharmonic vibration, and confinement. Additionally, our study has shown that the energy level spacing does not change as n takes on large values, this is because the deformation parameter $\bar{\theta}$ efficiently preserves the distance between energy levels. The same observation was also made in the reference mentioned. [89]. In analysis we clarify that, under specific conditions when $m\bar{\omega} \to m\omega_c/2$ and $c \to V_F$, the behavior of the DO system in the presence of a uniform magnetic field within the SdS algebra closely resembles to the dynamics of the monolayer Graphene problem within the same algebraic framework. Moreover, we have investigated all the different scenarios and special cases of the DO problem in the presence of a uniform magnetic field, using the framework of the SdS model.

Finally, at high temperatures, we used the Euler-MacLaurin formula to calculate the system's various thermodynamic properties up to the first order of (α, β). These properties include the partition function Z, the Helmholtz free energy F, the mean energy Ξ, the entropy S, and the heat capacity C. Through graphical representations of the SdS terms in these thermodynamic functions against temperature $k_B T$, we have illustrated that the influence of the α-deformation parameter is more significant than that of the β-parameter. It is important to note that, currently, these effects cannot be experimentally detected.

Chapter 6

General Conclusion

This thesis aims to investigate the Feynman approach within the context of the non-relativistic SdS model applied both to one and two Dirac oscillators subjected to constant electric and magnetic fields, respectively. Additionally, we formulate the D-dimensional momentum space path integral transition amplitude for both the harmonic oscillator and the free particle. These issues were previously addressed by [27, 34, 94], where the authors employed differential equations methods.

In the second chapter, we attempted to provide a penetrating insight into the concept of Feynman's formulation and what it depends on. Then, we developed the mathematical approach to this finfing as formulated by Trotter in various dimensions of space. Ultimately, our research culminated in deriving the non-relativistic propagator within polar coordinates, a primary objective of this thesis.

The third chapter, we were able to successfully find a model of Feynman path integral for D-dimensional non-relativistic quantum mechanical systems with two basic deformation parameters, built on the basis of the generalized Snyder model aSdS by means of our use of the coordinate transformation method in momentum space. As applications to our model, we investigated both the free particle and the harmonic oscillator potential, employing D-dimensional spherical coordinates for momentum variables. This approach simplifies the problem to one that is purely radial, facilitating a more straightforward analysis of the system's dynamics. Then, employing the method of coordinate transformation with the δ-point discretization interval, this maps problem to the one of a particle in the symmetric Pöschel–Teller potential. Moreover, it is noted that this choice is consistent with the approach used with a single parameter, suggesting that the discretization is similarly dependent on the δ-point discretization as demonstrated by [35, 37]. Through the application of the radial spectral decomposition of the transition amplitude the energy levels and the momentum space wave functions have been identified. Notably, the energy levels depend on $(2n+\ell)^2$, which is similar to the energy levels of a particle trapped in a potential well.

On the other hand, the fourth chapter covered the SdS model using the path integral formalism to study the Green function of the Dirac oscillator propagator in (1+1)-dimensional energy-momentum space. Through the application of the coordinate transformation method, the exact causal Green function and associated propagator were determined. As a result, we were able to derive the relevant energy values. The Green function and its associated propagator are expressed using Romonovski polynomials in the case of positives (α,β) and Jacobi polynomials in the case of negatives (α,β). Both cases involve sign deformation parameters. Furthermore, we have demonstrated that within the aSdS space framework, dependencies of energy on n^2 persist in cases where oscillation and electric fields are absent. In addition, we have derived limit cases for deformation parameters as well as created the non-relativistic energy levels with and without spin. Lastly, by employing the Euler-MacLaurin formula at high temperatures, we determined all thermodynamic quantities of our system to the first order of $(\alpha,\ \beta)$, including the partition function Z, the Helmholtz free energy F, the mean energy Ξ, the heat capacity C and the entropy S. By plotting the thermodynamic functions (GEUP) terms against the temperaturek_BT, we proved that the $\alpha-$deformation parameter has a greater effect than the $\beta-$parameter. Nevertheless, it is crucial to remember, though, that these effects can still not be detected using the current experimental means.

The fifth chapter covered the study of relativistic particles with spin $1/2$, under the effect of a constant magnetic field on the behavior of 2D-DO in representing the momentum space within the framwork of SdS model. Initially, Initially, for the generalised SdS algebra, we presented a new model for the Green function. This approach allowed us to directly integrate over even trajectories, resulting in an accurate calculation of Green's function in polar coordinates. The use of polar coordinates led to the simplification of the process in identifying the energy spectrum and corresponding wave functions. We shown that the SdS framework induces an energy dependency on both quantum numbers n and ℓ, even in the absence of oscillations and magnetic fields. As a result, phenomena like confinement, anharmonic vibration and also harmonic oscillation arise. Our study has also uncovered that as the quantum number n grows larger, the spacing between energy remains constant, with the deformation parameter $\bar{\theta}$ effectively maintaining the separation between energy levels, a finding consistent with observations reported in reference [89]. We also explain that in some cases, namely at $m\bar{\omega} \rightarrow m\omega_c/2$ and $c \rightarrow V_F$, the behavior of the Dirac oscillator under the action of a uniform magnetic field in the SdS algebra is very similar to the dynamics seen in graphene problem, within the same algebraic framework. In addition, we have applied the SdS model framework to all the distinct scenarios and special cases of the Dirac oscillator problem in the presence of a uniform magnetic field. Ultimately, at increased temperatures, we used the Euler-MacLaurin method to calculate the thermodynamic characteristics of our system up to the first order of (α, β). Properties include partition function Z, Helmholtz free energy F, mean energy Ξ, heat capacity C, and entropy S. Graphically representing the SdS terms inside these thermodynamic functions versus the temperature $k_B T$, we proved that the influence of the $\alpha-$deformation parameter is more important than that of the $\beta-$parameter. It is important to note that these effects cannot be detected experimentally at this time.

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Eur. Phys. J. Plus (2024) 139:204
https://doi.org/10.1140/epjp/s13360-024-04998-3

THE EUROPEAN PHYSICAL JOURNAL PLUS

Review

Thermal properties of a one-dimensional dirac oscillator in a homogeneous electric field with generalized snyder model: path integral treatment

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Received: 19 September 2023 / Accepted: 13 February 2024

Abstract In this paper, we derive the relativistic Green function using a path integral formulation for a (1 + 1)-Dirac oscillator system under a homogeneous electric field within the framework of the Snyder de Sitter model. Consequently, we calculate the propagator function and the corresponding spectral energies. The thermodynamic properties of a single electron are then extracted under high-temperature conditions for four sets of deformation parameters. We examine the impact of the deformation parameters on these properties and infer the limit cases for small parameters.

1 Introduction

After Snyder's work in 1947 [1, 2], where he introduced the Heisenberg generalization principle in quantum field theory to address the issue of divergence, this generalization has become crucial in quantum physics. Examples include dynamics based on variable masses in semiconductor heterostructures formulated by the generalized displacement operator [3], the movement of a 3He impurity atom in a Bose liquid by [4], and the description of low-energy excitations of graphene and Fermi velocity using the generalized Heisenberg algebra, making the commutator of momentum proportional to pseudo-spin [5]. Moreover, it plays a fundamental role in string theory [6], non-commutative geometries [7], black hole physics [8], and quantum gravity [9]. These theories require the existence of a minimum length on the order of the Planck mass ($m_P = \sqrt{\hbar c/G}$), $\sqrt{\beta} \sim 10^8 \text{kg}^{-1}$ (i.e., $\beta \sim (m_P)^{-2}$), under the concept of the generalized uncertainty principle (GUP), or the existence of a minimum momentum on the order of the square root of the cosmological constant, $\sqrt{\alpha} \sim 10^{-24} \text{cm}^{-1}$, as in the context of the extended uncertainty principle (EUP) [10]. Their combined presence gives the Snyder de Sitter (SdS) model, or in other words, triply special relativity (TSR), which relates three invariant scales: the speed of light in vacuum c, the Snyder parameter β, and the cosmological constant Λ [11]. These theories have solved several problems in quantum mechanics using different methods [12–19].

On the other hand, the Feynman path integral formalism is a mathematical framework used to understand the quantum mechanics starting from the notion of classical trajectories. The effective application of this mathematical technique depends on the choice of the discretization interval. In the realm of usual Heisenberg commutation relations, opting for the midpoint as the discretization schema gives an exact result for curved spaces, for all details, refer to the reference [20]. However, this choice swiftly becomes problematic when the Heisenberg principle is generalized, as exemplified by cases involving a nonzero minimal length [21, 22], and a nonzero minimal momentum [23, 24]. Furthermore, in [25], the authors have formulated the path integral approach in D-dimensional quantum mechanics, considering the coexistence of both minimal position and momentum uncertainty. In this paper, we extend this study to the relativistic case, focusing on the system of one-dimensional Dirac oscillator within a homogeneous electric field. Subsequently, we confirm the existence of a difference in the midpoint discretization interval in the Snyder (anti-)de Sitter ((a)SdS) model. Following [11], the generalization of the commutation relation between the position and momentum operators in one dimension is expressed as,

$$\left[\hat{X}, \hat{P}\right] = \imath\hbar\left(1 + \beta\hat{P}^2 + \alpha\hat{X}^2 + \sqrt{\alpha\beta}\left(\hat{X}\hat{P} + \hat{P}\hat{X}\right)\right). \tag{1}$$

If we consider ($\langle X\rangle = \langle P\rangle = 0$), Eq. (1) results in the following generalized uncertainty relation:

$$(\Delta X)(\Delta P) \geq \frac{\hbar}{2}\frac{\left(1 + \alpha(\Delta X)^2 + \beta(\Delta P)^2\right)}{1 + \hbar\sqrt{\alpha\beta}}. \tag{2}$$

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Published online: 02 March 2024

Consequently, modifying this deformed algebra gives minimal uncertainty both in position and momentum,

$$(\Delta X)_{\min} = \frac{\hbar\sqrt{\beta}}{1+2\hbar\sqrt{\alpha\beta}}, \quad (\Delta P)_{\min} = \frac{\hbar\sqrt{\alpha}}{1+2\hbar\sqrt{\alpha\beta}}. \tag{3}$$

In momentum representation, we can express the position $\hat{X}$ and momentum $\hat{P}$ operators obeying the algebra (1) in the following way

$$\hat{X} = \hat{\mathcal{X}} + \sqrt{\tfrac{\beta}{\alpha}}\kappa\hat{\mathcal{P}}, \qquad \hat{P} = -\sqrt{\tfrac{\alpha}{\beta}}\hat{\mathcal{X}} + (1-\kappa)\hat{\mathcal{P}}, \tag{4}$$

here, (α, β) are small and positive parameters. Meanwhile, κ is a free parameter that can be selected in each case to ensure the Hermiticity of the Hamiltonian in the dynamics. The operators $(\hat{X}, \hat{P})$ satisfy the following commutation relation [26]:

$$\left[\hat{\mathcal{X}}, \hat{\mathcal{P}}\right] = \imath\hbar\left(1+\beta\hat{\mathcal{P}}^2\right). \tag{5}$$

On another side, it is possible to write these position and momentum coordinate operators as to satisfy the Snyder-Heisenberg commutation relation (5) by using auxiliary operators $\hat{x}$ and $\hat{p}$, obeying standard commutation relation (i.e., $\left[\hat{x}, \hat{p}\right] = \imath\hbar$), defined by the following relations

$$\hat{\mathcal{X}} = \sqrt{1-\beta\hat{p}^2}\hat{x}, \quad \hat{\mathcal{P}} = \frac{\hat{p}}{\sqrt{1-\beta\hat{p}^2}}. \tag{6}$$

If $\alpha, \beta > 0$, the range of allowed values of p is bound by $p^2 < 1/\beta$ in the (SdS)-model and otherwise all real values of p are allowed in (aSdS)-model case. For the (SdS)-model, the operators of $\hat{X}$ and $\hat{P}$ are symmetric only in subspace $L^2(R, dp/\sqrt{1-\beta p^2})$, where the scalar product is defined as follows

$$\langle\psi \mid \phi\rangle = \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \frac{dp}{\sqrt{1-\beta p^2}}\psi^*(p)\phi(p), \tag{7}$$

here the wave function satisfies the periodic boundary conditions, $\psi\left(-1/\sqrt{\beta}\right) = \psi\left(1/\sqrt{\beta}\right)$, and this leads to the following closure relation:

$$\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \frac{dp}{\sqrt{1-\beta p^2}}|p\rangle\langle p| = 1. \tag{8}$$

We note that for (aSdS)-model, the (α, β) paramaters are negative, and we change the limits of integration in Eq. (8) in all the space. Additionally, the associate formal eigenvectors are those of the $\hat{X}$-position operator as given by [11],

$$\langle p|x\rangle_{\alpha,\beta} = \frac{1}{\sqrt{2\pi\hbar}}\left(1-\beta p^2\right)^{-\frac{\gamma}{2}} \exp\left(-\frac{\imath x}{\hbar}\frac{\arcsin\sqrt{\beta}p}{\sqrt{\beta}}\right), \ \gamma = \imath\kappa/\hbar\sqrt{\alpha\beta}. \tag{9}$$

Then, we apply the closure relation for the maximally localized states to Eq. (9) and use the properties of the delta function $\delta f(x) = \sum_i \delta(x-x_i)/f'(x_i)$, where x_i are the roots of $f(x)$ [25]. Finally, we obtain:

$$\left\langle p_j|p_{j-1}\right\rangle_{\alpha,\beta} = \int \frac{dx_j}{2\pi\hbar}\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta p_j^2}\exp\left(-\frac{\imath x_j}{\hbar}\left(p_j-p_{j-1}\right)\right). \tag{10}$$

These delta functions (10) are valid for the subspace $L^2(R, dp/\sqrt{1-\beta p^2})$ when α and β take both signs (i.e., SdS or aSdS). In the case of α and β being equal to zero, we recover the usual projection relation $\left\langle p|p'\right\rangle_{(\alpha,\beta)\to 0} = \delta\left(p-p'\right)$.

Otherwise, for the time p_0-component, there is no deformation applied to it, and thus we express it as follows:

$$\left\langle p_0|p_0'\right\rangle = \delta\left(p_0-p_0'\right) = \int \frac{dt}{2\pi\hbar}e^{-\frac{\imath}{\hbar}t\left(p_0-p_0'\right)}. \tag{11}$$

As a consequence, the elements matrix of the operators $\hat{\mathcal{X}}$ and $\hat{\mathcal{X}}^2$ are, respectively, given by

$$\left\langle p_j\left|\hat{\mathcal{X}}\right|p_{j-1}\right\rangle_{\alpha,\beta} = \left\langle p_j|p_{j-1}\right\rangle_{\alpha,\beta}\left[(\gamma-1)\frac{\imath\hbar\beta p}{\sqrt{1-\beta p_j^2}} + \sqrt{1-\beta p^2}x_j\right], \tag{12}$$

$$\begin{aligned}\left\langle p_j\left|\hat{\mathcal{X}}^2\right|p_{j-1}\right\rangle_{\alpha,\beta} \\ = \left\langle p_j|p_{j-1}\right\rangle_{\alpha,\beta}\left[-\gamma(\gamma-1)\frac{\hbar^2\beta^2 p_j^2}{1-\beta p_j^2} - \hbar^2\beta(\gamma-1) + \left(1-\beta p_j^2\right)x_j^2 + 2\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)p_j x_j\right].\end{aligned} \tag{13}$$

In Sect. 2, we present the formulation of the path integral for Dirac oscillator particles subjected to a uniform electric field within the Snyder–de Sitter model in one-dimensional momentum space, avoiding the use of Grassmann variables as proven in [21, 27]. This approach involves performing the path integration over the elements of the Green function matrix for 1D-Dirac oscillators particles subjected to the uniform electric field with Snyder–de Sitter model, a methodology previously employed in [28, 29]. In Sect. 3, we employ the coordinate transformation method to obtain the local kinetic part, resulting in the Rosen-Morse type I and II propagator [30]. The exact propagator and the corresponding energy eigenvalues are deduced. In Sect. 4, we evaluate the thermodynamic properties and provide a physical discussion of the corresponding plotted graphs for this system.

2 Path integral formulation in (anti) Snyder de Sitter

In the absence of electric field interaction, the Green function for relativistic Dirac oscillator particle in one-dimensional space is defined as the inverse of the Dirac operator

$$\left(\gamma^0 \imath\hbar\partial_t - \hat{H}\right)\hat{S} = -I, \tag{14}$$

where $\hat{H}$ is the Hamiltonian operator of the Dirac oscillator equation is given by

$$\hat{H} = c\gamma^1\left(\hat{P} - \imath m\omega\gamma^0\hat{X}\right) + mc^2. \tag{15}$$

where mc^2 is the rest mass and ω is the classical frequency of the oscillator. For detailed consideration, we choose that there is no deformation for the time component ($\hat{P}_0 = \imath\hbar\partial_0 = \imath\hbar\partial/\partial ct,\ \hat{X}_0 = \hat{x}^0 \equiv ct$) and that the momenta $\hat{P}$ and position $\hat{X}$ operators verify Eq. (6). Based on this, we can generalize the equation of the Green's function (14) for the $(1+1)$-dimensional Dirac oscillator under the influence of a uniform electric field $\mathcal{E}$ as follows:

$$\left[\gamma^0\left(\imath\hbar\partial_t + e\mathcal{E}\hat{X}\right) - c\gamma^1\left(\hat{P} - \imath m\omega\gamma^0\hat{X}\right) - mc^2\right]\hat{S} = -I. \tag{16}$$

In the $(1+1)$ dimension, the γ^μ-Dirac matrices are represented by the Pauli matrices following the choice

$$\gamma^0 = \sigma_3,\ \gamma^1 = \imath\sigma_2,\ \gamma^2 = -\imath\sigma_1. \tag{17}$$

Then, the solution of Eq. (14) is written as

$$\hat{S} = \left[\mathcal{O}_-^D\right]^{-1} = \left[\mathcal{O}_+^D\right]\left[\mathcal{O}_-^D\mathcal{O}_+^D\right]^{-1}, \tag{18}$$

with the operator $\mathcal{O}_\pm^D$ defined by

$$\mathcal{O}_\pm^D = \left[\gamma^0\left(\imath\hbar\partial_t + e\mathcal{E}\hat{X}\right) - c\gamma^1\left(\hat{P} - \imath m\omega\gamma^0\hat{X}\right) \pm mc^2\right]. \tag{19}$$

According to the Schwinger proper-time method [31] and noting that $\hat{S} = [\mathcal{O}_+^D][\mathcal{O}_-^D\mathcal{O}_+^D]^{-1}$, it is convenient to write the $\hat{S}$ Green's function as follows

$$\hat{S} = \left[\mathcal{O}_+^D\right]\left[\mathcal{O}_-^D\mathcal{O}_+^D\right]^{-1} = (\imath/\hbar)\left[\mathcal{O}_+^D\right]\int_0^\infty d\lambda \exp\left(\frac{\imath}{\hbar}\lambda\left[\mathcal{O}_-^D\mathcal{O}_+^D\right]\right), \tag{20}$$

here λ represents an invariant parameter and is an even variable, and the operator $[\mathcal{O}_-^D\mathcal{O}_+^D]$ playing the role of an Hamiltonian is expressed by the following expression

$$\left[\mathcal{O}_-^D\mathcal{O}_+^D\right] = \left\{\left(\hat{P}_0 + e\mathcal{E}\hat{X}\right)^2 - c^2\hat{P}^2 - c^2m^2\omega^2\hat{X}^2 - m^2c^4 - \left(ce\mathcal{E}\gamma^0\gamma^1 - \imath m\omega c^2\gamma^0\right)\left[\hat{X}, \hat{P}\right]\right\}. \tag{21}$$

By following the SdS algebra given by Eq. (1), we have:

$$\begin{aligned}\left[\mathcal{O}_-^D\mathcal{O}_+^D\right] = \Big\{&\hat{P}_0^2 - m^2c^4 + 2e\mathcal{E}\hat{P}_0\hat{X} - \varpi^2\hat{X}^2 - c^2\hat{P}^2 \\ &-\imath\hbar\left(ec\mathcal{E}\gamma^0\gamma^1 - \imath c^2m\omega\gamma^0\right)\left(1 + \beta\hat{P}^2 + \alpha\hat{X}^2 + \sqrt{\alpha\beta}\left(\hat{X}\hat{P} + \hat{P}\hat{X}\right)\right)\Big\},\end{aligned} \tag{22}$$

with $\varpi^2 = \left(c^2m^2\omega^2 - e^2\mathcal{E}^2\right)$.

Furthermore, we have to write this Hamiltonian in terms of position and momentum operators that satisfy the flat Snyder model based on the modified commutation relationship, and defined by Eq. (5) [11]. By substituting operators $\left(\hat{\mathcal{X}}, \hat{\mathcal{P}}\right)$ into an expression $[\mathcal{O}_-^D\mathcal{O}_+^D]$, Eq. (22) becomes,

$$\begin{aligned}\left[\mathcal{O}_-^D\mathcal{O}_+^D\right] = \Big\{&\hat{P}_0^2 - m^2c^4 + 2e\mathcal{E}\hat{P}_0\hat{\mathcal{X}} + 2e\mathcal{E}\hat{P}_0\kappa\sqrt{\tfrac{\beta}{\alpha}}\hat{\mathcal{P}} + c^2\left(-\frac{\varpi^2}{c^2}\tfrac{\beta}{\alpha}\kappa^2 - (1-\kappa)^2\right)\hat{\mathcal{P}}^2 \\ &+ \left(c^2(1-\kappa)\sqrt{\tfrac{\alpha}{\beta}} - \kappa\varpi^2\sqrt{\tfrac{\beta}{\alpha}}\right)\left(\hat{\mathcal{X}}\hat{\mathcal{P}} + \hat{\mathcal{P}}\hat{\mathcal{X}}\right)\end{aligned}$$

$$-\left(\varpi^2+c^2\tfrac{\alpha}{\beta}\right)\hat{\mathcal{X}}^2-\imath\hbar\left(ec\mathcal{E}\gamma^0\gamma^1-\imath c^2 m\omega\gamma^0\right)\left(1+\beta\hat{\mathcal{P}}^2\right)\Big\}. \tag{23}$$

To make the term $\left(\hat{\mathcal{X}}\hat{\mathcal{P}}+\hat{\mathcal{P}}\hat{\mathcal{X}}\right)$ mentioned above end to zero, we put the condition on κ,

$$\kappa=\left(1-\frac{\beta}{\alpha}\varpi^2/c^2\right)^{-1}. \tag{24}$$

As a result, the Hamiltonian operator becomes as

$$\left[\mathcal{O}_-^D\mathcal{O}_+^D\right]=\Bigg\{\hat{P}_0^2-m^2c^4+2e\mathcal{E}\hat{P}_0\hat{\mathcal{X}}+\frac{2e\mathcal{E}\hat{P}_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\hat{\mathcal{P}}-\left(\varpi^2+c^2\tfrac{\alpha}{\beta}\right)\hat{\mathcal{X}}^2$$
$$-\frac{\frac{\beta}{\alpha}\varpi^2}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\hat{\mathcal{P}}^2-\imath\hbar\left(ec\mathcal{E}\gamma^0\gamma^1-\imath c^2m\omega\gamma^0\right)\hat{F}\left(\hat{\mathcal{P}}\right)\Bigg\}, \tag{25}$$

with

$$\hat{F}\left(\hat{\mathcal{P}}\right)=1+\beta\hat{\mathcal{P}}^2. \tag{26}$$

The corresponding element matrix of $\left[\mathcal{O}_-^D\mathcal{O}_+^D\right]$ in momentum representation is

$$\mathcal{G}(p_b,p_a,p_{0b},p_{0a})=(\imath/\hbar)\int_0^\infty d\lambda\left\langle p_b,p_{0b}\right|\exp\left(-\frac{\imath}{\hbar}\lambda\left[-\mathcal{O}_-^D\mathcal{O}_+^D\right]\right)\left|p_a,p_{0a}\right\rangle. \tag{27}$$

Before proceeding further, it is appropriate to avoid the calculation of their Feynman path integral expression for matrices by introducing the following exponential matrix and simplifying its form as follows

$$e^{\lambda\left(ec\mathcal{E}\gamma^0\gamma^1-\imath c^2m\omega\gamma^0\right)\hat{F}\left(\hat{\mathcal{P}}\right)}=\frac{1}{2}\sum_{s=\pm1}\left\{\mathbb{I}-s\begin{pmatrix}\frac{cm\omega}{\varpi} & \imath\frac{e\mathcal{E}}{c\varpi}\\ \imath\frac{e\mathcal{E}}{c\varpi} & -\frac{cm\omega}{\varpi}\end{pmatrix}\right\}e^{\imath s\lambda c\varpi\hat{F}\left(\hat{\mathcal{P}}\right)}. \tag{28}$$

Then, we perform the following equality [28]:

$$\cosh(\delta)=\frac{cm\omega}{\varpi},\quad \sinh(\delta)=\frac{e\mathcal{E}}{c\varpi}, \tag{29}$$

after performing some calculations, we obtain:

$$e^{-\lambda\left(ec\mathcal{E}\gamma^0\gamma^1-\imath c^2m\omega\gamma^0\right)\hat{F}\left(\hat{\mathcal{P}}\right)}=\sum_{s=\pm1}\exp\left(-\frac{\delta}{2}\sigma_2\right)\mathbb{X}_s\mathbb{X}_s^+\exp\left(\frac{\delta}{2}\sigma_2\right)e^{\imath s\lambda c\varpi\hat{F}\left(\hat{\mathcal{P}}\right)}. \tag{30}$$

Here, $\mathbb{X}_s=\frac{1}{2}\left((1+s)\ (1-s)\right)^T$ and $\mathbb{X}_s^+$ is the transpose of the vector $\mathbb{X}_s$, denoted as $\mathbb{X}_s^+=\mathbb{X}_s^T$.

Hence, the expression (27) can be formulated as follows:

$$\mathcal{G}(p_b,p_a,p_{0b},p_{0a})=(\imath/\hbar)\sum_{s=\pm1}\exp\left(-\frac{\delta}{2}\sigma_2\right)\mathbb{X}_s\mathbb{X}_s^+\exp\left(\frac{\delta}{2}\sigma_2\right)\int_0^\infty d\lambda\langle p_b,p_{0b}|\exp\left(-\frac{\imath}{\hbar}\hat{\mathcal{H}}^{(s)}\right)|p_a,p_{0a}\rangle, \tag{31}$$

with

$$\hat{\mathcal{H}}^{(s)}=-\lambda\Bigg\{\hat{P}_0^2-m^2c^4+2e\mathcal{E}\hat{P}_0\hat{\mathcal{X}}+\frac{2e\mathcal{E}\hat{P}_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\hat{\mathcal{P}}-\left(\varpi^2+c^2\tfrac{\alpha}{\beta}\right)\hat{\mathcal{X}}^2$$
$$-\frac{\varpi^2\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\hat{\mathcal{P}}^2+s\hbar c\varpi\hat{F}\left(\hat{\mathcal{P}}\right)\Bigg\}. \tag{32}$$

After this stage, we will construct the Green function using path integral framework, and thus we decompose the exponential $\exp(-\imath\lambda\hat{\mathcal{H}}^{(s)})$ into $(N+1)$ exponential $\exp(-\imath\varepsilon\hat{\mathcal{H}}^{(s)})$, with $\varepsilon=\tau_j-\tau_{j-1}=1/(N+1)$. Then, we insert N times resolution identity (8) between each pair of infinitesimal operator $\exp(-\imath\varepsilon\hat{\mathcal{H}}^{(s)})$. Indeed, we will obtain

$$\mathcal{G}(p_b,p_a,p_{0b},p_{0a})=(\imath/\hbar)\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^+e^{\frac{\delta}{2}\sigma_2}\right]\lim_{N\longrightarrow\infty}\int_0^\infty d\lambda\prod_{j=1}^N\left[\int_{-\infty}^{+\infty}dp_{0j}\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_j}{\sqrt{1-\beta p_j^2}}\right]$$
$$\times\prod_{j=1}^{N+1}\langle p_j,p_{0j}|\exp(-\frac{\imath\varepsilon}{\hbar}\hat{\mathcal{H}}^{(s)})|p_{j-1},p_{0j-1}\rangle_{\alpha,\beta}. \tag{33}$$

To go further, it is convenient to develop the exponential up to the first order of ε. Thus, we write

$$\lim_{N\longrightarrow\infty,\varepsilon\to0}\left\langle p_j,p_{0j}\right|e^{-\frac{\imath\varepsilon}{\hbar}\hat{\mathcal{H}}^{(s)}}\left|p_{j-1},p_{0j-1}\right\rangle_{\alpha,\beta}$$
$$=\lim_{N\longrightarrow\infty,\varepsilon\to0}\Big[\left\langle p_j,p_{0j}\mid p_{j-1},p_{0j-1}\right\rangle_{\alpha,\beta}-\frac{\imath\varepsilon}{\hbar}\left\langle p_j,p_{0j}\right|\hat{\mathcal{H}}^{(s)}\left|p_{j-1},p_{0j-1}\right\rangle_{\alpha,\beta}\Big]. \quad (34)$$

Then, to eliminate the Hamiltonian operator which represened in the (SdS)-framework, we substitute all the operators ($\hat{\mathcal{X}}$, $\hat{\mathcal{P}}$, $\hat{\mathcal{X}}^2$, $\hat{\mathcal{P}}^2$) on the projection relation $\left\langle p_j\mid p_{j-1}\right\rangle_{\alpha,\beta}$ given in Eq. (10). Consequently, the expression $\mathcal{G}(p_b,\ p_a,\ p_{0b},\ p_{0a})$ is transformed into the following path integral in phase-space

$$\mathcal{G}(p_b,p_a,p_{0b},p_{0a})=(\imath/\hbar)\lim_{\substack{N\to\infty\\ 1\varepsilon\to0}}\sum_{s=\pm1}\Big[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^+e^{\frac{\delta}{2}\sigma_2}\Big]\int_0^\infty d\lambda\prod_{j=1}^{N}\Bigg[\int_{-\infty}^{+\infty}dp_{0j}\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_j}{\sqrt{1-\beta p_j^2}}\Bigg]$$
$$\times\prod_{j=1}^{N+1}\Bigg[\left(\tfrac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta p_j^2}\int\frac{dx_j}{2\pi\hbar}\frac{dt_j}{2\pi\hbar}e^{\frac{\imath}{\hbar}t_j\Delta p_{0j}}\Bigg]\exp\Bigg\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\Big[-x_j\Delta p_j+\lambda\varepsilon\left(p_{0j}^2-m^2c^4\right)$$
$$+\lambda\varepsilon\gamma(\gamma-1)\left(\varpi^2+c^2\tfrac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2p_j^2}{1-\beta p_j^2}+\lambda\varepsilon\hbar^2\beta(\gamma-1)\left(\varpi^2+c^2\tfrac{\alpha}{\beta}\right)$$
$$-\lambda\varepsilon\left(\varpi^2+c^2\tfrac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)x_j^2-\lambda\varepsilon2\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)\left(\varpi^2+c^2\tfrac{\alpha}{\beta}\right)p_jx_j$$
$$+\lambda\varepsilon2e\mathcal{E}p_{0j}\sqrt{1-\beta p^2}x_j+\lambda\varepsilon\frac{2e\mathcal{E}p_{0j}\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}}+2\lambda\varepsilon e\mathcal{E}p_{0j}(\gamma-1)\frac{\imath\hbar\beta p_j}{\sqrt{1-\beta p_j^2}}$$
$$-\lambda\varepsilon\frac{\varpi^2\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2}+\varepsilon s\hbar\lambda c\varpi\left(1+\frac{\beta p_j^2}{1-\beta p_j^2}\right)\Big]\Bigg\}. \quad (35)$$

