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The Error Estimates Between The Solution of The Classical Problem of Homogenization and The Solution of Its Homogenized Problem

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DEDICATION

I dedicate my thesis to ***my father*** who worked tirelessly all his life .l've never said this but i'am indebted to you for everything you've done. I want you to know that your support mattered immensely to me , i hope l've made you proud.

To ***my mom*** *whose words were a remedy and who never stopped believing in me , always prayed and wished the best for me , i always loved you and i'll love you forever i pray that i'll make you proud of me .*

To the pillars of strength that hold me up and the source of endless love and joy of my life ***my grandma*** *and* ***my grandpa****.*

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CONTENTS

NOTATIONS AND CONVENTIONS

The symbols:

- ε : a small parameter representing the size of the period
- $\omega \subset \subset \Omega$:ω strongly included in Ω, that's to say ,$\bar{\omega}$ is compact and $\bar{\omega} \subset \Omega$.

The operators:

- $\langle . \rangle$:moyenne operator : $\langle \mathrm{f} \rangle = \frac{1}{|y|} \int_Y f dy$.
- |Y| : The measurement of Y.
- ∇ : denotes the full gradient operator.
- ∇_x : denotes the gradient in the slow variable.
- ∇_y : denotes the gradient in the fast variable.
- div: denotes the full divergence operator.

- div_x : denotes the divergence in the slow variable.

- div_y : denotes the divergence in the fast variable.

- curl $_x$: denotes the rotation vector in the slow variable in two dimensions, such that:

$$\text{curl}_x = \begin{pmatrix} -\frac{\partial}{\partial x_2} \\ \frac{\partial}{\partial x_1} \end{pmatrix}$$

- curl $_y$: denotes the rotation vector in the fast variable in two dimensions, such that:

$$\text{curl}_y = \begin{pmatrix} -\frac{\partial}{\partial y_2} \\ \frac{\partial}{\partial y_1} \end{pmatrix}$$

- $\partial_i = \frac{\partial}{\partial x_i}$: Partial differentiation with respect to x_i.
- $\partial_i^\varepsilon = \frac{\partial}{\partial x_i^\varepsilon}$: Partial differentiation with respect to x_i^ε.
- $\partial_\nu = \nu_\alpha \partial_\alpha$: The directional derivative along the outer normal $\boldsymbol{\nu} = (\nu_\alpha)$.

The spaces:

- $L^2_\sharp(Y)$: denotes the subspace of functions in $L^2_{loc}(\mathbb{R}^n)$, which are Y-periodic.

- $H^1_\sharp(Y)$: denotes the subspace of functions in $H^1_{loc}(\mathbb{R}^n$, which are Y-periodic.

- $\mathrm{L}^2(\Omega)$: The space of square integrable functions for the Lebesgue dx measure.

- $M_s^{n\times n}$: denotes the set of $n \times n$ symmetric matrices.

$$\mathcal{M}_s(\alpha, \beta, \Omega) = \left\{ A \in L^\infty\left(\Omega; M_s^{n\times n}\right); \alpha|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2 \text{ for any } \xi \in \mathbb{R}^n \right\}$$

- $\mathrm{D}(\Omega)$:The class C^∞ function space , with compact support in Ω.

- $H^m(\Omega) := \{v \in L^2(\Omega), D^\alpha v \in L^2(\Omega), \forall |\alpha| \leq m\}$

- $W^{m,\infty}(\Omega) := \{f \in L^\infty(\Omega) : D^\alpha f \in L^\infty(\Omega), \forall |\alpha| \leq m\}$

- $L^\infty(\Omega)$: Space of bounded functions on Ω.

- $\mathcal{C}^m(\Omega)$: Space of $m-$times continuously differentiable functions on Ω, for $m \in \mathbb{N}_0$.

INTRODUCTION

Periodic homogenization is a mathematical technique used to analyze composite materials with periodic micro structures. These materials can be thought of as having a repeating pattern of inclusions or holes on a microscopic scale. While the material properties may vary significantly at this small scale, homogenization aims to find effective, constant material properties that can be used to model the overall behavior of the composite on a larger scale.

However, there's a gap between the exact solution for a heterogeneous material and the solution obtained using the homogenized model. Here's where error estimates come in,They quantify the difference between the solutions obtained using the homogenized model and the solutions of the original problem with the composite material.

Homogenization theory for second-order elliptic equations in divergence form with rapidly oscillating periodic coefficients is well-developed. Among several basic techniques in homogenization theory we are concerned in this thesis with the two-scale asymptotic expansions method.

The introduction of error estimates in periodic homogenization involves quantifying the discrepancy between the solutions obtained from the homogenized model and those from the original heterogeneous system.

The error estimate in periodic homogenization problems was presented for the first time in Bensoussan,Lions and Papanicolaou [**3**], Oleinik, Shamaev, and Yosifian [**8**], and Cioranescu and Donato [**11**, **16**, **23**] These results typically assumed a certain level of regularity for the material properties within the micro-structure.Piezothermoelasticity has gained a lot of attention over the past decades thanks to its importance in the industrial section. Our purpose in this thesis to present the error estimates of the-order with or without boundary layer terms in the periodic homogenization of elliptic equations in divergence form with Dirichlet boundary conditions comes as a mathematical-oriented study to build a better understanding of the theoretical justification of this phenomenon. The thesis is divided into three chapters, each has a main focus and purpose [**26**].

Let us shortly describe the contents of this dissertation. After this introduction, in chapter one, we give the general setting of the problem. In chapter tow we present an overview of some results obtained in the first and the second order corrections with and without boundary layer terms beside to establishing the second order error estimates. Finally in chapter three, we present all the results obtained in the third order corrections with and without boundary layer terms beside to establishing establish the third order error estimates.

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CHAPTER 1

INTRODUCTION TO HOMOGENIZATION AND CORRECTORS

Many problems encountered in different scientific fields depend on parameters having great spatial variability. Solving this type of problem on the scale of variation of these parameters can be very difficult due to the size of the meshes used. The aim of homogenization is to reformulate these problems in the form of a so-called homogenized problem by introducing defined effective parameters on a coarser spatial scale. The resolution of this second problem on this scale is then less expensive. In this chapter,we consider a classic boundary problem in homogenization, posed on a periodic structure of period ε, which presents the small parameter scale of the problem.

1.1 Periodic Homogenization

This section aims to examine classical homogenization results for periodic media, with a focus on the role of boundary layers.

1.1.1 Setting of the problem

Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz continuous boundary. Let $A(y)$ be a square symmetric matrix with entries $a_{ij}(y)$ which are Y-periodic functions belonging to $L^\infty(Y)$. We assume that there exist two constants $0 < \lambda < \Lambda < +\infty$ such that, for a.e. $y \in Y$,

$$\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2, \forall \xi \in R^n.$$

Let A_ε (x) = A($\dfrac{x}{\varepsilon}$) be a periodically oscillating matrix of coefficients. For a given function f $\in$ $\mathrm{L}^2(\Omega)$,We consider the following well posed problem:

$$(P_\varepsilon)\begin{cases} -\mathrm{div} A_\varepsilon \nabla u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

1.1.2 Existence and uniqueness

The desired goal out of this subsection is to prove existence and uniqueness of weak solutionu_ε of the preblem (p_ε),the proof will be done in several steps.

Step 1:

Variational problem:

To determine an appropriate weak formulation to our problem, we find firstly a variational formulation of the problem .

Let us introduce the Hilbert space V such that:

$$V = H_0^1(\Omega) := \left\{ v \in H^1(\Omega), v|_{\partial\Omega} = 0. \right\}$$

By multiplying the first equation(1.1) by v belonging to the space V, we obtain the following variational formulation after using Green's formula.

$$\begin{cases} \text{Find } u^\varepsilon \in H_0^1(\Omega) \text{ such as ,} \\ a\left(u^\varepsilon, v\right) = L(v), \forall v \in H_0^1(\Omega) \end{cases}$$

With:

$$a\left(u^{\varepsilon}, v\right)=\int_{\Omega} A^{\varepsilon} \nabla u \nabla v d x=\sum_{i, j=1}^{3} \int_{\Omega} a_{i j} \cdot(x) \frac{\partial u^{\varepsilon} \partial v}{\partial x_{j} \partial x_{i}} d x .$$

and,

$$L(v)=\int_{\Omega} f v d x .$$

Step 2:

continuty of a(.,.) :

- It is clear that a(.,.) is a symmetric and bilinear form on $H^1{}_0(\Omega)\times H^1{}_0(\Omega)$.
- we show that a(.,.) is continuous .

$$\begin{aligned}
|a(u, v))| & \leqslant \int_{\Omega}\left|A^{\varepsilon}(x) \nabla u^{\varepsilon} \nabla v\right| d x \\
& \leqslant\left\|A^{\varepsilon}(x)\right\|_{L^{\infty}} \int\left|\nabla u^{\varepsilon} \nabla v\right| d x \\
& \leqslant c\left\|\nabla u^{\varepsilon}\right\|_{L^{2}}\|\nabla v\|_{L^{2}} \\
& \leqslant c\left\|u^{\varepsilon}\right\|_{H_{0}^{1}}\|v\|_{H_{0}^{1}} .
\end{aligned}$$

So a(.,.) is continuous.

Step 3:

coercivity of a(.,.) :

$$a(v, v)=\int_{\Omega} A^{\varepsilon}(x)(\nabla v)^{2} d x \geqslant \alpha \int_{\Omega}|(\nabla v)|^{2} d x=\alpha||\nabla v||_{L^{2}}^{2}=||v||_{H_{0}^{1}(\Omega)}^{2} .$$

From this we conclude that a(.,.) coercive.

Step 4:

continuity of L(.,.) :

Now we prove that L(.,.) is a continuous bilinear form:

$$\begin{aligned}
|L(v)| \leqslant \int_{\Omega}|f v| d x \leqslant\|f\|\, \|_{L^{2}} \| v \|_{L^{2}} & \\
& \leqslant c_{p}\|f\|_{L^{2}}\|\nabla v\|_{L^{2}} \qquad (\text {Poincare} \quad \text {inequality}) \\
& =c_{p}\|f\|_{L^{2}}\|v\|_{H_{0}^{1}(\Omega)} .
\end{aligned}$$

Where C_p stands for the Poincare constant.

-So according to the *Theorem of Lax Milligram*, the problem(1.1) admits a unique solution u_ε in $H_0^1(\Omega)$.

1.2 Setting of the problem:

To find the homogenized problem we use an asymptotic expansion method, the principle of this method is to write the solution of (1.1) in the form of the following ansatz:

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^3 u_3\left(x, \frac{x}{\varepsilon}\right) + \dots . \tag{1.2}$$

Where each function u$_i$(x, y) is Y-periodic with respect to the fast variable y.

Suppose that a function $\phi^\varepsilon(x) = \phi(x, y)$, with $y = \frac{x}{\varepsilon}$, so we obtain the following relations :

$$\begin{cases} \dfrac{\partial \phi^\varepsilon(x,y)}{\partial x} = \dfrac{\partial \phi(x,y)}{\partial x} + \dfrac{1}{\varepsilon}\dfrac{\partial \phi(x,y)}{dy}; y = \dfrac{x}{\varepsilon} \\ \operatorname{div}\phi^\varepsilon(x) = \operatorname{div}_x \phi(x,y) + \dfrac{1}{\varepsilon}\operatorname{div}_y \phi(x,y), \\ \nabla\phi^\varepsilon = \nabla_x \phi(x,y) + \dfrac{1}{\varepsilon}\nabla_y \phi(x,y). \end{cases} \tag{1.3}$$

By substituting the asymptotic expansion (1.2) into (1.3), while considering (1.3) and discerning the various powers of ε, we get a cascade of equations .We Defining an operator L_ε by:

$$L_\varepsilon \phi = -div A_\varepsilon \nabla.$$

we may write :

$$L_\varepsilon = \varepsilon^{-2} L_0 + \varepsilon^{-1} L_1 + L_2.$$

where;

$$\begin{aligned} L_0 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) \\ L_1 &= -\frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) - \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) \\ L_2 &= -\frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right). \end{aligned}$$

The two space variables x and y are taken as independent, and at the end of the computation y is replaced by $\frac{x}{\varepsilon}$. Equation (1.1) is therefore equivalent to the following system.

$$\begin{aligned} & L_0 u_0 = 0 \\ & L_0 u_1 + L_1 u_0 = 0 \\ & L_0 u_2 + L_1 u_1 + L_2 u_0 = f \\ & L_0 u_3 + L_1 u_2 + L_2 u_1 = 0 \\ & L_0 u_4 + L_1 u_3 + L_2 u_2 = 0. \\ & \quad \dots\dots \end{aligned} \tag{1.4}$$

we obtain at successive powers of ε, the following equations:

$$\text{Ordre } \varepsilon^{-2} \begin{cases} L_0 u_0 = 0 \quad \text{on Y ,} \\ u_0 \quad is \quad Y-periodic. \end{cases} \qquad (P1)$$

$$\text{Ordre } \varepsilon^{-1} \begin{cases} L_0 u_1 = -L_1 u_0 \quad \text{on Y} \\ u_1 \quad is \quad Y-periodic. \end{cases} \qquad (P2)$$

$$\text{Ordre } \varepsilon^{0} : \begin{cases} L_0 u_2 = f - L_1 u_1 - L_2 u_0 \quad \text{on Y ,} \\ u_2 \text{ is Y-periodic .} \end{cases} \qquad (P3)$$

To solve the preceding system of equations,we need to recall the Fredholm Alternative lemma.

Fredholm Alternative lemma

Lemma 1.2.1

[6]
Let $f \in L^2(\Omega)$ y-periodic function we consider the following problem:

$$\begin{cases} L_0 \phi = f(y) \text{ on } Y \\ \phi \text{ is Y-periodic.} \end{cases}$$

So, there is a solution ϕ if and only if: $< f >= \frac{1}{|y|} \int_y f(y)dy = 0$.

- If there is a solution, then it is unique to an additive constant.

1.2.1 The cell and the homogenized problems

In using the Freedholm alternative lemma for periodic elliptic problem on (1.4), we are able to establish that every equation within (1.4) has a unique solution u_i(x, y) (up to a constant $\widetilde{u}_i$ that depends on x only).

✓For the initial problem (P1) :

$$\begin{cases} -\text{div} A(y)(\nabla_y u_0(x,y)) = 0 \\ u_0 \quad \text{is Y-periodic} \quad in \quad Y. \end{cases}$$

Indicates that $u_0(x, y) \equiv u_0(x)$, does not depend on the y variable.

