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Mathematical and numerical analysis of the Euler-Bernoulli obstacle type problem

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Dedication

I am forever grateful to my father, "Abd Elhakim", for being an unwavering pillar of support, encouragement, and inspiration.

I also express my gratitude to my mother, "Akila", who has been a constant source of light and love in my life.

I also extend my heartfelt appreciation to all my family members espacially my siblings, "Fatima", "Mostafa", "Khaled", "Amira" and "Houssam", for being a part of my life and supporting me in all my endeavors.

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Contents

Introduction

Partial differential equations are used to simulate a wide range of phenomena in the physical and engineering disciplines. However, However, various more complex nonlinear problems can be described by variational inequalities (VIs), such as contact problems in mechanics, heat control problems in engineering, leak phenomena in fluid dynamics and in financial mathematics. The first problem involving VI was studied In 1933 by Signorini [Sig33], which describes the contact of a deformable elastic body with a rigid frictionless foundation. In [Fic64] (1964) the rigorous analysis of the Signorini problem was given by Fichera in the form of an inequality. Duvaut & Lions (1976) formulated and studied many problems in mechanics and physics in the framework of VIs [DL76].

One of these variational inequalities is the obstacle problem of Euler Bernoulli beam which is modeled by a fourth-order elliptic VI of the first kind. For these types of problem, the existence and uniqueness of the problem solution was studied in various works, for example by Stampacchia. However when it comes to finding analytical solutions, it is difficult to find them, and this is why numerical methods like finite elements, finite differences... etc, have great importance in applications.

For solving fourth-order elliptic problems the conforming FE spaces need to be contained in C^1, which what our work will focus on, particularly Hermite $\mathbb{P}_3$ finite elements. And instead of focusing on the classical formulation we'll study an alternative variational formulation based on Lagrange multipliers, which has the advantage of providing a physical value by being an approximation for the contact force and the unknown contact domain, plus it lends to leads naturally to use semi-smooth methods like primal dual active set (PDAS).

Our work will be composed of four chapters. In the first chapter we introduce the variational form of the obstacle problem of Euler Bernoulli beam, we apply asymptotic analysis on the three dimensional Signorini problem for a beam to derive and justify the model. This allows us to obtain a one-dimensional approximate model which includes the obstacle problem for Euler Bernoulli beam.

In the second chapter we'll first show the problem is well posed, and next we study the regularity of the solution, which we'll show that while the solution is in the space $C^2 \cap H^3$, but the full regularity or H^4 regularity isn't true in general.

In the third chapter we'll present the classical finite element formulation and we try to find a prior error estimate, but we'll find out that the lack of full regularity makes it very hard to derive error estimates. And that's why we introduce new continuous and discrete formulations more suitable for numerical analysis. The continuous formulation is a mixed formulation based on Lagrange multipliers, and the discrete formulations are stable mixed formulations which verify inf-sup condition, one of is the mixed formulation based on biorthogonal dual space, which will be very useful to implement PDAS method later. Plus we give priori and posteriori error estimates of the alternative formulation.

Finally in the fourth chapter, we introduce the two methods used to find the discrete solution: Uzawa Method and the primal dual active set method, and later we implement them in Freefem++ and we present numerical tests to validate the results.

Chapter 1

Asymptotic Modelling of the obstacle problem for Euler-Bernoulli beam

1.1 Introduction

The classical models of elastic beams, rods, and plates in solid mechanics are derived from a priori theories about displacements and/or stress fields [Via85]. These theories, when substituted in the constitutive and equilibrium equations of three-dimensional elasticity, result in helpful simplifications. However, most of the models derived this manner require mathematical justification for their validity from both a constitutive and geometric perspective. So in this chapter we introduce the obstacle problem for the classical Euler-Bernoulli beam, and then we motivate and justify it mathematically following the application of the asymptotic method to the three-dimensional Signorini problem.

1.2 The classical model of elastic beams on a rigid foundation

The most known one-dimensional model for bending of clamped elastic beams above an obstacle corresponds to assume that a beam of length L which starts at $x = 0$, and that each point x of the central line of the beam is situated initially to a distance $s\,(x)$ of the obstacle, and the total loading applied at the same point is $F\,(x)$. Then the model can be written as follows (see [Cim73]):

$$\begin{aligned} &EI\,(u)^{(4)} = F + \lambda, \text{ in } (0, L) \\ &u(0) = u(L) = 0 \\ &(u)'\,(0) = (u)'\,(L) = 0 \\ &u \geq s, \lambda \geq 0, (u - s)\,\lambda = 0, \text{ in } (0, L) \end{aligned} \tag{1.1}$$

where

- u is the bending of the central line,
- λ is the (unknown) reaction of the foundation ,
- I is the inertia moment
- E is Young's modulus

The final sentence in (1.1) translates to the requirement for non-penetration and states that the reaction is only strictly positive in the event that contact with the obstacle is made.

The variational formulation of problem (1.1) can be easily obtained and is as follows:

$$\left\{ \begin{array}{l} u \in U := \{v \in H_0^2(0, L) : v \geq s \text{ a.e. in } (0, L)\} \\ EI \int_0^L (u)''\,(v - u)''\,dx \geq \int_0^L F\,(v - u)\,dx, \text{ for all } v \in U \end{array} \right. \tag{1.2}$$

1.3 MOTIVATION

We'll show that the previous model can be seen as an approximation of the three-dimensional Signorini problem using the asymptotic method, and this is based on the article [Via85]. Also, in what follows we will use the summation convention on repeated indices of tensor calculus; moreover, Latin indices take their values in the set $\{1,2,3\}$ and Greek indices (except ε) in $\{2,3\}$.

1.3.1 The Signorini problem

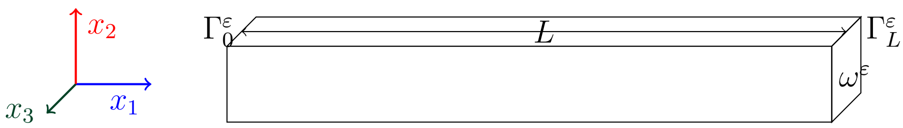

Figure 1.1: 3d beam Ω^ε

Let ω be an open, bounded and connected set in $\mathbb{R}^2$ with area $A(\omega)$. Given $\varepsilon \in \mathbb{R}, 0 < \varepsilon \leq 1$, and $L > 0$, we define

$$\omega^\varepsilon = \varepsilon\omega, \quad \gamma^\varepsilon = \partial\omega^\varepsilon = \varepsilon\partial\omega$$

and we also define $\Omega^\varepsilon = (0,L)\times\omega^\varepsilon$ which we will identify as the reference configuration of the actual beam.

We note by $x^\varepsilon = (x_1^\varepsilon, x_2^\varepsilon, x_3^\varepsilon) = (x_1, \varepsilon x_2, \varepsilon x_3)$ an arbitrary point in Ω^ε and by $n^\varepsilon = (n_i^\varepsilon)$ the outer unit normal vector on $\partial\Omega^\varepsilon$. The parameter ε represents the diameter of the transversal section ω^ε, that has area $A(\omega^\varepsilon) = \varepsilon^2 A(\omega)$. We denote the edges of Ω^ε by:

$$\Gamma_0^\varepsilon = \{0\} \times \omega^\varepsilon, \quad \Gamma_L^\varepsilon = \{L\} \times \omega^\varepsilon$$

We also assume the boundary γ^ε is divided into two nonempty disjoint parts denoted by γ_C^ε and γ_N^ε. Accordingly, we denote $\Gamma^\varepsilon = \Gamma_N^\varepsilon \cup \Gamma_C^\varepsilon$, with $\Gamma_N^\varepsilon = (0,L) \times \gamma_N^\varepsilon$ and $\Gamma_C^\varepsilon = (0,L) \times \gamma_C^\varepsilon$. The part Γ_C^ε of the boundary can have a contact without friction with an obstacle. We denote by $s^\varepsilon(x^\varepsilon)$ the distance of the point $x^\varepsilon \in \Gamma_C^\varepsilon$ to the obstacle measured in the normal direction of vector n^ε. We assume $s^\varepsilon : \Gamma_C^\varepsilon \to \mathbb{R}^+ \in L^\infty(\Gamma_C^\varepsilon)$. For convenience, we drop the superindex ε when $\varepsilon = 1$, i.e.:

$$\Omega = \Omega^1, \quad \Gamma_0 = \Gamma_0^1, \ldots$$

The beam is assumed to be made from homogeneous and isotropic material with Young's modulus E and Poisson's ratio ν. Also, we'll use Lamé's coefficients λ and μ, related with E and ν by the formulae

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = \frac{E}{2(1+\nu)} \tag{1.3}$$

We suppose that the beam is clamped in both ends Γ_0^ε and Γ_L^ε, and under the action of body forces of volume density $f^\varepsilon = (f_i^\varepsilon)$ and surface forces acting on Γ_N^ε of density $g^\varepsilon = (g_i^\varepsilon)$. We assume the following regularity for the forces:

$$f_i^\varepsilon \in L^2(\Omega^\varepsilon), \quad g_i^\varepsilon \in L^2(\Gamma_N^\varepsilon) \tag{1.4}$$

In linear elasticity, the classical model used for this situation is known as the Signorini problem and it is written as (see [DL76]): Find $u^\varepsilon : \bar{\Omega}^\varepsilon \to \mathbb{R}^3$ such that:

$$\begin{aligned} -\partial_j \sigma_{ij}(u^\varepsilon) := f_i^\varepsilon, &\quad \text{in } \Omega^\varepsilon \\ \sigma_{ij}(u^\varepsilon)\, n_j^\varepsilon = g_i^\varepsilon, &\quad \text{on } \Gamma_N^\varepsilon \\ u_i^\varepsilon = 0, &\quad \text{on } \Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon \\ u_n^\varepsilon \le s^\varepsilon, \quad \sigma_n^\varepsilon \le 0, \quad \sigma_{ti}^\varepsilon = 0, &\quad \text{on } \Gamma_C^\varepsilon \\ \sigma_n^\varepsilon (u_n^\varepsilon - s^\varepsilon) = 0, &\quad \text{on } \Gamma_C^\varepsilon \end{aligned} \tag{1.5}$$

where

- $\sigma^\varepsilon = \sigma(u^\varepsilon) = (\sigma_{ij}(u^\varepsilon))$ is the stress tensor, related with the displacement field $u^\varepsilon = (u_i^\varepsilon)$ by the Hooke's generalized law

$$\sigma_{ij}(u^\varepsilon) = \frac{\nu E}{(1+\nu)(1-2\nu)} e_{pp}(u^\varepsilon)\, \delta_{ij} + \frac{E}{1+\nu} e_{ij}(u^\varepsilon)$$

- $e(u^\varepsilon) = (e_{ij}(u^\varepsilon))$ is the linearized strain tensor

$$e_{ij}(u^\varepsilon) = \frac{1}{2}\left(\partial_i u_j^\varepsilon + \partial_j u_i^\varepsilon\right)$$

- $u_n^\varepsilon = u_i^\varepsilon n_i^\varepsilon, \sigma_n^\varepsilon = \sigma_{ij}(u^\varepsilon)\, n_i^\varepsilon n_j^\varepsilon$ and $\sigma_{ti}^\varepsilon = \sigma_{ij}(u^\varepsilon)\, n_j^\varepsilon - \sigma_n^\varepsilon n_i^\varepsilon$.

The two last conditions in (1.5) describes the well-known unilateral contact without friction. When introducing the space of admissible displacements,

$$V(\Omega^\varepsilon) = \left\{ v^\varepsilon = (v_i^\varepsilon) \in \left[H^1(\Omega^\varepsilon)\right]^3 : v^\varepsilon = 0 \text{ in } \Gamma_0^\varepsilon \cup \Gamma_L^\varepsilon \right\}$$

the following variational formulation of problem (1.5) can be easily obtained:

$$\begin{aligned} & u^\varepsilon \in K(\Omega^\varepsilon) := \{ v^\varepsilon \in V(\Omega^\varepsilon) : v_n^\varepsilon \le s^\varepsilon \text{ a.t. on } \Gamma_C^\varepsilon \} \\ & \int_{\Omega^\varepsilon} \sigma_{ij}(u^\varepsilon)\, e_{ij}(v^\varepsilon - u^\varepsilon)\, dx^\varepsilon \ge \int_{\Omega^\varepsilon} f_i^\varepsilon (v_i^\varepsilon - u_i^\varepsilon)\, dx^\varepsilon + \int_{\Gamma_N^\varepsilon} g_i^\varepsilon (v_i^\varepsilon - u_i^\varepsilon)\, da^\varepsilon \\ & \text{for all } v^\varepsilon \in K(\Omega^\varepsilon) \end{aligned} \tag{1.6}$$

The problem (1.6) is written as a classical variational inequality of the following form:

$$\left\{ \begin{array}{l} u^\varepsilon \in K\left(\Omega^\varepsilon\right) \\ a_\varepsilon\left(u^\varepsilon, v^\varepsilon - u^\varepsilon\right) \geq l_\varepsilon\left(v^\varepsilon - u^\varepsilon\right), \text{ for all } v^\varepsilon \in K\left(\Omega^\varepsilon\right) \end{array} \right. \tag{1.7}$$

where, for all $w^\varepsilon, v^\varepsilon \in \left[H^1\left(\Omega^\varepsilon\right)\right]^3$ we note:

$$a_\varepsilon\left(w^\varepsilon, v^\varepsilon\right) = \int_{\Omega^\varepsilon} \sigma_{ij}\left(w^\varepsilon\right) e_{ij}\left(v^\varepsilon\right) dx^\varepsilon, \quad l_\varepsilon\left(v^\varepsilon\right) = \int_{\Omega^\varepsilon} f_i^\varepsilon v_i^\varepsilon dx^\varepsilon + \int_{\Gamma_N^\varepsilon} g_i^\varepsilon v_i^\varepsilon da^\varepsilon$$

Because of the continuity of the linear form l_ε, the continuity and the coercivity of the bilinear form a_ε (from the Korn's inequality), the problem (1.7) has a unique solution for each ε (see [LS67]).

1.3.2 The asymptotic method

We introduce the change of variable

$$\Pi^\varepsilon : \Omega \longrightarrow \Omega^\varepsilon, \quad (x_1, x_2, x_3) \to x^\varepsilon = (x_1, \varepsilon x_2, \varepsilon x_3) \tag{1.8}$$

Also, we scale the unknown, test functions s^ε:

$$u_\alpha(\varepsilon)(x) = \varepsilon u_\alpha^\varepsilon\left(x^\varepsilon\right), u_1(\varepsilon)(x) = u_1^\varepsilon\left(x^\varepsilon\right), \tag{1.9}$$

$$v_\alpha(\varepsilon)(x) = \varepsilon v_\alpha^\varepsilon\left(x^\varepsilon\right), v_1(\varepsilon)(x) = v_1^\varepsilon\left(x^\varepsilon\right), \text{ for all } v^\varepsilon : \bar{\Omega}^\varepsilon \to \mathbb{R}^3. \tag{1.10}$$

We also assume

$$\begin{array}{l} f_\alpha^\varepsilon\left(x^\varepsilon\right) = \varepsilon f_\alpha(x), f_1^\varepsilon\left(x^\varepsilon\right) = f_1(x) \\ g_\alpha^\varepsilon\left(x^\varepsilon\right) = \varepsilon^2 g_\alpha(x), g_1^\varepsilon\left(x^\varepsilon\right) = \varepsilon g_1(x) \\ s^\varepsilon = \varepsilon^{-1} s(x), \end{array} \tag{1.11}$$

where the functions

$$f_i \in L^2(\Omega), g_i \in L^2\left(\Gamma_N\right), s \in L^\infty\left(\Gamma_C\right) \tag{1.12}$$

are independent of the parameter ε.

Consequently, the following result can be obtained via a simple computations.

Theorem 1.1 *The scaled displacement $u(\varepsilon)$ obtained by means the transforma tion* (1.9) *of the solution u^ε of problem* (1.7) *is the unique solution of the following variational problem in Ω :*

$$\left\{ \begin{array}{l} u(\varepsilon) \in K(\Omega) = \{v \in V(\Omega) : v_n \leq s \ a.e. \ on \ \Gamma_C\} \\ c_0(u(\varepsilon), v - u(\varepsilon)) + \varepsilon^2 c_2(u(\varepsilon), v - u(\varepsilon)) + \varepsilon^4 c_4(u(\varepsilon), v - u(\varepsilon)) \\ \geq \varepsilon^4 \left[\int_\Omega f_i\left(v_i - u_i(\varepsilon)\right) dx + \int_{\Gamma_N} g_i\left(v_i - u_i(\varepsilon)\right) du\right] \\ \textit{for all } v \in K(\Omega) \end{array} \right. \tag{1.13}$$

where for all $w, v \in V(\Omega)$ the bilinear forms c_0, c_2 and c_4 are defined by

$$\begin{aligned}
c_0(w,v) &= \int_\Omega \left[\lambda e_{\alpha\alpha}(w) e_{\beta\beta}(v) + 2\mu e_{\alpha\beta}(w) e_{\alpha\beta}(v)\right] dx \\
c_2(w,v) &= \int_\Omega \left[\lambda e_{\alpha\alpha}(w) e_{11}(v) + 4\mu e_{1\alpha}(w) e_{1\alpha}(v) + \lambda e_{11}(w) e_{\alpha\alpha}(v)\right] dx \\
c_4(w,v) &= \int_\Omega (\lambda + 2\mu) e_{11}(u) e_{11}(v) dx
\end{aligned}$$

We notice that the powers of ε^2 in (1.13), so it's natural to use asymptotical techniques to approximate $u(\varepsilon)$, when ε is small, by means of an expansion of the form

$$u(\varepsilon) = u^{(0)} + \varepsilon^2 u^{(2)} + \varepsilon^4 u^{(4)} + \dots \tag{1.14}$$

Such that

$$u^{(2p)} \in V(\Omega),\ u^{(0)} \in K(\Omega),\ u_n^{(2p)} \leq 0,\ p = 1, 2, \dots \tag{1.15}$$

In general, for a symmetric bilinear form $c : V(\Omega) \times V(\Omega) \to \mathbb{R}$, and for all $v \in V(\Omega)$ we obtain the following decomposition according to the expansion (1.14)-(1.15):

$$\begin{aligned}
c(u(\varepsilon), v - u(\varepsilon)) = c\left(u^{(0)}, v - u^{(0)}\right) + \varepsilon^2 c\left(u^{(2)}, v - 2u^{(0)}\right) + \\
+ \varepsilon^4 \left[c\left(u^{(4)}, v - 2u^{(0)}\right) - c\left(u^{(2)}, u^{(2)}\right)\right] + O\left(\varepsilon^6\right).
\end{aligned} \tag{1.16}$$

Substituting the expansion (1.14) into (1.13) while taking into account (1.16) for the bilinear forms $c_0,\ c_2,\ c_4$, we obtain the following inequality:

$$\begin{aligned}
& c_0\left(u^{(0)}, v - u^{(0)}\right) + \varepsilon^2 c_0\left(u^{(2)}, v - 2u^{(0)}\right) + \varepsilon^4 \left[c_0\left(u^{(4)}, v - 2u^{(0)}\right) - c_0\left(u^{(2)}, u^{(2)}\right)\right] \\
& + \varepsilon^2 c_2\left(u^{(0)}, v - u^{(0)}\right) + \varepsilon^4 c_2\left(u^{(2)}, v - 2u^{(0)}\right) + \varepsilon^4 c_4\left(u^{(0)}, v - u^{(0)}\right) \\
& \geq \varepsilon^4 \int_\Omega f_i\left(v_i - u_i^0\right) dx + \varepsilon^4 \int_{\Gamma_N} g_i\left(v_i - u_i^0\right) da + O\left(\varepsilon^6\right), \text{ for all } v \in K(\Omega).
\end{aligned} \tag{1.17}$$

1.3.3 The first order terms in the asymptotic expansion

Next we introduce some constants and functions which only depend on the geometry of the transversal section ω^ε. For simplicity, we assume $\varepsilon = 1$.