As usually done and after performing the Gaussian integration over t_j and x_j, we find the Lagrangian expression for this system

$$\mathcal{G}(p_b,p_a,p_{0b},p_{0a})=(\imath/\hbar)\lim_{\substack{N\to\infty\\ 1\varepsilon\to0}}\sum_{s=\pm1}\Big[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^+e^{\frac{\delta}{2}\sigma_2}\Big]\int_0^\infty d\lambda\prod_{j=1}^{N}\Bigg[\int_{-\infty}^{+\infty}dp_{0j}\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_j}{\sqrt{1-\beta p_j^2}}\Bigg]$$
$$\times\prod_{j=1}^{N+1}\Bigg[\delta\left(p_{0j}-p_{0j-1}\right)\left(\tfrac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta p_j^2}\sqrt{\frac{1}{4\imath\pi\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)}}\Bigg]$$
$$\exp\Bigg\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\Bigg[\frac{\Delta p_j^2}{4\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)}+\frac{\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)}{\left(1-\beta p_j^2\right)}p_j\Delta p_j+\lambda\varepsilon\left(p_{0j}^2-m^2c^4\right)$$
$$-\frac{\lambda\varepsilon\hbar^2\beta^2\left(\gamma-\frac{3}{2}\right)^2\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}{\left(1-\beta p_j^2\right)}p_j^2+\lambda\varepsilon\gamma(\gamma-1)\left(\varpi^2+c^2\tfrac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2p_j^2}{1-\beta p_j^2}$$
$$+\lambda\varepsilon\hbar^2\beta(\gamma-1)\left(\varpi^2+c^2\tfrac{\alpha}{\beta}\right)-2\imath\hbar\beta\lambda\varepsilon e\mathcal{E}p_{0j}\left(\gamma-\frac{3}{2}\right)\frac{p_j}{\sqrt{1-\beta p_j^2}}$$
$$-\frac{e\mathcal{E}p_{0j}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\Delta p_j}{\sqrt{1-\beta p_j^2}}+\frac{\lambda\varepsilon e^2\mathcal{E}^2p_{0j}^2}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}+\lambda\varepsilon\frac{2e\mathcal{E}p_{0j}\sqrt{\frac{\beta}{\alpha}}}{\left(1-\frac{\beta}{\alpha}\frac{e^2\mathcal{E}^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}}$$
$$+\lambda\varepsilon2e\mathcal{E}p_{0j}(\gamma-1)\frac{\imath\hbar\beta p_j}{\sqrt{1-\beta p_j^2}}-\lambda\varepsilon\frac{\varpi^2\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2}+\varepsilon s\hbar\lambda c\varpi\left(1+\frac{\beta p_j^2}{1-\beta p_j^2}\right)\Bigg]\Bigg\}. \quad (36)$$

In order to simplify the above expression, we perform the following equality to the first order of ε,

$$\frac{e\mathcal{E}p_{0j}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\Delta p_j}{\sqrt{1-\beta p_j^2}}=\frac{e\mathcal{E}p_{0j}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\Delta\arcsin\left(\sqrt{\beta}p_j\right)}{\sqrt{\beta}}$$

$$+\frac{e\mathcal{E}p_{0j}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\left(\Delta p_j\right)^2}{2}\frac{\beta p_j}{\left(1-\beta p_j^2\right)^{3/2}}, \tag{37}$$

where

$$\left(\Delta p_j\right)^2 \sim 2\imath\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right). \tag{38}$$

By substituting Eq. (38) in Eq. (37) and then into Eq. (36), we can write

$$\begin{aligned}
\mathcal{G}(p_b,p_a,p_{0b},p_{0a}) &= (\imath/\hbar)\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\delta(p_{0b}-p_{0a})\int_0^\infty d\lambda \\
&\times e^{\frac{\imath}{\hbar}\lambda\left[p_{0b}^2-m^2c^4+\lambda\varepsilon\hbar^2\beta(\gamma-1)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\right]}\times e^{-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_{0b}}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\arcsin(\sqrt{\beta}p_b)-\arcsin(\sqrt{\beta}p_a)}{\sqrt{\beta}}} \\
&\times\lim_{\substack{N\to\infty\\ 1\varepsilon\to0}}\prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_j}{\sqrt{1-\beta p_j^2}}\right]\prod_{j=1}^{N+1}\left[\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}\sqrt{\frac{1}{4\pi\imath\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}}\right] \\
&\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\Delta p_j^2}{4\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)}+\frac{\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)}{\left(1-\beta p_j^2\right)}p_j\Delta p_j+\right.\right. \\
&-\lambda\varepsilon\left(\gamma-\frac{3}{2}\right)^2\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2p_j^2}{\left(1-\beta p_j^2\right)}+\lambda\varepsilon\gamma(\gamma-1)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2p_j^2}{1-\beta p_j^2} \\
&-2\imath\hbar\beta\lambda\varepsilon\left(\gamma-\frac{3}{2}\right)\frac{e\mathcal{E}p_0p_j}{\sqrt{1-\beta p_j^2}}-\imath\hbar\lambda\varepsilon\frac{e\mathcal{E}p_0\beta p_j}{\sqrt{1-\beta p_j^2}}+2\imath\hbar\lambda\varepsilon(\gamma-1)\frac{e\mathcal{E}p_0\beta p_j}{\sqrt{1-\beta p_j^2}} \\
&+\lambda\varepsilon\frac{2e\mathcal{E}p_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}}-\lambda\varepsilon\frac{\beta}{\alpha}\frac{\left(c^2m^2\omega^2-e^2\mathcal{E}^2\right)}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2} \\
&\left.\left.+\lambda\varepsilon\frac{e^2\mathcal{E}^2p_0^2}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}+\lambda\varepsilon s\hbar c\varpi\left(1+\frac{\beta p_j^2}{1-\beta p_j^2}\right)\right]\right\},
\end{aligned} \tag{39}$$

and with some simplifications we will obtain

$$\begin{aligned}
\mathcal{G}(p_b,p_a,p_{0b},p_{0a}) &= (\imath/\hbar)\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\delta(p_{0b}-p_{0a})\int_0^\infty d\lambda\prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_j}{\sqrt{1-\beta p_j^2}}\right] \\
&\times e^{-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\arcsin(\sqrt{\beta}p_b)-\arcsin(\sqrt{\beta}p_a)}{\sqrt{\beta}}}\, e^{\frac{\imath}{\hbar}\lambda\left(p_{0j}^2+\frac{e^2\mathcal{E}^2p_0^2}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}-m^2c^4+\hbar^2\beta(\gamma-1)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\right)} \\
&\times\lim_{\substack{N\to\infty\\ 1\varepsilon\to0}}\prod_{j=1}^{N+1}\left[\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}}\sqrt{\frac{1}{4\pi\imath\hbar\varepsilon\lambda\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}}\right] \\
&\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\Delta p_j^2}{4\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)}+\frac{\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)}{1-\beta p_j^2}p_j\Delta p_j\right.\right. \\
&-\lambda\varepsilon\hbar^2\beta^2\left(\gamma^2-3\gamma+\frac{9}{4}\right)^2\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{p_j^2}{1-\beta p_j^2} \\
&+\lambda\varepsilon\left(\gamma^2-\gamma\right)\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{\hbar^2\beta^2p_j^2}{1-\beta p_j^2}+\lambda\varepsilon\frac{2e\mathcal{E}p_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}} \\
&\left.\left.-\lambda\varepsilon\frac{\left(c^2m^2\omega^2-e^2\mathcal{E}^2\right)\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2}+\varepsilon s\hbar\lambda c\varpi\left(1+\frac{\beta p_j^2}{1-\beta p_j^2}\right)\right]\right\}.
\end{aligned} \tag{40}$$

Additionally, all terms related to (γ) in Eq. (40) will be invalidated by the term $\left(\left(1-\beta p_{j-1}^2\right)/\left(1-\beta p_j^2\right)\right)^{\frac{\gamma}{2}}$ [25],

$$\left(\frac{1-\beta p_{j-1}^2}{1-\beta p_j^2}\right)^{\frac{\gamma}{2}} = \exp\left[\left(-\frac{\gamma}{2}\Delta p_j \frac{-2\beta p_j}{\left(1-\beta p_j^2\right)} + \frac{\gamma}{2}\imath\hbar\lambda\varepsilon\left(\varpi^2 + c^2\frac{\alpha}{\beta}\right)\left[-2\beta - \frac{4\beta^2 p_j^2}{1-\beta p_j^2}\right]\right)\right]. \tag{41}$$

After substituting the above result (41) into Eq. (40), we obtain

$$\mathcal{G}(p_b, p_a, p_{0b}, p_{0a}) = (\imath/\hbar)\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\delta(p_{0b}-p_{0a})\int_0^\infty d\lambda$$
$$\times\, e^{-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\varpi^2+c^2\frac{\alpha}{\beta}}\frac{\arcsin(\sqrt{\beta}p_b)-\arcsin(\sqrt{\beta}p_a)}{\sqrt{\beta}}}\, e^{\frac{\imath}{\hbar}\lambda\left(\frac{\left(c^2m^2\omega^2+c^2\frac{\alpha}{\beta}\right)p_0^2}{\varpi^2+c^2\frac{\alpha}{\beta}} - m^2c^4 - \hbar^2\beta\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\right)} \times \mathbb{K}(p_b, p_a, \lambda), \tag{42}$$

where the kernel propagator $\mathbb{K}(p_b,\ p_a,\ \lambda)$ is defined by the following path integral

$$\mathbb{K}(p_b, p_a, \lambda) = \lim_{N\to\infty}\prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_j}{\sqrt{1-\beta p_j^2}}\right]\prod_{j=1}^{N+1}\left[\sqrt{\frac{1}{4\pi\imath\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}}\right]$$
$$\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\left(\Delta p_j\right)^2}{4\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left(1-\beta p_j^2\right)} - \frac{3}{2}\frac{\imath\hbar\beta}{\left(1-\beta p_j^2\right)}p_j\Delta p_j\right.\right.$$
$$-\lambda\varepsilon\hbar^2\beta^2\frac{9}{4}\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\frac{p_j^2}{\left(1-\beta p_j^2\right)} + \lambda\varepsilon\frac{2e\mathcal{E}p_0\sqrt{\frac{\beta}{\alpha}}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j}{\sqrt{1-\beta p_j^2}}$$
$$\left.\left.-\lambda\varepsilon\frac{\varpi^2\frac{\beta}{\alpha}}{\left(1+\frac{\beta}{\alpha}\frac{\varpi^2}{c^2}\right)}\frac{p_j^2}{1-\beta p_j^2} + \lambda\varepsilon s\hbar c\varpi\left(1+\frac{\beta p_j^2}{1-\beta p_j^2}\right)\right]\right\}. \tag{43}$$

As usual for systems based on the principle of generalization, three quantum corrections must be applied: the measure term ($dp_j/\sqrt{1-\beta p_j^2}$), the action term ($\left(\Delta p_j\right)^2/2\varepsilon\left(1-\beta p_j^2\right)$), and the factor term ($p_j\Delta p_j/\left(1-\beta p_j^2\right)$) to achieve the conventional form of the Feynman path integral. Following [21–24], we can calculate the quantum corrections from these three terms through two-step process. The first one is to write this Kernel at the η-point discretization interval ($p_j^{(\eta)} = \eta p_j + \left(1-\eta p_{j-1}\right)$ because the midpoint interval is not suitable in the presence of the SdS model [21–24]. In the second step, to obtain the usual kinetic term ($\left(\Delta\tilde{q}_j\right)^2/2\varepsilon$), we must utilize the momentum coordinate transformation method defined by ($\sqrt{\beta}p = \sin\sqrt{\beta}\tilde{q}$). The formal treatment of the choice of the η-point discretization interval in the presence of the deformation coefficient has been formally addressed in previous references [21, 24], and after straightforward calculations, we obtain the total quantum correction,

$$C_T = \imath\hbar\lambda\varepsilon\beta\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)\left[1+\frac{9}{4}\tan^2\left(\sqrt{\beta}\tilde{q}\right)\right]. \tag{44}$$

and this corresponds to fixing $\eta = \frac{1}{2}(1\pm 1/\sqrt{2})$.

Substituting Eq. (44) in Eq. (43) and then into Eq. (42) we get:

$$\mathcal{G}(p_b, p_a, p_{0b}, p_{0a}) = (\imath/\hbar)\sum_{s=\pm1}\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^{+}e^{\frac{\delta}{2}\sigma_2}\right]\delta(p_{0b}-p_{0a})\int_0^\infty d\lambda$$
$$\times\, e^{-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\arcsin(\sqrt{\beta}p_b)-\arcsin(\sqrt{\beta}p_a)}{\sqrt{\beta}}}\, e^{\frac{\imath}{\hbar}\lambda\left(\frac{\left(\beta m^2\omega^2+\alpha\right)p_0^2}{\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)} - \frac{\varpi^2}{\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)} - m^2c^4\right)} \times \bar{K}(\tilde{q}_b, \tilde{q}_a, \lambda), \tag{45}$$

where the kernel propagator $\bar{K}(\tilde{q}_b,\ \tilde{q}_a,\ \lambda)$ becomes exactly the path integral representation of the transition amplitude relative to the Rosen–Morse of kind (I) potential [32]:

$$\bar{K}(\tilde{q}_b, \tilde{q}_a, \lambda) = \lim_{N\to\infty}\prod_{j=1}^{N}\left[\int d\tilde{q}_j\right]\prod_{j=1}^{N+1}\left[\sqrt{\frac{1}{4\pi\imath\hbar\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}}\right]$$
$$\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\left(\Delta\tilde{q}_j\right)^2}{4\lambda\varepsilon\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)} + \lambda\varepsilon\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)}\tan\left(\sqrt{\beta}\tilde{q}_j\right)\right.\right.$$

$$-\lambda\varepsilon\left(\frac{\varpi^2}{\alpha+\beta\frac{\varpi^2}{c^2}}-s\hbar c\varpi\right)\frac{1}{\cos^2(\sqrt{\beta}\tilde{q}_j)}\Bigg]\Bigg]\Bigg\}. \tag{46}$$

Immediately, let us study the case when $(\alpha,\ \beta)$ are negative, Eq. (46) transforms as,

$$\bar{K}(q_b,q_a,\lambda)=\sqrt{\beta}\int D[q(t)]\exp\left\{\frac{\imath}{\hbar}\int_0^\lambda\left[\frac{M}{2}\dot{q}^2(t)-A\tanh(q(t))+\frac{B}{\cosh^2(q(t))}\right]dt\right\}, \tag{47}$$

with $q(t),\ M, A$ and B defined by

$$q(t)=\sqrt{\beta}\tilde{q}(t),\ \ M=\frac{1}{2\lambda c^2\bar{\theta}}\ \text{ and } A=\lambda\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\bar{\theta}}, B=\lambda\hbar c\varpi\left(\frac{\varpi}{\hbar c\bar{\theta}}+s\right), \tag{48}$$

here $\bar{\theta}=\left(\alpha+\beta\varpi^2/c^2\right)$. Following Ref. [30, 33], we can write,

$$\bar{K}(q_b,q_a,\lambda)=\sqrt{\beta}\sum_{n=0}^{\infty}\Psi_n(q_b)\Psi_n^*(q_a)\exp\left(-\frac{\imath}{\hbar}\lambda\bar{E}_n\right), \tag{49}$$

where

$$\Psi(q)=\left[\left(1-\frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\,\Gamma(s-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]^{1/2}2^{n+(1-\bar{s})/2}$$
$$\times(1-\tanh q)^{\frac{\bar{s}}{2}-k_2-n}(1+\tanh q)^{k_2-\frac{1}{2}}P_n^{(\bar{s}-2k_2-2n,2k_2-1)}(\tanh q). \tag{50}$$

$P_n^{(\eta_1,\eta_2)}(z)$ denotes the Jacobi polynomial, and

$$\bar{E}_n=-\left[\frac{\hbar^2(\bar{s}-2n-1)^2}{8M}+\frac{2MA^2}{\hbar^2(\bar{s}-2n-1)^2}\right], \tag{51}$$

with

$$\bar{s}=\sqrt{1+8MB/\hbar^2},\ \ k_1=\frac{1}{2}(1+\bar{s}),\ \ k_2=\frac{1}{2}\left(1+\frac{1}{2}(\bar{s}-2n-1)-\frac{2MA}{\hbar(\bar{s}-2n-1)}\right). \tag{52}$$

Compensating for each of the values $(M,\ A,\ B,\ \bar{s})$ in Eq. (51), we obtain

$$\bar{E}_n=-\lambda\hbar^2c^2\bar{\theta}\left[\left(\nu_s-n-\frac{1}{2}\right)^2+\frac{\alpha e^2\mathcal{E}^2p_0^2}{\hbar^4c^4\bar{\theta}^4\left(\nu_s-n-\frac{1}{2}\right)^2}\right], \tag{53}$$

with

$$\nu_s=\frac{\sqrt{m^2\omega^2-\frac{e^2\mathcal{E}^2}{c^2}}}{\hbar\bar{\theta}}+\frac{s}{2}. \tag{54}$$

As consequence, the values $(\bar{s},\ k_2,\ 2k_2-1$ and $(\bar{s}-2k_2-2n))$ transform into the following formulas:

$$\bar{s}=2\nu_s,\ \ k_2=\frac{1}{2}\left(1+\frac{1}{2}(2\nu_s-2n-1)-\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\hbar c^2\bar{\theta}^2}\frac{1}{(2\nu_s-2n-1)}\right), \tag{55}$$

and

$$2k_2-1=\frac{1}{2}(2\nu_s-2n-1)-\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\hbar c^2\bar{\theta}^2}\frac{1}{(2\nu_s-2n-1)}=\eta_{n,s}^-, \tag{56}$$

$$\bar{s}-2k_2-2n=\frac{1}{2}(2\nu_s-2n-1)+\frac{2e\mathcal{E}p_0\sqrt{\alpha}}{\hbar c^2\bar{\theta}^2}\frac{1}{(2\nu_s-2n-1)}=\eta_{n,s}^+. \tag{57}$$

At this stage, we can write

$$\mathcal{G}(p_b,p_a,p_{0b},p_{0a})=(\imath/\hbar)\delta(p_{0b}-p_{0a})\sqrt{\beta}\sum_{s=\pm1}\sum_n\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^+e^{\frac{\delta}{2}\sigma_2}\right]\int_0^\infty d\lambda$$
$$\times\exp\Bigg[\frac{\imath\lambda}{\hbar}\Bigg(\left(\frac{\beta m^2\omega^2+\alpha}{\alpha+\beta\frac{\varpi^2}{c^2}}p_0^2-\frac{\varpi^2}{\alpha+\beta\frac{\varpi^2}{c^2}}-m^2c^4\right)$$
$$+\hbar^2c^2\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)\left[\left(\nu_s-n-\frac{1}{2}\right)^2+\frac{\alpha e^2\mathcal{E}^2p_0^2}{\hbar^4c^4\left(\alpha+\beta\frac{\varpi^2}{c^2}\right)^4\left(\nu_s-n-\frac{1}{2}\right)^2}\right]\Bigg)\Bigg]$$

$$\times \frac{(\bar{s}-2k_2-2n)n!\,\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)} 2^{2n+(1-2\nu_s)} e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}(\sqrt{\beta}p_b)-\sinh^{-1}(\sqrt{\beta}p_a)}{\hbar\sqrt{\beta}}\right]}$$
$$\times (1-\tanh q_b)^{\frac{\eta^+_{n,s}}{2}}(1+\tanh q_b)^{\frac{\eta^-_{n,s}}{2}} P_n^{(\eta^+_{n,s},\eta^-_{n,s})}(\tanh q_b)$$
$$\times (1-\tanh q_a)^{\frac{\eta^+_{n,s}}{2}}(1+\tanh q_a)^{\frac{\eta^-_{n,s}}{2}} P_n^{(\eta^+_{n,s},\eta^-_{n,s})}(\tanh q_a). \quad (58)$$

In the following section, we will calculate the propagator within the framework of anti-Snyder de Sitter model. Subsequently, we will extract the energy levels with their mapping in special cases of deformation parameters.

3 Extracting energy levels for (1D-DO) in a homogeneous electric field

In order to evaluate exactly the propagator expression, it is convenient to integrate on the proper time λ and to write the Fourier transformation of Eq. (58), and after simple calculation we find

$$\mathcal{G}(p_b,p_a,t_b,t_a)=\sqrt{\beta}\sum_{s=\pm1}\sum_n\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^+e^{\frac{\delta}{2}\sigma_2}\right]\int\frac{dp_0}{2\pi\hbar}\frac{e^{-\frac{\imath}{\hbar}p_0(t_b-t_a)}}{p_0^2-\left(E_{n,s}^{(\alpha,\beta)}\right)^2}$$
$$\times \frac{(\bar{s}-2k_2-2n)n!\,\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)} 2^{2n+(1-\bar{s})} e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\hbar\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}(\sqrt{\beta}p_b)-\sinh^{-1}(\sqrt{\beta}p_a)}{\sqrt{\beta}}\right]}$$
$$\times (1-\tanh q_b)^{\frac{\eta^+_{n,s}}{2}}(1+\tanh q_b)^{\frac{\eta^-_{n,s}}{2}} P_n^{(\eta^+_{n,s},\eta^-_{n,s})}(\tanh q_b)$$
$$\times (1-\tanh q_a)^{\frac{\eta^+_{n,s}}{2}}(1+\tanh q_a)^{\frac{\eta^-_{n,s}}{2}} P_n^{(\eta^+_{n,s},\eta^-_{n,s})}(\tanh q_a). \quad (59)$$

where

$$\left(E_{n,s}^{(\alpha,\beta)}\right)^2=\frac{\bar{\theta}}{\beta m^2\omega^2+\alpha\left(1+\frac{e^2\mathcal{E}^2/\hbar^2c^2}{\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}\right)}\left[m^2c^4+\hbar c\varpi(2n+1-s)-\hbar^2c^2\bar{\theta}\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]. \quad (60)$$

To determine the energy spectrum, let us integrate over the p_0 variable. This can be done by converting the problem to a complex integral along the special choice of the contour C, using the residue theorem, we get

$$\int_{-\infty}^{+\infty}f(p_0)\frac{dp_0}{2\pi\hbar}\frac{e^{-\frac{\imath}{\hbar}p_0(t_b-t_a)}}{p_0^2-\left(E_{n,s}^{(\alpha,\beta)}\right)^2}=-\imath\sum_{\epsilon=\pm1}f(\epsilon E_{n,s}^{(\alpha,\beta)})\frac{e^{\frac{-\imath}{\hbar}\epsilon E_{n,s,\epsilon}^{(\alpha,\beta)}(t_b-t_a)}}{2E_{n,s}^{(\alpha,\beta)}}\Theta(\epsilon(t_b-t_a)), \quad (61)$$

which has the poles

$$\epsilon E_{n,s}^{(\alpha,\beta)}=E_{n,s,\epsilon}^{(\alpha,\beta)}=\epsilon\sqrt{\frac{\bar{\theta}}{\beta m^2\omega^2+\alpha\left(1+\frac{e^2\mathcal{E}^2}{\varpi^2-2\bar{\theta}\hbar c\varpi\left(n+\frac{1}{2}-\frac{s}{2}\right)+\bar{\theta}^2\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2}\right)}}$$
$$\times\left[m^2c^4+\hbar c\varpi(2n+1-s)-\hbar^2c^2\bar{\theta}\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]^{\frac{1}{2}}, \quad (62)$$

where Θ is the Heaviside function. In Eq. (62) n is a quantum number, and the parameter $s=\pm1$ describes the two components of the Dirac spinor. $\epsilon=+1$ corresponds to the positive energy states, $\epsilon=-1$ corresponds to the negative energy states. The parameter $e=\mp1$ describes a negatively ($e=-1$) or positively ($e=+1$) charged particle, where $\mathcal{E}$ is the strength of the uniform electric field, and ω is the angular frequency of the oscillator. When the electric field $\mathcal{E}$ is set to zero in the context of the aSdS model applied to the Dirac oscillator, the corresponding spectral energy becomes:

$$E_{n,s}^{(\alpha,\beta)}(\mathcal{E}=0)=\pm\left[m^2c^4+\hbar c^2m\omega(2n+1-s)-\hbar^2c^2\left(\alpha+\beta m^2\omega^2\right)\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]^{\frac{1}{2}}. \quad (63)$$

In addition, for $\omega = \mathcal{E} = 0$, the corresponding energy levels reduce to

$$E_{n,s}^{(\alpha,\beta)}(\omega = \mathcal{E} = 0) = \pm\left[m^2c^4 - \hbar^2c^2\alpha\left(n + \frac{1}{2} - \frac{s}{2}\right)^2\right]^{\frac{1}{2}}. \tag{64}$$

This result demonstrates that, within the framework of Snyder (anti)-de Sitter model, the energy levels dependency on n^2 persist even in the absence of ω-oscillation and $\mathcal{E}$-electric fields.

At this stage, we have successfully extracted the spectral energy for the Dirac oscillator coupled to a uniform electric field. Although the corresponding normalized eigenspinors are lengthy and complex, we will proceed solely to find the Green function in momentum space. Consequently, from Eqs. (20) and (27) we can write the elements matrix of $S^{(\alpha,\beta)}(p_b, p_a, t_b, t_a)$ as:

$$S^{(\alpha,\beta)}(p_b, p_a, t_b, t_a) = \left[\mathcal{O}_+^D\right]_b \mathcal{G}(p_b, p_a, t_b, t_a), \tag{65}$$

here, $\mathcal{O}_+^D$ is defined in Eq. (19) and $\mathcal{G}(p_b, p_a, t_b, t_a)$ is calculated exactly in Eq. (59), thus, we obtain,

$$\begin{aligned}
S^{(\alpha,\beta)}(p_b, p_a, t_b, t_a) &= -\imath\sqrt{\beta}\sum_{s=\pm 1}\sum_n\left[\gamma^0\left(\hat{p}_{0b} + e\mathcal{E}\hat{X}_b\right) - c\gamma^1\left(\hat{P}_b - \imath m\omega\gamma^0\hat{X}_b\right) + mc^2\right]\left[e^{-\frac{\delta}{2}\sigma_2}\mathbb{X}_s\mathbb{X}_s^+ e^{\frac{\delta}{2}\sigma_2}\right]\\
&\times\sum_{\epsilon=\pm 1}\frac{e^{-\frac{\imath}{\hbar}\epsilon E_{n,s}^{(\alpha,\beta)}(t_b-t_a)}}{2E_{n,s}^{(\alpha,\beta)}}\Theta(\epsilon(t_b - t_a))\frac{\bar{\theta}\left[\left(1 - \frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{\left(\beta m^2\omega^2 + \alpha\right) + \frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}}\\
&\times 2^{2n+(1-\bar{s})}e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}p_0}{\hbar\left(\varpi^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}(\sqrt{\beta}p_b)-\sinh^{-1}(\sqrt{\beta}p_a)}{\sqrt{\beta}}\right]}\\
&\times(1 - \tanh q_b)^{\frac{\eta_{n,s}^+}{2}}(1 + \tanh q_b)^{\frac{\eta_{n,s}^-}{2}}P_n^{(\eta_{n,s}^+,\eta_{n,s}^-)}(\tanh q_b)\\
&\times(1 - \tanh q_a)^{\frac{\eta_{n,s}^+}{2}}(1 + \tanh q_a)^{\frac{\eta_{n,s}^-}{2}}P_n^{(\eta_{n,s}^+,\eta_{n,s}^-)}(\tanh q_a).
\end{aligned} \tag{66}$$

After that, writing the relations

$$\sum_{\epsilon=\pm 1} f(\epsilon)\Theta(\epsilon(t_b - t_a)) = f(s)\Theta(s(t_b - t_a)) + f(-s)\Theta(-s(t_b - t_a)), \tag{67}$$

and

$$\gamma^3 e^{A\sigma_2} = \sigma_3 e^{A\sigma_2} = -e^{A\sigma_2}\sigma_3,\ \ \sigma_3\chi_s = s\chi_s, \tag{68}$$

$$\gamma^1 e^{A\sigma_2} = \imath\sigma_2 e^{A\sigma_2} = \imath e^{A\sigma_2}\sigma_2,\ \ \sigma_2\chi_s = \imath s\chi_{-s}, \tag{69}$$

$$\gamma^2 e^{A\sigma_2} = -\imath\sigma_1 e^{A\sigma_2} = \imath\sigma_1 e^{A\sigma_2}, \sigma_1\chi_s = \chi_{-s}. \tag{70}$$

