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Present By: Meriem Chaachoue

Theme:

Study of certain system of difference equations

limb for jury :

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Résumé

Cette recherche porte sur la notion d'équation aux différences et ses modèles, en plus de ses solutions liées à la suite de Fibonacci. Nous avons ensuite étudié la stabilité et la périodicité de ces solutions.

Le système d'équations de différence est donné comme suit :

$$\begin{cases} x_{n+1} = \dfrac{1}{1+y_{n-1}} \\ y_{n+1} = \dfrac{1}{1+x_{n-1}} \end{cases} \quad n \in \mathbb{N}_0, k = 2$$

Les conditions initiales x_0, x_{-1}, y_0, y_{-1} sont des nombres réels.

Mot clés système d'équation aux différences, suite de Fibonacci, stabilité et périodicité des solutions.

Abstract

This research deals with the concept of the difference equation and its models, in addition to its solution related to the Fibonacci sequence. We then studied the stability and periodicity of these solutions.

The difference equation system is given as follows :

$$\begin{cases} x_{n+1} = \frac{1}{1+y_{n-1}} \\ y_{n+1} = \frac{1}{1+x_{n-1}} \end{cases} \quad n \in \mathbb{N}_0, k = 2$$

The initial conditions x_0, x_{-1}, y_0, y_{-1} are real numbers

Keywords : system of difference equations, Fibonacci sequences, stability, and periodicity of solutions.

ملخص

يتناول هذا البحث مفهوم معادلة الفروق ونماذجها، بالإضافة إلى حلولها المرتبطة بمتتالية فيبوناتشي.
بعدها درسنا استقرار ودورية هذه الحلول لبعض نظام معادلة الفروق على الشكل التالي:

$$\begin{cases} x_{n+1} & = \frac{1}{1+y_{n-1}} \qquad n \in \mathbb{N}_0, \quad K = 2 \\ y_{n+1} & = \frac{1}{1+x_{n-1}} \end{cases}$$

والشروط الأولية هي أعداد حقيقية x_0, x_{-1}, y_0, y_{-1}

الكلمات المفتاحية : حل معادلات الفروق، متتالية فيبوناتشي، استقرار، دورية.

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Thank you all.

Dedication

✎ ..... I dedicate this mode

To

The greate

To My dear mother

To

source of my happine

To

My dear sister

To

My friends Talha ibtissam

To

All my honorable profe

To

All the student 2^{nd}year Master's 2024

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Chaachoue Meriem

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Introduction

THE theory of difference equations has developed extensively in recent times since the twentieth century. It is a critical tool for analysis and modeling in several fields, including:

- **Population models (biology):** Difference equations model the changes in population structures such as migration and population growth.
- **Medicine:** These equations are used to model the progression of certain diseases, such as cancer tumors.
- **Computer science:** Difference equations assist in the design and analysis of the complexity of algorithms that are often based on iteration. They are also used to identify virus-infected devices over time.
- **Economics:** These equations contribute to modeling investment and consumption equations over time based on economic factors.

Hence, the theory of difference equations has received significant and well-deserved attention from scientists and researchers, as it plays a crucial role in modeling situations and challenges in our daily lives due to its accuracy. It is considered the cornerstone of applied mathematics.

Difference equations can be divided into different types based on their characteristics, such as linear or nonlinear.

In this paper, we will present a special case of a system of second-order difference equations derived from a general study of difference equation models [2].

One of the most important things we did in our research was to find solutions to the system of difference equations (for $k = 2$) and then study their stability. In addition to providing more details to study the solution of the system of difference equations periodically [13].

Thus, the theory of difference equations plays a fundamental and vital role in various fields due to its ability to represent temporal changes. It provides practical and effective solutions to many problems encountered in the natural sciences, engineering, and economics.

We have divided our work into three chapters:

- In the **first chapter**, we provided some definitions about difference equations, the system of difference equations and Fibonacci sequences. We also presented two simple models of difference equations derived from our daily lives.
- In the **second chapter**, we presented the solutions of a specific system of 2^{nd}-order difference equations in terms of Fibonacci sequences, and we also determined the equilibrium point for our previous system.
- In the **third chapter**, we studied the stability and periodicity of solutions of the system of difference equations.

Chapter 1

Preliminaries

1.1. Definition of the Difference Equation

A difference equation is a mathematical equation that links values of Δx_i to each other or to y_i.

They are repeatedly defined sequences in the model:

$$x_{n+1} = P(n, x_n), \quad n \in \mathbb{N}_0$$

In other words, the difference equation is a mathematical equation that expresses the relationship between a function at different points in time. It is often used in sequences. It is similar to differential equations, but the difference between them is that we will be dealing with discrete sequences instead of functions of continuous variables [1] .

1.2. Definition of a System of Difference Equations

It is a system of equations that contain unknowns. They are also sequences. It has two parts [2]:

The first part: It consists of a linear difference equation, which is considered somewhat understandable and easy due to its reliance on the principles of linearity, which have a role in finding a solution of this type.

Second part: The nonlinear difference equation. Here lies the problem because there are no direct methods to solve it.

1.3. Fibonacci Sequence

1.3.1. Definition [3]

Definition 1.1.
The Fibonacci sequence $\{F_n\}_{n\geqslant 0}$ is defined by:

$$\begin{cases} F_n & = F_{n-1} + F_{n-2} \\ F_0 & = 0 \quad \text{and} \quad F_1 = 1 \end{cases} \tag{1.3.1}$$

The solution of this equation (1.3.1) is given by the following formula:

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta} \tag{1.3.2}$$

This is called the Binet formula for Fibonacci numbers, where:

$$\alpha = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad \beta = \frac{1-\sqrt{5}}{2}$$

Then we can give the following limit:

$$\lim_{n\to\infty} \frac{F_{n+m}}{F_n} = \alpha^m \quad \text{for } m \in \mathbb{Z} \tag{1.3.3}$$

1.4. Simple models about difference equation

1.4.1. Definition (Compound Interest)

Compound interest is commonly used in long-term loans and deposits. Interest is added to the amount at regular intervals called *conversion periods*. Then the new amount will be used instead of the initial amount to calculate the conversion period.

- The part of the year in which the one-month conversion period occurs is referred to as $\tau = \frac{1}{12}$.
- Annual interest rate estimated at $\sigma\%$ and τ is the conversion period.
- The interest earned during the period is estimated at $\sigma\tau\%$ of the amount deposited at the beginning of the period.

$$\left\{\begin{matrix}\text{amount deposit after } t+1\\ \text{conversion periods}\end{matrix}\right\} = \left\{\begin{matrix}\{\text{amount on deposit after } t\\ \text{conversion periods}\end{matrix}\right\} + \frac{\tau\sigma}{100} \times \left\{\begin{matrix}\text{amount on deposit after } t\\ \text{conversion periods}\end{matrix}\right\}$$

To express this with a difference equation, for each t, A_t the amount deposited after t conversion periods, where,

$$A_{t+1} = A_t + \frac{\tau\,\sigma}{100}A_t = A_t\left(1 + \frac{\tau\,\sigma}{100}\right)$$

So, it is a simple first-order linear difference equation.