- Second moments of area of ω : $I_\alpha = \int_\omega x_\alpha^2 d\omega$.
- Functions $\Phi_{\alpha\beta}$ and δ_α :

$$\begin{aligned}
&\Phi_{22}(x_2,x_3)=\frac{1}{2}\left(x_2^2-x_3^2\right)=-\Phi_{33}(x_2,x_3)\\
&\Phi_{23}(x_2,x_3)=\Phi_{32}(x_2,x_3)=x_2x_3\\
&\delta_2(x_2,x_3)=x_3,\quad \delta_3(x_2,x_3)=-x_2
\end{aligned}\tag{1.18}$$

- Warping function w is the unique solution of the following problern:

$$\begin{aligned}
&w\in H^1(\omega),\quad \int_\omega w d\omega=0\\
&\int_\omega \partial_\alpha w\partial_\alpha\varphi d\omega=\int_\omega\left(x_3\partial_2\varphi-x_2\partial_3\varphi\right)d\omega,\ \text{for all } \varphi\in H^1(\omega)
\end{aligned}\tag{1.19}$$

- Timoshenko's functions η_β and θ_β are the unique solution of the following problems, respectively:

$$\begin{aligned}
&\eta_\beta\in H^1(\omega),\quad \int_\omega \eta_\beta d\omega=0,\\
&\int_\omega \partial_\alpha\eta_\beta\partial_\alpha\varphi d\omega=-2\int_\omega x_\beta\varphi d\omega,\ \text{for all } \varphi\in H^1(\omega).
\end{aligned}\tag{1.20}$$

$$\begin{aligned}
&\theta_\beta\in H^1(\omega),\quad \int_\omega \theta_\beta d\omega=0,\\
&\int_\omega\left(\partial_\alpha\theta_\beta+\Phi_{\alpha\beta}\right)\partial_\alpha\varphi d\omega=0,\ \text{for all } \varphi\in H^{\mathrm{L}}(\omega)
\end{aligned}\tag{1.21}$$

We also introduce the spaces:

$$V_1(\Omega)=\{v\in V(\Omega):e_{\alpha\beta}(v)=0\}\tag{1.22}$$

$$V_2(\Omega)=V_{BN}(\Omega)=\{v\in V(\Omega):e_{\alpha\beta}(v)=e_{1\alpha}(v)=0\}\tag{1.23}$$

The elements of $V_{BN}(\Omega)$ are called the Bernoulli-Navier displacements. We have the following equivalent definitions for for $V_1(\Omega)$ and $V_2(\Omega)$.

Lemma 1.2 *([TV96]). The following characterization for the spaces $V_1(\Omega)$ and $V_2(\Omega)$ hold:*

$$\begin{aligned}
V_1(\Omega)=\Big\{&v\in\left[H^1(\Omega)\right]^3:\\
&v_\alpha(x_1,x_2,x_3)=\chi_\alpha(x_1)+\delta_\alpha(x_2,x_3)\chi_1(x_1),\chi_1,\chi_\alpha\in H_0^1(0,L)\Big\}
\end{aligned}\tag{1.24}$$

$$\begin{aligned}
V_2(\Omega)=V_{BN}(\Omega)=\Big\{&v\in\left[H^1(\Omega)\right]^3:v_\alpha(x_1,x_2,x_3)=\chi_\alpha(x_1)\\
&v_1(x_1,x_2,x_3)=\chi_1(x_1)-x_\alpha\chi_\alpha'(x_1),\chi_1\in H_0^1(0,L),\chi_\alpha\in H_0^2(0,L)\Big\}
\end{aligned}\tag{1.25}$$

We also use the following sets of functions with separated variables.

$$
\begin{aligned}
W_T(\Omega) = \{ & v = (0, v_2, v_3) \in \left[H^1(\Omega)\right]^3 : \\
& v_\alpha (x_1, x_2, x_3) = \varphi_\alpha (x_2, x_3)\, \chi (x_1) : \varphi_\alpha \in H^1(\omega), \chi \in H^1_0(0, L)\} \\
\equiv \Big\{ & (v_2, v_3) \in \left[H^1(\Omega)\right]^2 : \\
& v_\alpha (x_1, x_2, x_3) = \varphi_\alpha (x_2, x_3)\, \chi (x_1), \varphi_\alpha \in H^1(\omega), \chi \in H^1_0(0, L)\}
\end{aligned}
\tag{1.26}
$$

$$
\begin{aligned}
W_L(\Omega) = \{ & v = (v_1, 0, 0) \in \left[H^1(\Omega)\right]^3 : \\
& v_1 (x_1, x_2, x_3) = \varphi (x_2, x_3)\, \chi (x_1), \varphi \in H^1(\omega), \chi \in H^1_0(0, L)\} \\
\equiv \{ & v_1 \in H^1(\Omega) : \\
& v_1 (x_1, x_2, x_3) = \varphi (x_2, x_3)\, \chi (x_1), \varphi \in H^1(\omega), \chi \in H^1_0(0, L)\}.
\end{aligned}
\tag{1.27}
$$

Knowing that the outward unit normal vector in $\Gamma = \Gamma_C \cup \Gamma_N$ is of the form $(0, n_2, n_3)$, we have

$$
K(\Omega) = W_1(\Omega) \times K_2(\Omega) \tag{1.28}
$$

where

$$
K_2(\Omega) = \Big\{ (v_\beta) \in \left[H^1(\Omega)\right]^2 : v_\beta = 0 \text{ on } \Gamma_0 \cup \Gamma_L, v_\beta n_\beta \leq s \text{ a.e. on } \Gamma_C \Big\} \tag{1.29}
$$

$$
W_1(\Omega) = \{ v_1 \in H^1(\Omega) : v_1 = 0 \text{ on } \Gamma_0 \cup \Gamma_L \} \tag{1.30}
$$

We also define the following transversal forces F_i^ε and moments M_i^ε, and the function $w^{\varepsilon(0)}$ (for simplicity, $\varepsilon = 1$ is assumed):

$$
F_i = \int_\omega f_i d\omega + \int_{\gamma_N} g_i d\gamma, \quad M_\alpha = \int_\omega x_\alpha f_1 d\omega + \int_{\gamma_N} x_\alpha g_1 d\gamma, \tag{1.31}
$$

$$
\left\{
\begin{array}{l}
w^{(0)} \in L^2 (0, L; H^1(\omega)) \text{ and a.e. in } (0, L), \quad \int_\omega w^{(0)} d\omega = 0 \\
\int_\omega \partial_\alpha w^{(0)} \partial_\alpha \varphi d\omega = \int_\omega f_1 \varphi d\omega + \int_{\gamma_N} g_1 \varphi d\gamma - \frac{1}{A(\omega)} F_1 \int_\omega \varphi d\omega \\
\text{for all } \varphi \in H^1(\omega).
\end{array}
\right. \tag{1.32}
$$

Theorem 1.3 *Let us suppose the forces satisfy the condition* (1.12) *and also*

$$
f_1 \in H^1 \left(0, L; L^2(\omega)\right), g_1 \in H^1 \left(0, L; L^2 (\gamma_N)\right) \tag{1.33}
$$

Then, the variational inequality (1.17) *yields the following inequalities involving* $u^{(0)}$ *:*

$$
c_0 \left(u^{(0)}, v - u^{(0)}\right) \geq 0, \text{ for all } v \in K(\Omega) \tag{1.34}
$$

$$
c_0 \left(u^{(2)}, v - 2u^{(0)}\right) + c_2 \left(u^{(0)}, v - u^{(0)}\right) \geq 0, \text{ for all } v \in K(\Omega). \tag{1.35}
$$

$$
\begin{aligned}
& c_0 \left(u^{(4)}, v - 2u^{(0)}\right) - c_0 \left(u^{(2)}, u^{(2)}\right) + c_2 \left(u^{(2)}, v - 2u^{(0)}\right) + c_4 \left(u^{(0)}, v - u^{(0)}\right) \\
& \geq \int_\Omega f_i \left(v_i - u_i^{(0)}\right) dx + \int_{\Gamma_N} g_i \left(v_i - u_i^{(0)}\right) da, \text{ for all } v \in K(\Omega).
\end{aligned}
\tag{1.36}
$$

(i) The displacement $u^{(0)} \in K(\Omega)$ is uniquely determined, it belongs to the space $V_{BN}(\Omega)$ and it has the following form

$$u_{\alpha}^{(0)}\left(x_1, x_2, x_3\right)=\xi_{\alpha}\left(x_1\right), \quad u_1^{(0)}\left(x_1, x_2, x_3\right)=\xi_1\left(x_1\right)-x_{\alpha} \xi_{\alpha}^{\prime}\left(x_1\right) \tag{1.37}$$

where the flexions (ξ_α) are the only solution of the following coupled elliptic variational inequality:

$$\left\{\begin{array}{l}(\xi_\alpha) \in [H_0^2(0,L)]^2 \cap [H_{loc}^3(0,L)]^2 \cap K_2(\Omega) \\ EI_\alpha \int_0^L \xi_\alpha''\left(\chi_\alpha-\xi_\alpha\right)'' dx_1 \geq \int_0^L F_\alpha\left(\chi_\alpha-\xi_\alpha\right) dx_1 \\ -\int_0^{L^2} M_\alpha\left(\chi_\alpha-\xi_\alpha\right)' dx_1, \text{ for all } (\chi_\alpha) \in [H_0^2(0,L)]^2 \cap K_2(\Omega)\end{array}\right. \tag{1.38}$$

and the stretching ξ_1 is the only solution of the following variational problem:

$$\left\{\begin{array}{l}\xi_1 \in H_0^1(0,L) \cap H^2(0,L) \\ EA(\omega) \int_0^L \xi_1' \chi' dx_1 = \int_0^L F_3 \chi dx_1, \text{ for all } \chi \in H_0^1(0,L)\end{array}\right. \tag{1.39}$$

(ii) The term $u^{(2)} \in [H^1(\Omega)]^3$ with $u_n^{(2)} \leq 0$ and it is characterized as follows:

$$u_{\alpha}^{(2)}\left(x_1, x_2, x_3\right)=z_{\alpha}\left(x_1\right)+U_{\alpha}^{(2)}\left(x_1, x_2, x_3\right) \tag{1.40}$$

$$u_1^{(2)}\left(x_1, x_2, x_1\right)=z_1\left(x_1\right)-x_{\alpha} z_{\alpha}^{\prime}\left(x_1\right)+U_1^{(2)}\left(x_1, x_2, x_3\right) \tag{1.41}$$

where $U^{(2)}=\left(U_i^{(2)}\right)$ has the following form

$$U_{\alpha}^{(2)}\left(x_1, x_2, x_1\right)=\delta_{\alpha} r\left(x_1\right)-\nu\left[x_{\alpha} \xi_1^{\prime}(x)-\Phi_{\alpha\beta} \xi_{\beta}^{\prime\prime}\left(x_1\right)\right] \tag{1.42}$$

$$\begin{aligned} U_1^{(2)}\left(x_1, x_2, x_3\right)= & -w r^{\prime}\left(x_1\right)+\nu\left\{\frac{1}{2}\left(x_2^2+x_3^2\right)-\frac{1}{2A(\omega)}\left(I_2+I_3\right)\right\} \xi_1^{\prime\prime}\left(x_1\right) \\ & +\left[(1+\nu) \eta_{\alpha}+\nu \theta_{\alpha}\right] \xi_{\alpha}^{\prime\prime\prime}\left(x_1\right)+\frac{2(1+\nu)}{E} w^{(0)}\end{aligned} \tag{1.43}$$

with $z_\alpha \in H^2(0,L), r \in H^1(0,L)$ and $z_1 \in H_0^1(0,L)$.

Proof. We will present the proof in several steps.

Step 1. Passing to the limit as ε tends to zero in inequality (1.17), we obtain (1.34), which is: for all $v \in K(\Omega)$:

$$\int_{\Omega}\left[\lambda e_{\alpha\alpha}\left(u^{(0)}\right) e_{\beta\beta}\left(v-u^{(0)}\right)+2 \mu e_{\alpha\beta}\left(u^{(0)}\right) e_{\alpha\beta}\left(v-u^{(0)}\right)\right] dx \geq 0 \tag{1.44}$$

Taking successively $v=2u^{(0)}$ and $v=0 \in K(\Omega)$ in (1.44) we have:

$$\int_{\Omega}\left[\lambda e_{\alpha\alpha}\left(u^{(0)}\right) e_{\beta\beta}\left(u^{(0)}\right)+2 \mu e_{\alpha\beta}\left(u^{(0)}\right) e_{\alpha\beta}\left(u^{(0)}\right)\right] dx=0$$

and, consequently, $u^{(0)} \in V_1(\Omega) \cap K(\Omega)$:

$$e_{\alpha\beta}\left(u^{(0)}\right)=0, \quad u_n^{(0)} \leq s \text{ on } \Gamma_C. \tag{1.45}$$

Condition (1.45) restricts the form of the transversal components of $u^{(0)}$ to the following one (from lemma (1.2)):

$$u_\alpha^{(0)}\left(x_1, x_2, x_3\right)=\xi_\alpha\left(x_1\right)+\delta_\alpha\left(x_2, x_3\right) \xi\left(x_1\right), \quad \xi_\alpha, \xi \in H_0^1(0, L) \tag{1.46}$$

Hence, inequality (1.34) is equivalent to the equation

$$c_0\left(u^{(0)}, v-u^{(0)}\right)=0, \text{ for all } v \in K(\Omega) \tag{1.47}$$

Step 2. Taking the limit as $\varepsilon \to 0$ on the combination of inequalities $\frac{1}{\varepsilon^2}$[(1.17)- (1.47)] we obtain the inequality (1.35). Taking into account the conditions (1.45), the inequality (1.30) can be written as:

$$\begin{aligned} & \int_\Omega\left[\lambda e_{\alpha\alpha}\left(u^{(2)}\right) e_{\beta\beta}(v)+2\mu e_{\alpha\beta}\left(u^{(2)}\right) e_{\alpha\beta}(v)\right] dx \\ & +\int_\Omega\left[\lambda e_{11}\left(u^{(0)}\right) e_{\alpha\alpha}(v)+4\mu e_{1\alpha}\left(u^{(0)}\right) e_{1\alpha}\left(v-u^{(0)}\right) dx \geq 0\right. \\ & \text{for all } v \in K(\Omega) \end{aligned} \tag{1.48}$$

Equation (1.48) evaluated successively in $v=2u^{(0)}$ and $v=0$ produces:

$$\int_\Omega 4\mu e_{1\alpha}\left(u^{(0)}\right) e_{1\alpha}\left(u^{(0)}\right) dx=0$$

which gives us

$$e_{1\alpha}\left(u^{(0)}\right)=0 \tag{1.49}$$

Properties (1.45) and (1.49) mean the term $u^{(0)}$ belongs to the space $V_2(\Omega)=V_{BN}(\Omega)$ of Bernoulli-Navier displacements and also to $K(\Omega)$. Specifically,

$$\begin{aligned} & u^{(0)} \in V_{BN}(\Omega) \cap K(\Omega) \\ & u_\alpha^{(0)}(x)=\xi_\alpha\left(x_1\right), \quad \xi_\alpha \in H_0^2(0, L) \\ & u_3^{(0)}(x)=\xi_1\left(x_1\right)-x_\alpha \xi_\alpha^{\prime}\left(x_1\right), \quad \xi_1 \in H_0^1(0, L), \\ & \xi_\alpha\left(x_1\right) n_\alpha\left(x_2, x_3\right) \leq s\left(x_1, x_2, x_3\right) \text{ a.e. on } \Gamma_C \end{aligned} \tag{1.50}$$

Then, by the corresponding identifications, we have

$$\left(\xi_\alpha\right) \in\left[H_0^2(0, L)\right]^2 \cap K_2(\Omega) \tag{1.51}$$

Now, by substituting (1.49) in inequality (1.48), we deduce, for all $v \in K(\Omega)$:

$$\int_\Omega\left[\lambda e_{\alpha\alpha}\left(u^{(2)}\right) e_{\beta\beta}(v)+2\mu e_{\alpha\beta}\left(u^{(2)}\right) e_{\alpha\beta}(v)\right] dx+\int_\Omega \lambda e_{11}\left(u^{(0)}\right) e_{\alpha\alpha}(v) dx \geq 0 \tag{1.52}$$

Hence, taking test functions $v \in W_T(\Omega) \cap K(\Omega)$ in (1.52) we have:

$$
\begin{gathered}
\int_0^L \left\{ \int_\omega \left[\lambda e_{\alpha\alpha}\left(u^{(2)}\right) e_{\beta\beta}(\varphi) + 2\mu e_{\alpha\beta}\left(u^{(2)}\right) e_{\alpha\beta}(\varphi) \right] d\omega \right. \\
\left. + \int_\omega \lambda e_{11}\left(u^{(0)}\right) e_{\beta\beta}(\varphi) d\omega \right\} \chi dx_1 \geq 0, \text{ for all } \varphi = (\varphi_\alpha) \in \left[H^1(\omega)\right]^2, \\
\text{and } \chi \in H_0^1(0,L) \text{ s.t. } \varphi_\alpha n_\alpha \leq s \text{ a.e. on } \Gamma_C, \chi \geq 0 \text{ a.e. in } (0,L).
\end{gathered}
$$

We conclude that the following equation holds a.e. in $(0, L)$:

$$
\begin{gathered}
\int_\omega \left[\lambda e_{\alpha\alpha}\left(u^{(2)}\right) e_{\beta\beta}(\varphi) + 2\mu e_{\alpha\beta}\left(u^{(2)}\right) e_{\alpha\beta}(\varphi) + \lambda e_{11}\left(u^{(0)}\right) e_{\alpha\alpha}(\varphi) \right] d\omega \geq 0 \\
\text{for all } \varphi = (\varphi_\alpha) \in \left[H^1(\omega)\right]^2 \text{ s.t. } \varphi_\alpha n_\alpha \leq s \text{ a.e. on } \Gamma_C
\end{gathered}
\tag{1.53}
$$

Taking $\varphi_\alpha \in \mathcal{D}(\omega)$ in (1.53) we conclude the following equality:

$$
\lambda e_{\rho\rho}\left(u^{(2)}\right) \delta_{\alpha\beta} + 2\mu e_{\alpha\beta}\left(u^{(2)}\right) = -\lambda e_{11}\left(u^{(0)}\right) \delta_{\alpha\beta}, \text{ a.e. in } \Omega = (0,L) \times \omega \tag{1.54}
$$

Since $e_{11}\left(u^{(0)}\right) = \xi_1' - x_\alpha \xi_\alpha''$ (see (1.50)), from (1.54) the expressions (1.40) and (1.42) of $u_\alpha^{(2)}$ are deduced (see [TV96],Th. 4.5). We note that conditions $z_\alpha, r \in H^1(0,L)$ are necessary but not sufficient in order to have $u_\alpha^{(2)} \in W_1(\Omega)$. Then $u_\alpha^{(2)} \in H^1(\Omega)$ but, in general, $u_\alpha^{(2)} \notin W_1(\Omega)$ and $\left(u_\alpha^{(2)}\right) \notin K_2(\Omega)$.