This leads to the following expression of the propagator $S^{(\alpha,\beta)}(p_b, p_a)$ in the momentum space

$$\begin{aligned}
S^{(\alpha,\beta)}(p_b, p_a) &= -\imath\sqrt{\beta}\sum_{s=\pm 1}\sum_n\left\{\left[\frac{e^{\frac{-\imath}{\hbar}E_{n,s}^{(s)}(t_b-t_a)}}{2E_{n,s}^{(s)}}\Theta(s(t_b - t_a))\right.\right.\\
&\times e^{-\frac{\delta}{2}\sigma_2}\left[\left(-\left(E_{n,s}^{(s)} + se\mathcal{E}\hat{X}_b\right) + mc^2\right)\mathbb{X}_s\mathbb{X}_s^+ - \imath sc\left(\hat{P}_b - \imath m\omega\hat{X}_b\right)\mathbb{X}_{-s}\mathbb{X}_s^+\right]e^{\frac{\delta}{2}\sigma_2}\\
&\times\bar{\theta}\frac{\left[\left(1 - \frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{\left(\beta m^2\omega^2 + \alpha\right) + \frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}}2^{2n+(1-\bar{s})}\\
&\times e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}E_{n,s}^{(s)}}{\left(c^2m^2\omega^2-e^2\mathcal{E}^2+c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}(\sqrt{\beta}p_b)-\sinh^{-1}(\sqrt{\beta}p_a)}{\hbar\sqrt{\beta}}\right]}\\
&\times(1 - \tanh q_b)^{\frac{\eta_{n,s}^+}{2}}(1 + \tanh q_b)^{\frac{\eta_{n,s}^-}{2}}P_n^{(\eta_{n,s}^+,\eta_{n,s}^-)}(\tanh q_b)\\
&\times(1 - \tanh q_a)^{\frac{\eta_{n,s}^+}{2}}(1 + \tanh q_a)^{\frac{\eta_{n,s}^-}{2}}P_n^{(\eta_{n,s}^+,\eta_{n,s}^-)}(\tanh q_a)\\
&+\left[\frac{e^{\frac{-\imath}{\hbar}E_{n,s}^{(-s)}(t_b-t_a)}}{2E_{n,s}^{(-s)}}\Theta(-s(t_b - t_a))\,\bar{\theta}\frac{\left[\left(1 - \frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{\left(\beta m^2\omega^2 + \alpha\right) + \frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}}2^{2n+(1-\bar{s})}\right.
\end{aligned}$$

$$\times \left[e^{-\frac{\delta}{2}\sigma_2}\left[\left[\left(E_{n,s}^{(-s)} - se\mathcal{E}\hat{X}_b\right) + mc^2\right]\mathbb{X}_s\mathbb{X}_s^+ - \imath sc\left(\hat{P}_b - \imath m\omega\gamma^0\hat{X}_b\right)\mathbb{X}_{-s}\mathbb{X}_s^+\right]e^{\frac{\delta}{2}\sigma_2}\right]$$

$$\times e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}E_{n,s}^{(-s)}}{\left(c^2m^2\omega^2 - e^2\mathcal{E}^2 + c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}(\sqrt{\beta}p_b) - \sinh^{-1}(\sqrt{\beta}p_a)}{\hbar\sqrt{\beta}}\right]}$$

$$\times (1-\tanh q_b)^{\frac{\eta_{n,s}^+}{2}}(1+\tanh q_b)^{\frac{\eta_{n,s}^-}{2}}P_n^{\left(\eta_{n,s}^+,\eta_{n,s}^-\right)}(\tanh q_b)$$

$$\times (1-\tanh q_a)^{\frac{\eta_{n,s}^+}{2}}(1+\tanh q_a)^{\frac{\eta_{n,s}^-}{2}}P_n^{\left(\eta_{n,s}^+,\eta_{n,s}^-\right)}(\tanh q_a)\Bigg\}. \tag{71}$$

To unify the expression of Heaviside function $\Theta(-s(t_b - t_a))$ by $\Theta(s(t_b - t_a))$, one must replace all term which are multiplied by $\Theta(-s(t_b - t_a))$ by s to $(-s)$. Additionaly, to unify the same energy we make the the following mapping

$$n \to n - s. \tag{72}$$

Therefore, the propagator $S^{(\alpha,\beta)}(p_b,\ p_a)$ of the $(1+1)$-dimensional Dirac oscillators subjected to an electric field in the context of the Snyder (anti-)de Sitter model in the momentum space becomes as

$$S^{(\alpha,\beta)}(p_b, p_a, T) = -\imath\sqrt{\beta}\sum_{s=\pm1}\sum_n\left\{\frac{e^{\frac{-\imath}{\hbar}E_{n,s}^{(s)}(t_b-t_a)}}{2E_{n,s}^{(s)}}\Theta(s(t_b-t_a))\right.$$

$$\times\left[\bar{\theta}\frac{\left[\left(1-\frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{\left(\beta m^2\omega^2+\alpha\right)+\frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}}2^{2n+(1-\bar{s})}\right.$$

$$\times e^{-\frac{\delta}{2}\sigma_2}\left[\left(-\left(E_{n,s}^{(s)} + se\mathcal{E}\hat{X}_b\right) + mc^2\right)\mathbb{X}_s\mathbb{X}_s^+ - \imath sc\left(\hat{P}_b - \imath m\omega\hat{X}_b\right)\mathbb{X}_{-s}\mathbb{X}_s^+\right]e^{\frac{\delta}{2}\sigma_2}$$

$$\times e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}E_{n,s}^{(s)}}{\hbar\left(c^2m^2\omega^2 - e^2\mathcal{E}^2 + c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}(\sqrt{\beta}p_b) - \sinh^{-1}(\sqrt{\beta}p_a)}{\sqrt{\beta}}\right]}$$

$$\times (1-\tanh q_b)^{\frac{\eta_{n,s}^+}{2}}(1+\tanh q_b)^{\frac{\eta_{n,s}^-}{2}}P_n^{\left(\eta_{n,s}^+,\eta_{n,s}^-\right)}(\tanh q_b)$$

$$\times (1-\tanh q_a)^{\frac{\eta_{n,s}^+}{2}}(1+\tanh q_a)^{\frac{\eta_{n,s}^-}{2}}P_n^{\left(\eta_{n,s}^+,\eta_{n,s}^-\right)}(\tanh q_a)$$

$$+\left[\bar{\theta}\frac{\left[\left(1-\frac{4MA}{\hbar(\bar{s}-2n-1)^2}\right)\frac{(\bar{s}-2k_2-2n)n!\Gamma(\bar{s}-n)}{\Gamma(\bar{s}+1-n-2k_2)\Gamma(2k_2+n)}\right]}{\left(\beta m^2\omega^2+\alpha\right)+\frac{e^2\mathcal{E}^2\alpha}{\hbar^2c^2\bar{\theta}^2\left(\nu_s-n-\frac{1}{2}\right)^2}}2^{2n+(1-\bar{s})}\right.$$

$$\times\left[e^{-\frac{\delta}{2}\sigma_2}\left[\left(\left(E_{n,s}^{(s)} + se\mathcal{E}\hat{X}_b\right) + mc^2\right)\mathbb{X}_{-s}\mathbb{X}_{-s}^+ + \imath sc\left(\hat{P}_b - \imath m\omega\gamma^0\hat{X}_b\right)\mathbb{X}_s\mathbb{X}_{-s}^+\right]e^{\frac{\delta}{2}\sigma_2}\right]$$

$$\times e^{\left[-\frac{\imath}{\hbar}\frac{e\mathcal{E}E_{n,s}^{(s)}}{\hbar\left(c^2m^2\omega^2 - e^2\mathcal{E}^2 + c^2\frac{\alpha}{\beta}\right)}\frac{\sinh^{-1}(\sqrt{\beta}p_b) - \sinh^{-1}(\sqrt{\beta}p_a)}{\sqrt{\beta}}\right]}$$

$$\times (1-\tanh q_b)^{\frac{\eta_{n,s}^+}{2}}(1+\tanh q_b)^{\frac{\eta_{n,s}^-}{2}}P_{n-s}^{\left(\eta_{n,s}^+,\eta_{n,s}^-\right)}(\tanh q_b)$$

$$\times (1-\tanh q_a)^{\frac{\eta_{n,s}^+}{2}}(1+\tanh q_a)^{\frac{\eta_{n,s}^-}{2}}P_{n-s}^{\left(\eta_{n,s}^+,\eta_{n,s}^-\right)}(\tanh q_a)\Bigg\}. \tag{73}$$

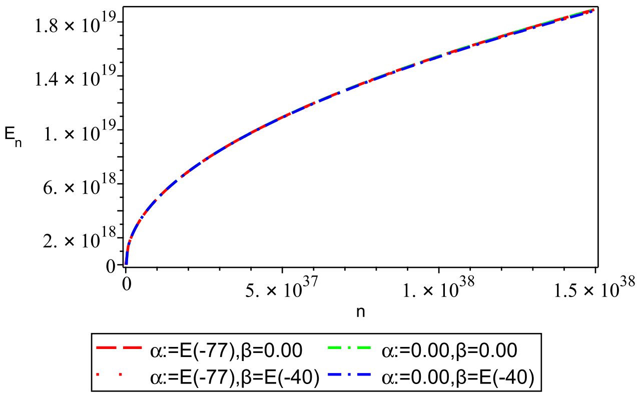
Furthermore, in the case of Snyder de-Sitter space, which can be constructed from propagator's function and spectral energies defined, respectively, in Eq. (73) and (62) by replacing α and β by $(-\alpha,\ -\beta)$. Also, the Jacobi polynomial is replaced by Romanovski polynomials [34],

$$P_n^{\left(\eta_{n,s}^+,\eta_{n,s}^-\right)}(\imath\tan q) \to R_n^{\left(\eta_{n,s}^+,\eta_{n,s}^-\right)}(\tan q). \tag{74}$$

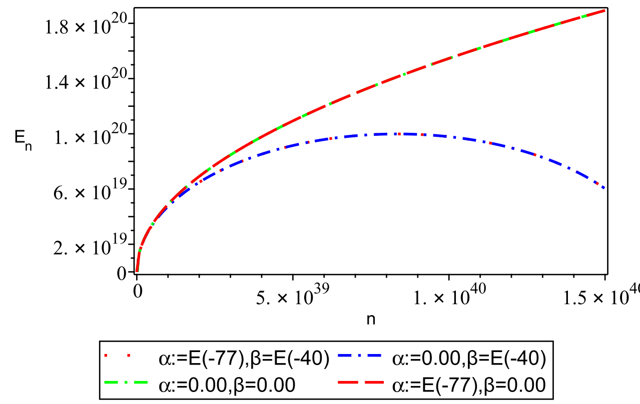
Moreover, in both cases for the sign parameters α and β, the expression for energy levels is related to n^2.

Usually, from theory of deformation the value of α and β is very small, so we expand (62) to first order in α and β, thus we find

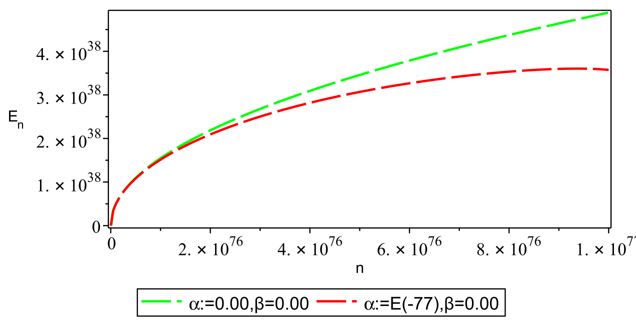
$$E_{n,s}^{(\alpha,\beta)} = \pm\sqrt{\frac{c^2m^2\omega^2 - e^2\mathcal{E}^2}{c^2m^2\omega^2}\left[m^2c^4 + \hbar c\sqrt{c^2m^2\omega^2 - e^2\mathcal{E}^2}(2n+1-s)\right]}$$



(a) First observation .



(b) Second observation.



(c) Third observation.

Fig. 1 $E_{n,\alpha,\beta}$ versus the quantum number n for different values of the deformation parameters

$$\mp \frac{\bar{\theta}}{2} \frac{\sqrt{\frac{c^2m^2\omega^2-e^2\mathcal{E}^2}{c^2m^2\omega^2}}\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2}{\left[m^2c^4+\hbar c\sqrt{c^2m^2\omega^2-e^2\mathcal{E}^2}(2n+1-s)\right]^{\frac{1}{2}}}. \tag{75}$$

Here, the first term represents the Landau levels of Dirac oscillator in homogeneous electric field without deformation, while the second term is the quantum gravity correction. It is interesting to note that when the value of electric field is large then critical one $\frac{e\mathcal{E}}{c} > m\omega$ the bounded eigenstates are absent. Now, let us consider the following particular cases.

1- In limit case $\alpha \to 0$, the expression of Eq. (62) reduces to that of the flat Snyder model,

$$E_{n,s}^{(\alpha=0)} = \pm \frac{\varpi}{cm\omega}\left[m^2c^4 + \hbar c\varpi\,(2n+1-s) - \hbar^2\beta\varpi^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]^{\frac{1}{2}}. \tag{76}$$

1. In limit case $\beta \to 0$, one recovers the spectral energies for the Heisenberg algebra in an (anti-)de Sitter background [26],

$$E_{n,s}^{(\beta=0)} = \pm\left(1 + \frac{e^2\mathcal{E}^2}{\left(\varpi^2 - 2\bar{\theta}\hbar c\varpi\left(n+\frac{1}{2}-\frac{s}{2}\right) + \bar{\theta}^2\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right)}\right)^{-1/2} \times\left[m^2c^4 - \hbar^2c^2\alpha\left(n+\frac{1}{2}-\frac{s}{2}\right)^2 + \hbar c\varpi\,(2n+1-s)\right]^{\frac{1}{2}}. \tag{77}$$

To explore the distinction between the presence and absence of the aSdS algebra, as well as the effect of having the one but not the other on energy levels, we graph the energy levels $E_{n,s=+1}^{(\alpha,\beta)}$ against the quantum numbers, n. To facilitate this presentation, we adopt the natural unit system, where $\hbar$, c, are all set to 1, resulting in dimensionless parameters, and the electron mass set at $m = 0.5\,\text{MeV}$ and an electric field $\mathcal{E}$ of $0.2\,\text{MeV}^2$, $e = 0.303$, $\omega = 2\,\text{MeV}$. We use four different deformation parameter values for this: (i.e., ($\alpha = 10^{-77}\,\text{MeV}$, $\beta = 10^{-40}\,\text{MeV}$), ($\alpha = 10^{-77}\,\text{MeV}$, $\beta = 0.0\,\text{MeV}$), ($\alpha = 0.0\,\text{MeV}$, $\beta = 10^{-40}\,\text{MeV}$) and ($\alpha = 0.0$, $\beta = 0.0$)), as illustrated in Fig. 1. This figure is broken down into three sub-figures (Fig. 1a–c).

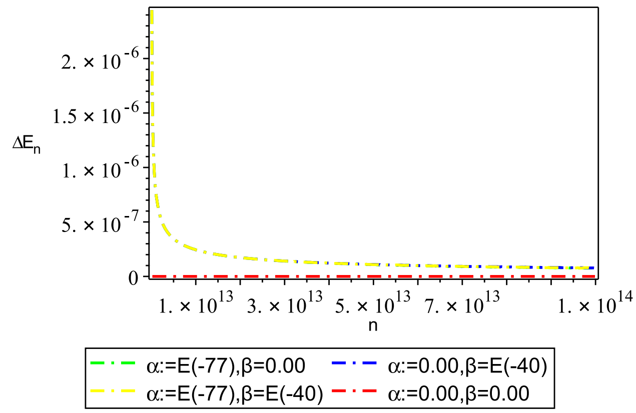


Fig. 2 The energy spacing between adjacent levels as a function of n

We note that in Fig. 1a, all energy levels cases are the same when the quantum number principle n between 0 and 2×10^{38}. Then, the separation occurs from $n = 10^{38}$ to $n = 10^{40}$, and the appearance state of curves $\beta \neq 0$ disappear when $\dot{n} > 10^{41}$, and is shown in Fig. 1b. Whereas in Fig. 1c, the plot of the case ($\alpha = 10^{-77}$, $\beta = 0.0$) appears at $\dot{n} > 10^{76}$, and disappears at $\dot{n} > 10^{77}$. From this data, it is evident that the α-parameter has a more pronounced influence than the β-parameter.

It is also shown in Fig. 2 that the energy spacing between adjacent levels is constant, which is a sign of hard confinement.

Likewise, we can plot all the energy levels curves in the case SdS algebra, where we will find, for example, in Fig. 1c that the energy spectrum curve for the HUP algebra is below the case ($\alpha = 10^{-77}$, $\beta = 0.0$).

Moreover, the nonrelativistic energy level is obtained considering that greater part of the total energy of the system lies in the rest energy (mc^2) of the particle [35], i.e., $E^{(\alpha,\beta)}_{n,s} = mc^2 + E^{(NR)}_{n,s,\alpha,\beta}$, where $mc^2 \gg E^{(NR)}_{n,s,\alpha,\beta}$ and $mc^2 \gg \sqrt{m^2\omega^2 - e^2\mathcal{E}^2/c^2}$. So, applying this prescription in Eq. (62), we obtain the following energy spectrum for a nonrelativistic particle in the presence of an uniform electric field and in the context of the Snyder (anti-)de Sitter model at the first order approximation,

$$E^{(NR)}_{n,s,\alpha,\beta} = \sqrt{\frac{\bar{\theta}}{\beta m^2\omega^2 + \alpha\left(1 + \frac{e^2\mathcal{E}^2}{\left(\varpi^2 - 2\bar{\theta}\hbar c\varpi\left(n+\frac{1}{2}-\frac{s}{2}\right) + \bar{\theta}^2\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right)}\right)}} \times \left[\frac{\hbar}{2m}(\varpi/c)(2n+1-s) - \frac{\hbar^2}{2m}\bar{\theta}\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]. \quad (78)$$

In limit case $\alpha \to 0$, Eq. (78) becomes as,

$$E^{(NR)}_{n,s,\alpha=0,\beta} = \sqrt{\frac{m^2\omega^2 - e^2\mathcal{E}^2/c^2}{m^2\omega^2}}\left[\frac{\hbar}{2m}\sqrt{m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}}(2n+1-s) - \frac{\hbar^2}{2m}\beta\left(m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}\right)\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]. \quad (79)$$

In limit case $\beta \to 0$, Eq. (78) transforms as,

$$E^{(NR)}_{n,s,\alpha,\beta=0} = \pm\left(1 + \frac{e^2\mathcal{E}^2}{\left(\varpi^2 - 2\alpha\hbar c\varpi\left(n+\frac{1}{2}-\frac{s}{2}\right) + \alpha^2\hbar^2c^2\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right)}\right)^{-1/2} \times \left[\frac{\hbar}{2m}(\varpi/c)(2n+1-s) - \frac{\hbar^2}{2m}\alpha\left(n+\frac{1}{2}-\frac{s}{2}\right)^2\right]. \quad (80)$$

From Eq. (78) and in first order of α, β, we can find the energy spectrum for a spinless nonrelativistic particle ($s = 0$) in the presence of an uniform electric field

$$E^{(NR)}_{n,s,\alpha,\beta} = \left[\hbar\left(\frac{m^2\omega^2 - e^2\mathcal{E}^2/c^2}{m^2\omega}\right)\left(n+\frac{1}{2}\right) - \frac{\hbar^2}{2m}\bar{\theta}\sqrt{\frac{m^2\omega^2 - e^2\mathcal{E}^2/c^2}{m^2\omega^2}}\left(n+\frac{1}{2}\right)^2\right]. \quad (81)$$

The first and second terms in Eq. (81) represent, respectively, the energy level for a spinless non-relativistic oscillator of frequency ω particle interacting with a uniform electric field in usual quantum mechanics (HUP), and the relativistic correction both in the context of the modification of the Heisenberg algebra. Also, if we take the limit $\mathcal{E} \to 0$, Eq. (81) transforms to

$$E^{(NR)}_{n,s,\alpha,\beta} = \left[\hbar\omega\left(n+\frac{1}{2}\right) - \frac{\hbar^2}{2m}\bar{\theta}\left(n+\frac{1}{2}\right)^2\right]. \tag{82}$$

Here, the first term is the energy level for a spinless non-relativistic oscillator of frequency ω particle in HUP, and the second term is the first correction of deformation in non relativistic case.

4 Thermodynamic functions

Now, let us study the thermodynamical properties for the problem of the Dirac oscillator particle interacting with a uniform electric field in the modified algebra (1) in the context of the aSdS model. To get these thermodynamic properties, we must first find the corresponding partition function. Indeed, we have,

$$Z = \sum_{n=0}^{\infty} e^{-\bar{\beta}E_n}, \tag{83}$$

where $\bar{\beta} = 1/(k_B T)$, k_B is the Boltzmann constant and T is the equilibrium temperature of system. For simplicity, we take the positive energy level for spin up ($s = +1$) at the first order of $(\alpha,\ \beta)$ given by Eq. (75). So, the sum (83) reads,

$$Z(T,\alpha,\beta) = \sum_{n=0}^{\infty} \exp\left[-\bar{\beta}\sqrt{b+an} - \bar{\beta}\frac{\bar{\theta}}{2}\frac{(\varpi/c)^2}{m^2\omega^2}\frac{\hbar^2c^2n^2}{\sqrt{b+an}}\right], \tag{84}$$

with $a = 2\frac{(\hbar c^2)(\varpi/c)^3}{m^2\omega^2}$, $b = \frac{(\varpi/c)^2 m^2c^4}{m^2\omega^2}$. At the first order of $(\alpha,\ \beta)$, the partition function (83) becomes

$$Z(T,\alpha,\beta) = Z^0(\bar{\beta}) + \bar{\theta}\Delta Z^{(1)}(\bar{\beta}), \tag{85}$$

where

$$Z^0(\bar{\beta}) = \sum_{n=0}^{\infty} e^{-\bar{\beta}\sqrt{b+an}} \quad \text{and} \quad \Delta Z^{(1)}(\bar{\beta}) = -\bar{\beta}\frac{\hbar^2c^2}{2}\frac{(\varpi/c)^2}{m^2\omega^2}\sum_{n=0}^{\infty}\frac{n^2}{\sqrt{b+an}}e^{-\bar{\beta}\sqrt{b+an}}. \tag{86}$$

We can evaluate the sums in (85) by using the Euler–Maclaurin summation formula [36]

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)dx - \sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0), \tag{87}$$

where B_{2p} are the Bernoulli numbers, with $B_2 = 1/6$, $B_4 = -1/30$, ..., and $f^{(2k-1)}(0)$ is the derivative of order $(2k-1)$ at $x = 0$, which are given as follows

$$f(0) = e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2}, \quad f^{(1)}(0) = -(\hbar c^2)\bar{\beta}\frac{(\varpi/c)^2}{m\omega}\frac{e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2}}{mc^2}, \tag{88}$$

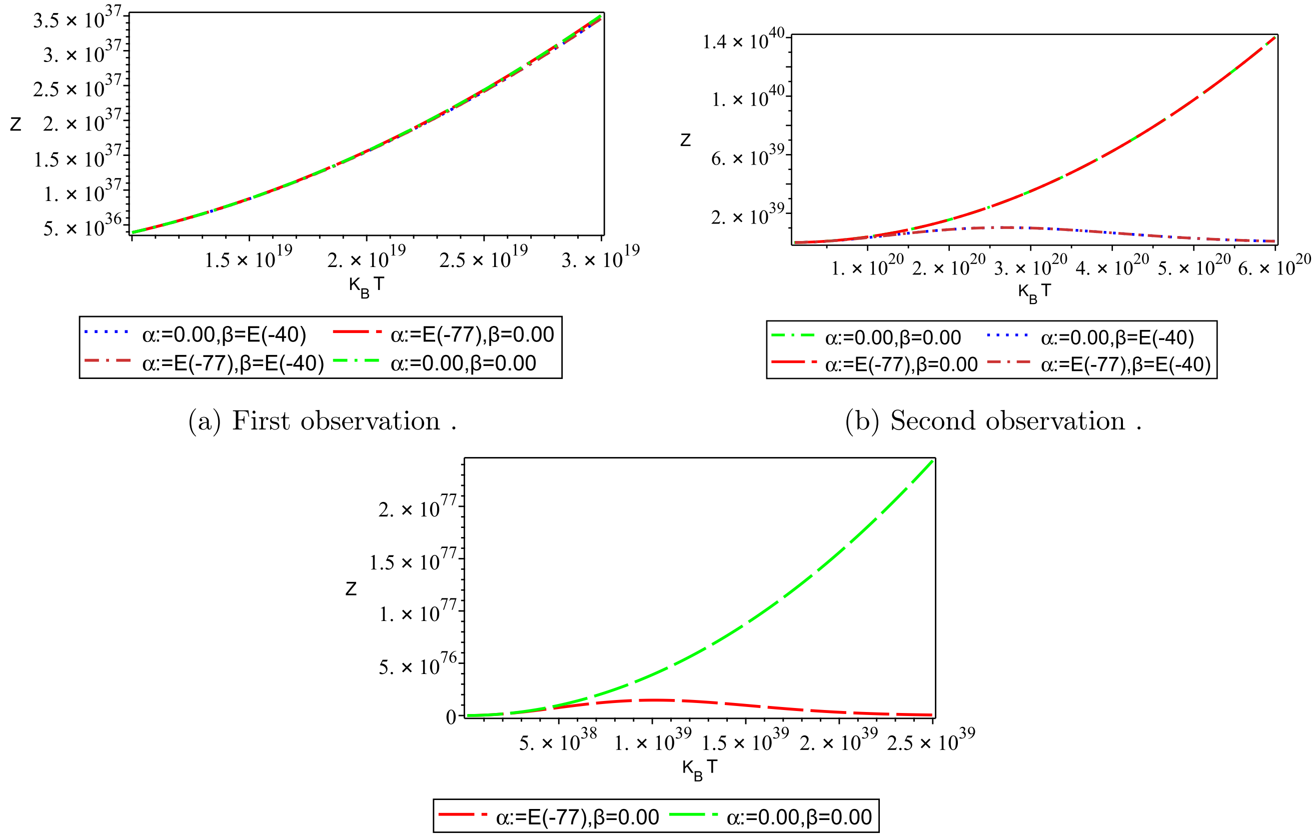
$$f^{(3)}(0) = \left\{-\frac{(\hbar c^2)^3(\varpi/c)^6}{(mc^2)^3(m\omega)^3}\bar{\beta}^3 - \frac{3(\hbar c)^3\varpi^5}{(cm\omega)^4(mc^2)^4}\bar{\beta}^2 - \frac{3(\hbar c^2)^3(\varpi/c)^4}{m\omega(mc^2)^5}\bar{\beta}\right.$$
$$\left. + 6\bar{\theta}\bar{\beta}^2\frac{\hbar^3c^3}{2}\frac{\varpi^3}{(c^2m^2\omega^2)(m^2c^4)} + 6\bar{\theta}\bar{\beta}\frac{\hbar^3c^3}{2}\frac{\varpi^2}{2cm\omega(mc^2)^3}\right\}e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2}. \tag{89}$$

Then, the integral over x in Eq. (87) is written as

$$\int_0^{\infty} f(x)dx = \left\{\frac{2\sqrt{b}}{a\bar{\beta}} + \frac{2}{a\bar{\beta}^2} - \frac{\bar{\theta}}{2}\frac{\hbar^2c^2(\varpi/c)^2}{m^2\omega^2}\left[\frac{16b}{a^3\bar{\beta}^2} + \frac{48\sqrt{b}}{a^3\bar{\beta}^3} + \frac{48}{a^3\bar{\beta}^4}\right]\right\}e^{-\bar{\beta}\sqrt{b}}. \tag{90}$$

Consequently, the partition function is now written as

$$Z(T,\alpha,\beta) = \left\{\frac{1}{2} + \frac{(m\omega)(mc^2)}{(\hbar c^2)(\varpi^2/c^2)}\frac{1}{\bar{\beta}} + \frac{m^2\omega^2}{(\hbar c^2)(\varpi^2/c^2)^{3/2}}\frac{1}{\bar{\beta}^2}\right.$$
$$- \frac{\bar{\theta}}{c^2}\left[\frac{(m^2\omega^2)(m^2c^4)}{(\hbar c^2)(\varpi/c)^5\bar{\beta}^2} + \frac{3(m^3\omega^3)(mc^2)}{(\hbar c^2)(\varpi/c)^6\bar{\beta}^3}\right.$$



(a) First observation .

(b) Second observation .

(c) Third observation.

Fig. 3 Partition function for the DO with uniform electric field as a function of temperature T for different values of the deformation parameters

$$+\frac{3m^4\omega^4}{(\hbar c^2)(\varpi/c)^7\bar{\beta}^4}\Bigg]\Bigg\}e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2}-\sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0). \tag{91}$$

To compute this partition function, we need to calculate the sum in the above expression, and for our case, it can be done only by numerical methods. Up to $k = 2$, this sum can be written as

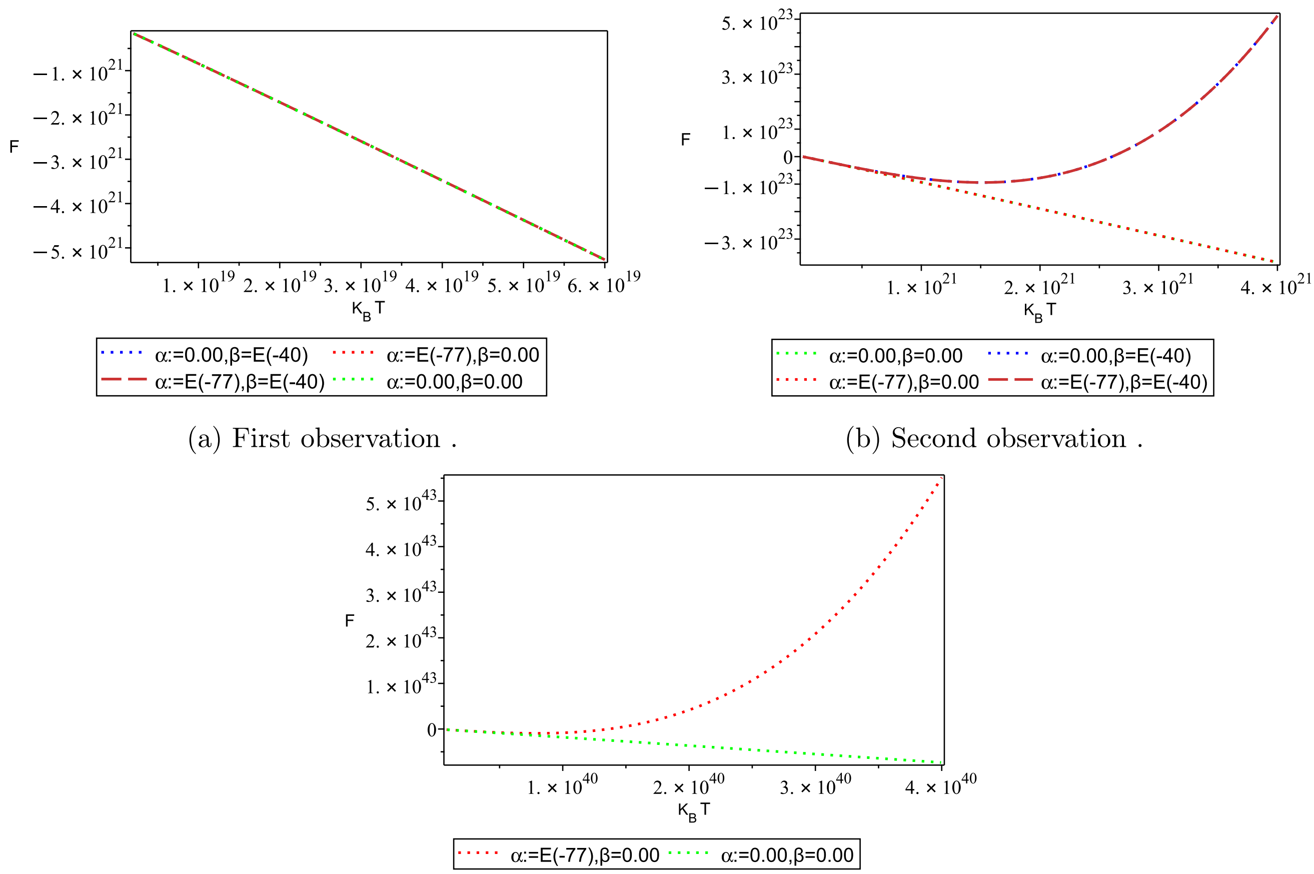
$$\sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0)=-\frac{\bar{\beta}}{6}\frac{\hbar c^2(\varpi/c)^2}{m\omega}\frac{e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2}}{mc^2}-\frac{1}{180}\Bigg[-\bar{\beta}\frac{3\left(\hbar c^2\right)^3(\varpi/c)^4}{m\omega\left(mc^2\right)^5}$$
$$-\bar{\beta}^2\frac{3\left(\hbar c^2\right)^3(\varpi/c)^5}{c^2\left(m^4\omega^4\right)\left(m^2c^4\right)^2}+6\bar{\theta}\bar{\beta}^2\frac{\left(\hbar c^2\right)^3}{2}\frac{(\varpi/c)^3}{c^2\left(m^2\omega^2\right)\left(m^2c^4\right)}-\bar{\beta}^3\frac{\left(\hbar c^2\right)^3(\varpi/c)^6}{\left(mc^2\right)^3(m\omega)^3}$$
$$+3\bar{\theta}\bar{\beta}\frac{\left(\hbar c^2\right)^3}{2}\frac{(\varpi/c)^2}{c^2(m\omega)\left(mc^2\right)^3}\Bigg]e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2}. \tag{92}$$

At high temperature ($\bar{\beta} \ll 1$), all terms in the sum of Eq. (92) have a positive power in $\bar{\beta}$, which are very small compared with the other term in Eq. (91). Hence, we can neglect the terms with $\bar{\beta}^n$ and the terms without $\bar{\beta}$. In addition to this, we also expand the function ($e^{-\bar{\beta}\frac{(\varpi/c)}{m\omega}mc^2}$) to the orders of $\bar{\beta}$ in Eq. (91), and then, with some simplifications, we neglect all the positive exponents of $\bar{\beta}$. The result of Eq. (91) becomes as:

$$Z(T,\alpha,\beta)\simeq\frac{m^2\omega^2}{(\hbar c^2)(\varpi/c)^3}\frac{1}{\bar{\beta}^2}-\frac{\bar{\theta}}{c^2}\left[\frac{3m^4\omega^4}{\left(\hbar c^2\right)(\varpi/c)^7}\frac{1}{\bar{\beta}^4}-\frac{\left(m^2\omega^2\right)\left(mc^2\right)^2}{2\left(\hbar c^2\right)(\varpi/c)^5}\frac{1}{\bar{\beta}^2}\right]. \tag{93}$$

As the $\bar{\theta}$-deformation parameter is very small, we can rewrite the partition function as follows:

$$Z(T,\alpha,\beta)\simeq\frac{m^2\omega^2}{\left(\hbar c^2\right)(\varpi/c)^3}(k_{\mathrm{B}}T)^2e^{-\bar{\theta}\left[\frac{3m^2\omega^2}{(\varpi/c)^4}(k_{\mathrm{B}}T)^2-\frac{1}{2}\frac{m^2c^4}{(\varpi/c)^2}\right]}. \tag{94}$$



(a) First observation .

