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The general decay of the wave equation solution with density and memory term in R^n

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Dedications

To my dear Parents and my brothers , who have helped me a lot with their patience and their prayers...my dedications to my dear educators from the university... To my colleagues in the Numerical Analysis... To all my colleagues in the Math department... To all who love me and who I love. Thank you all.

NAHLA

Acknowledgement

Above all, we thank **ALLAH** , the Allmighty ,for giving us the courage and will to accomplish this work.

Our sincere gratitude to all those who helped us directly or indirectly to carry out this work all of my **family** and **friends** .

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Abstract

The study's goal is to improve comprehension of viscoelastic wave equations by investigating the rate at which solutions decay in weighted spaces. We will achieve this by incorporating density and memory terms to address the absence of Poincare's inequality. Ultimately, the memory will make a valuable contribution to the existing literature on wave equations and their decay properties.

Keywords : Lyapunov function, Relaxation function, Density, Decay rate, Weighted spaces, Viscoelastic, Kirchhoff type.

Résumé

L'objectif de l'étude est d'améliorer la compréhension des équations d'ondes viscoélastiques en étudiant la vitesse à laquelle les solutions se désintégrent dans les espaces pondérés. Nous y parviendrons en incorporant des termes de densité et de mémoire pour remédier à l'absence d'inégalité de poincaré. En fin de compte, ce mémoire apportera une contribution précieuse à la littérature existante sur les équations des ondes et leurs propriétés de désintégration.

Mots-clés: Fonction Lyapunov, Détente, Densité, Taux de décomposition, Espaces pondérés, Viscoélastique, type Kirchhoff.

ملخص:

يهدف البحث إلى تحديد خصائص معدل الانحلال ودراسة اضمحلال حلول معادلات الموجات اللزجة المرنة, و تقديم مساحات موزونة للحلول باستخدام الكثافة. في الفصل الأول, نقدم بعض التعريفات الأساسية و المبرهنات و النظريات التي سنحتاجها في العمل. أما في الفصل الثاني و الثالث, ننشأ تكافئ وظيفي بين لابونوف و الطاقة و ذالك لتحليل سلوك الانحلال.

الكلمات المفتاحية: دالة لابونوف، دالة الاسترخاء، الكثافة، معدل الاضمحلال، المساحات الموزونة، اللزوجة المرنة، نوع كيرشوف.

Intoduction

Partial differential equations with time t as one of the independent variables are examples of nonlinear evolution equations. These equations are not only found in a wide variety of mathematical subjects but also in other scientific fields such as physics, mechanics, and material science. Among the notable examples of nonlinear evolution equations are the Navier-Stokes and Euler equations in fluid mechanics, the nonlinear reaction-diffusion equations in heat transfers and biological sciences, the nonlinear Klein-Gorden equations and the nonlinear Schrodinger equations in quantum mechanics, and the Cahn-Hilliard equations in material science. These are just a few examples. Many mathematicians and scientists working in the nonlinear sciences have shown a great deal of interest in the nonlinear evolution equations due to their complexity and the difficulties that arise when studying them theoretically.

When seen from a physical vantage point, problems of this kind often manifest themselves in viscoelasticity characteristics. Dafermos [15], who did so in 1970, was the first individual to investigate difficulties of this kind. Dafermos addressed the overall deterioration in his work. Since then, researchers have focused heavily on this topic, publishing numerous discoveries about the existence of solutions and their long-term behavior. This information may be obtained from the following sources: eight, six, twelve, eleven, and twenty-two. Due to the fact that these materials have a wide variety of applications in the natural sciences, the dynamics of these materials are not only intriguing but also of high importance. As a consequence of this, issues concerning the operations of the solutions for the partial differential equations (PDE) system have attracted a considerable amount of attention throughout the course of the two most recent years. I would like to use this opportunity to bring to your attention some previous discoveries about the viscoelastic wave equation. An example of this would be the investigation that Cavalcanti et al. [[12], [13]] conducted into the problem of the form in order to lay the groundwork for our present research.

The primary of the asymptotic behavior of the solutions as time progresses is the primary objective that we have planned. In fact, we show that, depending on the parameters in the systems and the starting data size, we may either prove the solutions to be global in time or explode in limited time (that is, some norms of the solution will become unbounded). This is something that we demonstrate.

The issue that naturally arises in the event that the solutions are global in time is about the pace at which they converge to the steady state and the convergence of the solutions. Examining how solutions to nonlinear evolution equations change over time is a big part of the research that looks at how partial differential equations and physics interact with each other. Included in this category are theories that explain the dynamics of gases, quantum theory, and thermoelasticity.