From (1.53), we see that $u^{(2)}$ is a solution of the following equation, for all $v \in K(\Omega)$:

$$
\int_\Omega \left[\lambda e_{\alpha\alpha}\left(u^{(2)}\right) e_{\beta\beta}(v) + 2\mu e_{\alpha\beta}\left(u^{(2)}\right) e_{\alpha\beta}(v) \right] dx + \int_\Omega \lambda e_{11}\left(u^{(0)}\right) e_{\alpha\alpha}(v) dx = 0 \tag{1.55}
$$

and, from (1.49) and (1.55) we deduce that

$$
c_0\left(u^{(2)}, v - 2u^{(0)}\right) + c_2\left(u^{(0)}, v - u^{(0)}\right) = 0, \text{ for all } v \in K(\Omega) \tag{1.56}
$$

Step 3. Passing to the limit as $\varepsilon \to 0$ in the combination of inequalities $\frac{1}{\varepsilon^4}[$ (1.17) - (1.47) $- \varepsilon^2$(1.56)] we deduce the following inequality (see (1.36)):

$$
\begin{gathered}
c_0\left(u^{(4)}, v - 2u^{(0)}\right) - c_0\left(u^{(2)}, u^{(2)}\right) + c_2\left(u^{(2)}, v - 2u^{(0)}\right) + c_4\left(u^{(0)}, v - u^{(0)}\right) \\
\geq \int_\Omega f_i\left(v_i - u_i^{(0)}\right) dx + \int_{\Gamma_N} g_i\left(v_i - u_i^{(0)}\right) da, \text{ for all } v \in K(\Omega)
\end{gathered}
\tag{1.57}
$$

So, taking into account the properties deduced in the previous steps, mainly (1.54), the equation (1.57) is written as:

$$
\begin{gathered}
\int_\Omega \left[\lambda e_{\alpha\alpha}\left(u^{(4)}\right) e_{\beta\beta}(v) + 2\mu e_{\alpha\beta}\left(u^{(4)}\right) e_{\alpha\beta}(v) \right] dx \\
+ \int_\beta \left[\lambda e_{\alpha\alpha}\left(u^{(2)}\right) e_{11}\left(v - u^{(0)}\right) + \lambda e_{11}\left(u^{(2)}\right) e_{\alpha\alpha}(v) \right] dx \\
+ \int_\Omega 4\mu e_{1\alpha}\left(u^{(2)}\right) e_{1\alpha}(v) dx + \int_\Omega (\lambda + 2\mu) e_{11}\left(u^{(0)}\right) e_{11}\left(v - u^{(0)}\right) \\
\geq \int_\Omega f_i\left(v_i - u_i^{(0)}\right) dx + \int_{\Gamma_N} g_i\left(v_i - u_i^{(0)}\right) da, \text{ for all } v \in K(\Omega).
\end{gathered}
\tag{1.58}
$$

Evaluating (1.58) in $v \in V_{BN}(\Omega) \cap K(\Omega)$ we obtain:

$$\begin{aligned} &\int_\Omega \lambda e_{\alpha\alpha}\left(u^{(2)}\right) e_{11}\left(v-u^{(0)}\right) dx + \int_\Omega (\lambda+2\mu) e_{11}\left(u^{(0)}\right) e_{11}\left(v-u^{(0)}\right) \\ &\geq \int_\Omega f_i\left(v_i-u_i^{(0)}\right) dx + \int_{\Gamma_N} g_i\left(v_i-u_i^{(0)}\right) da, \text{ for all } v \in V_{BN}(\Omega)\cap K(\Omega) \end{aligned} \tag{1.59}$$

We notice that for any $v \in V_{BN}(\Omega) \cap K(\Omega)$ we have

$$\begin{aligned} &v_\alpha\left(x_1,x_2,x_3\right) = \chi_\alpha\left(x_1\right), \left(\chi_\alpha\right) \in \left[H^2(0,L)\right]^2 \cap K_2(\Omega) \\ &v_3\left(x_1,x_2,x_3\right) = \chi_1\left(x_1\right) - x_\alpha \chi_\alpha'\left(x_1\right), \chi_1 \in H_0^1(0,L) \end{aligned} \tag{1.60}$$

Now, as a consequence of (1.37) and (1.40), we get

$$\lambda e_{\alpha\alpha}\left(u^{(2)}\right) + (\lambda+2\mu) e_{11}\left(u^{(0)}\right) = E\left[\xi_1' - x_\alpha \xi_\alpha''\right] \tag{1.61}$$

Then, by substituting (1.61) into (1.59), the problems (1.38) and (1.39) are derived. Existence, unicity and regularity of solution of problem (1.38) are exhibited in [BS68]:

Step 4. We restrict now (1.58) to $v \in K(\Omega) \cap V_1(\Omega)$, or, $e_{\alpha\beta}(v) = 0$. We have:

$$\begin{aligned} &\int_\Omega \left[\lambda e_{\alpha\alpha}\left(u^{(2)}\right) e_{11}\left(v-u^{(0)}\right) + 4\mu e_{1\alpha}\left(u^{(2)}\right) e_{1\alpha}\left(v-u^{(0)}\right)\right] dx \\ &+ \int_\Omega (\lambda+2\mu) e_{11}\left(u^{(0)}\right) e_{11}\left(v-u^{(0)}\right) \\ &\geq \int_\Omega f_i\left(v_i-u_i^{(0)}\right) dx + \int_{\Gamma_N} g_i\left(v_i-u_i^{(0)}\right) da \\ &\text{for all } v \in K(\Omega) \text{ s.t. } e_{\alpha\beta}(v) = 0 \end{aligned} \tag{1.62}$$

By taking in (1.62) respectively $v = \left(u_1^0 + v_1, 0, 0\right)$ and $v = \left(u_1^0 - v_1, 0, 0\right), v_1 \in W_1(\Omega)$, we get

$$\begin{aligned} &\int_\Omega \mu \partial_\alpha u_1^{(2)} \partial_\alpha v_1 dx = -\int_\Omega \lambda e_{\alpha\alpha}\left(u^{(2)}\right) \partial_1 v_1 dx - \int_\Omega \mu \partial_1 u_\alpha^{(2)} \partial_\alpha v_1 dx \\ &- \int_\Omega (\lambda+2\mu) \partial_1 u_1^{(0)} \partial_1 v_1 dx + \int_\Omega f_1 v_1 dx + \int_{\Gamma_N} g_1 v_1 da \\ &\text{for all } v_1 \in W_1(\Omega) \end{aligned} \tag{1.63}$$

Evaluating (1.63) in $v \in W_L(\Omega)$ we find:

$$\begin{aligned} &\int_0^L \left[\int_\omega \mu \partial_\alpha u_1^{(2)} \partial_\alpha \varphi d\omega\right] \chi dx_1 = -\int_0^L \left[\int_\omega \lambda e_{\alpha\alpha}\left(u^{(2)}\right) \varphi d\omega\right] \chi' dx_1 \\ &- \int_0^L \left[\int_\omega \mu \partial_1 u_\alpha^{(2)} \partial_\alpha \varphi d\omega\right] \chi dx_1 - \int_0^L \left[\int_\omega^L (\lambda+2\mu) \partial_1 u_1^{(0)} \varphi d\omega\right] \chi' dx_1 \\ &+ \int_0^L \left[\int_\omega f_1 \varphi d\omega\right] \chi dx_1 + \int_0^L \left[\int_{\gamma_N} g_1 \varphi d\gamma\right] \chi dx_1 \\ &\text{for all } \varphi \in H^1(\omega), \text{ for all } \chi \in H_0^1(0,L) \end{aligned} \tag{1.64}$$

Using now equalities (1.61), (1.40) and (1.42), the following equation in the sense of distributions in $(0, L)$ is derived:

$$\int_\omega \partial_\alpha u_1^{(2)} \partial_\alpha \varphi d\omega = \frac{E}{\mu}\left[\xi_1'' \int_\omega \varphi d\omega - \xi_\alpha''' \int_\omega x_\alpha \varphi d\omega\right] - z_\alpha' \int_\omega \partial_\alpha \varphi d\omega$$
$$- r' \int_\omega \delta_\alpha \partial_\alpha \varphi d\omega + \nu \xi_1'' \int_\omega x_\alpha \partial_\alpha \varphi d\omega - \nu \xi_\beta''' \int_\omega \Phi_{\alpha\beta} \partial_\alpha \varphi d\omega \tag{1.65}$$
$$+ \frac{1}{\mu} \int_\omega f_1 \varphi d\omega + \frac{1}{\mu} \int_{\gamma_N} g_1 \varphi d\gamma, \text{ for all } \varphi \in H^1(\omega)$$

For each $x_1 \in (0, L)$ the problem (1.65) is a Laplacian problem in ω with Neumann conditions in all boundary γ. The compatibility condition for φ such that $\partial_\alpha \varphi = 0$ is verified because of the equation (1.39) for the traction. Then, there exists a (non-unique) solution of (1.65) and it has the form given by expressions (1.41) and (1.43) (see [TV96], Sect. 8). ■

1.3.4 The limit model to the current beam

Having in mind that $u^{(0)}$ is a first order approximation of $u(\varepsilon)$ in Ω, we propose a first order approximation, $u^{0\varepsilon}$, of u^ε in Ω^ε, obtained by undoing the change of variable (1.8) and the scalings (1.9), (1.10) and (1.11):

$$u_\alpha^\varepsilon(x^\varepsilon) = \varepsilon^{-1} u_\alpha(\varepsilon)(x) \sim \varepsilon^{-1} u_\alpha^{(0)}(x) =: u_\alpha^{0\varepsilon}(x^\varepsilon) \tag{1.66}$$
$$u_1^\varepsilon(x^\varepsilon) = u_1(\varepsilon)(x) \sim u_1^{(0)}(x) =: u_1^{0\varepsilon}(x^\varepsilon) \tag{1.67}$$

From (1.37) we immediately deduce that $u_\alpha^{0\varepsilon}$ and $u_1^{0\varepsilon}$ are of the following form:

$$u_\alpha^{0\varepsilon}(x^\varepsilon) = \varepsilon^{-1} u_\alpha^{(0)}(x) = \varepsilon^{-1} \xi_\alpha(x_1) =: \xi_\alpha^\varepsilon(x_1) \tag{1.68}$$
$$u_1^{0\varepsilon}(x^\varepsilon) = u_1^{(0)}(x) = \xi_1(x_1) - x_\alpha \xi_\alpha'(x_1) = \xi_1^\varepsilon(x_1) - x_\alpha^\varepsilon (\xi_\alpha^\varepsilon)'(x_1) \tag{1.69}$$

where we put $\xi_1^\varepsilon = \xi_1$.

Using now problems (1.38) and (1.39) we obtain a complete characterization of the first order displacements $u^{0\varepsilon}$ by means a well-posed "one-dimensional" model.

Theorem 1.4 *The first order displacements field $u^{0\varepsilon}$ defined by* (1.68)-(1.69) *is a Bernoulli-Navier displacement, i.e.:*

$$u_\alpha^{0\varepsilon}(x^\varepsilon) = \xi_\alpha^\varepsilon(x_1), \quad \xi_\alpha^\varepsilon \in H_0^2(0, L) \tag{1.70}$$
$$u_1^{0\varepsilon}(x^\varepsilon) = \xi_1^\varepsilon(x_1) - x_\alpha^\varepsilon (\xi_\alpha^\varepsilon)'(x_1), \quad \xi_3^\varepsilon \in H_0^1(0, L) \tag{1.71}$$

where

(i) The flexions $(\xi_1^\varepsilon, \xi_2^\varepsilon)$ are the only solution of the following coupled variational inequality:

$$\left\{\begin{array}{l} (\xi_\alpha^\varepsilon) \in K^\varepsilon(0, L), \\ EI_\alpha^\varepsilon \int_0^L (\xi_\alpha^\varepsilon)'' (\chi_\alpha^\varepsilon - \xi_\alpha^\varepsilon)'' dx_1 \geq \int_0^L F_\alpha^\varepsilon (\chi_\alpha^\varepsilon - \xi_\alpha^\varepsilon) dx_1 \\ - \int_0^L M_\alpha^\varepsilon (\chi_\alpha^\varepsilon - \xi_\alpha^\varepsilon)' dx_1, \ \text{for all } (\chi_\alpha^\varepsilon) \in K^\varepsilon(0, L), \end{array}\right. \tag{1.72}$$

where

$$K^{\varepsilon}(0,L) := \left\{ (\chi_{\alpha}^{\varepsilon}) \in \left[H_0^2(0,L)\right]^2 : \chi_{\alpha}^{\varepsilon} n_{\alpha}^{\varepsilon} \leq s^{\varepsilon} \text{ a.e. on } \Gamma_C^{\varepsilon} = \gamma_C^{\varepsilon} \times (0,L) \right\} \tag{1.73}$$

(ii) The stretching ξ_1^{ε} is the only solution of the following problem:

$$\begin{cases} \xi_1^{\varepsilon} \in H_0^1(0,L) \cap H^2(0,L), \\ EA(\omega^{\varepsilon}) \int_0^L (\xi_1^{\varepsilon})' (\chi^{\varepsilon})' \, dx_1 = \int_0^L F_1^{\varepsilon} \chi^{\varepsilon} dx_1, \text{ for all } \chi^{\varepsilon} \in H_0^1(0,L). \end{cases} \tag{1.74}$$

Proof. It is a direct consequence of equations (1.38)-(1.39) and definitions (1.68)-(1.69) and (1.31). ■

Equation (1.74) is the classical model for the stretching of a clamped beam without any obstacles. The problem (1.72) represents a general bending model for a beam which may become in contact with a rigid obstacle. We notice that we can also define $K^{\varepsilon}(0,L)$ as follows:

$$K^{\varepsilon}(0,L) = \left\{ (\chi_{\alpha}^{\varepsilon}) \in \left[H_0^2(0,L)\right]^2 : \chi_{\alpha}^{\varepsilon}(x_1)\, n_{\alpha}^{\varepsilon}(x_1^{\varepsilon}, x_1^{\varepsilon}) \leq s^{\varepsilon}(x_1, x_2^{\varepsilon}, x_3^{\varepsilon}), \right. \\ \left. \text{for all } x_1 \in (0,L) \text{ and a.e. } (x_2^{\varepsilon}, x_3^{\varepsilon}) \in \gamma_C^{\varepsilon} \right\} \tag{1.75}$$

If we take the contact surface Γ_C^{ε} is plane and normal to one of the inertia axes of the beam (Ox_2, to fix the ideas). Consequently, the outward unit normal vector to Γ_C^{ε} is constant and it has one of the form $(0,+1,0)$ or $(0,-1,0)$. Let us assume $n = (0,-1,0)$.

From (1.75) one deduces that the convex set $K^{\varepsilon}(0,L)$ for this case is:

$$K^{\varepsilon}(0,L) = U_2^{\varepsilon}(0,L) \times H_0^2(0,L) \tag{1.76}$$

where

$$U_2^{\varepsilon}(0,L) = \{ \varphi^{\varepsilon} \in H_0^2(0,L) : \varphi^{\varepsilon}(x_1) \geq s^{\varepsilon}(x_1, x_2^{\varepsilon}, x_3^{\varepsilon}) \\ \text{for all } x_1 \in (0,L) \text{ and a.e. } (x_2^{\varepsilon}, x_3^{\varepsilon}) \in \gamma_C^{\varepsilon} \} \tag{1.77}$$

We assume that the beam and the obstacle are regular enough in such a way the following function $\hat{s}^{\varepsilon} : [0,L] \to \mathbb{R}$, is well defined and $\hat{s}^{\varepsilon} \in L^{\infty}(0,L)$:

$$\hat{s}^{\varepsilon}(x_1) = \inf_{(x_2^{\varepsilon}, x_3^{\varepsilon}) \in \gamma_C^{\varepsilon}} s^{\varepsilon}(x_1, x_2^{\varepsilon}, x_3^{\varepsilon}), \quad x_1 \in (0,L)$$

Then, we have an equivalent definition of $U^{\varepsilon}(0,L)$

$$U_2^{\varepsilon}(0,L) = \left\{ \varphi^{\varepsilon} \in H_0^2(0,L) : \varphi^{\varepsilon} \geq \hat{s}^{\varepsilon} \text{ a.e in } (0,L) \right\} \tag{1.78}$$

Setting in (1.72) successively $(\chi_2^{\varepsilon}, \chi_3^{\varepsilon}) = (\chi_2^{\varepsilon}, \xi_3^{\varepsilon})$ and $(\chi_2^{\varepsilon}, \chi_3^{\varepsilon}) = (\xi_2^{\varepsilon}, \chi_3^{\varepsilon})$, with $\chi_2^{\varepsilon} \in U_2^{\varepsilon}(0,L)$ and $\chi_3^{\varepsilon} \in H_0^2(0,L)$, we prove that, in this case, the limit problem (1.72) is equivalent to the following two problems:

$$\begin{cases} \xi_2^\varepsilon \in U_2^\varepsilon(0,L) \\ EI_2^\varepsilon \int_0^L (\xi_2^\varepsilon)'' (\chi_2^\varepsilon - \xi_2^\varepsilon)'' \, dx_1 \geq \int_0^L F_2^\varepsilon (\chi_2^\varepsilon - \xi_2^\varepsilon) \, dx_1 \\ - \int_0^L M_2^\varepsilon (\chi_2^\varepsilon - \xi_2^\varepsilon)' \, dx_1, \text{ for all } \chi_2^\varepsilon \in U_2^\varepsilon(0,L), \end{cases} \tag{1.79}$$

$$\begin{cases} \xi_3^\varepsilon \in H_0^2(0,L) \\ EI_3^\varepsilon \int_0^L (\xi_3^\varepsilon)'' (\chi_3^\varepsilon)'' \, dx_1 = \int_0^L F_3^\varepsilon \chi_3^\varepsilon dx_1 \\ - \int_0^L M_3^\varepsilon (\chi_3^\varepsilon)' \, dx_1, \text{ for all } \chi_3^\varepsilon \in H_0^2(0,L). \end{cases} \tag{1.80}$$

We observe that (1.80) is the usual variational model for bending in the direction Ox_2 and (1.79) is the classical one-dimensional obstacle problem (1.2). So we have mathematically justified this classical model as the first order approximation of the three-dimensional Signorini problem for an elastic beam when the boundary of contact is assumed to be plane and normal to one inertia axis.

1.4 Conclusion

Our work in this chapter was two folds: We justified mathematically the Euler Bernoulli obstacle problem and at the same time we got an approximate problem of the Signorini problem which should be far easier to solve than the classic one. Thus it is of interest to further study Euler Bernoulli obstacle problem and try solve it numerically and efficiently.

Chapter 2

Mathematical Analysis of the obstacle problem for Euler-Bernoulli beam

2.1 Introduction

In the last chapter we approximated Signorini problem and that lead us to find that it involves find the solution of obstacle problem for the Euler-Bernoulli beam, but before doing that we need to figure out some mathematical proprieties of the solution which are important the numerical analysis. In chapter we investigate the existence of the solution and its regularity, and later we introduce an alternative variational form appropriate for finite element analysis.

2.2 Existence, Uniqueness and Proprieties of The Solution

Let $\Omega = (0, L)$, $L > 0$, and ψ represents the obstacle. We reintroduce the obstacle problem of Euler-Bernoulli Beam but this time we omit the constants since they don't affect the mathematical analysis:

$$\begin{cases} u \in K := \{v \in H_0^2(\Omega) : v \geq s \text{ a.e. in } \Omega\} \\ \int_0^L (u)'' \, (v-u)'' \, dx \geq \int_0^L f \, (v-u) \, dx, \text{ for all } v \in K \end{cases} \tag{2.1}$$

2.2.1 Existence and Uniqueness

We first provide some useful definitions

Definition 2.1 *Let V be a Hilbert space. The bilinear form $a(.,.)$ is continuous on $V \times V$ if there exists $M > 0$ such that :*

$$|a(u, v)| \leq M\|u\|_V \|v\|_V$$

Definition 2.2 *Let V be a Hilbert space. The bilinear form $a(.,.)$ is coercive on $V \times V$ if there exists $m > 0$ such that :*

$$a(v, v) \geq m\|v\|_V^2$$

We also need the following theorem

Theorem 2.3 ***(Stampacchia's Theorem)***
Assume that $a(u, v)$ is a continuous coercive bilinear form on Hilbert space H. Let $K \subseteq H$ be a nonempty closed and convex subset. Then, given any $\phi \in H^\star$, there exists a unique element $u \in K$ such that

$$a(u, v-u) \geq \langle \phi, v-u \rangle \quad \forall v \in K \tag{2.2}$$

Moreover, if a is symmetric, then u is characterized by the property:

$$u \in K \text{ and } \frac{1}{2}a(u,v) - \langle \phi, u \rangle = \min_{v \in K}\{a(v,v)/2 - \langle \phi, u \rangle\} \tag{2.3}$$

Proof. See [Bre11] ■
Now we can prove the well posedness of the problem

Theorem 2.4 *The problem* (2.1) *has a unique solution*

Proof. We apply the previous theorem (2.3)

1. K is convex: Indeed, let $u, v \in K$ and $t \in [0,1]$
   - $tu + (1-t)v \geq t\psi + (1-t)\psi = \psi$
   - $tu(0) + (1-t)v(0) = t \cdot 0 + (1-t) \cdot 0 = 0$
   - $tu(L) + (1-t)v(L) = t \cdot 0 + (1-t) \cdot 0 = 0$

   Thus for all $t \in [0,1]$, we have $tu + (1-t)v \in K$

2. K is closed. Let $g(v) = |v(0)| + |v(L)| + \|(v-\psi)_-\|$. g is continuous on $H_0^2(\Omega)$ and $g^{-1}(\{0\}) = K$, thus K is closed

3. a is continuous :

$$a(u,v) = \left| \int_\Omega u''v''dx \right| \leq |u|_{2,\Omega}|v|_{2,\Omega} \leq \|u\|_{2,\Omega}\|v\|_{2,\Omega}$$

4. a is coersive : We apply Poincaré inequqlity multiple times

$$\begin{aligned} a(v,v) = &= \|v''\|_{0,\Omega}^2 \\ &= \frac{1}{3}\|v''\|_{0,\Omega}^2 + \frac{1}{3}\|v''\|_{0,\Omega}^2 + \frac{1}{3}\|v''\|_{0,\Omega}^2 \\ &\geq \frac{1}{3}\|v''\|_{0,\Omega}^2 + \frac{1}{3} \cdot \frac{1}{C_p}\|v'\|_{0,\Omega}^2 + \frac{1}{3} \cdot \frac{1}{C_p^2}\|v\|_{0,\Omega}^2 \\ &\geq m\|v\|_{H_0^2(\Omega)}^2 \end{aligned}$$

Thus the existence and uniqueness are proven. ■

Remark 2.5 *Because $a(u,v)$ is symmetric, Stampacchia Thoerem gives us the equivalent minimization problem of* (2.1) *as well*

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ \quad J(u) \leq J(v)\ \forall v \in K \end{cases} \tag{2.4}$$

Where $J : H_0^2 \longrightarrow \mathbb{R}$ is defined as follows :

$$J(v) = \frac{1}{2}\int_\Omega |v''|^2 dx - \int_\Omega f \cdot v\ dx \tag{2.5}$$

Remark 2.6 *We also notice that the functional* (2.5) *looks similar to the energy of the elastic obstacle. Indeed, in the elastic obstacle the displacement is constrained by an obstacle, denoted by* ψ*, which is allowed to be elastic. The energy resulting from contact with an elastic obstacle can be written as*

$$\frac{1}{2\varepsilon}\int_{\omega}(u-\psi)_{-}^{2}dx, \tag{2.6}$$

where $\varepsilon > 0$ *is the inverse of an appropriately scaled "spring constant" and*

$$(u-\psi)_{-} = \min(u-\psi, 0).$$

The loading consists of a distributed load $f \in L^2(\Omega)$ *with the energy*

$$\ell(v) = \int_{\Omega} f v dx. \tag{2.7}$$

The total energy thus reads as

$$J(v) = \frac{1}{2}a(v,v) + \frac{1}{2\varepsilon}\int_{\Omega}(v-\psi)_{-}^{2}dx - \ell(v). \tag{2.8}$$

The space of kinematically admissible displacements is $V = H_0^2(\Omega)$ *and the displacement function* u *is thus obtained minimising the energy, viz.*

$$J(u) \leq J(v) \tag{2.9}$$

or by solving the weak formulation : find $u \in V$ *such that*

$$a(u,v) + \frac{1}{\varepsilon}((u-\psi)_{-}, v) = \ell(v), \quad \forall v \in V, \tag{2.10}$$

From this last problem, we can see the relationship between the elastic and rigid obstacle. When $\varepsilon \to 0$ *in the problem* (2.10)*, we get the the rigid obstacle problem* (2.1)*.*

2.2.2 Proprieties of the solution

We introduce the following definition:

Definition 2.7 *Let* v *in* $H^2(\Omega)$*.* v *is called a supersolution if for all* $w \in H_0^2(\Omega)$ *such that* $w \geq 0$ *we have*

$$\int_0^L v''w''dx - \langle f, w\rangle \geq 0 \tag{2.11}$$

Then we can prove the following theorem.