A_t geometric sequence representing compound interest, it is given in the form [4]

$$A_t = \left(1 + \frac{\tau\,\sigma}{100}\right)^t A_0$$

1.4.2. Loan repayments

> **Definition 1.2** ([4]).
> To express the repayment of the above loans using the equation, some modifications were added to the above.

Below we will present a scheme used to repay car and home loans.

This process is carried out by paying at equal intervals of time to reduce loans and pay interest on the amounts due.

According to a study by several scientists, they found that debt increases due to the interest resulting from the debts that always remain after the last payment.

$$\begin{Bmatrix} \text{debt after} \\ t+1 \end{Bmatrix} = \begin{Bmatrix} \text{debt after} \\ t \text{ payments} \end{Bmatrix} + \begin{Bmatrix} \text{interest on this debt} \end{Bmatrix} - \begin{Bmatrix} \text{payment} \end{Bmatrix} \tag{1.4.1}$$

To model all of this in the form of a difference equation, let:

- P_0 be the initial debt that will be paid for each t,
- P_t be the debt due after payment t,
- K be the payment after each period conversion.

The difference equation can be written as:

$$P_{t+1} = P_t + \frac{\tau\sigma}{100} P_t - K = P_t \left(1 + \frac{\tau\sigma}{100}\right) - K \tag{i}$$

Notation. *The equation (i) is an equation that is difficult to solve.*

In the last section, we have provided two examples as an easy model of difference equation taken from the rules of some banks and then translated mathematically into equations.

Form solutions of a system of difference equations using specific sequences

Chapter 2

2.1. Difference equations

2.1.1. Non-linear difference equations

Let $\mathcal{K}$ be a continuously differentiable function with

$$\mathcal{K} : I^2 \to I, \quad I \subseteq \mathbb{R}$$

Definition 2.1 ([5]).
The 2-order nonlinear difference equation is given as

$$x_{n+1} = \mathcal{K}(x_n, x_{n-1}) \tag{2.1.1}$$

and $n \in \mathbb{N}_0$, let the initial conditions be as follows

$$(x_{-1}, x_0) \in I^2$$

Definition 2.2 ([5]).
We say that $\bar{x} \in I$ is an equilibrium point of the equation (2.1.1), if it comes true $\bar{x} = \mathcal{K}(\bar{x}, \bar{x})$ and if we have

$$x_n = \bar{x} \quad \forall n \geqslant -2 = -k \quad n, k \in \mathbb{N}_0.$$

Definition 2.3 ([5]).
Let it be

$$y_{n+1} = p_0 y_n + p_1 y_{n-1} \tag{2.1.2}$$

We can say that the linear difference equation related to the equation (2.1.1) with

$$p_i = \frac{\partial \mathcal{K}}{\partial x_i}(\bar{x}, \bar{x}) \quad i = 0, 1, 2.$$

$$P(\lambda) = \lambda^3 - p_0\lambda^2 - p_1\lambda - p_2 \tag{2.1.3}$$

with $P(\lambda)$ is the characteristic polynomial associated to (2.1.2).

Theorem 1 (Rouché's Theorem [6]).
Let $f(z)$ and $g(z)$ be two holomorphic functions in the open set Ω of the complex plane $\mathbb{C}$ and suppose that $|f(z)| > |g(z)|$ at each point on $\mathbb{C}$.
Then, $f(z)$ and $f(z) + g(z)$ have the same number of zeros in $\mathbb{C}$.

2.2. System of Non-Linear Difference Equations

Definition 2.4 ([7]).
Let P and Φ be two continuously differentiable functions as given by:

$$P : I^2 \times J^2 \to I, \quad \Phi : I^2 \times J^2 \to J \quad I, J \subseteq \mathbb{R}$$

So, be it the following difference equations systems

$$\begin{cases} x_{n+1} = P(x_n, x_{n-1}, y_n, y_{n-1}) \\ y_{n+1} = \Phi(x_n, x_{n-1}, y_n, y_{n-1}) \end{cases} \tag{2.2.1}$$

$n \in \mathbb{N}_0$, let the initial conditions be as follows $(x_1, x_0) \in I^2$ and $(y_1, y_0) \in J^2$.

Now, we define the function

$$W : I^2 \times J^2 \to I^2 \times J^2$$

and

$$W(x) = (P_0(x), P_1(x), \Phi_0(x), \Phi_1(x))^T$$

with

$$X = (u_0, u_1, v_0, v_1)^T$$

$$P_0(x) = P(x), \quad P_1(x) = u_0$$

Another way, we have

$$\Phi_0(x) = \Phi(x), \quad \Phi_1(x) = v_0$$

Let

$$X_n = (x_n, x_{n-1}, y_n, y_{n-1})^T \tag{2.2.2}$$

and therefore, the system (2.2.1) is equivalent to the following system

$$X_{n+1} = W(X_n) \quad n \in \mathbb{N} \tag{I}$$

Which

$$\begin{cases} x_{n+1} = P(x_n, x_{n-1}, y_n, y_{n-1}) \\ x_n = x_n \\ x_{n-1} = x_{n-1} \\ y_{n+1} = \Phi(x_n, x_{n-1}, y_n, y_{n-1}) \\ y_n = y_n \\ y_{n-1} = y_{n-1} \end{cases}$$

Definition 2.5 ([5]).

1. The point $(\bar{x}, \bar{y})$ is the equilibrium point of system (2.2.1), if we have

$$\begin{cases} \bar{x} = P(\bar{x}, \bar{x}, \bar{y}, \bar{y}) \\ \bar{y} = \Phi(\bar{x}, \bar{x}, \bar{y}, \bar{y}) \end{cases}$$

2. The point $\bar{X} = (\bar{x}, \bar{x}, \bar{y}, \bar{y}) \in I^2 \times J^2$ is the equilibrium point of system (I) if we have

$$X = W(X)$$

Remarque. We can say that the point $(\bar{x}, \bar{y}) \in I \times J$ is an equilibrium point for the system (2.2.1) if we have

$$\bar{X} = (\bar{x}, \bar{x}, \bar{y}, \bar{y}) \in I^2 \times J^2$$

is an equilibrium point of system (I).[7]

Definition 2.6 ([7]).
The linear system related to system (I) around the equilibrium point

$$\bar{X} = (\bar{x}, \bar{x}, \bar{y}, \bar{y})$$

is given as follows:

$$X_{n+1} = BX_n \quad h \in \mathbb{N}$$

Where B is the Jacobian matrix of function W at the equilibrium point $\bar{X}$:

$$B = \begin{pmatrix} \frac{\partial P_0}{\partial u_0}(\bar{X}) & \frac{\partial P_0}{\partial u_1}(\bar{X}) & \frac{\partial P_0}{\partial v_0}(\bar{X}) & \frac{\partial P_0}{\partial v_1}(\bar{X}) \\ \frac{\partial P_1}{\partial u_0}(\bar{X}) & \frac{\partial P_1}{\partial u_1}(\bar{X}) & \frac{\partial P_1}{\partial v_0}(\bar{X}) & \frac{\partial P_1}{\partial v_1}(\bar{X}) \\ \frac{\partial \phi_0}{\partial u_0}(\bar{X}) & \frac{\partial \phi_0}{\partial u_1}(\bar{X}) & \frac{\partial \phi_0}{\partial v_0}(\bar{X}) & \frac{\partial \phi_0}{\partial v_1}(\bar{X}) \\ \frac{\partial \phi_1}{\partial u_0}(\bar{X}) & \frac{\partial \phi_1}{\partial u_1}(\bar{X}) & \frac{\partial \phi_1}{\partial v_0}(\bar{X}) & \frac{\partial \phi_1}{\partial v_1}(\bar{X}) \end{pmatrix}$$

Theorem 2 ([7]).
If all eigenvalues of the Jacobian matrix A satisfy the condition $|\lambda| < 1$, then the equilibrium point $\bar{X}$ of system (I) is asymptotically stable (or locally asymptotically stable).