The second equation in (1.4) gives u_1:

$$u_1(x,y) = -\chi^j(y)\frac{\partial u_0}{\partial x_j}(x) + \tilde{u_1}(x). \qquad (1.5)$$

Where $\chi^j(y)$ are the unique solutions in $H^1_\sharp(Y)$ of the first cell problem:

$$\begin{cases} L_0\chi^j(y) = -\dfrac{\partial a_{ij}}{\partial y_i}(y) \quad \text{in } Y \\ \displaystyle\int_Y \chi^j(y)dy = 0 \end{cases} \tag{1.6}$$

The third equation in (1.4) gives u$_2$:

$$u_2\left(x, \frac{x}{\varepsilon}\right) = \chi^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) - \chi^j\left(\frac{x}{\varepsilon}\right)\frac{\partial \tilde{u}_1}{\partial x_j}(x) + \tilde{u}_2(x). \tag{1.7}$$

where $\tilde{u}_2(x)$ is an additive constant and $\chi^j(y), j = 1, \ldots, n$, are the unique solutions in $H^1_\sharp(Y)$ with zero average of the cell equation:

$$\begin{cases} L_0\chi^j = b_{ij} - \fint_Y b_{ij}(y)dy \quad \text{in } Y; \\ \fint_Y \chi^j(y)dy = 0 \qquad y \to \chi^j(y) Y\text{-periodic.} \end{cases} \tag{1.8}$$

With:

$$b_{ij}(y) = a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial_{yk}} - \frac{\partial}{\partial_{yk}}(a_{ki}(y)\chi^j).$$

The fourth equation in (1.4) gives u$_3$:

$$u_3(x, y) = \chi^{ijk}(y)\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} + \chi^{ij}(y)\frac{\partial^2 \tilde{u}_1(x)}{\partial x_i \partial x_j} - \chi^j(y)\frac{\partial \tilde{u}_2}{\partial x_j}(x) + \tilde{u}_3(x). \tag{1.9}$$

Where $\chi^{ijk}(y)$ are the unique solutions in $H^1_\sharp(Y)$ of the first cell problem:

$$\begin{cases} L_0\chi^{ijk}(y) = c_{ijk} - \displaystyle\int_Y c_{ijk}(y)dy \quad \text{in} Y \\ \displaystyle\int_Y \chi^{ijk}(y)dy = 0. \end{cases} \tag{1.10}$$

With :

$$c_{ijk} = -a_{ij}\chi^k + \frac{\partial}{\partial_{ym}}(a_{im}\chi^{jk}) + a_{im}\frac{\partial \chi^{jk}}{\partial_{ym}}.$$

The fifth equation in (1.4) gives u$_4$:

$$u_4(x, y) = \chi^{ijmp}(y)\frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_m \partial x_p} + \chi^{ijk}(y)\frac{\partial^3 \tilde{u}_1(x)}{\partial x_i \partial x_j \partial x_k} + \chi^{ij}(y)\frac{\partial^2 \tilde{u}_2(x)}{\partial x_i \partial x_j} - \chi^j(y)\frac{\partial \tilde{u}_3}{\partial x_j}(x) + \tilde{u}_4(x). \tag{1.11}$$

where : $\chi^{ijmp} \in H^1_\sharp(Y)$ are the unique solutions of the fourth cell problem:

$$\begin{cases} L_0\chi^{ijmp} = d_{ijmp} - \displaystyle\int_Y d_{ijmp}(y)dy \quad \text{in } Y \\ \displaystyle\int_Y \chi^{ijmp}(y)dy = 0. \end{cases} \tag{1.12}$$

With:

$$d_{ijmp} = a_{ij}\chi^{mp} + \frac{\partial}{\partial y_k}\left(a_{ik}\chi^{jmp}\right) + a_{ik}\frac{\partial \chi^{jmp}}{\partial y_k}.$$

The homogenized problem of (P^ε) is obtained by averaging the third equation in (1.4). It is given by:

$$(P_H)\begin{cases} -\mathrm{div} A^*\nabla u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.13}$$

where the coefficients of the homogenized matrix A^* are given by:

$$a^*_{ij} = \int_Y \left[a_{ij}(y) - a_{ik}\frac{\partial \chi^j}{\partial y_k}(y)\right] dy. \tag{1.14}$$

such that $\left(a^*_{ij}\right)$ is bounded, symmetric and uniformly elliptic. The problem (P_H) is well-posed in $H^1_0(\Omega)$.

The functions $\tilde{u}_1, \tilde{u}_2, \tilde{u}_3$ and $\tilde{u}_4$ are non-oscillating functions which represent the average of u_1, u_2, u_3 and u_4 respectively and are solutions in Ω of the equations:

$$-\mathrm{div}\left[A^*\nabla \tilde{u}_1(x)\right] = < c_{ijk} > \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}. \tag{1.15}$$

$$-\mathrm{div}\left[A^*\nabla \tilde{u}_2(x)\right] = < d_{ijkl} > \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l} + < c_{ijk} > \frac{\partial^3 \tilde{u}_1}{\partial x_i \partial x_j \partial x_k}. \tag{1.16}$$

$$\begin{aligned} -\mathrm{div}\left[A^*\nabla \tilde{u}_3(x)\right] = < e_{ijklm} > \frac{\partial^5 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l \partial x_m} + < d_{ijkl} > \frac{\partial^4 \tilde{u}_1}{\partial x_i \partial x_j \partial x_k \partial x_l} \\ + < c_{ijk} > \frac{\partial^3 \tilde{u}_2}{\partial x_i \partial x_j \partial x_k}. \end{aligned} \tag{1.17}$$

where

$$e_{ijklm} = a_{ij}\chi^{klm} + \frac{\partial}{\partial y_r}\left(a_{ir}\chi^{jklm}\right) + a_{ir}\frac{\partial}{\partial y_r}\left(\chi^{jklm}\right).$$

and

$$\begin{aligned} -\mathrm{div}\left[A^*\nabla \tilde{u}_4(x)\right] = < h_{ijklmn} > \frac{\partial^6 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l \partial x_m \partial x_n} + < e_{ijklm} > \frac{\partial^5 \tilde{u}_1}{\partial x_i \partial x_j \partial x_k \partial x_l \partial x_m} \\ + < d_{ijkl} > \frac{\partial^4 \tilde{u}_2}{\partial x_i \partial x_j \partial x_k \partial x_l} + < c_{ijk} > \frac{\partial^3 \tilde{u}_3}{\partial x_i \partial x_j \partial x_k}. \end{aligned} \tag{1.18}$$

where

$$h_{ijklmn} = a_{ij}\chi^{klmn} + \frac{\partial}{\partial y_r}\left(a_{ir}\chi^{jklmn}\right) + a_{ir}\frac{\partial}{\partial_y}(\chi^{jklmn}).$$

such that $\chi^{jklmn} \in H^1_\sharp(Y)$ are the unique solutions of the fifth cell problem:

$$\begin{cases} L_0\chi^{jklmn}(y) = e_{ijklm} - < e_{ijklm} > \\ \int_Y \chi^{jklmn}(y)dy = 0 \end{cases} \tag{1.19}$$

Remark 1.2.1

*[**24**]*

i) The functions $\tilde{u}_1$,$\tilde{u}_2$,$\tilde{u}_3$ and $\tilde{u}_4$ are not uniquely defined since the equations (1.15), (1.16),(1.17), and (1.18) haven't any boundary conditions, and it is very difficult to determine them. However, there is a special geometric case allows us to find out the boundary conditions for only $\tilde{u}_1$.

ii) It is technically complicated to keep track of boundary conditions when seeking u in the form(1.2), especially near the boundary, so we expect u_ε to behave like:

$$u_\varepsilon(x) = u_0(x) + \varepsilon\left[u_1(x,y) + u_1^{bl,\varepsilon}(x)\right] + \varepsilon^2\left[u_2(x,y) + u_2^{bl,\varepsilon}(x)\right] + \ldots\ldots. \tag{1.20}$$

Where each boundary layer term $u_i^{bl,\varepsilon}$ satisfies

$$\begin{cases} -\operatorname{div} A_\varepsilon \nabla u_i^{bl,\varepsilon} = 0 & \text{in } \Omega, \\ u_i^{bl,\varepsilon} = -u_i\left(x, \dfrac{x}{\varepsilon}\right) & \text{on } \partial\Omega. \end{cases} \tag{1.21}$$

Remark 1.2.2

*[**24**]*

1. *Each term $\left(u_i + u_i^{bl,\varepsilon}\right)$ in the new ansatz (1.20) satisfies a homogeneous Dirichlet boundary condition, which is the advantage of this approach.*

2. *The coefficients and Dirichlet boundary data in equation (1.21) exhibit periodic and fast oscillations.*

3. *As $\partial\Omega$ is Lipschitz continuous , and $u_i^{bl,\varepsilon}$ belongs to $H^1{}_0(\Omega)$, the equation (1.21) is guaranteed to have a unique solution.*

4. *Both the coefficients and the Dirichlet boundary data in (1.21) are periodic and rapidly oscillating.*

5. *The case where the boundary data in (1.21) is not oscillating and belongs to $L^p(\partial\Omega)$, $1 < p < \infty$, was studied by Avellaneda and Lin [5].*

6. *The asymptotic analysis of (1.21) turns out to be more difficult than that of (P_ε) since $u_i^{bl,\varepsilon}$ is not uniformly bounded in the usual energy space $H^1(\Omega)$.*

The problem of analyzing the asymptotic behavior of equation (1.21) is known to be highly challenging and has only been studied extensively for a specific type of domain. This particular type of domain is characterized by having hyperplanes as boundaries, as mentioned in reference [12] and the sources cited therein. A significant breakthrough in this field was achieved through the groundbreaking work of Gérard-Varet and Masmoudi [19]. They focused on solutions to elliptic systems of divergence type, assuming that the domain Ω is smooth, bounded, and uniformly convex in $\mathbb{R}^n$ (where n is greater than or equal to 2). Their research demonstrated that as ε approaches zero, the unique solution $u_i^{bl,*}$ of equation (1.21) converges strongly in the $L^2(\Omega)$ space to a function denoted as $u_i^{bl,*}$, which itself satisfies a certain equation:

$$\begin{cases} -\text{div} A_\varepsilon^* \nabla u_i^{bl,*}(x) = 0, \text{ in } \Omega \\ \qquad\qquad u_i^{bl,*} = -\bar{u}_i(x) \text{ on } \partial\Omega, \end{cases} \tag{1.22}$$

-where A^*=a_{ij} defined in (1.14),and $\bar{u}_i$ is the homogenized Dirichlet boundary data that depends non trivially on u_i.

CHAPTER 2

FIRST AND SECOND ORDER ERROR ESTIMATES

2.1 An overview of error estimates

Within this section,we present some known results on the error estimates in periodic homogenization ,of the first and second order with and without boundary layer terms , such as interested by the elliptic equation in divergence form with Dirichlet boundary conditions.

Firstly we start with the error estimate between u_ε and u_0 the unique solutions of (P_ε) and (P_H) respectively, to do this, one common approach is to use the two scale convergence technique or homogenization convergence framework. This framework allows us to establish convergence results between u_ε and u_0, As ε tends to zero [See chapter one]. Secondly we study the error estimate between u_ε and u_1 and u_2 with and without boundary layers [**24**] .

$$||u_\varepsilon - u_0||_{L^\infty(\Omega)} \leq C_\varepsilon. \tag{2.1}$$

And for $\chi^j \in \mathrm{L}^\infty(\mathrm{y})$,we obtain :

$$||u_\varepsilon - u_0||_{L^2(\Omega)} \leq C_\varepsilon. \tag{2.2}$$

2.2 First order error estimate :

2.2.1 First order error estimate without boundary layer:

The error approximation using a **first order corrector** in the periodic homogenization of problem (P_ε) was provided with further regularity conditions on u_0 or the cell functions χ^j. Assuming that χ^j belongs to $W^{1,\infty}(\mathrm{Y})$, Bensoussan et aL. [**2**] derived the estimation:

Theorem 2.1 [2]

Let u_ε be the solution of the problem (P_ε),And Let u_0 be the solution of the problem (P_H).

Then u_ε converge weakly to u_0 in $H^0{}_1(\Omega)$.If furthermore $u_0 \in W^{2,\infty}(\Omega)$,then:

$$||u_\varepsilon - u_0 - \varepsilon u_1||_{H^1(\Omega)} \leq C\sqrt{\varepsilon}. \tag{2.3}$$

Where u_1 given by (1.5).

This estimate is obtained by Jikov and AL [8],Under the asymptions that $u_0 \in C^2(\bar{\Omega})$, and $\nabla_y \chi^j \in L^\infty(Y)$, and by Allair And Amar [**10**] Under the asymption that $u_0 \in W^{2,\infty}(\Omega)$

The estimate of this theorem has a general character since it holds for a wide range of boundary value problems , and not only for the Dirichlet problem.

- The proof of this theorem is completely standard (see e.g. [24]).

Remark: [24]

Without any regularity assumptions on χ^j, and under the hypothesis that $u_0 \in H^2(\Omega)$ where Ω is bounded domain in $\mathbb{R}^n$ with $C^{1,1}$ regularity Griso [14] using the periodic unfolding method introduced in [13] and further developed in [16], proved this estimates :

$$||u_\varepsilon - u_0 - \varepsilon u_1||_{H_1(\Omega)} \leq C\sqrt{\varepsilon}||u_0||_{H^2(\Omega)}. \tag{2.4}$$

2.2.2 First order estimate with boundary layer

In this part, we will look for minimal hypotheses on u_0 necessary to prove classical error estimates.
Our interest lies in the first boundary layer $u_1^{bl,\varepsilon}$, which is equivalently defined by:

$$\begin{cases} -\mathrm{div} A_\varepsilon \nabla u_1^{bl,\varepsilon} = 0 & \text{in } \Omega, \\ u_1^{bl,\varepsilon}(x) = \chi^j\left(\frac{x}{\varepsilon}\right)\frac{\partial u}{\partial x_j}(x) - \tilde{u}_1(x) & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

Theorem 2.2

Let u_ε and u_0 be the unique solutions of (1.1) and (1.13) respectively [See Chapter 1]. Assume that $u \in W^{2,\infty}(\Omega)$. Let $u_1, u_1^{bl,\varepsilon}$ be defined by (1.5) and (1.21). Then:

$$\left\| u_\varepsilon(x) - u_0(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) - \varepsilon u_1^{bl,\varepsilon}(x) \right\|_{H_0^1(\Omega)} \leq C\varepsilon \tag{2.6}$$

Allair And Amar proved this theorem in [10].

Proof [10]

As in [**9**], defining :$r_\varepsilon(x) = \varepsilon^{-1}\left(u_\varepsilon(x) - u(x) - \varepsilon u_1(x, x/\varepsilon) - \varepsilon u_1^{bl,\varepsilon}(x)\right)$, it satisfies

$$\begin{cases} -\mathrm{div} A_\varepsilon \nabla r_\varepsilon = \varepsilon^{-1}\left(f + \mathrm{div} A_\varepsilon \nabla u\right) + \mathrm{div} A_\varepsilon \nabla u_1 & \text{in } \Omega \\ r_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

Taking into account system (1.4), for any $\phi \in H_0^1(\Omega)$, we have:

$$\begin{aligned}
&\left|\int_\Omega \left[\frac{1}{\varepsilon}\left(f + \mathrm{div} A_\varepsilon \nabla u\right) + \mathrm{div} A_\varepsilon \nabla u_1\right]\phi dx\right| \\
&= \left|\int_\Omega \left[-\frac{1}{\varepsilon}\mathrm{div}_y A_\varepsilon \nabla_y u_2\left(x, \frac{x}{\varepsilon}\right) + \mathrm{div}_x A_\varepsilon \nabla_x u_1\left(x, \frac{x}{\varepsilon}\right)\right]\phi dx\right| \\
&\quad \leq \left|\int_\Omega - \left[\mathrm{div}_x A_\varepsilon \nabla_y u_2\left(x, \frac{x}{\varepsilon}\right) + \frac{1}{\varepsilon}\mathrm{div}_y A_\varepsilon \nabla_y u_2\left(x, \frac{x}{\varepsilon}\right)\right]\phi dx\right| \\
&\quad + \left|\int_\Omega \left[\mathrm{div}_x A_\varepsilon \nabla_y u_2\left(x, \frac{x}{\varepsilon}\right) + \mathrm{div}_x A_\varepsilon \nabla_x u_1\left(x, \frac{x}{\varepsilon}\right)\right]\phi dx\right| \\
&\quad \leq \left|\int_\Omega A_\varepsilon \nabla_y u_2\left(x, \frac{x}{\varepsilon}\right)\nabla\phi dx\right| + C\|\phi\|_{H_0^1(\Omega)} \leq C\|\phi\|_{H_0^1(\Omega)}.
\end{aligned}$$

Passing to the supremum when $\|\phi\|_{H_0^1(\Omega)} = 1$, we obtain that:

$$\|r_\varepsilon\|_{H_0^1(\Omega)} \leq \frac{1}{\lambda}\left\|\frac{1}{\varepsilon}\left(f + \mathrm{div} A_\varepsilon \nabla u\right) + \mathrm{div} A_\varepsilon \nabla u_1\right\|_{H^{-1}(\Omega)} \leq C.$$

which implies the desired result.