Theorem 2.8 *The solution of the problem* (2.1) *is a supersolution.*

Proof. In fact, if $w \in H_0^2$, such that $w \geq 0$ then $u + w \in K$. Taking $v = u + w \in K$, and replacing it in (2.1) we find

$$\int_0^L u'' w'' dx \geq \langle f, w \rangle$$

Therefore $\int_0^L u'' w'' dx - \langle f, w \rangle \geqslant 0$ for all $w \in H_0^2,\ w \geq 0$. And u is a supersolution. ■

Theorem 2.9 *Let u be a supersolution then it satisfies the following equation in the distribution sense*

$$u^{(4)} - f = \mu_u$$

where μ_u is a positive measure on Ω.

Proof. From theorem (2.8), we obtain that for any positive test function $\eta \in D(\Omega)$ we have:

$$\int u'' \eta'' - \langle f, \eta \rangle \geq 0$$

so $(u'')'' - f \geq 0$ in the sense of distributions. Let us consider the following linear operator on the space $D(\Omega)$,

$$\Lambda(\eta) = \int u'' \eta'' - \langle f, \eta \rangle$$

Then Λ is a continuous linear operator on $C_0^\infty(\Omega)$, therefore it is a distribution. According to Riesz-Schwartz theorem [Sch66], Λ represents a positive measure. let us denote this measure by μ_u. Then $(u'')'' - f = \mu_u$ in the sense that

$$\int_\Omega u'' \eta'' - \langle f, \eta \rangle = \int_\Omega \eta d\mu_u$$

for every $\eta \in D(\Omega)$. ■

Another interesting point is the contact set, or where the solution touches the obstacle. While in general the contact set can't be known a priori, the following theorem can be helpful to find it in specific circumstances

Theorem 2.10 *Let u be a solution of the problem* (2.1) *and suppose ψ is a supersolution. If $a \leq b \in C = \{x | x \in [0, L]\}, u(x) = \psi(x)\}$, then $[a, b] \subset A$.*

Let $a, b \in A$, and let

$$v = \begin{cases} \psi & \text{on } [a, b] \\ u & \text{on } \Omega -]a, b[\end{cases}$$

It's clear that $v \in K$, $u - v \geqq 0$ on $[a, b]$ and $u - v = 0$ on $\Omega -]a, b[$.

We take $w = u - v$, then $w \in H_0^2(\Omega)$ and $w \geq 0$ on Ω. Since ψ is a supersolution, so will be v and we obtain

$$\int_0^L v'' \left(u'' - v''\right) dx - \langle f, u_0 - v \rangle \geq 0 \tag{2.12}$$

Taking $v = v$ in (2.1) we get

$$\int_0^L (u'')\,(v'' - u'')\,dx + \langle f, v - u\rangle \geq 0 \tag{2.13}$$

Adding (2.12) and (2.13) term by term addition, we obtain

$$\int_0^L -\left(v'' - u_0''\right)^2 dx \geq 0$$

Which means $u = v = \psi$ on $[a, b]$. Therefore $[a, b] \subset A$.

Remark 2.11 *From the last theorem we deduce that if ψ is a supersolution, the beam touches the obstacle in either exactly one point or over one closed interval. This observation can be useful to find the solution and the contact set, and we'll use it in next section to make a counterexample.*

2.3 Regularity of The Solution

2.3.1 H^3_{loc} regularity

Lemma 2.12 *Suppose that $f \in L^2(]0, L[)$. Then any supersolution $v(x)$ we have:*

- $v'' \in C^2(]0, L[)$
- $v''' \in L^2_{loc}(]0, L[)$ *and the limits $v'''(x+)$, $v'''(x-)$ exist for all $x \in]0, L[$, plus*

$$v'''(x+) \geq v'''(x-) \text{ for all } x \in]0, L[$$

Proof. Since v is a supersolution, then by theorem (2.9), the distribution $\mu_v = v^{(4)} - f$, is a positive measure. Let $a \in]0, L[$, we define

$$\phi\colon]0, L[\to \mathbb{R},\ \phi(x) = \begin{cases} -\mu_v([x, a)) & \text{if } x < a \\ \mu_v((a, x]) & \text{if } x \geq a \end{cases}$$

We can easily verify that ϕ is non decreasing, locally integrable and $(\phi)' = \mu_v$. On top of that we have

$$f \in L^2(]0, L[) \subset L^1(]0, L[) \Rightarrow \exists F = \int_0^x f(t)dt \in C^0([0, L]) : f(x) = (F(x))'$$

Taking all that into consideration we obtain:

$$v^{(4)} = \mu_v + f \Rightarrow v''' = \phi + F + C_1,$$

where C_1 is some constant. Then v''' is bounded on $[x_1, x_2]$ for all $x_1, x_2 \in]0, L[$, which implies $v''' \in L^2_{loc}(]0, L[)$ and the limits $v'''(x+)$, $v'''(x-)$ exists for all $x \in]0, L[$, such that

$$v'''(x+) \geq v'''(x-) \text{ for all } x \in]0, L[$$

We continue integrating

$$v''(x) = \int_a^x [\phi(t) + F(t) + C_1]\, dt + C_2$$

then v'' is continuous for $x \in] - L, L[$. ∎

Theorem 2.13 *Assuming $f \in L^2(]0, L[)$ and*

$$\psi(x) \in C^0([0, L]), \quad \psi(0) < 0, \quad \psi(L) < 0$$

.

Then the solution of problem (2.1) $u \in C^2([0, L]) \cap H^3_{loc}([0, L])$

Proof. $u \in C^2(]0, L[) \cap H^3_{loc}(]0, L[)$ follows immediately from theorem (2.8) and lemma (2.12). From the continuity of u, ψ we obtain:

$$\begin{aligned}
u(0) = 0 > \psi(0) &\Rightarrow \exists \epsilon > 0, \forall x \in [0, \epsilon] : u(x) > \psi(x) \\
&\Rightarrow \forall x \in [0, \epsilon] : \mu_u = 0 \\
&\Rightarrow \forall x \in [0, \epsilon] : \phi(x) \text{ is } \textit{constant} \\
&\Rightarrow u''(0) = \int_a^\epsilon [\phi(t) + F(t) + C_1]\, dt + \int_\epsilon^0 [\phi(t) + F(t) + C_1]\, dt + C_2 \\
&\qquad = \lim_{x \to 0} u(x)
\end{aligned}$$

Therefore u'' is continuous on $x = 0$ and $u^{(3)}$ is bounded in the neighberhood of 0. In similar fashion we can show u'' is continuous on $x = L$ and $u^{(3)}$ is bounded in the neighberhood of L, thus proving the theorem. ∎

2.3.2 The lack of H^4 regularity

The question to ask next: Can we have higher regularity for u than H^3_{loc}?

Unfortunately, even with smooth f and ψ, u in general does not belong to H^4, not even H^4_{loc}. As the next example will show

Example 2.14 *Let $f = 0$, $L = 1$ and $\psi = -3(2x - 1)^2 + 1$ in the problem* (2.1). *We'll exploit the fact that $\psi(1 - x) = \psi(x)$, let*

$$\begin{aligned}
T :& V \to V \\
& v \mapsto Tv(x) = v(1 - x)
\end{aligned}$$

It's easy to verify $T(K) = K$ and $T = T^{-1}$. Let $w = Tu$ and and $v \in K$

$$a(w, v) = \int_0^1 w''v'' = -\int_1^0 (Tw)''(Tv)'' = \int_0^1 u''(Tv)'' \geq 0$$

Which means w is a solution to (2.1), hence $w(x) = u(1-x) = u(x)$, because the problem has a unique solution. We also notice that ψ is supersolution as well which means according to theorem (2.10) and the fact that $u(1-x) = u(x)$, either the contact zone is in the form of $[\alpha, 1-\alpha]$ such that $\alpha \in]0, \frac{1}{2}[$, or the contact zone is one single point $x = \frac{1}{2}$. lets assume the contact zone is a closed interval, then

$$u(x) = \begin{cases} ax^3 + bx^2 & x \in [0, \alpha] \\ -3(2x-1)^2 + 1 & x \in]\alpha, 1-\alpha] \\ a(1-x)^3 + b(1-x)^2 & x \in]1-\alpha, 1] \end{cases}$$

For some $a, b \in \mathbb{R}$. However taking into account the theorem (2.12), that u is C^2 in the neighborhood of α we find $\alpha = \frac{1}{2}$; contradiction.
Hence the contact zone is $\{\frac{1}{2}\}$, and we obtain

$$u(x) = \begin{cases} -16x^3 + 12x^2 & x \in [0, \frac{1}{2}] \\ -16(1-x)^3 + 12(1-x)^2 & x \in [\frac{1}{2}, 1] \end{cases}$$

Therefore

$$u^{(3)} = \begin{cases} -96 & x \in [0, \frac{1}{2}[\\ 96 & x \in]\frac{1}{2}, 1] \end{cases}$$

So $u^{(3)}$ is not continuous in Ω, hence $u \notin H_loc^4(\Omega)$.

The optimal regularity is still an open question [Ale19], and at the time of publication of this thesis no example of H^4 regularity can found publicly in the case where the obstacle intersects but is not tangent on the solution. This lack of H^4 has big implication when performing finite element which we'll discuss in the next chapter.

Remark 2.15 *If we go back to the elastic obstacle problem (2.10), we can see that the reaction force between the obstacle and the plate is given by*

$$u^{(4)} - f = \lambda = -\frac{1}{\varepsilon}(u - \psi)_-. \tag{2.14}$$

Hence the Lagrange multiplier λ belongs to $L^2(\Omega)$ and the solution u belongs to $H^4(\Omega)$. So even though the solution $u_\varepsilon \in H^4(\Omega)$ approaches u when $\varepsilon \to 0$, the limit doesn't have the same regularity of that of u_ε.

2.4 Conclusion

In this chapter we presented mathematical analysis of the problem: we proved it's well posed so it has a unique solution, we investigated some proprieties of this solution and and showed that in general it doesn't belong to H^4 space. Now we're to apply the finite element method on the problem.

Chapter 3

Finite Element Method on the obstacle problem for Euler-Bernoulli beam

3.1 Introduction

In this chapter we're interested in having a finite element formulation suitable for numerical analysis. In previous chapter we hinted at the importance of H^4 regularity and here we'll show it. And then later we'll discuss a different formulation of problem of which we'll apply finite element method on it and discuss the error estimates, priori and posteriori.

3.2 Classic Formulation

Let's consider the uniform discretisation of the intervalle $[0, L]$,

$$0 = x_0 < x_1 < \cdots < x_n = 1, \quad x_i - x_{i-1} = h = 1/n; \quad n \geq 2$$

and the finite space

$$V_h := \left\{v_h \in V : v_{h|[x_{i-1},x_i]} \in \mathbb{P}_3\left([x_{i-1}, x_i]\right), \forall i = 1, \ldots, n\right\} = \mathcal{P}_3^{\text{Hermit}} \cap H_0^2(\Omega)$$

Such that P_3^{Hermit} is the Hermite finite element space generated by the basis:

$$B_h = \{\Phi_i, \Psi_i : i = 1...n-1\} \tag{3.1}$$

$$\Phi_i(x) = \Phi\left(\frac{2x - x_{i-1} + x_{i+1}}{2h}\right) \tag{3.2}$$

$$\Psi_i(x) = \Psi\left(\frac{2x - x_{i-1} + x_{i+1}}{2h}\right) \tag{3.3}$$

With Φ, Ψ being the reference basis in the interval $[-1, 1]$ defined as follows:

$$\Phi(x) = \begin{cases} 1 - 3x^2 - 2x^3 & \text{if } -1 \leq x \leq 0 \\ 1 - 3x^2 + 2x^3 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.4}$$

$$\Psi(x) = \begin{cases} x + 2x^2 + x^3 & \text{if } -1 \leq x \leq 0 \\ x - 2x^2 + x^3 & \text{if } 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.5}$$

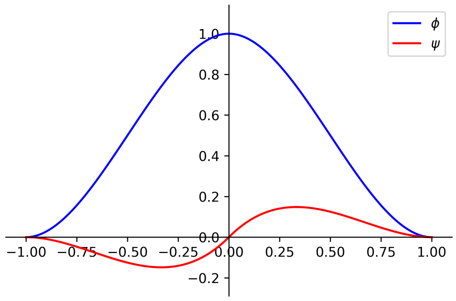

Figure 3.1: Hermite P3 reference element

Let's assume $\psi \in H^4(\Omega)$, then the convex K can approximated by :

$$K_h = \{v_h \in V_h, v_h \geq \psi_h\}$$

Such that $\psi_h = \mathcal{I}_h^{her}(\psi)$ is hermite interpolant of ψ in V_h. It's easy to verify K_h is non-empty closed convex in V_h.
We define the following discrete problem

$$\begin{cases} \text{Find } u_h \in K_h \text{ s.t} \\ a(u_h, v_h - u_h) \geq \langle f, v_h - u_h \rangle \quad \forall v_h \in K_h \end{cases} \tag{3.6}$$

Theorem 3.1 *The problem* (3.6) *has a unique solution.*

3.2.1 Priori error estimate in case of full regularity

Let u be solution of continous problem (2.1) and u_h the solution of the discrete problem. We'll assume $u \in H^4(\Omega)$.

Theorem 3.2 *If $f \in L^2(\Omega)$ and $\psi \in H^4(\Omega)$, we have :*

$$\|u - u_h\|_{H^2(\Omega)} \leq Ch^4 \left(|u|_{H^4(\Omega)} + \|f\|_{L^2(\Omega)} + |\psi|_{H^4(\Omega)}\right)$$

Proof. If $f \in L^2(\Omega)$ and $u \in H^4(\Omega)$, the la solution u will satisfy the following complementary problem in the strong sense:

$$\begin{cases} u^{(4)} - f \geq 0 & \text{a.e in } \Omega \\ u - \psi \geq 0 & \text{a.e in } \Omega \\ (u^{(4)} - f)(u - \psi) = 0 & \text{a.e in } \Omega \\ u = 0 & \text{in } \delta\Omega \end{cases} \tag{3.7}$$

Using the characterization of u in (3.7), we proceed to prove the error estimate.

$$
\begin{aligned}
a(u-u_h,u-u_h) &= a(u-u_h,u-v_h)+a(u-u_h,v_h-u_h)\\
&\le a(u-u_h,u-v_h)-\left(u^{(4)},v_h-u_h\right)-(f,v_h-u_h)\\
&= a(u-u_h,u-v_h)-\left(u^{(4)}+f,v_h-\psi_h+\psi_h-u_h\right)\\
&= a(u-u_h,u-v_h)-\left(u^{(4)}+f,v_h-\psi_h\right)-\left(u^{(4)}+f,\psi_h-u_h\right)
\end{aligned}
$$

We use the fact that $\left(u^{(4)}+f,\psi_h-u_h\right)\ge 0$, we get

$$
\begin{aligned}
a(u-u_h,u-u_h) &\le a(u-u_h,u-v_h)-\left(u^{(4)}+f,v_h-\psi_h\right)\\
&= a(u-u_h,u-v_h)-\left(u^{(4)}+f,v_h-u+u-\psi+\psi-\psi_h\right)\\
&= a(u-u_h,u-v_h)-\left(u^{(4)}+f,v_h-u\right)-(u^{(4)}+f,u-\psi)-\left(u^{(4)}+f,\psi-\psi_h\right)\\
&= a(u-u_h,u-v_h)-\left(u^{(4)}+f,v_h-u\right)-\left(u^{(4)}+f,\psi-\psi_h\right)
\end{aligned}
$$

by using this inequality

$$
ab\le \frac{a^2}{2\varepsilon}+\frac{\varepsilon}{2}b^2,\quad \forall\varepsilon>0
$$

With $\varepsilon=\frac{M^2}{m}$, $a=M\|u-u_h\|_{H^2(\Omega)}$, $b=\|u-v_h\|_{H^2(\Omega)}$ we obtain:

$$
\begin{aligned}
m\|u-u_h\|^2_{H^2(\Omega)} &\le a(u-u_h,u-u_h)\\
&\le \frac{m}{2}\|u-u_h\|^2_{H^2(\Omega)}+\frac{M^2}{2\alpha}\|u-v_h\|^2_{H^2(\Omega)}\\
&+\|u^{(4)}+f\|_{L^2(\Omega)}\left(\|v_h-u\|_{L^2(\Omega)}+\|\psi-\psi_h\|_{L^2(\Omega)}\right)
\end{aligned}
\tag{3.8}
$$

And recalling Hermite interpolant characteristics :

$$
\|v-\mathcal{I}_h^{her}(v)\|_{H^2(\Omega)}\le C\,h^2|v|_{H^4(\Omega)},\qquad \forall v\in H^4(\Omega) \tag{3.9}
$$

$$
\|v-\mathcal{I}_h^{her}(v)\|_{L^2(\Omega)}\le C\,h^4|v|_{H^4(\Omega)},\qquad \forall v\in H^4(\Omega) \tag{3.10}
$$

Then by inserting $v_h=\mathcal{I}_h^{her}(u)$ in (3.8) we obtain

$$
\begin{aligned}
\|u-u_h\|^2_{H^2(\Omega)} &\le C\left(\left\|u-\mathcal{I}_h^{her}(u)\right\|^2_{H^2(\Omega)}+\|u^{(4)}+f\|_{L^2}\left(\left\|\mathcal{I}_h^{her}(u)-u\right\|_{L^2(\Omega)}+\|\psi_h-\psi\|_{L^2(\Omega)}\right)\right)\\
&\le C\left(h^4|u|^2_{H^4(\Omega)}+\left(\|u^{(4)}\|_{L^2(\Omega)}+\|f\|_{L^2(\Omega)}\right)h^4\left(|u|_{H^4(\Omega)}+|\psi|_{H^4(\Omega)}\right)\right)\\
&\le Ch^4\left(|u|_{H^4(\Omega)}+\|f\|_{L^2(\Omega)}+|\psi|_{H^4(\Omega)}\right)^2
\end{aligned}
$$

■

3.2.2 The implication of the lack of full regularity

As seen in the previous chapter, we can't assume $u\in H^4(\Omega)$, so the previous error estimate can't be used in practice. Hence the lack of H^4 is serious issue and the main difficulty for deriving the optimal $O(h)$ rate, and numerical analysis on the problem may became challenging. So different ideas and formulations are needed.