(b) Second observation .

(c) Third observation.

Fig. 4 The Helmholtz free energy function for the DO with uniform electric field as a function of temperature T for different values of the deformation parameters

The limit $\bar{\theta} \to 0$ gives the partition function for the (1 + 1)-dimensional Dirac oscillator subjected to uniform electric field in HUP algebra. The term related to $\bar{\theta}$ represents a contribution of the SdS algebra on Z-function. Now, the partition function in Eq. (94) will help us getting all thermodynamic functions, such as, the F-Helmholtz free energy, the Ξ-mean energy, the S-entropy and the C-heat capacity. For example, the Helmholtz free energy for our problem in high temperature becomes as

$$F(T, \alpha, \beta) = -T \ln(Z) = F_0(\bar{\beta}) + \bar{\theta} \Delta F^1(\bar{\beta}), \tag{95}$$

with $F_0(\bar{\beta})$ begin the Helmholtz free energy for $(1+1)$-dimensional Dirac oscillator with homogenous electric field in HUP algebra.

$$F_0(\bar{\beta}) = -2T \ln\left(\frac{m\omega(k_B T)}{\sqrt{\hbar c^2 \left(m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}\right)^{3/2}}}\right), \tag{96}$$

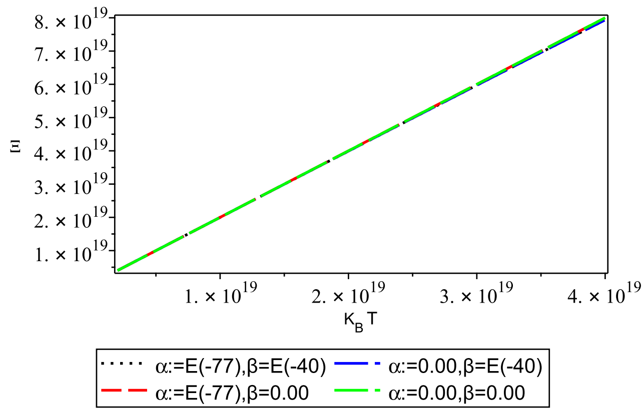
and $\Delta F^1(\bar{\beta})$ represents the first-order correction for the SdS deformation

$$\Delta F^1(\bar{\beta}) = -\frac{1}{2k_B} \frac{m^2c^2}{\left(m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}\right)}(k_B T) + \frac{3m^2\omega^2}{c^2 k_B \left(m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}\right)^2}(k_B T)^3. \tag{97}$$

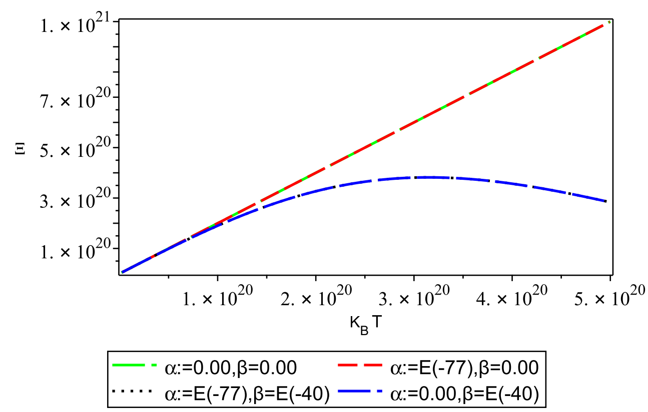
The relation between mean energy and partition function gives us the following expression

$$\Xi(T, \alpha, \beta) = -\frac{\partial \ln(Z)}{\partial \bar{\beta}} = 2k_B T \exp\left(-3\bar{\theta} \frac{m^2\omega^2}{c^2\left(m^2\omega^2 - \frac{e^2\mathcal{E}^2}{c^2}\right)^2}(k_B T)^2\right). \tag{98}$$

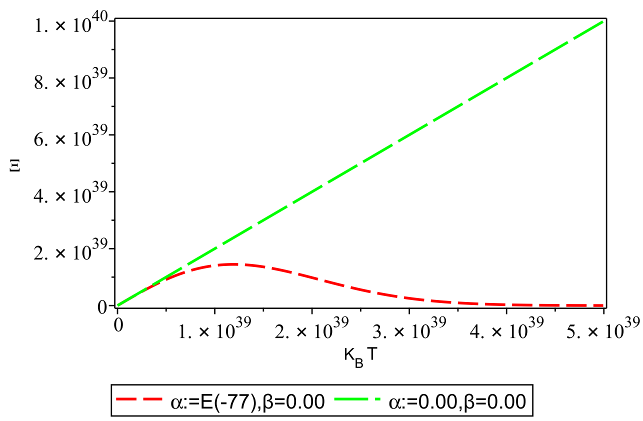
When $\bar{\theta} \to 0$, we recover the usual case of mean energy in HUP algebra.



(a) First observation .



(b) Second observation .



(c) Third observation.

Fig. 5 The mean energy function for the DO with uniform electric field as a function of temperature T for different values of the deformation parameters

For the heat capacity, we have

$$C(T,\alpha,\beta)=\frac{\partial\,\Xi}{\partial T}=C_0(\bar{\beta})+\bar{\theta}\Delta C^1(\bar{\beta}), \tag{99}$$

where $C_0(\bar{\beta})=2k_{\rm B}$ is constant in the absence of (a)SdS algebra, whereas $\Delta C^1(\bar{\beta})$ represents the first correction of the heat capacity, which dependent of T^2.

$$\Delta C^1(\bar{\beta})=18\frac{m^2\omega^2k_{\rm B}^3T^2}{c^2\left(m^2\omega^2-\frac{e^2\mathcal{E}^2}{c^2}\right)^2}. \tag{100}$$

Finally, the entropy is given as

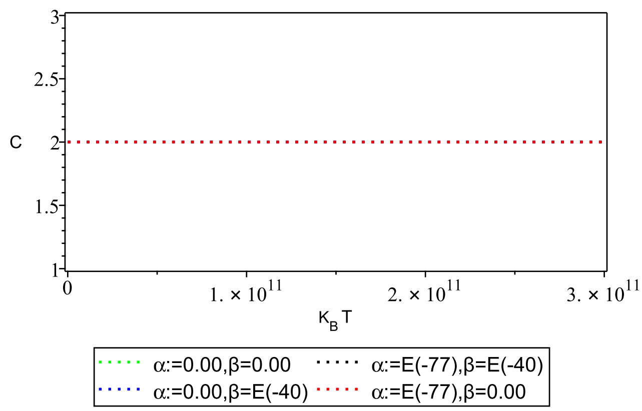
$$S(T,\alpha,\beta)=k_{\rm B}\ln(Z)-k_{\rm B}\bar{\beta}\frac{\partial\ln(Z)}{\partial\bar{\beta}}=S_0(\bar{\beta})+\bar{\theta}\Delta S^1(\bar{\beta}), \tag{101}$$

where $S_0(\bar{\beta})$ stands to the entropy for $(1+1)$-dimensional Dirac oscillator under the uniform electric field in HUP algebra and reads as

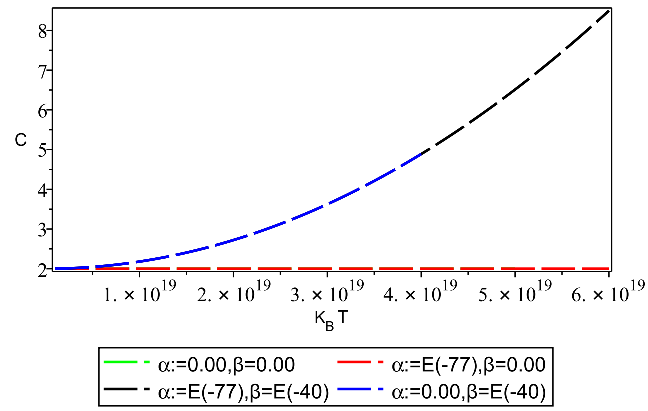
$$S_0(\bar{\beta})=2k_{\rm B}+2k_{\rm B}\ln\left(\frac{m\omega}{\sqrt{\hbar c^2\left(m^2\omega^2-\frac{e^2\mathcal{E}^2}{c^2}\right)^{3/2}}}(k_{\rm B}T)\right), \tag{102}$$

while $\Delta S^1(\bar{\beta})$ is the term correction of entropy in first order of $(\alpha,\ \beta)$ and is written as,

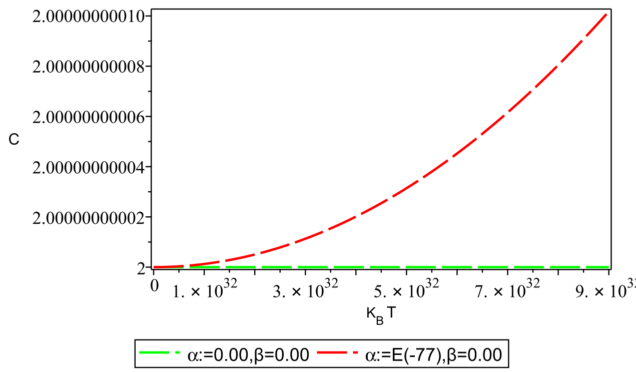
$$\Delta S^1(\bar{\beta})=k_{\rm B}\left[\frac{m^2c^2}{2\left(m^2\omega^2-\frac{e^2\mathcal{E}^2}{c^2}\right)}-\frac{9m^2\omega^2}{c^2\left(m^2\omega^2-\frac{e^2\mathcal{E}^2}{c^2}\right)^2}(k_{\rm B}T)^2\right]. \tag{103}$$



(a) First observation .



(b) Second observation .



(c) Third observation.

Fig. 6 The heat capacity function for the DO with uniform electric field as a function of temperature T for different values of the deformation parameters

In the subsequent figures, we juxtapose the thermodynamic properties of our system across varying deformation parameters. To streamline our presentation, we have employed the natural unit system, where $\hbar$, c, and k_B are all set to 1, rendering all parameters dimensionless. This demands accurate estimations of the relevant physical quantities. As such, we have selected the oscillator value at roughly 2 MeV within the high-temperature range, the electron mass as $m = 0.5\,\text{MeV}$, and the electric field $\mathcal{E}$ at $0.2\,\text{MeV}^2$. Consequently, the thermodynamic properties are illustrated in Figs. 3, 4, 5, 6 and 7, as functions of temperature (T), with four different values of deformation parameters, namely, ($\alpha = 10^{-77}$ MeV, $\beta = 10^{-35}$ MeV), ($\alpha = 0.0$ MeV, $\beta = 10^{-35}$ MeV), ($\alpha = 10^{-77}$ MeV, $\beta = 0.0$ MeV) and ($\alpha = 0.0$ MeV, $\beta = 0.0$ MeV).

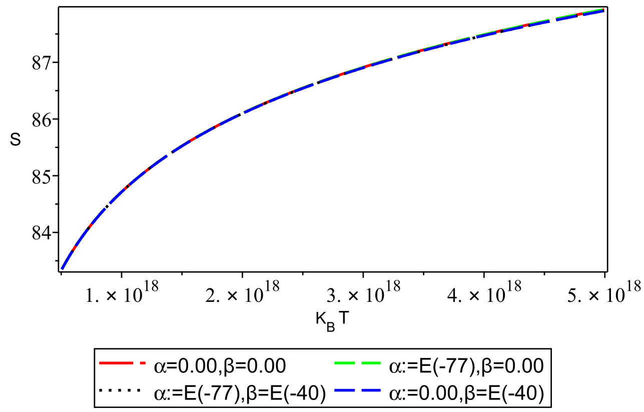
Notably, Fig. 3a demonstrates that the aSdS algebra leads to a surge in the partition function from $k_B T = 1 \times 10^{19}$ to approximately $k_B T \sim 2.5 \times 10^{19}$ MeV. Subsequently, the curves Fig. 3b corresponding to ($\alpha = 10^{-77}$ MeV, $\beta = 10^{-35}$ MeV) and ($\alpha = 0.0$ MeV, $\beta = 10^{-35}$ MeV) goes down to zero after the temperature $k_B T \sim 10^{20}$ MeV. However, the other two curves line up closely up to $k_B T \sim 5 \times 10^{38}$ MeV, after which the curve for ($\alpha = 10^{-77}$ MeV, $\beta = 0.0$ MeV) collapses to zero when $k_B T$ surpasses 10^{39} MeV in Fig. 3c.

In Fig. 4, we have the Helmholtz free energy for the one-dimensional Dirac oscillator within the aSdS context as a function of $k_B T$ and this depiction indicates that the aSdS algebra leads to a decline in the F-function, spanning from $k_B T = 1 \times 10^{19}$ to $k_B T \sim 6 \times 10^{19}$ MeV across all four cases of deformation parameters in Fig. 4a. Beyond $k_B T > 10^{39}$, the curves Fig. 4b for both (($\alpha = 10^{-77}$ MeV, $\beta = 10^{-35}$ MeV) and ($\alpha = 0.0$ MeV, $\beta = 10^{-35}$ MeV)) vanish when $\beta \neq 0$. Meanwhile, the case characterized by (($\alpha = 10^{-77}$ MeV, $\beta = 0.0$ MeV)) has an effect up to temperature $k_B T > 10^{21}$ MeV in Fig. 4c.

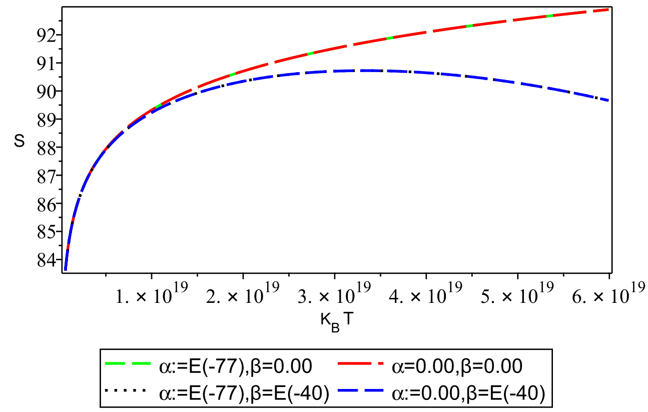
Furthermore, within the aSdS model, the mean energy exhibits a growth as the temperature rises, as depicted in Fig. 5a.

In Fig. 5b, it is shown that for the cases ($\alpha = 10^{-77}$ MeV, $\beta = 10^{-35}$ MeV) and ($\alpha = 0.0$ MeV, $\beta = 10^{-35}$ MeV), the curves decline to zero after reaching the temperature $k_B T \sim 10^{21}$ MeV. However, for the case ($\alpha = 10^{-77}$ MeV, $\beta = 0.0$ MeV), the curve Fig. 5c goes down to zero when $k_B T$ surpasses 5×10^{39} MeV.

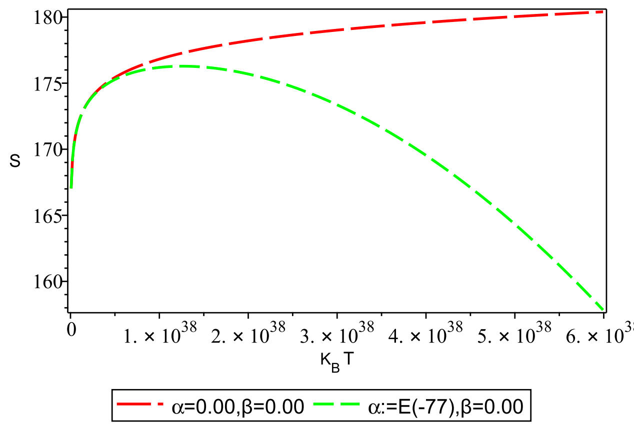
Also, the heat capacity in Fig. 6a is a constant $C = 2k_B$ when $k_B T < 10^{11}$. Then, when $k_B T > 10^{19}$ MeV the cases ($\alpha = 10^{-77}$ MeV, $\beta = 10^{-35}$ MeV) and ($\alpha = 0.0$ MeV, $\beta = 10^{-35}$ MeV) exhibit an increase with temperature, as presented in Fig. 6b. Figure 6c shows the increasing of the capacity for the case ($\alpha = 10^{-77}$ MeV, $\beta = 0.0$ MeV) with the increasing temperature at $k_B T > 10^{32}$.



(a) First observation .



(b) Second observation .



(c) Third observation.

Fig. 7 The entropy function for the DO with uniform electric field as a function of temperature T for different values of the deformation parameters

Finally, in Fig. 7a, we plot the effect of aSdS on entropy function in three graphs. According to Fig. 7b, the aSdS makes the values of entropy smaller with temperature for the cases (($\alpha = 10^{-77}$ MeV, $\beta = 10^{-35}$ MeV) and ($\alpha = 0.0$ MeV, $\beta = 10^{-35}$ MeV)) at temperature $k_B T > 10^{19}$, whereas, in Fig. 7c, the entropy function of the case ($\alpha = 10^{-77}$ MeV, $\beta = 0.0$ MeV) decreases with temperature $k_B T > 10^{38}$.

Just as we observed, the aSdS algebra has a more significant impact on energy eigenvalues when the α-parameter is present compared to the β -parameter, a similar pattern holds true for thermodynamic functions. Likewise, we can deduce the thermodynamic properties and appropriate curves for the SdS model case simply by substituting (α and β) by ($-\alpha$, $-\beta$). Finally, when (a)SdS parameters $\alpha = \beta = 0$ and electric field $\mathcal{E} \to 0$ our results align exactly with that of Ref. [37].

5 Conclusion

In the present paper, we have constructed the 1D Dirac oscillator subjected to the uniform electric field in the momentum space representation and in the presence of Snyder (anti-)de Sitter model. Using the coordinate transformation method, the exact casual Green function and its corresponding propagator are calculated, and then appropriate energy values are derived from it. In both cases for the sign deformation parameters, the Green function and its corresponding propagator are expressed in terms of Romonovski polynomials when (α, β) are positive, and in terms of Jacobi polynomials when (α, β) are negative. Furthermore, we have demonstrated that within the framework of Snyder (anti)-de Sitter space, energy dependencies on n^2 persist even in the absence of oscillation and electric fields. Additionally, we have derived limit cases for deformation parameters and constructed the non-relativistic energy level in this context of aSdS algebra with and without spin.

Finally, at high temperatures, we use the Euler–MacLaurin formula, all thermodynamic quantities of our system have been determined in first order of (α, β), such as, the partition function Z, the Helmholtz free energy F, the mean energy Ξ, the entropy S and the heat capacity C. By plotting the EUP terms of thermodynamic functions with temperature $k_B T$, we have shown the influence of the α-deformation parameter important than the β-parameter. However, these effects cannot be detected by current experimental means.

Data Availability Statement No data associated in the manuscript.

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IOP Publishing Phys. Scr. 99 (2024) 045223 https://doi.org/10.1088/1402-4896/ad2f91

Physica Scripta

RECEIVED
22 November 2023

REVISED
20 February 2024

ACCEPTED FOR PUBLICATION
1 March 2024

PUBLISHED
13 March 2024

PAPER

Exact Green's function for 2D dirac oscillator in constant magnetic field within curved snyder space, and its thermal properties

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Keywords: path integral formalism, curved Snyder space, Relativistic green function, thermodynamic properties, Dirac oscillator

Abstract

Following the path integral approach, and in the context of curved Snyder space, we formulate the Green function for a (1+1)-dimensional Dirac oscillator system subject to a homogeneous magnetic field. Using the radial coordinates transformation the Green function and the electron propagator are calculated. Consequently, the exact bound states and their corresponding spectral energies are extracted. Our analysis has revealed that, under specific conditions when $m\bar{\omega} \longrightarrow m\omega_c/2$ and $c \longrightarrow V_F$, the behavior of the Dirac oscillator system in the presence of a uniform magnetic field within the SdS algebra closely resembles the dynamics of the monolayer graphene problem in the same algebraic framework. At high temperatures, the thermodynamic properties of the electron gas in the four cases of deformation parameters were extracted. The effect of the deformation parameters on these properties are tested, and also the limit cases for small parameters were inferred.

1. Introduction

The Dirac oscillator (DO) model describes a relativistic quantum mechanical system that combines the aspects of the Dirac equation and those of the harmonic oscillator. It describes the behavior of a relativistic particle with spin 1/2 in the presence of a harmonic potential type which is obtained by the following transformation on the momentum vector ($\vec{p} \rightarrow \vec{p} - im\omega\gamma^0\vec{x}$), here γ^0 refers to the Dirac matrix. Several versions of this system have been introduced in various forms due to its close connection with multiple physical phenomena in quantum physics. At first it was introduced by Ito *et al* [1] and developed by Moshinsky and Szczepaniak in [2]. Taking the non-relativistic limit into account, the behavior of the quantum harmonic oscillator can be then recovered; however, a spin–orbit coupling term also arises in this limit. Many examples have been studied in different fields of physics and let us cite for example the references [3–8].

In addition, with the emergence of deformation theories based on Heisenberg's generalization principle [9–11], numerous researchers promptly look for investigating its influence on relativistic oscillators. In [12, 13], this Dirac oscillator model in the presence of a minimal length in one and two dimensions is presented by using the Green's function technique. Also the high-temperature thermodynamic properties of a Dirac oscillator in one dimension are determined in [14]. Moreover, the anti-de Sitter commutation relations give rise to the appearance of minimal uncertainty and then in [15] it is described the Dirac oscillator in one dimension using the position space representation, where analysis is performed on the thermodynamic properties of relativistic harmonic oscillators at high-temperatures. Later, Benzair *et al* computed the energy spectrum of the Dirac Oscillator (DO) using the path integral formulation in one and two dimensions, respectively, within the extended uncertainty principle framework [16, 17]. Moreover, the investigation of thermodynamic properties for relativistic oscillator particles under deformed algebra resonated in this aspect, as evidenced by the references [14, 17–22]. Also [23] where the authors have studied the relativistic spinning massless particle in Graphene layer in the presence of an homogeneous magnetic field. Additionally, de Montigny *et al* [24] investigated the behavior of the Dirac oscillator in the Som–Raychaudhuri space-time, focusing on the influence of its frequency and the vorticity parameter. This study was then extended to the DKP oscillator for a spin-zero field in a cosmic-string background space-time, characterized by a stationary cylindrical symmetric metric, as discussed in [25]. However, in spite of that this DO has been much discussed, there are only a few studies that have been established through the path integral approach. These latter applications are based on three deformed algebras. The first, doubly special relativity theories (DSR), is referred to as generalized uncertainty principle (GUP) [9, 26], confirming the existence of a minimum measurable length. Furthermore, the second involves the existence of a minimum measurable momentum, necessitating the modification of the Heisenberg uncertainty principle into an extended uncertainty principle (EUP) [11, 27, 28]. Meanwhile, the third is obtained by combining GUP and EUP, derived from a model of DSR on a (anti)-de Sitter background, giving rise to triply special relativity (TSR) or named the Snyder de Sitter (SdS) model [10, 23].

The algebra of the SdS model is built using operators for position $\hat{X}_\mu$, momentum $\hat{P}_\mu$, and Lorentz generator $\hat{J}_{\mu\nu}$, which adhere to the following algebraic relationship

$$\begin{aligned} \left[\hat{J}_{\mu\nu}, \hat{X}_\sigma\right] &= i\hbar\left(\eta_{\mu\sigma}\hat{X}_\nu - \eta_{\nu\sigma}\hat{X}_\mu\right), \quad \left[\hat{J}_{\mu\nu}, \hat{P}_\sigma\right] = i\hbar\left(\eta_{\mu\sigma}\hat{P}_\nu - \eta_{\nu\sigma}\hat{P}_\mu\right),\\ \left[\hat{X}_\mu, \hat{P}_\nu\right] &= i\hbar\left(\eta_{\mu\nu} + \alpha\hat{X}_\mu\hat{X}_\nu + \beta\hat{P}_\mu\hat{P}_\nu + \sqrt{\alpha\beta}\left(\hat{P}_\mu\hat{X}_\nu + \hat{X}_\nu\hat{P}_\mu - \hat{J}_{\mu\nu}\right)\right),\\ \left[\hat{X}_\mu, \hat{X}_\nu\right] &= i\hbar\beta\hat{J}_{\mu\nu}; \qquad \left[\hat{P}_\mu, \hat{P}_\nu\right] = i\hbar\alpha\hat{J}_{\mu\nu}. \end{aligned} \tag{1}$$

Here, $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ is the flat Minkowski space-time metric and $\hat{J}_{\mu\nu} = \hat{X}_\mu\hat{P}_\nu - \hat{X}_\nu\hat{P}_\mu$ are the generators of the Lorentz symmetry. While α and β are the coupling constants have dimensions of inverse length and inverse mass, respectively. In the limit $\alpha \to 0$, the algebra (1) reduces to the Snyder model in flat space [11]. Additionally, as $\beta \to 0$, the algebra (1) becomes the de Sitter algebra, with this parameter playing a role proportional to the cosmological constant $\Lambda = -3\alpha$ [11, 27, 28].

