People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

KASDI MERBAH UNIVERSITY, OUARGLA

FACULTY of MATHEMATICS and MATERIAL SCIENCE

DEPARTMENT of MATHEMATICS

The memory submitted in partial execution of the requirements of :

The Master Degree

Option : **Probability and Statistic**

Presented by : Berrabah Kaouthar

Title :

Fractional backward stochastic differential equations

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Juin 2024

Dedication

We did not reach beginnings except through His guidance we Allah not reach endings except through His success and we Allah not achieve goals except through His grace.

Praise be to Allah for love and gratitude Praise be to Allah for the beginning and the end.

To the one who taught me morals before letters to the bridge that takes me up to the one under whose feet Allah made heaven to the compassionate heart to the dearest and most precious person in existence.

My dear mother DALILA.

To the one who supported me without limits and gave me something in return.To the one who taught me that the world is a struggle and its weapon is knowledge and knowledge supporting me in my path and my strength Allah after.

My dear father SADEK.

To my sister HANAA and my brothers HAROUN and NAIM and MOHAMED ISLAM.

To my aunts uncle uncle and all relatives.

To all colleagues and friends.

To all the professors and students of the University of Kasdi-Merbah Ouargla.

Thanks

First of all, I thank Allah for allowing me to complete this work.

I extend my thanks and sincere appreciation to my professor and supervisor of this work **Dr : Saouli Mostapha Abdelouahab** for guiding me to complete this project.

And to all my friends in the field of probability and statistics.

I express my sincere thanks and respect to the professors and all the teaching staff at the Faculty of Mathematics in Ouargla.

And to everyone who helped me from near or far in carrying out this work.

Thank you for everything.

Abbreviations and Notations

The different abbreviations and notation used throughout this dissertation are explained below :

$(\Omega, \mathcal{F}, \mathbb{P})$	Probability space
SDE	Stochastic differential equation.
$BSDE$	Backward stochastic differential equation.
BM	Brownian motion.
FBM	Fractional Brownian motion.
$\mathbb{P} - a.s$	Almost certainly for the probability measure $\mathbb{P}$.
B_t	Brownian motion.
B_t^H	Fractional Brownian motion.
$\mathcal{F}_t$	The filtration generated.
$\mathcal{F}_t^W$	The filtration generated by the Brownian motion.
$\mathcal{F}_t^W \vee \mathcal{N}$	The sup between filtration generated by the Brownian motion and negligable set.
$\mathbb{R}^n$	Enclidean real space of n-dimensional.
$\mathbb{R}^{n\times d}$	Set of real matrice $n \times d$.
$\mathbb{L}^1$	Space of integrable processes.
$\mathbb{N}^*$	The set of natural numbers that do not contain zero.
$L^2(\Omega, \mathcal{F}_T, \mathbb{P})$	Set of random variables, $\mathcal{F}_T$-measurable and square integrable.
$\mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$	Set of twice differentiable function.

$\mathcal{C}^2\left(\mathbb{R}^d\right)$	Set of twice differentiable functiont the real space of d-dimensional
$\mathbb{R}^d$	Enclidean real space of d-dimensional
$\mathbb{L}^2\left(\Omega, \mathcal{F}, \mathbb{P}, \mathcal{H}\right)$	Set of random variables, a the completion of the measurable functions.
$\mathbb{L}^2\left(\Omega, \mathcal{F}, \mathbb{P}\right)$	Set of random variables, $\mathcal{F}$-measurable and square integrable.
$\mathbb{L}^2\left(\Omega, \mathcal{F}, \mathcal{H}\right)$	Set of random variables, a the completion of the measurable functions.
$\mathbb{R}$	Real numbers set.
$\mathbb{R}^+$	The set of positive real numbers.
$\mathbb{R}^-$	The set of negative real numbers.
$\mathbb{R}_+^*$	The set of positive real numbers that do not include zero.
$\mathbb{Z}$	Integer numbers set.
$\mathbb{N}$	Natural numbers set.
$\mathbb{E}^{\mathcal{F}_t}$	The conditional expectation with respect to $\mathcal{F}_t$.
$\mathbb{D}_s^H$	The Malliavin derivative operator.
$\mathcal{N}$	Set of negligible N.
$:=$	Equal by definition.
$\langle \cdot, \cdot \rangle$	Scalar product.

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General introduction

The processes have exhibited self-similarity across various fields such as physics, communications networks, and finance. Fractional Brownian Motion (FBM in short), characterized by the Hurst parameter $H \in (0,1)$, is a self-similar process. Specifically, $B^H_{\alpha t}$ shares the same distribution as $\alpha^H B^H_t$ for any $\alpha > 0$. When $H = 1/2$, it corresponds to the standard Wiener process. For $H > 1/2$, FBM exhibits long-term dependence, which is advantageous in emerging models. However, classical stochastic calculus theorems cannot straightforwardly determine fractional integrals when B^H. Consequently, two distinct types of integrals have been defined for FBM.

The first type is the Riemann–Stieltjes path integral, which is applicable when paths are continuous (Young, 1936) [13]. This integral behaves akin to Stratonovich integration but presents challenges in practical applications.

The second type, introduced by Decreusefond and Üstünel (1998) [2], is the Skorokhod integral, also known as the adjoint derivative integral within the framework of stochastic calculus. It possesses the zero-mean property and can be expressed as a limit of Riemann sums defined using Wick products. Its development was influenced by advancements in backward stochastic differential equations (BSDEs) during the 1990s.

BSDEs were initially explored by Pardoux and Peng (1990) [10], who provided a probabilistic interpretation of certain partial differential equations (PDEs). Pardoux and Zhang (1998) [11] extended BSDEs, and Hu (2005) [4] and Hu and Peng (2009) [5] first investigated BSDEs in relation to FBM, establishing existence and uniqueness under specific assumptions.

Maticiuc and Nie (2012) [8] improved upon these results by removing some of these restrictive assumptions. They also introduced a theory of backward stochastic variational inequalities, further proving existence and uniqueness of solutions for reflected BSDEs driven by FBM.

Our work aims to delve into fractional generalized backward stochastic differential equations driven by FBM, structured into three chapters :

The first chapter introduces concepts and fundamental properties of fractional Brownian motion.

The second chapter establishes the existence and uniqueness of solutions for generalized backward stochastic differential equations driven by standard Brownian motion.

The third chapter focuses on studying generalized BSDEs with respect to FBM, culminating in proofs of existence and uniqueness for solutions of generalized backward stochastic differential equations with FBM.

Chapitre 1

Fractional Brownian motion and their properties

1.1 Fractional Brownian motion

Définition 1.1 *A fractional Brownian motion with parameter* $H \in (0,1)$ *is real centered gaussian process noted* $\{B_t^H : t \in \mathbb{R}\}$ *defined on a probabilite space* $(\Omega, \mathbb{F}, \mathbb{P})$ *and verifying :*

i) $B_0^H = 0;\ \mathbb{P} - a.s.$

ii) $\mathbb{E}\left[\left[B_t^H\right]^2\right] = |t|^{2H}, \forall t \in \mathbb{R}.$

iii) B^H has stationary increases.

Remark 1.1 *The parameter* H *is called the hurst parameter.*

Définition 1.2 ***Proposition 1.1*** *The fractional Brownian motion admits the function* R_H *of* $\mathbb{R}^2$ *in* $\mathbb{R}$ *defined by*

$$R_H(t,s) = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right),$$

as a covariance function.

Proof. We have

$$\mathbb{E}\left[\left[B_t^H - B_s^H\right]^2\right] = \mathbb{E}\left[\left[B_t^H\right]^2\right] + \mathbb{E}\left[\left[B_s^H\right]^2\right] - 2\mathbb{E}\left[B_t^H B_s^H\right],$$

and like

$$B_t^H - B_s^H \overset{\mathcal{L}}{=} B_{t-s}^H,$$

finally, we have

$$\mathbb{E}\left[\left[B_t^H - B_s^H\right]^2\right] = \frac{1}{2}\left(|s|^{2H} + |t|^{2H} - |t-s|^{2H}\right).$$

■

1.2 Existence of fractional Brownian motion

Définition 1.3 *A function $c : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is semi-definite and positive if for all $(s_1, ..., s_m) \in \mathbb{R}^m$ and all $(u_1, ..., u_m) \in \mathbb{R}^m$ we have :*

$$\sum_{i=1}^{m}\sum_{j=1}^{m} c(s_i, s_j) u_i u_j \geq 0. \tag{1.1}$$

Theorem 1.1 *Let $m : \mathbb{R} \longrightarrow \mathbb{R}$ and $c : \mathbb{R}^2 \longrightarrow \mathbb{R}$ symmetric and positive semi-definite then there exists a unique real gaussian process up to an equivalence of mean m and covariance function c.*

- *Two real gaussian processes with the same mean and the same covariance function are equivalent.*
- *Two real gaussian processes with the same mean and the same covariance function with $\mathbb{P} - a.s$ trajectories. continuesto the right are indistinguishable.*

Proposition 1.2 *The function R_H is symmetric positive semi-definite and continuous.*

Proof. Continuity and symmetry are immediate to demonstrate.

Let $(s_1, ..., s_m) \in \mathbb{R}^m$ and $(u_1, ..., u_m) \in \mathbb{R}^m$ it is a eqution of showing 1.1for this we will use the fact that the function $s \longrightarrow \phi(s) = \exp\left(-c|s|^{2H}\right)$ is the characteristic function of a sub-gaussian $S\alpha S$ random variable.therefore it is positive semi-definite function in s and therefore :

$$\forall \left(u_i, u_j, s_i, s_j\right) \in \mathbb{R}^4,\ \sum_{i=0}^{m}\sum_{j=0}^{m}\phi\left(s_i - s_j\right) u_i u_j \geq 0, \tag{1.2}$$

for this, consider a mass at the origin (s_0) equal to $u_0 = -\sum_{i=1}^{m} u_i$,we then have :

$$\sum_{i=0}^{m}\sum_{j=0}^{m}\phi\left(s_i - s_j\right) u_i u_j = -\sum_{i=0}^{m}\sum_{j=0}^{m}\left|s_i - s_j\right|^{2H} u_i u_j, \tag{1.3}$$

in fact, we have

$$\begin{aligned}\sum_{i=1}^{m}\sum_{j=1}^{m}\left|s_i\right|^{2H} u_i u_j &= \sum_{i=1}^{m}\left|s_i\right|^{2H} u_i \sum_{j=1}^{m} u_j \\ &= -\sum_{i=1}^{m}\left|s_i\right|^{2H} u_i u_0 \\ &= -\sum_{i=0}^{m}\left|s_i - s_0\right|^{2H} u_i u_0,\end{aligned}$$

likewise we have :

$$\sum_{i=1}^{m}\sum_{j=1}^{m}\left|s_j\right|^{2H} u_i u_j = -\sum_{j=0}^{m}\left|s_i - s_0\right|^{2H} u_i u_{0,}$$

which show 1.3.consider $c > 0$ sufficiently, small, like$\sum_{i=0}^{m}\sum_{j=0}^{m} u_i u_j = 0$,we have :

$$\begin{aligned}\sum_{i=0}^{m}\sum_{j=0}^{m}\exp\left(-c\left|s_i - s_j\right|^{2H}\right) u_i u_j &= \sum_{i=0}^{m}\sum_{j=0}^{m}\left(\exp\left(-c\left|s_i - s_j\right|^{2H}\right) - 1\right) u_i u_j \\ &= -c\sum_{i=0}^{m}\sum_{j=0}^{m}\left|s_i - s_j\right|^{2H} u_i u_j + \circ\left(c\right),\end{aligned}$$

the resulta is demonstrated using 1.2. ■

1.3 Main properties for the trajectories of FBM

1.3.1 Self-similarity of fractional Brownian motion

Définition 1.4 *A process $\{X_t : t \in \mathbb{R}\}$ is said to be self-similarity of order $\beta > 0$ if exists $\beta > 0$ such that, for all $\alpha > 0$, process :*

$$\{X_{\alpha t}, t \in \mathbb{R}\} \text{ and } \left\{\alpha^{\beta} X_t, t \in \mathbb{R}\right\},$$

have the same law.

Theorem 1.2 *The fractional Brownian motion $\left\{B_t^H : t \in \mathbb{R}\right\}$ with self-similarity H is parameter of order H.*

Proof. Let us set $\alpha > 0$.it is obvious that $\left\{B_{\alpha t}^H : t \in \mathbb{R}\right\}$ and $\left\{\alpha^H B_t^H : t \in \mathbb{R}\right\}$ are two centered gaussian processes. it is therefore sufficient to show that they have the same covariance function

$$\begin{aligned}
\mathbb{E}\left[B_{\alpha t}^H B_{\alpha s}^H\right] &= \frac{1}{2}\left(|\alpha s|^{2H} + |\alpha t|^{2H} - |\alpha t - \alpha s|^{2H}\right) \\
&= \frac{1}{2}\alpha^{2H}\left(|s|^{2H} + |t|^{2H} - |t - s|^{2H}\right). \\
\mathbb{E}\left[\alpha^H B_t^H \alpha^H B_s^H\right] &= \frac{1}{2}\alpha^{2H}\mathbb{E}\left[B_t^H B_s^H\right] \\
&= \frac{1}{2}\alpha^{2H}\left(|s|^{2H} + |t|^{2H} - |t - s|^{2H}\right).
\end{aligned}$$

The following property shows that among gaussian processes characters increase stationary ments and self-similarity are characteristic of fractional Brownian motion it also provides the description of the $H = 0$ and $H = 1$. ■

Proposition 1.3 *Let $\{X_t : t \in \mathbb{R}\}$ be a self-similar non-degenerate process of order H with stationary increases and finite variance then :*

– $X_0 = 0, \mathbb{P} - a.s.$.