3.3 Mixed Formulation

We introduce a new formulation for the problem which is based on [GSV17, GSV19] which exploits the Lagrange multiplier from the strong complementary form (3.7). However the strong form isn't correct since $u^{(4)} - f \in H^{-2}(\Omega)$ is a positive measure and it doesn't belong in general to $L^2(\Omega)$, and the actual equivalent (weak) complementary form can represented as follows

$$\begin{cases} \langle u^{(4)} - f, v\rangle \geq 0 & \forall v \geq 0 \in H_0^2(\Omega) \\ u - \psi \geq 0 & \text{a.e in } \Omega \\ \langle u^{(4)} - f, u - \psi\rangle = 0 & \\ u = 0 & \text{in } \delta\Omega \end{cases} \tag{3.11}$$

We introduce the Lagrange Multiplier λ defined as

$$\lambda = u^{(4)} - f \in H^{-2}(\Omega) = (H_0^2(\Omega))'$$

Therefore we obtain

$$\begin{cases} u^{(4)} - f = \lambda & \\ \langle \lambda, v\rangle \geq 0 & \forall v \geq 0 \in H_0^2(\Omega) \\ u - \psi \geq 0 & \text{a.e in } \Omega \\ \langle \lambda, u - \psi\rangle = 0 & \\ u = 0 & \text{in } \delta\Omega \end{cases} \tag{3.12}$$

We introduce the following spaces

$$\begin{aligned} & V := H_0^2 \\ & Q := H^{-2}(\Omega) = V' \\ & \Lambda := \{\mu \in Q \quad | \quad \langle \mu, v\rangle \geq 0, \quad \forall v \in V, \quad v \geq 0 \text{ a.e in } \Omega\} \end{aligned}$$

Using the usual math manipulations we can easily verify (3.12) is equivalent to the following mixed variational inequality:

$$\begin{cases} \text{Find } (u, \lambda) \in V \times \Lambda \text{ s.t:} \\ a(u, v) - \langle v, \lambda\rangle = (f, v) \quad \forall v \in V \\ \langle u, \mu - \lambda\rangle \geq \langle \psi, \mu - \lambda\rangle \quad \forall \mu \in \Lambda \end{cases} \tag{3.13}$$

To establish the equivalence between (2.1) and (3.13), we need the following theorem

Theorem 3.3 *[BHR78]*
Suppose that there exists a constant $\beta > 0$ such that

$$\inf_{\mu \in Q} \sup_{v \in V} \frac{\langle v, \mu\rangle}{\|v\|_V \|\mu\|_Q} \geq \beta, \quad \mu, q \neq 0.$$

Then problems (2.1) *and* (3.13) *have at most one solution. If either problem has a solution, then they both have solutions. Furthermore if* (u, λ) *solves* (3.13)*, then* u *solves* (2.1)*.*

And know we're ready to show well posedness of the mixed variational inequality.

Theorem 3.4 *The problem* (3.13) *has a unique solution* (u, λ)*, and* u *is the solution of* (2.1)

Proof. We know that the problem (2.1) has a unique solution, so by theorem (3.3) to complete the proof we only need to show such $\beta > 0$ exists.
Let $\mu \neq 0 \in Q = (H_0^2(\Omega))'$. By Riesz representation theorem, there exists $v_\mu \in H_0^2(\Omega) = V$ such that:

$$\begin{aligned} &\forall v \in V : \langle v, \mu \rangle = (v, v_\mu)_V \\ &\|v_\mu\|_V = \|\mu\|_Q \end{aligned}$$

So

$$\sup_{v \in V} \langle v, \mu \rangle \geq \langle v_\mu, \mu \rangle = (v_\mu, v_\mu)_V = \|v_\mu\|_V^2 = \|v_\mu\|_V \|\mu\|_Q$$

Therefore

$$\inf_{\mu \in Q} \sup_{v \in V} \frac{\langle v, \mu \rangle}{\|v\|_V \|\mu\|_Q} \geq 1, \quad \mu, q \neq 0.$$

■

Let $\mathcal{H} := V \times Q$, and define the bilinear form $\mathscr{A} : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ and the linear form $\mathscr{L} : \mathcal{H} \to \mathbb{R}$ through

$$\begin{aligned} \mathscr{A}((v, \xi); (w, \mu)) &= a(v, w) - \langle \xi, w \rangle - \langle \mu, v \rangle = (v'', w'') - \langle \xi, w \rangle - \langle \mu, v \rangle \\ \mathscr{L}(w, \mu) &= (f, w) - \langle \psi, \mu \rangle \end{aligned}$$

Problem (3.13) can now be written in a compact way as follows

$$\begin{cases} \text{Find } (u, \lambda) \in V \times \Lambda \quad \text{such that} \\ \mathscr{A}((u, \lambda); (w, \mu - \lambda)) \leq \mathscr{L}(w, \mu - \lambda), \quad \forall (w, \mu) \in V \times \Lambda \end{cases} \tag{3.14}$$

Theorem 3.5 *For all* $(v, \xi) \in V \times Q$ *there exists* $w \in V$ *such that :*

$$\mathscr{A}((v, \xi); (w, -\xi)) \gtrsim (\|v\|_2 + \|\xi\|_{-2})^2 \tag{3.15}$$

$$\|w\|_2 \lesssim \|v\|_2 + \|\xi\|_{-2} \tag{3.16}$$

Proof. Let the pair $(v, \xi) \in \mathscr{H}$. By Riesz representation theorem, there exists a unique $q \in V$ which satisfies

$$\langle z, \xi \rangle = (q, z)_V = (q'', z'') + (q', z') + (q, z) \quad \forall z \in V. \tag{3.17}$$

$$\|q\|_2 = \|\xi\|_{-2}. \tag{3.18}$$

Let $w = v - q$. Using (3.17), (3.18), Poincaré's and Cauchy-Schwarz inequalities, we obtain

$$\begin{aligned}\mathcal{A}(v,\xi;w,-\xi) &= (v'',w'') - \langle w,\xi\rangle + \langle v,\xi\rangle \\ &= (v'',v'') - (v'',q'') + \langle q,\xi\rangle \\ &\geq \|v''\|_0^2 - \|v''\|_0\|q''\|_0 + (q,q)_V \\ &\geq |v|_2^2 - |v|_2\|q\|_2 + \|q\|_2^2 \\ &\geq \frac{1}{2}\left(|v|_2 + \|\xi\|_{-2}\right)^2 \\ &\geq C\left(|v|_2 + \|\xi\|_{-2}\right)^2\end{aligned}$$

And finally, it follows from the triangle inequality that

$$\|w\|_2 = \|v - q\|_2 \leq \|v\|_2 + \|q\|_2 = \|v\|_2 + \|\xi\|_{-2}.$$

■

3.3.1 The mixed finite element

Let consider the the uniform discretisation of the interval $\overline{\Omega} = [0, L]$,

$$0 = x_0 < x_1 < \cdots < x_n = L, \quad x_i - x_{i-1} = h = L/n; \quad n \geq 2$$

Let also define the finite spaces V_h, Q_h and the set Λ_h as follows:

$$\begin{cases} V_h := \left\{v_h \in V : v_{h|[x_{i-1},x_i]} \in \mathbb{P}_3\left([x_{i-1},x_i]\right), \forall i = 1,\ldots,n\right\} = \mathcal{P}_3^{\text{Hermit}} \cap H_0^2(\Omega) \\ Q_h := \left\{\mu_h \in Q : \mu_{h|[x_{i-1},x_i]} \in \mathbb{P}_0\left([x_{i-1},x_i]\right), \forall i = 1,\ldots,n\right\} \\ \Lambda_h := \left\{\mu_h \in Q_h : \mu_h \geq 0 \quad \text{dans } \Omega\right\}. \end{cases} \tag{3.19}$$

We're ready to present the corresponding discrete problem of (3.14)

$$\begin{cases} \text{Find } \left(u_h,\lambda_h\right) \in V_h \times \Lambda_h \text{ s.t:} \\ \mathscr{A}\left(\left(u_h,\lambda_h\right);\left(v_h,\mu_h - \lambda_h\right)\right) \leq \mathscr{L}\left(v_h,\mu_h - \lambda_h\right) \quad \forall\left(v_h,\mu_h\right) \in V_h \times \Lambda_h \end{cases} \tag{3.20}$$

For this mixed finite element method of variational inequality, the finite element spaces must satisfy the "Babuska-Brezzi" condition, also called inf-sup condition:

$$\forall \xi_h \in Q_h : \sup_{v_h \in V_h} \frac{\langle v_h,\xi_h\rangle}{\|v_h\|_2} \gtrsim \|\xi_h\|_{-2} \tag{3.21}$$

In order to prove the inf-sup condition is verified for the choosen spaces, we introduce the following discrete H^{-2} norm:

$$\|\xi_h\|_{-2,h}^2 = \sum_{i=1}^{n} h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 = \left(h^2\|\xi_h\|_0\right)^2, \quad \forall \xi_h \in Q_h$$

Lemma 3.6

$$\exists C > 0, \quad \|v_h\|_2 \leq Ch^{-2} \|v_h\|_{0,\Omega}, \quad \forall v \in V_h \tag{3.22}$$

Proof. We know that there exists c $>$ 0 such that

$$\|v_h'\| \leq ch^{-1} \|v_h\|$$

Therefore

$$\|v_h\|_2 \leq c' |v_h|_2 = c' \|v_h''\| \leq cc'h^{-1} \|v_h'\| \leq Ch^{-2} \|v_h\|$$

■

Lemma 3.7 *There exists two strictly positive constants C_1, C_2 such that:*

$$\forall \xi_h \in Q_h : \sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} \geq C_1 \|\xi_h\|_{-2} - C_2 \|\xi_h\|_{-2,h} \tag{3.23}$$

Proof. Recall the definition of negative norm

$$\forall \xi \in Q, \quad \|\xi\|_{-2} = \sup_{w \in V} \frac{\langle \xi, w \rangle}{\|w\|_V} \tag{3.24}$$

But $Q_h \subset Q$, therefore for all $\xi_h \in Q_h$ (3.24) imply the existence of $w \neq 0 \in H_0^2(\Omega)$ and $C > 0$ such that :

$$\langle w, \xi_h \rangle \geq C\|w\|_2 \|\xi_h\|_{-2}$$

Let w_h be the hermit interpolant of w, then we obtain :

$$\begin{aligned} \langle w_h, \xi_h \rangle &= \langle w_h - w, \xi_h \rangle + \langle w, \xi_h \rangle \\ &\geq \langle w_h - w, \xi_h \rangle + C\|w\|_2 \|\xi_h\|_{-2} \\ &\geq -\left(h^{-2} \|w_h - w\|_0\right)\left(h^2 \|\xi_h\|_0\right) + C\|w\|_2 \|\xi_h\|_{-2} \\ &= -\|w\|_2 \left(\frac{h^{-2} \|w_h - w\|_0}{\|w\|_2} \|\xi_h\|_{-2,h} + C \|\xi_h\|_{-2} \right) \end{aligned}$$

Recall the following proprieties of hermit interpolant:

$$\|w_h - w\|_{0,\Omega} \leq C'h^2 \|w\|_{2,\Omega}, \quad \|w_h\|_{2,\Omega} \leq C''\|w\|_{2,\Omega}$$

Then, we deduce

$$\frac{\langle w_h, \xi_h \rangle}{\|w_h\|_2} \geq C_1 \|\xi_h\|_{-2} - C_2 \|\xi_h\|_{-2,h}$$

Therefore

$$\sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} \geq C_1 \|\xi_h\|_{-2} - C_2 \|\xi_h\|_{-2,h}$$

■

Lemma 3.8 *There exists a constant $C_3 > 0$ such that*

$$\forall \xi_h \in Q_h : \sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} \geq C_3 \|\xi_h\|_{-2,h} \tag{3.25}$$

Proof. Let $\xi_h = \sum_{k=1}^{n} \xi_h^k \chi_k$ where $\{\chi_i = 1_{]x_{i-1},x_i[} : i = 1...n\}$ is the basis of Q_h

- **Case 1: n = 2k**
  Let $v_h \in V_h$ such that

$$v_h = h^4 \sum_{i=1}^{k} (\xi_h^{2i-1} + \xi_h^{2i}) \Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \Psi_{2i-1}$$

  Hence we obtain

$$\begin{aligned}
\langle v_h, \xi_h \rangle &= h^4 \sum_{i=1}^{k} \langle (\xi_h^{2i-1} + \xi_h^{2i}) \Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \Psi_{2i-1}, \xi_h^{2i-1} \chi_{2i-1} + \xi_h^{2i} \chi_{2i} \rangle \\
&= h^4 \sum_{i=1}^{k} \xi_h^{2i-1} \int_{x_{2i-2}}^{x_{2i-1}} (\xi_h^{2i-1} + \xi_h^{2i}) \Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \Psi_{2i-1} dx \\
&\qquad + \xi_h^{2i} \int_{x_{2i-1}}^{x_{2i}} (\xi_h^{2i-1} + \xi_h^{2i}) \Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h} \Psi_{2i-1} dx \\
&= h^4 \sum_{i=1}^{k} \xi_h^{2i-1} \int_{x_{2i-2}}^{x_{2i-1}} \frac{1}{2} (\xi_h^{2i-1} + \xi_h^{2i}) - \frac{1}{12} (\xi_h^{2i} - \xi_h^{2i-1}) dx \\
&\qquad + \xi_h^{2i} \int_{x_{2i-1}}^{x_{2i}} \frac{1}{2} (\xi_h^{2i-1} + \xi_h^{2i}) + \frac{1}{12} (\xi_h^{2i} - \xi_h^{2i-1}) dx \\
&\gtrsim h^4 \sum_{i=1}^{k} h (\xi_h^{2i-1} + \xi_h^{2i})^2 + h (\xi_h^{2i} - \xi_h^{2i-1})^2 \\
&= h^4 \sum_{i=1}^{k} h \left((\xi_h^{2i-1})^2 + (\xi_h^{2i})^2 \right) \\
&= h^4 \sum_{i=1}^{k} \int_{x_{2i-2}}^{x_{2i}} (\xi_h)^2 dx \\
&= \|\xi_h\|_{-2,h}^2
\end{aligned}$$

At the same time

$$
\begin{aligned}
\|v_h\|_0^2 &= h^8 \sum_{i=1}^{k} \int_{x_{2i-2}}^{x_{2i}} \left((\xi_h^{2i-1} + \xi_h^{2i})\Phi_{2i-1} + \frac{\xi_h^{2i} - \xi_h^{2i-1}}{h}\Psi_{2i-1} \right)^2 dx \\
&= h^8 \sum_{i=1}^{k} \int_{x_{2i-2}}^{x_{2i}} (\xi_h^{2i-1} + \xi_h^{2i})^2 (\Phi_{2i-1})^2 + (\frac{\xi_h^{2i} - \xi_h^{2i-1}}{h})^2 (\Psi_{2i-1})^2 dx \\
&= h^8 \sum_{i=1}^{k} \int_{x_{2i-2}}^{x_{2i}} \frac{26}{35}(\xi_h^{2i-1} + \xi_h^{2i})^2 + \frac{2}{105}(\xi_h^{2i} - \xi_h^{2i-1})^2 dx \\
&\lesssim h^8 \sum_{i=1}^{k} \int_{x_{2i-2}}^{x_{2i}} (\xi_h^{2i-1} + \xi_h^{2i})^2 + (\xi_h^{2i} - \xi_h^{2i-1})^2 dx \\
&\lesssim h^4 \|\xi_h\|_{-2,h}^2
\end{aligned}
$$

By using the estimate from (3.22), we get

$$
\|v_h\|_2 \leq Ch^{-2} \|v_h\|_0 \lesssim \|\xi_h\|_{-2,h}^2
$$

Hence we deduce

$$
\sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_{2,\Omega}} \geq C \|\xi_h\|_{-2,h}^2
$$

- **Case 2: n = 2k+1**
  For all $\xi_h \in Q_h$ we have either

$$
\sum_{i=1}^{2k} h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 \geq \frac{1}{2} \|\xi_h\|_{-2,h}^2 \quad \text{or} \quad \sum_{i=2}^{2k+1} h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 \geq \frac{1}{2} \|\xi_h\|_{-2,h}^2
$$

  In the first case we take the same v_h as earlier, and the second we take

$$
v_h = h^4 \sum_{i=1}^{k} (\xi_h^{2i} + \xi_h^{2i+1})\Phi_{2i} + \frac{\xi_h^{2i+1} - \xi_h^{2i}}{h}\Psi_{2i}
$$

  Either way, we obtain

$$
\sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_{2,\Omega}} \geq \frac{1}{2} C \|\xi_h\|_{-2,h}^2
$$

Finally taking $C_3 = \frac{1}{2}C$, then for all $n \geq 2$

$$
\sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_{2,\Omega}} \geq C_3 \|\xi_h\|_{-2,h}^2
$$

■

Now we have all the prerequisites to prove inf-sup condition

Theorem 3.9 *For all $\xi_h \in Q_h$, there exists $C > 0$*

$$\sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} \geq C \|\xi_h\|_{-2}$$

Proof. Let $t > 0$. Using the results of lemma (3.7) and lemma (3.8) we get:

$$\begin{aligned}
\sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} &= t \sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} + (1-t) \sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} \\
&\geq t \left(C_1 \|\xi_h\|_{-2} - C_2 \|\xi_h\|_{-2,h} \right) + (1-t) C_3 \|\xi_h\|_{-2,h} \\
&= tC_1 \|\xi_h\|_{-2} + (C_3 - t(C_2 + C_1)) \|\xi_h\|_{-2,h}
\end{aligned}$$

If we choose $t = \frac{C_3}{C_1 + C_2}$, the second term vanishes and we obtain

$$\sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} \geq \frac{C_1 C_3}{C_1 + C_2} \|\xi_h\|_{-2}$$

■

3.3.2 Existence, Uniqueness and Stability

Here we'll see the importance of inf-sup condition

Theorem 3.10 *The problem* (3.20) *is well posed*

Proof. Thanks to stability of the pair (V_h, Q_h) proven earlier, we deduce

$$\inf_{\xi_h \in Q_h} \sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2 \|\xi_h\|_{-2}} \geq C$$

Then as direct consequence of theorem (3.3) and (3.10), the mixed finite element problem has a unique solution (u_h, λ_h) and u_h is a solution of the problem discrete naive problem (3.6). ■
The inf-sup condition also implies the following discrete stability estimate.