In this paper, our aim is to formulate exactly the path integral approach in momentum space representation for the two dimensions DO (2D-DO) in the context of curved Snyder model (SdS) and in the presence of a homogeneous magnetic field. Furthermore, in the non-relativistic Snyder-de Sitter model, the deformed Heisenberg algebra in 3-dimensional case is given by [10],

$$\begin{aligned} \left[\hat{X}_i, \hat{P}_j\right] &= \imath\hbar\left(\delta_{ij} + \alpha\hat{X}_i\hat{X}_j + \beta\hat{P}_i\hat{P}_j + \sqrt{\alpha\beta}\left(\hat{P}_i\hat{X}_j + \hat{X}_j\hat{P}_i\right)\right),\\ \left[\hat{X}_i, \hat{X}_j\right] &= \imath\hbar\beta\hat{J}_{ij}, \left[\hat{P}_i, \hat{P}_j\right] = \imath\hbar\alpha\hat{J}_{ij}. \end{aligned} \tag{2}$$

Here, (α, β) are small and positive parameters, and $\hat{J}_{ij} = \hat{X}_i\hat{P}_j - \hat{X}_j\hat{P}_i$. In the limits $\alpha \to 0$, $\beta \to 0$ and $(\alpha \to 0, \beta \to 0)$ one recovers the Snyder model in flat space, to the de Sitter algebra, and the undeformed Heisenberg algebra, respectively [11]. Therefore, given these commutation relations, it becomes crucial to examine the transformation that connects this deformed algebra with the Snyder algebra. This transformation was originally introduced by Mignemi in [10] and is defined as,

$$\hat{X}_i = \hat{\mathcal{X}}_i + \sqrt{\frac{\beta}{\alpha}}\kappa\hat{\mathcal{P}}_i = \imath\hbar\sqrt{1-\beta\mathbf{p}^2}\frac{\partial}{\partial p_i} + \sqrt{\frac{\beta}{\alpha}}\kappa\frac{p_i}{\sqrt{1-\beta\mathbf{p}^2}}, \tag{3}$$

$$\hat{P}_i = -\sqrt{\frac{\alpha}{\beta}}\hat{\mathcal{X}}_i + (1-\kappa)\hat{\mathcal{P}}_i = -\imath\hbar\sqrt{\frac{\alpha}{\beta}}\sqrt{1-\beta\mathbf{p}^2}\frac{\partial}{\partial p_i} + (1-\kappa)\frac{p_i}{\sqrt{1-\beta\mathbf{p}^2}}. \tag{4}$$

The index (i=1,2) for $(\hat{X}_i \equiv (\hat{X}, \hat{Y}), \hat{P}_i \equiv (\hat{P}_X, \hat{P}_Y))$ denotes the coordinates and momentum components operators. κ is a free parameter that can be chosen in each case to ensure that the Hamiltonian is symmetric and that $(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i)$ satisfy the following commutation relations [11],

$$\left[\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_j\right] = \imath\hbar\left(\delta_{ij} + \beta\hat{\mathcal{P}}_i\hat{\mathcal{P}}_j\right), \left[\hat{\mathcal{X}}_i, \hat{\mathcal{X}}_j\right] = \beta\left(\hat{\mathcal{X}}_i\hat{\mathcal{P}}_j - \hat{\mathcal{X}}_j\hat{\mathcal{P}}_i\right), \left[\hat{\mathcal{P}}_i, \hat{\mathcal{P}}_j\right] = 0. \tag{5}$$

Hence, it becomes feasible to express the position $\hat{\mathcal{X}}_i$ and momentum $\hat{\mathcal{P}}_i$ coordinate operators of the Snyder Heisenberg brackets (5) in terms of auxiliary operators $\hat{x}_i = \imath\hbar\,\partial/\partial p_i$ and $\hat{p}_i = p_i$, which keep to the following relationships:

$$\hat{\mathcal{X}}_i = \sqrt{1-\beta\mathbf{p}^2}\,\hat{x}_i, \; \hat{\mathcal{P}}_i = \frac{\hat{p}_i}{\sqrt{1-\beta\mathbf{p}^2}}. \tag{6}$$

It is important to emphasize that when $\alpha, \beta > 0$, the momentum operator p_i is constrained within the interval of $(-1/\sqrt{\beta})$ to $(1/\sqrt{\beta})$. In particular, in a case where both $\langle P_i\rangle$ and $\langle X_i\rangle$ are equal to zero, the uncertainty relation takes the following form:

$$(\Delta X)_i(\Delta P)_j \geqslant \frac{\hbar}{2}(\delta_{ij} + \alpha(\Delta X)_i(\Delta X)_j + \beta(\Delta P)_i(\Delta P)_j + \sqrt{\alpha\beta}((\Delta P)_i(\Delta X)_j + (\Delta X)_i(\Delta P)_j)). \tag{7}$$

It is worth highlighting that in the cases where $\alpha, \beta < 0$ (i.e., aSdS), the concept of minimal uncertainties does not arise, and all real values of p_i remain permissible. Before we to go to the details which are in the next subsequent section, it is important to note a change in the definition of the scalar product. As seems that the operators of $\hat{X}_i$ and $\hat{P}_i$ are symmetric only in subspace $L^2(R^2, d\vec{p}/\sqrt{1-\beta\mathbf{p}^2})$), we adopt the following form as presented in [10],

$$\langle\psi|\phi\rangle = \int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \frac{d\vec{p}}{\sqrt{1-\beta\mathbf{p}^2}} \psi^*(p)\phi(p), \tag{8}$$

where the wave-functions satisfied the periodic boundary conditions, $\psi(-1/\sqrt{\beta}) = \psi(1/\sqrt{\beta})$, and thus the modified closure relation is given by [29],

$$\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \frac{d\vec{p}}{\sqrt{1-\beta\mathbf{p}^2}} |\vec{p}\rangle\langle\vec{p}| = 1. \tag{9}$$

Now, by employing the closure relation for the maximally localized states, we derive the following expression [29]:

$$\langle\vec{p}|\vec{p}\,'\rangle_{\alpha,\beta} = \left(\frac{1-\beta\mathbf{p}'^2}{1-\beta\mathbf{p}^2}\right)^{\frac{\gamma}{2}} \sqrt{1-\beta\mathbf{p}^2}\,\delta^2(\vec{p} - \vec{p}\,'), \text{ and } \gamma = i\kappa/\hbar\sqrt{\alpha\beta}. \tag{10}$$

It is important to notice that for $(\alpha < 0, \beta < 0)$, we modify the limits of integration in the above equation to all the space. In addition, where both α and β are equal to zero, we regain the standard projection relation, denoted by $\langle\vec{p}|\vec{p}\,'\rangle_{(\alpha,\beta)\to 0} = \delta^2(\vec{p} - \vec{p}\,')$. However, it is important to note that in the time component, there is no deformation observed in the time-momentum relationship.

$$\left\langle p_0|p_0'\right\rangle = \delta\left(p_0 - p_0'\right) = \int \frac{dt}{2\pi\hbar} e^{-\frac{i}{\hbar}t(p_0 - p_0')}. \tag{11}$$

As a result, the matrix elements of the operators $\mathcal{P}_i^2$, $(\hat{\mathcal{P}}_1\hat{\mathcal{X}}_2 - \hat{\mathcal{P}}_2\hat{\mathcal{X}}_1)$ and $\mathcal{X}_i^2$ are respectively given as follows,

$$\mathcal{P}_i^2\langle\vec{p}|\vec{p}\,'\rangle_{\alpha,\beta} = \langle\vec{p}|\vec{p}\,'\rangle_{\alpha,\beta}\left[\frac{p_i^2}{1-\beta p_i^2}\right], \tag{12}$$

and

$$\left(\hat{\mathcal{P}}_1\hat{\mathcal{X}}_2 - \hat{\mathcal{P}}_2\hat{\mathcal{X}}_1\right)\langle\vec{p}|\vec{p}\,'\rangle_{\alpha,\beta} = \langle\vec{p}|\vec{p}\,'\rangle_{\alpha,\beta}(p_y x - p_x y), \tag{13}$$

and

$$\hat{\mathcal{X}}_i^2\langle\vec{p}|\vec{p}\,'\rangle_{\alpha,\beta} = \langle\vec{p}|\vec{p}\,'\rangle_{\alpha,\beta}\left[-\gamma(\gamma-1)\frac{\hbar^2\beta^2 p_i^2}{1-\beta p_i^2} - \hbar^2 2\beta(\gamma-1) - 2\imath\hbar\beta\left(\gamma - \frac{3}{2}\right)(p_x x + p_y y) + \left(1-\beta p_i^2\right)x_i^2\right]. \tag{14}$$

This paper is organized as follows: In section 2, we are interested into the formulation of the path integral for spinorial particles within curved Snyder space-time. It is important to notice that this formulation is here achieved without relying on Grassmann variables as substantiated in [12, 30]. This approach is based on performing path integral calculation on the elements matrix of the Green function. A similar technique has been applied in previous studies [17, 31]. In section 3, the use of the polar coordinate transformation allows us to separate the angular part from the radial one. This separation led to the derivation of the Poschl–Teller radial propagator [13, 32]. Section 4 is dedicated to deriving the precise bound states and their associated energy eigenvalues. Under specific conditions when $m\bar{\omega} \to m\omega_c/2$ and $c \to V_F$, the behavior of the Dirac oscillator system in the presence of a uniform magnetic field within the SdS algebra closely resembles the dynamics of the monolayer Graphene problem within the same and this has been demonstrated in section 5. In section 6, we examine and discuss the special cases arising from these studies. Finally, the thermodynamic functions are tested and plotted for this system in section 7.

2. Path integral formulation of 2DDO in curved snyder space-time

We proceed to derive the Green function $\hat{S}$ for the problem of the 2D Dirac oscillator (DO) in the presence of the uniform magnetic field ($\vec{B} = \mathcal{B}\vec{k}$), which is given by the following equation [7],

$$(\hat{H} - \imath\hbar\partial_t)\hat{S} = \mathbb{I}. \tag{15}$$

Here, $\mathbb{I}$ is the unit matrix. In the absence of electromagnetic interaction, the Hamiltonian expression for the Dirac oscillator is as follows [7],

$$\hat{H} = c\vec{\alpha}.\left(\vec{\hat{P}} - \imath m\omega\beta\vec{\hat{X}}\right) + \beta mc^2, \tag{16}$$

here, the $\vec{\hat{P}}$ – momenta and $\vec{\hat{X}}$ – position operators are verified the equation (6). While c denotes the speed of light, m is the mass of the particle, and ω is the angular frequency of the oscillator. The $\vec{\alpha}$ and β matrices are represented by the σ_i – Pauli matrices as

$$\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \alpha_2 = \begin{pmatrix} 0 & -\imath \\ \imath & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{17}$$

Equation (15) reduces after applying minimal electromagnetic coupling in equation (16) [7]:

$$\left[\gamma^0\hat{P}_0 - \vec{\gamma}.(\vec{\hat{P}} - \frac{e}{c}\vec{A}) + \imath m\omega\gamma^0\vec{\gamma}.\vec{\hat{X}} - mc\right]\hat{S} = -\mathbb{I}. \tag{18}$$

The parameter $e = \mp|e|$ describes a particle with negative charge ($e = -|e|$) or positively ($e = |e|$) positive charge. The γ^μ – Dirac matrices are then represented by the Pauli matrices in the two dimensions

$$\gamma^0 = \sigma_3, \ \ \gamma^1 = \imath\sigma_2, \ \ \gamma^2 = -\imath\sigma_1. \tag{19}$$

Note that there is no deformation occurring in the time component ($\hat{P}_0 = \imath\hbar\partial_0 = \imath\hbar\partial/\partial ct$, $\hat{X}_0 = \hat{x}^0 \equiv ct$). The vector potential $\vec{A}$ is the potential of a constant magnetic field $\mathcal{B}$ and has the two components ($\vec{A} = \frac{\mathcal{B}}{2c}(-\hat{X}_2, \hat{X}_1)$). Thus, equation (18) is written as [7],

$$[\gamma^0\imath\hbar\partial/\partial t - c\gamma^1(\hat{P}_1 + m\bar{\omega}\hat{X}_2) - c\gamma^2(\hat{P}_2 - m\bar{\omega}\hat{X}_1) - mc^2]\hat{S} = -\mathbb{I}, \tag{20}$$

where $\bar{\omega}$ denotes to $\bar{\omega} = \omega \mp \omega_c/2$, with $\omega_c = \frac{|e|\mathcal{B}}{mc}$ is the cyclotron frequency. The formal solution of equation (20) is

$$\hat{S} = -[\mathcal{O}_-^D]^{-1} = -\mathcal{O}_+^D\left[\mathcal{O}_-^D\mathcal{O}_+^D\right]^{-1}, \tag{21}$$

with the operators $\mathcal{O}_\pm^D$ defined by

$$\mathcal{O}_\pm^D = [\gamma^0\imath\hbar\partial/\partial t - c\gamma^1(\hat{P}_1 + m\bar{\omega}\hat{X}_2) - c\gamma^2(\hat{P}_2 - m\bar{\omega}\hat{X}_1) \pm mc^2]. \tag{22}$$

According to the Schwinger proper-time method [33] and noting that $\hat{S} = -[\mathcal{O}_+^D][\mathcal{O}_-^D\mathcal{O}_+^D]^{-1}$, it is convenient to write the Green's matrix operator $\hat{S}$ as follows

$$\hat{S} = \left[\mathcal{O}_+^D\right]\hat{\mathcal{G}}, \tag{23}$$

where

$$\hat{\mathcal{G}} = \frac{\imath}{\hbar}\int_0^\infty d\lambda \exp\left(-\frac{\imath}{\hbar}\lambda\hat{\mathcal{H}}\right), \tag{24}$$

and here λ represents an even variable. As for $\hat{\mathcal{H}}$– operator, it is expressed by the following equation:

$$\begin{aligned}\hat{\mathcal{H}} = &-[\gamma^0\imath\hbar\partial/\partial t - c\gamma^1(\hat{P}_1 + m\bar{\omega}\hat{X}_2) - c\gamma^2(\hat{P}_2 - m\bar{\omega}\hat{X}_1) - mc^2] \\ &\times [\gamma^0\imath\hbar\partial/\partial t - c\gamma^1(\hat{P}_1 + m\bar{\omega}\hat{X}_2) - c\gamma^2(\hat{P}_2 - m\bar{\omega}\hat{X}_1) + mc^2].\end{aligned} \tag{25}$$

After simplifying, we will find

$$\begin{aligned}\hat{\mathcal{H}} = -\Big[&-\hbar^2\partial_t^2 - m^2c^4 - c^2\left(\hat{P}_1^2 + \hat{P}_2^2\right) - c^2(m\bar{\omega})^2\left(\hat{X}_1^2 + \hat{X}_2^2\right) \\ &- c^2m\bar{\omega}[(\hat{X}_2\hat{P}_1 + \hat{P}_1\hat{X}_2) - (\hat{X}_1\hat{P}_2 + \hat{P}_2\hat{X}_1)] \\ &+ c^2\gamma^1\gamma^2\{[\hat{P}_1, \hat{P}_2] + (m\bar{\omega})^2[\hat{X}_1, \hat{X}_2] + m\bar{\omega}[\hat{X}_2, \hat{P}_2] + m\bar{\omega}[\hat{X}_1, \hat{P}_1]\}\Big].\end{aligned} \tag{26}$$

Further, we have to write this Hamiltonian by the position and momentum operators which achieve the deformed quantum algebra introduced by Snyder and are based on the modified commutation relation defined in previous section (see, equation (5)) [10]. Performing the operators ($\hat{\mathcal{X}}_i$, $\hat{\mathcal{P}}_j$) on $\hat{\mathcal{H}}$– expression, as a result, equation (26) becomes

$$\begin{aligned}\hat{\mathcal{H}} = -\Bigg[&-\hbar^2\partial_t^2 - m^2c^4 - c^2\left((m\bar{\omega})^2 + \frac{\alpha}{\beta}\right)\left(\hat{\mathcal{X}}_1^2 + \hat{\mathcal{X}}_2^2\right) - 2c^2m\bar{\omega}\left(\hat{\mathcal{X}}_2\hat{\mathcal{P}}_1 - \hat{\mathcal{X}}_1\hat{\mathcal{P}}_2\right) \\ &- c^2\left((1-\kappa)^2 + \kappa^2\frac{\beta}{\alpha}(m\bar{\omega})^2\right)\left(\hat{\mathcal{P}}_1^2 + \hat{\mathcal{P}}_2^2\right) - \imath\hbar\gamma^1\gamma^2\hat{\mathsf{F}}(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i)\Bigg],\end{aligned} \tag{27}$$

with

$$\hat{\mathsf{F}}(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i) = c^2\beta\left(\frac{\alpha}{\beta} + (m\bar{\omega})^2\right)\left(\hat{\mathcal{X}}_2\hat{\mathcal{P}}_1 - \hat{\mathcal{X}}_1\hat{\mathcal{P}}_2\right) - c^2m\bar{\omega}\left(2 + \beta\left(\hat{\mathcal{P}}_1^2 + \hat{\mathcal{P}}_2^2\right)\right). \tag{28}$$

The equation above reveals that the term $(\hat{\mathcal{P}}_i\hat{\mathcal{X}}_i + \hat{\mathcal{P}}_i\hat{\mathcal{X}}_i)$ will be obviously absent according to the provided value of κ

$$\kappa = \frac{1}{1 + \frac{\beta}{\alpha}(m\bar{\omega})^2}. \tag{29}$$

The corresponding element matrix of $\hat{\mathcal{G}}$ in momentum representation is

$$\mathcal{G}(\vec{p}_b, \vec{p}_a, p_{0b}, p_{0a}) = -\frac{\imath}{\hbar}\int_0^\infty d\lambda \langle \vec{p}_b,, , p_{0b} \mid \exp\left(-\frac{\imath}{\hbar}\lambda\hat{\mathcal{H}}\right) \mid \vec{p}_a,, , p_{0a}\rangle. \tag{30}$$

Before, we start to construct the Green's function using path integral formalism, we must get rid of the matrices that do not settle with the formalism by making the following exponential matrix. Evidently, we can simplify it as

$$\exp\left(\lambda\gamma^1\gamma^2\hat{\mathsf{F}}\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i\right)\right) = \cos\left(\lambda\hat{\mathsf{F}}\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i\right)\right) + \gamma^1\gamma^2\sin\left(\lambda\hat{\mathsf{F}}\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i\right)\right), \tag{31}$$

and this is done taking into account the properties of Dirac's matrices $(\gamma^1\gamma^2)^2 = -1$. Hence, the equation (31) becomes in another form

$$\exp\left(\lambda\gamma^1\gamma^2\hat{\mathsf{F}}\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i\right)\right) = \frac{1}{2}\sum_{s=\pm1}[1 + \imath s\gamma^1\gamma^2]\exp\left(-\frac{\imath}{\hbar}s\lambda\hbar\hat{\mathsf{F}}\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i\right)\right). \tag{32}$$

As a result, equation (30) can be written as follow

$$\mathcal{G}(\vec{p}_b, \vec{p}_a, p_{0b}, p_{0a}) = \frac{\imath}{2\hbar}\sum_{s=\pm1}[1 + \imath s\gamma^1\gamma^2]\int_0^\infty d\lambda\langle\vec{p}_b, p_{0b} \mid \exp\left(-\frac{\imath}{\hbar}\hat{\mathcal{H}}^{(s)}\right) \mid \vec{p}_a, p_{0a}\rangle, \tag{33}$$

with

$$\begin{aligned}\hat{\mathcal{H}}^s = -\lambda\Bigg[&-\hbar^2\partial_t^2 - c^2\left(\frac{\alpha}{\beta} + (m\bar{\omega})^2\right)\left(\hat{\mathcal{X}}_1^2 + \hat{\mathcal{X}}_2^2\right) - \frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta} + (m\bar{\omega})^2}\left(\hat{\mathcal{P}}_1^2 + \hat{\mathcal{P}}_2^2\right)\\ &- 2c^2m\bar{\omega}\left(\hat{\mathcal{P}}_1\hat{\mathcal{X}}_2 - \hat{\mathcal{P}}_2\hat{\mathcal{X}}_1\right) - s\hbar\hat{\mathsf{F}}\left(\hat{\mathcal{X}}_i, \hat{\mathcal{P}}_i\right) - m^2c^4\Bigg].\end{aligned} \tag{34}$$

For the kernel of (33), we decompose the exponential $\exp(-\imath\lambda\hat{\mathcal{H}}^{(s)})$ into $(N+1)$ exponential $\exp(-\imath\varepsilon\hat{\mathcal{H}}^{(s)})$, with $\varepsilon = \tau_j - \tau_{j-1} = 1/(N+1)$. Then we insert N resolutions of identities (9) between each pair of infinitesimal operator $\exp(-\imath\varepsilon\hat{\mathcal{H}}^{(s)})$. Indeed we have [29],

$$\begin{aligned}\mathcal{G}(\vec{p}_b, \vec{p}_a, p_{0b}, p_{0a}) = &\frac{\imath}{2\hbar}\sum_{s=\pm1}[1 + \imath s\gamma^1\gamma^2]\lim_{N\longrightarrow\infty}\int_0^\infty d\lambda\prod_{j=1}^{N}\int\frac{dp_{0j}d\vec{p}_j}{\sqrt{1-\beta\vec{p}_j^2}}\prod_{j=1}^{N+1}\\ &\times\left\langle\vec{p}_j, p_{0j}\,\middle|\, e^{-\frac{\imath\varepsilon}{\hbar}\hat{\mathcal{H}}^{(s)}}\,\middle|\,\vec{p}_{j-1}, p_{0j-1}\right\rangle_{\alpha,\beta}.\end{aligned} \tag{35}$$

To go further, it is convenient to develop the exponential up to the first order of ε. Thus, we find

$$\begin{aligned}&\lim_{N\longrightarrow\infty,\varepsilon\to0}\left\langle\vec{p}_j, p_{0j}\,\middle|\, e^{-\frac{\imath\varepsilon}{\hbar}\hat{\mathcal{H}}^{(s)}}\,\middle|\,\vec{p}_{j-1}, p_{0j-1}\right\rangle_{\alpha,\beta}\\ &= \lim_{N\longrightarrow\infty,\varepsilon\to0}\left[\left\langle\vec{p}_j, p_{0j}|\vec{p}_{j-1}, p_{0j-1}\right\rangle_{\alpha,\beta} - \frac{\imath\varepsilon}{\hbar}\left\langle\vec{p}_j, p_{0j}\,\middle|\,\hat{\mathcal{H}}^{(s)}\,\middle|\,\vec{p}_{j-1}, p_{0j-1}\right\rangle_{\alpha,\beta}\right].\end{aligned} \tag{36}$$

After this stage for eliminating the Hamiltonian operator which represent in the (SdS)-framework, we inject all the operators $(\mathcal{X}_i^2, \mathcal{P}_i^2, \hat{\mathcal{P}}_1\hat{\mathcal{X}}_2 - \hat{\mathcal{P}}_2\hat{\mathcal{X}}_1)$ in the projection relation $\langle\vec{p}_j|(.)|\vec{p}_{j-1}\rangle_{\alpha,\beta}$ given in equation (10). Consequently, the expression $\mathcal{G}(\vec{p}_b, \vec{p}_a, p_{0b}, p_{0a})$ is transformed into the following path integral in phase-space

$$\mathcal{G}\left(\vec{p}_b, \vec{p}_a, p_{0b}, p_{0a}\right) = -\frac{\imath}{2\hbar}\lim_{\substack{N\to\infty\\ \varepsilon\to 0}}\sum_{s=\pm 1}[1 + \imath s\gamma^1\gamma^2]\int_0^\infty d\lambda \prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{dp_{0j}d\vec{p}_j}{\sqrt{1-\beta\vec{p}_j^2}}\right]$$
$$\times\prod_{j=1}^{N+1}\left[\left(\frac{1-\beta\vec{p}_{j-1}^2}{1-\beta\vec{p}_j^2}\right)^{\frac{\gamma}{2}}\sqrt{1-\beta\vec{p}_j^2}\int\frac{dx_j^\mu}{(2\pi\hbar)^3}\right]\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[-x_j^\mu\Delta p_{j_\mu} + \lambda\varepsilon\left[p_{0j}^2 - m^2c^4\right.\right.\right.$$
$$-c^2\left(\frac{\alpha}{\beta} + (m\varpi)^2\right)\left(-\gamma(\gamma-1)\frac{\hbar^2\beta^2\vec{p}_j^2}{1-\beta\vec{p}_j^2} - 2\beta\hbar^2(\gamma-1)\right.$$
$$+2\imath\hbar\beta\left(\gamma-\frac{3}{2}\right)\left(x_jp_{x_j} + y_jp_{y_j}\right) + \left.\left(1-\beta\vec{p}_j^2\right)\left(x_j^2+y_j^2\right)\right) + s\hbar c^2m\varpi\left(2 + \frac{\beta\vec{p}_j^2}{1-\beta\vec{p}_j^2}\right)$$
$$\left.\left.\left.-\frac{c^2(m\varpi)^2}{\frac{\alpha}{\beta}+(m\varpi)^2}\frac{\vec{p}_j^2}{1-\beta\vec{p}_j^2} + c^2\left(2m\varpi + s\hbar\beta\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)\right)\left(p_{y_j}x_j - p_{x_j}y_j\right)\right]\right]\right\}. \quad (37)$$

It is worth observing that all terms scaled by the $\gamma-$ parameter can be removed by utilizing the term $((1-\beta\vec{p}_{j-1}^2)/(1-\beta\vec{p}_j^2))^{\frac{\gamma}{2}}$. This can be understood through the subsequent analysis [29],

$$\ln\left(\frac{1-\beta\vec{p}_{j-1}^2}{1-\beta\vec{p}_j^2}\right)^{\gamma/2} = -\frac{\gamma}{2}\ln\left(\frac{1-\beta\vec{p}_j^2}{1-\beta\vec{p}_{j-1}^2}\right)$$
$$=\beta\gamma\frac{(p_{xj}\Delta p_{xj} + p_{yj}\Delta p_{yj})}{1-\beta\vec{p}_j^2} - \frac{2\imath\varepsilon}{\hbar}\gamma\beta c^2\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)$$
$$-\frac{\imath\varepsilon}{\hbar}2\beta^2\gamma c^2\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)\frac{\vec{p}_j^2}{1-\beta\vec{p}_j^2}. \quad (38)$$

Furthermore, after performing the multiple Gaussian integrations over (x, y, t), the Green function $\mathcal{G}(\vec{p}_b, \vec{p}_a, p_{0b}, p_{0a})$ will be transformed to the Lagrangian path integral representation as follows:

$$\mathcal{G}\left(\vec{p}_b, \vec{p}_a, p_{0b}, p_{0a}\right) = -\frac{\imath}{2\hbar}\lim_{N\longrightarrow\infty}\sum_{s=\pm 1}[1 + \imath s\gamma^1\gamma^2]\delta(p_{0b}-p_{0a})\int_0^\infty d\lambda\prod_{j=1}^{N}\left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}}\frac{d\vec{p}_j}{\sqrt{1-\beta\vec{p}_j^2}}\right]$$
$$\times\prod_{j=1}^{N+1}\left[\sqrt{1-\beta\vec{p}_j^2}\left(\frac{1}{(2\pi\hbar)}\sqrt{\frac{\pi}{\frac{\imath}{\hbar}\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)\left(1-\beta\vec{p}_j^2\right)}}\right)^2\right]$$
$$\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\left(\Delta\vec{p}_j\right)^2}{4\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)\left(1-\beta\vec{p}_j^2\right)} + \lambda\varepsilon\left(p_0^2 - m^2c^4\right)\right.\right.$$
$$-\frac{9}{4}\hbar^2\beta^2c^2\lambda\varepsilon\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)\frac{\vec{p}_j^2}{1-\beta\vec{p}_j^2} - \frac{3}{2}\imath\hbar\beta\frac{\left(p_{y_j}\Delta p_{y_j} + p_{x_j}\Delta p_{x_j}\right)}{\left(1-\beta\vec{p}_j^2\right)}$$
$$+\lambda\varepsilon\frac{c^2\left(2m\varpi + s\hbar\beta\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)\right)^2}{4\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)}\frac{\vec{p}_j^2}{1-\beta\vec{p}_j^2} - 2\beta c^2\hbar^2\lambda\varepsilon\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)$$
$$+\frac{2m\varpi + s\hbar\beta\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)}{2\left(\frac{\alpha}{\beta}+(m\varpi)^2\right)}\frac{\left(p_{x_j}\Delta p_{y_j} - p_{y_j}\Delta p_{x_j}\right)}{\left(1-\beta\vec{p}_j^2\right)}$$
$$\left.\left.-\lambda\varepsilon\frac{c^2(m\varpi)^2}{\frac{\alpha}{\beta}+(m\varpi)^2}\frac{\vec{p}_j^2}{1-\beta\vec{p}_j^2} + s\hbar c^2m\varpi\lambda\varepsilon\left(2 + \frac{\beta\vec{p}_j^2}{1-\beta\vec{p}_j^2}\right)\right]\right\}. \quad (39)$$

In the next section, we will successfully complete the calculations if we use 2D spherical coordinates. Because it is known the symmetries play a preponderant role in the preservation of the physical quantities of the system which requires us to seek the best way of taking them into account.