Remark: [25]

- Holds for any choice of :$\tilde{u}_1$, since the $H^1(\Omega)$-norm of $\varepsilon\tilde{u}_1(x)$ is precisely of order ε. In truth, Theorem 2.3 is not satisfactory since the boundary layer $u_1^{bl,\varepsilon}$ is not explicit with respect to ε. To find the asymptotic behavior of $u_1^{bl,\varepsilon}$, i.e. to homogenize (1.12), is a very difficult problem that has been addressed only for very special domains Ω .

- To improve Theorem 2.3 by removing the boundary layer term at the price of getting merely interior estimates. Indeed if $u_1^{bl,\varepsilon}$ is really oscillating only near the boundary, one can expect that it does not play any role for interior estimates.

- Unfortunately, it is impossible to achieve the problem without further assumptions. Indeed, obtained optimal interior estimates mainly in two different cases: first, for a general domain Ω with either the maximum principle or smooth coefficients, and second, for general L^∞-coefficients with a rectangular domain Ω.We now treat the former case.

- Namely we assume one of the following assumptions:

✓(**H1**) Equation (1.2) is a scalar equation, i.e. it's solution u_ε is a real-valued function, and therefore the maximum principle applies.

✓(**H2**) The boundary of Ω is smooth, say C^2, and the coefficients $a_{ij}(y)$ in equation (1.2) are Hölder continuous, i.e. there exists $\gamma \in (0,1]$ such that:

$$\sup_{x,y\in\mathbb{R}^n} \frac{|a_{ij}(x)-a_{ij}(y)|}{|x-y|^\gamma} = \|a_{ij}\|_{C^\gamma(\mathbb{R}^n)} < +\infty \quad \forall i,j=1,\ldots,n.$$

Therefore the results of Avellaneda-Lin apply (weaker assumptions on the boundary of Ω are possible).

The error estimates in the H^1 norm:

For the study carried out in this part we need the following results.

Lemma 2.2.1

A function $v \in L^2_\sharp(Y)^2$,($v \in L^2_\sharp(Y)^3$) , satisfies:

$$div v = 0, and \int_Y v = 0.$$

iff there exists a function $\phi \in H^1_\sharp(Y)^2$, $(H^1_\sharp(Y)^3)$,such that,

$$v = curl\phi.$$

In the sequel of this section, we assume that $f \in H^2(\Omega)$, which implies, according to the regularity theory that $u_0 \in H^4(\Omega)$. It is straightforward to verify that (P_ε) can be written as:

$$\begin{cases} A_\varepsilon \nabla u_\varepsilon - v_\varepsilon = 0 \\ -div v_\varepsilon = f \end{cases}$$

We expected that v_ε behaves like:

$$v_\varepsilon = v_0\left(x,\frac{x}{\varepsilon}\right) + \varepsilon v_1\left(x,\frac{x}{\varepsilon}\right) + .. + \varepsilon^j v_j\left(x,\frac{x}{\varepsilon}\right) + \ldots.$$

where each v_j is Y-periodic in the fast variable " $y = \frac{x}{\varepsilon}$ ".

Remark: [25]

The benefit of finding an equivalent problem to (P_ε) is to compute v_j which are very important in the proof of our first main result.

By taking into account that $\nabla = \nabla_x + \frac{1}{\varepsilon}\nabla_y$ and $div = div_x + \frac{1}{\varepsilon}div_y$ together with identifying the different powers of ε we get:

$$\left(\varepsilon^{-1}\right)\begin{cases} a(y)\nabla_y u_0 = 0 \\ -div_y v_0 = 0, \end{cases}$$

$$\left(\varepsilon^{0}\right)\begin{cases} a(y)\nabla_x u_0 + a(y)\nabla_y u_1 - v_0 = 0 \\ -div_x v_0 - div_y v_1 = f, \end{cases}$$

$$\left(\varepsilon^{1}\right)\begin{cases} a(y)\nabla_x u_1 + a(y)\nabla_y u_2 - v_1 = 0 \\ -div_x v_1 - div_y v_2 = 0, \end{cases}$$

$$\left(\varepsilon^{2}\right)\begin{cases} a(y)\nabla_x u_2 + a(y)\nabla_y u_3 - v_2 = 0 \\ -div_x v_2 - div_y v_3 = 0. \end{cases}$$

$$v_0 = a(y)\nabla_x u_0 + a(y)\nabla_y u_1$$

By formally identifying powers of ε,we obtain:

$$a(y)\nabla u_0 = 0,$$

$$-\nabla_y v_0 = 0, \tag{2.7}$$

$$a(y)\nabla_y u_1 + a(y)\nabla_x u_0 - v_0 = 0, \tag{2.8}$$

$$-\nabla_y v_1 - \nabla_x v_0 = f, \tag{2.9}$$

Under the assymption that Ω is bounded domain in $\mathbb{R}^n$, $u_0 \in H^2(\Omega)$,Shari Moskow and Michael Vogelius[9] proved the following proposition:

Proposition 2.2.1

Let u_ε and u_0 be the unique solution of (1.1) and (1.13)respectively ,set $u_1(x,y) = -\chi^j(y)\frac{\partial u_0}{\partial x_j}(x)$,and let $u_1^{bl,\varepsilon}(x) \in H^1(\Omega)$,There exists a constant C, independent of u_0 and ε such that:

$$\left\| u_\varepsilon(x) - u_0(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1^{bl,\varepsilon}\left(x, \frac{x}{\varepsilon}\right)\right\|_{H_0^1(\Omega)} \leq C\varepsilon ||u_0||_{H^2(\Omega)} \tag{2.10}$$

Proof: [9]

Define v_0 by:

$$v_0(x,y) = a(y)\nabla_x u_0(x) + a(y)\nabla_y u_1(x,y), \tag{2.11}$$

i.e.
$$(v_0(x,y))_i = a_{ij}(y)\frac{\partial u_0}{\partial x_j}(x) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k}(y)\frac{\partial u_0}{\partial x_j}(x) = \left(a_{ij}(y) - a_{ik}(y)\frac{\partial \chi^j}{\partial y_k}\right)\frac{\partial u_0}{\partial x_j}.$$

Note that this definition ensures that (2.11) and (2.8) are satisfied,It is easy to see that:

$$\begin{aligned}
-div_y\,(v_0)_i = -div_y\left\{a_{ij} - a_{ik}\frac{\partial \chi^j}{\partial y_k}\right\}\frac{\partial u_0}{\partial x_j} &= -div_y\,\{a_{ij}\}\frac{\partial u_0}{\partial x_j} + div_y\left\{a_{ik}\frac{\partial \chi^j}{\partial y_k}\right\}\frac{\partial u_0}{\partial x_j}\\
&= \{-div_y a_{ij} + div_y a_{ij}\}\frac{\partial u_0}{\partial x_j}\\
&= 0
\end{aligned}$$

since e we are in two dimensions, it follows immediately that there exists q(x, y) so that:

$$(\nabla_y q)^{\perp} = v_0 - A\nabla u_0,$$

where $\perp$ indicates rotation by angle $\frac{\pi}{2}$ counterclockwise. Owing to the fact that each entry of $v_0 - A\nabla u_0$,has Y-integral zero ((due to the definition of A) simple manipulations immediately give that q(x, y) is periodic in y. Since the operations to construct q are operations in the y-variable entirely, it is clear that we may select q such that:

$$\sup_{y\in Y}|\nabla_x q(x,y)| \le C\sum_{ij}\left|\frac{\partial^2 u_0}{\partial x_i \partial x_j}(x)\right| \quad a.e. x \in \Omega.$$

If we take $v_1(x,y) = (\nabla_x q(x,y))^{\perp}$, then

$$\begin{aligned}
\nabla_y v_1 &= \nabla_y(\nabla_x q)^{\perp}\\
&= -\nabla_x(\nabla_y q)^{\perp}\\
&= -\nabla_x v_0 - f,
\end{aligned} \tag{2.12}$$

in other words, the pair v_0, v_1 solves (2.9). Due to our previous estimate of $\nabla_x q$, it follows that

$$\sup_{y\in Y}|v_1(x,y)| \le C\sum_{ij}\left|\frac{\partial^2 u_0}{\partial x_i \partial x_j}(x)\right| \quad a.e. x \in \Omega.$$

From the definition of v_1, we also immediately get that;

$$\nabla_x v_1(x,y) = 0 in \Omega.$$

Now define;

$$Z_\varepsilon(x) = u_\varepsilon(x) - u_0(x) - \varepsilon u_1(x, \frac{x}{\varepsilon}), and$$

$$\mu_\varepsilon(x) = a(\frac{x}{\varepsilon})\nabla u_\varepsilon(x) - v_0(x, \frac{x}{\varepsilon}) - \varepsilon v_1(x, \frac{x}{\varepsilon}).$$

A simple calculation then gives:

$$
\begin{aligned}
&a(\frac{x}{\varepsilon})\nabla Z_\varepsilon(x) - \mu_\varepsilon(x) \\
&= a(\frac{x}{\varepsilon})\nabla u_\varepsilon(x) - a(\frac{x}{\varepsilon})\nabla_x u_0(x) - \varepsilon a(\frac{x}{\varepsilon})\nabla_x u_1(x,\frac{x}{\varepsilon}) - a(\frac{x}{\varepsilon})\nabla_y u_1(x,\frac{x}{\varepsilon}) \\
&- a(\frac{x}{\varepsilon})\nabla u_\varepsilon(x) + v_0(x,\frac{x}{\varepsilon}) + \varepsilon v_1(x,\frac{x}{\varepsilon}) \\
&= -a(\frac{x}{\varepsilon})\nabla_x u_0(x) - \varepsilon a(\frac{x}{\varepsilon})\nabla_x u_1(x,\frac{x}{\varepsilon}) - a(\frac{x}{\varepsilon})\nabla_y u_1(x,\frac{x}{\varepsilon}) + v_0(x,\frac{x}{\varepsilon}) + \varepsilon v_1(x,\frac{x}{\varepsilon}) \\
&= \varepsilon \left(v_1(x,\frac{x}{\varepsilon}) - a(\frac{x}{\varepsilon})\nabla_x u_1(x,\frac{x}{\varepsilon}) \right.
\end{aligned}
$$

Here we have again used the notation ∇ for the full gradient and ∇_x and ∇_y for derivatives in the first and second variables, respectively. In the last identity we used (2.11). From the above identities and our estimate of $\sup_{y\in Y}\left|v_1(x,\frac{x}{\varepsilon})\right|$,it follows that:

$$
||a(\frac{x}{\varepsilon})\nabla Z_\varepsilon - \mu_\varepsilon||_{L^2} \leq C\varepsilon ||u_0||_{H^2(\Omega)}. \tag{2.13}
$$

We may also calculate:

$$
\begin{aligned}
\nabla\mu_\varepsilon(x) &= \nabla a(\frac{x}{\varepsilon})\nabla u_\varepsilon(x) - \nabla_x v_0(x,\frac{x}{\varepsilon}) - \frac{1}{\varepsilon}\nabla_y v_0(\frac{x}{\varepsilon}) \\
&- \varepsilon\nabla_x v_1(\frac{x}{\varepsilon}) - \nabla_y v_1(\frac{x}{\varepsilon}) \\
&= -f(x) - \nabla_x v_0(x,\frac{x}{\varepsilon}) - \nabla_y v_1(x,\frac{x}{\varepsilon}) - \varepsilon\nabla_x v_1(x,\frac{x}{\varepsilon}) \\
&= \varepsilon\nabla_x v_1(x,\frac{x}{\varepsilon}) \\
&= 0.
\end{aligned} \tag{2.14}
$$

In the second identity, we used the equation for ue and the fact that $\nabla_y v_0 = 0$; in the last two identities we used the relation (2.12) and the fact that $\nabla_x v_1 = 0$,
given $g \in L^2(\Omega)$, let $w_\varepsilon \in H^1_0(\Omega)$ denote the solution to:

$$
\begin{aligned}
-\nabla a(\frac{x}{\varepsilon})\nabla w_\varepsilon &= g \quad in \quad \Omega \\
w_\varepsilon &= 0 \quad in \quad \partial\Omega.
\end{aligned} \tag{2.15}
$$

Using the facts that $Z_\varepsilon + \varepsilon u_1^{bl,\varepsilon} \in H_1^0(\Omega)$,and that the matrix a is symmetric, we now obtain:

$$
\begin{aligned}
\int_\Omega (Z_\varepsilon + \varepsilon u_1^{bl,\varepsilon}) g dx &= \int_\Omega a(\frac{x}{\varepsilon})(\nabla Z_\varepsilon + \varepsilon\nabla u_1^{bl,\varepsilon})\nabla w_\varepsilon dx \\
&= \int_\Omega a(\frac{x}{\varepsilon})\nabla w_\varepsilon dx + \varepsilon\int_\Omega a(\frac{x}{\varepsilon})\nabla u_1^{bl,\varepsilon}\nabla w_\varepsilon dx \\
&= \int_\Omega (a(\frac{x}{\varepsilon})\nabla Z_\varepsilon - \mu))\nabla w_\varepsilon dx.
\end{aligned} \tag{2.16}
$$

A combination of (2.13) and(2.15),(2.16) now yields:

$$\left|\int_\Omega (Z_\varepsilon + \varepsilon u_1^{bl,\varepsilon}) g dx\right| \leq C||a(\frac{x}{\varepsilon})(\nabla Z_\varepsilon - \mu_\varepsilon)||_{L^2(\Omega)}||w_\varepsilon||_{H^1(\Omega)} \leq C\varepsilon||u_0||_{H^2}||g||_{H^{-1}}.$$

By dividing by $||g||_{H^{-1}}$ and taking the supremum over all $g \neq 0$,we immediately conclude that:

$$||Z_\varepsilon + \varepsilon u_1^{bl,\varepsilon}||_{H^1(\Omega)} \leq C\varepsilon||u_0||_{H^2(\Omega)},$$

which is exactly the estimate claimed in the statement of this proposition. ■

the error estimate in the L^2 norm:

Using the first-order boundary layer corrector $u_1^{bl,\varepsilon}$,under the assumptions that $u_0 \in H^3(\Omega)$,Shari Moskow and Michael Vogelius[9] obtained this estimate:

Theorem 2.3 [9]

Let u_ε and u_0 be the unique solutions of (1.1) and (1.13) respectively,let $u_1(x, y) = -\chi^j(y)\frac{\partial u_0}{\partial x_j}(x)$,and let $u_1^{bl,\varepsilon}(x) \in H^1(\Omega)$,There exists a constant C, independent of u_0 and ε such that:

$$\left\|u_\varepsilon(x) - u(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1^{bl,\varepsilon}\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leq C\varepsilon||u_0||_{H^3(\Omega)} \tag{2.17}$$

Proof: [9]
Define:

$$\Psi_\varepsilon(x) = u_\varepsilon(x) - u_0(x) - \varepsilon u_1(\frac{x}{\varepsilon}) + \varepsilon u_1^{bl,\varepsilon}(\frac{x}{\varepsilon}) - \varepsilon^2 u_2(\frac{x}{\varepsilon}) \leq C\varepsilon^2||u_0||_{H^3(\Omega)}.$$

and,

$$\xi_\varepsilon(x) = a(\frac{x}{\varepsilon})\nabla u_\varepsilon(x) - v_\varepsilon(\frac{x}{\varepsilon}) - \varepsilon v_1(\frac{x}{\varepsilon}) - \varepsilon^2 v_2(\frac{x}{\varepsilon}).$$