– $0 < H \leq 1$.

– For every thing $t \in \mathbb{R}$ and $s \in \mathbb{R}$,

$$Cov\left(X_t,\ X_s\right) = \frac{Var\left(X_1\right)}{2}\left\{|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right\}.$$

– For every thing $t \in \mathbb{R}$,for every thing $0 < H < 1$,

$$\mathbb{E}\left[X_t\right] = 0.$$

– For every thing $t \in \mathbb{R}$, for $H = 1$, $X_t = tX_1$ $\mathbb{P} - a.s.$

– If moreover X is gaussian then it is indistinguishable from a fractional Brownian motion.

Proof. For every thing $a > 0$ we have :

$$X\left(a.0\right) \stackrel{\mathcal{L}}{=} a^H X\left(0\right),\left(a^H - 1\right) X\left(0\right) \stackrel{\mathcal{L}}{=} 0.$$

– So $X\left(0\right) = 0$ $\mathbb{P} - a.s.$

– By stationarity we have for all $s > 0$ and every thing $t > s$:

$$\begin{aligned}\mathbb{E}\left[X_t.X_s\right] &= \frac{1}{2}\left[\mathbb{E}\left[X_t^2\right] + \mathbb{E}\left[X_s^2\right] - \mathbb{E}\left[\left(X_t - X_s\right)^2\right]\right],\\ &= \frac{1}{2}\left[\mathbb{E}\left[X_t^2\right] + \mathbb{E}\left[X_s^2\right] - \mathbb{E}\left[X_{t-s}^2\right]\right].\end{aligned}$$

– By self-similarity we then have :

$$\begin{aligned}\mathbb{E}\left[X_t.X_s\right] &= \frac{1}{2}\left[t^{2H}\mathbb{E}\left[X_1^2\right] + s^{2H}\mathbb{E}\left[X_1^2\right] - \left(t-s\right)^{2H}\mathbb{E}\left[X_1^2\right]\right],\\ &= \frac{1}{2}R_H\left(t,s\right)\mathbb{E}\left[X_1^2\right].\end{aligned}$$

- Let $s>0, t_1>0$ and $t_2>0$.by minkowski we have :

$$\begin{aligned}\mathbb{E}\left[\left[X_{s+t_1+t_2}-X_s\right]^2\right]^{\frac{1}{2}} &\leq \mathbb{E}\left[\left[X_{s+t_1+t_2}-X_{s+t_1}\right]^2\right]^{\frac{1}{2}}+\mathbb{E}\left[\left[X_{s+t_1}-X_s\right]^2\right]^{\frac{1}{2}}.\\ \mathbb{E}\left[X_{t_1+t_2}^2\right]^{\frac{1}{2}} &\leq \mathbb{E}\left[X_{t_2}^2\right]^{\frac{1}{2}}+\mathbb{E}\left[X_{t_1}^2\right]^{\frac{1}{2}}.\\ \frac{\mathbb{E}\left[X_1^2\right]}{2}\left[t_1+t_2\right]^H &\leq \frac{\mathbb{E}\left[X_1^2\right]}{2}\left[t_2^H+t_1^H\right].\end{aligned}$$

- Consequently we have $H \leq 1$. moreover, the finite variance implies $H>0$.
- Let $0<H<1$, we have :

$$\mathbb{E}\left[X_1\right]=\mathbb{E}\left[X_2-X_1\right]=\left(2^H-1\right) \mathbb{E}\left[X_1\right].$$

Consequently, we have $\mathbb{E}\left[X_1\right]=0$ by self-similarity it is the same for all $t>0$ and as $\mathbb{E}\left[X_{-1}\right]=\mathbb{E}\left[X_{1-2}\right]=\mathbb{E}\left[X_1\right]-\mathbb{E}\left[X_2\right]=0$ the result is true, by self-similarity, for all $t \in \mathbb{R}$.
- Let $t>0$ and $s>0$ be $H=1$, we have :

$$\begin{aligned}\mathbb{E}\left[X_t X_s\right] &= \mathbb{E}\left[X_1^2\right].t.s.\\ \mathbb{E}\left[X_t-t X_1\right] &= \mathbb{E}\left[X_t^2\right]-2.t.\mathbb{E}\left[X_t X_1\right]+t^2 \mathbb{E}\left[X_1^2\right].\\ &= \left(t^2-2 t^2+t^2\right) E\left[X_1^2\right]=0.\end{aligned}$$

- So $X_t=t X_1 \ \mathbb{P}-a.s.$for all t by continuity of trajectories we conclude that for all t, $X_t=t X_1 \ \mathbb{P}-a.s.$
- We apply theorem these are teo centered gaussian processes having the same covariance function they are therefore indistinguishable.

■

1.3.2 Hölder continuity and the modification of FBM

Theorem 1.3 *Any fractional Brownian motion admits a modification whose trajectories have a Hölder continuity of order* $\gamma>H$ *on any interval* $[0, p]$ *with* $p>0$.

Proof. *It suffices to show that, for all* $\alpha>0$ *there exists a constant* C_α *such that , for all*

$(s,t) \in [0,p]^2$:

$$\mathbb{E}\left[\left|B_t^H - B_s^H\right|^\alpha\right] \le C_\alpha \left|t-s\right|^{\alpha H}. \tag{1.4}$$

Indeed, condition 1.4 ensurse, by kolmogorov's regularlty theorem that $\left\{B_t^H : t \in [0,p]\right\}$ *admits a modification whose trajectories are Hölder continuous of order* $\gamma \in \left[0, \frac{\alpha H - 1}{\alpha}\right]$ *for all* $\alpha > 0$ *which shows the result condition 1.4 follows from the stationarity of increments and self-similarity :*

$$\begin{aligned}\mathbb{E}\left[\left|B_t^H - B_s^H\right|^\alpha\right] &= \mathbb{E}\left[\left|B_{t-s}^H\right|^\alpha\right], \\ &= \left|t-s\right|^{\alpha H} \mathbb{E}\left[\left|B_1^H\right|^\alpha\right],\end{aligned}$$

hence the result with $C_\alpha = \mathbb{E}\left[\left|B_1^H\right|^\alpha\right] < +\infty$. ■

Theorem 1.4 *The trajectories of fractional Brownian motion have* $\mathbb{P}-a.s$ *no Hölder continuity of order higher than* H *on any bounded interval.*

1.3.3 Non-differentiability for the trajectories of FBM

Theorem 1.5 *Let* $t_0 \in \mathbb{R}$. *The trajectories of fractional Brownian motion are* $\mathbb{P}-a.s.not$ *differentiable in* t_0.

Proof. We want to show that $\forall t_0 \in \mathbb{R}$, $P\left[\lim\sup_{t \longrightarrow t_0} \left|\frac{B_t^H - B_{t_0}^H}{t - t_0}\right| = +\infty\right] = 1$. We return to the case$t_0 = 0$ thanks to stationarity. We will therefore study the behavior of$\left|\frac{B_t^H}{t}\right|$ when t tends towards t_0.

In fact we will demonstrate non-differentiability on the right : We set : $A(t) = \left[\sup_{0 \le s \le t} \left|\frac{B_t^H}{t}\right| \ge M\right]$ with $M > 0$

$$\begin{aligned}\mathbb{P}\left[A(t)\right] &\ge \mathbb{P}\left[\left|\frac{B_t^H}{t}\right| \ge M\right], \\ &\ge \mathbb{P}\left[\frac{t^H}{t}\left|B_1^H\right| \ge M\right], \qquad \text{by self-similarity} \\ &\ge \mathbb{P}\left[\left|B_1^H\right| \ge M.t^{1-H}\right] \underset{t \to 0}{\to} \mathbb{P}\left[\left|B_1^H\right| \ge 0\right].\end{aligned}$$

Thus we have $\forall M$, $\mathbb{P}\left[A\left(t\right)\right] \underset{t\to 0^{+}}{\longrightarrow} 1$

$$\overline{\lim_{t\to 0^{+}}}\left|\frac{B_t^H}{t}\right| = +\infty \quad \mathbb{P}-a.s.$$
$$\overline{\lim_{t\to 0}}\left|\frac{B_t^H}{t}\right| = +\infty \quad \mathbb{P}-a.s..$$

Hence the theorem. ■

1.3.4 The variation of orders p of fractional Brownian motion

Theorem 1.6 *Consider the variation of order p of the fractional Brownian motion defined by :*

$$V_p = \mathbb{P} - \lim_{n\to\infty} V_{n,p},$$

with

$$V_{n,p} = \sum_{j=1}^{2^n} \left|B^H\left(j.2^{-n}\right) - B^H\left(\left(j-1\right).2^{-n}\right)\right|^p.$$

So we have :

$$Vp = \begin{cases} 0 & \text{if } pH > 1, \\ +\infty & \text{if } pH < 1, \\ \mathbb{E}\left[\left|B_1^H\right|^p\right] & \text{if } pH = 1. \end{cases}$$

Proof. Let $p \in \mathbb{R}^{+*}$, consider the following sequences of random variables :

$$\left\{Y_{n,p} = \left[2^{-n}\right]^{pH-1} \sum_{j=1}^{2^n} \left|B^H\left(j.2^{-n}\right) - B^H\left(\left(j-1\right).2^{-n}\right)\right|^p : n \in \mathbb{N}^*\right\}.$$

And

$$\left\{\widetilde{Y}_{n,p} = 2^{-n} \sum_{j=1}^{2^n} \left|\left(B^H\left(j\right) - B^H\left(j-1\right)\right)\right|^p : n \in \mathbb{N}^*\right\}.$$

Self-similarity ensures that $B^H\left(j.2^{-n}\right) \overset{\mathcal{L}}{=} 2^{-nH}.B^H\left(j\right)$.therefore , it is clear that for all $n \in \mathbb{N}^*$, $Y_{n,p} \overset{\mathcal{L}}{=} .\ \widetilde{Y}_{n,p}$.it is now sufficient to notice that the squence $\left\{B^H\left(j\right) - B^H\left(j-1\right) : j \in \mathbb{Z}\right\}$is stationary and ergodic (like any sequence resulting from a gaussian process with continuous spectral measurement) has

$$\mathbb{E}\left[\widetilde{Y}_{n,p}\right] = \mathbb{E}\left[\left|B_1^H\right|^p\right] := c_{p,H} \qquad \text{for all } n \in \mathbb{N}^*. \tag{1.5}$$

The ergodic theorem tells us that we have :

$$\widetilde{Y}_{n,p} \overset{\mathbb{L}^1}{\to} c_{p,H} \quad \text{and} \quad \widetilde{Y}_{n,p} \overset{a.s}{\to} c_{p,H} \quad \text{so} \quad {}_{n,p} \overset{\mathcal{L}}{\to} c_{p,H}. \tag{1.6}$$

To demonstrate 1.5, it is enough to evoke stationarity :

$$\begin{aligned} \mathbb{E}\left[\widetilde{Y}_{n,p}\right] &= 2^{-n}\sum_{j=1}^{2^n}\mathbb{E}\left[\left|\left(B^H(j) - B^H(j-1)\right)\right|^p\right], \\ &= 2^{-n}\sum_{j=1}^{2^n}\mathbb{E}\left[\left|B^H(1)\right|^p\right], \\ &= 2^{-n}2^n\mathbb{E}\left[\left|B^H(1)\right|^p\right], \end{aligned}$$

we have 1.6 and like $Y_{n,p} \overset{\mathcal{L}}{=} \widetilde{Y}_{n,p}$we therfore have $Y_{n,p} \overset{\mathcal{L}}{\to} c_{p,H}$ as con is a deterministic constant, this implies that $Y_{n,p} \overset{\mathbb{P}}{\to} c_{p,H}$ therefore the $[2^{-n}]^{pH-1}.V_{n,p} \overset{\mathbb{P}}{\to} c_{p,H}$ which demonstrates result.

■

Corollary 1.1 *The fractional brownian motion is $\mathbb{P}-a.s.$with unbounded variations on any compact of $\mathbb{R}$.*

Proof. *By self-similarity and by stationarity of the increments, it suffices to consider the compact $[0,1]$.considering the particular subdivision of $[0,1] : \{0, 2^{-n}, ..., j.2^{-n}, ..., 1\}$, to have the property of bounded variation (by b) it is necessary that $V_{n,1} \underset{n\to}{\to} b$ $\mathbb{P}-a.s.$with $b < \infty$.but this is not possible because theorem 1.6 provides us.with subsequence which almost surely towards infinity $(p = 1,\ H < 1)$.* ■

1.4 Main properties of fractional Brownian motion

1.4.1 The increments of fractional Brownian motion

Définition 1.5 *Given a stationary stochastic process $\{X_t : t \in \mathbb{R}\}$, the sequence*

$$\{r(n) = \mathbb{E}[X_{n+s}X_s] : n \in \mathbb{N}^*\},$$

does not depend on s we then say that X .is long-term dependent if

$$\sum_{n\in\mathbb{N}^*} r(n) = +\infty.$$

Proposition 1.4 *The increases of B^H are long-term dependent if and only if $H > \frac{1}{2}$.*

Proof. For all $n \in \mathbb{N}^*$ we have :

$$\begin{aligned} r(n) &= \mathbb{E}\left[B_1^H\left(B_{n+1}^H - B_n^H\right)\right], \\ &= \frac{1}{2}\left[(n+1)^{2H} - 2n^{2H}(n-1)^{2H}\right], \quad (1.7) \\ &= 2H(2H-1)n^{2H-2} + \underset{n\to\infty}{O}\left(n^{2H-2}\right), \quad (1.8) \end{aligned}$$

by 1.8,we see that $r(n)$ is the general term of a divergent series if and only if $2H-2 > -1$ negatively correlated if $H > \frac{1}{2}$. ■

Proposition 1.5 *The increments of fractional Brownian motion are positively correlated if $\frac{1}{2} < H < 1$, negatively correlated if $0 < H < \frac{1}{2}$ (we speak of anti-persistence) and independent if $H = \frac{1}{2}$.*

Proof. From 1.7 we see that , if $H = \frac{1}{2}$, $r(n) = 0$ for all $n \in \mathbb{N}^*$ and therefore the increases are independent.on the other hand, by 1.8 we also see, at least for large n that $r(n) < 0$ as soon as $2H(2H-1) < 0$ that is to say $H < \frac{1}{2}$. ■

1.4.2 Non-markovian of fractional Brownian motion

Définition 1.6 *Let $\{X_t : t \in \mathbb{R}\}$ be a gaussian process center.if X is a markov process , then $\forall s < t < u$ with $\Gamma(t,t) > 0$,*

$$\Gamma(s,u)\,\Gamma(t,t) = \Gamma(s,t)\,\Gamma(t,u), \tag{1.9}$$

where Γ is the covariance function of X , futhermore.if $\Gamma(t,t) = 0$ then $\{X_s : s \leq t\}$and $\{X_s : s \geq t\}$ are independent.