3. Green function in polar coordinates

Let us develop the above path integrals (39) in the relative polar coordinates (p_ρ, p_θ) where the two dimensions spherical coordinates for momentum variables $\vec{p}$ are defined by

$$p_x = p_\rho \cos(p_\theta), \;\; p_y = p_\rho \sin(p_\theta), \tag{40}$$

here $0 < p_\theta < \pi$ and $\vec{p}^2 = p_x^2 + p_y^2$. This leads to the transformation over measure term, kinetic term and the rest action terms [29],

$$\prod_{j=1}^{N} \left[\int_{-1/\sqrt{\beta}}^{1/\sqrt{\beta}} \frac{d\vec{p}_j}{\sqrt{1-\beta\vec{p}_j^2}}\right] = \prod_{j=1}^{N} \left[\int \frac{p_{\rho_j} dp_{\rho_j}}{\sqrt{1-\beta p_{\rho_j}^2}} dp_{\theta_j}\right]. \tag{41}$$

$$\left(\Delta\vec{p}_j\right)^2 = p_{\rho_j}^2 + p_{\rho_{j-1}}^2 - 2p_{\rho_j}p_{\rho_{j-1}}\cos\left(\Delta p_{\theta_j}\right). \tag{42}$$

$$\vec{p}_j \Delta\vec{p}_j = p_{\rho_j}\Delta p_{\rho_j} + p_{\rho_j}p_{\rho_{j-1}}\left(\frac{1}{2}\left(\Delta p_{\theta_j}\right)^2 + \ldots\right). \tag{43}$$

$$p_{x_j}\Delta p_{y_j} - p_{y_j}\Delta p_{x_j} = p_{\rho_j}p_{\rho_{j-1}}\sin\left(\Delta p_{\theta_j}\right), \tag{44}$$

where the correction $(\Delta p_{\theta_j})^2$ is determined from the kinetic energy term and is equal to [29],

$$\left(\Delta p_{\theta_j}\right)^2 \sim 2\imath\hbar\lambda\varepsilon c^2\left(\frac{\alpha}{\beta} + (m\bar{\omega})^2\right)\frac{1-\beta p_{\rho_j}^2}{p_{\rho_j}p_{\rho_{j-1}}}. \tag{45}$$

Then inserting this in equation (39), the Green function will transform as

$$\begin{aligned}
&\mathcal{G}\left(p_{\rho_b}, p_{\rho_a}, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right) = -\frac{\imath}{2\hbar}\lim_{N\longrightarrow\infty}\sum_{s=\pm1}[1 + \imath s\gamma^1\gamma^2]\delta(p_{0b} - p_{0a})\\
&\times\int_0^\infty d\lambda \prod_{j=1}^{N}\left[\int\frac{p_{\rho_j}dp_{\rho_j}}{\sqrt{1-\beta p_{\rho_j}^2}}dp_{\theta_j}\right]\prod_{j=1}^{N+1}\left[\frac{\sqrt{1-\beta p_{\rho_j}^2}}{4\pi\imath\hbar\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}\right]\\
&\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{p_{\rho_j}^2 + p_{\rho_{j-1}}^2 - 2p_{\rho_j}p_{\rho_{j-1}}\cos\left(\Delta p_{\theta_j}\right)}{4\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)} - \imath\hbar\frac{3}{2}\frac{\beta p_{\rho_j}\Delta p_{\rho_j}}{\left(1-\beta p_{\rho_j}^2\right)} + \lambda\varepsilon\left(p_0^2 - m^2c^4\right)\right.\right.\\
&-\frac{9}{4}\hbar^2\beta^2\lambda\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{p_{\rho_j}^2}{\left(1-\beta p_{\rho_j}^2\right)} - 2\beta\hbar^2\lambda\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\\
&+\frac{3}{2}\beta\hbar^2c^2\lambda\varepsilon\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right) + \lambda\varepsilon\frac{c^2\left(2m\bar{\omega} + s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right)^2 p_{\rho_j}^2}{4\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}\\
&+\frac{2m\bar{\omega} + s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)}{2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)}\frac{p_{\rho_j}p_{\rho_{j-1}}\sin\left(\Delta p_{\theta_j}\right)}{\left(1-\beta p_{\rho_j}^2\right)} + s\hbar c^2 m\bar{\omega}\lambda\varepsilon\left(2 + \frac{\beta p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\right)\\
&\left.\left.-\lambda\varepsilon\frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta}+(m\bar{\omega})^2}\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\right]\right\}.
\end{aligned} \tag{46}$$

The third term in kinetic energy together with the last term in the procedure indicates the possibility of an angle shift according to the following relation

$$p_{\theta_j} \to p_{\theta_j} + \tau_j\lambda c^2\left(2m\bar{\omega} + s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right), \tag{47}$$

here τ is the time physics. After this step to perform the path integral over angle p_{θ_j}, we will use the well-known relation [34]

$$\exp(a\cos p_\theta) = \sum_{\ell=-\infty}^{+\infty} I_\ell(a)\exp(\imath\ell p_\theta), \tag{48}$$

where $I_\ell(a)$ are the modified Bessel functions and after straightforward calculations, equation (46) can be written as

$$\mathcal{G}\left(p_{\rho_b}, p_{\rho_a}, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right) = -\frac{\imath}{2\hbar}\lim_{N\longrightarrow\infty}\sum_{s=\pm1}[1+\imath s\gamma^1\gamma^2]\delta(p_{0b}-p_{0a})\int_0^\infty d\lambda\prod_{j=1}^{N}\left[\int\frac{p_{\rho_j}dp_{\rho_j}}{\sqrt{1-\beta p_{\rho_j}^2}}\right]$$
$$\times\prod_{j=1}^{N+1}\left[\left(\frac{\sqrt{1-\beta p_{\rho_j}^2}}{4\pi\imath\hbar\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}\right)\sum_{\ell_j=-\infty}^{+\infty}I_{\ell_j}\left(-\frac{\imath}{\hbar}\frac{p_{\rho_j}p_{\rho_{j-1}}}{2\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}\right)\right]$$
$$\times\prod_{j=1}^{N}\left[\int dp_{\theta_j}\right]\prod_{j=1}^{N+1}\left[e^{\imath\ell_j\left(\Delta p_{\theta_j}+\lambda\varepsilon c^2\left(2m\bar{\omega}+s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right)\right)}\right]$$
$$\times\exp\left\{\frac{\imath}{\hbar}\left[\frac{p_{\rho_j}^2+p_{\rho_{j-1}}^2}{4\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}+\lambda\varepsilon\left(p_0^2-m^2c^4\right)-\frac{9}{4}\lambda\varepsilon\hbar^2c^2\beta^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\right.\right.$$
$$-\imath\hbar\frac{3}{2}\frac{\beta p_{\rho_j}\Delta p_{\rho_j}}{1-\beta p_j^2}+\frac{3}{2}\lambda\varepsilon\hbar^2c^2\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)-2\lambda\varepsilon\hbar^2c^2\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)$$
$$\left.\left.-\lambda\varepsilon\frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta}+(m\bar{\omega})^2}\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}+\lambda\varepsilon c^2s\hbar m\bar{\omega}\left(2+\frac{\beta p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\right)\right]\right\}, \tag{49}$$

where $I_\ell(z)$ are the modified Bessel functions. The $N-$ integrations over the $p_{\theta_j}-$ variables can now be performed and produce the N symbols of Kronecker [34],

$$\prod_{j=1}^{N}\left[\int_0^{2\pi}dp_{\theta_j}\right]\prod_{j=1}^{N+1}\left[e^{\imath\ell_j\Delta p_{\theta_j}}\right]=\prod_{j=1}^{N}\left(2\pi\delta_{\ell_j,\ell_{j+1}}\right)e^{\imath\ell_{N+1}p_{\theta_{N+1}}-\imath\ell_1p_{\theta_0}}. \tag{50}$$

These symbols can eliminate all the summations except one which is noted ℓ. We now define the radial time evolution amplitudes by the following expression with respect to the azimuthal quantum numbers ℓ [34]:

$$\mathcal{G}\left(p_{\rho_b}, p_{\rho_a}, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right)=\frac{1}{2\pi}\frac{1}{\sqrt{p_{\rho_b}p_{\rho_a}}}\sum_{\ell=-\infty}^{+\infty}e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\mathcal{G}_\ell\left(p_{\rho_b}, p_{\rho_a}; p_{0b}, p_{0a}\right), \tag{51}$$

with

$$\mathcal{G}_\ell\left(p_{\rho_b}, p_{\rho_a}; p_{0b}, p_{0a}\right)=-\frac{\imath}{2\hbar}\delta(p_{b0}-p_{a0})\sum_{s=\pm1}[1+\imath s\gamma^1\gamma^2]\lim_{N\longrightarrow\infty}\int_0^\infty d\lambda e^{\frac{\imath}{\hbar}\lambda(p_0^2-m^2c^4)}$$
$$\times\prod_{j=1}^{N}\int_{-1/\beta}^{1/\beta}\left[\frac{dp_{\rho_j}}{\sqrt{1-\beta p_{\rho_j}^2}}\right]\prod_{j=1}^{N+1}\left[4\pi\imath\hbar\lambda\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right]^{-1/2}$$
$$\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{\left(\Delta p_{\rho_j}\right)^2}{4\lambda\varepsilon c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\left(1-\beta p_{\rho_j}^2\right)}-\imath\hbar\frac{3}{2}\frac{\beta p_{\rho_j}\Delta p_{\rho_j}}{\left(1-\beta p_{\rho_j}^2\right)}\right.\right.$$
$$+\lambda\varepsilon\left(-\frac{9}{4}\hbar^2\beta^2c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}-\frac{c^2(m\bar{\omega})^2}{\frac{\alpha}{\beta}+(m\bar{\omega})^2}\frac{p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}\right.$$
$$+s\hbar c^2m\bar{\omega}\frac{\beta p_{\rho_j}^2}{1-\beta p_{\rho_j}^2}-\hbar^2c^2\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\frac{(\ell^2-1/4)\left(1-\beta p_{\rho_j}^2\right)}{p_{\rho_j}p_{\rho_{j-1}}}$$
$$\left.\left.\left.-\frac{c^2\hbar^2\beta}{2}\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)+2c^2s\hbar m\bar{\omega}+\hbar\ell c^2\left[2m\bar{\omega}+s\hbar\beta\left(\frac{\alpha}{\beta}+(m\bar{\omega})^2\right)\right]\right)\right]\right\}. \tag{52}$$

This is done taking into account the following relation

$$I_\ell(z)=e^z(2\pi z)^{-1/2}\tilde{I}_\ell(z). \tag{53}$$

The asymptotic equality of modified Bessel functions $\tilde{I}_\ell(z)=\exp\left(-\frac{\ell^2-1/4}{2z}\right)$ holds as $|z|\to\infty$, and $|\arg z|<0$ [34].

Due to the presence of term measure in this expression of the propagator (52), it assumes a more complicated form. For simplify this, we will use the point transformation method (see, [12]) where the Υ-point discretization interval is defined as

$$p_{\rho_j}^{(\delta)} = \Upsilon p_{\rho_j} + (1 - \Upsilon) p_{\rho_{j-1}}. \tag{54}$$

In accordance with [12, 13], this yields three quantum corrections: the momentum term measure $(dp_{\rho_j}\big/\sqrt{1 - \beta p_{\rho_j}^2})$, the kinetic energy term, and the second term in action (52). To further analyze these corrections, we expand them using the discretization interval of $p_{\rho_j}^{(\Upsilon)}$-points. Then, in order to recover the conventional kinetic term $(\frac{(\Delta x_j)^2}{4\varepsilon c^2\left(\frac{\alpha}{\beta} + (m\varpi)^2\right)})$, we employ a coordinate transformation $\sqrt{\beta}p_\rho = f(x)$. The choice of $f(x)$ is determined by the following condition:

$$df(x)\big/dx = \sqrt{1 - \beta p_\rho^2} \Rightarrow \sqrt{\beta}p_\rho = \sin x. \tag{55}$$

Following this, we proceed to formulate the kinetic energy and measurement terms using the discretization interval represented by Υ-points. This allows us to arrive at the total correction denoted as C_T

$$C_T = \imath\hbar\varepsilon c^2(\alpha + \beta(m\varpi)^2)\left[\frac{5}{4}\tan^2 x - (2\Upsilon^2 - \Upsilon - 1)\frac{1}{\cos^2 x}\right]. \tag{56}$$

By using the predetermined Υ-values outlined in [12, 13] (specifically, $\Upsilon = 0, 1/2$), C_T takes on the following form:

$$C_T = \imath\hbar\varepsilon\beta c^2\left(\frac{\alpha}{\beta} + (m\varpi)^2\right)\left[1 + \frac{9}{4}\tan^2 x\right]. \tag{57}$$

This transforms the radial function $\mathcal{G}_\ell(x_b, x_a; p_{0b}, p_{0a})$ into the form

$$\begin{aligned}\mathcal{G}_\ell(x_b, x_a; p_{0b}, p_{0a}) &= -\frac{\imath}{2\hbar}\delta(p_{b0} - p_{a0})\sum_{s=\pm1}[1 + \imath s\gamma^1\gamma^2]\\ &\times\int_0^\infty d\lambda\exp\left\{\frac{\imath\lambda}{\hbar}\left[p_0^2 - m^2c^4 + \frac{1}{2}\hbar^2c^2(\alpha + \beta(m\varpi)^2) + \hbar\ell c^2[2m\varpi + s\hbar(\alpha + \beta(m\varpi)^2)]\right.\right.\\ &\left.\left.+2c^2 s\hbar m\varpi + \hbar^2c^2(\alpha + \beta(m\varpi)^2)(\ell^2 - 1/4) - c^2\left[s\hbar m\varpi - \frac{(m\varpi)^2}{(\alpha + \beta(m\varpi)^2)}\right]\right]\right\}\mathcal{K}_\ell(x_b, x_a, \lambda).\end{aligned} \tag{58}$$

The kernel propagator $\mathcal{K}_\ell(x_b, x_a, \lambda)$ corresponds precisely to the path integral of a particle in the Poschl-Teller (PT) potential as presented in reference [32],

$$\begin{aligned}\mathcal{K}_\ell(x_b, x_a, \lambda) &= \lim N \longrightarrow \infty\prod_{j=1}^{N}\left[\int dx_j\right]\prod_{j=1}^{N+1}\left[4\pi\imath\hbar\varepsilon\beta c^2\left(\frac{\alpha}{\beta} + (m\varpi)^2\right)\right]^{-1/2}\\ &\times\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{(\Delta x_j)^2}{4\varepsilon c^2(\alpha + \beta(m\varpi)^2)} - \varepsilon\hbar^2c^2(\alpha + \beta(m\varpi)^2)\left[\frac{\frac{(m\varpi)^2}{\alpha + \beta(m\varpi)^2} - s\hbar m\varpi}{\hbar^2(\alpha + \beta(m\varpi)^2)}\frac{1}{\cos^2 x} + \frac{\ell^2 - 1/4}{\sin^2 x}\right]\right]\right\}.\end{aligned} \tag{59}$$

As reported in [32], the transition amplitude concerning the Poschl-Teller potential yields the following outcome:

$$\begin{aligned}K &= \lim_{N\longrightarrow\infty}\prod_{j=1}^{N}\int dx_j\prod_{j=1}^{N+1}\left(\sqrt{\frac{M}{2\pi\imath\hbar\varepsilon}}\right)\exp\left\{\frac{\imath}{\hbar}\sum_{j=1}^{N+1}\left[\frac{M}{2\varepsilon}(\Delta x_j)^2 - \varepsilon\frac{\hbar^2}{2M}\left[\frac{(\nu^2 - 1/4)}{\cos^2 x} + \frac{(\delta^2 - 1/4)}{\sin^2 x}\right]\right]\right\}\\ &= \sum_n\Phi_n(x_b)\Phi_n(x_a)\exp\left[-\frac{\imath}{\hbar}\left(\frac{\hbar^2}{2M}(\delta + \nu + 2n + 1)^2\right)\right],\end{aligned} \tag{60}$$

where

$$\begin{aligned}\Phi_n^{(\delta,\nu)}(x) &= \left[2(\delta + \nu + 2n + 1)\frac{n!\Gamma(\delta + \nu + n + 1)}{\Gamma(\delta + n + 1)\Gamma(\nu + n + 1)}\right]^{1/2}\\ &\times(\sin x)^{\delta+1/2}(\cos x)^{\nu+1/2}P_n^{(\delta,\nu)}(\cos 2x).\end{aligned} \tag{61}$$

By comparison we can identify both values of M, δ and ν, respectively as

$$M \equiv \frac{1}{2c^2(\alpha + \beta(m\varpi)^2)},\quad \delta = |\ell|, \tag{62}$$

and

$$\nu_s = \pm\left(\frac{1}{2} - s\frac{m\bar{\omega}/\hbar}{\alpha + \beta(m\bar{\omega})^2}\right). \quad (63)$$

By adhering to the condition of the generalized uncertainty principle outlined in the introduction, we accept the values

$$\nu_+ = -\frac{1}{2} + \frac{m\bar{\omega}}{\hbar(\alpha + \beta(m\bar{\omega})^2)}, \quad \nu_- = \frac{1}{2} + \frac{m\bar{\omega}}{\hbar(\alpha + \beta(m\bar{\omega})^2)}, \quad (64)$$

while rejecting other negative values. Consequently, it leads to the following result

$$\nu_{-s} = \nu_s + s, \quad \text{with } \nu_s = -\frac{s}{2} + \frac{m\bar{\omega}}{\hbar\bar{\theta}}, \quad (65)$$

with $\bar{\theta} = \alpha + \beta(m\bar{\omega})^2$. Thereafter according to the following $\gamma^\mu-$ properties, $\frac{1}{2}\sum_{s=\pm1}[1 + \imath s\gamma^1\gamma^2] = \sum_{s=\pm1}\chi_s\chi_s^+$ with $\chi_s^+ = \frac{1}{2}(1 + s \quad 1 - s)$. Equation (52) transforms into the following form

$$\begin{aligned} \mathcal{G}_\ell(x_b, x_a; p_{0b}, p_{0a}) = &\frac{1}{\hbar}\delta(p_{0b} - p_{0a})\sum_n\sum_{s=\pm1}\chi_s\chi_s^+\int d\lambda e^{\frac{\imath\lambda}{\hbar}(p_0^2 - \omega_{s,\ell,n}^2)} \\ &\times\left[2(|\ell| + \nu_s + 2n + 1)\frac{n!\Gamma(|\ell| + \nu_s + n + 1)}{\Gamma(|\ell| + n + 1)\Gamma(\nu_s + n + 1)}\right] \\ &\times(\sin x_b)^{|\ell|+1/2}(\cos x_b)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}(\cos 2x_b) \\ &\times(\sin x_a)^{|\ell|+1/2}(\cos x_a)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}(\cos 2x_a), \end{aligned} \quad (66)$$

with

$$\omega_{s,\ell,n}^2 = m^2c^4 + \hbar^2c^2\bar{\theta}[2n + 1 - s + |\ell| - \ell][|\ell| + \ell + 2\nu_s + 2n + 1 + s]. \quad (67)$$

In order to evaluate exactly the propagator expression, we will formulate the Fourier transformation of (66) with respect to p_{0b} and p_{0a} variables. After integration over λ at this step gives the expression,

$$\begin{aligned} \mathcal{G}\left(p_{\rho_b}, p_{\rho_a}, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right) = &\sum_n\sum_{s=\pm1}\sum_{\ell=-\infty}^{+\infty}\frac{e^{\imath\ell\left(p_{\theta_b} - p_{\theta_a}\right)}}{2\pi}\chi_s\chi_s^+\int\frac{dp_0}{2\pi\hbar}\frac{e^{-\frac{\imath}{\hbar}p_0(t_b - t_a)}}{p_0^2 - \omega_{s,\ell,n}^2} \\ &\times\left[2(|\ell| + \nu_s + 2n + 1)\frac{n!\Gamma(|\ell| + \nu_s + n + 1)}{\Gamma(|\ell| + n + 1)\Gamma(\nu_s + n + 1)}\right] \\ &\times(\sin x_b)^{|\ell|}(\cos x_b)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}(\cos 2x_b) \\ &\times(\sin x_a)^{|\ell|}(\cos x_a)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}(\cos 2x_a). \end{aligned} \quad (68)$$

By employing the residue theorem at the pole p_0 enables us to express

$$\int\frac{dp_0}{2\pi\hbar}\frac{e^{-\frac{\imath}{\hbar}p_0(t_b - t_a)}}{p_0^2 - \omega_{n,s,\ell}^2} = -\imath\sum_{\epsilon=\pm1}\frac{e^{-\frac{\imath}{\hbar}\epsilon E_{n,s,\ell}(t_b - t_a)}}{2E_{n,s,\ell}}\Theta(\epsilon(t_b - t_a)), \quad (69)$$

which has the poles

$$E_{n,s,\ell} = \pm\sqrt{m^2c^4 + 4\hbar^2c^2\bar{\theta}\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]\left[\frac{|\ell|}{2} + \frac{\ell}{2} + \nu_s + n + \frac{1}{2} + \frac{s}{2}\right]}. \quad (70)$$

This also validates the subsequent identity for any arbitrary function

$$\sum_{\epsilon=\pm1}f(\epsilon)\Theta(\epsilon(t_b - t_a)) = f(s)\Theta(s(t_b - t_a)) + f(-s)\Theta(-s(t_b - t_a)), \quad (71)$$

where $\Theta(x)$ is the Heaviside function. This leads to the following expression:

$$\begin{aligned} &\mathcal{G}\left(x_b, x_a, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right) = \imath\sum_n\sum_{s=\pm1}\sum_{\ell=-\infty}^{+\infty}\frac{e^{\imath\ell\left(p_{\theta_b} - p_{\theta_a}\right)}}{2\pi}\chi_s\chi_s^+\left\{\left[\frac{e^{-\frac{\imath}{\hbar}sE_{n,s}(t_b - t_a)}}{2E_{n,s}}\Theta(s(t_b - t_a))\right]\right. \\ &\times\left[2(|\ell| + \nu_s + 2n + 1)\frac{n!\Gamma(|\ell| + \nu_s + n + 1)}{\Gamma(|\ell| + n + 1)\Gamma(\nu_s + n + 1)}\right] \\ &\times(\sin x_b)^{|\ell|}(\cos x_b)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}(\cos 2x_b)(\sin x_a)^{|\ell|}(\cos x_a)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}(\cos 2x_a) \\ &+\left[\frac{e^{\imath sE_{n,s}(t_b - t_a)}}{2E_{n,s}}\Theta(-s(t_b - t_a))\right]\left[2(|\ell| + \nu_{-s} + 2n + 1)\frac{n!\Gamma(|\ell| + \nu_{-s} + n + 1)}{\Gamma(|\ell| + n + 1)\Gamma(\nu_{-s} + n + 1)}\right] \\ &\left.\times(\sin x_b)^{|\ell|}(\cos x_b)^{\nu_{-s}+1/2}P_n^{(|\ell|,\nu_{-s})}(\cos 2x_b)(\sin x_a)^{|\ell|}(\cos x_a)^{\nu_{-s}+1/2}P_n^{(|\ell|,\nu_{-s})}(\cos 2x_a)\right\}. \end{aligned} \quad (72)$$

Moreover, to unify the expression of energy between the terms $\Theta(s(t_b - t_a))$ and $\Theta(-s(t_b - t_a))$, we make the following change $(s \rightarrow -s)$ in the terms that multiplied by $\Theta(-s(t_b - t_a))$, these lead to

$$n \to n - s,\ |\ell| \to |\ell| + s,\ \nu_{-s} = \nu_s + s. \tag{73}$$

Hence, the representation of the Green function takes the form

$$\begin{aligned}\mathcal{G}\left(x_b, x_a, p_{\theta_b}, p_{\theta_a}; p_{0b}, p_{0a}\right) &= \imath\sum_{n}\sum_{s=\pm1}\sum_{\ell=-\infty}^{+\infty}\frac{e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}}{2\pi}\frac{e^{-\frac{\imath}{\hbar}sE_{n,s}(t_b-t_a)}}{2E_{n,s}}\Theta(s(t_b - t_a))\\ &\times\left\{\left[2(|\ell| + \nu_s + 2n + 1)\frac{n!\Gamma(|\ell| + \nu_s + n + 1)}{\Gamma(|\ell| + n + 1)\Gamma(\nu_s + n + 1)}\right]\mathsf{F}_n^{(|\ell|,\nu_s)}(x_b)\mathsf{F}_n^{(|\ell|,\nu_s)}(x_a)\chi_s\varkappa_s^{+}\right.\\ &\left.+\left[2(|\ell| + \nu_s + 2n + 1)\frac{(n - s)!\Gamma(|\ell| + \nu_s + s + n + 1)}{\Gamma(|\ell| + n + 1)\Gamma(\nu_s + n + 1)}\right]\mathsf{F}_{n-s}^{(|\ell|+s,\nu_s+s)}(x_b)\mathsf{F}_{n-s}^{(|\ell|+s,\nu_s+s)}(x_a)\chi_{-s}\varkappa_{-s}^{+}\right\},\end{aligned} \tag{74}$$

where

$$\mathsf{F}_n^{(|\ell|,\nu_s)}(x) = (u)^{|\ell|}(\upsilon)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}(1 - 2u^2). \tag{75}$$

$$\mathsf{F}_{n-s}^{(|\ell|+s,\nu_s+s)}(x) = (u)^{|\ell|+s}(\upsilon)^{\nu_s+s+1/2}P_{n-s}^{(|\ell|,\nu_s+s)}(1 - 2u^2), \tag{76}$$

and $u = \sin x$, $\upsilon = \cos x$.

In the following section, we provide a precise solution for our problem by benefiting from various symmetry properties of the propagator to calculate the normalized eigenspinors and their corresponding energy spectrum.

4. Eigenspinors and energy spectrum

To obtain an exact evaluation of the propagator expression, we apply the operator $[\mathcal{O}_+^D]_b$ to the function (72). Utilizing the provided relationships, we can then apply the operator $[\mathcal{O}_+^D]_b$ to $\chi_s\chi_s^+$, which is expressed as follows:

$$\left[\mathcal{O}_+^D\right]_b\chi_s\chi_s^+ = \left[\chi_s\chi_s^+(s\imath\hbar\,\partial_{tb} + mc^2) + \chi_{-s}\chi_s^+\{(s\hat{P}_{1b} + \imath\hat{P}_{2b}) + m\bar{\omega}(s\hat{X}_{2b} - \imath\hat{X}_{1b})\}\right]. \tag{77}$$

In polar coordinates is written as

$$[\mathcal{O}_+^D]_b\chi_s\chi_s^+ = \chi_s\chi_s^+(s\imath\hbar\,\partial_{tb} + mc^2) + \chi_{-s}\chi_s^+\left\{\begin{aligned}&\left[s\hbar\sqrt{\frac{\alpha}{\beta}}\sqrt{1 - \beta p_b^2}\,e^{\imath s p_{\theta_b}}\left[-\imath\frac{\partial}{\partial p_b} + \frac{s}{p_{\rho_b}}\frac{\partial}{\partial p_{\theta_b}}\right] + s\frac{(1-\kappa)pe^{\imath s p_{\theta_b}}}{\sqrt{1-\beta p_b^2}}\right]\\ &+m\bar{\omega}\left[\imath\hbar\sqrt{1 - \beta p_b^2}\,e^{\imath s p_{\theta_b}}\left[-\imath\frac{\partial}{\partial p_b} + \frac{s}{p_b}\frac{\partial}{\partial p_{\theta_b}}\right] - \imath\frac{\kappa\sqrt{\frac{\beta}{\alpha}}pe^{\imath s p_{\theta_b}}}{\sqrt{1-\beta p_b^2}}\right]\end{aligned}\right\}. \tag{78}$$

In conclusion, we arrive at the spectral decomposition of the Green function $S(p_b, p_a)$ as presented below:

$$\begin{aligned}S\left(\vec{p}_b, \vec{p}_a; t_b, t_a\right) &= \frac{\imath}{\sqrt{2\pi}}\sum_{s=\pm1}\sum_{n}\frac{e^{-\frac{\imath}{\hbar}sE_{n,s}(t_b-t_a)}}{2E_{n,s}}\Theta(s(t_b - t_a))\\ &\times\left\{\left[2(|\ell| + \nu_s + 2n + 1)\frac{n!\Gamma(|\ell| + \nu_s + n + 1)}{\Gamma(|\ell| + n + 1)\Gamma(\nu_s + n + 1)}\right]\mathsf{F}_n^{(\ell,\nu_s)}(x_a)\right.\\ &\times\left[\begin{aligned}&\chi_s\chi_s^+(E_{n,s,\ell} + mc^2) + e^{\imath s\theta_b}\chi_{-s}\chi_s^+\\ &\times\left\{\begin{aligned}&\left[s\hbar\sqrt{\frac{\alpha}{\beta}}\sqrt{1 - \beta p_{\rho_b}^2}\left[-\imath\frac{\partial}{\partial p_{\rho_b}} + \frac{s}{p_{\rho_b}}\frac{\partial}{\partial p_{\theta_b}}\right] + \frac{\frac{\beta}{\alpha}(m\bar{\omega})^2}{1 + \frac{\beta}{\alpha}(m\bar{\omega})^2}\frac{sp_{\rho_b}}{\sqrt{1-\beta p_{\rho_b}^2}}\right]\\ &+\imath m\bar{\omega}\left[\hbar\sqrt{1 - \beta p_{\rho_b}^2}\left[-\imath\frac{\partial}{\partial p_{\rho_b}} + \frac{s}{p_{\rho_b}}\frac{\partial}{\partial p_{\theta_b}}\right] - \frac{\sqrt{\frac{\beta}{\alpha}}}{1 + \frac{\beta}{\alpha}(m\bar{\omega})^2}\frac{p_{\rho_b}}{\sqrt{1-\beta p_{\rho_b}^2}}\right]\end{aligned}\right\}\end{aligned}\right]e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\mathsf{F}_n^{(\ell,\nu_s)}(x_b)\\ &+\left[2(|\ell| + \nu_s + 2n + 1)\frac{(n - s)!\Gamma(|\ell| + \nu_s + s + n + 1)}{\Gamma(|\ell| + n + 1)\Gamma(\nu_s + n + 1)}\right]\mathsf{F}_{n-s}^{(\ell+s,\nu_s+s)}(x_a)\\ &\times\left[\begin{aligned}&\chi_{-s}\chi_{-s}^+(-E_{n,s,\ell} + mc^2) + e^{-\imath s p_{\theta_b}}\chi_s\chi_{-s}^+\\ &\times\left\{\begin{aligned}&\left[-s\hbar\sqrt{\frac{\alpha}{\beta}}\sqrt{1 - \beta p_{\rho_b}^2}\left(-\imath\frac{\partial}{\partial p_{\rho_b}} - \frac{s}{p_{\rho_b}}\frac{\partial}{\partial p_{\theta_b}}\right) - s\frac{\frac{\beta}{\alpha}(m\bar{\omega})^2}{1 + \frac{\beta}{\alpha}(m\bar{\omega})^2}\frac{p_{\rho_b}}{\sqrt{1-\beta p_{\rho_b}^2}}\right]\\ &+m\bar{\omega}\left[\hbar\sqrt{1 - \beta p_{\rho_b}^2}\left(\frac{\partial}{\partial p_{\rho_b}} - \imath\frac{s}{p_{\rho_b}}\frac{\partial}{\partial p_{\theta_b}}\right) - \imath\frac{\sqrt{\frac{\beta}{\alpha}}}{1 + \frac{\beta}{\alpha}(m\bar{\omega})^2}\frac{p_{\rho_b}}{\sqrt{1-\beta p_{\rho_b}^2}}\right]\end{aligned}\right\}\end{aligned}\right]\\ &\left.\times e^{\imath(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}\mathsf{F}_{n-s}^{(\ell+s,\beta_s+s)}(x_b)\right\}.\end{aligned} \tag{79}$$

The Green function can be expressed through a straightforward calculation as outlined below

$$
\begin{aligned}
&S\left(\vec{p}_b, \vec{p}_a; t_b, t_a\right) = \frac{\imath\beta}{\sqrt{2\pi}}\sum_{s=\pm1}\sum_{n}\frac{e^{-\frac{\imath}{\hbar}sE_{n,s}(t_b-t_a)}}{2E_{n,s}}\Theta(s(t_b-t_a))\\
&\times\Bigg\{\left[2(|\ell|+\nu_s+2n+1)\frac{n!\Gamma(|\ell|+\nu_s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}\right]\\
&\times\left[\begin{array}{l}\chi_s\chi_s^{+}(E_{n,s,\ell}+mc^2)e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}+e^{\imath sp_{\theta_b}}e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_{-s}\chi_s^{+}\dfrac{\hbar\sqrt{\beta}}{\sqrt{1-\eta_b^2}}\\
\times\left\{\begin{array}{l}\imath\sqrt{\dfrac{\alpha}{\beta}}\left\{(1-s)[(|\ell|-\nu_s)+(|\ell|+\nu_s)\eta_b]P_n^{(\ell,\nu_s)}(\eta_b)+2s\left(1-\eta_b^2\right)\dfrac{dP_n^{(\ell,\nu_s)}(\eta_b)}{d\eta_b}\right\}\\
+m\varpi\left\{(1-s)[(|\ell|-\nu_s)+(|\ell|+\nu_s)\eta_b]P_n^{(\ell,\nu_s)}(\eta_b)-2\left(1-\eta_b^2\right)\dfrac{dP_n^{(\ell,\nu_s)}(\eta_b)}{d\eta_b}\right\}\end{array}\right.\end{array}\right]\\
&\times(\sin x_b)^{|\ell|}(\cos x_b)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}(\cos 2x_b)(\sin x_a)^{|\ell|}(\cos x_a)^{\nu_s+1/2}P_n^{(|\ell|,\nu_s)}(\cos 2x_a)\\
&+\left[2(|\ell|+\nu_s+2n+1)\frac{(n-s)!\Gamma(|\ell|+\nu_s+s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}\right]\\
&\times\left[\begin{array}{l}-\chi_{-s}\chi_{-s}^{+}(E_{n,s,\ell}-mc^2)e^{\imath(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}+e^{-\imath sp_{\theta_b}}e^{\imath(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_s\chi_{-s}^{+}\dfrac{\hbar\sqrt{\beta}}{\sqrt{\left(1-\eta_b^2\right)}}\\
\times\left\{\begin{array}{l}\imath\sqrt{\dfrac{\alpha}{\beta}}\left\{\begin{array}{l}(1+s)[(|\ell|+1)-(\nu_s+1)+((|\ell|+1)+(\nu_s+1))\eta_b]P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)\\-2s\left(1-\eta_b^2\right)\dfrac{d}{d\eta_b}P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)\end{array}\right\}\\
+m\varpi\left\{\begin{array}{l}(1+s)[(|\ell|+1)-(\nu_s+1)+((|\ell|+1)+(\nu_s+1))\eta_b]P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)\\-2\left(1-\eta_b^2\right)\dfrac{d}{d\eta_b}P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)\end{array}\right\}\end{array}\right.\end{array}\right]\\
&\times(\sin x_b)^{|\ell|+s}(\cos x_b)^{\nu_s+s+1/2}P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_b)(\sin x_a)^{|\ell|+s}(\cos x_a)^{\nu_s+s+1/2}P_{n-s}^{(|\ell|+s,\nu_s+s)}(\eta_a)\Bigg\}.
\end{aligned}
\tag{80}
$$