Here the functions u_2, v_1, v_2,are as defined just prior to the statement of this theorem. A simple calculation gives:

$$\begin{aligned}
&a(x/\varepsilon)\nabla\psi_\varepsilon(x) - \xi_\varepsilon(x) \\
&= a(x/\varepsilon)\nabla u_\varepsilon(x) - a(x/\varepsilon)\nabla u_0(x) - \varepsilon a(x/\varepsilon)\nabla u_1(x, x/\varepsilon) \\
&\quad - \varepsilon^2 a(x/\varepsilon)\nabla u_2(x, x/\varepsilon) - a(x/\varepsilon)\nabla u_\varepsilon(x) + v_0(x, x/\varepsilon) \\
&\quad + \varepsilon v_1(x, x/\varepsilon) + \varepsilon^2 v_2(x, x/\varepsilon) \\
&= - a(x/\varepsilon)\nabla_x u_0(x) - \varepsilon a(x/\varepsilon)\nabla_x u_1(x, x/\varepsilon) - a(x/\varepsilon)\nabla_y u_1(x, x/\varepsilon) \\
&\quad - \varepsilon^2 a(x/\varepsilon)\nabla_x u_2(x, x/\varepsilon) - \varepsilon a(x/\varepsilon)\nabla_y u_2(x, x/\varepsilon) \\
&\quad + v_0(x, x/\varepsilon) + \varepsilon v_1(x, x/\varepsilon) + \varepsilon^2 v_2(x, x/\varepsilon) \\
&= \varepsilon^2 (v_2(x, x/\varepsilon) - a(x/\varepsilon)\nabla_x u_2(x, x/\varepsilon)).
\end{aligned}$$

In the last identity, we used that :
$a(y)\nabla_y u_1(x) + a(y)\nabla_x u_0 - v_0 = 0,$ and $a\nabla_x u_1 + a\nabla_y u_2 = v_1$, are satisfied. From the estimates. we have on $\nabla_x u_2(x, y)$ and $v_2(x, y)$,it follows immediately that.

$$||a(\frac{x}{\varepsilon})\nabla\Psi_\varepsilon - \xi_\varepsilon||_{L^2} \leq C\varepsilon^\varepsilon ||u_0||_{H^3}. \tag{2.18}$$

We may also calculate:

$$\begin{aligned}
\nabla\cdot \xi_\varepsilon(x) &= \nabla\cdot a(x/\varepsilon)\nabla u_\varepsilon(x) - \nabla_x\cdot v_0(x/\varepsilon) - \varepsilon^{-1}\nabla_y\cdot v_0(x/\varepsilon) \\
&- \varepsilon\nabla_x\cdot v_1(x/\varepsilon) - \nabla_y\cdot v_1(x/\varepsilon) \\
&- \varepsilon^2\nabla_x\cdot v_2(x/\varepsilon) - \varepsilon\nabla_y\cdot v_2(x/\varepsilon) \\
&= -\varepsilon^2\nabla_x\cdot v_2(x/\varepsilon) \\
&= 0.
\end{aligned}$$

In the second identity, we used the equation for u_ε, plus the fact that (2.8), (2.9) and $\nabla_x v_1 - \nabla_y v_2 = 0$ are all satisfied.
In addition to $u_1^{bl,\varepsilon}$, we define $\varphi_\varepsilon \in H^1(\Omega)$ to be the solution to:

$$\begin{aligned}
\nabla\cdot a(\frac{x}{\varepsilon})\nabla\varphi_\varepsilon = 0 \quad in \quad \Omega, \\
\varphi_\varepsilon = u_2(\frac{x}{\varepsilon}) \quad on \quad \partial\Omega
\end{aligned}$$

Here we use that u_0 is in $H^3(\Omega)$, so that has atrace in$H^{\frac{1}{2}}(\partial\Omega)$.From the the formula for u_2 it follows that:

$$||\varphi_\varepsilon||_{L^2} \leq C||u_2(\frac{x}{\varepsilon})||_{L^2(\partial\Omega} \leq C||u_0||_{H^3}$$

Given g in $L^2(\Omega)$,let $w_\varepsilon \in H^1(\Omega)$, denote the solution to

$$\nabla\cdot a(\frac{x}{\varepsilon})\nabla w_\varepsilon = g \quad in \quad \Omega,$$

$$w_\varepsilon = 0 \quad on \quad \partial\Omega.$$

Since $\Psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon}\varepsilon + \varepsilon^2\varphi_\varepsilon \in H_1^0$, and since a is symmetric, integration by parts yield:

$$\begin{aligned}\int_\Omega (\Psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon}\varepsilon + \varepsilon^2\varphi_\varepsilon) g dx &= \int_\Omega a(\frac{x}{\varepsilon})(\nabla(\Psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon}\varepsilon))\cdot \nabla w_\varepsilon dx \\ &= \int_\Omega a(\frac{x}{\varepsilon})\nabla\Psi_\varepsilon\cdot \nabla w_\varepsilon dx \\ &+ a(\frac{x}{\varepsilon})(\varepsilon u_1^{bl,\varepsilon}\varepsilon + \varepsilon^2\varphi_\varepsilon)\cdot \nabla w_\varepsilon dx \\ &= \int_\Omega a(\frac{x}{\varepsilon})\nabla\Psi_\varepsilon\cdot \nabla w_\varepsilon dx.\end{aligned} \tag{2.19}$$

Here we used the equations for $u_1^{bl,\varepsilon}$ and φ_ε , to arrive at the last identity. At the same time, due to [**2.19**].

$$\begin{aligned}\int_\Omega a(\frac{x}{\varepsilon})\nabla\Psi_\varepsilon\cdot \nabla w_\varepsilon dx &= \int_\Omega (a(\frac{x}{\varepsilon})\nabla\Psi_\varepsilon - \xi_\varepsilon)\cdot \nabla w_\varepsilon dx + \int_\Omega \xi_\varepsilon\cdot \nabla w_\varepsilon dx \\ &= \int_\Omega (a(\frac{x}{\varepsilon})\nabla\Psi_\varepsilon - \xi_\varepsilon)\cdot \nabla w_\varepsilon dx - \int_\Omega \nabla\xi_\varepsilon w_\varepsilon dx \\ &= \int_\Omega (a(\frac{x}{\varepsilon})\nabla\Psi_\varepsilon - \xi_\varepsilon)\cdot \nabla w_\varepsilon dx.\end{aligned} \tag{2.20}$$

Now yields:

$$\left|\int_\Omega (\Psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon}\varepsilon + \varepsilon^2\varphi_\varepsilon) g dx\right| \leq ||a(\frac{x}{\varepsilon})\nabla\Psi_\varepsilon - \xi_\varepsilon||_{L^2(\Omega}||x_\varepsilon||_{H^1} \leq C\varepsilon^2||u_0||_{H^3}||g||_{H^{-1}}.$$

After deviding by$||g||_{H^{-1}}$ and taking the supremum over g, we may rewrite this estimate as:

$$||u_\varepsilon(x) - u_0 - \varepsilon u_1(\frac{x}{\varepsilon} + \varepsilon u_1^{bl,\varepsilon}(x) - \varepsilon^2 u_2(\frac{x}{\varepsilon} + \varepsilon^2\varphi_(x)||_{H^1} \leq C\varepsilon^2||u_0||_{H^3}$$

Since $u_2 and \varphi$ are bounded in $L^2(\Omega)$ by$||u_0||_{H^3}$, independently of ε,it follows immediately that

$$\left\|u_\varepsilon(x) - u(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1^{bl,\varepsilon}\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\Omega)} \leq C\varepsilon||u_0||_{H^3(\Omega)}$$

■

Shari Moskow and Michael Vogelius[9], obtained also the same estimate but under another assumptions conditions :

Theorem 2.4

Let $\Omega \subset \mathbb{R}^2$, be a bounded, convex curvilinear polygon of class C^∞ ,andlet u_0 denote the solution to the homogenized problem . Suppose that $u_0 \in H^{2+\omega}(\Omega)$ for some $0 \leq \omega \leq 1$.
and let $u_1^{bl,\varepsilon}(x) \in H^1(\Omega)$ defined in (1.21),such that:

$$\left\| u_\varepsilon(x) - u(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1^{bl,\varepsilon}\left(x, \frac{x}{\varepsilon}\right) \right\|_{L^2(\Omega)} \leq C_\omega \varepsilon^{1+\omega} ||u_0||_H^{2+\omega}(\Omega). \tag{2.21}$$

Where the constant C_ω independen of u_0 and ε.

2.2.3 Second order estimate without boundary layers

In this part we see an estimation that obtained by Doina Cioranescu and Patrizia Donato[11] .

Theorem 2.5

Let f $\in H^{-1}(\Omega)$ and let u_ε the solution of (1.1),where u_0 is solution of (1.13) ,$u_1 \in W_{per}(Y)$ is defined in (1.5), and u_2 is defined in (1.7).
Moreover if f $\in C^\infty(\bar{\Omega})$, $\partial\Omega$ is of class C^∞ and , furthermore.
$\chi^j, \chi^{ij} \in W^{1,\infty}(\Omega)$, then, there exists a constant C independent of ε such that :

$$\left\| u_\varepsilon - u_0 + \varepsilon u_1 - \varepsilon^2 u_2 \right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{1}{2}}. \tag{2.22}$$

Proof: (See page 133 in [11]) ■

2.2.4 Second order estimate with boundary layers terms

Theorem 2.6

Let Ω is a cubic domain and $u_0 \in W^{2,\infty}(\Omega)$, where u_1 is defined by (1.5) and $\tilde{u}_1$ satisfies (1.22),Such that:

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 \right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{3}{2}}. \tag{2.23}$$

In the case of a smooth enough boundary for a convex bounded domain Ω, and assuming that: $u_0 \in H^3(\Omega), \tilde{u}_1 = \tilde{u}_2 = 0$ and χ^j, χ^{ij} in $W^{1,p}(Y)$ for some $p > n$, Onofrei and Vernescu [20] proved the estimate:

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 \right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{3}{2}} \left\| u_0 \right\|_{H^3(\Omega)}$$

Theorem 2.7

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H) respectively, with $\Omega \subset \mathbb{R}^n$ is a strictly convex bounded domain with $\partial\Omega \in C^\infty$. Assume that $f \in C^\infty(\bar{\Omega})$ and $\chi^{ijk} \in W^{1,\infty}(Y)$. Then:

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} \right\|_{H_0^1(\Omega)} \leq C\varepsilon^2 \left\| u_0 \right\|_{H^3(\Omega)}. \tag{2.24}$$

Proof: The domain Ω is strictly convex if the open straight segment joining any two points of $\partial\Omega$ lies entirely in Ω.
Let us define : $r_\varepsilon(x) = \frac{1}{\varepsilon^2}\left(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon}\right)$, it satisfies:

$$\begin{cases} -\text{div} A_\varepsilon \nabla r_\varepsilon = \dfrac{1}{\varepsilon^2}\left(f + \text{div} A_\varepsilon \nabla u_0\right) + \dfrac{1}{\varepsilon}\text{div} A_\varepsilon \nabla u_1 + \text{div} A_\varepsilon \nabla u_2 & \text{in } \Omega \\ r_\varepsilon = 0 & \text{on } \partial\Omega \end{cases} \tag{2.25}$$

Using the relations (1.4), (1.2) and the fact that u_0 is independent of y, we get:

$$f + \text{div} A_\varepsilon \nabla u_0 = f - L_2 u_0 - \frac{1}{\varepsilon} L_1 u_0 = L_0 u_2 + L_1 u_1 - \frac{1}{\varepsilon} L_1 u_0$$
$$\text{div} A_\varepsilon \nabla u_1 = -L_2 u_1 - \frac{1}{\varepsilon} L_1 u_1 - \frac{1}{\varepsilon^2} L_0 u_1$$
$$\text{div} A_\varepsilon \nabla u_2 = -L_2 u_2 - \frac{1}{\varepsilon} L_1 u_2 - \frac{1}{\varepsilon^2} L_0 u_2$$
$$\text{div} A_\varepsilon \nabla u_2 = -L_2 u_2 - \frac{1}{\varepsilon} L_1 u_2 - \frac{1}{\varepsilon^2} L_0 u_2$$

So the equation(2.25)is reduced to:

$$\begin{aligned} -\text{div} A_\varepsilon \nabla r_\varepsilon &= \frac{1}{\varepsilon^2}\left(L_0 u_2 + L_1 u_1 - \frac{1}{\varepsilon} L_1 u_0\right) + \frac{1}{\varepsilon}\left(-L_2 u_1 - \frac{1}{\varepsilon} L_1 u_1 - \frac{1}{\varepsilon^2} L_0 u_1\right) \\ &+ \left(-L_2 u_2 - \frac{1}{\varepsilon} L_1 u_2 - \frac{1}{\varepsilon^2} L_0 u_2\right) \\ &= -\frac{1}{\varepsilon^3}\left(L_1 u_0 + L_0 u_1\right) + \frac{1}{\varepsilon^2}\left(L_0 u_2 + L_1 u_1 - L_1 u_1 - L_0 u_2\right) - \frac{1}{\varepsilon}\left(L_2 u_1 + L_1 u_2\right) - L_2 u_2 \\ &= \frac{1}{\varepsilon} L_0 u_3 - L_2 u_2 \end{aligned}$$

Then the variational formulation of(2.25)is:

$$\begin{cases} \text{Find } r_\varepsilon \in H_0^1(\Omega) \text{ such that} \\ \int_\Omega A_\varepsilon \nabla r_\varepsilon \nabla \phi dx = \frac{1}{\varepsilon}\int_\Omega \left(L_0 u_3\right)\phi dx - \int_\Omega \left(L_2 u_2\right)\phi dx, \quad \forall \phi \in H_0^1(\Omega) \end{cases}$$

We have for all $\phi \in H_0^1(\Omega)$ the estimate:

$$\begin{aligned}\left|\int_\Omega A_\varepsilon \nabla r_\varepsilon \nabla \phi dx\right| &= \left|\frac{1}{\varepsilon}\int_\Omega (L_0 u_3)\,\phi dx - \int_\Omega (\mathrm{div}_x A_\varepsilon \nabla_y u_3)\,\phi dx + \int_\Omega (\mathrm{div}_x A_\varepsilon \nabla_y u_3)\,\phi dx - \int_\Omega (L_2 u_2)\,\phi dx\right| \\ &\leq \left|\frac{1}{\varepsilon}\int_\Omega (L_0 u_3)\,\phi dx - \int_\Omega (\mathrm{div}_x A_\varepsilon \nabla_y u_3)\,\phi dx\right| + \left|\int_\Omega (\mathrm{div}_x A_\varepsilon \nabla_y u_3)\,\phi dx - \int_\Omega (L_2 u_2)\,\phi dx\right| \\ &= \left|\int_\Omega (\mathrm{div} A_\varepsilon \nabla_y u_3)\,\phi dx\right| + \left|\int_\Omega (\mathrm{div}_x A_\varepsilon (\nabla_x u_2 + \nabla_y u_3))\,\phi dx\right| \\ &= \left|\int_\Omega A_\varepsilon \nabla_y u_3 \nabla\phi dx\right| + \left|-\int_\Omega A_\varepsilon (\nabla_x u_2 + \nabla_y u_3)\nabla\phi dx\right| \\ &\leq 2\left|\int_\Omega A_\varepsilon \nabla_y u_3 \nabla\phi dx\right| + \left|\int_\Omega A_\varepsilon \nabla_x u_2 \nabla\phi dx\right|.\end{aligned}$$