2.3. Form of the Solution of the System of Difference Equations

2.3.1. The First System

In this part, we will present solutions to the system of difference equations associated with Fibonacci sequences.

Let the following system be a difference equation system:

$$\begin{cases} x_{n+1} = \dfrac{1}{1+y_{n-1}} \quad n \in \mathbb{N}_0,\ k=2 \\ \\ y_{n+1} = \dfrac{1}{1+x_{n-1}} \end{cases} \tag{2.3.1}$$

where the initial condition of negative index terms $x_{-2}, x_{-1}, x_0, y_{-2}, y_{-1}, y_0 \in \mathbb{R} - F$ and

$$F = \left\{ -\frac{F_{n+1}}{F_n}\ ;\ n = 1, 2, \ldots \right\}.$$

Theorem 3 ([8]).
The solution of system (2.3.1) are given as follows:

$$\begin{cases} x_{4n+1} = \dfrac{F_{2n+1} + F_{2n} y_{-1}}{F_{2n+2} + F_{2n+1} y_{-1}} \quad ; x_{4n+2} = \dfrac{F_{2n+1} + F_{2n} y_0}{F_{2n+2} + F_{2n+1} y_0} \\ \\ y_{4n+1} = \dfrac{F_{2n+1} + F_{2n} x_{-1}}{F_{2n+2} + F_{2n+1} x_{-1}} \quad ; y_{4n+2} = \dfrac{F_{2n+1} + F_{2n} x_0}{F_{2n+2} + F_{2n+1} x_0} \end{cases} \tag{2.3.2}$$

From $i = 3, 4$ we have

$$\begin{cases} x_{4n+3} = \dfrac{F_{2n+2} + F_{2n+1} x_{-1}}{F_{2n+3} + F_{2n+2}\, x_{-1}} \quad ; x_{4n+4} = \dfrac{F_{2n+2} + F_{2n+1} x_0}{F_{2n+3} + F_{2n+2}\, x_0} \\ \\ y_{4n+3} = \dfrac{F_{2n+2} + F_{2n+1} y_{-1}}{F_{2n+3} + F_{2n+2} y_{-1}} \quad ; y_{4n+4} = \dfrac{F_{2n+2} + F_{2n+1} y_0}{F_{2n+3} + F_{2n+2}\, y_0} \end{cases} \tag{2.3.3}$$

proof.

From (2.3.6) we have

For $n=0$ and when $i=1,2$ we have

$$x_1 = \frac{1}{1+y_{-1}} \quad ; x_2 = \frac{1}{1+y_0} \quad ; \quad x_3 = \frac{1}{1+y_1}$$
$$y_1 = \frac{1}{1+x_{-1}} \quad ; y_2 = \frac{1}{1+x_0} \quad \quad y_3 = \frac{1}{1+x_1}$$

On the other hand, for $n=0$ and $i=3,4$ we have

$$x_3 = \frac{1+x_{-1}}{2+x_{-1}} \quad ; x_4 = \frac{1+x_0}{2+x_0}; \frac{1+x_1}{2+x_0}$$
$$y_3 = \frac{1+y_{-1}}{2+y_{-1}} \quad ; y_4 = \frac{1+y_0}{2+y_1}; \frac{1+y_1}{2+y_1}$$

So, the result for $n=0$.

Suppose now that $n \geqslant 1$, and our assumption holds for $n-1$.

For $i=1,2$ we have:

$$x_{4(n-1)+1} = \frac{F_{2n-1}+F_{2n-2}y_{-1}}{F_{2n}+F_{2n-1}y_{-1}}; x_{4(n-1)+2} = \frac{F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0} \tag{2.3.4}$$

$$y_{4(n-1)+1} = \frac{F_{2n-1}+F_{2n-2}x_{-1}}{F_{2n}+F_{2n-1}x_{-1}}; y_{4(n-1)+2} = \frac{F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0} \tag{2.3.5}$$

■

For $i = 3, 4$ we find:

$$x_{4(n-1)+3} = \frac{F_{2n} + F_{2n-1}x_{-1}}{F_{2n+1}F_{2n}x_{-1}} \qquad x_{4(n-1)+4} = \frac{F_{2n} + F_{2n-1}x_0}{F_{2n+1}F_{2n}x_0} \tag{2.3.6}$$

$$y_{4(n-1)+3} = \frac{F_{2n} + F_{2n-1}y_{-1}}{F_{2n+1}F_{2n}\ y_1} \qquad y_{4(n-1)+4} = \frac{F_{2n} + F_{2n-1}y_0}{F_{2n+1}F_{2n}\ y_0} \tag{2.3.7}$$

By following (2.3.1), (2.3.4), (2.3.5) and for $i = 1$ that:

$$\begin{aligned}
x_{4n+1} &= \frac{1}{y_{4n-1}} \\
&= \frac{1}{1 + \dfrac{1}{1 + x_{4n-3}}} \\
&= \frac{1}{\dfrac{2 + x_{4n-3}}{1 + x_{4n-3}}} \\
&= \frac{1 + x_{4n-3}}{2 + x_{4n-3}} \\
&= \frac{1 + \dfrac{F_{2n-1} + F_{2n-2}y_{-1}}{F_{2n} + F_{2n-1}y_{-1}}}{2 + \dfrac{F_{2n-1} + F_{2n-2}y_{-1}}{F_{2n} + F_{2n-1}y_{-1}}} \\
&= \frac{\dfrac{F_{2n} + F_{2n-1}y_{-1} + F_{2n-1} + F_{2n-2}y_{-1}}{F_{2n} + F_{2n-1}y_{-1}}}{\dfrac{2F_{2n} + 2F_{2n-1}y_{-1} + F_{2n-1} + F_{2n-2}y_{-1}}{F_{2n} + F_{2n-1}y_{-1}}}
\end{aligned}$$

$$= \frac{F_{2n} + F_{2n-1}y_{-1} + F_{2n-1} + F_{2n-2}y_{-1}}{F_{2n} + F_{2n-1}y_{-1}} \cdot \frac{F_{2n} + F_{2n-1}y_{-1}}{2F_{2n} + 2F_{2n-1}y_{-1} + F_{2n-1} + F_{2n-2}y_{-1}}$$