The main aim of this research is to study some hyperbolic systems with the presence of different mechanisms of damping and under assumptions on initial data and boundary conditions. Our major objective is to study the asymptotic behavior of solutions when the time evolves. In fact, we prove that under some assumptions on the parameters in the systems and on the size of the initial data, the solutions can be proved to be either global in time or may blow up in finite time (i.e some norms of the solution will be unbounded in finite time). If the solutions are global in time, then the natural question is about their convergence to the steady state and the rate of convergence. The study of the asymptotic behavior of solutions of nonlinear evolution equations, particularly those governing gas dynamics, quantum theory and thermoelasticity, has been an important area for the interaction between the partial differential equations and physics.

Contents

Notations

- Ω : bounded domain in $\mathbb{R}^n$.
- Γ : topological bouundary of Ω.
- $x = (x_1, x_2)$: generic point of $\mathbb{R}^n$.
- $dx = dx_1 dx_2$: Lebesgue measuring on Ω.
- ∇u : gradient of u.
- Δu : Laplacien of u.
- $D(\Omega)$: space of differntiable functions with compact support in Ω.
- $D'(\Omega)$: distribution space.
- $C^k(\Omega)$: space of functions k-times continuously differentiable in Ω.
- $L^p(\Omega)$: space of functions p-th power integrated on with measure of dx.
- $W^{1,p}(\Omega) = \{u \in L^p(\Omega), \nabla u \in L^p(\Omega)\}$.
- H: Hilbert space.
- $H_0^1(\Omega) = W_0^{1,2}$.

- If X is a Banach space:

  $L^p(0,T;X) = \{f : (0,T) \longrightarrow X \text{is measurable}; \int_0^t \|f(t)\|_X^p dt < \infty\}$.

  $L^\infty(0,T;X) = \{f : (0,T) \longrightarrow X \text{is measurable}; ess - \sup_{t\in[0,T]} \|f(t)\|_X^p < \infty$.

  $C^k([0,T];X)$:Space of functions $k - times$ continuously differentiable from $[0;T] \longrightarrow X$.

  $D([0,T];X)$: space of functions continuously differentiable with compact support $in[0;T]$.

Chapter 1

Preliminaries

In this chapter, present the elemantary symbols, definitions and provide many tools on the basic concepts of inequalities and spaces, we will use later.

1.1 Functional spaces

1.1.1 Lebesgue spaces

Definition 1.1

[4] Let Ω be a domain in $\mathbb{R}^n (n \in \mathbb{N})$, for $1 \leqslant p < \infty$, the Lebesgue space $L^p(\Omega)$ is defined by :

$L^p(\Omega) = \{u : \Omega \longrightarrow \mathbb{R},\ u \text{ is measurable and } \int_\Omega |u(x)|^p dx < \infty\}$,

with the norm

$$\|u\|_p = (\int_\Omega |u(x)|^p dx)^{\frac{1}{p}}$$

In addition, we define $L^\infty(\Omega)$ by:

$L^\infty(\Omega) = \{u : \Omega \longrightarrow \mathbb{R},\ u \text{ is measurable and } \exists c > 0 \text{ such that } |u(x)| \leqslant c\ a.e\ on\ \Omega\}$,

equipped with the norm
$\|u\|_\infty = ess \sup_{x \in \Omega} |u(x)| = \inf\{c : |u(x)| \leqslant c\ a.e\ on\ \Omega\}$

1.1.2 Hilbert spaces

Definition 1.2

An inner product on *a* complex linear space X is a map

$$(.,.) : X \times X \longrightarrow \mathbb{C}.$$

such that, for all $x, y, z \in X$ and $\lambda, \mu \in \mathbb{C}$: $(x, \lambda y + \mu z) = \lambda(x, y) + \mu(x, z)$ (linear in the second argument):

1.$(y, x) = \overline{(x, y)}$ (symmetrical);

2.$(x, x) \geqslant 0$ (Positive);

3.$(x, x) = 0$ if and only if $x = 0$ (positive definite).

We call a linear space with an inner product a pre-Hilbert space.

If X is a linear space with an inner product (.,.), then we can define an norm in X by :

$$\|x\| = \sqrt{(x, x)}. \tag{1.1}$$

Definition 1.3

A Hilbert space is a complete inner product space.

Example 1.1

The stander inner product on $\mathbb{C}^n$ is given by

$$(x, y) = \sum_{j=1}^{n} x_j \bar{y}_j. \tag{1.2}$$

where $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, with $x_j, y_j \in \mathbb{C}$.

Example 1.2

Let $C([a, b])$ denote the space of all complex-*valued* continuous functions defined on the interval $[a, b]$. We define an inner product on $C([a, b])$ by

$$(f, g) = \int_a^b f(x)\overline{g(x)}dx, \tag{1.3}$$

Where $f, g : [a, b] \longrightarrow \mathbb{C}$ are continuous functions.

Example 1.3

let $u, v \in L^2(\Omega)$ the inner product is defined by

$$(u, v) = \int_\Omega u\bar{v}d\Omega, \tag{1.4}$$

with respect to the associated norm,

$$\|u\|_2 = \left(\int_\Omega |u(x)|d\Omega\right)^{\frac{1}{2}}. \tag{1.5}$$

Remark 1.1

The spaces $L^p([a, b])$ are Banach spaces but they are not Hilbert spaces when $p \neq 2$.