Using the L^∞ boundedness of A_ε, and that $\|\nabla_y u_3\|_{L^2(\Omega)} \leq C_{13}\|u_0\|_{H^3(\Omega)}$ and $\|\nabla_x u_2\|_{L^2(\Omega)} \leq C\|u_0\|_{H^3(\Omega)}$.

we get:

$$\left|\int_\Omega A_\varepsilon \nabla r_\varepsilon \nabla\phi dx\right| \leq C\|u_0\|_{H^3(\Omega)}\|\phi\|_{H_0^1(\Omega)}, \forall\phi \in H_0^1(\Omega)$$

By taking $\phi = r_\varepsilon$ and using the ellipticity of A_ε, we obtain:

$$\lambda\|r_\varepsilon\|^2_{H_0^1(\Omega)} \leq \int_\Omega A_\varepsilon \nabla r_\varepsilon \nabla r_\varepsilon dx \leq C\|u_0\|_{H^3(\Omega)}\|r_\varepsilon\|_{H_0^1(\Omega)}$$

which implies that:

$$\|r_\varepsilon\|_{H_0^1(\Omega)} \leq C\|u_0\|_{H^3(\Omega)}$$

■

Consider a second-order corrector, assuming $f \in C^\infty(\bar{\Omega}), \tilde{u}_1 = \tilde{u}_2 = 0$ and χ^j, χ^{ij} in $W^{1,\infty}(Y)$, Cioranescu and Donato[11] obtained the estimate:

$$\left\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2\right\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon} \tag{2.26}$$

From the proof of the theorem (2.2) that proved by Shari Moskow and Michael Vogelius[9],they obteined this estimate under an assumptions conditions :

$$\left\|u_\varepsilon - u_0 - \varepsilon u_1 + \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 + \varepsilon^2 u_2^{bl,\varepsilon}\right\|_{H^1(\Omega)} \leq C\varepsilon^2 ||u_0||_{H^3}(\Omega) \tag{2.27}$$

2.3 Interior Error Estimates

2.3.1 First order estimate without boundary layer

Theorem 2.8

Let u_ε and u_0 be the unique solutions of (1.1) and (1.13) respectively. Let u_1 be defined by (1.5). Assume that either hypothesis (H1) or (H2) holds true. Assume also that $u \in W^{3,\infty}(\Omega)$. Then, for any open set $\omega \subset\subset \Omega$ compactly embedded in Ω, there exists a constant C, depending on ω but not on ε, such that:

$$\left\| u_\varepsilon(x) - u_0(x) - \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right)\right\|_{H^1(\omega)} \leq C\varepsilon. \tag{2.28}$$

Where C depends on ω

The proof of the theorem is based on this Lemma :

Lemma 2.3.1

For a sequence ϕ_ε in $H^1(\Omega)$ we define the sequence of solutions $z_\varepsilon \in H^1(\Omega)$ of

$$\begin{cases} -\text{div} A_\varepsilon \nabla z_\varepsilon = 0 & \text{in } \Omega \\ z_\varepsilon = \phi_\varepsilon & \text{on } \partial\Omega \end{cases}$$

Assume that there exists a constant C such that, either (H1) holds and $\|\phi_\varepsilon\|_{L^\infty(\partial\Omega)} \leq C$, or (H2) holds and $\|\phi_\varepsilon\|_{L^2(\partial\Omega)} \leq C$. Then, for any open set $\omega \subset\subset \Omega$, there exists a positive constant C such that

$$\|z_\varepsilon\|_{H^1(\omega)} \leq C$$

proof of lemma 2.3.1:(See[10])

Proof: [10]
For $\omega \subset\subset \Omega$, we observe that:

$$\|u_\varepsilon - u - \varepsilon u_1\|_{H^1(\omega)} \leq \left\|u_\varepsilon - u - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon}\right\|_{H_0^1(\Omega)} + \varepsilon \left\|u_1^{bl,\varepsilon}\right\|_{H^1(\omega)}.$$

Since $u_1(x, x/\varepsilon)$ is a bounded sequence in either $L^\infty(\partial\Omega)$ or $L^2(\partial\Omega)$, application of Lemma (2.3) yields the desired result. ∎

CHAPTER 3

THIRD ORDER ERROR ESTIMATES

In this chapter, we proceed our study of the error estimates in the periodic homogenization of elliptic equations by present the error estimates of the third-order with and without boundary layers terms, So in the first we deals to the third error estimate without the boundary layers correctors , Then we prove two theories related to the interior error estimate for periodic homogenization with the third error estimate with the boundary layers correctors
Lastly we compare the previous results with the results obtained.

3.1 Third Order Error Estimates

3.1.1 Without the Boundary-Layers

The results obtained in this section are taken from[2]. In this section we need more regularity for u_0 the solution of (P_H) which requires more regularity on the data, and we suppose that the functions $\widetilde{u}_i = \langle u_i \rangle \equiv 0, i = 1,2,3,4$. Since we will not try to compute the minimal regularity required for Ω and f, we simply assume in the sequel that Ω is a bounded domain with $\partial\Omega \in C^\infty$ and $f \in C^\infty(\bar{\Omega})$ which implies, according to the regularity theory (see Evans [49]), that $u_0 \in C^\infty(\bar{\Omega})$. Using the density of $C^\infty(\bar{\Omega})$ in $W^{m,p}(\Omega)$ for all $m \in \mathbb{N}^*$ and $1 \leq p < \infty$, we have $u_0 \in W^{m,p}(\Omega)$.

The first result concerns the second-order error estimate with boundary layers correctors. In this case, we need the regularity $H^3(\Omega)$ for u_0.

Theorem 3.1

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H) respectively, with $\Omega \subset \mathbb{R}^n$ is a bounded domain with $\partial\Omega \in C^\infty$. Assume that $f \in C^\infty(\bar{\Omega})$ and χ^j, χ^{ij} and $\chi^{ijk} \in W^{1,\infty}(Y)$. Then

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 \right\|_{H^1(\Omega)} \leq C\sqrt{\varepsilon} \tag{3.1}$$

To proof this theorem ,we need to use the following tools:

Proposition 3.1.1

Let F be in $H^{-1}(\Omega)$. Then, there exist $n+1$ functions $f_0, f_1, \ldots, f_n$ in $L^2(\Omega)$ such that

$$F = f_0 + \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

in the sense of distributions. Moreover

$$\|F\|^2_{H^{-1}(\Omega)} = \inf \sum_{i=0}^{n} \|f_i\|^2_{L^2(\Omega)}$$

where the infimum is taken over all the vectors $(f_0, f_1, \ldots f_n) \in [L^2(\Omega)]^{n+1}$. Conversely, if $(f_0, f_1, \ldots, f_n)$ is a vector in $[L^2(\Omega)]^{n+1}$, then $F \in H^{-1}(\Omega)$ and it satisfies

$$\|F\|^2_{H^{-1}(\Omega)} \leq \sum_{i=0}^{n} \|f_i\|^2_{L^2(\Omega)}$$

Lemma 3.1.1

Let Ω be a bounded domain with a smooth boundary and

$$B_\delta = \{x \in \Omega, \rho(x, \partial\Omega) < \delta\} \text{ with } \delta > 0$$

Then there exists $\delta_0 > 0$ such that for every $\delta \in (0, \delta_0)$ and every $v \in H^1(\Omega)$ we have

$$\|v\|_{L^2(B_\delta)} \leq C\delta^{\frac{1}{2}}\|v\|_{H^1(\Omega)}$$

where $\rho(x, \partial\Omega)$ denotes the distance of $x \in \Omega$ from the set $\partial\Omega$, and C_{18} is a constant independent of δ and v

Proof (See [Chapter 1, Lemma 1.5, [85]][2]).

Theorem 3.2

Let $A\left(\frac{x}{\varepsilon}\right)$ be an uniformly elliptic bounded matrix and $\partial\Omega$ be Lipschitz continuous. Suppose that

$f \in H^{-1}(\Omega)$ and $g \in H^{\frac{1}{2}}(\partial\Omega)$ then, there exists a unique $u_\varepsilon \in H^1(\Omega)$ solution to

$$\begin{cases} -\operatorname{div}\left(A\left(\dfrac{x}{\varepsilon}\right)\nabla u_\varepsilon\right) = f \text{ in } \Omega \\ u_\varepsilon = g \text{ on } \partial\Omega \end{cases}$$

and

$$\|u_\varepsilon\|_{H^1(\Omega)} \leq C\|f\|_{H^{-1}(\Omega)} + C\|g\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

Proof. (See [Theorem 23.4,Lectures on linear partial differential equations book])[25]
We give the proof of Theorem 3.1

Proof: We set:

$$\begin{aligned}
Z_\varepsilon &= u_\varepsilon - \left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3\right) \\
u_0 &= u_0(x) \\
u_1 &= -\chi^j \frac{\partial u_0}{\partial x_j} \\
u_2 &= \chi^{ij} \frac{\partial^2 u_0}{\partial x_i \partial x_j} \\
u_3 &= \chi^{ijk} \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k}
\end{aligned}$$

then,

$$\begin{aligned}
L_\varepsilon Z_\varepsilon &= L_\varepsilon u_\varepsilon - L_\varepsilon\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3\right) \\
&= L_\varepsilon u_\varepsilon - \left(\varepsilon^{-2} L_0 + \varepsilon^{-1} L_1 + L_2\right)\left(u_0 + \varepsilon u_1 + \varepsilon^2 u_2 + \varepsilon^3 u_3\right) \\
&= L_\varepsilon u_\varepsilon - \varepsilon^{-2} L_0 u_0 - \varepsilon^{-1}\left(L_0 u_1 + L_1 u_0\right) - \left(L_0 u_2 + L_1 u_1 + L_2 u_0\right) \\
&\quad - \varepsilon\left(L_0 u_3 + L_2 u_1 + L_1 u_2\right) - \varepsilon^2\left(L_1 u_3 + L_2 u_2\right) - \varepsilon^3\left(L_2 u_3\right)
\end{aligned}$$

We using the equations of (1.4), we get;

$$L_\varepsilon Z_\varepsilon = -\varepsilon^2 \left(L_1 u_3 + L_2 u_2\right) - \varepsilon^3 \left(L_2 u_3\right)$$

Since

$$\frac{\partial}{\partial x_i} = \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}, \qquad \text{and } \frac{\partial}{\partial y_i} = \varepsilon \frac{\partial}{\partial x_i}$$

A simple computation shows that:

$$\begin{aligned} L_1 u_3 &= -a_{lm}\left(\frac{x}{\varepsilon}\right)\frac{\partial \chi^{ijk}}{\partial y_m}\frac{\partial^4 u_0}{\partial x_l \partial x_i \partial x_j \partial x_k} - \varepsilon \frac{\partial}{\partial x_l}\left(a_{lm}\left(\frac{x}{\varepsilon}\right)\chi^{ijk}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0}{\partial x_m \partial x_i \partial x_j \partial x_k}\right) \\ &- \varepsilon L_2 u_3 \\ L_2 u_2 &= -a_{lm}\left(\frac{x}{\varepsilon}\right)\chi^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0}{\partial x_l \partial x_m \partial x_i \partial x_j} \end{aligned}$$

then:

$$\begin{aligned} L_\varepsilon Z_\varepsilon = &-\varepsilon^2\left(a_{lm}\left(\frac{x}{\varepsilon}\right)\frac{\partial \chi^{ijk}}{\partial y_m}\frac{\partial^4 u_0}{\partial x_l \partial x_i \partial x_j \partial x_k}\right) - \varepsilon^3\left(\frac{\partial}{\partial x_l}\left(a_{lm}\left(\frac{x}{\varepsilon}\right)\chi^{ijk}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0}{\partial x_m \partial x_i \partial x_j \partial x_k}\right)\right) \\ &- \varepsilon^2 a_{lm}\left(\frac{x}{\varepsilon}\right)\chi^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0}{\partial x_l \partial x_m \partial x_i \partial x_j} \end{aligned}$$

Taking into account that u_ε and u_0 vanish on the boundary $\partial\Omega$, then it follows easily that Z_ε satisfies

$$\begin{cases} L_\varepsilon Z_\varepsilon = \varepsilon^2 F^\varepsilon & \text{in } \Omega \\ Z_\varepsilon = \varepsilon G^\varepsilon & \text{on } \partial\Omega \end{cases}$$

where

$$\begin{cases} F^\varepsilon = -a_{lm}\left(\frac{x}{\varepsilon}\right)\frac{\partial \chi^{ijk}}{\partial y_m}\frac{\partial^4 u_0}{\partial x_l \partial x_i \partial x_j \partial x_k} - a_{lm}\left(\frac{x}{\varepsilon}\right)\chi^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0}{\partial x_l \partial x_m \partial x_i \partial x_j} \\ \quad - \varepsilon\left(\frac{\partial}{\partial x_l}\left(a_{lm}\left(\frac{x}{\varepsilon}\right)\chi^{ijk}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0}{\partial x_m \partial x_i \partial x_j \partial x_k}\right)\right) \\ G^\varepsilon = -u_1 - \varepsilon u_2 - \varepsilon^2 u_3 \end{cases}$$

We put
$F_0 = -a_{lm}\left(\frac{x}{\varepsilon}\right)\frac{\partial \chi^{ijk}}{\partial y_m}\frac{\partial^4 u_0}{\partial x_l \partial x_i \partial x_j \partial x_k} - a_{lm}\left(\frac{x}{\varepsilon}\right)\chi^{ij}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0}{\partial x_l \partial x_m \partial x_i \partial x_j}$,
$F_l = -a_{lm}\left(\frac{x}{\varepsilon}\right)\chi^{ijk}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0}{\partial x_m \partial x_i \partial x_j \partial x_k}$.
Under the assumptions on a_{lm}, u_0, χ^{ij} and χ^{ijk} we get

$$\|F_0\|_{L^2(\Omega)} \leq C \tag{3.2}$$

$$\|F_l\|_{L^2(\Omega)} \leq C \tag{3.3}$$

we Using the Proposition 3.1.1 then from (3.2) and (3.3) we obtain $F^\varepsilon \in H^{-1}(\Omega)$.
Let's look at the function G_ε. We prove the following estimate:

$$\|G_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\varepsilon^{\frac{-1}{2}}$$

At this point, we need to introduce the function $m_\varepsilon \in D(\Omega)$ defined as follows

$$\begin{cases} m_\varepsilon = 1 \text{ if } \rho(x,\partial\Omega) \leq \varepsilon \\ m_\varepsilon = 0 \text{ if } \rho(x,\partial\Omega) \geq 2\varepsilon \\ \|\nabla m_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon} \end{cases}$$

For the existence of such kind of functions see [**22**] and the references therein.