Corollary 1.2 *Let $0 < H < 1$ and $H \neq \frac{1}{2}$.*

1. *The fractional brownian motion $\left\{B_t^H : t \in \mathbb{R}\right\}$ is not markovian.*
2. *The fractional brownian motion $\left\{B_t^H : t \in \mathbb{R}^+\right\}$ is not markovian.*

Proof. If it were markovian , as we have $R_H(0,0) = 0$,the processes $\left\{B_t^H : t \in \mathbb{R}^+\right\}$ and $\left\{B_t^H : t \in \mathbb{R}^-\right\}$ would be independent , which is absurd.

If it were markovian , its covariance function would satisfy 1.9 and in particular , like $1 < 2 < 3$, we would have :

$$\begin{cases} R_H(1,3)\,R_H(2,2) = R_H(1,2)\,R_H(2,3), \\ \frac{1}{2}\left(1+3^{2H}-2^{2H}\right).2^{2H} = \frac{1}{2}\left(1+2^{2H}-1\right).\frac{1}{2}\left(2^{2H}+3^{2H}-1\right), \\ 2+3^{2H}-3.2^{2H} = 0, \end{cases}$$

after studying the function $H \mapsto 3 = 3^{2H} - 3.2^{2H}$, we see that this function only vanishes for het $H = \frac{1}{2}$and for $H = 1$(case excluded by definition).the only possible case$\left(H = \frac{1}{2}\right)$corresponds to that of ordinary brownian motion which is markovian. ■

1.4.3 Quadratic and semi-martingale variation of FBM

Définition 1.7 *A process X it is xith finite quadratic variation if there exists a process denoted $< X >$ such that , for all t for a series of subdivision Δ_n of $[0,t]$ such that the step*

$|\Delta_n| \to 0$ we have :

$$\mathbb{P} - \lim_{n\to\infty} \sum_{(t_i,t_{i+1})\in\Delta_n} (X_{t_i+1} - X_{t_i})^2 =< X >_t .$$

Theorem 1.7 *Let $\left\{B_t^H : t \in \mathbb{R}\right\}$ be a with parameter H we have :*

$$\begin{aligned} \langle B^H \rangle_t &= 0, \forall t \in \mathbb{R} \qquad \textit{for } H > \frac{1}{2}, \\ \langle B^{\frac{1}{2}} \rangle_t &= t, \forall t \in \mathbb{R}, \\ \langle B^H \rangle_t &= +\infty, \forall t \in \mathbb{R}^* \qquad \textit{for } H < \frac{1}{2}. \end{aligned}$$

Proof. *Let $t \in \mathbb{R}$ be assumed to be strictly positive fo fixe the ideas.*

Let $\{\Delta_n : 0 = t_0 < t_1 < ... < t_n = t,\ n \in \mathbb{N}^*\}$ be a sequence of subdivisions of $[0,t]$ whose step $|\Delta n|$ antends towards 0. Consider $T_t^{\Delta n} = \sum_{k=0}^{n-1} \left(B_{t_{k+1}}^H - B_{t_k}^H\right)^2$.

First case: $H > \frac{1}{2}$. We will therefore show the convergence in $\mathbb{L}^1$ of $T_t^{\Delta_n}$ towards 0.

By stationarity of the inctements, we have :

$$\begin{aligned} \mathbb{E}\left[T_t^{\Delta_n}\right] &= \sum_{k=0}^{n-1} \mathbb{E}\left[\left(B_{t_{k+1}}^H - B_{t_k}^H\right)^2\right] \\ &= \sum_{k=0}^{n-1} |t_{k+1} - t_k|^{2H} \\ &\le \sum_{k=0}^{n-1} |t_{k+1} - t_k|\, |t_{k+1} - t_k|^{2H-1} \\ &\le |\Delta_n|^{2H-1} \sum_{k=0}^{n-1} |t_{k+1} - t_k| \\ &\le |\Delta_n|^{2H-1}\, t, \end{aligned}$$

as $2H - 1 > 0$, we therefore have $\lim_{n\to\infty} |\Delta_n|^{2H-1} t = 0$ and the resulta follows.

2nd case :$H < \frac{1}{2}$. Let us shoz the divergence of $T_t^{\Delta_n}$ towards infinity. Let us call A the set of subdivisions of $[0,t]$ whose step tends towards 0 and consider :

$$E = \sup_A \mathbb{E}\left[\sum_{k=0}^{n-1} \left(B_{t_{k+1}}^H - B_{t_k}^H\right)^2\right].$$

Therefore reduced by the subdivision $\tau_i = \frac{it}{2^n}$ we therefore have :

$$\begin{aligned} E &\geq \mathbb{E}\left[\sum_{i=0}^{n-1}\left(B^H_{\tau_i} - B^H_{\tau_{i-1}}\right)^2\right] \\ &\geq (2^n+1)\left(\frac{t}{2^n}\right)^{2H} \\ &\geq \left(t^{2H}\right)\cdot\left(\frac{1}{2^{n(2H-1)}} + \frac{1}{2^{(2nH)}}\right), \end{aligned}$$

as we have $2H-1<0$ and $2H>0$, we therefore :

$$\lim_{n\to\infty}\frac{1}{2^{n(2H-1)}} = \infty \quad \textit{and} \quad \lim_{n\to\infty}\frac{1}{2^{(2nH)}} = 0,$$

which leads to the expected result. As a corollary we have the following result. ■

Theorem 1.8 *The fractional Brownian motion is not a semi-martingale relative to its natural filtration.*

Proof. *Let's assume it's a semi-martingale. It is therefore continuous and zero at $0.B^H$ is therefore written uniquely in the form $B^H = M+V$ where M is a continuous local martingale zero at 0 and V a continuous process with finite variation zero in 0.*

1st case :$H>\frac{1}{2}$. We have $<M>_t=<B^H_t>=0 \quad \forall t\in\mathbb{R}$. So by virtue of the Doob-Meyer decomposition $M^2-<M>$ is a continuous local martingale zero at 0 that is to say there exists an increasing sequence $\{T_n : n\in\mathbb{N}\}$ of stopping times such that

$$\lim_{n\to\infty} T_n = +\infty \quad \mathbb{P}-a.s..$$

And

$$\forall n,\ \forall t,\ \mathbb{E}\left[M^2_{t\wedge T_n}\right] = \mathbb{E}\left[M^2_{0\wedge T_n}\right] = 0.$$

$$\forall n,\ \forall t,\ M^2_{t\wedge T_n} = 0,\ \mathbb{P}-a.s..$$

As T_n tend to increase towards $+\infty$ $\mathbb{P}-a.s.$, we have :

$$\forall t,\ M_t^2 = 0,\ \mathbb{P}-a.s.$$

Therefore M^2 is indistingulshable from the null process.

Finally, $\forall t \quad B_t^H = V_t \quad \mathbb{P}-a.s.$and therefore B^H is $\mathbb{P}-a.s.$with finite quadratic variation, absurd.

2nd case :$H < \frac{1}{2}$. The quadratic variation of M would only be defined at 0, which contradicts the hypothesis of continuity, absurd. The direct consequence of this theorem is the impossibility of directly defining an Itô type integral for the fractional Brownian motion. ■

Définition 1.8 *We call a Dirichlet process X a process which is decomposed as follows :*

$$X = M + A,$$

with M an integrable square martingale and A a process with zero quadratic variation.

Proposition 1.6 *The fractional Brownian motion with parameter H is a Dirichlet process.*

Chapitre 2

Generalized backward stochastic differential equations

2.1 Notation and assumptions

Let T be a fixed final time Throughout this paper $\{W_t, 0 \leq t \leq T\}$ will denote d-dimensional Brownian motions $(d \geq 1)$,defined on the complete probability spaces $(\Omega, \mathcal{F}, \mathbb{P})$. In addition, we put

$$\mathcal{F}_t \triangleq \mathcal{F}_t^W \vee \mathcal{N},$$

where $\mathcal{N}$ is the collection of $\mathbb{P}$-null sets.In other words, the σ-fields $\mathcal{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$, are $\mathbb{P}$-complete.

Let $\{k_t, 0 \leq t \leq T\}$ be a continuous, increasing and $\mathcal{F}_t$-adapted real-valued process such that $k_0 = 0$. For any $n \geq 1$, we consider the following spaces of processes :

– The Banach space $\mathcal{M}^2(\mathcal{F}, [0, T]; \mathbb{R}^n)$ of all equivalence classes (with respect to the measure $d\mathbb{P}\times\, dt$) where each equivalence class contains an d-dimensional jointly measurable random process $\{\varphi_t, t \in [0, T]\}$ which satisfies:

1. **(i)** $\mathbb{E} \int_0^T |\varphi_t|^2 dt < \infty$;

   (ii) φ_t is $\mathcal{F}_t$-measurable, for almost all $t \in [0, T]$ Usually an equivalence class will beidentified with (one of) its members.

– The Banach space $\mathcal{K}^2(\mathcal{F},[0,T];R^n)$ of all (equivalence classes of) n-dimensional jointly measurable random processes $\{\varphi_t, \in [0,T]\}$ which satisfy :

1. **(i)** $\mathbb{E}\int_0^T |\varphi_t|^2 dk_t < \infty$;

   (ii) φ_t is $\mathcal{F}_t$-measurable, for almost all $t \in [0,T]$.

   Here equivalence is taken with respect to the measure $d\mathbb{P} \times dk_t$.

– The set $\mathcal{S}^2(\mathcal{F},[0,T];\mathbb{R}^n)$ of continuous d-dimensional random processes which satisfy :

1. **(i)** $\mathbb{E}\left(\sup\limits_{0\leq t\leq T} |\phi_t|^2\right) < \infty$;

   (ii) ϕ_t is $\mathcal{F}_t$-measurable, for almost all $t \in [0,T]$.

We consider coefficients f and h with the following properties :

$$f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \to \mathbb{R}^n,$$
$$h : \Omega \times [0,T] \times \mathbb{R}^n \to \mathbb{R}^n,$$

such that there exist $\mathcal{F}_t$-adapted processes $\{f_t, h_t : 0 \leq t \leq T\}$ with values in $[1,+\infty)$ and with the property that for any $(t,y,z) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^{n\times d}$, and $\mu > 0$, the following hypotheses are satisfied for some strictly positive finite constant C :

$$(\mathbf{H}_1)\begin{cases} f(t,y,z) \text{ and } h(t,y) \text{ are } \mathcal{F}_t - \text{measurable processes,} \\ |f(t,y,z)| \leq f_t + C(|y| + \|z\|), \\ |h(t,y)| \leq h_t + C|y|, \\ \mathbb{E}\left(\int_0^T e^{\mu k_t} f_t^2 dt + \int_0^T e^{\mu k_t} h_t^2 dk_t < \infty\right). \end{cases}$$

Moreover, we assume that there exist constants $C > 0$, $\beta_1 > 0$ such that for any (y_1,z_1), $(y_2,z_2) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$

$$(\mathbf{H}_2)\begin{cases} (i)\ |f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \leq c(|y_1-y_2|^2 + \|z_1-z_2\|^2), \\ (ii)\ |h(t,y_1) - h(t,y_2)| \leq \beta_1|y_1-y_2|. \end{cases}$$

Throughout this work, $\langle\cdot,\cdot\rangle$ will denote the scalar product on $\mathbb{R}^n$, i.e.$\langle x,y\rangle := \sum_{i=1}^n x_i y_i$, for all $(x,y) \in \mathbb{R}^n \times\ \mathbb{R}^n$, Sometimes, we will also use the notation x^*y to designate $\langle x,y\rangle$

Remark 2.1 *C will always denote a finite constant whose value may change from one line to the next, and which usually is (strictly) positive.*

2.2 Existence and uniqueness theorem

Suppose that we are given a terminal condition $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_T, \mathbb{P})$ such that, for all $\mu > 0$, $\mathbb{E}(e^{\mu k_T}|\xi|^2) < \infty$.