Theorem 1.1 (Lax-Milgram)

[4] Assume that $a(u, v)$ is a continuous coercive bilinear form on H. Then, given an $y\phi \in H'$ there exists a unique element $u \in H$ such that

$$a(u, v) =< \phi, v >, \forall v \in H.$$

Moreover, if a symmetric, then u is characterized by the property

$$\frac{1}{2}a(u, v)- < \phi, U >= \min_{v \in H}\{\frac{1}{2}a(u, v)- < \phi, U >\}.$$

1.1.3 Sobolev spaces

Difinition 1.4

[5]For $k \in \mathbb{N}$ and $1 \leqslant p \leqslant \infty$. We define the *Sobolev* space

$$W^{p,k}(\Omega) = \{u \in L^p(\Omega), D^\alpha u \in L^p(\Omega) \forall \alpha \in \mathbb{N}^n with |\alpha| \leqslant k, \}$$

equipped with the norm

$$\|u\|_{k,p} = \left(\sum_{|\alpha| \leqslant k} \|D^\alpha u\|_p^p \right)^{\frac{1}{p}}, 1 \leqslant p < \infty$$

$$\|u\|_{k,\infty} = \max_{|\alpha| \leqslant k} \|D^\alpha u\|_\infty,$$

where $D^\alpha u$ is the α-th weak derivative of u which is defined as

$$\int_\Omega u(x) D^\alpha \varphi(x) = -1^{|\alpha|} \int_\Omega v(x)\varphi(x), \forall \varphi \in C_c^\infty(\Omega),$$

$$|\alpha| = \alpha_1 + ... + \alpha_n,$$

and

$$v = D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial_{x_1^{\alpha 1}} .....\partial_{x_n^{\alpha n}}}.$$

The space $W^{k,2}(\Omega)$ is denoted by $H^k(\Omega)$, which is a Hilbert space with respect to the inner product

$$(u, v)_{H^k} = \int_\Omega \sum_{\alpha \leqslant k} D^\alpha u(x) D^\alpha v(x) dx, \forall u, v \in H^k(\Omega).$$

Definition 1.5

[5] We denote by $W_0^{k,p}(\Omega)$ the closure of $C_c^\propto(\Omega)$ in $W^{k,p}(\Omega)$.

1.2 Some inequalities

Theorem 1.2(Cauchy-Schwarz inequality)

Let $u, v \in L^2(\Omega)$ and $v \in L^2(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\|uv\|_1 \leqslant \|u\|_2 \|v\|_2$$

Theorem 1.3(Hölder's inequality)

Let $1 \leqslant p \leqslant \infty$, if $u \in L^p(\Omega)$ and $v \in L^{p'}(\Omega)$, then $uv \in L^1(\Omega)$ and

$$\|uv\|_1 \leqslant \|u\|_p \|v\|_{p'},$$

where $\dfrac{1}{p} + \dfrac{1}{p'} = 1$

Theorem 1.4(Young's inequality)

Let $1 \leq p \leq \infty$. Then $a, b > 0$, Then for any $\epsilon > 0$, we have

$$ab \leq \epsilon a^p + C_\epsilon b^{p'},$$

where $C_\epsilon = \dfrac{1}{p'(\epsilon p)^{\frac{p'}{p}}}$. For $p = p' = 2$, we have $ab \leqslant \epsilon a^2 + \dfrac{b^2}{4\epsilon}$.

1.2.1 Some results about Soboleve spaces

In this Section, we list a few pertinent qualities that Sobolev space-related functions benefit from without providing any supporting evidence.

Theorem 1.5 (Trace theorem [20])

Let Ω be a bounded open set of find with Lipschtiz continuous boundary and let $s > 1/2$.

1. There exists a unique linear continuous map $\gamma_0 : H^s(\Omega) \longrightarrow H^{s-1}(\partial\Omega)$ such that $\gamma_0 v = v|_{\partial\Omega}$ for each $v \in H^s(\Omega) \bigcap C^0(\bar{\Omega})$.

2.There exists a linear continuous map $R_0 : H^{s-1}(\partial\Omega) \longrightarrow H^s(\Omega)$ such that $\gamma_0 r_0 \phi = \phi$ for each $\phi \in H^{s-1}(\partial\Omega)$. Analogous results also hold true if we consider the trace γ_Σ over a Lipshitz continuous subset Σ of the boundary $\partial\Omega$.

The so-called Poincare inequality is a crucial finding that will be widely applied in the sequel.