$$= \frac{F_{2n} + F_{2n-1} + F_{2n-1} + F_{2n-2}y_{-1}}{2F_{2n} + 2F_{2n-1}y_{-1} + F_{2n-1} + F_{2n-2}y_{-1}}$$

$$x_{4n+1} = \frac{F_{2n+1} + F_{2n}y_{-1}}{F_{2n+2} + F_{2n+1}y_{-1}}$$

For $i = 2$: Form (2.3.1),(2.3.4) we get

$$
\begin{aligned}
x_{4n+2} &= \frac{1}{1+y_{4n}} \\
&= \frac{1}{1+\dfrac{1}{1+x_{4n-2}}} \\
&= \frac{1}{\dfrac{1+x_{4n-2}+1}{1+x_{4n-2}}} \\
&= \frac{1+x_{4n-2}}{2+x_{4n-2}} \\
&= \frac{1+\dfrac{F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0}}{2+\dfrac{F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0}} \\
&= \frac{\dfrac{F_{2n}+F_{2n-1}y_0+F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0}}{\dfrac{2F_{2n}+2F_{2n-1}y_0+F_{2n-1}+F_{2n-2}y_0}{F_{2n}+F_{2n-1}y_0}} \\
x_{4n+2} &= \frac{F_{2n}+F_{2n-1}y_0+F_{2n-1}+F_{2n-2}y_0}{2F_{2n}+2F_{2n-1}y_0+F_{2n-1}+F_{2n-2}y_0} \\
x_{4n+2} &= \frac{F_{2n+1}+F_{2n}y_0}{F_{2n+2}+F_{2n+1}y_0}
\end{aligned}
$$

and for $i = 1, k = 2$, Form (2.3.1), (2.3.5) we find

$$
\begin{aligned}
y_{4n+1} &= \frac{1}{1+x_{4n-1}} \\
&= \frac{1}{1+\dfrac{1}{1+y_{4n-3}}} \\
&= \frac{1}{\dfrac{1+y_{4n-3}+1}{1+y_{4n-3}}} \\
&= \frac{1}{\dfrac{2+y_{4n-3}}{1+y_{4n-3}}} \\
&= \frac{1+y_{4n-3}}{2+y_{4n-3}} = \frac{1+\dfrac{F_{2n-1}+F_{2n-2}x_{-1}}{F_{2n}+F_{2n-1}x_{-1}}}{2+\dfrac{F_{2n-1}+F_{2n-2}x_{-1}}{F_{2n}+F_{2n-1}x_{-1}}} \\
&= \frac{\dfrac{F_{2n}+F_{2n-1}x_{-1}+F_{2n-1}+F_{2n-2}x_{-1}}{F_{2n}+F_{2n-1}x_{-1}}}{\dfrac{2F_{2n}+2F_{2n}x_{-1}+F_{2n-1}+F_{2n-2}x_{-1}}{F_{2n}+F_{2n-1}x_{-1}}} \\
&= \frac{F_{2n}+F_{2n-1}x_{-1}+F_{2n-1}+F_{2n-2}x_{-1}}{2F_{2n}+2F_{2n}x_{-1}+F_{2n-1}+F_{2n-2}x_{-1}} \\
y_{4n+1} &= \frac{F_{2n+1}+F_{2n}x_{-1}}{F_{2n+2}+F_{2n+1}x_{-1}}
\end{aligned}
$$

For $i = 2$ We have

$$\begin{aligned}
y_{4n+2} &= \frac{1}{1+x_{4n}} \\
&= \frac{1}{1+\dfrac{1}{1+y_{4n-2}}} \\
&= \frac{1}{\dfrac{2+y_{4n-2}}{1+y_{4n-2}}} \\
&= \frac{1+y_{4n-2}}{2+y_{4n-2}} \\
&= \frac{1+\dfrac{F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0}}{2+\dfrac{F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0}} \\
&= \frac{\dfrac{F_{2n}+F_{2n-1}x_0+F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0}}{\dfrac{2F_{2n}+2F_{2n-1}x_0+F_{2n-1}+F_{2n-2}x_0}{F_{2n}+F_{2n-1}x_0}} \\
&= \frac{F_{2n}+F_{2n-1}x_0+F_{2n-1}+F_{2n-2}x_0}{2F_{2n}+2F_{2n-1}x_0+F_{2n-1}+F_{2n-2}x_0} \\
y_{4n+2} &= \frac{F_{2n+1}+F_{2n}x_0}{F_{2n+2}+F_{2n+1}x_0}
\end{aligned}$$

Similarly for $i = 3, 4$ and from (2.3.1), (2.3.6), (2.3.7) we get

$$\begin{aligned}
x_{4n+3} &= \frac{1}{1+x_{4n+1}} = \frac{1}{1+\dfrac{1}{1+x_{4n-1}}} \\
&= \frac{1}{\dfrac{2+x_{4n-1}}{1+x_{4n-1}}} \\
&= \frac{1+x_{4n-1}}{2+x_{4n-1}} \\
&= \frac{1+\dfrac{F_{2n}+F_{2n-1}x_{-1}}{F_{2n+1}+F_{2n}x_{-1}}}{2+\dfrac{F_{2n}+F_{2n-1}x_{-1}}{F_{2n+1}+F_{2n}x_{-1}}} \\
&= \frac{\dfrac{F_{2n+1}+F_{2n}x_{-1}+F_{2n}+F_{2n-1}x_{-1}}{F_{2n+1}+F_{2n}x_{-1}}}{\dfrac{2F_{2n+1}+2F_{2n}x_{-1}+F_{2n}+F_{2n-1}x_{-1}}{F_{2n+1}+F_{2n}x_{-1}}}
\end{aligned}$$

$$x_{4n+3} = \frac{F_{2n+1} + F_{2n}x_{-1} + F_{2n} + F_{2n-1}x_{-1}}{2F_{2n+1} + 2F_{2n}x_{-1} + F_{2n} + F_{2n-1}x_{-1}}$$

$$x_{4n+3} = \frac{F_{2n+2} + F_{2n+1}x_{-1}}{F_{2n+3} + F_{2n+2}x_{-1}}$$

For $i = 4$ (2.3.1), (2.3.6) we have

$$\begin{aligned}
x_{4n+4} &= \frac{1}{1 + x_{4n+2}} \\
&= \frac{1}{1 + \frac{1}{1 + x_{4n}}} \\
&= \frac{1}{\frac{2 + x_{4n}}{1 + x_{4n}}} \\
&= \frac{1 + x_{4n}}{2 + x_{4n}} \\
&= \frac{1 + \frac{F_{2n} + F_{2n-1}x_0}{F_{2n+1} + F_{2n}x_0}}{2 + \frac{F_{2n} + F_{2n-1}x_0}{F_{2n+1} + F_{2n}x_0}} \\
&= \frac{F_{2n+1} + F_{2n}x_0 + F_{2n} + F_{2n-1}x_0}{2F_{2n+1} + 2F_{2n}x_0 + F_{2n} + F_{2n-1}x_0} \\
x_{4n+4} &= \frac{F_{2n+2} + F_{2n+1}x_0}{F_{2n+3} + F_{2n+2}x_0}
\end{aligned}$$