Theorem 3.11 *For all $(v_h, \xi_h) \in \mathscr{H}$, there exists $w_h \in V_h$ such that :*

$$\mathscr{A}\left((v_h, \xi_h); (w_h, -\xi_h)\right) \gtrsim \left(\|v_h\|_2 + \|\xi_h\|_{-2}\right)^2 \tag{3.26}$$

$$\|w_h\|_2 \lesssim \|v_h\|_2 + \|\xi_h\|_{-2} \tag{3.27}$$

Proof.
Let $\xi_h \in Q_h$. Let's consider the following auxiliary problem :

$$\begin{cases} \text{Find } q_h \in V_h \text{ s.t} \\ a(q_h, z_h) + (q_h, z_h)_{H^1} = \langle \xi_h, z_h \rangle, \quad \forall z_h \in V_h \end{cases}$$

This is a typical elliptic variational equality, with a unique solution which satisfies

$$\|q_h\|_2^2 = \langle \xi_h, q_h \rangle$$

Because the inf-sup condition (3.21) is verified, then

$$\|\xi_h\|_{-2} \lesssim \sup_{z_h \in V_h} \frac{\langle \xi_h, z_h \rangle}{\|z_h\|_V} = \sup_{z_h \in V_h} \frac{a\left(q_h, z_h\right) + \left(q_h, z_h\right)_{H^1}}{\|z_h\|_V} \leq \|q_h\|_2$$

However

$$\|q_h\|_2 = \frac{\|q_h\|_2^2}{\|q_h\|_2} = \frac{\langle \xi_h, q_h \rangle}{\|q_h\|_2} \leq \frac{\|\xi_h\|_{-2}\|q_h\|_2}{\|q_h\|_2} = \|\xi_h\|_{-2}$$

Now, if we take $w_h = v_h - q_h$,

$$\begin{aligned}
\mathcal{A}\left(\left(v_h, \xi_h\right); \left(w_h, -\xi_h\right)\right) &= \mathcal{A}\left(\left(v_h, \xi_h\right); \left(v_h - q_h, -\xi_h\right)\right) \\
&= a\left(v_h, v_h\right) - a\left(v_h, q_h\right) - \langle \xi_h, v_h - q_h \rangle + \langle \xi_h, v_h \rangle \\
&= a\left(v_h, v_h\right) - a\left(v_h, q_h\right) + \langle \xi_h, q_h \rangle \\
&\geq \|v_h''\|_0^2 - \int_\Omega v_h'' q_h'' + \|q_h\|_2^2 \\
&\geq \frac{1}{2}\left(\|v_h\|_0^2 + \|q_h\|_0^2\right) \\
&\gtrsim \frac{1}{2}\left(\|v_h\|_2^2 + \|q_h\|_2^2\right) \\
&\gtrsim \left(\|v_h\|_2^2 + \|\xi_h\|_{-2}^2\right)
\end{aligned}$$

And we he have

$$\|w_h\|_2 \leq \|v_h\|_2 + \|q_h\|_2 \lesssim \|v_h\|_2 + \|\xi_h\|_{-2}$$

■

3.3.3 A priori error estimate

Theorem 3.12 *Let* (u, λ) *be the solution of the continuous problem* (3.14) *and* (u_h, λ_h) *the solution of the discrete problem* (3.20), *then the following estimate error holds*

$$\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_2 + \inf_{\mu_h \in \Lambda_h} \left(\|\lambda - \mu_h\|_{-2} + \sqrt{\langle u - \psi, \mu_h \rangle}\right) \quad (3.28)$$

Proof. Let $(v_h, \mu_h) \in V_h \times \Lambda_h$. By the previous theorem, there exists $w_h \in V_h$ such that

$$\|w_h\|_1 \lesssim \|u_h - v_h\|_1 + \|\lambda_h - \mu_h\|_{-1}$$

and

$$\left(\|u_h - v_h\|_1 + \|\lambda_h - \mu_h\|_{-1}\right)^2 \lesssim \mathcal{A}\left(\left(u_h - v_h, \lambda_h - \mu_h\right); \left(w_h, \mu_h - \lambda_h\right)\right)$$

Considering the discrete problem statement and by exploiting the bilinearity of $\mathcal{A}$, we obtain

$$\begin{aligned}
\mathcal{A}\left((u_h - v_h, \lambda_h - \mu_h); (w_h, \mu_h - \lambda_h)\right) &= \mathcal{A}\left((u_h, \lambda_h); (w_h, \mu_h - \lambda_h)\right) - \mathcal{A}\left((v_h, \mu_h); (w_h, \mu_h - \lambda_h)\right) \\
&\leq \mathcal{A}\left((u - v_h, \lambda - \mu_h); (w_h, \mu_h - \lambda_h)\right) + \mathcal{L}\left(w_h, \mu_h - \lambda_h\right) \\
&\quad - \mathcal{A}\left((u, \lambda); (w_h, \mu_h - \lambda_h)\right) \\
&= \mathcal{A}\left((u - v_h, \lambda - \mu_h); (w_h, \mu_h - \lambda_h)\right) + \langle u - \psi, \mu_h - \lambda_h \rangle \\
&= \mathcal{A}\left((u - v_h, \lambda - \mu_h); (w_h, \mu_h - \lambda_h)\right) + \langle u - \psi, \mu_h \rangle \\
&\quad + \underbrace{\langle u - \psi, -\lambda_h \rangle}_{\leq 0} \\
&\leq \mathcal{A}\left((u - v_h, \lambda - \mu_h); (w_h, \mu_h - \lambda_h)\right) + \langle u - \psi, \mu_h \rangle
\end{aligned}$$

The bilinear form $\mathcal{A}$ is continuous, therefore

$$\begin{aligned}
&\mathcal{A}\left(u - v_h, \lambda - \mu_h; w_h, \mu_h - \lambda_h\right) \\
&\quad \leq \left(\|u - v_h\|_2 + \|\lambda - \mu_h\|_{-2}\right)\left(\|w_h\|_2 + \|\lambda_h - \mu_h\|_{-2}\right) \\
&\quad \lesssim \left(\|u - v_h\|_2 + \|\lambda - \mu_h\|_{-2}\right)\left(\|u_h - v_h\|_2 + \|\lambda_h - \mu_h\|_{-2}\right)
\end{aligned}$$

Combining the previous estimates, we obtain

$$\begin{aligned}
&\left(\|u_h - v_h\|_2 + \|\lambda_h - \mu_h\|_{-2}\right)^2 \\
&\quad \lesssim \mathcal{A}\left((u - v_h, \lambda - \mu_h); (w_h, \mu_h - \lambda_h)\right) + \langle u - \psi, \mu_h \rangle \\
&\quad \lesssim \left(\|u - v_h\|_2 + \|\lambda - \mu_h\|_{-2}\right)\left(\|u_h - v_h\|_2 + \|\lambda_h - \mu_h\|_{-2}\right) + \langle u - \psi, \mu_h \rangle \\
&\quad \leq \frac{\left(\|u - v_h\|_2 + \|\lambda - \mu_h\|_{-2}\right)^2}{2\epsilon} + \frac{\epsilon\left(\|u_h - v_h\|_2 + \|\lambda_h - \mu_h\|_{-2}\right)^2}{2} + \langle u - \psi, \mu_h \rangle \\
&\quad \lesssim \frac{\left(\|u_h - v_h\|_2 + \|\lambda_h - \mu_h\|_{-2}\right)^2}{2\epsilon} + \left(\|u_h - v_h\|_2 + \|\lambda_h - \mu_h\|_{-2} + \sqrt{\langle u - \psi, \mu_h \rangle}\right)^2
\end{aligned}$$

This implies

$$\|u_h - v_h\|_2 + \|\lambda_h - \mu_h\|_{-2} \lesssim \|u - v_h\|_2 + \|\lambda - \mu_h\|_{-2} + \sqrt{\langle u - g, \mu_h \rangle}$$

The triangle inequality gives:

$$\begin{aligned}
\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} &\leq \|u - v_h\|_2 + \|u_h - v_h\|_2 + \|\lambda - \mu_h\|_{-1} + \|\lambda_h - \mu_h\|_{-2} \\
&\lesssim \|u - v_h\|_2 + \|\lambda - \mu_h\|_{-1} + \sqrt{\langle u - g, \mu_h \rangle}
\end{aligned}$$

Finally, since the choice of (v_h, μ_h) was arbitrary we conclude

$$\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} \lesssim \inf_{v_h \in V_h} \|u - v_h\|_2 + \inf_{\mu_h \in \Lambda_h} \left(\|\lambda - \mu_h\|_{-2} + \sqrt{\langle u - g, \mu_h \rangle}\right)$$

■

3.3.4 A posteriori Error Estimate

Next we derive the a posteriori estimate. We define the following error indicators

$$\eta_i^2 = h^4 \left\| \lambda_h + f \right\|^2_{0,]x_{i-1},x_i[}\,, \quad \eta^2 = \sum_{i=1}^{n-1} \eta_i^2$$

$$S^2 = h^{-4} \left\| (\psi - u_h)_+ \right\|_0^2 + \left\langle (\psi - u_h)_+\,, \lambda_h \right\rangle, \text{ where } (\psi - u_h)_+ = \max\{\psi - u_h, 0\}$$

We also introduce and prove some useful lemmas

Lemma 3.13 *There exists $w \in V$ and $w_h \in V_h$ such that*

$$\|u - u_h\|_2^2 + \|\lambda - \lambda_h\|_{-2}^2 \lesssim \mathscr{L}(w - w_h, \lambda_h - \lambda) - \mathscr{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) \quad (3.29)$$

Proof. We know from theorem (3.5), there exists $w \in V$ such that:

$$\mathcal{A}((u - u_h, \lambda - \lambda_h); (w, \lambda_h - \lambda)) \gtrsim \left(\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2}\right)^2$$

Let w_h be the Hermit interpolant of w, hence we obtain

$$\begin{aligned}
&\left(\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2}\right)^2 \\
&\quad \lesssim \mathcal{A}((u - u_h, \lambda - \lambda_h); (w, \lambda_h - \lambda)) \\
&\quad = \mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{A}((u_h, \lambda_h); (w, \lambda_h - \lambda)) \\
&\quad \lesssim \mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{A}((u_h, \lambda_h); (w, \lambda_h - \lambda)) + \underbrace{\mathcal{L}(-w_h, 0) - \mathcal{A}((u_h, \lambda_h); (-w_h, 0))}_{\geq 0} \\
&\quad = \mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) + \mathcal{L}(-w_h, 0) \\
&\quad = \mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) + \mathcal{L}(-w_h, 0) \\
&\qquad - \mathcal{L}(w, \lambda_h - \lambda) + \mathcal{L}(w, \lambda_h - \lambda) \\
&\quad = \underbrace{\mathcal{A}((u, \lambda); (w, \lambda_h - \lambda)) - \mathcal{L}(w, \lambda_h - \lambda)}_{\leq 0} - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) + \mathcal{L}(w - w_h, 0) \\
&\quad \leq \mathcal{L}(w - w_h, \lambda_h - \lambda) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda))
\end{aligned}$$

■

Lemma 3.14 *The following estimate holds:*

$$\langle u_h - \psi, \lambda_h - \lambda \rangle \leq \left\| (\psi - u_h)_+ \right\|_2 \|\lambda - \lambda_h\|_{-2} + \left\langle (\psi - u_h)_+\,, \lambda_h \right\rangle \quad (3.30)$$

Proof.
If we take $v_h = 0$ in the discrete problem (3.20) we get:

$$-\langle u_h, \mu_h - \lambda_h \rangle \leq -\langle \psi_h, \mu_h - \lambda_h \rangle \quad (3.31)$$

For the choices $\mu_h = 0$ and $\mu_h = 2\lambda_h$ in (3.31) we find that : $\langle u_h - \psi, \lambda_h\rangle = 0$. Hence,

$$\begin{aligned}\langle u_h - \psi, \lambda_h - \lambda\rangle &= \langle \psi - u_h, \lambda\rangle \\ &\leq \left\langle (\psi - u_h)_+, \lambda - \lambda_h\right\rangle + \left\langle (\psi - u_h)_+, \lambda_h\right\rangle \\ &\leq \left\|(\psi - u_h)_+\right\|_2 \|\lambda - \lambda_h\|_{-2} + \left\langle (\psi - u_h)_+, \lambda_h\right\rangle\end{aligned}$$

■

We proceed to prove the reliability and efficiency of the proposed error indicators.

Theorem 3.15 ***(Reliability)***

The following posteriori error estimate holds

$$\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} \lesssim \eta + S \tag{3.32}$$

Proof. From Lemma (3.13)

$$\begin{aligned}&\left(\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2}\right)^2 \\ &\lesssim \mathcal{L}(w - w_h, \lambda_h - \lambda) - \mathcal{A}((u_h, \lambda_h); (w - w_h, \lambda_h - \lambda)) \\ &= (f, w - w_h) - \langle \psi, \lambda_h - \lambda\rangle - a(u_h, w - w_h) + \langle \lambda_h, w - w_h\rangle + \langle \lambda_h - \lambda, u_h\rangle\end{aligned}$$

We have:

$$\begin{aligned}\int_{x_i}^{x_{i+1}} u^{(4)} v dx = \int_{x_i}^{x_{i+1}} \left(u^{(3)}\right)' v dx &= \left[u^{(3)} v\right]_{x_i}^{x_{i+1}} - \int_{x_i}^{x_{i+1}} u^{(3)} v' dx \\ &= \left[u^{(3)} v\right]_{x_i}^{x_{i+1}} - \left[u^{(2)} v'\right]_{x_i}^{x_{i+1}} + \int_{x_i}^{x_{i+1}} u'' v'' dx\end{aligned}$$

We also have $w(x_i) = w_h(x_i), w'(x_i) = w'_h(x_i)$ et $u_h|_{[x_i, x_{i+1}]} \in \mathbb{P}^3$ Therefore

$$a(u_h, w - w_h) = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} u''_h (w - w_h)'' dx = \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} u_h^{(4)} (w - w_h) dx = 0$$

Consequently, by using lemma (3.14)

$$\begin{aligned}\left(\|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2}\right)^2 &\lesssim (f + \lambda_h, w - w_h) + \langle \lambda_h - \lambda, u_h - \psi\rangle \\ &= \sum_{i=0}^{n-1} \int_{x_i}^{x_{i+1}} h^2 (f + \lambda_h) h^{-2} (w - w_h) dx + \langle \lambda_h - \lambda, u_h - \psi\rangle \\ &\leq \left(\sum_{i=0}^{n-1} \eta_i^2\right)^{1/2} \|w - w_h\|_0 + \left\|(\psi - u_h)_+\right\|_2 \|\lambda - \lambda_h\|_{-2} + \left\langle (\psi - u_h)_+, \lambda_h\right\rangle \\ &\lesssim \left(\sum_{i=0}^{n-1} \eta_i^2\right)^{1/2} \|w\|_2 + \frac{\left\|(\psi - u_h)_+\right\|_2^2}{2\varepsilon} + \frac{\varepsilon \|\lambda - \lambda_h\|_{-2}^2}{2} + \left\langle (\psi - u_h)_+, \lambda_h\right\rangle\end{aligned}$$

Reminder of estimates of Hermite interpolant

$$\|w_h\|_2 \le C\|w\|_{2,\Omega}$$
$$h^{-2}\|w - w_h\|_{0,\Omega} \le C\|w\|_{2,\Omega}$$

And we also have

$$\|w\|_2 \lesssim \|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2}$$

By using this last inequality we obtain the desired result. ■

Theorem 3.16 ***(efficiency)***
The following estimate holds:

$$\eta \lesssim \|u - u_h\|_2 + \|\lambda - \lambda_h\|_{-2} + \operatorname{osc}(f) \tag{3.33}$$

Proof. We use the the bubble functions $b_i \in H_0^2([x_{i-1}, x_i])$, and we define γ_i by

$$\gamma_i = h^4 b_i (\lambda_h + f_h) \text{ in } [x_{i-1}, x_i] \quad \text{and} \quad \gamma_i = 0 \quad \text{in } \Omega \backslash [x_{i-1}, x_i]$$

Taking $v_h = \gamma_i$ in (3.13) we get:

$$(u'', \gamma_i'') - \langle \gamma_i, \lambda \rangle = (f, \gamma_i) \tag{3.34}$$

Using the above and the characteristics of bubble functions we get

$$\begin{aligned} h^4 \|\lambda_h + f_h\|^2_{0,[x_{i-1},x_i]} &\lesssim h^4 \left\| \sqrt{b_i}\,(\lambda_h + f_h) \right\|^2_{0,[x_{i-1},x_i]} \\ &= (\lambda_h + f_h, \gamma_i) \\ &= (\lambda_h, \gamma_i) + (f, \gamma_i) + (f_h - f, \gamma_i) \\ &= \left(u_h^{(4)} + \lambda_h, \gamma_i\right) + (u'', \gamma_i'') - \langle \gamma_i, \lambda \rangle + (f_h - f, \gamma_i) \\ &= \left((u - u_h)'', \gamma_i''\right) + \langle \gamma_i, \lambda_h - \lambda \rangle + (f_h - f, \gamma_i) \end{aligned}$$

Defining $\gamma = \sum_{i=1}^{n-1} \gamma_i$, and summing over all intervals we get:

$$\begin{aligned} &\sum_{i=1}^{n-1} h^4 \|\mu_h + f_h\|_0^2 \\ &\quad \lesssim \sum_{i=1}^{n-1} \left\{ \left((u - v_h)'', \gamma_i''\right) + \langle \gamma_i, \lambda_h - \lambda \rangle + (f_h - f, \gamma_i) \right\} \\ &\quad = \left((u - v_h)'', \gamma''\right) + \langle \gamma, \lambda_h - \lambda \rangle + (f_h - f, \gamma) \\ &\quad \le \|u - u_h\|_2 \|\gamma\|_2 + \|\lambda_h - \lambda\|_{-2} \|\gamma\|_2 + \operatorname{osc}(f) \left(\sum_{i=1}^{n-1} h^{-4} \|\gamma\|^2_{0,[x_{i-1},x_i]} \right)^{\frac{1}{2}} \\ &\quad \lesssim \left(\|u - u_h\|_2 + \|\lambda_h - \lambda\|_{-2} + \operatorname{osc}(f) \right) h^{-2} \|\gamma\|_0 \end{aligned} \tag{3.35}$$

However:

$$\begin{aligned}
h^{-4}\|\gamma\|_0^2 &= h^{-4}\sum_{i=1}^{n-1}\|\gamma_i\|_0^2 \\
&= h^4\sum_{i=1}^{n-1}\|b_i\left(\lambda_h + f_h\right)\|_0^2 \\
&\lesssim \sum_{i=1}^{n-1} h^4\left\|\lambda_h + f_h\right\|_0^2
\end{aligned} \tag{3.36}$$

Thus we conclude

$$\eta \lesssim \|u - u_h\|_2 + \|\lambda_h - \lambda\|_{-2} + \mathrm{osc}(f)$$

■

3.4 Mixed Formulation using Bi-orthogonal dual

In this section we introduce new space of the discrete Lagrangian, based on the works of Barbara Wohlmuth [Woh11, Woh01]. We apply similar idea of deriving bi-orthogonal basis of $\mathbb{P}_n$ elements, and we derive a bi-orthogonal basis for $C^1 \cap \mathbb{P}_3$ Hermite elements. Let V_h be Hermite finite element space generated by $B_{PH3} = \{\phi_i, \psi_i : i = 1...n-1\}$, and Q_h the dual space generated by $B = \{\overline{\Phi}_i, \overline{\Psi}_i : i = 1...n-1\}$. We want the basis to satisfy the following properties :

- Locality of the support

$$\begin{aligned}
\text{supp }\overline{\Phi}_p &= \text{ supp }\Phi_p \\
\text{supp }\overline{\Psi}_p &= \text{ supp }\Psi_p
\end{aligned} \tag{3.37}$$

- Local biorthogonality relation

$$\begin{aligned}
\int \Phi_i\overline{\Phi}_j &= \delta_{ij} \\
\int \Psi_i\overline{\Psi}_j &= \delta_{ij} \\
\int \Phi_i\overline{\Psi}_j &= 0 \\
\int \Psi_i\overline{\Phi}_j &= 0
\end{aligned} \tag{3.38}$$

- Best approximation property:

$$\inf_{\mu_h \in Q_h}\|\mu - \mu_h\| \lesssim h^2|\mu|_2 \tag{3.39}$$

- Uniform inf-sup condition

$$\sup_{v_h \in X_h}\frac{b(v_h, \mu_h)}{\|v_h\|_2} \gtrsim \|\mu_h\|_{-2} \tag{3.40}$$

The key idea is to write $\overline{\Phi}_i$ as linear combination of elements B_{PH3}, and because we want supp $\overline{\Phi}_p =$ supp Φ_p, we obtain:

$$\overline{\Phi}_i = (a_1\Phi_i + a_2\Psi_i + a_3\Phi_{i-1} + a_4\Psi_{i-1} + a_5\Phi_{i+1} + a_6\Psi_{i+1}) \text{ in supp } \Phi_i, \quad 0 \text{ Otherwise.}$$

We define:

$$V = \begin{bmatrix} \Phi_i \\ \Psi_i \\ \Phi_{i-1} \\ \Psi_{i-1} \\ \Phi_{i+1} \\ \Phi_{i+1} \end{bmatrix} [\Phi_i \ \Psi_i \ \Phi_{i-1} \ \Psi_{i-1} \ \Phi_{i+1} \ \Psi_{i+1}] \tag{3.41}$$

taking (3.38) into account we get the following matrix equation:

$$M \begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3.42}$$

Where M is 6×6 matrix, and

$$M_{ij} = \int_{x_{i-1}}^{x_{i+1}} V_{ij} dx \tag{3.43}$$

Or

$$M = h \begin{pmatrix} \frac{26}{35} & 0 & \frac{9}{70} & \frac{13h}{420} & \frac{9}{70} & \frac{-13h}{420} \\ 0 & \frac{2h^2}{105} & \frac{-13h}{420} & \frac{-h^2}{140} & \frac{13h}{420} & \frac{-h^2}{140} \\ \frac{9}{70} & \frac{-13h}{420} & \frac{13}{35} & \frac{11h}{210} & 0 & 0 \\ \frac{13h}{420} & \frac{-h^2}{140} & \frac{11h}{210} & \frac{h^2}{105} & 0 & 0 \\ \frac{9}{70} & \frac{13h}{420} & 0 & 0 & \frac{13}{35} & \frac{-11h}{210} \\ \frac{-13h}{420} & \frac{-h^2}{140} & 0 & 0 & \frac{-11h}{210} & \frac{h^2}{105} \end{pmatrix} \tag{3.44}$$

We easily calculate its inverse

$$M^{-1} = h^{-1} \begin{pmatrix} 2 & 0 & 1 & \frac{-12}{h} & 1 & \frac{12}{h} \\ 0 & \frac{150}{h^2} & \frac{-15}{h} & \frac{195}{h^2} & \frac{15}{h} & \frac{195}{h^2} \\ 1 & \frac{-15}{h} & 14 & \frac{-183}{2h} & -1 & \frac{-27}{2h} \\ \frac{-12}{h} & \frac{195}{h^2} & \frac{-183}{2h} & \frac{1587}{2h^2} & \frac{27}{2h} & \frac{363}{2h^2} \\ 1 & \frac{15}{h} & -1 & \frac{27}{2h} & 14 & \frac{183}{2h} \\ \frac{12}{h} & \frac{195}{h^2} & \frac{-27}{2h} & \frac{363}{2h^2} & \frac{183}{2h} & \frac{1587}{2h^2} \end{pmatrix} \tag{3.45}$$