Therorem 1.6 (Poincaré inequality [1])

Assume that Ω is a bounded connected open set of $\mathbb{R}^d$ and that Σ is a (non-empty) Lipschitz continuous subset of the boundary $\partial\Omega$.Then there exists a constant $C_\Omega > 0$ such that

$$\int_\Omega v^2(X)dX \leqslant C_\Omega \int_\Omega |\nabla v(X)|^2 dX, \tag{1.6}$$

for each $v \in H^1_\Sigma(\Omega)$

Lemma 1.1(Sobolev-Poincare inequality)

Let q be a number with

$$2 \leqslant q < \infty, (n = 1, 2), 2 \leqslant q \leqslant \frac{2n}{n-2}(n \geqslant 3),$$

then there exists a constant $C_s = C_s(\Omega, q)$ such that

$$\|u\|_q \leqslant c\|\nabla u\|_2 for u \in H^1_0(\Omega). \tag{1.7}$$

Theorem 1.7(Sobolev embedding theorem [1])

Assume that Ω is a (bounded or unbounded) open set of $\mathbb{R}^d$ with a Lipschitz continuous boundary, and that. Then the following continuous embeddings hold :

1. If $1 \leqslant p < d$, then $W^{s,p}(\Omega) \subset L^{p*}(\Omega)$ for $p* = dp/(d - sp)$.

2. If $sp = d$, then $W^{s,p}(\Omega) \subset L^q(\Omega)$ for ay q such that $p \leqslant q < \infty$.

3. If $sp > d$, then $W^{s,p}(\Omega) \subset C^0\bar{\Omega}$.

Lemma 1.2(Korn's inequality)

Let Ω be an open, connected domain in n-dimensional Euclidean space R^n ,$n \geqslant 2$. Let $H^1(\Omega)$ be the Sobolev spaces of all vector filed $v = (v^1, ..., v^n)$ on Ω that, along with their (first) weak derivatives, lie in the Lebesgue spaces $L^1(\Omega)$.

Denoting the partial derivative with respect to the ith component by ∂_i, the norm in $H^1(\Omega)$ is given by

$$\|v\|_{H^1(\Omega)} = (\int_\Omega \sum_{i=1}^n |v^i(x)|^2 dx + \int_\Omega \sum_{i=1}^n |\partial_i v^i(x)|^2 dx)^{1/2}$$

Then there is a constant $C \geqslant 0$,Known as the Korn constant of Ω, such that, for all $v \in H^1(\Omega)$.

$$\|v\|_{H^1}^2 \leqslant C \int_\Omega \sum_{i,j=1}^n (|v^i(x)|^2 + |(e_{ij}v)(x)|^2) dx$$

where e denotes the symmetrized gradient given by

$$e_{ij}v = \frac{1}{2}(\partial_i v^j + \partial_j v^i)$$

1.2.2 Green's formula

Proposition 1.1

[6] Let Ω be an open subset of $\mathbb{R}^d$, with a Lipschitz boundary.Then for all $u, v \in H^1(\Omega)$, we have

$$\int_\Omega (\frac{\partial u}{\partial x_i} v + \frac{\partial v}{\partial x_i} u) dx = \int_{\partial\Omega} \gamma_0(u)\gamma_0(v)\eta_i ds, i = 1, ..., d.$$

Where η_i the i-th component of the outward normal vector η.

1.3 Logarithmic Hölder Conttinuty

In this section we introduce the most important condition on the exponent in the study of variable exponent spaces, the log-Hölder continuity condition.

Definition 1.6([8],page 100)

We say that function $\alpha : \Omega \longrightarrow \mathbb{R}$ is locally log-Hölder continuous on Ω if there exists $c_1 > 0$ such that

$$|\alpha(x) - \alpha(y)| \leq \frac{c_1}{\log(e + 1/|x+y|)} \tag{1.8}$$

for all $x, y \in \Omega$ we say that α satisfies the log-Hölder decay condition if there exist $\alpha_\infty \in \mathbb{R}$ and constant $c_2 > 0$ such that

$$|\alpha(x) - \alpha_\infty| \leq \frac{c_2}{\log(e/|x|)}$$

for all $x \in \Omega$ we say that α is globally log-Hölder continuous in Ω if it is locally log-Hölder continuous and satisfies the log-Hölderdecay condition.

The constant c_1 and c_2 are called the local log-Hölder constant and the log-Hölder decay constant, respectively. The maximum $\max c_1, c_2$ is just called the log-Hölder constant of α.

1.3.1 $L^{p(.)}, W^{1,p(.)}$ spaces

We define the spaces

$$C^+(\bar{\Omega}) = \{ \text{ continuous function } p(.) : \bar{\Omega} \longrightarrow \mathbb{R}_+ \text{ such that } 2 < p^- < p^+ < \infty\}$$

where,

$$p^- = \min_{x\in\bar{\Omega}} p(x) \text{ and } p^+ = \max_{x\in\bar{\Omega}} p(x)$$

We define the Lebesgue space with variable exponent

$$L^{p(.)} = \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \int_\Omega |u(x)^{p(x)} dx\}$$

endowed with Luxembourg norm :

$$\|u\|_{p(.)} = \|u\|_{L^{p(.)}} = \inf\{\epsilon > 0, \int_\Omega |\frac{u(x)}{\epsilon}|^{p(x)} dx \leq 1\}$$

. The space $(L^{p(.)}, \|.\|_{p(.)})$ is a reflexive Banach space, uniformly convex and its dual space is isomorphic to $(L^{p(.)}(\Omega), \|.|_{q(x)})$ where

$$\frac{1}{p(x)} + \frac{1}{q(x)} = 1,$$

and

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega), |\nabla u| \in L^{p(x)}(\Omega)\}$$

With the norm

$$\|u\| = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}, u \in W^{1,p(x)}(\Omega)$$

. **Remark 1.2** We denote by $W_0^{1,p(x)}(\Omega)$ the closure of C_0^{∞} in $W^{1,p(x)}(\Omega)$.