Taking help from the properties of Jacobi's polynomials, as elucidated in reference [35],

$$
\frac{dP_n^{(|\ell|,\nu_s)}(\eta)}{d\eta}=\frac{1}{2}\frac{\Gamma(n+|\ell|+\nu_s+2)}{\Gamma(n+|\ell|+\nu_s+1)}P_{n-1}^{(|\ell|+1,\nu_s+1)}(\eta),
\tag{81}
$$

and

$$
\begin{aligned}
(1-\eta)^{\alpha_1}(1+\eta)^{\beta_1}\frac{d}{d\eta}P_n^{(\alpha_1,\beta_1)}(\eta) &= -2(n+1)(1-\eta)^{\alpha_1-1}(1+\eta)^{\beta_1-1}P_{n+1}^{(\alpha_1-1,\beta_1-1)}(\eta)\\
&+(\alpha(1-\eta)^{\alpha_1-1}(1+\eta)^{\beta_1}-\beta_1(1-\eta)^{\alpha_1}(1+\eta)^{\beta_1-1})P_n^{(\alpha_1,\beta_1)}.
\end{aligned}
\tag{82}
$$

We can reformulate the Green's function in the following expression:

$$
\begin{aligned}
&S\left(\vec{p}_b, \vec{p}_a; t_b, t_a\right) = \frac{\imath\beta}{\sqrt{2\pi}}\sum_{s=\pm1}\sum_{n}\frac{e^{-\frac{\imath}{\hbar}sE_{n,s}(t_b-t_a)}}{2E_{n,s}}\Theta(s(t_b-t_a))\\
&\times\left\{\left[\begin{array}{l}\left[\dfrac{2(E_{n,s,\ell}+mc^2)n!(|\ell|+\nu_s+2n+1)\Gamma(|\ell|+\nu_s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}\right]F_n^{(|\ell|,\nu_s)}(\eta_b)F_n^{(\ell,\nu_s)}(\eta_a)e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_s\chi_s^{+}\\
+\left[\dfrac{2\hbar\sqrt{\beta}\left(\imath\sqrt{\frac{\alpha}{\beta}}-sm\varpi\right)\left(n-\frac{s}{2}+\frac{1}{2}\right)!}{n!}\dfrac{n!2(|\ell|+\nu_s+2n+1)\Gamma(|\ell|+\nu_s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}\dfrac{\Gamma\left(n+|\ell|+\nu_s+\frac{s}{2}+\frac{1}{2}+1\right)}{\Gamma(n+|\ell|+\nu_s+1)}\right]\\
\times F_{n-s}^{(\ell+s,\nu_s+s)}(\eta_b)F_n^{(\ell,\nu_s)}(\eta_a)e^{\imath sp_{\theta_b}}e^{\imath\ell\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_{-s}\chi_s^{+}\end{array}\right]\right.\\
&+\left.\left[\begin{array}{l}-\dfrac{2(E_{n,s,\ell}-mc^2)(n-s)!(|\ell|+\nu_s+2n+1)\Gamma(|\ell|+\nu_s+s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}F_{n-s}^{(\ell+s,\nu_s+s)}(\eta_b)F_{n-s}^{(\ell+s,\nu_s+s)}(\eta_a)e^{\imath(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_{-s}\chi_{-s}^{+}\\
+\left[\dfrac{2s\hbar\sqrt{\beta}\left(\imath\sqrt{\frac{\alpha}{\beta}}+sm\varpi\right)\left(n-\frac{s}{2}+\frac{1}{2}\right)!}{(n-s)!}\dfrac{2(n-s)!(|\ell|+\nu_s+2n+1)\Gamma(|\ell|+\nu_s+s+n+1)}{\Gamma(|\ell|+n+1)\Gamma(\nu_s+n+1)}\dfrac{\Gamma(n+|\ell|+\nu_{-s}+1)}{\Gamma\left(n+|\ell|+\nu_{-s}-\frac{s}{2}+\frac{1}{2}\right)}\right]\\
\times F_n^{(\ell,\nu_s)}(\eta_b)F_{n-s}^{(\ell+s,\nu_s+s)}(\eta_a)e^{-\imath sp_{\theta_b}}e^{\imath(\ell+s)\left(p_{\theta_b}-p_{\theta_a}\right)}\chi_s\chi_{-s}^{+}\end{array}\right]\right\}.
\end{aligned}
\tag{83}
$$

By utilizing the symmetry properties of the propagator, we can express it in compressed form:

$$S^c\left(\vec{p}_b, \vec{p}_a; t_b, t_a\right) = \imath\sum_{n=0}^{\infty}\sum_{\ell}\sum_{s=\pm1} \times\left[s\Phi^s_{n,\ell}\left(p_{\rho_b}, p_{\theta_b}; t_b\right)\left(\Phi^s_{n,\ell}\left(p_{\rho_a}, p_{\theta_a}; t_a\right)\right)^{\dagger}\sigma_3 e^{-\frac{\imath}{\hbar}sE_{n,s,\ell}(t_b-t_a)}\Theta(s(t_b - t_a))\right]. \quad (84)$$

From this perspective, we can conclude that the normalized eigenspinors of our system are defined by:

$$\Phi^s_{n,\ell}(p_\rho, p_\theta) = \frac{\sqrt{\beta}}{\sqrt{2E_{n,s,\ell}}}\left\{\sqrt{2(E_{n,s,\ell} + mc^2)\frac{n!(\ell+\nu_s+2n+1)\Gamma(\ell+\nu_s+n+1)}{\Gamma(\ell+n+1)\Gamma(\nu_s+n+1)}}\mathsf{F}_n^{(\ell,\nu_s)}(\eta)e^{\imath\ell p_\theta}\chi_s + \sqrt{-2(E_{n,s,\ell} - mc^2)\frac{(n-s)!(\ell+\nu_s+2n+1)\Gamma(\ell+\nu_s+s+n+1)}{\Gamma(\ell+n+1)\Gamma(\nu_s+n+1)}}\mathsf{F}_{n-s}^{(\ell+s,\nu_s+s)}(\eta)e^{-\imath(\ell+s)p_\theta}\chi_{-s}\right\}, \quad (85)$$

and we can return to the old variables by means of the following relations $\eta = \cos 2x = 1 - 2\beta p_\rho^2$.

While the spectral energies remain as the pole expressions given in equation (70), it is observed that for $\omega = 0$, we can substitute $m\bar{\omega}$ with $(e\mathcal{B}/2c)$and $\bar{\theta}$ with $(\alpha + \beta(e\mathcal{B}/2c)^2)$ in equation (70). Meanwhile, as $\mathcal{B}$ approaches 0, we obtain the following result:

$$E^{\pm}_{s,\ell,n} = \pm\sqrt{m^2c^4 + 4\hbar^2c^2\alpha\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]\left[n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right]}. \quad (86)$$

This finding underscores that, in the context of the Snyder (anti)-de Sitter model, the energy levels still exhibit a dependence on n^2 even in the absence of $\omega -$ oscillation and magnetic fields $\mathcal{B}$. Typically, in deformation theory, the values of α and β are very small. Therefore, we can perform a first-order expansion of (70) with respect to α and β. This leads us to the following result:

$$E_{s,\ell,n} = E^0_{s,\ell,n} + \bar{\theta}\Delta E^1_{s,\ell,n}. \quad (87)$$

Here, the first term represent the Landau levels of (2+1)-dimensional Dirac oscillator in the presence of homogeneous magnetic field without deformation (HUP),

$$E^0_{s,\ell,n} = \pm\sqrt{m^2c^4 + 4\hbar c^2 m\bar{\omega}\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]}, \quad (88)$$

and the second term is the quantum gravity correction.

$$\Delta E^1_{s,\ell,n} = 2\hbar^2c^2\frac{\left(n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right)\left(n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right)}{\sqrt{m^2c^4 + 4\hbar c^2 m\bar{\omega}\left(n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right)}}. \quad (89)$$

1-

In limit case $\alpha \to 0$, the expression of equation (70) reduces to that of the flat Snyder model,

$$E_{s,\ell,n} = E^0_{s,\ell,n} + \beta\Delta E^{\alpha=0}_{s,\ell,n}, \quad (90)$$

with

$$\Delta E^{\alpha=0}_{s,\ell,n} = 2\hbar^2c^2(m\bar{\omega})^2\frac{\left(n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right)\left(n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right)}{\sqrt{m^2c^4 + 4\hbar c^2 m\bar{\omega}\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]}}. \quad (91)$$

1. In limit case $\beta \to 0$, one recovers the spectral energies for the Heisenberg algebra in an (anti-)de Sitter background [11],

$$E_{s,\ell,n} = E^0_{s,\ell,n} + \alpha\Delta E^{\beta=0}_{s,\ell,n}, \quad (92)$$

with

$$\Delta E^{\beta=0}_{s,\ell,n} = 2\hbar^2c^2\frac{\left(n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right)\left(n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right)}{\sqrt{m^2c^4 + 4\hbar c^2 m\bar{\omega}\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]}}. \quad (93)$$

Using natural units ($\hbar = c = 1$), we compute the conventional energy eigenvalues of the DO and the corrections introduced within in the context of SdS model for a single electron. This calculation is carried out

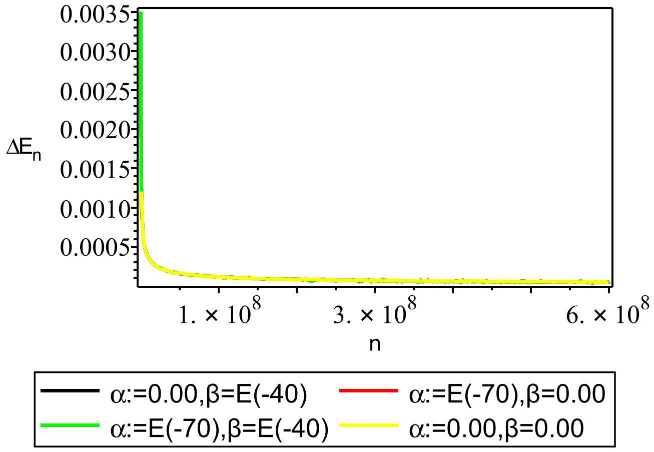


Figure 1. The energy spacing between adjacent levels as a function of n and $\ell = 0$ with $s = 1$.

Table 1. The ordinary energy eigenvalues and the corrected ones of the 2D Dirac oscillator in the presence of homogenuse magnetic field (in MeV) for a single electron at different values of n and with $s = +1$.

state n	ℓ	$E^0_{s,\ell,n}$	$\Delta E^1_{s,\ell,n} \times (10^{-70} + 0.9 \times 10^{-43})$	$\Delta E^{\alpha=0}_{s,\ell,n} \times (10^{-70})$	$\Delta E^{\beta=0}_{s,\ell,n} \times (10^{-40})$
0	0	0.510 999	0	0	0
1	−1	0.709 591	8.455 57	0.007 763	8.455 57
	0	0.618 32	4.851 854	0.004 454	4.851 854
	1	0.618 32	8.086 424	0.007 424	8.086 424
2	−2	0.863 667	23.157 084	0.021 26	23.157 084
	−1	0.790 392	18.977 918	0.017 423	18.977 918
	0	0.709 591	14.092 617	0.012 938	14.092 617
	1	0.709 591	19.729 664	0.018 114	19.729 664
	2	0.709 591	25.366 71	0.023 289	25.366 71
3	−3	0.994 143	42.2474	0.038 787	42.247 45
	−2	0.931 193	37.586 202	0.034 508	37.586 202
	−1	0.863 667	32.419 918	0.029 764	32.419 918
	0	0.790 392	26.569 085	0.024 393	26.569 085
	1	0.790 392	34.160 252	0.031 362	34.160 252
	2	0.790 392	41.751 419	0.038 332	41.751 419
	3	0.790 392	49.342 586	0.045 301	49.342 586

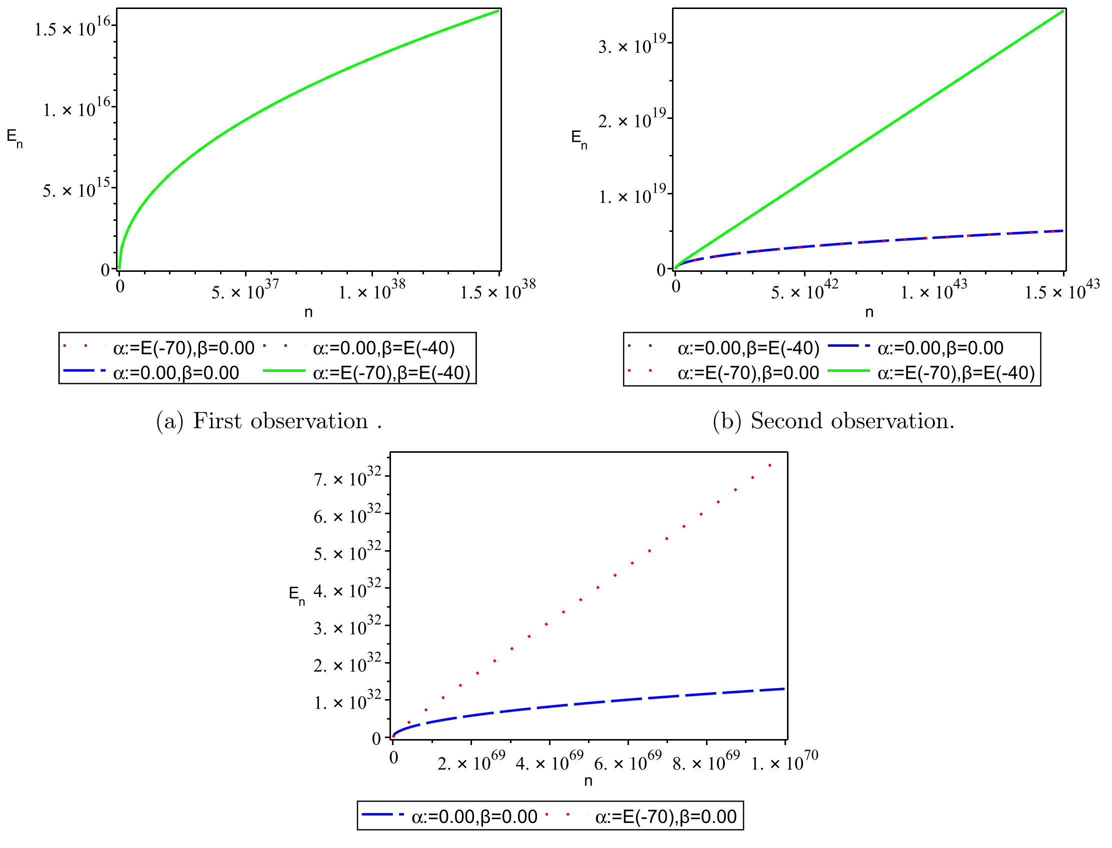
using equations (88), (89), (91) and (93), with $\alpha = 10^{-70}$ and $\beta = 10^{-40}$, $m = 0.5MeV$, and $m\omega = 1MeV^2$, while considering the case $s = +1$. Thus, table 1 displays the specific energy spectrum values corresponding to various combinations of n and ℓ. It is worth mentioning that the ground energy values in table 1 remain unaltered due to the SdS model.

Moreover, we can observe that the spacing of the energy levels gives a stable result in the following figure 1

Comparable outcomes were achieved for the two-dimensional Dirac Oscillator (2D-DO) in the presence of the Extended Uncertainty Principle (EUP) [17]. Additionally, similar results were obtained in the case of the one-dimensional DO within anti-de Sitter space [15] and in scenarios involving minimal lengths [14]. It is evident that in the absence of the SdS (Snyder-de Sitter) algebra, the energy level spacing for the 2D-DO is zero. This implies that in regular space, energy levels tend to become continuous for large values of n, while the deformation coefficient continues to maintain the separation of energy levels.

5. Massless dirac particle in graphene layer

In this context, we are considering massless Dirac fermions within a Graphene layer situated for the SdS mode and subjected to an external uniform magnetic field. We obtain the expressions for energy and wave functions by setting $m\bar{\omega} \to m\omega_c/2$ and $c \to V_F$ in equations (70) and (85). Thus, the resulting energy spectra and corresponding eigenspinors are given by:



(a) First observation .

(b) Second observation.

(c) Third observation.

Figure 2. $E_{n,\alpha,\beta}$ versus the quantum number n for different values of the deformation parameters.

$$E_{s,\ell,n} = \pm 2\hbar V_F \sqrt{\alpha + \beta\left(\frac{e\mathcal{B}}{2c}\right)^2} \sqrt{\left(n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right)\left(n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2} + \frac{\frac{e\mathcal{B}}{2\hbar c}}{\alpha + \beta\left(\frac{e\mathcal{B}}{2c}\right)^2}\right)}, \quad (94)$$

and

$$\Phi^{s}_{n,\ell}(p_\rho, p_\theta) = \sqrt{\beta}\left\{\sqrt{\frac{n!(\ell + \nu_s + 2n + 1)\Gamma(\ell + \nu_s + n + 1)}{\Gamma(\ell + n + 1)\Gamma(\nu_s + n + 1)}}\, \mathrm{F}_n^{(\ell,\nu_s)}(\eta) e^{\imath \ell p_\theta} \chi_s \right.$$
$$\left. + \sqrt{\frac{(n-s)!(\ell + \nu_s + 2n + 1)\Gamma(\ell + \nu_s + s + n + 1)}{\Gamma(\ell + n + 1)\Gamma(\nu_s + n + 1)}}\, \mathrm{F}_{n-s}^{(\ell + s,\nu_s + s)}(\eta) e^{-\imath(\ell + s)p_\theta} \chi_{-s}\right\}. \quad (95)$$

These results is in accordance with those of Graphene in curved Snyder space as reported in reference [23]. It's important to highlight that these authors did not provide an exact solution due to the omission of calculating the wave function expressions. Moreover, we can generate plots illustrating the energy levels for a single electron using equations (94) , for $\alpha = 10^{-70}$, $\beta = 10^{-40}$, $V_F = 0.00373$, while considering the case $s = +1$.

We note that in figure 2(a), all cases of curves energy levels are the same when the quantum number principle n between 0 and 1.5×10^{38}. Then the curves of the two cases $((\alpha \neq 0, \beta \neq 0)$ and $(\alpha = 0, \beta \neq 0))$ separates from the two cases $((\alpha \neq 0, \beta = 0)$ and $(\alpha = 0, \beta = 0))$ when $n > 10^{41}$, which is shown in the figure 2(b). Whereas in figure 2(c), the plot of the state $(\alpha \neq 0, \beta = 0)$ is separated from state $(\alpha = 0, \beta = 0)$ if the quantum number $n > 10^{69}$.

6. Non-relativistic limit

To derive the energy levels in the non-relativistic limit for the 2D-DO within the framework of a uniform magnetic field and the anti-de Sitter space system, we take the limit as $mc^2 \to \infty$. Employing a second-order Taylor expansion of equation (70), we obtain the following result:

$$E_{s,\ell,n} = mc^2 + \hbar\bar{\omega}[2n + 1 - s + |\ell| - \ell] + 2\bar{\theta}(\hbar^2/m)\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]\left[n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right], \quad (96)$$

with mc^2 represents the rest energy of the particle, the second and third terms represent, respectively, the energy of the non-relativistic 2D harmonic oscillator of frequency $\bar{\omega}$ and the correction in the context of the curved Snyder space. In the non-relativistic limit, the normalized wave functions with spin 1/2 are given by

$$\Phi^s_{n,\ell}(p_\rho, p_\theta) = \sqrt{\beta}e^{\imath\ell p_\theta}\sqrt{\frac{n!(\ell + \nu_s + 2n + 1)\Gamma(\ell + \nu_s + n + 1)}{\Gamma(\ell + n + 1)\Gamma(\nu_s + n + 1)}} \times (\sqrt{\beta}p_\rho)^{|\ell|}\left(1 - \beta p_\rho^2\right)^{\nu_s + 1/2} P_n^{(|\ell|,\nu_s)}\left(1 - 2\beta p_\rho^2\right)\chi_s, \quad (97)$$

where we have used the following limits:

$$\lim_{m\to\infty}\frac{E_{n,s,\ell} + mc^2}{E_{n,s,\ell}} = 2, \quad \lim_{m\to\infty}\frac{E_{n,s,\ell} - mc^2}{E_{n,s,\ell}} = 0. \quad (98)$$

In limit case $\alpha \to 0$, equation (96) becomes as,

$$E_{s,\ell,n} = mc^2 + \hbar\bar{\omega}[2n + 1 - s + |\ell| - \ell] + 2\beta(m\bar{\omega})^2(\hbar^2/m)\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]\left[n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right]. \quad (99)$$

In limit case $\beta \to 0$, equation (96) transforms as,

$$E_{s,\ell,n} = mc^2 + \hbar\bar{\omega}[2n + 1 - s + |\ell| - \ell] + 2\alpha(\hbar^2/m)\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]\left[n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right]. \quad (100)$$

From equation (96) and in first order of (α, β), we can find the energy spectrum for a spinless nonrelativistic particle ($s = 0$) in the presence of an uniform magnetic field

$$E^{(NR)}_{n,s=0} = \hbar\bar{\omega}(2n + 1 + |\ell| - \ell) + 2\bar{\theta}(\hbar^2/m)\left[n + \frac{1}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]\left[n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right]. \quad (101)$$

The first and second terms in equation (101) represent respectively the energy level for a spinless non-relativistic oscillator of frequency ω particle interacting with a uniform magnetic field in usual quantum mechanics (HUP), and the relativistic correction both in the context of the modification of the Heisenberg algebra. Also, if we take the limit $\mathcal{B} \to 0$, equation (101) transforms to

$$E^{(NR)}_{n,s=0} = \hbar\omega[2n + 1 + |\ell| - \ell] + 2(\alpha + \beta(m\omega)^2)(\hbar^2/m) \left[n + \frac{1}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]\left[n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2}\right]. \quad (102)$$

Here, the first term is the energy level for a spinless non-relativistic oscillator of frequency ω particle in HUP, and the second is the first correction of deformation in non relativistic case.

7. Without deformation case

In order to obtain the ordinary case, we discuss the two limits:

1- **Limit** $\alpha \to 0, \beta \neq 0$:

To get the usual wave functions for the 2*D* Dirac oscillator in the presence of uniform magnetic field, we replace $\alpha = 0$, equation (85) becomes as

$$\Phi^s_{n,\ell}(p_\rho, p_\theta) = \frac{\sqrt{\beta}}{\sqrt{2E^{\alpha=0}_{n,s,\ell}}}\left\{\sqrt{2\left(E^{\alpha=0}_{n,s,\ell} + mc^2\right)\frac{n!\left(\ell + \nu^\beta_s + 2n + 1\right)\Gamma\left(\ell + \nu^\beta_s + n + 1\right)}{\Gamma(\ell + n + 1)\Gamma\left(\nu^\beta_s + n + 1\right)}}\,\mathsf{F}_n^{(\ell,\nu^\beta_s)}(\eta)e^{\imath\ell p_\theta}\chi_s + \sqrt{-2\left(E^{\alpha=0}_{n,s,\ell} - mc^2\right)\frac{(n-s)!\left(\ell + \nu^\beta_s + 2n + 1\right)\Gamma\left(\ell + \nu^\beta_s + s + n + 1\right)}{\Gamma(\ell + n + 1)\Gamma\left(\nu^\beta_s + n + 1\right)}}\,\mathsf{F}_{n-s}^{(\ell+s,\nu^\beta_s+s)}(\eta)e^{-\imath(\ell+s)p_\theta}\chi_{-s}\right\}, \quad (103)$$

where $E^{(\alpha=0)}_{n,\ell,s}$ is the energy spectrum for the Dirac oscillator in two dimensions and in Snyder space

$$E^{\alpha=0,\pm}_{s,\ell,n} = \pm\sqrt{m^2c^4 + 4\hbar^2c^2\beta(m\bar{\omega})^2\left[n + \frac{1}{2} - \frac{s}{2} + \frac{|\ell|}{2} - \frac{\ell}{2}\right]\left[n + \frac{1}{2} + \frac{|\ell|}{2} + \frac{\ell}{2} + \frac{m\bar{\omega}/\hbar}{\beta(m\bar{\omega})^2}\right]}. \quad (104)$$

While ν_s^β and $\mathsf{F}_n^{(\ell,\nu_s^\beta)}(\eta)$ are given by, respectively,

$$\nu_s^\beta = -\frac{s}{2} + \frac{1}{\beta\hbar m\bar{\omega}},\quad \mathsf{F}_n^{(\ell,\nu_s^\beta)}(\eta) = (u)^{|\ell|}(\upsilon)^{\nu_s^\beta+1/2}P_n^{(|\ell|,\nu_s^\beta)}(1 - 2u^2). \quad (105)$$

2- **Limit** $\beta \to 0, \alpha \to 0$:

In order to obtain the ordinary case, let us derive the spinorial wave functions in momentum space representation of the usual Dirac oscillator by putting $\beta \to 0$ and $\alpha \to 0$ (i.e., $\bar{\theta} \to 0$), and we can write ν_s^β in equation (105) as

$$\nu_s^\beta = \frac{1}{\beta\hbar m\bar{\omega}}. \quad (106)$$

Indeed using the [35] we have

$$L_n^\mu(x) = \lim_{\nu_s\to\infty} P_n^{(\mu,\nu_s)}\left(1 - \frac{2x}{\nu_s}\right),\quad \lim_{\bar{\mu}\to+\infty}\frac{\Gamma(\bar{\mu}+\mu)}{\Gamma(\bar{\mu})}e^{-\mu\ln(\bar{\mu})} = 1, \quad (107)$$

with $x = \frac{p_\rho^2}{\hbar m\bar{\omega}}$ and $\bar{\mu} = \nu_s + n + 1$, by observing that (to $\mathcal{O}(\beta)$)

$$\lim_{\alpha,\beta\to 0}\left(1 - \beta p_\rho^2\right)^{\frac{m\bar{\omega}/\hbar}{2(\beta(m\bar{\omega})^2)}} = e^{-\frac{p_\rho^2}{2\hbar m\bar{\omega}}}. \quad (108)$$

$L_k^\gamma(x)$ are Laguerre polynomials. Therefore, in the limit $\bar{\theta} \to 0$, the spinorial wave functions become

$$\lim_{\alpha\to0,\beta\to0}\Psi_{n,\ell,s}(p_\rho, p_\theta) = (-1)^n e^{\imath\ell p_\theta}\sqrt{\frac{m\bar{\omega}n!}{2\pi\Gamma(n+\ell+1)}}\sqrt{\frac{E^{(\bar{\theta}=0)}_{n,\ell,s}+m}{E^{(\bar{\theta}=0)}_{n,\ell,s}}}(p_\rho/\sqrt{m\bar{\omega}})^\ell e^{-\frac{p_\rho^2}{2m\bar{\omega}\hbar}}L_n^\ell\left(\frac{p_\rho^2}{m\bar{\omega}\hbar}\right)\chi_s$$
$$-(-1)^n\imath e^{\imath(\ell+s)p_\theta}\sqrt{\frac{m\bar{\omega}(n-s)!}{2\pi\Gamma(n+\ell+1)}}\sqrt{\frac{E^{(\bar{\theta}=0)}_{n,\ell,s}-m}{E^{(\bar{\theta}=0)}_{n,\ell,s}}}(p_\rho/\sqrt{m\bar{\omega}})^{\ell+s}e^{-\frac{p_\rho^2}{2m\bar{\omega}\hbar}}L_{n-s}^{\ell+s}\left(\frac{p_\rho^2}{m\bar{\omega}\hbar}\right)\chi_{-s}, \quad (109)$$

where $E^{(\bar{\theta}=0)}_{n,\ell,s}$ is the usual energy spectrum for the Dirac oscillator in two dimensions (see [4, 36, 37])

$$E^{(\bar{\theta}=0)}_{n,\ell,s} = \pm\sqrt{m^2 + 2m\bar{\omega}(2n + 1 - s + |\ell| - \ell)}. \quad (110)$$

In addition, in the absence of deformation parameters, the authors in [7] have formulated the same problem using the path integral method, deriving the same formula for the standard energy levels of the 2D Dirac oscillator as given in equation (110). However, the spinorial eigenfunction, as presented in equation (109), is discovered to be the Fourier transform of the eigenfunction computed in a previous paper.