Définition 2.1 *By definition, a solution to a generalized BSDE (ξ, f, h, k) is a pair (Y, Z) $\in \mathcal{S}^2(F, [0,T]; \mathbb{R}^n) \times \mathcal{M}^2(F, [0,T]; \mathbb{R}^{n\times d})$,such that,for any $0 \leq t \leq T$*

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T h(s, Y_s)dk_s - \int_t^T Z_s dW_s. \tag{2.1}$$

Remark 2.2 *If h satisfies$(H_2)(ii)$ then, by changing the solutions and the coefficients f and h, we may and do suppose that h satisfies a stronger condition of the form*

(iv) $(y_1 - y_2, h(t, y_1) - h(t, y_2)) \leq \beta_2 |y_1 - y_2|^2,$ *where* $\beta_2 < 0.$

Indeed,(Y_t, Z_t) solves the generalized BSDE in 2.1 if and only if for every (some) $\eta > 0$ the pair $(\bar{Y}_t, \bar{Z}_t) = (e^{\eta k_t} Y_t, e^{\eta k_t} Z_t)$ solves an analogous generalized BSDE, with f and h replaced respectively by :

$$\bar{f}\,(t, y, z) = e^{\eta k_t} f\left(t, e^{-\eta k_t} y, e^{-\eta k_t} z\right),$$
$$\bar{h}\,(t, y) = e^{\eta k_t} h\left(t, e^{-\eta k_t} y\right) - \eta y.$$

Then we can always choose η such that the function $\bar{h}$ satisfies (iv) with a strictly negative β_2.Our main goal in this section is to prove the following theorem

Theorem 2.1 *Under the above hypotheses (H_1) and (H_2) there exists a unique solution for the generalized BSDE in 2.1*

We will follow the same line of arguments as Pardoux and Peng [10] did. So let us first establish the result in 2.1 for BSDEs where the coefficients f and h do not depend on (y, z)

More precisely, let $f, h : \Omega \times [0,T] \to \mathbb{R}^n$ satisfy (H_1), and let ξ and k be as before.Consider the equation :

$$Y_t = \xi + \int_t^T f(s)ds + \int_t^T h(s)dk_s - \int_t^T Z_s dW_s. \tag{2.2}$$

Then we have the following result.

Theorem 2.2 *Under hypothesis (H_1), there exists a unique solution to equation 2.2.*

Proof. *To show the existence, we consider the martingale*

$$M_t = \mathbb{E}\left[\xi + \int_0^T f(s)ds + \int_0^T h(s)dk_s / \mathcal{F}_t\right], \tag{2.3}$$

which is clearly a square integrable martingale by (H_1).As in Pardoux and Peng [10], an extension of Itô's martingale representation theorem yields the existence of a $\mathcal{F}_t-$progressively measurable process (Z_t) with values in $\mathbb{R}^{n\times d}$ such that

$$\mathbb{E}\left(\int_0^T \|Z_t\|^2 dt\right) < \infty \qquad \text{and} \qquad M_T = M_t + \int_t^T Z_s dW_s. \tag{2.4}$$

We subtract the quantity $\int_0^T f(s)ds + \int_0^T h(s)dk_s$ from both sides of the martingale in 2.3 and employ the martingale representation in 2.4 to obtain

$$Y_t = \xi + \int_t^T f(s)ds + \int_t^T h(s)dk_s - \int_t^T Z_s dW,$$

where

$$Y_t = \mathbb{E}\left[\xi + \int_0^T f(s)ds + \int_0^T h(s)dk_s / \mathcal{F}_t\right].$$

It remains to prove the uniqueness and to show that Y_t and Z_t are $\mathcal{F}_t$-measurable, the proof is analogous to that of Pardoux and Peng [10] , and is therefore omitted. ■

We will also need the following generalized Itô formula. In the proof we use arguments which are similar to those used by Pardoux and Peng in [10].

Lemma 2.1 *Let* $\alpha \in \mathcal{S}^2(\mathcal{F},[0,T];\mathbb{R}^n)$, $\beta \in \mathcal{M}^2(\mathcal{F},[0,T];\mathbb{R}^n)$, $\gamma \in \mathcal{M}^2(\mathcal{F},[0,T];\mathbb{R}^{n\times d})$, $\theta \in \mathcal{K}^2(\mathcal{F},[0,T];\mathbb{R}^n)$ *and* $\delta \in \mathcal{M}^2(\mathcal{F},[0,T];\mathbb{R}^{n\times d})$ *be such that*

$$\alpha_1 = \alpha_0 + \int_0^T \beta_s ds + \int_0^T \theta_s dk_s - \int_t^T \delta_s dW_{s.}$$

Then, for any function $\phi \in \mathcal{C}^2(\mathbb{R}^n,\mathbb{R})$

$$\begin{aligned}\phi(\alpha_1) &= \phi(\alpha_0) + \int_0^t \langle \nabla\phi(\alpha_s), \beta_s\rangle ds + \int_0^t \langle \nabla\phi(\alpha_s), \theta_s\rangle dk_s \\ &+ \int_0^t \langle \nabla\phi(\alpha_s), \delta_s dW_s\rangle + \frac{1}{2}\int_0^t Tr\left[\phi'(\alpha_s)\delta_s\delta_s^*\right] ds\end{aligned}$$

In particular

$$|\alpha|_t^2 = |\alpha_0|^2 + 2\int_0^t \langle \alpha_s, \beta_s\rangle ds + 2\int_0^t \langle \alpha_s, \theta_s\rangle dk_s + 2\int_0^t \langle \alpha_s, \delta_s dW_s\rangle + \int_0^t \|\delta_s\|^2 ds.$$

Next, we establish an a priori estimate for the solution of the BSDE in 2.1

Proposition 2.1 *Let the conditions* (H_1) *and* (H_2) *be satisfied. If* $\{(Y_t, Z_t); 0 \le t \le T\}$ *is a solution of BSDE 2.1, then there exists a finite constant* C*, which depends on* K*,* T *and* β_2*, such that for all* $\mu \in \mathbb{R}$ *and* $\lambda > 0$ *the following inequality holds*

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le T} e^{\mu t+\lambda k_t}|Y_t|^2 + \int_0^T e^{\mu t+\lambda k_t}|Y_t|^2 dk_t + \int_0^T e^{\mu t+\lambda k_t}\|Z_t\|^2 dt\right) \\ &\le C\mathbb{E}\left(e^{\mu T+\lambda k_T}|\xi|^2 + \int_0^T e^{\mu t+\lambda k_t}|f_t|^2 dt + \int_0^T e^{\mu t+\lambda k_t}|h_t|^2 dk_t\right).\end{aligned}$$

Proof. Classical arguments, such as Doob's inequality, justify the fact that the processe $\int_0^t e^{\mu s+\lambda k_s}\langle Y_s, Z_s dW_s)\rangle$ is uniformly integrable martingale, By 2.1, we then have :

$$\begin{aligned}&\mathbb{E}\left[e^{\mu_t+\lambda k_t}|Y_t|^2 + \int_t^T e^{\mu s+\lambda k_s}\|Z_s\|^2 ds + \lambda\int_t^T e^{\mu s+\lambda k_s}|Y_s|^2 dk_s\right] \\ &\le \mathbb{E}\left(e^{\mu T+\lambda k_T}|\xi|^2 + 2\int_t^T e^{\mu s+\lambda k_s}\langle Y_s, f(s,Y_s,Z_s)\rangle ds\right) \\ &+ \mathbb{E}\left(2\int_t^T e^{\mu s+\lambda k_s}\langle Y_s, h(s,Y_s)\rangle dk_s + \mu\int_t^T e^{\mu s+\lambda k_s}|Y_s|^2 ds\right)\end{aligned} \tag{2.5}$$

But from (H_1), (H_2) and the fact that

$$2ab \leq \frac{1-\alpha}{2c}a^2 + \frac{2c}{1-\alpha}b^2, \qquad c > 0,$$

it follows that there exists a constant $c(\alpha)$ such that

$$2\langle y, f(s,y,z)\rangle \leq c\,|f_s|^2 + c(\alpha)\,|y|^2 + \frac{1-\alpha}{2}\,\|z\|^2, \tag{2.6}$$

$$2\langle y, h(s,y)\rangle \leq 2\beta_2\,|y|^2 + |y| \times |h_s|^2 \leq (2\beta_2 + |\beta_2|)\,|y|^2 + \frac{1}{|\beta_2|}h_s^2, \tag{2.7}$$

Then, from Gronwall's lemma, we obtain

$$\sup_{0\leq t\leq T} \mathbb{E}\left(e^{\mu t+\lambda k_t}\,|Y_t|^2 + \int_0^T e^{\mu s+\lambda k_s}\,|Y_t|^2\,dk_t + \int_0^T e^{\mu s+\lambda k_s}\,\|Z_t\|^2\,ds\right) \tag{2.8}$$

$$\leq C\mathbb{E}\left(e^{\mu T+\lambda k_T}\,|\xi|^2 + \int_0^T e^{\mu s+\lambda k_s}\,|f_s|^2\,ds + \int_0^T e^{\mu s+\lambda k_s}\,|h_s|^2\,dk_s\right). \tag{2.9}$$

Finally, 2.1 follows from the Burkhölder–Davis–Gundy inequality and 2.8. ■

Next, let (ξ, f, h, k) and be two sets of data, each satisfying conditions (H_1) and (H_2).Then we have the following result :

Proposition 2.2 *Let (Y,Z) (or (Y',Z')) denote a solution of the BSDE(ξ,f,h,k) (or BSDE(ξ',f',h',k') With the notation*

$$(\bar{Y},\bar{Z},\bar{\xi},\bar{f},,\bar{h},\bar{k}) = (Y-Y', Z-Z', \xi-\xi', f-f', h-h', k-k'),$$

it follows that for every $\mu > 0$, there exists a constant $C > 0$ such that

$$\mathbb{E}\left(\sup_{0\leq t\leq T} e^{\mu A_t}|\bar{Y}_t|^2 + \int_0^T e^{\mu A_t}\|\bar{Z}_t\|^2 dt\right)$$
$$\leq C\mathbb{E}(e^{\mu A_T}|\bar{\xi}|^2 + \int_0^T e^{\mu A_t}\,|f(t,Y_t,Z_t) - f'(t,Y_t,Z_t)|^2\,dt + \int_0^T e^{\mu A_t}\,|h(t,Y_t)|^2\,d\left|\bar{k}\right|_t$$
$$+ \int_0^T e^{\mu A_t}\,|h(t,Y_t) - h'(t,Y_t)|^2\,dk_t'),$$

here $A_t \triangleq |\bar{k}|_t + k'_t$ *and* $|\bar{k}|_t$*is the total variation of the process* $\bar{k}$

Proof. The proof follows the same ideas and arguments as in Pardoux and Zhang [11], 2.1, so we just repeat the main steps. From 2.1 we obtain

$$e^{\mu A_t}|\bar{Y_t}|^2 + \int_t^T e^{\mu A_s}\|\bar{Z_s}\|^2 ds + \mu\int_t^T e^{\mu A_s}|\bar{Y_t}|^2 dA_s$$
$$= e^{\mu A_T}|\bar{\xi}|^2 + 2\int_t^T e^{\mu A_s}\langle \bar{Y_s} - f(s,Y_s,Z_s) - f'(s,Y'_s,Z'_s)\,ds\rangle + 2\int_t^T e^{\mu A_s}\langle \bar{Y_s}, h(s,Y_s)\rangle dk_s \tag{2.10}$$
$$+ 2\int_t^T e^{\mu A_s}\langle \bar{Y_s}, h(s,Y_s) - h'(s,Y'_s)\rangle dk'_s - 2\int_t^T e^{\mu A_s}\langle \bar{Y_s}, \bar{Z_s} dW_s\rangle$$

Using conditions $(H_1),(H_2)$, and the algebraic inequality $2ab \leq a^2/\varepsilon + \varepsilon b^2$, then from 2.10 we obtain

$$\begin{aligned}
&\mathbb{E}\left(e^{\mu A_t}|\bar{Y_t}|^2 + \int_t^T e^{\mu A_s}\|\bar{Z_s}\|^2 ds + \mu\int_t^T e^{\mu A_s}|\bar{Y_t}|^2 dA_s\right)\\
&\leq \mathbb{E}(^{\mu A_T}|\bar{\xi}|^2 + C\int_t^T e^{\mu A_s}|\bar{Y_t}|^2 ds + \int_t^T e^{\mu A_s}\left|\bar{f}(s,Y_s,Z_s)\right|^2 ds\\
&+ \frac{1}{\varepsilon}\int_t^T e^{\mu A_s}\left|\bar{h}(s,Y_s)\right|^2 dk'_s + \frac{1}{\mu}\int_t^T e^{\mu A_s}|h(s,Y_s)|^2 d\left|\bar{k_s}\right|\\
&+ \mu\int_t^T e^{\mu A_s}|\bar{Y_s}|^2 d\left|\bar{k}\right|_s + (2\beta_2+\varepsilon)\int_t^T e^{\mu A_s}|\bar{Y_s}|^2 dk'_s).
\end{aligned} \tag{2.11}$$

By choosing $\varepsilon = \mu + 2|\beta_2|$,and using Gronwall's lemma, from 2.11 we infer that

$$\begin{aligned}
&\mathbb{E}\left(e^{\mu A_t}|\bar{Y_t}|^2 + \int_0^T e^{\mu A_s}\|\bar{Z_s}\|^2 dt\right)\\
&\leq C\mathbb{E}(e^{\mu A_T}|\bar{\xi}|^2 + \int_0^T e^{\mu A_t}\left|\bar{f}(s,Y_s,Z_s)\right|^2 ds\\
&+ \int_0^T e^{\mu A_s}|h(s,Y_s)|^2 d\left|\bar{k_s}\right| + \int_0^T e^{\mu A_s}\left|\bar{h}(s,Y_s)\right|^2 dk'_s)
\end{aligned} \tag{2.12}$$