1.3.2 $L^p(0, T; X)$ spaces

Let x be a banach space , denote by $L^p(0, T; X)$ the space of measurable functions

$$f :]0, T[\longrightarrow X$$

$$t \longrightarrow f(t),$$

such that

$$\int_0^T (\|f(t)\|_X^p)^{\frac{1}{p}} dt = \|f\|_{L^p(0,T,X)} < \infty.$$

If $p = \infty$

$$\|f\|_{L^\infty(0,T,X)} = \sup_{t \in]0,T[} ess \|f(t)\|_X.$$

Theorem 1.8 The space $L^p(0, T; X)$ is a banach space.

Lemma 1.3

Let $f \in L^p(0, T; X)$ and $\frac{\partial f}{\partial t} \in L^p(0, T; X)$, $(1 \leqslant p \leqslant \infty)$, then, the function f is continuous from $[0, T]$ to X . i.e. $f \in C^1(0, T, X)$.

1.4 Results in spaces with exponents variables

Proposition 1.2 (see , [24],[25])

Let $u_n, u \in L^{p(x)}(\Omega)$ and $p^+ < +\infty$, then

1)$\|u\|_{L^{p(x)}}(\Omega) < 1 \Longleftrightarrow \int_\Omega |u|^{p(x)} dx < 1;$

2)$\|u\|_{L^{p(x)}}(\Omega) > 1 \Longrightarrow \|u\|^{p^-}_{L^{p(x)}(\Omega)} \leqslant \int_\Omega |u|^{p(x)} dx \leqslant \|u\|^{p^+}_{L^{p(x)}(\Omega)};$

3)$\|u\|_{L^{p(x)}}(\Omega) < 1 \Longrightarrow \|u\|^{p^+}_{L^{p(x)}(\Omega)} \leqslant \int_\Omega |u|^{p(x)} dx \leqslant \|u\|^{p^-}_{L^{p(x)}(\Omega)};$

4)$\|u\|_{L^{p(x)}}(\Omega) \longrightarrow 0 \Longleftrightarrow \int_\Omega |u|^{p(x)}dx \longrightarrow 0.$

Lemma 1.4 (Poincare inequality [24],[25])

Let Ω be a bounded domain of $\mathbb{R}^n$ an suppose that $p(.)$ satisfies (1.8). Then,

$$\|u\|_{p(.)} \leqslant c(\Omega)\|\nabla u\|_{p(.)}, \forall u \in W_0^{1,p(.)}(\Omega), \tag{1.9}$$

where $c = c(p_1, p_2, |\Omega|) > 0$.

Next we have a Sobolev-Poincare inequality

Lemma 1.5 (Generalized Hölder inequality [24],[25])

For any function $u \in L^{p(x)}(\Omega)$ and $v \in L^{q(x)}(\Omega)$, we have

$$|\int_\Omega u(x)v(x)dx| \leqslant (\frac{1}{p^-} + \frac{1}{q^-})\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{q(x)}(\Omega)} \leqslant 2\|u\|_{L^{p(x)}(\Omega)}\|v\|_{L^{q(x)}(\Omega)}, \tag{1.10}$$

where

$$q(x) = \frac{p(x)}{p(x)-1}.$$

Lemma 1.6

If $p : \bar{\Omega} \longrightarrow [1, \infty)$ is continuous,

$$2 \leqslant p_1 \leqslant p(x) \leqslant p_2 \leqslant \frac{2n}{n-2}, n \geqslant 3, \tag{1.11}$$

satisfies, thenthe embedding $H_0^1(\Omega) \hookrightarrow L^{p(;)}(\Omega)$ is continuous.

Lemma 1.7(see [21])

If $p_2 < \infty$ and $p : \bar{\Omega} \longrightarrow [1, \infty)$ is a measurable function, then $C_0^\infty(\Omega)$ is dense in $L^{p(\cdot)}(\Omega)$.