and for $i = 3$ by (2.3.1), (2.3.7) we have

$$\begin{aligned}
y_{4n+3} &= \frac{1}{1 + y_{4n+1}} \\
&= \frac{1}{1 + \frac{1}{1 + y_{4n-1}}} \\
&= \frac{1}{\frac{2 + y_{4n-1}}{1 + y_{4n-1}}} \\
&= \frac{1 + y_{4n-1}}{2 + y_{4n-1}} \\
&= \frac{1 + \frac{F_{2n} + F_{2n-1}y_{-1}}{F_{2n+1} + F_{2n}y_{-1}}}{2 + \frac{F_{2n} + F_{2n-1}y_{-1}}{F_{2n+1} + F_{2n}y_{-1}}} \\
y_{4n+3} &= \frac{F_{2n+2} + F_{2n+1}y_{-1}}{F_{2n+3} + F_{2n+2}y_{-1}}
\end{aligned}$$

and for $i = 4$ by (2.3.1), (2.3.7) we get:

$$\begin{aligned}
y_{4n+4} &= \frac{1}{1 + y_{4n+2}} \\
&= \frac{1}{1 + \dfrac{1}{1 + y_{4n}}} \\
&= \frac{1}{\dfrac{2 + y_{4n}}{1 + y_{4n}}} \\
&= \frac{1 + y_{4n}}{2 + y_{4n}} \\
&= \frac{1 + \dfrac{F_{2n} + F_{2n-1}y_0}{F_{2n+1} + F_{2n}y_0}}{2 + \dfrac{F_{2n} + F_{2n-1}y_0}{F_{2n+1}F_{2n}y_0}} \\
&= \frac{F_{2n+1} + F_{2n}y_0 + F_{2n} + F_{2n-1}y_0}{2F_{2n+1} + 2F_{2n}y_0 + F_{2n} + F_{2n-1}y_0} \\
y_{4n+4} &= \frac{F_{2n+2} + F_{2n+1}y_0}{F_{2n+3} + F_{2n+2}y_0}
\end{aligned}$$

This completes the proof

Stability and periodicity of solutions of the 2-order difference equation system

Chapter 3

3.1. Linearized stability

We have the following system

$$\begin{cases} x_{n+1} = \dfrac{1}{1+y_{n-1}} \\ y_{n+1} = \dfrac{1}{1+x_{n-1}} \end{cases} \qquad n \in \mathbb{N},\ K = 2$$

The initial conditions of the negative index terms x_{-1}, x_0, y_0, y_1 are in $\mathbb{R}$ with $F = -\dfrac{F_{n+1}}{F_n}, n = 1, 2, \cdots$[2].

Theorem 4 ([8]).
The equilibrium point S is locally asymptotically stable.

We use the following lemma to prove Theorem 4.

Lemma 1 ([8]).
The system (2.3.1) has a unique positive solution that is namely the equilibrium point in $I^2 \times J^2$.

$$S = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2} \right)$$

Proof. Clearly the system

$$\begin{cases} \bar{x} = \dfrac{1}{1+\bar{y}} \\ \bar{y} = \dfrac{1}{1+\bar{x}} \end{cases} \tag{3.1.1}$$

■

From system (3.1.1) we find

$$\begin{cases} \bar{x} + \bar{x}\bar{y} - 1 = 0 \\ \bar{y} + \bar{y}\bar{x} - 1 = 0 \end{cases} \tag{3.1.2}$$

After solving the set of two equations, we get

$$\bar{x} = \bar{y} \tag{3.1.3}$$

Substituting (3.1.3) into equation (3.1.2), we find

$$\bar{x}^2 + \bar{x} - 1 = 0$$

$$\Delta = (1)^2 - 4(1)(-1) = 5$$

Which gives

$$\bar{x} = \frac{-1+\sqrt{5}}{2}$$

$$\bar{y} = \frac{-1+\sqrt{5}}{2}$$

The system (3.1.1) has a unique positive solution in $I \times J$

$$(\bar{x}, \bar{y}) = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$$

Proof of Theorem 4.

From Definition 2.4 and from the equilibrium point

$$\bar{S} = \left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right) \in I^2 \times J^2$$

We obtain the following linear system

■

$$X_{n+1} = BX_n$$

$$X_n = (x_n, x_{n-1}, y_n, y_{n-1})^T$$

Where B is a 4×4 Jacobian matrix given as follows

$$B = \begin{pmatrix} 0 & 0 & 0 & \frac{-3+\sqrt{5}}{2} \\ 1 & 0 & 0 & 0 \\ 0 & \frac{-3+\sqrt{5}}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

We calculate the characteristic polynomial:

$$\begin{aligned} P(\lambda) &= \det(B - \lambda I_4) \\ &= \begin{vmatrix} -\lambda & 0 & 0 & \frac{-3+\sqrt{5}}{2} \\ 1 & -\lambda & 0 & 0 \\ 0 & \frac{-3+\sqrt{5}}{2} & -\lambda & 0 \\ 0 & 0 & 1 & -\lambda \end{vmatrix} \\ &= (-\lambda) \begin{vmatrix} -\lambda & 0 & 0 \\ \frac{-3+\sqrt{5}}{2} & -\lambda & 0 \\ 0 & 1 & -\lambda \end{vmatrix} - \left(\frac{-3+\sqrt{5}}{2}\right) \begin{vmatrix} 1 & -\lambda & 0 \\ 0 & \frac{-3+\sqrt{5}}{2} & -\lambda \\ 0 & 0 & 1 \end{vmatrix} \\ &= (-\lambda) \cdot (-\lambda)^3 - \left(\frac{-3+\sqrt{5}}{2}\right) \begin{vmatrix} 1 & -\lambda \\ 0 & \frac{-3+\sqrt{5}}{2} \end{vmatrix} \\ &= (-\lambda)^4 - \left(\frac{-3+\sqrt{5}}{2}\right) \cdot \left(\frac{-3+\sqrt{5}}{2}\right) \\ &= \lambda^4 - \left(\frac{-3+\sqrt{5}}{2}\right)^2 \end{aligned}$$

Now, we can get the eigenvalues of the matrix B according to the characteristic polynomial

$$P(\lambda) = \det(B - \lambda I_4) = \lambda^4 - \left(\frac{-3+\sqrt{5}}{2}\right)^2 = 0$$

Then, consider the two functions defined by $f(\lambda) = \lambda^4$ and $g(\lambda) = \left(\frac{-3+\sqrt{5}}{2}\right)^2$.

We have

$$\left|\frac{-3+\sqrt{5}}{2}\right| < 1$$

So, $|f(\lambda)| > |g(\lambda)|$ for all $|\lambda| = 1$.

According to Rouché's Theorem, all zeros of $P(\lambda) = f(\lambda) - g(\lambda) = 0$ lie in the open unit disk $|\lambda| < 1$. So from the theorem 2 We get that S is locally asymptotically stable.