Finally, the the coordinates of $\overline{\Phi}_i$ are

$$\begin{bmatrix} a_1 \\ \vdots \\ a_6 \end{bmatrix} = M^{-1} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = h^{-1} \begin{pmatrix} 2 \\ 0 \\ 1 \\ \frac{-12}{h} \\ 1 \\ \frac{12}{h} \end{pmatrix} \tag{3.46}$$

Similarly, the coordinates of $\overline{\Psi}_i$ are

$$\begin{bmatrix} b_1 \\ \vdots \\ b_6 \end{bmatrix} = M^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = h^{-2} \begin{pmatrix} 0 \\ \frac{150}{h} \\ -15 \\ \frac{195}{h} \\ 15 \\ \frac{195}{h} \end{pmatrix} \tag{3.47}$$

Figure 3.2: Graph of the biorthogonal basis of Hermite elements

With this construction of Q_h basis, the conditions (3.37), (3.38) are clearly verified, and so is the condition (3.39). Indeed, since $\overline{\Phi}$ and $\overline{\Psi}$ are locally linear combination of Hermite finite element basis we deduce:

$$\inf_{\mu \in Q_h} \|\mu - \mu_h\| \leq \|\mu - \mathcal{I}_h^{her}\mu\| \lesssim h^2|\mu|_2 \tag{3.48}$$

For the inf-sup condition we use similar method like we used to prove the stability of the pair $(\mathcal{P}_3^{\text{Hermit}}, \mathbb{P}_0)$ in the previous section. We introduce the following negative discrete norm:

$$\|\xi_h\|_{-2,h}^2 = \sum_{i=1}^{n} h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 = \left(h^2\|\xi_h\|_0\right)^2, \quad \forall \xi_h \in Q_h$$

Now we prove the following:

Lemma 3.17 *There exists a constant $C_4 > 0$ such that*

$$\forall \xi_h \in Q_h : \sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} \geq C_4 \|\xi_h\|_{-2,h} \tag{3.49}$$

Proof. We need the next formula:

$$\forall a, b, c, d \in \mathbb{R} : (a + b + c + d)^2 \leq 4(a^2 + b^2 + c^2 + d^2).$$

And let $\Phi, \Psi, \overline{\Phi}, \overline{\Psi}$ the reference basis on $[-1, 1]$ of $\Phi_i, \Psi_i, \overline{\Phi}_i, \overline{\Psi}_i$, then we can easily verify:

$$\begin{aligned} \|\Phi_i\|_0^2 &= h\|\Phi\|_0^2, & \|\Psi_i\|_0^2 &= h^2\|\Psi\|_0^2 \\ \|\overline{\Phi}_i\|_0^2 &= h^{-1}\|\overline{\Phi}\|_0^2, & \|\overline{\Psi}_i\|_0^2 &= h^{-2}\|\overline{\Psi}\|_0^2 \end{aligned}$$

Let $\xi_h = \sum_{i=1}^{n-1} \alpha_i \overline{\Phi}_i + \beta_i \overline{\Psi}_i \in Q_h$, we take:

$$v_h = \sum_{i=1}^{n-1} \frac{\alpha_i}{h} \Phi_i + \frac{\beta_i}{h^2} \Psi_i$$

Then it follows immediately that:

$$\langle \xi_h, v_h \rangle = h^4 \sum_{i=1}^{n-1} \frac{\alpha_i^2}{h} + \frac{\beta_i^2}{h^2}$$

Next we estimate the negative discrete norm of ξ_h. For convenience we assume $\alpha_0 = \beta_0 = 0$ and $\overline{\Phi}_0 = \overline{\Psi}_0 = 0$.

$$\begin{aligned}
\|\xi_h\|_{-2,h}^2 &= \sum_{i=1}^{n} h^4 \int_{x_{i-1}}^{x_i} \xi_h^2 \\
&= \sum_{i=1}^{n} h^4 \int_{x_{i-1}}^{x_i} \left(\alpha_{i-1}\overline{\Phi}_{i-1} + \beta_{i-1}\overline{\Psi}_{i-1} + \alpha_i \overline{\Phi}_i + \beta_i \overline{\Psi}_i\right)^2 \\
&\leq 4h^4 \sum_{i=1}^{n-1} \int_{x_{i-1}}^{x_i} \left(\alpha_{i-1}\overline{\Phi}_{i-1}\right)^2 + \left(\beta_{i-1}\overline{\Psi}_{i-1}\right)^2 + \left(\alpha_i \overline{\Phi}_i\right)^2 + \left(\beta_i \overline{\Psi}_i\right)^2 \\
&= 4h^4 \sum \alpha_i^2 \|\overline{\Phi}_i\|^2 + \beta_i^2 \|\overline{\Psi}_i\|^2 \\
&= 4h^4 \sum \frac{\alpha_i^2}{h} \|\overline{\Phi}\|^2 + \frac{\beta_i^2}{h^2} \|\overline{\Psi}\|^2 \\
&\lesssim h^4 \sum_{i=1}^{n-1} \frac{\alpha_i^2}{h} + \frac{\beta_i^2}{h^2} \\
&= \langle \xi_h, v_h \rangle
\end{aligned}$$

Similarly we find

$$\|v_h\|_0^2 \lesssim h^4 \langle \xi_h, v_h \rangle$$

And by employing the inverse inequality (3.22), we get

$$\|v_h\|_2^2 \leq C h^{-4} \|v_h\|_0^2 \lesssim (\langle \xi_h, v_h \rangle)^2$$

Therefore we conclude

$$\sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} \geq C_4 \|\xi_h\|_{-2,h}$$

■

For the rest, we use similar proof of theorem (3.21) since the detail of proof are largely independent on the space choosen for the dual. Hence by introducing the following spaces and sets:

$$\begin{cases} V_h := \{v_h \in V : v_{h|[x_{i-1},x_i]} \in \mathbb{P}_3\left([x_{i-1}, x_i]\right), \forall i = 1, \ldots, n\} = \mathcal{P}_3^{\text{Hermit}} \cap H_0^2(\Omega) \\ Q_h := < B > \\ \Lambda_h := \{\mu_h \in Q_h : (\mu_h)_i \geq 0 \quad \forall i = 1 \ldots n-1\}. \end{cases} \tag{3.50}$$

We obtain

Theorem 3.18 *The finite element space pair* (V_h, Q_h) *in* (3.50) *is stable and satisfy inf-sup condition*

$$\sup_{v_h \in V_h} \frac{\langle v_h, \xi_h \rangle}{\|v_h\|_2} \geq C \|\xi_h\|_{-2}$$

Likewise all results about existence, error estimates from the previous section stays the same (albeit with different constants) since the proofs are independent from the finite discrete spaces chosen as long as these spaces are stable and verify inf-sup condition.

3.5 Conclusion

As we've seen, we derived the naive finite element formulation of the problem and we showed its inadequacy for numerical analysis. Hence we used another more suitable stable mixed finite element formulation, which we derived its error estimates, priori and posteriori. It rests now to test and validate this formulation numerically, which what will be done in the next chapter.

Chapter 4

Numerical tests

4.1 Introduction

In this chapter we'll give brief description of two methods to solve variational inequalities: Uzawa Method and Primal Dual Active Set (PDAS). Later we implement Euler Bernoulli obstacle problems using these methods in Freefem++, and we numerically analyze the results.

4.2 Uzawa Method

Let $N = dim(V_h),\ L = dim(Q_h)$ the discrete problem can formulated as follows: Find $u \in \mathbb{R}^N, \lambda \in \Lambda$ such that

$$Au + B^T\lambda = f \tag{4.1}$$

$$(\mu - \lambda)^T Bu \leq (\mu - \lambda)^T g \quad \forall \mu \in \Lambda \tag{4.2}$$

where

$$\Lambda = \left\{\mu \in \mathbb{R}^L : \quad \mu_k \geq 0\right\}$$

Using equation (4.1), we have

$$u = -A^{-1}\left(B^T\lambda - f\right) \tag{4.3}$$

Inserting this into our inequality system (4.2), we get

$$-(\mu - \lambda)^T BA^{-1}B^T\lambda \leq (\mu - \lambda)^T\left(g - BA^{-1}f\right)$$

This leaves us with a problem for λ only. Setting the Schur complement $S = BA^{-1}B^T$ and right hand side $h = BA^{-1}f - g$, we obtain

$$(\mu - \lambda)^T(S\lambda - h) \geq 0 \quad \forall \mu \in \Lambda \tag{4.4}$$

The Schur complement S is symmetric as A is symmetric. We also have that A is positive definite and the rank of $B = L$, therefore also S is positive definite and there exists numbers s_1, s_2 such that

$$\langle S\mu, \mu\rangle \geq s_1\|\mu\|^2(S\mu, \lambda) \leq s_2\|\mu\|\|\lambda\|$$

Due to this, equation (4.4) is equivalent to minimization problem

$$\mathcal{F}(\lambda) = \min_{\mu\in\Lambda} \mathcal{F}(\mu) \tag{4.5}$$

Where $\mathcal{F}(\mu) = \frac{1}{2}\mu^T S\mu - h^T\mu$. Therefore

$$\nabla(\mathcal{F}(\mu)) = S\mu - h$$

If we have an algorithm to minimize $\mathcal{F}$ over Λ, then we can determine u from the equation (4.3). In the following, we will need the projection operator P_Λ, which acts from R^L onto Λ.

Lemma 4.1 *The components of $P_\Lambda(\mu)$ are given by:*

$$P_\Lambda(\mu)_k = \max(\mu_k, 0)$$

4.2.1 Uzawa's algorithm

In this subsection, we'll briefly describe the classical Uzawa algorithm for solving the inequalities of the form (4.1,4.2). See [Arr58] more details about the algorithm.
Starting with some initial guess, λ_h can be computed from the constrained minimization problem (4.5). Then equation (4.3) allows the computation of u_h. A classical method of this type is the Uzawa algorithm [Arr58], which relies on an exact solver for the equation (4.1) and a Jacobi-like iteration for the constrained minimization problem (4.5). Below we outline the algorithm for this method.

Algorithm 1 Uzawa

give some initial value $\lambda^{(0)}$
k = 0
repeat
$\lambda_*^{(k+1)} := \lambda^{(k)} - \alpha M^{-1}\left(S\lambda^{(k)} - h\right)$
Take $\lambda^{(k+1)}$ as the projection of $\lambda_*^{(k+1)}$ on $\Lambda : \lambda^{(k+1)} := P_\Lambda\left(\lambda_*^{(k+1)}\right)$
k = k + 1
until $\left\|\lambda^{(k+1)} - \lambda^{(k)}\right\| \leq \varepsilon \left\|\lambda^{(k+1)}\right\|$
$u_h = -A^{-1}\left(B^T\lambda^{(k+1)} - f\right)$

Such that M is the mass matrix of Q_h.
The convergence of Uzawa algorithm depends The choice of the parameter α, like shown in the following theorem

Theorem 4.2 *Let (u, λ) be a solution to the system (4.1,4.2). Let s_1, s_2 denote the smallest and the largest eigenvalues of $M^{-1}S$, and let $\left(u^{(k)}, \lambda^{(k)}\right)$ be defined by Uzawa's method. Then there exists a positive constant $\bar{\alpha} = \frac{2}{\lambda_2} > 0$ such that for each choice $\alpha \in (0, \bar{\alpha})$ there holds*

$$u^{(k)} \to u, \quad \lambda^{(k)} \to \lambda.$$

Proof. see [Arr58] ■

4.3 Primal Dual Active Set Method

In this section we'll discuss primal-dual active set strategy [HIK02] for the obstacle problem. Recall the mixed formulation (3.13), we make a slight change by introducing

$$b(v,\mu) = -\langle \mu, v\rangle,$$

Hence we can write the problem as

$$\begin{cases} \text{Find } (u,\lambda) \in V \times \Lambda \\ a(u,v) + b(v,\lambda) = f(v), & \forall v \in X \\ \quad b(u,\mu-\lambda) \le g(\mu-\lambda), & \forall \mu \in \Lambda \end{cases} \tag{*}$$

Where $\Lambda = \{\mu \in V' : \langle v,\mu\rangle \le 0, \quad \forall v \ge 0 \in V\}$. We observe that $(*)$ is an optimization problem under the constraint of the KKT triple:

$$u \ge \psi, \quad \lambda \ge 0, \quad b(u,\lambda) = g(\lambda) \tag{4.6}$$

Since the finite spaces can be written as $V_h = <\phi_p>$, $Q_h = <\Psi_p>$, then in the discrete problem we obtain

$$u_h = \sum \alpha_p\phi_p, \quad \lambda_h = \sum_p \beta_p\Psi_p, \quad \psi = \psi_h = \sum_p g_p\phi_p$$

Then the discrete version of the KKT conditions reads:

$$\alpha_p \ge g_p, \qquad \beta_p \le 0, \qquad (\alpha_p - g_p)\beta_p = 0 \tag{4.7}$$

Lemma 4.3 *The KKT [1] conditions*

$$\alpha_p \ge g_p, \qquad \beta_p \le 0, \qquad (\alpha_p - g_p)\beta_p = 0 \tag{4.8}$$

is equivalent to $C(\alpha_p,\beta_p) = 0$ *with*

$$C(x,y) = y - \min(0, y + c(x-g)), \quad c > 0 \textit{ fixed}$$

Consequently, we can rewrite the discret version of primal-dual variational inequality as a non-linear equality formulation.

$$\begin{cases} A_h\alpha_h + B_h\beta_h = f_h \\ \quad C_h(\alpha_h,\beta_h) = 0 \end{cases} \tag{4.9}$$

If the nonlinear function C_h is differentiable, applying Newton's method we get

$$\begin{pmatrix} \alpha_h^{\ell+1} \\ \beta_h^{\ell+1} \end{pmatrix} = \begin{pmatrix} \alpha_h^{\ell} \\ \beta_h^{\ell} \end{pmatrix} - \begin{pmatrix} A_h & B_h \\ \partial_{\alpha_h}C_h & \partial_{\beta_h}C_h \end{pmatrix}^{-1} \begin{pmatrix} A_h\alpha_h^\ell + B_h\beta_h^\ell - f_h \\ C_h(\alpha_h^\ell,\beta_h^\ell) \end{pmatrix}$$

[1] Karush-Kuhn-Tucker (KKT) conditions

Multiplying on both sides of the system with the Jacobian leads to a system of linear equations to be solved in each Newton step.

$$\begin{cases} A_h\alpha_h^{\ell+1} + B_h\beta_h^{\ell+1} = f_h \\ \partial_{\alpha_h}C_h\alpha_h^{\ell+1} + \partial_{\beta_h}C_h\beta_h^{\ell+1} = \partial_{\alpha_h}C_h\alpha_h^{\ell} + \partial_{\beta_h}C_h\beta_h^{\ell} - C_h(\alpha_h^{\ell}, \beta_h^{\ell}) \end{cases} \tag{4.10}$$

The nonlinear function C_h is called NCP function and it maps $\mathbb{R}^{n_h} \times \mathbb{R}^{n_h}$ onto $\mathbb{R}^{n_h}$ where n_h is the number of vertices

$$C_h(\alpha_h, \beta_h)_p = C_p(\alpha_p, \beta_p) = \beta_p - \min(0, \beta_p + c(\alpha_p - g_p))$$

Ignoring the fact that the min function is not differentiable we get:

1. Case $p \neq q$
$$\frac{\partial C_p}{\partial \alpha_q} = \frac{\partial C_p}{\partial \beta_q} = 0,$$
2. Case $\beta_p + c(\alpha_p - g_p) > 0$. Then, $C_p(\alpha_p, \beta_p) = \beta_p$ and thus
$$\frac{\partial C_p}{\partial \alpha_p} = 0 \quad \text{and} \quad \frac{\partial C_p}{\partial \beta_p} = 1$$
3. Case $\beta_p + c(\alpha_p - g_p) < 0$. Then, $C_p(\alpha_p, \beta_p) = -c(\alpha_p - g_p)$ and thus
$$\frac{\partial C_p}{\partial \alpha_p} = -c \quad \text{and} \quad \frac{\partial C_p}{\partial \beta_p} = 0$$
4. Case $\beta_p + c(\alpha_p - g_p) = 0$. This case is not well-defined but thanks to the next result we can safely treat it as case < 0.

   Lemma 4.4 *[HIK02]*
   The mapping $y \to \max(0, y)$ *from* $\mathbb{R}^n$ *to* $\mathbb{R}^n$ *is Newton differentiable on* $\mathbb{R}^n$

The system (4.10) reads:

$$\begin{pmatrix} A_h & B_h \\ \partial_{\alpha_h}C_h & \partial_{\beta_h}C_h \end{pmatrix} \begin{pmatrix} \alpha_h^{\ell+1} \\ \beta_h^{\ell+1} \end{pmatrix} = \begin{pmatrix} f_h \\ \partial_{\alpha_h}C_h\alpha_h^{\ell} + \partial_{\beta_h}C_h\beta_h^{\ell} - C_h(\alpha_h^{\ell}, \beta_h^{\ell}) \end{pmatrix}$$

1. Case $\beta_p + c(\alpha_p - g_p) > 0$. Then, $C_p(\alpha_p, \beta_p) = \beta_p$ and thus
$$\frac{\partial C_p}{\partial \alpha_p} = 0 \quad \text{and} \quad \frac{\partial C_p}{\partial \beta_p} = 1$$
   then from the second block line we find for node p
$$\beta^{\ell+1} = \beta^{\ell} - \beta^{\ell} = 0$$
2. Case $\beta_p + c(\alpha_p - g_p) \leq 0$. Then, $C_p(\alpha_p, \beta_p) = -c(\alpha_p - g_p)$ and thus
$$\frac{\partial C_p}{\partial \alpha_p} = -c \quad \text{and} \quad \frac{\partial C_p}{\partial \beta_p} = 0$$
   and thus,
$$-c\alpha_p^{\ell+1} = -c\alpha_p^{\ell} - (-c(\alpha_p^{\ell} - g_p)) = -cg_p$$

Algorithm 2 PDAS Algorithm

Let $\mathcal{A}_h^0 \subset \mathscr{P}_h = \{x_0, ..., x_n\}$ be given and $\mathcal{I}_h^0 = \mathscr{P}_h \setminus \mathcal{A}_h^0$.
For $\ell = 0, 1, 2, \ldots$

1. Set

$$\alpha_p^\ell = g_p \quad \text{in } \mathcal{A}_h^\ell \tag{4.11}$$
$$\beta_p^\ell = 0 \quad \text{in } \mathcal{I}_h^\ell \tag{4.12}$$

2. Solve

$$A_h \alpha^\ell + B_h \beta^\ell = f \tag{4.13}$$

3. Update the active set

$$\mathcal{A}_h^{\ell+1} = \{p \in \mathcal{A}_h^\ell; \quad \beta_p^\ell \le 0\} \cup \{p \in \mathcal{I}_h^\ell; \quad \alpha_p^\ell \le 0\} \tag{4.14}$$
$$\mathcal{I}_h^{\ell+1} = \mathscr{P}_h \setminus \mathcal{A}_h^{\ell+1} \tag{4.15}$$

4. Stop if $\mathcal{A}_h^{\ell+1} = \mathcal{A}_h^\ell$

It is convenient to arrange the coordinates in such a way that the active and inactive ones occur in consecutive order. In step 2, system (4.13) implies

$$\begin{pmatrix} A_\mathcal{A} & A_{\mathcal{A}\mathcal{I}} \\ A_{\mathcal{I}\mathcal{A}} & A_\mathcal{I} \end{pmatrix} \begin{pmatrix} g_\mathcal{A} \\ \alpha_\mathcal{I} \end{pmatrix} + \begin{pmatrix} D_\mathcal{A} & 0 \\ 0 & D_\mathcal{I} \end{pmatrix} \begin{pmatrix} \beta_\mathcal{A} \\ 0 \end{pmatrix} = \begin{pmatrix} f_\mathcal{A} \\ f_\mathcal{I} \end{pmatrix} \tag{4.16}$$