The proposition follows from 2.12 and the Burkhölder–Davis–Gundy inequality. ■

Remark 2.3 *If we denote by* $\mathbb{E}^{\mathcal{F}_t}$ *the conditional expectation with respect to* $\mathcal{F}_t$*, then we can* *show that for every $\mu, \lambda > 0$, there exists a constant $C > 0$ such that $\forall t \in [0,T]$*

$$\begin{aligned} e^{\mu A_t+\lambda t}|\bar{Y}_t|^2 &= \mathbb{E}^{\mathcal{F}_t}\left(e^{\mu A_t+\lambda t}|\bar{Y}_t|^2\right) \\ &\leq C\mathbb{E}(e^{\mu A_T+\lambda T}|\bar{\xi}|^2 + \int_0^T e^{\mu A_s+\lambda s}\left|\bar{f}(s,Y_s,Z_s)\right|^2 ds \\ &+ \int_0^T e^{\mu A_s+\lambda s}\left|\bar{h}(s,Y_s)\right|^2 dk'_s + \int_0^T e^{\mu A_s+\lambda s}|h(s,Y_s)|^2 d\left|\bar{k}_s\right|),\ \mathbb{P}-\textit{almos tsurely.} \end{aligned}$$

Theorem 2.3 *The uniqueness is a consequence of 2.2. We now turn to the existence. In the space $\mathcal{S}^2(\mathcal{F},[0,T];\mathbb{R}^n)\times\mathcal{M}^2(\mathcal{F},[0,T];\mathbb{R}^n)$ we define by recursion thesequence $\{(Y_t^i,Z_t^i)\}_{i=0,1,2,..}$. as follows.Put $Y_t^0=0, Z_t^0=0$.Given the pair, (Y_t^i,Z_t^i) we define $f^{i+1}(s)=f(s,Y_s^i,Z_s^i)$ and $h^{i+1}(s)=h(s,Y_s^i)$. Now, applying (H_1), we obtain*

$$|h^{i+1}(s)| \leq h_s + K|Y_s^i| \triangleq h_s^{i+1},$$

and by using 2.1, we obtain

$$\mathbb{E}\int_0^T e^{\mu k_s}\left(h_s^{i+1}\right)^2 dk_s \leq C\mathbb{E}\left(\int_0^T e^{\mu k_s}h_s^2 dk_s + \int_0^T e^{\mu k_s}\left|Y_s^i\right|^2 dk_s\right) < \infty.$$

By the same arguments one can show that f^{i+1} also satisfy (H_1). Using 2.2, we consider the process $\{(Y_t^{i+1},Z_t^{i+1})\}$ as being the unique solution to the equation

$$Y_t^{i+1} = \xi + \int_t^T f(s,Y_s^i,Z_s^i)ds + \int_t^T h(s,Y_s^i)dk_s - \int_t^T Z_s^{i+1}dW_s \tag{2.13}$$

We will show that the sequence $\{(Y_t^i,Z_t^i)\}$ converges in the space $\mathcal{S}^2(\mathcal{F},[0,T];\mathbb{R}^n)\times\mathcal{M}^2(\mathcal{F},[0,T];\mathbb{R}^n)$ to a pair of processes (Y_t,Z_t) which will be our solution. Indeed, let

$$\bar{Y}_t^{i+1} \triangleq Y_t^{i+1} - Y_t^i, \qquad \bar{Z}_t^{i+1} \triangleq Z_t^{i+1} - Z_t^i.$$

Let $\mu > 0, \lambda > 0$ using Itô's formula, we obtain

$$\begin{aligned}
&e^{\mu t+\lambda k_t}\left|\overline{Y}_t^{i+1}\right|^2 + \int_t^T e^{\mu s+\lambda k_s}\left\|\overline{Z}_s^{i+1}\right\|^2 ds \\
&= 2\int_t^T e^{\mu s+\lambda k_s}\langle \overline{Y}_s^{i+1}, f(s,Y_s^i,Z_s^i) - f(s,Y_s^{i-1},Z_s^{i-1})\rangle ds - \mu\int_t^T e^{\mu s+\lambda k_s}\left|\overline{Y}_s^{i+1}\right|^2 ds \\
&+ 2\int_t^T e^{\mu s+\lambda k_s}\langle \overline{Y}_s^{i+1}, h(s,Y_s^i) - h(s,Y_s^{i-1})\rangle dk_s - \mu\int_t^T e^{\mu s+\lambda k_s}\left|\overline{Y}_s^{i+1}\right|^2 dk_s \\
&- 2\int_t^T e^{\mu s+\lambda k_s}\langle \overline{Y}_s^{i+1}, Z_s^{i+1}\rangle dW_s.
\end{aligned}$$

Taking the expectation, we get

$$\begin{aligned}
&\mathbb{E}e^{\mu t+\lambda k_t}\left|\overline{Y}_t^{i+1}\right|^2 + \mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left\|\overline{Z}_s^{i+1}\right\|^2 ds \\
&= 2\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\langle \overline{Y}_s^{i+1}, f(s,Y_s^i,Z_s^i) - f(s,Y_s^{i-1},Z_s^{i-1})\rangle ds \\
&+ 2\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\langle \overline{Y}_s^{i+1}, h(s,Y_s^i) - h(s,Y_s^{i-1})\rangle dk_s - \mu\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left|\overline{Y}_s^{i+1}\right|^2 dk_s \\
&- \mu\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left|\overline{Y}_s^{i+1}\right|^2 ds.
\end{aligned}$$

With the same arguments as in the 2.2 of 2.2, one can show that there exist constant $C > 0$ such that

$$\begin{aligned}
&\mathbb{E}\left(e^{\mu t+\lambda k_t}\left|\overline{Y}_t^{i+1}\right|^2\right) + \mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left\|\overline{Z}_s^{i+1}\right\|^2 ds \\
&+ \mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left((\mu - C)\left|\overline{Y}_s^{i+1}\right|^2 ds + (\lambda - C)\left|\overline{Y}_s^{i+1}\right|^2 dk_s\right) \\
&\leq \frac{1+\alpha}{2}\left(C\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left|\overline{Y}_s^{i}\right|^2 ds + C\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left|\overline{Y}_s^{i}\right|^2 dk_s + \mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left\|\overline{Z}_s^{i}\right\|^2 ds\right).
\end{aligned}$$

Next, we choose μ *and* λ *in such a way that* $\mu - C = C$ *and* $\lambda - C = C$*, to obtain*

$$\mathbb{E}\left(e^{\mu t+\lambda k_t}\left|\bar{Y}_t^{i+1}\right|^2\right)+\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left\|\bar{Z}_s^{i+1}\right\|^2 ds$$
$$+C\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left|\bar{Y}_s^{i+1}\right|^2 dk_s+\bar{c}\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left|\bar{Y}_s^{i+1}\right|^2 ds$$
$$\leq\left(\frac{1+\alpha}{2}\right)^i\left[C\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left|\bar{Y}_s^{1}\right|^2 ds+C\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left|\bar{Y}_s^{1}\right|^2 dk_s+\mathbb{E}\int_t^T e^{\mu s+\lambda k_s}\left\|\bar{Z}_s^{1}\right\|^2 ds\right].$$

Since $(1+\alpha)/2<1$*, then* $\{(Y_t^i,Z_t^i)\}_{i=1,..}$ *is a Cauchy sequence in the space*

$$\mathbb{L}^2(\mathcal{F},[0,T];\mathbb{R}^n)\times\mathcal{M}^2(\mathcal{F},[0,T];\mathbb{R}^{n\times d}).$$

From the Burkhölder–Davis–Gundy inequality it follows that the sequence (Y_t^i) *is also a Cauchy sequence in the space* $\mathcal{S}^2(\mathcal{F},[0,T];\mathbb{R}^n)$ *By completeness its limit* $(Y_t,Z_t)=\lim_{i\to\infty}(Y_t^i,Z_t^i)$ *exists in the space* $\mathcal{S}^2(\mathcal{F},[0,T];\mathbb{R}^n)\times\mathcal{M}^2(\mathcal{F},[0,T];\mathbb{R}^{n\times d})$*.Passing to the limit in equation 2.13, we obtain the result in 2.2.*

Chapitre 3

Generalized BSDEs with respect to fractional Brownian motion

3.1 Assumptions and definition

Assume that

H$_1$) η_0 is a given constant

H$_2$) b, σ : $[0,T] \to \mathbb{R}$ are continuous deterministic functions, σ is differentiable and such that $\sigma(t) \neq 0, t \in [0,T]$ .Note that, since

$$\|\sigma\|_t^2 = H(2H-1)\int_0^t \int_0^t |u-v|^{2H-1} \sigma(u)\sigma(v)\,dudv,$$

we have

$$\frac{d}{dt}\left(\|\sigma\|_t^2\right) = \sigma(t)\hat{\sigma}(t) > 0,$$

where

$$\hat{\sigma}(t) = \int_0^t \phi(t-v)\sigma(v)\,dv.$$

Let $\mathcal{D}$ be an open connected sub set of $\mathbb{R}^d$ such that for some $l \in C^2\left(\mathbb{R}^d\right).D = \{x : l(x) > 0\}$ and $\partial D = \{x : l(x) = 0\}$ and $|\nabla l(x)| = 1$ for $x \in \partial D$.

Let $\eta_0 \in D$ and let (η_t, A_t) be solution of the following reflected SDE with respect to fractional brownian motion

$$\eta_t = \eta_0 + \int_0^t b(s)\, ds + \int_0^t \sigma(s)\, dB_s^H + \int_0^t \nabla l(\eta_s)\, dA_s. \tag{3.1}$$

By a solution of 3.1 we mean a pair of processes such that $\eta \in D$, A is a nondecreasing process, $A_0 = 0$ and $\int_0^t (\eta_t - a)\, dA_t \leq 0$ for any $a \in D$.

The existence of such a problem was shown in lions and sznitman (1984) [7] for a standard Brownian motion and in ferrante and rovira (2011) [3] for FBM and a set $D = (0, \infty)$

From Pardoux and Zhang (1998) [11] we know that for $H = 1/2$ and for each $v, t > 0$ there exists $C(v,t)$such that $Ee^{vA_t} \leq C(v,t)$.

We consider the following generalized BSDE with respect to FBM :

$$Y_t = \xi + \int_t^T f(s, \eta_s, Y_s, Z_s)\, ds + \int_t^T g(s, \eta_s, Y_s)\, d\Lambda_s - \int_t^T Z_s dB_s^H, t \in [0,T], \tag{3.2}$$

where Λ is an increasing process, $\Lambda_0 = 0$,we suppose that for some $v > 0$

H$_3$**)** $\xi = h(\eta_T)$ for some function h with bounded derivative, $\mathbb{E}e^{v\Lambda_T} |\xi|^2 < \infty$.

H$_4$**)** $f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $g : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions and there exists a constant $L > 0$ such that for all $t \in [0,T]$, $x, \acute{x}, y, \acute{y}, z, \acute{z} \in \mathbb{R}$,

$$|f(t,x,y,z) - f(t,\acute{x},\acute{y},\acute{z})| \leq L(|x-\acute{x}| + |y-\acute{y}| + |z-\acute{z}|),$$

$$|g(t,x,y) - g(t,\acute{x},\acute{y})| \leq L|y-\acute{y}|,$$

$$\mathbb{E}\left(\int_t^T e^{v\Lambda_t} |f(t,0,0,0)|^2 dt + \int_0^T e^{v\Lambda_t} |g(t,,\eta_t,0)|^2 d\Lambda_t\right) < \infty.$$

Now we introduce the following space

$$V_{[0,T]} = \left\{Y = \phi(.,\eta); \phi \in C_{pol}^{1.2}([0,T] \times \mathbb{R}), \frac{\partial \phi}{\partial t} \text{ is bounded}, t \in [0,T]\right\},$$

and by $\tilde{V}^{H}_{[0,T]}$denote the completion of $V_{[0,T]}$under the following norm

$$\|Y\|_v = \left(\int_0^T t^{2H-1}\mathbb{E}e^{v\Lambda_t}\left|Y_t\right|^2 dt\right)^{1/2} = \left(\int_0^T t^{2H-1}\mathbb{E}e^{v\Lambda_t}\left|\phi\left(t,\eta_t\right)\right|^2 dt\right)^{1/2}.$$

Définition 3.1 *A solution of a generalized BSDE with respect to FBM associated with date* (ξ, f, g, Λ) *is a pair* $(Y,Z) = (Y_t, Z_t)_{t\in[0,T]}$ *of processes satisfying 3.2 and such that* $Y \in \tilde{V}^{1/2}_{[0,T]}$ and $Z \in \tilde{V}^{H}_{[0,T]}$.

3.2 Existence and uniqueness result

Theorem 3.1 *Assume* (H_1) *and* (H_2)*. The exists a unique solution of 3.2.moreover,for all* $t \in [0,T]$,

$$\mathbb{E}\left(e^{v\Lambda_t}\left|Y_t\right|^2 + \int_t^T e^{v\Lambda_s}s^{2H-1}\left|Z_s\right|^2 ds + \int_t^T e^{v\Lambda_s}\left|Y_s\right|^2 d\Lambda_s\right) \leq C\Theta\left(t,T\right),$$

where

$$\Theta\left(t,T\right) = \mathbb{E}\left(e^{v\Lambda_T}\left|\xi\right|^2 + \int_t^T e^{v\Lambda_s}\left|f\left(s,0,0,0\right)\right|^2 ds + \int_t^T e^{v\Lambda_s}\left|\eta_s\right|^2 ds + \int_t^T e^{v\Lambda_s}\left|g\left(s,,\eta_s,0\right)\right|^2 d\Lambda_s\right).$$

Proof. First we will show the second part of above theorem.Assume that (Y,Z) is a solution of 3.2.by C we will denote a constant which may vary from line,

From the Itô formula

$$\begin{aligned}
e^{v\Lambda_t}\left|Y_t\right|^2 &= e^{v\Lambda_T}\left|\xi\right|^2 - \int_t^T 2e^{v\Lambda_s}Y_s\, dY_s - \int_t^T ve^{v\Lambda_s}\left|Y_s\right|^2 d\Lambda_s - \frac{1}{2}\int_t^T 2e^{v\Lambda_s}\frac{d}{ds}\left(\|f\|_s^2\, ds\right) \\
&= e^{v\Lambda_T}\left|\xi\right|^2 - 2\int_t^T e^{v\Lambda_s}Y_s\, dY_s - v\int_t^T e^{v\Lambda_s}\left|Y_s\right|^2 d\Lambda_s - \int_t^T e^{v\Lambda_s}\frac{d}{ds}\left(\|f\|_s^2\, ds\right) \\
&= e^{v\Lambda_T}\left|\xi\right|^2 + 2\int_t^T e^{v\Lambda_s}Y_s f\left(s,\eta_s,Y_s,Z_s\right) + 2\int_t^T e^{v\Lambda_s}Y_s g\left(s,\eta_s,Y_s\right) d\Lambda_s \\
&- 2\int_t^T e^{v\Lambda_s}Y_s Z_s dB_s^H - 2\int_t^T e^{v\Lambda_s}\mathbb{D}_s^H Y_s Z_s ds - v\int_t^T e^{v\Lambda_s}\left|Y_s\right|^2 d\Lambda_s.
\end{aligned}$$

It is known thaf $\mathbb{D}_s^H Y_s = (\hat{\sigma}(s)/\sigma(s)) Z_s$, 3.1.moreover by 3.1, there exists $M > 0$ such that for all $t \in [0,T]$, $t^{2H-1}/M \leq \hat{\sigma}(t)/\sigma(t) \leq Mt^{2H-1}$.