Theorem 5 ([8]).
The equilibrium point S is globally asymptotically stable.

Proof. Let $\{x_n, y_n\}_{n \geqslant e}$ be a solution of (2.3.1). By Theorem 4, we only need to prove that S is a global attractor, that is,

$$\lim_{n\to\infty} (x_n, y_n) = S \quad \text{or} \quad \lim_{n\to\infty} (x_n, x_{n-1}, y_n, y_{n-1}) = \bar{S}$$

We prove that for $i = 1, 2, 3, 4$, we have

$$\lim_{n\to\infty} x_{4n+i} = \lim_{n\to\infty} y_{4n+i} = \frac{-1+\sqrt{5}}{2}$$

From Theorem 1 we find

(i) For $i = 1$

$$\begin{aligned} \lim_{n\to\infty} x_{4n+1} &= \lim_{n\to\infty} \frac{F_{2n+1} + F_{2n}y_{-1}}{F_{2n+2} + F_{2n+1}y_{-1}} \\ &= \lim_{n\to\infty} \frac{1 + \dfrac{F_{2n}}{F_{2n+1}}y_{-1}}{\dfrac{F_{2n+2}}{F_{2n+1}} + y_{-1}} \end{aligned} \tag{3.1.4}$$

Using Binet's formula

$$F_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \in \mathbb{N}_0 \tag{3.1.5}$$

and $\alpha = \dfrac{1+\sqrt{5}}{2}, \quad \beta = \dfrac{-1-\sqrt{5}}{2}$, we get

$$\lim_{n\to\infty} \frac{F_{2n}}{F_{2n+1}} = \frac{1}{\alpha} \tag{3.1.6}$$

Similarly we get

$$\lim_{n\to\infty} \frac{F_{2n+2}}{F_{2n+1}} = \alpha \tag{3.1.7}$$

Then, from (3.1.4), (3.1.6), and (3.1.7), we get

$$\begin{aligned}
\lim_{n\to\infty} x_{4n+1} &= \lim_{n\to\infty} \frac{1+\dfrac{1}{\alpha}y_{-1}}{\alpha+y_{-1}} \\
&= \frac{\frac{\alpha+y_{-1}}{\alpha}}{\alpha+y_{-1}} \\
&= \frac{1}{\alpha} = \frac{1}{\dfrac{1+\sqrt{5}}{2}} = \frac{2}{1+\sqrt{5}} = \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} \\
&= \frac{-1+\sqrt{5}}{2}
\end{aligned}$$

(ii) For $i = 2$

$$\lim_{n\to\infty} x_{4n+2} = \lim_{n\to\infty} \frac{F_{2n+1}+F_{2n}y_0}{F_{2n+2}+F_{2n+1}y_0} \tag{3.1.8}$$

$$= \lim_{n\to\infty} \frac{1+\dfrac{F_{2n}y_0}{F_{2n+1}}}{\dfrac{F_{2n+2}}{F_{2n+1}}+y_0} \tag{3.1.9}$$

■

Then, from (3.1.4), (3.1.6), (3.1.7), (3.1.9) we get

$$\begin{aligned}
\lim_{n\to\infty} x_{4n+2} &= \frac{1+\dfrac{y_0}{\alpha}}{\alpha+y_0} = \frac{\frac{\alpha+y_0}{\alpha}}{\alpha+y_0} \\
&= \frac{1}{\alpha} = \frac{1}{\dfrac{1+\sqrt{5}}{2}} = \frac{2}{1+\sqrt{5}} = \frac{2(1-\sqrt{5})}{(1+\sqrt{5})(1-\sqrt{5})} \\
&= \frac{-1+\sqrt{5}}{2}
\end{aligned}$$

So, for $i = 1, 2$, $\lim_{n\to\infty} x_{4n+i} = \dfrac{-1+\sqrt{5}}{2}$

Now, for $i = 1$, we get

$$\lim_{n\to\infty} y_{4n+1} = \lim_{n\to\infty} \frac{F_{2n+1}+F_{2n}x_{-1}}{F_{2n+2}+F_{2n+1}x_{-1}}$$

$$= \lim_{n\to\infty} \frac{1+\dfrac{F_{2n}}{F_{2n+1}}x_{-1}}{\dfrac{F_{2n+2}}{F_{2n+1}}+x_{-1}} \tag{3.1.10}$$

Then, from (3.1.6), (3.1.7), (3.1.10), we have

$$\lim_{n\to\infty} y_{4n+1} = \frac{1+\dfrac{x_{-1}}{\alpha}}{\alpha+x_{-1}} = \frac{\frac{\alpha+x_{-1}}{\alpha}}{\alpha+x_{-1}}$$
$$= \frac{1}{\alpha} = \frac{1}{\dfrac{1+\sqrt{5}}{2}} = \frac{2}{1+\sqrt{5}} = \frac{2}{1+\sqrt{5}} = \frac{-1+\sqrt{5}}{2}$$
$$\lim_{n\to\infty} y_{4n+1} = \frac{-1+\sqrt{5}}{2}$$

For $i=2$ we have

$$\lim_{n\to\infty} y_{4n+2} = \lim_{n\to\infty} \frac{F_{2n+1}+F_{2n}x_0}{F_{2n+2}+F_{2n+1}x_0}$$
$$= \lim_{n\to\infty} \frac{1+\dfrac{F_{2n}x_0}{F_{2n+1}}}{\dfrac{F_{2n+2}}{F_{2n+1}}+x_0} \tag{3.1.11}$$

Then, from (3.1.6), (3.1.7), (3.1.11), we get

$$\lim_{n\to\infty} y_{4n+2} = \frac{1+\dfrac{x_0}{\alpha}}{\alpha+x_0} = \frac{\frac{\alpha+x_0}{\alpha}}{\alpha+x_0}$$
$$= \frac{1}{\alpha} = \frac{2}{1+\sqrt{5}} = \frac{-1+\sqrt{5}}{2}$$
$$\lim_{n\to\infty} y_{4n+2} = \frac{-1+\sqrt{5}}{2}$$

So, as a result, for $i=1,2$,

$$\lim_{n\to\infty} y_{4n+i} = \frac{-1+\sqrt{5}}{2}$$

Similarly, we get for $i=3,4$. Now for $i=3$

$$\lim_{n\to+\infty} x_{4n+3} = \lim_{n\to+\infty} \frac{F_{2n+2}+F_{2n+1}x_{-1}}{F_{2n+3}+F_{2n+2}x_{-1}}$$
$$= \lim_{n\to+\infty} \frac{1+\dfrac{F_{2n+1}}{F_{2n+2}}x_{-1}}{\dfrac{F_{2n+3}}{F_{2n+2}}+x_{-1}} \tag{3.1.12}$$

from (3.1.6), (3.1.7), (3.1.12) we get

$$\lim_{n\to+\infty} x_{4n+3} = \frac{1+\dfrac{1}{\alpha}x_{-1}}{\alpha+x_{-1}}$$

$$= \frac{\dfrac{\alpha+x_{-1}}{\alpha}}{\alpha+x_{-1}} = \frac{1}{\alpha}$$

$$= \frac{1}{\dfrac{1+\sqrt{5}}{2}}$$

$$\lim_{n\to+\infty} x_{4n+3} = \frac{2}{1+\sqrt{5}} = \frac{-1+\sqrt{5}}{2}$$

by the same arguments, we have

$$\lim_{n\to+\infty} x_{4n+4} = \lim_{n\to+\infty} \frac{F_{2n+2}+F_{2n+1}x_0}{F_{2n+3}+F_{2n+2}x_0} = \frac{-1+\sqrt{5}}{2}$$