Note that the diagonal property of B_h results from the fact that biorthogonal basis functions of V_h and Q_h have been used.
Then (4.16) implies that:

$$A_\mathcal{A} g_\mathcal{A} + A_{\mathcal{A}\mathcal{I}} \alpha_\mathcal{A} + D_\mathcal{A} \beta_\mathcal{A} = f_\mathcal{A} \implies D_\mathcal{A} \beta_\mathcal{A} = f_\mathcal{A} - A_\mathcal{A} g_\mathcal{A} - A_{\mathcal{A}\mathcal{I}} \alpha_\mathcal{A} \tag{4.17}$$

then we write

$$\begin{pmatrix} A_\mathcal{A} & A_{\mathcal{A}\mathcal{I}} \\ A_{\mathcal{I}\mathcal{A}} & A_\mathcal{I} \end{pmatrix} \begin{pmatrix} g_\mathcal{A} \\ \alpha_\mathcal{I} \end{pmatrix} + \begin{pmatrix} f_\mathcal{A} - A_\mathcal{A} g_\mathcal{A} - A_{\mathcal{A}\mathcal{I}} \alpha_\mathcal{A} \\ 0 \end{pmatrix} = \begin{pmatrix} f_\mathcal{A} \\ f_\mathcal{I} \end{pmatrix} \tag{4.18}$$

this implies

$$\begin{pmatrix} A_\mathcal{A} & A_{\mathcal{A}\mathcal{I}} \\ A_{\mathcal{I}\mathcal{A}} & A_\mathcal{I} \end{pmatrix} \begin{pmatrix} g_\mathcal{A} \\ \alpha_\mathcal{I} \end{pmatrix} - \begin{pmatrix} 0 & A_{\mathcal{A}\mathcal{I}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_\mathcal{A} \\ \alpha_\mathcal{I} \end{pmatrix} = \begin{pmatrix} A_\mathcal{A} g_\mathcal{A} \\ f_\mathcal{I} \end{pmatrix} \tag{4.19}$$

thus

$$\begin{pmatrix} A_\mathcal{A} & 0 \\ A_{\mathcal{I}\mathcal{A}} & A_\mathcal{I} \end{pmatrix} \begin{pmatrix} \alpha_\mathcal{A} \\ \alpha_\mathcal{I} \end{pmatrix} = \begin{pmatrix} A_\mathcal{A} g_\mathcal{A} \\ f_\mathcal{I} \end{pmatrix} \tag{4.20}$$

or more simply, (by multiplying both sides by $\begin{pmatrix} A_A^{-1} & 0 \\ 0 & I_{\mathcal{I}} \end{pmatrix}$) we get

$$\begin{pmatrix} I_{\mathcal{A}} & 0 \\ A_{\mathcal{I}\mathcal{A}} & A_{\mathcal{I}} \end{pmatrix} \begin{pmatrix} \alpha_{\mathcal{A}} \\ \alpha_{\mathcal{I}} \end{pmatrix} = \begin{pmatrix} g_{\mathcal{A}} \\ f_{\mathcal{I}} \end{pmatrix} \tag{4.21}$$

In freefem++ software, system (4.21) must written as :

$$\begin{pmatrix} \text{tgv } I_{\mathcal{A}} & 0 \\ A_{\mathcal{I}\mathcal{A}} & A_{\mathcal{I}} \end{pmatrix} \begin{pmatrix} \alpha_{\mathcal{A}} \\ \alpha_{\mathcal{I}} \end{pmatrix} = \begin{pmatrix} \text{tgv } g_{\mathcal{A}} \\ f_{\mathcal{I}} \end{pmatrix} \tag{4.22}$$

So an alternative algorithm is as follows

Algorithm 3 PDAS Algorithm 2

Let $\mathcal{A}_h^0 \subset \mathscr{P}_h = \{x_0, ..., x_n\}$ be given and $\mathcal{I}_h^0 = \mathscr{P}_h \setminus \mathcal{A}_h^0$.
For $\ell = 0, 1, 2, \ldots$

1. Set

$$\alpha_p^\ell = g_p \quad \text{in } \mathcal{A}_h^\ell$$

2. Solve equation (4.22) to get the value of α^ℓ
3. Calculate β^ℓ from (4.17)
4. Update the active set

$$\begin{aligned} \mathcal{A}_h^{\ell+1} &= \{p \in \mathcal{A}_h^\ell; \quad \beta_p^\ell \leq 0\} \cup \{p \in \mathcal{I}_h^\ell; \quad \alpha_p^\ell \leq 0\} \\ \mathcal{I}_h^{\ell+1} &= \mathscr{P}_h \setminus \mathcal{A}_h^{\ell+1} \end{aligned}$$

5. Stop if $\mathcal{A}_h^{\ell+1} = \mathcal{A}_h^\ell$

4.4 Tests and Results

In this section, we'll solve numerically the obstacle problem using Uzawa method and PDAS method which were presented earlier in the chapter. Uzawa method will be used for the stable mixed formulation with $(P_3^{\text{Hermite}}, P_0)$ couple, while PDAS method will be used for the mixed formulation with biorthogonal basis. The implementation was prepared by the FreeFem++, and the graphic representation was done using Python. We'll present the results for several examples, and for all these examples:

- We use the interval $\Omega =]0, 1[$,
- The discretisation is uniform,
- The initial value chosen to be the null vector,
- The error stopping criteria was chosen to be $\epsilon = 10^{-6}$.

4.4.1 Example 1: Contact Zone is a closed interval

We take $f = 0$ and $\psi = 4x^3 - 9x^2 + 6x - 1$

$$\begin{cases} \text{find } u \in K := \{v \in H_0^2(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \\ a(u, v - u) \geq 0, \quad \forall v \in K \end{cases}$$

The exact solution of this problem is

$$u(x) = \begin{cases} -4x^3 + 3x^2 & \text{if } x \in [0, \frac{1}{2}[\\ 4x^3 - 9x^2 + 6x - 1 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

First we investigate the value of *alpha* for the Uzawa algorithm. Because in general it's hard to find the exact values of eigenvalues of a matrix we ended up guessing some values and this is what we found

α	1	10	100	500	1000	1100	1150
Number of iterations	13585	6965	714	139	63	56	82

Table 4.1: Number of iterations by for some values α in Uzawa algorithm ($N = 10$)

Because we saw similar results for different values of M we ended up choosing $\alpha = 1100$ as the main value when testing Uzawa Algorithm.

N	5	10	50	100	500
Number of iterations (Uzawa)	32	56	18739	82195	40502
L^2 error (Uzawa)	1.099e-3	9.879e-4	6.670e-05	5.463e-05	4.640e-05
Number of iterations (PDAS)	4	8	44	71	378
L^2 error (PDAS)	4.524e-3	6.016e-4	6.162e-05	4.325e-05	1.693e-06

Table 4.2: Number of iterations and L^2 Error (Example 1)

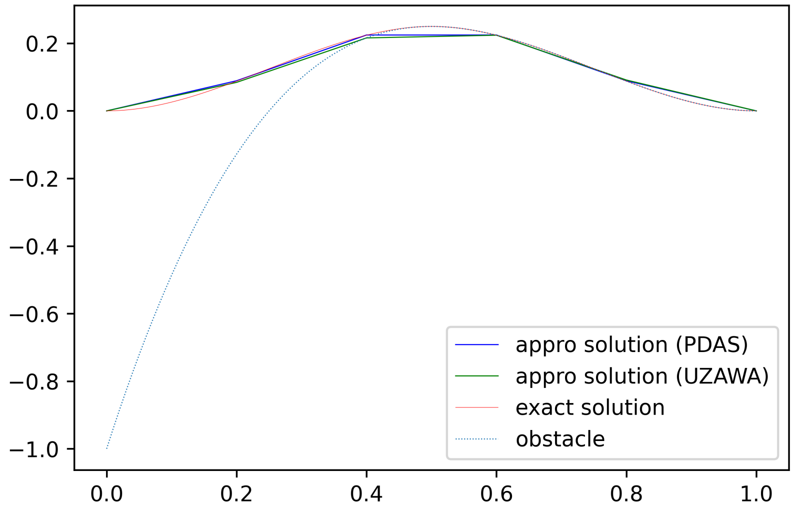

Figure 4.1: Contact zone with N = 5

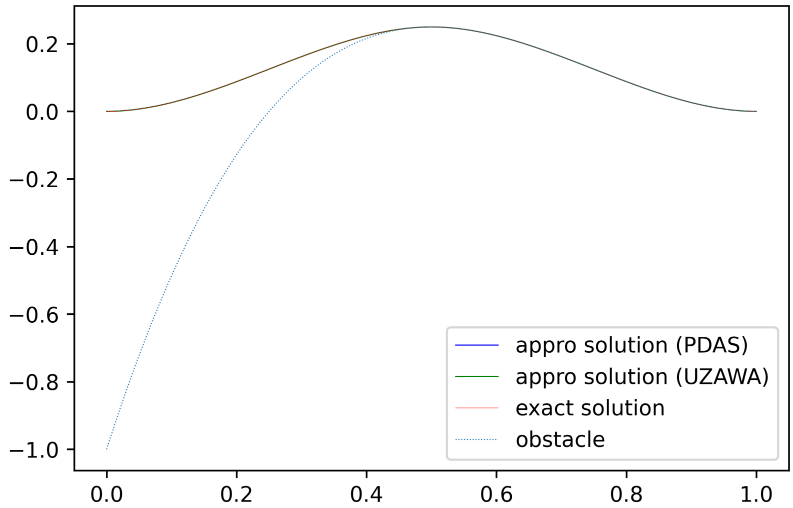

Figure 4.2: Contact zone with N = 100

4.4.2 Example 2: Contact Zone is a single point

We take $f = 0$ and $\psi = -3(2x-1)^2 + 1$

$$\begin{cases} \text{find } u \in K := \{v \in H_0^2(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \\ a(u, v-u) \geq 0, \quad \forall v \in K \end{cases}$$

The exact solution of this problem is

$$u(x) = \begin{cases} -16x^3 + 12x^2 & \text{if } x \in [0, \frac{1}{2}[\\ 16x^3 - 36x^2 + 24x - 4 & \text{if } x \in [\frac{1}{2}, 1] \end{cases}$$

Like in the previous example, we perform Uzawa Method using $\alpha = 1100$

N	5	10	50	100	500
Number of iterations (Uzawa)	14	20	6378	12277	5072
L^2 error (Uzawa)	1.992e-03	1.353e-03	2.181e-04	1.867e-04	2.721e-04
Number of iterations (PDAS)	2	5	18	34	162
L^2 error (PDAS)	1.236e-02	1.978e-15	3.289e-13	8.091e-12	1.1209e-08

Table 4.3: Number of iterations and L^2 Error (Example 2)

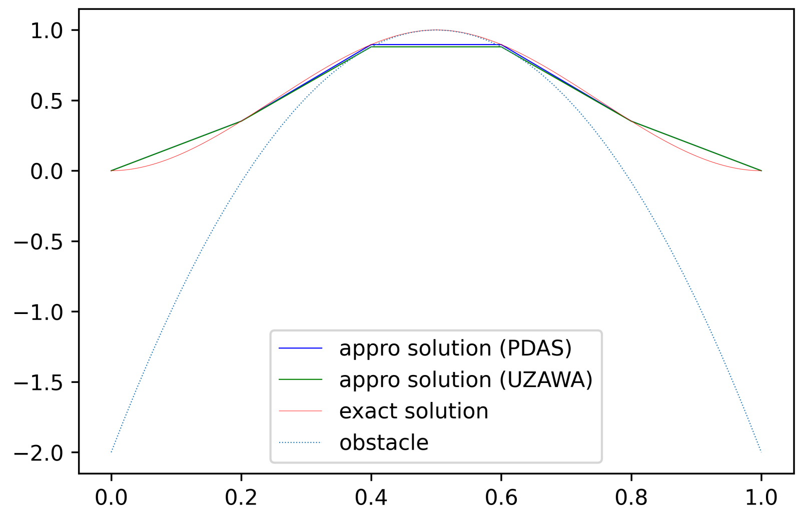

Figure 4.3: Contact zone with N = 5

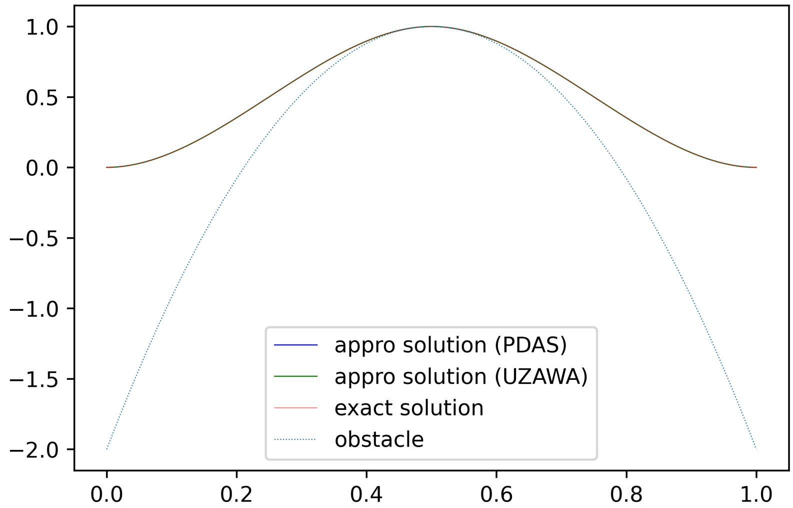

Figure 4.4: Contact zone with N = 100

Something peculiar happens in this example: the lower the number of discretisation, the less error the PDAS method achieves, but the error is still so small for almost all cases which means PDAS is still effective. Worth the investigation in the future to see whether this affects all the cases of single point contact zone or its something particular to this case.

4.4.3 Example 3: Obstacle is constant

We take $f = -18432$ and $\psi = -1$

$$\begin{cases} \text{find } u \in K := \{v \in H_0^2(\Omega) : v \geq \psi \text{ a.e. in } \Omega\} \\ a(u, v-u) \geq 0, \quad \forall v \in K \end{cases}$$

The exact solution of this problem is

$$u(x) = \begin{cases} -4x^2(192x^2 - 128x + 24) & \text{if } x \in [0, \dfrac{1}{4}[\\ -1 & \text{if } x \in [\dfrac{1}{4}, \dfrac{3}{4}[\\ -(2x-2)^2(192x^2 - 256x + 88) & \text{if } x \in [\dfrac{3}{4}, 1] \end{cases}$$

In this example we choose $\alpha = 1000$ because it was the fastest to converge (when we tried $\alpha = 1100$ like the previous examples, the method doesn't converge).

N	5	10	50	100	500
Number of iterations (Uzawa)	500	3444	2857	3202	6860
L^2 error (Uzawa)	3.867e-02	1.086e-02	1.214e-03	9.038e-04	5.083e-04
Number of iterations (PDAS)	2	4	14	26	111
L^2 error (PDAS)	8.657e-02	1.326e-02	1.274e-04	7.468e-06	6.082e-08

Table 4.4: Number of iterations and L^2 Error (Example 3)

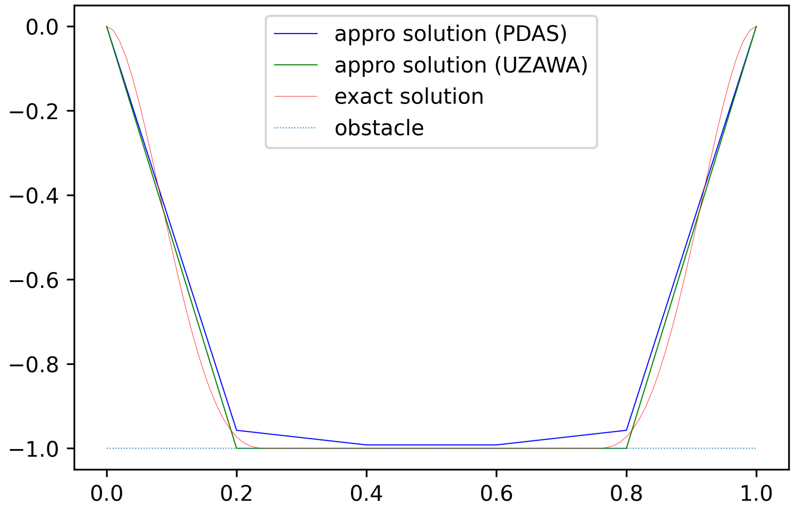

Figure 4.5: Contact zone with N = 5

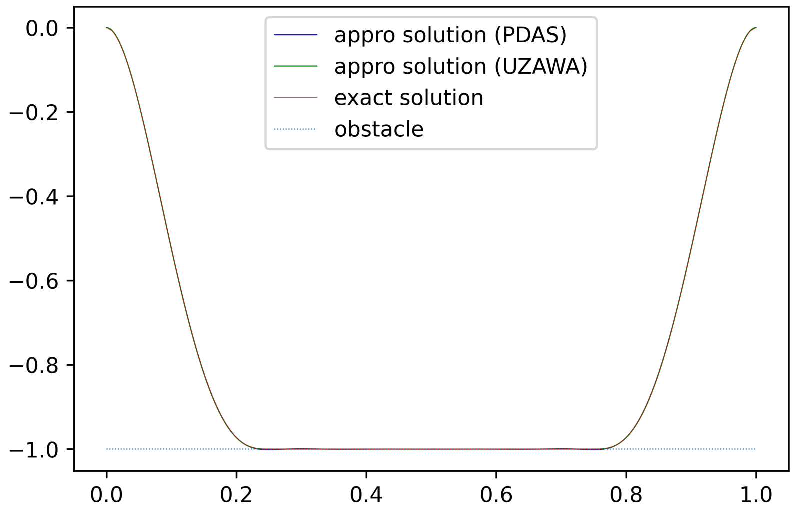

Figure 4.6: Contact zone with N = 100

Conclusion

We have derived a one dimensional unilateral contact model of obstacle type starting from the three dimensional Signorini problem by using the asymptotic analysis method. The obtained problem is governed by a fourth order differential operator. Then we have developed two new finite element methods for fourth order variational inequalities. For the case when the complementarity form of the variational inequality, exists in a strong sense, i.e the solution satisfies the full regularity H^4, quasi-optimal error estimate is derived in the same fashion for second order variational inequalities. For the case when the complementarity system exists only in weak sense, the key for the first method, is to introduce a new compact formulation to connect the continuous and discrete problems. When we have used the conforming P_3-Hermite element, our compact form doesn't need to any extra stabilisation term. For the second method, a biorthogonal dual basis is constructed and therefore, a variationaly consistent method is developed. For both methods, optimal a priori error estimate can be derived by mean of medius analysis and a reliable a posteriori error estimate of residual type is also obtained.
As perspectives or extensions of the present work, we believe that we can consider the following problems:

- Variational inequalities of second kind including friction law.
- The Koiter shell model.
- Piezoelectric effects.

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Abstract

In this work we study finite element methods for obstacle Problem of Euler Bernoulli beam. We begin by applying the asymptotic expansion method to the three-dimensional Signorini problem for an elastic beam, of which we obtain a one-dimensional model which includes the classical bending model of an elastic beam on a rigid obstacle also known as Euler Bernoulli obstacle Problem. We study the existence of the solution of the problem and its regularity, then we apply the finite element method on the classic formulation and later on the mixed formulation of the problem which based on Lagrange multiplier, and we give priori and posteriori error estimates. Finally a numerical test with Freefem++ is presented in which we use Uzawa method and PDAS method.

Keywords: Fourth Order Variational Inequalities, Euler Bernoulli Beam, Finite Element Method, PDAS.

Résumé

Dans ce travail, nous étudions les méthodes d'éléments finis pour le problème d'obstacle de la poutre d'Euler Bernoulli. Nous commençons par appliquant la méthode de développement asymptotique au problème tridimensionnel de Signorini pour une poutre élastique, dont nous obtenons un modèle unidimensionnel qui inclut le modèle classique de flexion d'une poutre élastique sur un obstacle rigid. Nous étudions l'existence de la solution du problème et sa régularité, puis nous appliquons la méthode des éléments finis sur la formulation classique et plus tard sur la formulation mixte du problème basée sur le multiplicateur de Lagrange, et nous donnons des estimations d'erreurs a priori et a posteriori. Enfin un test numérique avec Freefem++ est présenté dans lequel nous utilisons la méthode Uzawa et la méthode PDAS.

Mots clés : Inégalités variationnelles du quatrième ordre, poutre d'Euler Bernoulli, méthode des éléments finis, PDAS.

ملخص

في هذا العمل قمنا بدراسة طرق العناصر المنتهية لمشكلة عائق في عارضة أويلر برنولي. نبدأ بتطبيق طريقة التحليل المقارب على مسألة سينيوريني ثلاثية الأبعاد للعارضة المرنة، والتي نتحصل منها على نموذج أحادي البعد يحتوي على نموذج الانحناء الكلاسيكي للعارضة المرنة فوق عائق صلب او ما يعرف أيضاً بمسألة عائق أويلر برنولي . قمنا بدراسة وجود حل المشكلة وانتظامه، ثم طبقنا طريقة العناصر المنتهية على الصياغة الكلاسيكية ولاحقاً على الصياغة المختلطة للمسألة معتمدين على مضاعف لاغرانج، وأعطينا تقديرات الخطأ القبلية والبعدية. وأخيرا تم تقديم اختبار عددي مع ++Freefem نستخدم فيه طريقة Uzawa وطريقة PDAS

الكلمات المفتاحية: المتباينات التغايرية من الدرجة الرابعة، عارضة أويلر برنولي، طريقة العناصر المنتهية، PDAS