By lipschitz continuity of f and g we have

$$2yf(t,\eta,y,z) \leq +2|y| \ \ |f(t,\eta,y,z) - f(t,0,0,0) + f(t,0,0,0)|$$

$$2yf(t,\eta,y,z) \leq +2|y| \ \ L(|\eta| + |y| + |z|) + 2|y| |f(t,0,0,0)|$$

we hava $2ab \leq a^2 + b^2$ so:

$2L|y|\ |\eta| \leq L^2|y|^2 + |\eta|^2$ and $2L|y|\ |y| \leq 2L|y|^2$ and $2L|y|\ |z| = 2L\frac{\sqrt{s^{2H-1}}}{\sqrt{M}}|z|\frac{\sqrt{M}}{\sqrt{s^{2H-1}}}|y| \leq \frac{ML^2|y|^2}{s^{2H-1}} + \frac{s^{2H-1}}{M}|z|^2$ and $2|y||f(t,0,0,0)| \leq |y|^2 + |f(t,0,0,0)|^2$

We find

$$\begin{aligned} 2yf(t,\eta,y,z) &\leq L^2|y|^2 + |\eta|^2 + 2L|y|^2 \\ &\quad + \frac{ML^2|y|^2}{s^{2H-1}} + \frac{s^{2H-1}}{M}|z|^2 + |y|^2 + |f(t,0,0,0)|^2 \\ &\leq \left(L^2 + 2L + \frac{ML^2}{s^{2H-1}} + 1\right) \ |y|^2 + |\eta|^2 + \frac{1}{M}s^{2H-1}|z|^2 + \ |f(t,0,0,0)|^2 \end{aligned}$$

and

$$\begin{aligned} 2yg(t,\eta,y) &\leq +2|y| \ \ |g(t,\eta,y) - g(t,\eta,0)| + 2|y| \ \ |g(t,\eta,0)| \\ &\leq 2L|y|^2 \ \ + |y|^2 + |g(t,\eta,0)|^2 \\ 2yg(t,\eta,y) &\leq (2L+1)|y|^2 + |g(t,\eta,0)|^2 \end{aligned}$$

Therefore,we can write

$$\begin{aligned} &\mathbb{E}\left(e^{v\Lambda_t}|Y_t|^2 + v\int_t^T e^{v\Lambda_s}|Y_s|^2 d\Lambda_s + \frac{2}{M}\int_t^T e^{v\Lambda_s}s^{2H-1}|Z_s|^2 ds\right) \\ &\leq \Theta(t,T) + \mathbb{E}\int_t^T \left(L^2 + 2L + \frac{ML^2}{s^{2H-1}} + 1\right) e^{v\Lambda_s}|Y_s|^2 ds \\ &+ \frac{1}{M}\mathbb{E}\int_t^T e^{v\Lambda_s}s^{2H-1}|Z_s|^2 ds + (2L+1)\mathbb{E}\int_t^T e^{v\Lambda_s}|Y_s|^2 d\Lambda_s. \end{aligned}$$

Choosing $v-(2L-1)\geq 1\Rightarrow v\geq (2L+2)$ we get

$$\mathbb{E}\left(e^{v\Lambda_t}\left|Y_t\right|^2+\int_t^T e^{v\Lambda_s}\left|Y_s\right|^2 d\Lambda_s+\frac{1}{M}\int_t^T e^{v\Lambda_s}s^{2H-1}\left|Z_s\right|^2 ds\right)$$
$$\leq \Theta\left(t,T\right)+\mathbb{E}\int_t^T\left(L^2+2L+\frac{ML^2}{s^{2H-1}}+1\right)e^{v\Lambda_s}\left|Y_s\right|^2 ds. \tag{3.3}$$

By gronwall's inequality,

$$e^{v\Lambda_t}\left|Y_t\right|^2\leq \Theta\left(t,T\right)+\int_t^T\left(L^2+2L+\frac{ML^2}{s^{2H-1}}+1\right)e^{v\Lambda_s}\left|Y_s\right|^2 ds$$
$$\mathbb{E}e^{v\Lambda_t}\left|Y_t\right|^2\leq \Theta\left(t,T\right)\exp\left\{\left(L^2+2L+1\right)\left(T-t\right)+ML^2\frac{T^{2-2H}-t^{2-2H}}{2-2H}\right\},$$

and by 3.3 also

$$\mathbb{E}\left(\int_t^T e^{v\Lambda_s}s^{2H-1}\left|Z_s\right|^2 ds+\int_t^T e^{v\Lambda_s}\left|Y_s\right|^2 d\Lambda_s\right)\leq C\Theta\left(t,T\right).$$

Now we will prove the existence and uniqueness of the solution of 3.2.the method used here is similar to that in the 3.1 we will show that the the mapping $\Gamma:\tilde{V}{}^{1/2}_{[0,T]}\times\tilde{V}{}^{H}_{[0,T]}\rightarrow\tilde{V}{}^{1/2}_{[0,T]}\times\tilde{V}{}^{H}_{[0,T]}$given by $(U,V)\rightarrow\Gamma\left(U,V\right)=\left(Y,Z\right)$ is a contraction where$(Y,Z)$is a solution of the following generalized BSDE :

$$Y_t=\xi+\int_t^T f\left(s,\eta_s,U_s,V_s\right)ds+\int_t^T g\left(s,\eta_s,U_s\right)d\Lambda_s-\int_t^T Z_s dB_s^H.$$

Let $k\in\mathbb{N}$ and $t_i=\frac{i-1}{k}T, i=1,...,k+1$.first we will show that Γ is a contraction on $\tilde{V}{}^{1/2}_{[t_k,T]}\times\tilde{V}{}^{H}_{[t_k,T]}$.take $U,\acute{U}\in\tilde{v}{}^{1/2}_{[t_k,T]}$and $V,\acute{V}\in\tilde{v}{}^{H}_{[t_k,T]}$,let $\Gamma\left(U,V\right)=\left(Y,Z\right)$ and $\Gamma\left(\acute{U},\acute{V}\right)=\left(\acute{Y},\acute{Z}\right)$and let $\Delta Y=Y-\acute{Y}$, $\Delta Z=Z-\acute{Z}$, $\Delta U=U-\acute{U}$ and $\Delta U=V-\acute{V}$.

From the Itô formula, for $t \in [t_k, T\ ]$,

$$e^{v\Lambda_t} |Y_t - Y_t'|^2 = e^{v\Lambda_T} |\xi|^2 - 2\int_t^T e^{v\Lambda_s} Y_s - Y_s'\ dY_s$$
$$- v\int_t^T e^{v\Lambda_s} |Y_s - Y_s'|^2 d\Lambda_s - \int_t^T e^{v\Lambda_s} \frac{d}{ds}\left(\|f\|_s^2\, ds\right)$$
$$e^{v\Lambda_t} |\Delta Y_t|^2 = e^{v\Lambda_T} |\xi|^2 - 2\int_t^T e^{v\Lambda_s} \Delta Y_s\ dY_s - v\int_t^T e^{v\Lambda_s} |\Delta Y_s|^2 d\Lambda_s - \int_t^T e^{v\Lambda_s} \frac{d}{ds}\left(\|f\|_s^2\, ds\right)$$
$$e^{v\Lambda_T} |\xi|^2 - 2\int_t^T e^{v\Lambda_s} \Delta Y_s\ dY_s - v\int_t^T e^{v\Lambda_s} |\Delta Y_s|^2 d\Lambda_s - \int_t^T e^{v\Lambda_s} \frac{\hat{\sigma}(s)}{\sigma(s)} |\Delta Z_s|^2$$
$$= e^{v\Lambda_T} |\xi|^2 + 2\int_t^T e^{v\Lambda_s} \Delta Y_s f(s, \eta_s, U_s, V_s) + 2\int_t^T e^{v\Lambda_s} \Delta Y_s g(s, \eta_s, U_s)\, d\Lambda_s$$
$$- 2\int_t^T e^{v\Lambda_s} Y_s Z_s dB_s^H - 2\int_t^T e^{v\Lambda_s} \frac{\hat{\sigma}(s)}{\sigma(s)} |\Delta Z_s|^2 ds - v\int_t^T e^{v\Lambda_s} |\Delta Y_s|^2 d\Lambda_s.$$

similarly as before

$$\mathbb{E}\left(e^{v\Lambda_t} |\Delta Y_t|^2 + v\int_t^T e^{v\Lambda_s} |\Delta Y_s|^2 d\Lambda_s + 2\int_t^T e^{v\Lambda_s} \frac{\hat{\sigma}(s)}{\sigma(s)} |\Delta Z_s|^2 ds\right)$$
$$= 2\mathbb{E}\int_t^T e^{v\Lambda_s} \Delta Y_s \left(f(s, \eta_s, U_s, V_s) - f\left(s, \eta_s, \acute{U}_s, \acute{V}_s\right)\right) ds$$
$$+ 2E\int_t^T e^{v\Lambda_s} \Delta Y_s \left(g(s, \eta_s, U_s) - g(s, \eta_s, U_s)\right) d\Lambda_{s.}$$

Note that $2\Delta y\left(f(t, \eta, u, v) - f(t, \eta, \acute{u}, \acute{v})\right) \leq 2L|\Delta y|\left(|\Delta u| + |\Delta v|\right)$ and $2\Delta y\left(g(t, \eta, u) - g(t, \eta, \acute{u})\right) \leq 2L|\Delta y||\Delta u| \leq L^2/\beta|\Delta y|^2 + \beta|\Delta u|^2$.

choose $v - L^2/\beta \geq 1 \Rightarrow v = L^2/\beta + 1$.Then, and by the schwarz inequality we obtain

$$\mathbb{E}\left(e^{v\Lambda_t} |\Delta Y_t|^2 + \int_t^T e^{v\Lambda_s} |\Delta Y_s|^2 d\Lambda_s + \frac{2}{M}\int_t^T e^{v\Lambda_s} s^{2H-1} |\Delta Z_s|^2 ds\right)$$
$$\leq 2L\int_t^T \mathbb{E}e^{v\Lambda_s} |\Delta Y_s| \left(|\Delta U_s| + |\Delta V_s|\right) ds + \beta\mathbb{E}\int_t^T e^{v\Lambda_s} |\Delta U_s|^2 d\Lambda_s \tag{3.4}$$
$$\leq 2L\int_t^T \left(\mathbb{E}e^{v\Lambda_s} |\Delta Y_s|^2\right)^{1/2} \left(\mathbb{E}e^{v\Lambda_s}\left(|\Delta U_s| + |\Delta V_s|\right)^2\right)^{1/2} ds + \beta\mathbb{E}\int_t^T e^{v\Lambda_s} |\Delta U_s|^2 d\Lambda_s. \tag{3.5}$$

Denote $x(t) = \left(\mathbb{E}e^{v\Lambda_t} |\Delta Y_s|^2\right)^{1/2}$ and $a(t) = \beta\mathbb{E}\int_t^T e^{v\Lambda_s} |\Delta U_s|^2 d\Lambda_s$.then, by 3.5

$$x^2(t) \leq 2L\int_t^T x(s)\left(\mathbb{E}e^{v\Lambda_s}\left(|\Delta U_s| + |\Delta V_s|\right)^2\right)^{1/2} ds + a(t), t \in [t_k, T\ ].$$

Applying 3.2 to the above inequality we get

$$x(t) \leq \sqrt{a(t)} + \sqrt{2L} \int_t^T \left(\mathbb{E} e^{v\Lambda_s} \left(|\Delta U_s|^2 + |\Delta V_s|^2 \right) \right)^{1/2} ds,$$

and therefore for $t \in [t_k, T]$

$$\mathbb{E} e^{v\Lambda_t} |\Delta Y_t|^2 \leq 2a(t) + 4L^2 \left(\int_t^T \left(\mathbb{E} e^{v\Lambda_s} \left(|\Delta U_s|^2 + |\Delta V_s|^2 \right) \right)^{1/2} ds \right)^2.$$