Lemma 1.8([20] Hölder inequality)

Let $p, q, s \geqslant 1$ be measurable functions defined on Ω, and

$$\frac{1}{s(y)} = \frac{1}{p(y)} + \frac{1}{q(y)} for a.e y \in \Omega,$$

satisfies . If $f \in L^{P(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$ and

$$\|f.g\|_{s(.)} \leq \|f\|_{p(.)}\|g\|_{q(.)}.$$

Lemma 1.9(see [20])

If $p \geqslant 1$ is a measurable function on Ω, then

$$\min\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\} \leq \rho_{p(.)}(u) \leq \max\{\|u\|_{p(.)}^{p_1}, \|u\|_{p(.)}^{p_2}\},$$

for any $u \in L^{p(.)}(\Omega)$ and for a.e. $x \in \Omega$.

Lemma 1.10(see,[20]Gronwall inequality)

Let $C > 0, u(t)$ and $y(t)$ be continuous nonnegative function defind for $0 \leq t <\propto$ satisfying the inequality

$$u(t) \leq C + \int_0^t u(s)y(s)ds, 0 \leq t < \infty.$$

show that

$$u(t) \leq Cexp(\int_0^t y(s)ds), 0 \leq t < \infty.$$

Lemma 1.11 (Modified Gronwall inequality)

Let u and h be continuous nonnegative function defind for $0 \leq t < \infty$ satisfying the inequality

$$0 \leq u(t) \leq C + \int_0^t u(s)h(s)ds, 0 \leq t \leq < \infty$$

.

with $C > 0$

$$u(t) \leq (C^{-r} - r\int_0^t h(s)ds)^{-1/r}, 0 \leq t < \infty$$

.

as long as the right-hand side exists .

Chapter 2

Decay of a solution of the wave equation with density and memory term in R^n

2.1 Preliminaries and position of the problem

Let us consider the following problem

$$\begin{cases} \rho(x)(|u^{'}|^{q-2}u^{'})^{'} - \Delta_x u + \int_0^t g(t-s)\Delta_x u(s,x)ds = 0, \quad x \in \mathbb{R}^n, t > 0 \\ u(0,x) = u_0(x) \in \mathcal{H}(\mathbb{R}^n), \\ u^{'}(0,x) = u_1(x) \in L^q_\rho(\mathbb{R}^n), \end{cases} \tag{p}$$

In this paper we are going to consider the solutions in spaces weighted by the density ρ in order to compensate for the lack of Poincare's inequality.
In this framework, (see [10](Theorem 2.4),[11] (Proposition 2.1)), it is well known that, for any initial data $u_0 \in \mathcal{H}(\mathbb{R}^n), u_1 \in L^q_\rho(\mathbb{R}^n)$, then problem (P) has a unique solution $u \in C([0,T), \mathcal{H}(\mathbb{R}^n)), u^{'} \in C([0,T), L^q_\rho(\mathbb{R}^n))$ for T small enough, under hypothesis (A1)-(A4).

The energy of u at time t is defined by

$$2E(t) = \frac{2(q-1)}{q}\|u^{'}\|^q_{L^q_\rho(\mathbb{R}^\aleph)} + (1 - \int_0^t g(s)ds)\|\nabla_x u\|_2^2 + (g \circ \nabla_x u). \tag{2.1}$$

and the following energy functional law holds, which means that, our energy is uniformly bounded and decreasing along the trajectories.

$$2E^{'}(t) = (g^{'} \circ \nabla_x u)(t) - g(t)\|\nabla_x u(t)\|_2^2 < 0, \quad \forall t \geqslant 0. \tag{2.2}$$

The following notation will be used throughout this paper

$$(\Phi^s \circ \Psi)(t) = \int_0^t \Phi^s(t-\tau)\|\Psi(t) - \Psi(\tau)\|_2^2 d\tau. \tag{2.3}$$

The problem (P) for the case $q = 2, \rho(x) = 1$, in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive non-increasing function was considered in [11], where they established an explicit and general decay rate result for relaxation function satisfying :

$$g'(t) \leq -H(g(t)), t \geq 0, H(0) = 0, \tag{2.4}$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$,to improve conditions considered recently by Alabau-Boussouira and Cannarsa [3] on the relaxation function

$$g'(t) \leq -\mathcal{X}(g(t)), \mathcal{X}(0) = \mathcal{X}'(0) = 0, \tag{2.5}$$

where $\mathcal{X}$ is a non-negative function, strictly increasing and strictly convex on $(0, k_0], k_0 > 0$. They required that

$$\int_0^{k_0} \frac{dx}{\mathcal{X}(x)} = +\infty, \int_0^{k_0} \frac{xdx}{\mathcal{X}(x)} < 1, \lim_{s \longrightarrow 0^+} \inf \frac{\mathcal{X}(s)/s}{\mathcal{X}'(s)} > \frac{1}{2} \tag{2.6}$$

and proved a decay result for the energy of (P) with $q = 2, \rho(x) = 1$ in a bounded domain. In addition to these assumptions, if

$$\lim_{s \longrightarrow 0^+} \sup \frac{\mathcal{X}(s)/s}{\mathcal{X}'(s)} < 1, \tag{2.7}$$

then an explicit rate of decay was given.
We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$ when no confusion arises. We denote by
$|\nabla_x u|^2 = \sum_{i=1}^n (\frac{\partial u}{\partial x_i})^2, \Delta_x u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$. The constants c used throughout this paper are positive generic constants which may be different in various occurrences, also the functions considered are all real valued, here $u' = du(t)/dt$ and $u" = d^2u(t)/dt^2$.