Set

$$V_\varepsilon = m_\varepsilon G_\varepsilon.$$

$$V_\varepsilon = \overline{\{x, \rho(x,\partial\Omega) < 2\varepsilon\}}.$$

which will be denoted by U_ε

Using the$H^1-norm, we have \|V_\varepsilon\|_{H^1(U_\varepsilon)} = \|V_\varepsilon\|_{L^2(U_\varepsilon)} + \|\nabla V_\varepsilon\|_{L^2(U_\varepsilon)} .Clearly, from the definition of$ m$_\varepsilon$ and the regularity properties of $u_0, \chi^j, \chi\{ij, \chi^{ijk}$,one has that

$$\|V_\varepsilon\|_{L^2(U_\varepsilon)} \leq C.$$

On the other hand, we have

$$\begin{aligned} \frac{\partial V^\varepsilon}{\partial x_i}(x) =& m_\varepsilon(x)\left[\frac{1}{\varepsilon}\frac{\partial \chi^k}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0(x)}{\partial x_k} + \chi^k\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0(x)}{\partial x_i \partial x_k} - \frac{\partial \chi^{kl}}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0(x)}{\partial x_k \partial x_l} - \right. \\ & \left. \varepsilon\chi^{kl}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_i \partial x_k \partial x_l} - \varepsilon\frac{\partial \chi^{klm}}{\partial y_i}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_k \partial x_l \partial x_m} - \varepsilon^2\chi^{klm}\left(\frac{x}{\varepsilon}\right)\frac{\partial^4 u_0(x)}{\partial x_i \partial x_k \partial x_l \partial x_m}\right] \\ & + \frac{\partial m_\varepsilon}{\partial x_i}\left[\chi^k\left(\frac{x}{\varepsilon}\right)\frac{\partial u_0(x)}{\partial x_k} - \varepsilon\chi^{kl}\left(\frac{x}{\varepsilon}\right)\frac{\partial^2 u_0(x)}{\partial x_k \partial x_l} - \varepsilon^2\chi^{klm}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0(x)}{\partial x_k \partial x_l \partial x_m}\right] \end{aligned}$$

Again, on the account of the above definition of m_ε and the regularity properties of u_0, χ^k, χ^{kl} and χ^{klm}, it is easy to check that

$$\|\nabla V^\varepsilon\|_{L^2(U_\varepsilon)} \leq \frac{1}{\varepsilon}C\|u_0\|_{H^1(U_\varepsilon)} + C$$

and owing to Lemma 3.1.1, we derive that

$$\|u_0\|_{H^1(U_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}\|u_0\|_{H^2(\Omega)}$$

Then we conclude that:

$$\begin{aligned} \|V_\varepsilon\|_{H^1(U_\varepsilon)} &\leq C + \varepsilon^{-1}C_{26}\|u_0\|_{H^1(U_\varepsilon)} \\ &\leq C + \varepsilon^{-1}C\left(C\varepsilon^{\frac{1}{2}}\|u_0\|_{H^2(\Omega)}\right) \\ &\leq C\varepsilon^{\frac{-1}{2}} \end{aligned}$$

On $\partial\Omega, V_\varepsilon = G_\varepsilon$, this gives that

$$\|G_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} = \|V_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\,\|V_\varepsilon\|_{H^1(\Omega)} = C\,\|V_\varepsilon\|_{H^1(U_\varepsilon)} \leq C\varepsilon^{\frac{-1}{2}}$$

Using the regularity results of Theorem 4.3, we deduce that:

$$\|Z_\varepsilon\|_{H^1(\Omega)} \leq \varepsilon^2\,\|F^\varepsilon\|_{H^{-1}(\Omega)} + \varepsilon\,\|G_\varepsilon\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\varepsilon^{\frac{1}{2}}$$

which proves the theorem.

The third result is about the third-order error estimate without the third boundary-layer corrector.

3.1.2 With the Boudary-Layers

In this case, we need u_0 to be in $W^{4,\infty}(\Omega)$. Using the Sobolev embedding result (see Adams [1]): Let $l \in \mathbb{N}, m \in \mathbb{N}^*$ and $1 \leq p < \infty$. If either $(m-l)p > n$ or $m - l = n$ and $p = 1$, then $W^{m,p}(\Omega) \hookrightarrow W^{l,q}(\Omega)$, for $p \leq q \leq \infty$. Therefore, we have $W^{n+4,1}(\Omega) \hookrightarrow W^{4,\infty}(\Omega)$ and like $u_0 \in C^\infty(\bar{\Omega}) \subset W^{m,p}(\Omega)$ for all $m \in \mathbb{N}^*$ and $1 \leq p < \infty$, and then, $u_0 \in W^{4,\infty}(\Omega)$.

Theorem 3.3

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H), respectively, with $\Omega \subset \mathbb{R}^n$ is a strictly convex bounded domain with $\partial\Omega \in C^\infty$. Assume that $f \in C^\infty(\bar{\Omega})$ and $\chi^{ijk}, \chi^{ijkl} \in W^{1,\infty}(Y)$. Then:

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3 \right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{5}{2}} \tag{3.4}$$

To proof this theorem, we need the following Lemma :

Lemma 3.1.2

Let ϕ_ε be a sequence of functions in $W^{1,\infty}(\Omega)$, such that:

$$\|\phi_\varepsilon\|_{L^\infty(\Omega)} \leq C \quad \text{and} \quad \|\nabla\phi_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}$$

Let $z_\varepsilon \in H^1(\Omega)$ be the solution of:

$$\begin{cases} -\mathrm{div} A_\varepsilon \nabla z_\varepsilon = 0 & \text{in } \Omega \\ z_\varepsilon = \phi_\varepsilon & \text{on } \partial\Omega. \end{cases}$$

Then, it satisfies:

$$\|z_\varepsilon\|_{H^1(\Omega)} \leq \frac{C}{\sqrt{\varepsilon}}.$$

For the proof, we refer the reader to (Lemma 2.6, [10])

Proof: Defining:
$r_\varepsilon(x) = \frac{1}{\varepsilon^3}\left(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3\right)$
it satisfies:

$$\begin{cases} -\mathrm{div} A_\varepsilon \nabla r_\varepsilon = \frac{1}{\varepsilon^3}\left(f + \mathrm{div} A_\varepsilon \nabla u_0\right) + \frac{1}{\varepsilon^2}\mathrm{div} A_\varepsilon \nabla u_1 + \frac{1}{\varepsilon}\mathrm{div} A_\varepsilon \nabla u_2 + \mathrm{div} A_\varepsilon \nabla u_3 & \text{in } \Omega, \\ r_\varepsilon = -u_3\left(x, \frac{x}{\varepsilon}\right) & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

We decompose

$$r_\varepsilon = r_\varepsilon^1 + r_\varepsilon^2$$

, where r_ε^1 satisfies:

$$\begin{cases} -\mathrm{div} A_\varepsilon \nabla r_\varepsilon^1 = \frac{1}{\varepsilon^3}\left(f + \mathrm{div} A_\varepsilon \nabla u_0\right) + \frac{1}{\varepsilon^2}\mathrm{div} A_\varepsilon \nabla u_1 + \frac{1}{\varepsilon}\mathrm{div} A_\varepsilon \nabla u_2 + \mathrm{div} A_\varepsilon \nabla u_3 & \text{in } \Omega, \\ r_\varepsilon^1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

and r_ε^2 satisfies:

$$\begin{cases} -\mathrm{div} A_\varepsilon \nabla r_\varepsilon^2 = 0 & \text{in } \Omega, \\ r_\varepsilon^2 = -u_3\left(x, \frac{x}{\varepsilon}\right) = -\chi^{ijk}\left(\frac{x}{\varepsilon}\right)\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} & \text{on } \partial\Omega \end{cases} \tag{3.7}$$

Using the fact that $u_3\left(x, \frac{x}{\varepsilon}\right)$ satisfies:

$$\|u_3\|_{L^\infty(\Omega)} \leq C_{38} \quad \text{and} \quad \|\nabla u_3\|_{L^\infty(\Omega)} \leq \frac{C_{39}}{\varepsilon}$$

then Lemma 3.2 gives that $\|r_\varepsilon^2\|_{H^1(\Omega)} \leq \frac{C_{40}}{\sqrt{\varepsilon}}$. On the other hand, we will now estimate r_ε^1 the solution of the problem (4.5). Using the results obtained in the proof of Theorem 4.1 and the fact that:

$$\mathrm{div} A_\varepsilon \nabla u_3 = -L_2 u_3 - \frac{1}{\varepsilon} L_1 u_3 - \frac{1}{\varepsilon^2} L_0 u_3$$

we get

$$-\mathrm{div} A_\varepsilon \nabla r_\varepsilon^1 = -L_2 u_3 - \frac{1}{\varepsilon}\left(L_1 u_3 + L_2 u_2\right) = -L_2 u_3 + \frac{1}{\varepsilon} L_0 u_4$$

The variational formulation of (3.6)is:

$$\begin{cases} \text{Find } r_\varepsilon^1 \in H_0^1(\Omega) \text{ such that} \\ \int_\Omega A_\varepsilon \nabla r_\varepsilon^1 \nabla \phi \mathrm{d}x = \frac{1}{\varepsilon}\int_\Omega \left(L_0 u_4\right)\phi \mathrm{d}x - \int_\Omega \left(L_2 u_3\right)\phi \mathrm{d}x, \quad \forall \phi \in H_0^1(\Omega) \end{cases}$$

We have for all $\phi \in H_0^1(\Omega)$ the estimate:

$$\left|\int_\Omega A_\varepsilon \nabla r_\varepsilon^1 \nabla \phi \mathrm{d}x\right|$$

$$
\begin{aligned}
&= \left| \frac{1}{\varepsilon} \int_{\Omega} (L_0 u_4)\, \phi \mathrm{d}x - \int_{\Omega} (\mathrm{div}_x A_\varepsilon \nabla_y u_3)\, \phi \mathrm{d}x + \int_{\Omega} (\mathrm{div}_x A_\varepsilon \nabla_y u_3)\, \phi \mathrm{d}x - \int_{\Omega} (L_2 u_3)\, \phi \mathrm{d}x \right| \\
&\leq \left| \frac{1}{\varepsilon} \int_{\Omega} (L_0 u_4)\, \phi \mathrm{d}x - \int_{\Omega} (\mathrm{div}_x A_\varepsilon \nabla_y u_4)\, \phi \mathrm{d}x \right| + \left| \int_{\Omega} (\mathrm{div}_x A_\varepsilon \nabla_y u_4)\, \phi \mathrm{d}x - \int_{\Omega} (L_2 u_3)\, \phi \mathrm{d}x \right| \\
&= \left| - \int_{\Omega} (\mathrm{div} A_\varepsilon \nabla_y u_4)\, \phi \mathrm{d}x \right| + \left| \int_{\Omega} (\mathrm{div}_x A_\varepsilon \left(\nabla_x u_3 + \nabla_y u_4 \right))\, \phi \mathrm{d}x \right| \\
&= \left| \int_{\Omega} A_\varepsilon \nabla_y u_4 \nabla \phi \mathrm{d}x \right| + \left| - \int_{\Omega} A_\varepsilon \left(\nabla_x u_3 + \nabla_y u_4 \right) \nabla \phi \mathrm{d}x \right| \\
&\leq 2 \left| \int_{\Omega} A_\varepsilon \nabla_y u_4 \nabla \phi \mathrm{d}x \right| + \left| \int_{\Omega} A_\varepsilon \nabla_x u_3 \nabla \phi \mathrm{d}x \right|
\end{aligned}
$$

Using the L^∞ boundedness of A_ε, $\nabla_y u_4$ and $\nabla_x u_3$, we get:

$$\left| \int_{\Omega} A_\varepsilon \nabla r_\varepsilon^1 \nabla \phi \mathrm{d}x \right| \leq C_{41} \|\phi\|_{H_0^1(\Omega)}, \forall \phi \in H_0^1(\Omega)$$

By taking $\phi = r_\varepsilon^1$ and using the ellipticity of A_ε, we obtain:

$$\lambda \left\| r_\varepsilon^1 \right\|_{H_0^1(\Omega)}^2 \leq \int_{\Omega} A_\varepsilon \nabla r_\varepsilon^1 \nabla r_\varepsilon^1 \, \mathrm{d}x \leq C_{41} \left\| r_\varepsilon^1 \right\|_{H_0^1(\Omega)}$$

which implies that:

$$\left\| r_\varepsilon^1 \right\|_{H_0^1(\Omega)} \leq C_{42}$$

Finally, we get $\varepsilon^3 \left\| r_\varepsilon \right\|_{H^1(\Omega)} \leq C_{43} \varepsilon^{\frac{5}{2}}$ which establishes the desired estimate. ■

The fourth result concerns the third-order error estimate with boundary layers correctors. In this case, we need u_0 to be in $W^{4,\infty}(\Omega)$.

Theorem 3.4

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H), respectively, with $\Omega \subset \mathbb{R}^n$ is a strictly convex bounded domain with $\partial\Omega \in C^\infty$. Assume that $f \in C^\infty(\bar{\Omega})$ and $\chi^{ijkl} \in W^{1,\infty}(Y)$. Then:

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3 - \varepsilon^3 u_3^{bl,\varepsilon} \right\|_{H_0^1(\Omega)} \leq C_{44} \varepsilon^3 \tag{3.8}$$

Proof

We Define: $r_\varepsilon(x) = \frac{1}{\varepsilon^3}\left(u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3 - \ \varepsilon^3 u_3^{bl,\varepsilon}\right)$, it satisfies:

$$\begin{cases} -\mathrm{div} A_\varepsilon \nabla r_\varepsilon = \frac{1}{\varepsilon^3}\left(f + \mathrm{div} A_\varepsilon \nabla u_0\right) + \frac{1}{\varepsilon^2}\mathrm{div} A_\varepsilon \nabla u_1 + \frac{1}{\varepsilon}\mathrm{div} A_\varepsilon \nabla u_2 + \mathrm{div} A_\varepsilon \nabla u_3 & \text{in } \Omega, \\ r_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases}$$

This problem is the same as problem in theorem (3.3), so the solution r_ε has the same estimate of r_ε^1 the solution of (4.5), that is:

$$\|r_\varepsilon\|_{H_0^1(\Omega)} = \left\|r_\varepsilon^1\right\|_{H_0^1(\Omega)} \leq C_{45}.$$

. thus :

$$\left\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3 - \varepsilon^3 u_3^{bl,\varepsilon}\right\|_{H_0^1(\Omega)} = \varepsilon^3 \|r_\varepsilon\|_{H_0^1(\Omega)} \leq C_{45}\varepsilon^3$$

which completes the proof.

Remark: I

n accordance with the results obtained in Theorems (2.8), (3.1), (3.3), (3.4)and the estimates (2.3)and (2.21), we infer that the correctors have no influence on the improvement of the order of the error in the estimates. However, the introduction of boundary layers terms improves these, estimates

The conditions posed on the homogenized solution u_0,and on the solutions of the cell-problems $\chi^{ijk} and \chi^{ijmp}$ in Theorems (3.3 and 3.4 in the above section, bring us to the following question : if we assume minimal regularity assumptions, can one prove differently and obtain the third-order error estimates as stated in theorems (3.3) and (3.4)?

3.2 Third-Order Corrections In Periodic Homogenization Using Mixed Method

- All the results presented in this section are taken from [25]
- in this section we answer the questions we asked previously, our study will be in dimension two.