For i=3

$$\lim_{n\to+\infty} y_{4n+3} = \lim_{n\to+\infty} \frac{F_{2n+2}+F_{2n+1}y_{-1}}{F_{2n+3}+F_{2n+2}y_{-1}}$$

$$= \lim_{n\to+\infty} \frac{1+\dfrac{F_{2n+1}}{F_{2n+2}}y_{-1}}{\dfrac{F_{2n+3}}{F_{2n+2}}+y_{-1}} \tag{3.1.13}$$

From (3.1.6), (3.1.7), (3.1.13), we find

$$\lim_{n\to+\infty} y_{4n+3} = \frac{1+\dfrac{1}{\alpha}y_{-1}}{\alpha+y_{-1}}$$

$$= \frac{\frac{\alpha+y_{-1}}{\alpha}}{\alpha+y_{-1}}$$

$$= \frac{1}{\alpha} = \frac{1}{\dfrac{1+\sqrt{5}}{2}}$$

$$\lim_{n\to+\infty} y_{4n+3} = \frac{-1+\sqrt{5}}{2}$$

By the same arguments, for $i = 4$ we have

$$\lim_{n\to+\infty} y_{4n+4} = \lim_{n\to+\infty} \frac{F_{2n+2}+F_{2n+1}y_0}{F_{2n+3}+F_{2n+2}y_0} = \frac{-1+\sqrt{5}}{2}$$

Finally, we find that for $i = 3, 4$,

$$\lim_{n\to+\infty} y_{4n+i} = \frac{-1+\sqrt{5}}{2}$$

This completes the proof.

Example 1.

To achieve the results of this section, we present the following numerical example.
Consider

$$x_{-1} = 1.22, \quad x_0 = 0.52, \quad y_{-1} = 0.50, \quad y_0 = 0.35$$

- In Figure (3.1), the sequences $(x_n)_{n=1}$ and $(y_n)_{n=1}$ of the solution to system (2.1) with the initial conditions converge to the equilibrium point $\left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$.

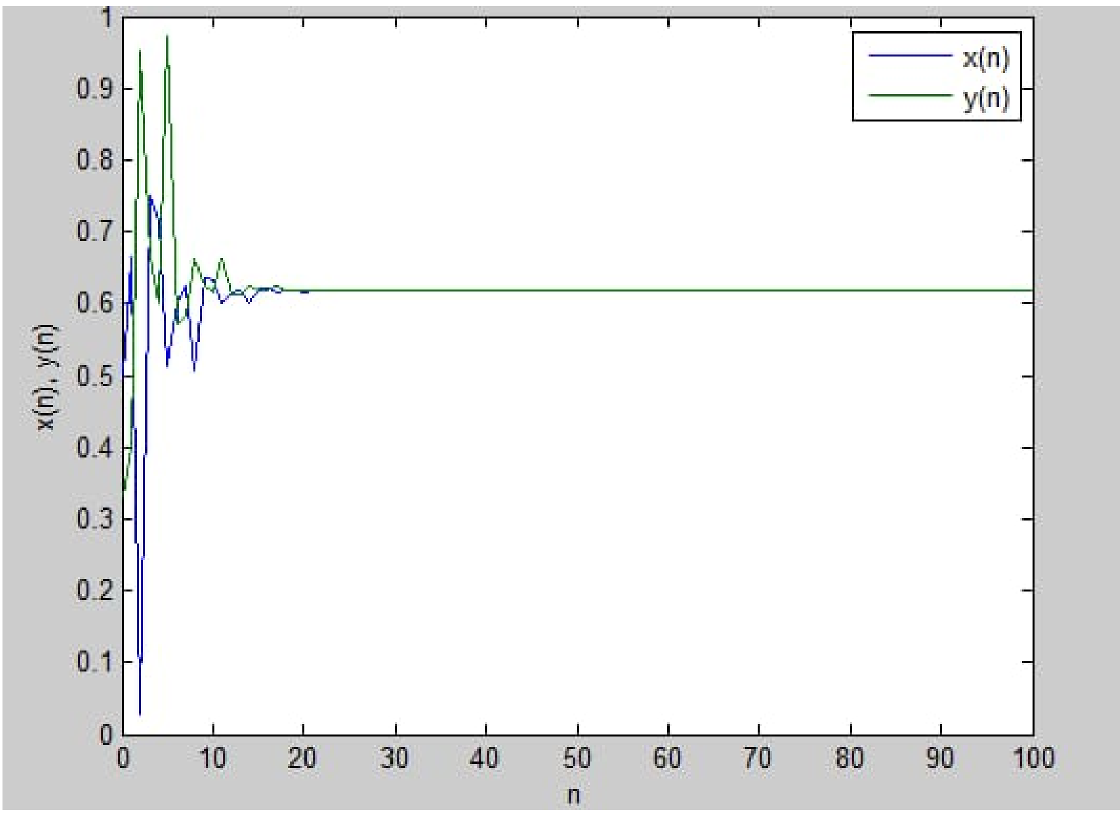

Figure 3.1: Plot of $(x_n, y_n)_{n=1}$ for the system (2.1) where $k = 2$

Figure (3.1): The sequences $(x_n)_{n=1}$ (blue) and $(y_n)_{n=1}$ (green) of the solution of system (2.3.1) with the initial conditions.

3.2. Periodicity of solutions

In this part, we will resort to studying the periodicity of the solution of the system of first-order difference equations, due to the difficulty of finding the periodicity of the solution of our previous system.

3.2.1. The second system

Now, we will present the following system of first-order difference equations.

$$\begin{cases} x_{n+1} = \dfrac{1}{1-y_n}, & n \in \mathbb{N}_0 \\ y_{n+1} = \dfrac{1}{1-x_n} \end{cases} \tag{3.2.1}$$

Where the initial conditions are real numbers $x_0, y_0 \in \mathbb{R} - \{0,1\}$[13].

Lemma 2 ([13]).
Let $(x_n)_{n\geqslant 0}, (y_n)_{n\geqslant 0}$ be a solution of system (3.2.1) and for $n = 0, 1, \dots$ we get

$$x_{n+6} = x_n, \quad y_{n+6} = y_n$$

That is $(x_n)_{n\geqslant 0}, (y_n)_{n\geqslant 0}$ will be periodic at period six.

Now, we will present proof of Lemma 2.