8. Thermodynamic functions

Next, let us delve into the thermodynamic properties of a solitary electron engaged with the Dirac oscillator operating within the modified algebra outlined in equation (2). To compute these properties, our first task is to derive the partition function for this particular system. The partition function is given by the following expression

$$Z = \sum_{n=0}^{\infty} e^{-\bar{\beta}E_n}. \quad (111)$$

Here, we introduce the parameter $\bar{\beta} = 1/(k_B T)$, where k_B represents the Boltzmann constant, and T signifies the temperature of the system. In this context, the energy levels E_n are determined by equation (87). We specifically concentrate on the positive energy levels since for negative energies, the summation in equation (111) becomes divergent. Additionally, we consider $s = +1$ and $\ell = 0$,

$$Z = \sum_{n=0}^{\infty} \exp\left[-\bar{\beta}\sqrt{b+an} - \bar{\theta}\bar{\beta}\frac{2\hbar^2c^2\left(n^2+\frac{n}{2}\right)}{\sqrt{b+an}}\right]. \quad (112)$$

In the first-order approximation with respect to $\bar{\theta}$, we obtain the following expression for the partition function:

$$Z(T, \alpha) = Z^0(\bar{\beta}) + \bar{\theta}\Delta Z^{(1)}(\bar{\beta}), \quad (113)$$

where

$$Z^0(\bar{\beta}) = \sum_{n=0}^{\infty} e^{-\bar{\beta}\sqrt{b+an}}, \quad \Delta Z^{(1)}(\bar{\beta}) = -2\hbar^2\bar{\beta}c^2\sum_{n=0}^{\infty}\frac{(n^2+n/2)}{\sqrt{b+an}}e^{-\bar{\beta}\sqrt{b+an}}, \quad (114)$$

with $a = 4\hbar m\bar{\omega}c^2$, $b = m^2c^4$. We can evaluate the sums in (113) by applying the Euler-Mclaurin summation formula, we have

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2}f(0) + \int_0^{\infty} f(x)dx - \sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0), \quad (115)$$

where B_{2p} are the Bernoulli numbers, $B_2 = 1/6, B_4 = -1/30, \ldots$, and $f^{(2k-1)}(0)$ is the derivative of order $(2k-1)$ at $x = 0$

$$f(0) = e^{-\bar{\beta}mc^2}, \quad f^{(1)}(0) = -2\bar{\beta}\hbar\bar{\omega}e^{-\bar{\beta}mc^2} - 2\bar{\theta}\bar{\beta}\hbar^2c^2\frac{e^{-\bar{\beta}mc^2}}{2}, \quad (116)$$

$$f^{(3)}(0) = -8\bar{\beta}(\hbar\bar{\omega})^3\frac{(\bar{\beta}mc^2)^2 + 3\bar{\beta}mc^2 + 3}{m^2c^4}e^{-\bar{\beta}mc^2} \quad (117)$$

$$+2\bar{\theta}\bar{\beta}\hbar^2c^2\frac{6\hbar\bar{\omega}\left(2mc^2(\bar{\beta}mc^2+1) - \hbar\bar{\omega}\left(m^2c^4\bar{\beta}^2 + 3\bar{\beta}mc^2 + 3\right)\right)}{(mc^2)^3}e^{-\bar{\beta}mc^2}. \quad (118)$$

The integral over x in equation (113) is given by

$$\int_0^{\infty} f(x)dx = \frac{e^{-\bar{\beta}mc^2}(\bar{\beta}mc^2+1)}{2\bar{\beta}^2\hbar\bar{\omega}mc^2} - 2\bar{\theta}\hbar^2c^2\bar{\beta}\left[\frac{3e^{-\bar{\beta}mc^2}}{4(\hbar\bar{\omega}mc^2)^3\bar{\beta}^5} + \frac{3e^{-\bar{\beta}mc^2}}{4(\hbar\bar{\omega})^3m^2c^4\bar{\beta}^4}\right.$$
$$\left. + \frac{e^{-\bar{\beta}mc^2}}{2(\hbar\bar{\omega})^3mc^2\bar{\beta}^3} + \frac{e^{-\bar{\beta}mc^2}}{8(\hbar\bar{\omega}mc^2)^2\bar{\beta}^3} + \frac{e^{-\bar{\beta}mc^2}}{8(\hbar\bar{\omega})^2mc^2\bar{\beta}^2}\right]. \quad (119)$$

Thus, the partition function can be expressed as follows:

$$Z(T, \alpha) = \frac{1}{2}e^{-\bar{\beta}mc^2} + \frac{1+\bar{\beta}mc^2}{2\bar{\beta}^2\hbar\bar{\omega}mc^2}e^{-\bar{\beta}mc^2} - 2\bar{\theta}\hbar^2c^2\bar{\beta}\left[\frac{3}{4(\hbar\bar{\omega}mc^2)^3\bar{\beta}^5} + \frac{3}{4(\hbar\bar{\omega})^3m^2c^4\bar{\beta}^4}\right.$$
$$\left. + \frac{1}{2(\hbar\bar{\omega})^3mc^2\bar{\beta}^3} + \frac{1}{8(\hbar\bar{\omega}mc^2)^2\bar{\beta}^3} + \frac{1}{8(\hbar\bar{\omega})^2mc^2\bar{\beta}^2}\right]e^{-\bar{\beta}mc^2} - \sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0). \quad (120)$$

To compute the partition function, it is necessary to evaluate the sum presented in the expression above. In our particular case, this summation can only be accomplished through numerical techniques. Up to $k = 2$, the sum can be represented as follows:

$$\sum_{k=1}\frac{B_{2k}}{(2k-1)!}f^{(2k-1)}(0) = -\frac{\bar{\beta}}{6}(2\hbar\bar{\omega} + \bar{\theta}\hbar^2c^2)e^{-\bar{\beta}mc^2} - \frac{1}{180}\left[-8\bar{\beta}(\hbar\bar{\omega})^3\frac{(\bar{\beta}mc^2)^2 + 3\bar{\beta}mc^2 + 3}{m^2c^4}\right.$$
$$\left.+2\bar{\theta}\bar{\beta}\hbar^2c^2\frac{6\hbar\bar{\omega}\left(2mc^2(\bar{\beta}mc^2+1) - \hbar\bar{\omega}\left(m^2c^4\bar{\beta}^2 + 3\bar{\beta}mc^2 + 3\right)\right)}{(mc^2)^3}\right]e^{-\bar{\beta}mc^2}. \quad (121)$$

At high temperatures ($\bar{\beta} \ll 1$), it is important to note that all the terms within the sum presented in equation (121) exhibit positive powers of $\bar{\beta}$, and these terms are considerably smaller when compared to the remaining term in equation (120). As a result, we can safely omit the terms involving $\bar{\beta}^n$ and the terms that do not contain $\bar{\beta}$, leading to the following simplified form of the partition function:

$$Z(T, \alpha, \beta) \simeq \frac{(k_BT)^2}{2\hbar\bar{\omega}mc^2} - \bar{\theta}\left[\frac{3\hbar^2c^2(k_BT)^4}{2(\hbar\bar{\omega}mc^2)^3} + \frac{\hbar^2c^2(k_BT)^2}{4(\hbar\bar{\omega})^3mc^2} + \frac{\hbar^2c^2(k_BT)^2}{4(\hbar\bar{\omega})^2(mc^2)^2}\right]. \quad (122)$$

The first term in the partition function corresponds to the conventional two-dimensional Dirac oscillator in standard quantum mechanics. The terms involving represent the effects of spatial deformation due to the presence of SdS model. With the partition function established, we can derive various thermodynamic functions. For instance, the Helmholtz free energy of the 2D Dirac oscillator subject to a homogeneous magnetic field at high temperatures can be written as follows:

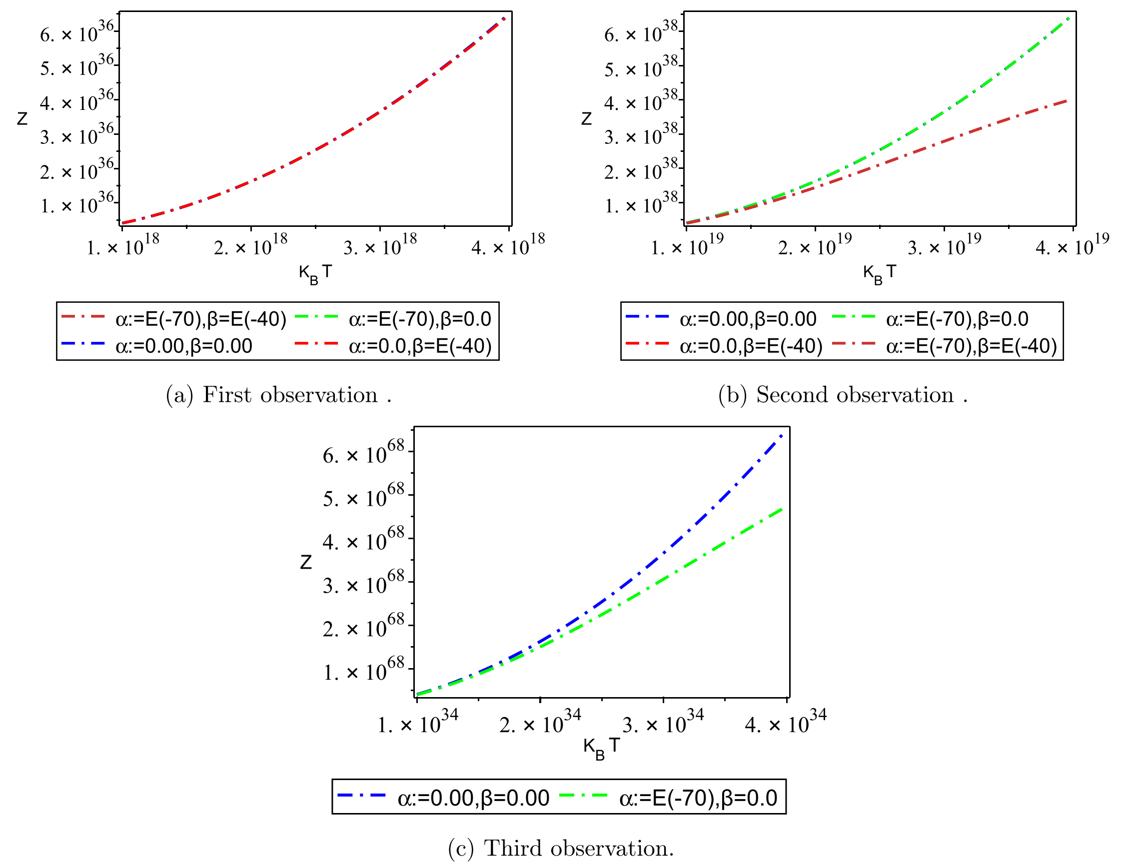


Figure 3. Partition function for the 2D-DO with uniform magnetic field as a function of temperature $k_B T$ for different values of the deformation parameters.

$$F(T,\,\alpha) = -T\ln(Z)$$
$$= -2T\ln\left(\frac{k_B T}{\sqrt{2\hbar\varpi mc^2}}\right) + \bar{\theta}\left[\frac{3\hbar^2c^2(k_B T)^3}{(\hbar\varpi)^2(mc^2)^2} + \frac{\hbar^2c^2T}{2(\hbar\varpi)^2} + \frac{\hbar^2c^2T}{2(\hbar\varpi)(mc^2)}\right]. \tag{123}$$

The connection between the mean energy and the partition function can be defined

$$\Xi(T,\,\alpha) = -\frac{\partial\ln(Z)}{\partial\bar{\beta}} = 2k_B T + 6\bar{\theta}\frac{\hbar^2c^2(k_B T)^3}{(\hbar\varpi)^2(mc^2)^2}. \tag{124}$$

As $\bar{\theta} \longrightarrow 0$, we regain the standard mean energy corresponding to the Heisenberg uncertainty principle (HUP) algebra.

For the heat capacity, we have

$$C(T,\,\alpha,\,\beta) = \frac{\partial\Xi}{\partial T} = 2k_B + 18\bar{\theta}\frac{\hbar^2c^2(k_B)^3T^2}{(\hbar\varpi)^2(mc^2)^2}. \tag{125}$$

In the limit $\bar{\theta} \longrightarrow 0$, which corresponds to the absence of the SdS algebra, the heat capacity remains constant, specifically $C = 2k_B$. However, in the presence of SdS algebra, it is evident that the heat capacity exhibits temperature-dependent variations due to the modification introduced by the standard Heisenberg algebra. Finally, the entropy is defined as

$$S(T,\,\alpha,\,\beta) = k_B\ln(Z) - k_B\bar{\beta}\frac{\partial\ln(Z)}{\partial\bar{\beta}} = S_0(\bar{\beta}) + \bar{\theta}\Delta S^1(\bar{\beta}). \tag{126}$$

Here, $S_0(\bar{\beta})$ represents the entropy for the $(2+1)-$dimensional Dirac oscillator under a uniform magnetic field within the HUP algebra. It is given by the following expression:

$$S_0(\bar{\beta}) = 2k_B + 2k_B\ln\left(\frac{k_B T}{\sqrt{2\hbar\varpi mc^2}}\right). \tag{127}$$

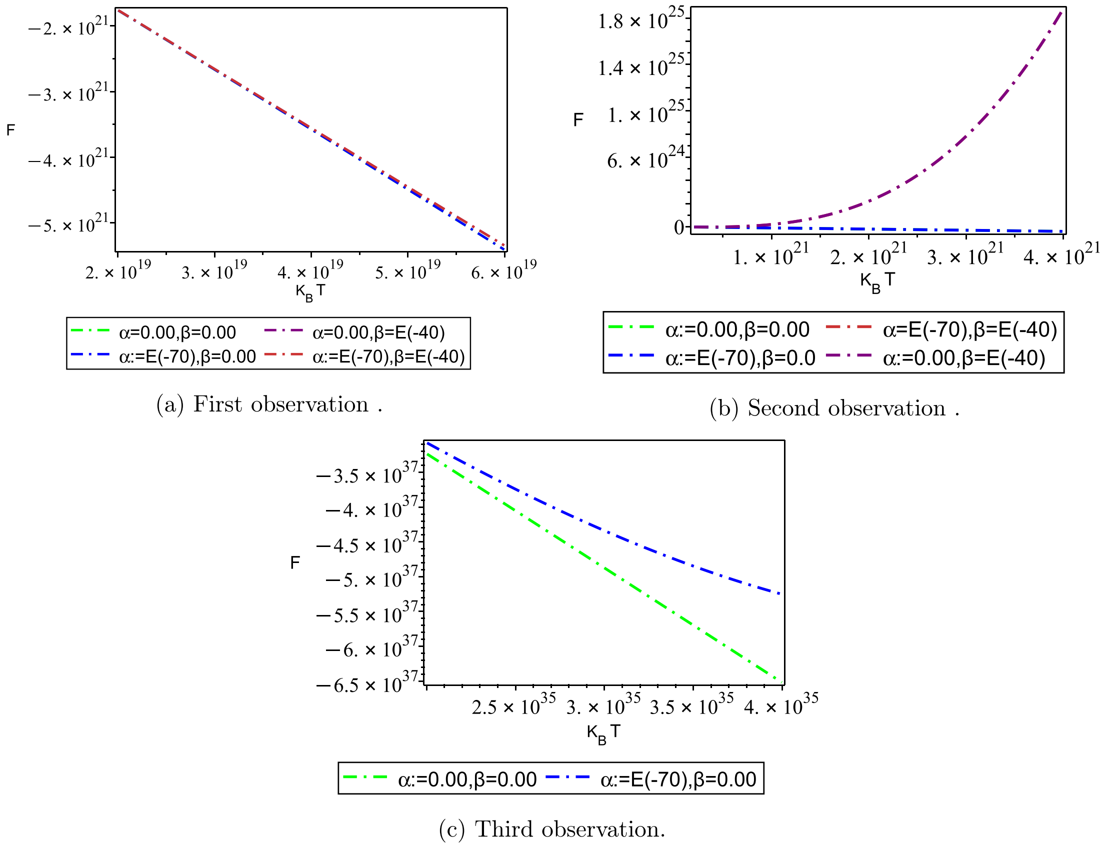


Figure 4. The Helmholtz free energy function for the 2D-DO with uniform magnetic field as a function of temperature $k_B T$ for different values of the deformation parameters.

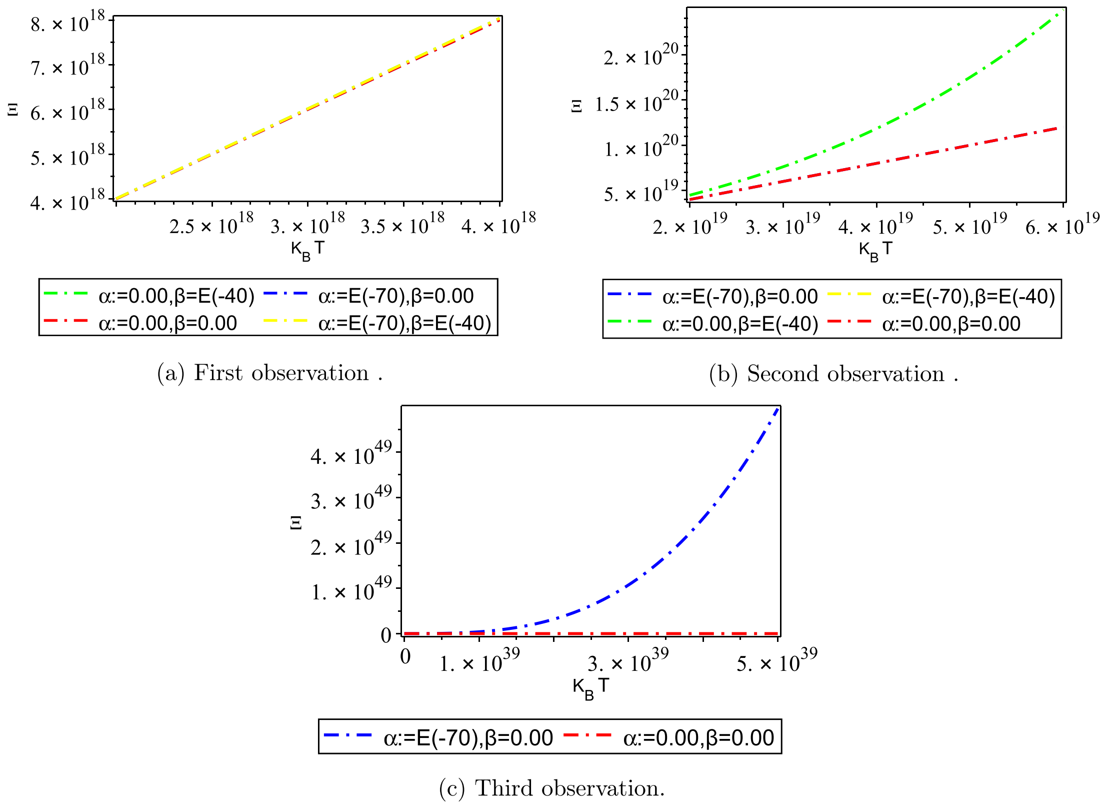
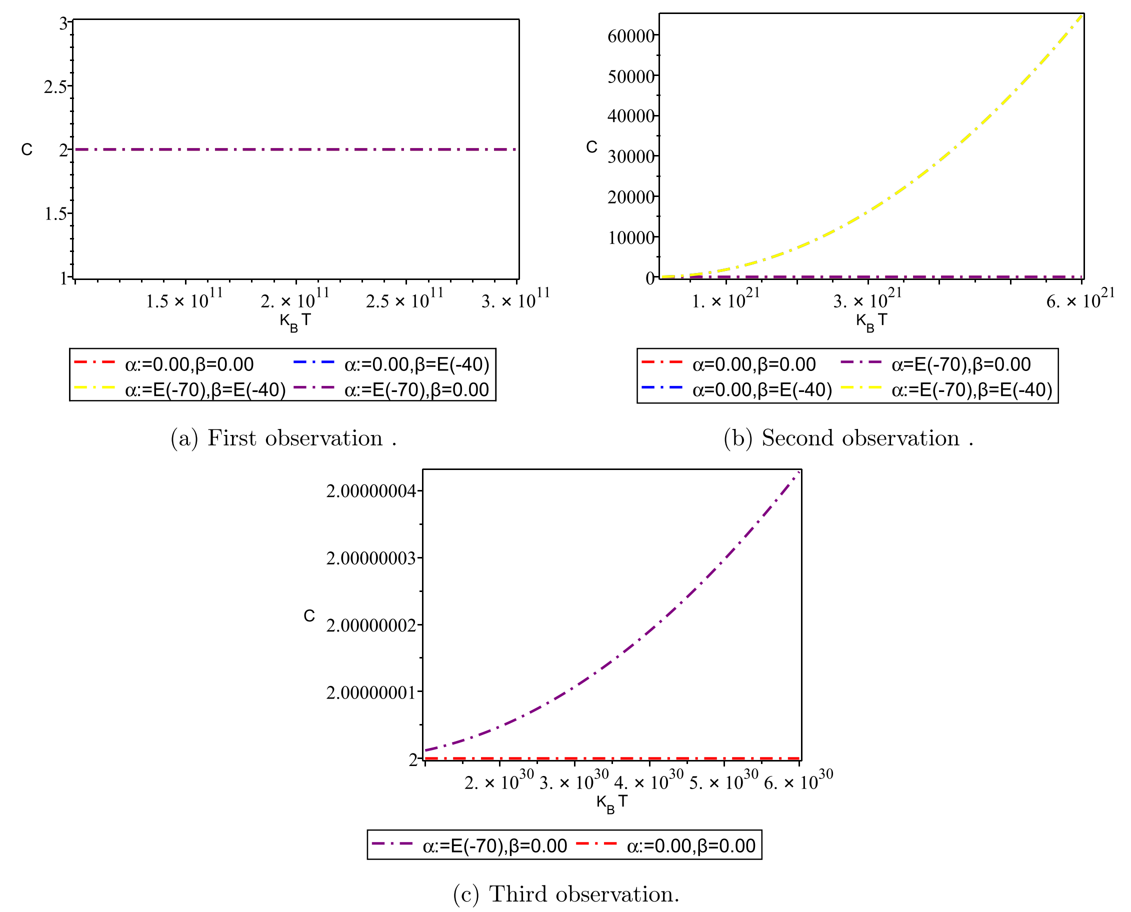


Figure 5. The mean energy function for the 2D-DO with uniform magnetic field as a function of temperature $k_B T$ for different values of the deformation parameters.



(a) First observation .

(b) Second observation .

(c) Third observation.

Figure 6. The heat capacity function for the 2D-DO with uniform magnetic field as a function of temperature $k_B T$ for different values of the deformation parameters.

Meanwhile, $\Delta S^1(\bar{\beta})$ denotes the first-order correction term for entropy in terms of (α, β), and it is expressed as,

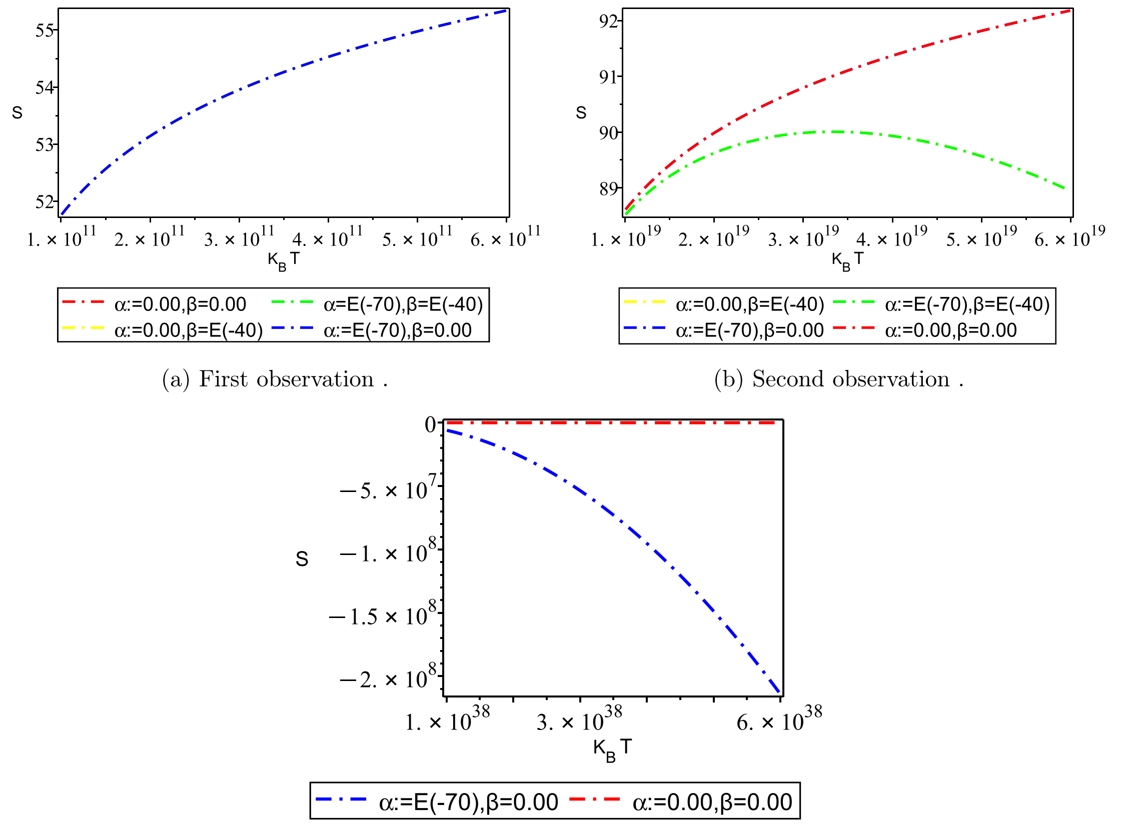
$$\Delta S^1(\bar{\beta}) = -k_B\left[\frac{9\hbar^2 c^2 (k_B T)^2}{(\hbar\varpi)^2 (mc^2)^2} + \frac{\hbar^2 c^2}{2(\hbar\varpi)^2} + \frac{\hbar^2 c^2}{2(\hbar\varpi)(mc^2)}\right]. \tag{128}$$

In the subsequent figures, we present a comparative analysis of the thermodynamic properties of our system across different deformation parameters. To facilitate this presentation, we adopt the natural unit system, where $\hbar$, c, and k_B are all set to 1, resulting in dimensionless parameters. To ensure accuracy, we have carefully chosen specific values for the relevant physical quantities. These values include the oscillator parameter at approximately 2 MeV within the high-temperature range, the electron mass set at $m = 0.5$ MeV, and an magnetic field $\mathcal{B}$ of 0.2 MeV^2. Consequently, we depict the thermodynamic properties in figures 3–7 as functions of temperature ($k_B T$). These figures showcase the behavior of these properties for four distinct sets of deformation parameters, specifically, $(\alpha = 10^{-70}, \beta = 10^{-40})$, $(\alpha = 0.0, \beta = 10^{-40})$, $(\alpha = 10^{-70}, \beta = 0.0)$ and $(\alpha = 0.0, \beta = 0.0)$.

Notably, the figure 3(a) demonstrates that the SdS algebra leads to a surge in the partition function from $k_B T = 1 \times 10^{19}$ to approximately $k_B T \sim 2.5 \times 10^{19}$ MeV. Subsequently, the curves 3(b) corresponding to $(\alpha \neq 0, \beta \neq 0)$ and $(\alpha = 0.0,\ \beta \neq 0)$ goes down to zero after the temperature $k_B T \sim 10^{20}$ MeV. However, the other two curves line up closely up to $k_B T\ \sim\ 5\ \times\ 10^{38}$ MeV, after which the curve for $(\alpha \neq 0, \beta = 0)$ collapses to zero when $k_B T$ surpasses 10^{35} MeV in figure 3(c).

In figure 4, we have the Helmholtz free energy for the one-dimensional Dirac oscillator within the SdS context as a function of $k_B T$ and this depiction indicates that the SdS algebra leads to a decline in the $F-$ function, spanning from $k_B T = 1 \times 10^{19}$ to $k_B T \sim 6 \times 10^{19} MeV$ across all four cases of deformation parameters in figure 4(a). Beyond $k_B T > 10^{39}$, the curves 4(b) for both $((\alpha \neq 0, \beta \neq 0)$ and $(\alpha = 0, \beta \neq 0))$ vanish when $\beta \neq 0$. Meanwhile, the case characterized by $((\alpha \neq 0, \beta = 0))$ has an effect up to temperature $k_B T > 10^{21} MeV$ in figure 4(c).

Furthermore, within the SdS model, the mean energy exhibits a growth as the temperature rises, as depicted in figure 5(a). In figure 5(b) it is shown that for the cases $(\alpha \neq 0, \beta \neq 0)$ and $(\alpha = 0, \beta \neq 0)$, the curves decline to



(a) First observation .

(b) Second observation .

(c) Third observation.

Figure 7. The entropy function for the 2D-DO with uniform magnetic field as a function of temperature k_BT for different values of the deformation parameters.

zero after reaching the temperature $k_BT \sim 10^{21} MeV$. However, for the case ($\alpha \neq 0, \beta = 0$), the curve 5(c) goes down to zero when k_BT surpasses 5×10^{39} MeV.

Also, the heat capacity in figure 6(a) is a constant $C = 2k_B$ when $k_BT < 10^{11}$. Then, when $k_BT > 10^{19}MeV$ the cases (($\alpha \neq 0, \beta \neq 0$) and ($\alpha = 0, \beta \neq 0$)) increase with the increasing temperature, which is presented in figure 6(b). The figure 6(c) shows the increasing of the capacity for the case ($\alpha \neq 0, \beta = 0$) with the increasing temperature at $k_BT > 10^{32}$.

Finally, in figure 7(a), we plot the effect of SdS model on entropy function in three images. According to the figure 7(b), the aSdS makes the values of entropy smaller with temperature for the cases (($\alpha \neq 0, \beta \neq 0$) and ($\alpha = 0, \beta \neq 0$)) at temperature $k_BT > 10^{19}$. Whereas, in figure 7(c), the entropy function of the case ($\alpha \neq 0$, $\beta = 0$) decreases with temperature $k_BT > 10^{38}$.

From the above figures, the effect SdS algebra on the thermodynamic functions have a more significant impact when the $\alpha -$ parameter is present compared to the β-parameter. Likewise, we can deduce the thermodynamic properties and appropriate curves for the aSdS model case simply by substituting (α and β) by ($-\alpha, -\beta$). Finally, when $\alpha = \beta = 0$ and magnetic field tends to zero ($\mathcal{B} \rightarrow 0$), our results are very accurate. The thermal properties of the three-dimensional Dirac oscillator without deformed commutation relation of the Heisenberg uncertainty principle are consider in [38]. Also we can recover all thermodynamic functions for the massless Dirac fermions in Graphene layer in a curved Snyder space when takes the limits $m\bar{\omega} \rightarrow m\omega_c/2$ and $c \rightarrow V_F$ (see [23]).

9. Conclusion

In this paper, we have investigated the behavior of the 2D Dirac oscillator subjected to a constant magnetic field, using the momentum space representation within the framework of the SdS model principle. In first, we introduced a novel model for the Green function that is applicable to the generalized SdS algebra. Subsequently, we straightforward integrate over even trajectories, leading to the precise calculation of the Green's function in polar coordinates. The passage to polar coordinates has facilitated the determination of the energy spectrum and the associated wave functions. It has been demonstrated that the SdS introduces a dependence of energies on both n and ℓ, even when oscillations and magnetic field are not present. This effect leads to the emergence of phenomena such as harmonic oscillation, anharmonic vibration, and confinement. Furthermore, our investigation has revealed that as n assumes large values, the energy level spacing remains constant, with the deformation parameter $\bar{\theta}$ effectively maintaining the separation between energy levels. The same observation was also made in the reference mentioned. [17]. In analysis we clarify that, under specific conditions when $m\bar{\omega} \rightarrow m\omega_c/2$ and $c \rightarrow V_F$, the behavior of the Dirac oscillator system in the presence of a uniform magnetic field within the SdS algebra closely resembles to the dynamics of the monolayer Graphene problem within the same algebraic framework. Furthermore, we have thoroughly examined all the distinct scenarios and special cases of the Dirac oscillator problem in the presence of a uniform magnetic field, using the framework of the SdS model.

Finally, when considering high temperatures, we applied the Euler-MacLaurin formula to compute various thermodynamic properties of our system up to the first order of (α, β). These properties include the partition function Z, the Helmholtz free energy F, the mean energy Ξ, the entropy S, and the heat capacity C. Through graphical representations of the SdS terms in these thermodynamic functions against temperature k_BT, we have illustrated that the influence of the α-deformation parameter is more significant than that of the β-parameter. It is important to note that, currently, these effects cannot be experimentally detected.

Data availability statement

The data cannot be made publicly available upon publication because no suitable repository exists for hosting data in this field of study. The data that support the findings of this study are available upon reasonable request from the authors.

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