Now we can compute

$$\begin{aligned}
\int_{t_k}^T x^2(s)\, ds &\leq (T - t_k) \left(2a(t_k) + 8L^2 \left(\int_{t_k}^T \left(\mathbb{E} e^{v\Lambda_s} |\Delta U_s|^2 \right)^{1/2} ds \right)^2 \right) \\
&\quad + 8L^2 (T - t_k) \left(\int_{t_k}^T \left(\frac{1}{s^{2H-1}} \cdot \mathbb{E} e^{v\Lambda_s} s^{2H-1} |\Delta V_s|^2 \right)^{1/2} ds \right)^2 \\
&\leq (T - t_k) \begin{pmatrix} 2a(t_k) + 8L^2 (T - t_k)\, \mathbb{E} \int_{t_k}^T e^{v\Lambda_s} |\Delta U_s|^2 ds \\ + \frac{8L^2 \left(T^{2-2H} - t_k^{2-2H} \right)}{2-2H} \mathbb{E} \int_{t_k}^T e^{v\Lambda_s} s^{2H-1} |\Delta V_s|^2 \end{pmatrix} = C (T - t_k)\, \Theta(t_k, T).
\end{aligned}$$

And similarly

$$\int_{t_k}^T \frac{1}{s^{2H-1}} x^2(s)\, ds \leq \frac{C}{2 - 2H} \cdot \left(T^{2-2H} - t_k^{2-2H} \right) \cdot \Theta(t_k, T),$$

Where

$$\Theta(t_k, T) = \mathbb{E} \left(\int_{t_k}^T e^{v\Lambda_s} |\Delta U_s|^2 (ds + d\Lambda_s) + \int_{t_k}^T e^{v\Lambda_s} s^{2H-1} |\Delta V_s|^2 ds \right).$$

Using above inequalities, from 2.4 we deduce

$$\begin{aligned}
&\mathbb{E}\left(\int_{t_k}^{T} e^{v\Lambda_s}\left|\Delta Y_s\right|^2(ds+d\Lambda_s)+\int_{t_k}^{T} e^{v\Lambda_s}s^{2H-1}\left|\Delta Z_s\right|^2 ds\right)\\
&\leq \mathbb{E}\left(\int_{t_k}^{T} e^{v\Lambda_s}\left|\Delta Y_s\right|^2 ds+C\beta\mathbb{E}\int_{t_k}^{T} e^{v\Lambda_s}\left|\Delta U_s\right|^2 d\Lambda_s\right)\\
&+C\mathbb{E}\int_{t}^{T} e^{v\Lambda_s}\frac{1}{\beta}\left|\Delta Y_s\right|^2\left(1+\frac{1}{s^{2H-1}}\right)+\beta\left(\left|\Delta U_s\right|^2+s^{2H-1}\left|\Delta V_s\right|^2\right)ds\\
&\leq C\left(T-t_k\right)\tilde{\Theta}\left(t_k,T\right)+C\beta\tilde{\Theta}\left(t_k,T\right)+\frac{C}{\beta}\int_{t_k}^{T} x^2(s)\left(1+\frac{1}{s^{2H-1}}\right)ds\\
&\leq C\left(\beta+\left(1+\frac{1}{\beta}\right)\left(T-t_k\right)+\frac{1}{\beta}\left(T^{2-2H}-t_k^{2-2H}\right)\right)\tilde{\Theta}\left(t_k,T\right).
\end{aligned}$$

Choosing β such that $C\beta\leq 1/4$ and taking k large enough that $C(\beta+1)(T-t_k)/\beta\leq 1/4$ and $C\left(T^{2-2H}-t_k^{2-2H}\right)/\beta\leq 1/4$ we obtain

$$\mathbb{E}\left(\int_{t_k}^{T} e^{v\Lambda_s}\left(\left|\Delta Y_s\right|^2(ds+d\Lambda_s)+s^{2H-1}\left|\Delta Z_s\right|^2 ds\right)\right)\leq\frac{3}{4}\tilde{\Theta}\left(t_k,T\right).$$

Since Γ is a contraction, (Y^n,Z^n) is a cauchy sequence in $\tilde{V}{}^{1/2}_{[t_k,T]}\times\tilde{V}{}^{H}_{[t_k,T],}$where $(Y^0,Z^0)\in\tilde{V}{}^{1/2}_{[t_k,T]}\times\tilde{V}{}^{H}_{[t_k,T],}$and for $n\geq 0$

$$Y_t^{n+1}=\xi+\int_t^T f\left(s,\eta_s,Y_s^n,Z_s^n\right)ds+\int_t^T g\left(s,\eta_s,Y_s^n\right)d\Lambda_s-\int_t^T Z_s^{n+1}dB_s^H.$$

Then there exists $(Y,Z)\in\tilde{V}{}^{1/2}_{[t_k,T]}\times\tilde{V}{}^{H}_{[t_k,T]}$being a limit of (Y^n,Z^n), i.e.

$$\lim_{n\to\infty}\mathbb{E}\left(e^{v\Lambda_t}\left|Y_t^n-Y_t\right|^2+\int_{t_k}^{T} e^{v\Lambda_s}\left(\left|Y_s^n-Y_s\right|^2+s^{2H-1}\left|Z_s^n-Z_s\right|^2\right)ds\right)=0.$$

Moreover $\lim\limits_{n\to\infty}\mathbb{E}\int_{t_k}^{T} e^{v\Lambda_s}\left|Y_s^n-Y_s\right|^2 d\Lambda_s=0.$therefore for any $t\in[t_k,T]$

$$\begin{aligned}
&\lim_{n\to\infty}\left(-Y^{n+1}+\xi+\int_t^T f\left(s,\eta_s,Y_s^n,Z_s^n\right)ds+\int_t^T g\left(s,\eta_s,Y_s^n\right)d\Lambda_s\right)\\
&=-Y+\xi+\int_t^T f\left(s,\eta_s,Y_s,Z_s\right)ds+\int_t^T g\left(s,\eta_s,Y_s\right)d\Lambda_s \text{ in } \mathbb{L}^2\left(\nleq,\mathbb{F},\mathbb{P}\right),
\end{aligned}$$

and $Z^n \mathbf{1}_{[t,T]} \to Z\ \mathbf{1}_{[t,T]}$ in $\mathbb{L}^2(\Omega, \mathcal{F}, \mathcal{H})$ .we show that (Y, Z) satisfies3.2 on $[t_k, T]$.the next step is to solve the equation on $[t_{k-1}, t_k]$ .with the same arguments, repeating the above technique we obtain a uniqueness of the solution of generalized BSDE with respect to FBM on the whole interval $[0, T]$. ■

Conclusion

In this work we study the generalized backward stochastic differential equation driven by fractional Brownian motion. First of all we defined fractional Brownian motion and studied its properties in all its details and quotes from the existence and uniqueness of solutions of fractional generalized backward stochastic differential equation. The idea of proving using the fixed point theorem thus showing that there exists a unique solution in the same space. We note that pretty much of the technical difficulties coming from the fractional brownien motion, since B^H with $H > \frac{1}{2}$ is not a semimartingale, we cannot use the classical theory of stochastic calculus.

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Annex : Some mathematical tools

Theorem 3.2 ***(Fixed point)*** *Let (E, d) be a complete metric space and $\varphi : E \to E$ a contractiog map, i.e lipschitzian with ratio $k < 1$.then, φ admit a unique point fixed $a \in E$ such that :$\varphi(a) = a$.*

Définition 3.2 ***Young inequality :*** *We say that two numbers $p, q > 0$, are conjugated in the sense of Young, if :*

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Young inequality says that if p and q are conjugate and if a,$a, b \geq 0$,So

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

with equality if and only if a $a^p = b^q$.

For example ,if $p = q = 2$ we find the inequality

$$2ab \leq a^2 + b^2.$$

Hölder inequality. Hölder inequality says that if $p, q > 0$,are conjugate in the sense of Young, then

$$\int_D (f(x)g(x))d\mu(x) \leq \left(\int_D |f(x)|^p d\mu(x)\right)^{1/p} \cdot \left(\int_D |g(x)|^p d\mu(x)\right)^{1/p}$$

Theorem 3.3 ***(Grönwall's inequality)*** *Let I denote an interval of the real line of the form $[a, \infty)$ or $[a, b]$ or $[a, b)$ with $a < b$. Let α, β and u be real-valued functions defined on*

I. Assume that β and u are continuous and that thenegative part of α is integrable on every closed and bounded subinterval of I.

If β is non-negative and if u satisfies the integral inequality

$$u(t) \leq \alpha(t) + \int_a^t \beta(s)\, u(s)\, ds \qquad \forall t \in I,$$

then

$$u(t) \leq \alpha(t) + \int_a^t \alpha(s)\, \beta(s) \exp\left(\int_s^t \beta(r)\, dr\right) ds \qquad t \in I.$$

Proof. Define

$$v(s) = \exp\left(-\int_a^t \beta(r)\, dr\right) \int_a^s \beta(r)\, u(r)\, dr, \qquad s \in I.$$

Using the product rule, the chain rule,the derivative of the exponential function and the fundamental theorem of calculus, we obtain for the derivative

$$v'(s) = \left(u(s) - \int_a^s \beta(r)\, u(r)\, dr\right) \beta(s) \exp\left(-\int_a^s \beta(r)\, dr\right), \qquad s \in I,$$

where we used the assumed integral inequality for the upper estimate.Since β and the exponential are non-negative, this gives an upper estimate for the derivative of $v(s)$ Since $v(a) = 0$, integration of this inequality from a to t gives

$$v(t) \leq \int_a^t \alpha(s)\, \beta(s) \exp\left(-\int_a^s \beta(r)\, dr\right) ds.$$

Using the definition of $v(t)$ from the firs step,and then this inequality and the functional equation of the exponential function,we obtain

$$\begin{aligned} \int_a^t \beta(s)\, u(s)\, ds &= \exp\left(\int_a^t \beta(r)\, dr\right) v(t) \\ &\leq \int_a^t \alpha(s)\, \beta(s) \exp\left(\int_a^t \beta(r)\, dr - \int_a^s \beta(r)\, dr\right) ds. \end{aligned}$$

Substituting this result into the assumed integral inequality gives Grönwall's inequality. ∎

Theorem 3.4 *(**Burkhölder-Davis-Gundy inequality "B-D-G inequality"**) For all $p > 0$, there exist positive constants to c_p and C_p, such that, for any continuous local martingale $X = (X_t)_{t\geq 0}$, zero at 0*

$$c_p = \mathbb{E}\left[\langle X, X\rangle_\infty^{\frac{p}{2}}\right] \leq \left[\sup_{t\geq 0}|X_t|^p\right] \leq C_p\mathbb{E}\left[\langle X, X\rangle_\infty^{\frac{p}{2}}\right].$$

Proposition 3.1 *Let $(Y, Z) \in V_T \times V_T$ be the solution of BSDE*

$$\begin{cases} -dY(t) = f(t, \eta(t), Y(t), Z(t))dt - Z(t)\delta B^H(t), t \in [0, T], \\ Y(T) = \xi, \end{cases}$$

constructed in the assumptions $(H_1) - (H_4)$ be satisfied Then the BSDE.

$$Y(t) = \xi + \int_t^T f(s, \eta(s), Y(s), Z(s))ds \int_t^T Z(s)dB^H(s),\ t \in [0, T].$$

has a solution $(Y, Z) \in \tilde{V}{}^{\frac{1}{2}}_{[0,T]} \times \tilde{V}{}^{H}_{[0,T]}$.Then for almost $t \in (0, T]$

$$\mathbb{D}_t^H Y(t) = \frac{\hat{\sigma}(t)}{\sigma(t)} Z(t).$$

Proof. From

$$Y_{k+1}(t) = \xi + \int_t^T f(s, \eta(s), Y_k(s), Z_k(s))ds - \int_t^T Z_{k+1}(s)dB^H(s), \quad t \in [0, T],$$

we know that $(Y_k, Z_k) \in \tilde{V}{}^{\frac{1}{2}}_{[0,T]} \times \tilde{V}{}^{H}_{[0,T]}$ satisfies

$$Y_{k+1}(t) = \xi + \int_t^T f(s, \eta(s), Y_k(s), Z_k(s))ds - \int_t^T Z_{k+1}(s)\delta B^H(s),\ t \in [0, T], k \geq 1.$$

We recall that $Y_k(t) = u_k(t, \eta(t))$, $Z_k(t) = v_k(t, \eta(t))$, $t \in [0, T]$ and $Z_k(t) = \sigma(t)\frac{\partial}{\partial x}u_k(t, \eta(t))$. Since $(Y_k, Z_k) \to (Y, Z)$ in $\tilde{V}{}^{\frac{1}{2}}_{[0,T]} \times \tilde{V}{}^{H}_{[0,T]}$, there exists a subsequence, by convenience still denoted by $\{(Y_k, Z_k)\}_{k\in\mathbb{N}}$, such that for arbitrary $\rho > 0$, we have that $\lim_{k\to\infty} \mathbb{E}\left|Y_k(s) - Y(s)\right|^2$

$= 0$ and $\lim_{k\to\infty} \mathbb{E}\left|Z_k(s) - Z(s)\right|^2 = 0$, for almost all $s \in [\rho, T]$.As a process with the parameter r,

$$\begin{aligned} D_r Y_k(t) &= \frac{\partial}{\partial x} u(t, \eta(t)) \sigma(r) \mathbf{1}_{[0,t]}(r) \\ &= \frac{\sigma(r)}{\sigma(t)} Z_k(t) \mathbf{1}_{[0,t]}(r) \overset{L^2([0,T]\times\Omega)}{\rightarrow} \frac{\sigma(r)}{\sigma(t)} Z(t) \mathbf{1}_{[0,t]}(r), \\ &\text{as } k \to \infty, \quad \text{for almost all } t \in [\rho, T]. \end{aligned}$$

On the other hand, since $\mathbb{L}^2([0,T]) \subset \mathcal{H}$, we conclude that the convergence also holds in $L^2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$.Consequently, in $\mathbb{L}^2(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$ $D_r Y(t) = \lim_{k\to\infty} D_r Y_k(t) = \lim_{k\to\infty} \frac{\sigma(r)}{\sigma(t)} Z_k(t) \mathbf{1}_{[0,t]}(r) = \frac{\sigma(r)}{\sigma(t)} Z(t) \mathbf{1}_{[0,t]}(r)$,a.e.$t \in [\rho, T]$, and, thus,

$$\mathbb{D}_t^H Y(t) = \int_0^T \phi(t-r) D_r Y(t) dr = \frac{\hat{\sigma}(t)}{\sigma(t)} Z(t), \text{ a.e.} t \in [\rho, T],$$

where $\hat{\sigma}(t)$ is defined by

$$\hat{\sigma}\,(t) := \int_0^t \phi(t-r)\sigma(r) dr, t \in [0, T].$$

Considering that $\rho > 0$ is arbitrary, we have

$$\mathbb{D}_t^H Y(t) = \frac{\hat{\sigma}(t)}{\sigma(t)} Z(t), \quad \text{a.e,} t \in [0, T],$$

which completes the proof. ■

Theorem 3.5 ***(Doob's inequality)*** $\forall p > 1$. *if* $\{X_N\}_{n=1,\ldots,N}$ *cat a martingale in* $\mathbb{L}^p$

$$\mathbb{E}\left[|X_N|^p\right] \le \mathbb{E}\left[X^{*p}\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[|X_N|^p\right],$$

with $X^* := \max\left(|X_1|, ..., |X_N|\right)$.