Proof.
By using the system (3.2.1) we find

$$\begin{aligned} x_{n+6} = x_{(n+5)+1} &= \frac{1}{1-y_{n+5}} \\ &= \frac{1}{1-\dfrac{1}{1-x_{n+4}}} \\ &= \frac{1}{\dfrac{1-x_{n+4}-1}{1-x_{n+4}}} \\ &= \frac{-1+x_{n+4}}{x_{n+4}} \\ &= \frac{-1+\dfrac{1}{1-y_{n+3}}}{\dfrac{1}{1-y_{n+3}}} \end{aligned}$$

$$
\begin{aligned}
x_{n+6} &= \frac{\dfrac{-1+y_{n+3}+1}{1-y_{n+3}}}{\dfrac{1}{1-y_{n+3}}} \\
&= y_{n+3} \\
&= \frac{1}{1-x_{n+2}} \\
&= \frac{1}{1-\dfrac{1}{1-y_{n+1}}} \\
&= \frac{1}{\frac{1-y_{n+1}-1}{1-y_{n+1}}} \\
&= \frac{1}{\frac{-y_{n+1}}{1-y_{n+1}}} \\
&= \frac{-1+y_{n+1}}{y_{n+1}} \\
&= \frac{-1+\frac{1}{1-x_n}}{\frac{1}{1-x_n}} \\
&= \frac{\frac{-1+x_{n+1}}{1-x_n}}{\frac{1}{1-x_n}} \\
&= \frac{\frac{x_n}{1-x_n}}{\frac{1}{1-x_n}} \\
&= \frac{x_n}{1-x_n}(1-x_n) \\
x_{n+6} &= x_n \qquad\qquad \text{with } n \in \mathbb{N}_0
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
y_{n+6} &= y_{(n+5)+1} \\
&= \frac{1}{1-x_{n+5}} \\
&= \frac{1}{1-\frac{1}{1-y_{n+4}}} \\
&= \frac{1}{\dfrac{1-y_{n+4}-1}{1-y_{n+4}}} \\
&= \frac{1}{\dfrac{-y_{n+4}}{1-y_{n+4}}} = \frac{-1+y_{n+4}}{y_{n+4}}
\end{aligned}
$$

$$
\begin{aligned}
y_{n+6} &= \frac{-1+\dfrac{1}{x_{n+3}}}{\dfrac{1}{1-x_{n+3}}} \\
&= \frac{\dfrac{-1+x_{n+3}+1}{1-x_{n+3}}}{\dfrac{1}{1-x_{n+3}}} \\
&= \frac{\dfrac{x_{n+3}}{1-x_{n+3}}}{\dfrac{1}{1-x_{n+3}}} = \frac{x_{n+3}}{1-x_{n+3}}\left(1-x_{n+3}\right) \\
&= x_{n+3} = \frac{1}{1-y_{n+2}} \\
&= \frac{1}{1-\dfrac{1}{1-x_{n+1}}} = \frac{1}{\dfrac{1-x_{n+1}-1}{1-x_{n+1}}} \\
&= \frac{1}{\frac{-x_{n+1}}{1-x_{n+1}}} \\
&= \frac{-1+x_{n+1}}{x_{n+1}} = \frac{-1+\frac{1}{1-y_n}}{\frac{1}{1-y_n}} \\
&= \frac{\frac{-1+y_n+1}{1-y_n}}{\frac{1}{1-y_n}} \\
y_{n+6} &= y_n
\end{aligned}
$$

So, we got

$$y_{n+6} = y_n, \qquad n \in \mathbb{N}_0$$

This is complets the proof ■

Example 2.

We present this numerical example to confirm the results of this section. Assume $x_0 = 0.2$ and $y_0 = 1.1$ (Figure 3.2).

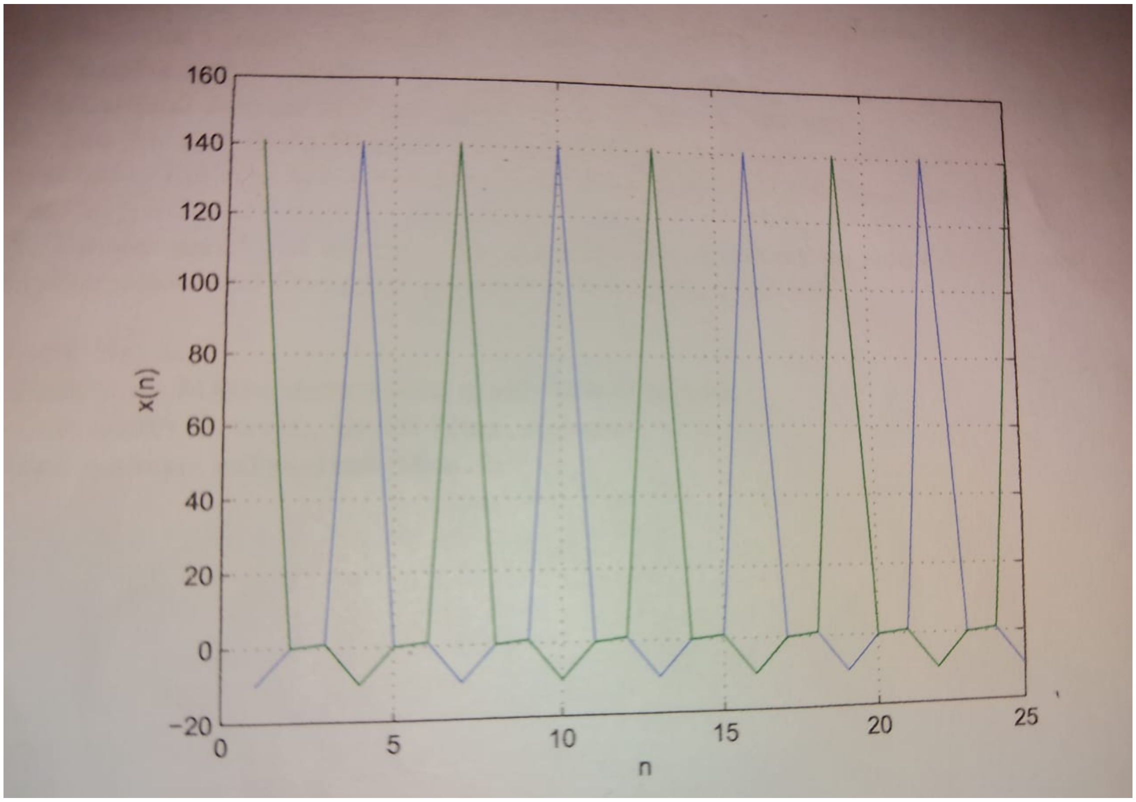

Figure 3.2: This figure represents the periodicity of system solution.

Conclusion

In this study, we reviewed some models of difference equations and provided detailed steps for finding solutions to the system of difference equations associated with the Fibonacci sequence.

Then, we studied the local and global behavior of those solutions reached, taking into account the initial conditions of this system. We also supported this part with numerical examples that confirm the validity and stability of those solutions and their convergence towards $\left(\frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}\right)$. Finally, we proved that the solutions of the system of difference equations (3.2.1) are periodic with the sixth period, and we attached a numerical example to confirm its results.

In short, our research contributes to enhancing the importance of studying the system of difference equations.

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