Our goal : The main purpose of this work is to allow a wider class of relaxation functions and improve earlier results in the literature. The basic mechanism behind the decay rates is the relation between the damping and the energy .
First we recall and make use the following assumption on the functions ρ and g as :

A1: To guarantee the hyperbolicity of the system, we assume that the function $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying :

$$1 - \bar{g} = l > 0; g(0) = g_0 > 0, \tag{2.8}$$

where $\bar{g} = \int_0^\infty g(t)dt$.

A2: There exists a positive function $H \in C^1(\mathbb{R}^+)$ such that

$$g'(t) + H(g(t)) \leq 0, t \geq 0, H(0) = 0. \tag{2.9}$$

A3: H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$.

Remark 1.1[11]

A- We can deduce that there exists $t_1 > 0$ large enough such that :

1)$\forall t \geq t_1$: We have $\lim_{s \longrightarrow +\infty} g(s) = 0$, which implies that $\lim_{s \longrightarrow +\infty} -g'(s)$ cannot be a positive, so $\lim_{s \longrightarrow +\infty} -g'(s) = 0$. Then $g(t_1) > 0$ and

$$\max\{g(s), -g'(s)\} < \min\{r, H(r), H_0(r)\}. \tag{2.10}$$

where $H_0(t) = H(D(t))$ provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex C^2 function on $(0, r]$ and

$$\int_0^{+\infty} g(s)H_0(-g'(s))ds < +\infty.$$

2)$\forall t \in [0, t_1]$:As g is nonincreasing, $g(0) > 0$ and $g(t_1) > 0$ then $g(t) > 0$ and

$$g(0) \geq g(t) \geq g(t_1) > 0.$$

Therefore, since H is a positive continuous function, then

$$a \leq H(g(t)) \leq b,$$

for some positive constants a and b. Consequently,

$$g'(t) \leq -H(g(t)) \leq -kg(t), k > 0,$$

then

$$g'(t) \leq -kg(t), k > 0. \tag{2.11}$$

B-Let H_0^*be the convex conjugate of H_0 in the sense of Young (see [15], page 61-64), then

$$H_0^*(s) = s(H_0^{'})^{-1}(s) - H_0[(H_0^{'})^{-1}(s)], s \in (0, H_0^{'}(r)),$$

and satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), A \in (0, H_0^{'}(r)), B \in (0, r]. \tag{2.12}$$

A4: The function $\rho : \mathbb{R}^n \longrightarrow \mathbb{R}_+^n, \rho(x) \in C^{0,\gamma}(\mathbb{R}^n)$ with $\gamma \in (0, 1)$ and $\rho \in L^s(\mathbb{R}^n) \bigcap L^\infty(\mathbb{R}^n)$, where $s = \dfrac{2n}{2n - qn + 2q}$.

Definition 1.2 ([10],[18]). We definit the function spaces of our problem and its norm as follows :

$$\mathcal{H}(\mathbb{R}^n) = \left\{ f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))^n \right\}. \tag{2.13}$$

and the space $L_\rho^2(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ function with respect to the inner product

$$(f, h)_{L_\rho^2(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < p < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{2.14}$$

The space$L_2^\rho(\mathbb{R}^n)$ is a separable Hilbert space.
The following technical lemma will play an important role in the sequel.

Lemma 1.3 [14] (Lemma 1.1) For any two functions $g, v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have

$$\begin{aligned} v'(t) \int_0^t g(t-s)v(s)ds = & -\frac{1}{2}\frac{d}{dt}\int_0^t g(t-s)|v(t)-v(s)|^2 ds \\ & + \frac{1}{2}\frac{d}{dt}\Big(\int_0^t g(s)ds \Big)|v(t)|^2 + \frac{1}{2}\int_0^t g'(t-s)|v(t)-v(s)|^2 ds \\ & - \frac{1}{2}g(t)|v(t)|^2. \end{aligned} \tag{2.15}$$

and

$$|\int_0^t g(t-s)(v(t)-v(s))ds|^2 \le \Big(\int_0^t |g(s)|^{2(1-\theta)} ds \Big)\Big(\int_0^t |g(t-s)|^{2\theta}|v(t)-v(s)|^2 ds \Big). \tag{2.16}$$

we are now ready to state and prove our main results

2.2 Decay rate results

Lemma2.1[17]
Let ρ satisfies (A4),then for any $u \in \mathcal{H}(\mathbb{R}^n)$

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \le \|\rho\|_{L^s(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)} \tag{2.17}$$

with $s = \dfrac{2n}{2n - qn + 2q}, 2 \leq q \leq \dfrac{2n}{n-2}$.