From the proof of proposition (2.2.1) and under the different powers of ε we get ;

$$(v_1)\begin{cases} v_1 = a(y)\nabla_x u_1 + a(y)\nabla_y u_2, \text{ i.e. } (v_1)_k = \left(-a_{ki}\chi^j + a_{kl}\dfrac{\partial \chi^{ij}}{\partial y_l}\right)\dfrac{\partial^2 u_0}{\partial x_i \partial x_j}, \\ \langle (v_1)_k \rangle = \langle c_{ijk}(y)\rangle \dfrac{\partial^2 u_0}{\partial x_i \partial x_j}, \\ \langle div_x v_1 \rangle = 0 \\ div_y v_1 = -div_x v_0 - f \end{cases}$$

And,

$$(v_2)\begin{cases} v_2 = a(y)\nabla_x u_2 + a(y)\nabla_y u_3, \text{ i.e. } (v_2)_m = \left(a_{mk}\chi^{ij} + a_{ml}\dfrac{\partial \chi^{ijk}}{\partial y_l}\right)\dfrac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \\ \langle (v_2)_m \rangle = \langle d_{mijk}\rangle \dfrac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \\ \langle div_x v_2 \rangle = 0 \\ div_y v_2 = -div_x v_1 \end{cases}$$

and,

$$(v_3)\begin{cases} v_3 = \operatorname{curl}_x K(x,y) \\ div_x v_3 = 0 \\ div_y v_3 = -div_x v_2 \end{cases}$$

$$\sup_{y\in Y} |v_3| \leq C \sum_{i,j,k,l} \left| \frac{\partial^4 u_0}{\partial x_i \partial x_j \partial x_k \partial x_l} \right| \tag{3.9}$$

3.2.1 The boundary layers terms

Under the assumption that $u_0 \in H^4(\Omega)$, so the functions u_1, u_2, u_3 have a traces in $H^{\frac{1}{2}}(\partial\Omega)$, consequently, and owing to theorem of trace we can extract the following estimates:

$$\begin{aligned} \|u_1\|_{H^{\frac{1}{2}}(\partial\Omega)} &\leq C\,\|u_0\|_{H^4(\Omega)} \\ \|u_2\|_{H^{\frac{1}{2}}(\partial\Omega)} &\leq C\,\|u_0\|_{H^4(\Omega)} \\ \|u_3\|_{H^{\frac{1}{2}}(\partial\Omega)} &\leq C\,\|u_0\|_{H^4(\Omega)} \end{aligned} \tag{3.10}$$

Therefore we can introduce the boundary layers functions $u_1^{bl,\varepsilon}, u_2^{b,\varepsilon}$ and $u_3^{b,\varepsilon}$ the unique solutions to $\left(P_{u_1^{b,\varepsilon}}\right), \left(P_{u_2^{bl,\varepsilon}}\right)$ and $\left(P_{u_3^{bl,\varepsilon}}\right)$ respectively, where

$$\left(P_{u_1^{bl,\varepsilon}}\right)\begin{cases} \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u_1^{bl,\varepsilon}\right)=0 & \text{in } \Omega \\ u_1^{bl,\varepsilon}=u_1 & \text{on } \partial\Omega \end{cases} \tag{3.11}$$

and

$$\left(P_{u_2^{bl,\varepsilon}}\right)\begin{cases} \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u_2^{bl,\varepsilon}\right)=0 & \text{in } \Omega \\ u_2^{bl,\varepsilon}=u_2 & \text{on } \partial\Omega \end{cases} \tag{3.12}$$

and

$$\left(P_{u_3^{b,\varepsilon}}\right)\begin{cases} \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla u_3^{bl,\varepsilon}\right)=0 & \text{in } \Omega \\ u_3^{bl,\varepsilon}=u_3 & \text{on } \partial\Omega \end{cases} \tag{3.13}$$

Remark 3.2.1

The existence and uniqueness of $u_1^{bl,\varepsilon}, u_2^{bl,\varepsilon}$ and $u_3^{bl,\varepsilon}$ can be deduced immediately from Theorem (3.2).

From the L^2-estimates proved in ([9]) and the formula for each $u_i(x,y)$, it follows that

$$\begin{aligned} \left\|u_1^{bl,\varepsilon}\right\|_{L^2(\Omega)} &\leq C\left\|u_1\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^2(\partial\Omega)} \leq C\left\|u_0\right\|_{H^4(\Omega)}, \\ \left\|u_2^{bl,\varepsilon}\right\|_{L^2(\Omega)} &\leq C\left\|u_2\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^2(\partial\Omega)} \leq C\left\|u_0\right\|_{H^4(\Omega)}, \\ \left\|u_3^{bl,\varepsilon}\right\|_{L^2(\Omega)} &\leq C\left\|u_3\left(x,\frac{x}{\varepsilon}\right)\right\|_{L^2(\partial\Omega)} \leq C\left\|u_0\right\|_{H^4(\Omega)} \end{aligned} \tag{3.14}$$

The first result concerns the third-order error estimate with the third-order boundary layer corrector. For this case we need the regularity $H^4(\Omega) for u_9$

Theorem 3.6

Let u_ε and u_0 denote the unique solutions of (P_ε) and (P_H) respectively, suppose that $f \in H^2(\Omega)$ then:

$$\left\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\right\|_{H_0^1(\Omega)} \leq C\varepsilon^3 \left\|u_0\right\|_{H^4(\Omega)} \tag{3.15}$$

Proof: [25] The proof will be divided into three steps.

Step 1: The definitions of ψ_ε and ξ_ε.Let;

$$\psi_\varepsilon = u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3$$
$$\xi_\varepsilon = a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon - v_0 - \varepsilon v_1 - \varepsilon^2 v_2 - \varepsilon^3 v_3$$

such that:

$$\begin{aligned}
a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon &= a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon - a\left(\frac{x}{\varepsilon}\right)\nabla u_0 - \varepsilon a\left(\frac{x}{\varepsilon}\right)\nabla u_1 - \varepsilon^2 a\left(\frac{x}{\varepsilon}\right)\nabla u_2 - \varepsilon^3 a\left(\frac{x}{\varepsilon}\right)\nabla u_3 \\
div\xi_\varepsilon &= \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\right)\nabla u_\varepsilon\Big) - div_x v_0 - \frac{1}{\varepsilon}div_y v_0 - \varepsilon div_x v_1 - div_y v_1 - \varepsilon^2 div_x v_2 - \varepsilon div_y v_2 \\
&- \varepsilon^3 div_x v_3 - \varepsilon^2 div_y v_3 \\
&= -f(x) - div_x v_0 - \varepsilon div_x v_1 + div_x v_0 + f(x) - \varepsilon^2 div_x v_2 - \varepsilon div_y v_2 - \varepsilon^3 div_x v_3 \\
&- \varepsilon^2 div_y v_3 \\
&= -\varepsilon div_x v_1 - \varepsilon div_y v_2 \\
&= 0. \\
a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon &= a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon - a\left(\frac{x}{\varepsilon}\right)\nabla u_0 - \varepsilon a\left(\frac{x}{\varepsilon}\right)\nabla u_1 - \varepsilon^2 a\left(\frac{x}{\varepsilon}\right)\nabla u_2 - \varepsilon^3 a\left(\frac{x}{\varepsilon}\right)\nabla u_3 \\
&- a\left(\frac{x}{\varepsilon}\right)\nabla u_\varepsilon + v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \varepsilon^3 v_3 \\
&= -a\left(\frac{x}{\varepsilon}\right)\nabla_x u_0 - \varepsilon a\left(\frac{x}{\varepsilon}\right)\nabla_x u_1 - a\left(\frac{x}{\varepsilon}\right)\nabla_y u_1 - \varepsilon^2 a\left(\frac{x}{\varepsilon}\right)\nabla_x u_2 - \varepsilon a\left(\frac{x}{\varepsilon}\right)\nabla_y u_2 \\
&- \varepsilon^3 a\left(\frac{x}{\varepsilon}\right)\nabla_x u_3 - \varepsilon^2 a\left(\frac{x}{\varepsilon}\right)\nabla_y u_3 + a\left(\frac{x}{\varepsilon}\right)\nabla_x u_0 + a\left(\frac{x}{\varepsilon}\right)\nabla_y u_1 + \varepsilon a\left(\frac{x}{\varepsilon}\right)\nabla_x u_1 \\
&+ \varepsilon a\left(\frac{x}{\varepsilon}\right)\nabla_y u_2 + \varepsilon^2 a\left(\frac{x}{\varepsilon}\right)\nabla_y u_3 + \varepsilon^2 a\left(\frac{x}{\varepsilon}\right)\nabla_x u_2 + \varepsilon^3 v_3 \\
&= \varepsilon^3\left(v_3 - a\left(\frac{x}{\varepsilon}\right)\nabla_x u_3\right).
\end{aligned} \tag{3.16}$$

Step 2: The estimation of $\left\|a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon\right\|_{L^2(\Omega)}$.
Since χ^{ijk} are in $C^\infty(Y)$ and $u_0 \in H^4(\Omega)$ we see that:

$$\sup_{y\in Y}|\nabla_x u_3| \le C\sum_{i,j,k,l}\left|\frac{\partial^4 u_0}{\partial x_i\partial x_j\partial x_k\partial x_l}\right| \tag{3.17}$$

Therefore from (3.9) and (3.17) we conclude that

$$\begin{aligned}
\left\|a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon\right\|_{L^2(\Omega)} &\le \varepsilon^3\|v_3\|_{L^2(\Omega)} + \varepsilon^3\left\|a\left(\frac{x}{\varepsilon}\right)\nabla_x u_3\right\|_{L^2(\Omega)} \\
&\le C\varepsilon^3\|u_0\|_{H^4(\Omega)}.
\end{aligned} \tag{3.18}$$

Step 3: The estimation of $\left\|\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\right\|_{H_0^1(\Omega)}$.
Let $g \in L^2(\Omega)$ and $\omega_\varepsilon \in H_0^1(\Omega)$ the solution to:

$$\begin{cases} -div(a(\frac{x}{\varepsilon})\nabla\omega_\varepsilon) = g \text{in } \Omega \\ \omega = 0 \text{on} \partial\Omega \end{cases}$$

Since $\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon} \in H_0^1(\Omega)$, so by using the Green Formula the integration yields

$$\begin{aligned}
\int_\Omega \left(\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\right) g dx &= \int_\Omega -\operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\nabla\omega_\varepsilon\right)\left(\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\right) dx \\
&= \int_\Omega a\left(\frac{x}{\varepsilon}\right)\left(\nabla\psi_\varepsilon + \varepsilon\nabla u_1^{bl,\varepsilon} + \varepsilon^2\nabla u_2^{bl,\varepsilon} + \varepsilon^3\nabla u_3^{bl,\varepsilon}\right)\cdot\nabla\omega_\varepsilon dx \\
&= \int_\Omega a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon\cdot\nabla\omega_\varepsilon \\
&- \int_\Omega \operatorname{div}\left(a\left(\frac{x}{\varepsilon}\right)\left(\varepsilon\nabla u_1^{bl,\varepsilon} + \varepsilon^2\nabla u_2^{bl,} + \varepsilon^3\nabla u_3^{bl,\varepsilon}\right)\right)\omega_\varepsilon dx \\
&= \int_\Omega a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon\cdot\nabla\omega_\varepsilon dx
\end{aligned} \tag{3.19}$$

Making use of (3.16) and taking advantage of the ellipticity of A_ε, we get:

$$\begin{aligned}
\int_\Omega a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon\cdot\nabla\omega_\varepsilon dx &= \int_\Omega \left(a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon\right)\cdot\nabla\omega_\varepsilon + \int_\Omega \xi_\varepsilon\cdot\nabla\omega_\varepsilon dx \\
&= \int_\Omega \left(a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon\right)\cdot\nabla\omega_\varepsilon - \int_\Omega div\xi_\varepsilon\omega_\varepsilon dx \\
&= \int_\Omega \left(a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon\right)\cdot\nabla\omega_\varepsilon dx \\
&\le \left\|a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon\right\|_{L^2(\Omega)}\|\omega_\varepsilon\|_{H_0^1(\Omega)} \\
&\le C\left\|a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon\right\|_{L^2(\Omega)}\|g\|_{H^{-1}(\Omega)}.
\end{aligned} \tag{3.20}$$

Using the estimate obtained in (3.18), it follows that:

$$\left|\int_\Omega \left(\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\right) g dx\right| \le C\left\|a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon\right\|_{L^2(\Omega)}\|g\|_{H^{-1}(\Omega)}$$

by dividing by $\|g\|_{H^{-1}(\Omega)}$ and taking the supremum over all $g \neq 0$, we immediately conclude that

$$\begin{aligned}
\sup \frac{\left|\int_\Omega \left(\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\right) g\right|}{\|g\|_{H^{-1}(\Omega)}} &\le C\left\|a\left(\frac{x}{\varepsilon}\right)\nabla\psi_\varepsilon - \xi_\varepsilon\right\|_{L^2(\Omega)} \\
&\le C\varepsilon^3\|u_0\|_{H^4(\Omega)}
\end{aligned}$$

Hence, it seems clear that

$$\left\|\psi_\varepsilon + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon}\right\|_{H_0^1(\Omega)} \le C\varepsilon^3\|u_0\|_{H^4(\Omega)}$$

which establishes the formula. ■

The second result is about the third-order error estimate without the third-order boundary layer corrector. Again, for this case we need the regularity $H^4(\Omega)$ for u_0.

Theorem 3.7

Let u_ε and u_0 denote the unique solutions of (P_ε) and (P_H) respectively, suppose that $f \in H^2(\Omega)$, then

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} \right\|_{H^1(\Omega)} \leq C\varepsilon^{\frac{5}{2}} \left\| u_0 \right\|_{H^4(\Omega)} \tag{3.21}$$

Proof: Using the result obtained in Theorem (3.6), we have

$$\begin{aligned} &\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} \right\|_{H^1(\Omega)} \\ &= \left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon} - \varepsilon^3 u_3^{bl,\varepsilon} \right\|_{H^1(\Omega)} \\ &\leq \left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon} + \varepsilon^3 u_3^{bl,\varepsilon} \right\|_{H_0^1} + \varepsilon^3 \left\| u_3^{bl,\varepsilon} \right\|_{H^1(\Omega)} \\ &\leq C\varepsilon^3 \left\| u_0 \right\|_{H^4(\Omega)} + \varepsilon^3 \left\| u_3^{bl,\varepsilon} \right\|_{H^1(\Omega)} \end{aligned}$$

The task is now to estimate $\left\| u_3^{bl,\varepsilon} \right\|_{H^1(\Omega)}$. Since u_3 has a trace in $H^{\frac{1}{2}}(\partial\Omega)$, consequently, owing to Theorem (3.2) we can conclude that

$$\left\| u_3^{bl,\varepsilon} \right\|_{H^1(\Omega)} \leq C_{33} \left\| u_3 \right\|_{H^{\frac{1}{2}}(\partial\Omega)}$$

The proof is completed by showing that:

$$\left\| u_3 \right\|_{H^{\frac{1}{2}}(\partial\Omega)} \leq C\varepsilon^{\frac{-1}{2}} \tag{3.22}$$

For this purpose, we define the function $\kappa_\varepsilon(x) \in D(\Omega)$, such that

$$\begin{cases} \kappa_\varepsilon = 1 \text{ if } \rho(x, \partial\Omega) \leq \varepsilon \\ \kappa_\varepsilon = 0 \text{ if } \rho(x, \partial\Omega) \geq 2\varepsilon \\ \left\| \nabla \kappa_\varepsilon \right\|_{L^\infty(\Omega)} \leq \dfrac{C}{\varepsilon} \end{cases}$$

For the existence of such kind of functions see [32] and the references therein.
Let us put

$$V_\varepsilon = \kappa_\varepsilon u_3$$

such that

$$\operatorname{supp} V_\varepsilon = \{x, \rho(x, \partial\Omega) \leq 2\varepsilon\}$$

which will be denoted by U_ε
At this stage, the only point remaining to get (3.22), is the estimation of $||V_\varepsilon||_{H^1}(\Omega)$.
Making use of H^1-norm, we get:

$$||V_\varepsilon||_{H^1}(\Omega) = ||V_\varepsilon||_{L^2}(\Omega) + ||\nabla V_\varepsilon||_{L^2}(\Omega).$$

Clearly, from the definition of κ_ε ,and the assumption that $u_0 \in H^4(\Omega)$ with taking advantage of