Remark 3.1 *The function $\hat{\sigma}$ defined by*

$$\hat{\sigma}\,(t) := \int_0^t \phi(t-r)\sigma(r)dr, t \in [0,T].$$

can be written in the following form :

$$\hat{\sigma}(t) = H(2H-1)t^{2H-1}\int_0^1 (1-u)^{2H-2}\sigma(tu)du, t \in [0,T].$$

Moreover, we observe that $\|\sigma\|_t^2$ is continuously differentiable with respect to t, and

(a) $\frac{d}{dt}\left(\|\sigma\|_t^2\right) = 2\sigma(t)\hat{\sigma}(t) > 0, t \in (0,T]$,

(b) *for a suitable constant $M > 0$, $\frac{1}{M}t^{2H-1} \leq \hat{\sigma}(t)\sigma(t) \leq Mt^{2H-1}, t \in [0,T]$.*

Lemma 3.1 *Let $a, \alpha, \beta : [0,T] \to \mathbb{R}_+$ be three nonnegative Borel functions such thata is decreasing and $\alpha, \beta \in \mathbb{L}^1_{loc}([0,\infty])$. If $x : [0,T] \to \mathbb{R}_+$ is a continuous function such that*

$$x^2(t) \leq a(t) + 2\int_t^T \alpha(s)x(s)ds + 2\int_t^T \beta(s)x^2(s)ds, \qquad t \in [0,T],$$

then

$$x(t) \leq \sqrt{a(t)}\exp\left(\int_t^T \beta(s)ds\right) + \int_t^T \alpha(s)\exp\left(\int_t^s \beta(r)dr\right)ds, \qquad t \in [0,T].$$

Remark 3.2 *Now from*

$$x^2(t) \leq 2\sqrt{3}\int_t^T x(s)\left[\mathbb{E}\left(L^2|U(s)|^2 + L^2|V(s)|^2 + |f(s,\eta(s),0,0)|^2\right)\right]^{1/2} ds, \qquad t \in [0,T]$$

and the above lemma, by setting

$$a(t) = 0, \beta(s) = 0,$$
$$\alpha(s) = \sqrt{3}\left[\mathbb{E}\left(L^2|U(s)|^2 + L^2|V(s)|^2 + |f(s,\eta(s),0,0)|^2\right)\right]^{1/2} ds, \quad s \in [0,T],$$

we have

$$x(t) \leq \sqrt{3} \int_t^T \left[\mathbb{E}\left(L^2|U(s)|^2 + L^2|V(s)|^2 + |f(s,\eta(s),0,0)|^2\right)\right]^{1/2} ds, \quad t \in [0,T],$$

and, hence, for any $\beta > 0$*,*

$$\begin{aligned}
\left[\mathbb{E}|Y(t)|^2\right]^{1/2} &\leq \sqrt{3} \int_t^T \left(L\left[\mathbb{E}|U(s)|^2\right]^{1/2} + L\left[\mathbb{E}|V(s)|^2\right]^{1/2} + \left[\mathbb{E}|f(s,\eta(s),0,0)|^2\right]^{1/2}\right) ds \\
&\leq \sqrt{3}L \int_t^T \left(e^{-\beta s}\left[e^{2\beta s}\mathbb{E}|U(s)|^2\right]^{1/2} + \frac{e^{-\beta s}}{s^{H-1/2}}\left[s^{2H-1}e^{2\beta s}\mathbb{E}|V(s)|^2\right]^{1/2}\right) ds \\
&+ \sqrt{3}\int_t^T e^{-\beta s}\left[e^{2\beta s}\mathbb{E}\left(|f(s,\eta(s),0,0)|^2\right)\right]^{1/2} ds \\
&\leq \sqrt{3}L\left(\int_t^T e^{-\beta s} ds\right)^{1/2}\left(\int_t^T e^{2\beta s}\mathbb{E}|U(s)|^2 ds\right)^{1/2} \\
&+ \sqrt{3}L\left(\int_t^T \frac{e^{-\beta s}}{s^{2H-1}} ds\right)^{1/2}\left(\int_t^T s^{2H-1}e^{2\beta s}\mathbb{E}|V(s)|^2 ds\right)^{1/2} \\
&+ \sqrt{3}\left(\int_t^T e^{-2\beta s} ds\right)^{1/2}\left(\int_t^T e^{2\beta s}\mathbb{E}|f(s,\eta(s),0,0)|^2 ds\right)^{1/2}. \qquad (3.6)
\end{aligned}$$

Let us use the following notations :

$$\begin{aligned}
&A_t := \left(\int_t^T e^{2\beta s}\mathbb{E}|U(s)|^2 ds\right)^{1/2}, \; B_t := \left(\int_t^T s^{2H-1}e^{2\beta s}\mathbb{E}|V(s)|^2 ds\right)^{1/2}, \; and \\
&C_t = \left(\int_t^T e^{2\beta s}\mathbb{E}|f(s,\eta(s),0,0)|^2 ds\right)^{1/2}, \qquad t \in [0,T].
\end{aligned}$$

Since $\int_t^T e^{-2\beta s} ds = \frac{1}{2\beta}\left(e^{-2\beta t} - e^{-2\beta T}\right)$ *we have for* $\alpha > 0$ *with* $0 < \alpha < 2 - 2H < 1$ *and* $\beta > 0$*,*

$$e^{2\beta t}\int_t^T \frac{e^{-2\beta s}}{s^{2H-1}} ds \leq \int_t^T \frac{(2\beta(s-t))^{-\alpha}}{s^{2H-1}} ds \leq \frac{1}{(2\beta)^\alpha}\int_0^T \frac{1}{s^{\alpha+2H-1}} ds < \infty.$$

This allows to conclude from 3.6 that

$$e^{2\beta t}\mathbb{E}|Y(t)|^2 \leq \frac{9L^2}{2\beta}A_t^2 + \frac{9L^2}{(2\beta)^\alpha}\int_t^T \frac{(s-t)^{-\alpha}}{s^{2H-1}} ds \; B_t^2 + \frac{9}{2\beta}C_t^2. \qquad (3.7)$$

Consequently, there exists $C(\beta)$ *with* $\lim_{\beta\to\infty} C(\beta) = 0$*, s.t.*

$$e^{2\beta t}\mathbb{E}|Y(t)|^2 dt \le C(\beta)\left(A_t^2 + B_t^2 + C_t^2\right), \qquad t \in [0,T]. \tag{3.8}$$

Applying the Itô formula to $|Y(t)|^2$*, taking the expectation* $\mathbb{E}|Y(t)|^2$ *and then deter mining the function de* $d\left(e^{2\beta t}\mathbb{E}|Y(t)|^2\right)$ *and using* $\mathbb{D}_t^H Y(t) = \frac{\hat{\sigma}(t)}{\sigma(t)} Z(t)$*, the Lipschitz property of* f *as well as 3.7 we obtain (Recall for a suitable constant* $M > 0$*,* $\frac{1}{M}t^{2H-1} \le \frac{\hat{\sigma}(t)}{\sigma(t)} \le M t^{2H-1}$*,* $t \in [0,T]$*.for the definition of* M*)*

$$\begin{aligned}
& e^{2\beta t}\mathbb{E}|Y(t)|^2 + 2\beta \int_t^T e^{2\beta s}\mathbb{E}|Y(s)|^2 ds + \frac{2}{M}\int_t^T s^{2H-1} e^{2\beta s}\mathbb{E}|Z(s)|^2 ds \\
& \le 2\int_t^T e^{2\beta s}\mathbb{E}\left[|Y(s)|\left(L|U(s)| + L|V(s)| + |f(s,\eta(s),0,0)|\right)\right] ds \\
& \le 2L\int_t^T \left[\mathbb{E}\left(e^{2\beta s}|Y(s)|^2\right)\right]^{1/2}\left[\mathbb{E}\left(e^{2\beta s}|U(s)|^2\right)\right]^{1/2} ds \\
& + 2L\int_t^T \left[\mathbb{E}\left(\frac{e^{2\beta s}}{s^{2H-1}}|Y(s)|^2\right)\right]^{1/2}\left[\mathbb{E}\left(e^{2\beta s}s^{2H-1}|V(s)|^2\right)\right]^{1/2} ds \\
& + 2L\int_t^T \left[\mathbb{E}\left(e^{2\beta s}|Y(s)|^2\right)\right]^{1/2}\left[\mathbb{E}\left(e^{2\beta s}|f(s,\eta(s),0,0)|^2\right)\right]^{1/2} ds \\
& \le 2L\int_t^T \left[C(\beta)\left(A_s^2 + B_s^2 + C_s^2\right)\right]^{1/2}\left[\mathbb{E}\left(e^{2\beta s}|U(s)|^2\right)\right]^{1/2} ds \\
& + 2L\int_t^T \left[\frac{1}{s^{2H-1}}C(\beta)\left(A_s^2 + B_s^2 + C_s^2\right)\right]^{1/2}\left[\mathbb{E}\left(e^{2\beta s}s^{2H-1}|V(s)|^2\right)\right]^{1/2} ds \\
& + 2\int_t^T \left[\left[C(\beta)\left(A_s^2 + B_s^2 + C_s^2\right)\right]^{1/2}\right]^{1/2}\left[\mathbb{E}\left(e^{2\beta s}|f(s,\eta(s),0,0)|^2\right)\right]^{1/2} ds \\
& \le 2L\sqrt{C(\beta)}\left(A_t + B_t + C_t\right)\left(\sqrt{T-t}A_t + \sqrt{\frac{T^{2-2H} - t^{2-2H}}{2-2H}}B_t + \sqrt{T-t}C_t\right).
\end{aligned}$$

Thus, the above inequality and 3.8 allow to conclude inequality

$$\begin{aligned}
& \sup_{t\in[0,T]} e^{2\beta t}\mathbb{E}|Y(t)|^2 + \int_0^T e^{2\beta s}\mathbb{E}|Y(s)|^2 ds + \int_0^T s^{2H-1}e^{2\beta s}\mathbb{E}|Z(s)|^2 ds \\
& \le C(\beta)\left(\int_0^T e^{2\beta s}\mathbb{E}|U(s)|^2 + \int_0^T s^{2H-1}e^{2\beta s}\mathbb{E}|V(s)|^2 + \int_0^T e^{2\beta s}|f(s,\eta(s),0,0)|^2 ds\right).
\end{aligned}$$

ملـخص

في هذا العمل قمنا بدراسة وجود وتفرد حلول المعادلة التفاضلية العشوائية التراجعية المعممة الموجهة بالحركة البراونية كسرية باستخدام نظرية النقطة الصامدة، معظم الصعوبات التقنية القادمة من الحركة البراونية الكسرية، نظرًا لأن B^H في حالة $H > \frac{1}{2}$ليست شبه مارتينجال، ولهذا لا يمكننا استخدام النظرية الكلاسيكية لحساب التفاضل والتكامل العشوائي.

الـكـلـمـات الـمـفـتـاحـيـة :

معادلة التفاضلية العشوائية التراجعية، الحركة البراونية الكسرية، معادلة التفاضلية العشوائية التراجعية المعممة، نظرية النقطة الصامدة.

Abstract

In this work, we studied the existence and uniqueness of solutions of the generalized backward stochastic differential equation with fractional Brownian motion, using fixed-point theory. The pretty much of the technical difficulties coming from the fractional brownien motion, since $B^H$with $H > \frac{1}{2}$ is not a semi martingale; we cannot use the classical theory of stochastic calculus.

Keywords:

Backward stochastic differential equation, fractional Brownian motion, generalized backward stochastic differential equation, fixed-point theory.

Résumé :

Dans ce travail, nous avons étudié l'existence et l'unicité des solutions de l'équation différentielle stochastique rétrograde généralisée avec mouvement Brownien fractionnaire, en utilisant la théorie du point fixe. La majeure partie des difficultés techniques proviennent du mouvement Brownien fractionnaire, puisque B^H avec $\mathrm{H} > \frac{1}{2}$ $\mathrm{n'est\ pas\ une\ semi-martingale,}$ nous ne pouvons pas utiliser la théorie classique du calcul stochastique.

Mots clés :

Équation différentielle stochastique rétrograde, mouvement Brownien fractionnaire, équation différentielle stochastique rétrograde généralisée, théorie du point fixe.