$a_{ij}, \chi^{ijk} \in C^{\infty}(Y)$,we obtain:

$$\begin{aligned} \|V_\varepsilon\|_L^2(U_\varepsilon) &= \left\| \kappa_\varepsilon(x)\chi^{ijk}(\frac{x}{\varepsilon})\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \right\|_{L^2(U_\varepsilon)} \\ &\leq \left\| \chi^{ijk}(\frac{x}{\varepsilon}) \right\|_{L^\infty(Y)} \left\| \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \right\|_{L^2(U_\varepsilon)} \\ &\leq C \left\| \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \right\|_{L^2(U_\varepsilon)} \\ &\leq C||u_0||_{H^3(U_\varepsilon)}. \end{aligned} \tag{3.23}$$

Hence ,

$$\|V_\varepsilon\|_{L^2(U_\varepsilon)} \leq C||u_0||_{H^3(U_\varepsilon)}. \tag{3.24}$$

Let us now estimate the gradient of V_ε, first we have:

$$\frac{\partial V^\varepsilon}{\partial x_l}(x) = \kappa_\varepsilon(x)\left\{ \frac{1}{\varepsilon}\frac{\chi^{ijk}}{\partial y_l}(\frac{x}{\varepsilon})\frac{\partial^3 u_0(x)}{\partial x_i \partial x_j \partial x_l} + \chi^{ijk}(\frac{x}{\varepsilon})\frac{\partial^4 u_0(x)}{\partial x_i \partial x_j \partial x_k \partial x_l} \right\} + \frac{\partial \kappa_\varepsilon(x)}{\partial x_l}\left\{ \chi^{ijk}\frac{x}{\varepsilon}\frac{\partial^3 u_0}{\partial x_i \partial x_j \partial xk} \right\}. \tag{3.25}$$

Again, from the above definition of κ_ε , and the assumption that$u_0 \in H^4(\Omega)$, with taking advantage of: $a_{ij}(y), \chi^{ijk} \in C^\infty(Y)$one can have;

$$\|V_\varepsilon\|_{L^2(U_\varepsilon)} \leq \frac{C}{\varepsilon}\left\| \frac{\partial^3 u_0}{\partial x_i \partial x_j \partial x_k} \right\|_{L^2(U_\varepsilon)} + C\left\| \frac{\partial^4 u_0}{\partial x_i \partial x_{jk} \partial x_l} \right\|_{L^2(U_\varepsilon)} \tag{3.26}$$

however,

$$\|V_\varepsilon\|_{L^2(U_\varepsilon)} \leq C\varepsilon^{-1}||u_0||_{H^3(U_\varepsilon)} + C||u_0||_{H^4(U_\varepsilon)}. \tag{3.27}$$

Furthermore, by applying Lemma we derive that:

$$||u_0||_{H^3(U_\varepsilon)} \leq C\varepsilon^{\frac{1}{2}}||u_0||_{H^4(U_\varepsilon)}. \tag{3.28}$$

Combining (3.24) with (3.27) and making use of (3.28),we conclude that:

$$\begin{aligned} \|V_\varepsilon\|_{H^1(U_\varepsilon)} &\leq C||u_0||_{H^3(U_\varepsilon)} + C\varepsilon^{-1}||u_0||_{H^3(U_\varepsilon)} + C||u_0||_{H^4(U_\varepsilon)} \\ &\leq C\varepsilon^{\frac{1}{2}}||u_0||_{H^4(\Omega)} + C\varepsilon^{-1}(C\varepsilon^{\frac{1}{2}}||u_0||_{H^4(\Omega) + C||u_0||_{H^4(\Omega)}} \\ &\leq C\varepsilon^{\frac{-1}{2}}||u_0||_{H^4(\Omega)}. \end{aligned} \tag{3.29}$$

On $\partial\Omega, V_\varepsilon = u_\varepsilon$,so;

$$||u_\varepsilon||_{H^{\frac{1}{2}}(\partial\Omega)} = ||V_\varepsilon||_{H^{\frac{1}{2}}(\partial\Omega)} \leq \|V_\varepsilon\|_{H^1(\Omega)} = \|V_\varepsilon\|_{H^1(U_\varepsilon)} \leq C\varepsilon^{\frac{-1}{2}}||u_0||_{H^4(\Omega)}. \tag{3.30}$$

Using the regularity results of Theorem 3.2, we deduce that

$$\left\| u_3^{bl,\varepsilon} \right\|_{H^1(\Omega)} \leq C||u_\varepsilon||_{H^{\frac{1}{2}}(\partial\Omega)} \leq \varepsilon^{\frac{-1}{2}}||u_0||_{H^4(\Omega)}. \tag{3.31}$$

Substituting (3.31) in (3.21), we get:

$$\begin{aligned}\left\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 + \varepsilon u_1^{bl,\varepsilon} + \varepsilon^2 u_2^{bl,\varepsilon}\right\|_{H^1(\Omega)} &\leq C\varepsilon^3 \|u_0\|_{H^4(\Omega)} + \varepsilon^3 \left\|u_3^{bl,\varepsilon}\right\|_{H^1(\Omega)} \\ &\leq C\varepsilon^3 \|u_0\|_{H^4(\Omega)} + C\varepsilon^{\frac{5}{2}} \|u_0\|_{H^4(\Omega)} \\ &\leq C\varepsilon^{\frac{5}{2}} \|u_0\|_{H^4(\Omega)},\end{aligned}$$

which is precisely the assertion of the theorem. ■

3.3 Interior Error Estimate

In this part we prove two theorems about second and third orders interior error estimates , to prove this two theorems we use the method that Allar adopted in his article (See [**10**])

3.3.1 Second Order Error Estimate

Theorem 3.8

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H) respectively,Assume that $u_0 \in W^{4,\infty}(\Omega)$, Let $u_1$be defined by (1.5),and u_2 satisfies equation (1.7) ,assume that either hypothesis H1 or H2 holds true , then for any open set $\omega \subset\subset \Omega$ compactly embedded in Ω there exists a constant C ,depending on ω but not ε,such that:

$$\left\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2\right\|_{H^1(\omega)} \leq C\varepsilon. \tag{3.32}$$

Proof

- The proof of this estimate (3.32), is based on Lemma 2.3 (See chapter 2).

For $\omega \subset\subset \Omega$, we observe that:

$$\left\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2\right\|_{H^1(\omega)} \leq \left\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon}\right\|_{H_0^1(\Omega)} + \varepsilon||u_1^{bl,\varepsilon}||_{H^1(\omega)} + \varepsilon^2||u_2^{bl,\varepsilon}||_{H^1(\omega)}.$$

from the proof of (2.24) we obtain:

$$\left\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon}\right\|_{H_0^1(\Omega)} \leq C\varepsilon^2 \left\|u_0\right\|_{H^3(\Omega)}$$

From the L^2 estimates proved in [**5**] , and the formula for each u_1, it follows that:

$$\begin{aligned}
\left\|u_1^{bl,\varepsilon}\right\|_{L^2(\Omega)} &\leq C \left\|u_1\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\partial\Omega)} \leq C \left\|u_0\right\|_{H^4(\Omega)}, \\
\left\|u_2^{bl,\varepsilon}\right\|_{L^2(\Omega)} &\leq C \left\|u_2\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\partial\Omega)} \leq C \left\|u_0\right\|_{H^4(\Omega)}, \\
\left\|u_3^{bl,\varepsilon}\right\|_{L^2(\Omega)} &\leq C \left\|u_3\left(x, \frac{x}{\varepsilon}\right)\right\|_{L^2(\partial\Omega)} \leq C \left\|u_0\right\|_{H^4(\Omega)}.
\end{aligned} \tag{3.33}$$

Since $u_1(x, x/\varepsilon), u_2(x, x/\varepsilon)$ are a bounded sequence in either $L^\infty(\partial\Omega)$ or $L^2(\partial\Omega)$, So according Lemma 2.3 , we conclude that :

$||u_1^{bl,\varepsilon}||_{H^1(\omega)}$ and $||u_2^{bl,\varepsilon}||_{H^1(\omega)}$, are bounded by C.

So,

$$\begin{aligned}
\left\|u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2\right\|_{H^1(\omega)} &\leq C\varepsilon^2 + \varepsilon C + \varepsilon^2 C \\
&\leq \varepsilon(\varepsilon C_1 + C) \\
&\leq C\varepsilon.
\end{aligned}$$

Wich completes the proof.

3.3.2 Third Order Error Estimate

Theorem 3.9

Let u_ε and u_0 be the unique solutions of (P_ε) and (P_H) respectively. Assume that $u_0 \in W^{4,\infty}(\Omega)$, Let u_1,u_2,u_3 be defined by (1.5),(1.7) and (1.9),assume that either hypothesis H1 or H2 holds true , then for any open set $\omega \subset\subset \Omega$ compactly embedded in Ω there exists a constant C ,depending on ω but not ε,such that:

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 \right\|_{H^1(\omega)} \leq C\varepsilon \tag{3.34}$$

Where the constant C depends only on ω which is any open set such that $\omega \subset\subset \Omega$

Proof

- The proof of this estimate (3.34), is based on Lemma 2.3 (See chapter 2).

For $\omega \subset\subset \Omega$, we observe that:

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon^2 u_2 - \varepsilon^3 u_3 \right\|_{H^1(\omega)} \leq \left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} \right\|_{H_0^1(\Omega)} - \varepsilon^3 u_3 \\ - \varepsilon^3 u_3^{bl,\varepsilon} + \varepsilon ||u_1^{bl,\varepsilon}||_{H^1(\omega) + \varepsilon^2 ||u_2^{bl,\varepsilon}||_{H^1(\omega)} - ||u_3^{bl,\varepsilon}||_{H^1(\omega)}}.$$

from the proof of theorem (3.4) we obtain:

$$\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^2 u_2^{bl,\varepsilon} - \varepsilon^3 u_3 - \varepsilon^3 u_3^{bl,\varepsilon} \right\|_{H_0^1(\Omega)} \leq C\varepsilon^3\varepsilon$$

From (3.33) , Since $u_1(x, x/\varepsilon), u_2(x, x/\varepsilon)$ and $u_3(x, x/\varepsilon)$ are a bounded sequence in either $L^\infty(\partial\Omega)$ or $L^2(\partial\Omega)$, So according Lemma 2.3 , we conclude that :

$||u_1^{bl,\varepsilon}||_{H^1(\omega)}$, $||u_2^{bl,\varepsilon}||_{H^1(\omega)}$ and $||u_3^{bl,\varepsilon}||_{H^1(\omega)}$, are bounded by C.

So,

$$\begin{aligned}\left\| u_\varepsilon - u_0 - \varepsilon u_1 - \varepsilon u_1^{bl,\varepsilon} - \varepsilon^2 u_2 - \varepsilon^3 u_3 \right\|_{H^1(\omega)} &\leq C\varepsilon^3 + \varepsilon C + \varepsilon^2 C + \varepsilon^3 C \\ &\leq \varepsilon(\varepsilon^2 C_1 + \varepsilon C + C) \\ &\leq C\varepsilon.\end{aligned}$$

Wich completes the proof.

APPENDIX I

In this appendix, we will give some preliminaries needed to carry on our study.

In all the rest, Ω designates an open open from $\mathbb{R}^N$ provided with the measurement of Lebesgue D_x, and border $\partial\Omega$ sufficiently regular.

Theorem 1. *(**Poincaré**'s Inequality)*

Let Ω be a bounded domain in $\mathbb{R}^N$ with LIPSCHITZ boundary Γ. There exists a positive constant C_P such that, for all $v \in H^1_{\tilde{\Gamma}} = \{v \in H^1(\Omega), \quad v = 0 \quad on \quad \tilde{\Gamma} \subset \Gamma\}$

$$||v||_{L^2(\Omega)} \leq C_P ||\nabla v||_{\mathbb{L}^2(\Omega)}.$$

POINCARÉ's inequality holds if Ω has finite measure or is bounded at least in one direction.

Theorem 2. *(**Trace Theorem**)*

Let Ω be a bounded open set in $\mathbb{R}^N$ with LIPSCHITZ boundary Γ. There exists a bounded linear operator called trace operator and denoted T such that

$$T : H^1(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) \to L^2(\Gamma) \cap \mathcal{C}^0(\overline{\Gamma})$$
$$v \mapsto Tv = v|_{\partial\Omega}$$

The continuity of T implies the existence of a positive constant C_t such that

$$||Tu||_{L^2(\Gamma)} \leq C_t ||v||_{H^1(\Omega)}, \forall v \in H^1(\Omega).$$

Theorem 3. *(**Green**'s Integration by Parts Formula)*

Let Ω *be a bounded open domain in* $\mathbb{R}^3$ *with a sufficiently smooth boundary* Γ *and* $\mathbf{n}$ *is the outward normal. Then for all* $u, v \in \mathcal{C}^1(\overline{\Omega})$

$$\int_{\Omega} \partial_i u(x) v(x) dx = -\int_{\Omega} u(x) \partial_i v(x) dx + \int_{\Gamma} u(x) v(x) n_i d\Gamma.$$

Theorem 4. *(**Young**'s Inequality)*

Let a *and* b *be two non-negative real numbers. If* $p, q \in]1, +\infty[$ *with* $\frac{1}{p} + \frac{1}{q} = 1$ *then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 5. *(**Hölder**'s Inequality)*

Let Ω *be a domain in* $\mathbb{R}^N$ *and* $p, q \in]1, +\infty[$ *with* $\frac{1}{p} + \frac{1}{q} = 1$. *If* $u \in L^p(\Omega)$ *and* $v \in L^q(\Omega)$ *then* $uv \in L^1(\Omega)$ *and:*

$$||uv||_{L^1(\Omega)} \leq ||u||_{L^p(\Omega)} ||v||_{L^q(\Omega)}.$$

CONCLUSION

Through this work, we notice that the assumptions on u_0, the cell solutions, the boundary layer terms and the geometry of the domain play an important role in the improvement of the error estimates, also we deduce that our proved interrior error estimates following Allaire's method do not improve the estimates order which leads us as a future work to think about new formulas for the second and the third boundary layers correctors which needs rigorous study.

Abstruct

This thesis aims to study the L^2, H^{1}_{0} and H1-norms error estimates of the first, second and third-order with or without boundary layer correctors, for the periodic homogenization of elliptic equations in divergence form with Dirichlet boundary conditions. Our study comes for two reasons, the first one is to show how hypothesis imposed on the data can influence on the improvement of the estimates order . The second reason is to show how we can differently prove the same results using different mathematical thechnicals.

Keywords : homogenization, asymptotic analysis, error estimates, boundary layers.

Résumé

Cet mémoire vise à étudier les estimations d'erreurs des normes L^2, H^{1}_{0} et H1 du premier, deuxième et troisième ordre avec ou sans correcteurs de couche limite, pour l'homogénéisation périodique des équations elliptiques.
Sous forme de divergence avec les conditions aux limites de Dirichlet. Notre étude vient pour deux raisons, la première est de montrer comment les hypothèses imposées sur les données peuvent influencer l'amélioration de l'ordre des estimations. La deuxième raison est de montrer comment nous pouvons prouver différemment les mêmes résultats en utilisant différents techniques mathématiques.

Mots clés : homogénéisation, analyse asymptotique, estimations d'erreurs, couches limites.

الملخص

تهدف هذه المدكرة إلى دراسة تقديرات الخطأ لمعايير من الرتبة الأولى و الثانية و الثالثة مع أو بدون مصححات الطبقة الحدودية، من أجل التجانس الدوري للمعادلات الإهليلجية
في شكل تباعد مع شروط حدود ديريشليت. وتأتي دراستنا لسببين، الأول هو إظهار كيف يمكن للافتراضات المفروضة على البيانات أن تؤثر على تحسين ترتيب التقديرات. السبب الثاني هو إظهار كيف يمكننا إثبات نفس النتائج بشكل مختلف باستخدام تقنيات رياضية مختلفة.

الكلمات المفتاحية: التجانس، التحليل التقاربي، تقديرات الخطأ، الطبقات الحدودية

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