For the purpose of constructing a Lyapunov function L equivalent to E, we introduce the next functionals

$$\psi_1(t) = \int_{\mathbb{R}^n} \rho(x) u|u'|^{q-2}u' dx. \tag{2.18}$$

$$\psi_2(t) = -\int_{\mathbb{R}^n} \rho(x) u|u'|^{q-2}u' \int_0^t g(t-s)(u(t)-u(s))dsdx. \tag{2.19}$$

Lemma2.2 Under the assumption (A1)-(A4), the functional ψ_1 satisfies, along the solution of (p)

$$\psi_1'(t) \leq \|u'\|^q_{L^p_\rho(\mathbb{R}^n)} + (\sigma - l)\|\nabla_x u\|_2^2 + \frac{(1-l)}{4\sigma}(g \circ \nabla_x u) \tag{2.20}$$

proof.

From Eq. (2.18), integrate by parts over $\mathbb{R}^n$, we have

$$\psi_1'(t) = \int_{\mathbb{R}^n} \rho(x) u' q dx + \int_{\mathbb{R}^n} \rho(x) u(|u'|^{q-2}u')' dx$$
$$= \int_{\mathbb{R}^n} \left(\rho(x) u'^q + u\Delta_x u - u \int_0^t g(t-s)\Delta_x u(s,x) ds \right) dx$$
$$\leq \|u'\|^q_{L^p_\rho(\mathbb{R}^n)} - l\|\nabla_x u\|_2^2 + \int_{\mathbb{R}^n} \nabla_x u int_0^t g(t-s)(\nabla_x u(s) - \nabla_x u(t))dsdx.$$

Usinge Young's inequality and Lemma 1.3, we obtain

$$\begin{aligned}
\psi_1'(t) &\leq \|u'\|^q_{L^p_\rho(\mathbb{R}^n)} - l\|\nabla_x u\|_2^2 \\
&+ \sigma\|\nabla_x u\|_2^2 + \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)|\nabla_x u(s) - \nabla_x u(t)|ds\right)^2 dx \\
&\leq \|u'\|^q_{L^q_\sigma(\mathbb{R}^n)} + (\sigma - l)\|\nabla_x u\|_2^2 + \frac{(1-l)}{4\sigma}(g \circ \nabla_x u).
\end{aligned} \tag{2.21}$$

■

Lemma 2.3 Under the assumption (A1)-(A4), the functional ψ_2 satisfies ,along the solution of (P), for any $\sigma \in (0,1)$

$$\begin{aligned}
\psi_2'(t) &\leq \left(\sigma - \int_0^t g(s)ds\right)\|u'\|^q_{L^q_\sigma(\mathbb{R}^n)} \\
&+ \sigma\|\nabla_x u\|_2^2 + \frac{c}{\rho}(g \circ \nabla_x u) - c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}(g' \circ \nabla_x u)^{q/2}
\end{aligned} \tag{2.22}$$

proof.

Exploiting in (P), to get

$$\begin{aligned}
\psi_2'(t) &= -\int_{\mathbb{R}^n} \rho(x)(|u'|^{q-2}u'' \int_0^t g(t-s)(u(t)-u(s))dsdx \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&= \int_{\mathbb{R}^n} \nabla_x u \int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))dsdx \\
&- \int_{\mathbb{R}^n} \left(\int_0^t g(t-s)\nabla_x u(s,x)ds\right)\left(\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))ds\right)dx \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&= \left(1-\int_0^t g(s)ds\right)\int_{\mathbb{R}^n} \nabla_x u \int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))dsdx \\
&+ \int_{\mathbb{R}^n} \left(\int_0^t g(t-s)(\nabla_x u(t)-\nabla_x u(s))ds\right)^2 dx \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}
\end{aligned}$$

By Holder's and Young's inequalities and Lemma 2.1 we estimate

$$
\begin{aligned}
&-\int_{\mathbb{R}^n}\rho(x)|u'|^{q-2}u'\int_0^t g'(t-s)(u(t)-u(s))dsdx\\
&\leq \left(\int_{\mathbb{R}^n}\rho(x)|u'|^q dx\right)^{(q-1)/q}\times\\
&\left(\int_{\mathbb{R}^n}\rho(x)|\int_0^t g'(t-s)(u(t)-u(s))ds|^q\right)^{1/q}\\
&\leq \sigma\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}+c_\sigma\|\int_0^t g'(t-s)(u(t)-u(s))ds\|^q_{L^q_\rho(\mathbb{R}^n)}\\
&\leq \sigma\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}-c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}(g'\circ\nabla_x u)^{q/2}(t).
\end{aligned}
$$

Using Young's and Poincare's inequalities and lemma 1.3, we obtain

$$
\begin{aligned}
\psi_2'(t) &\leq \sigma\|\nabla_x u\|_2^2(g\circ\nabla_x u)+\frac{c}{\sigma}(g\circ\nabla_x u)-c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}(g'\circ\nabla_x u)^{q/2}\\
&+(\sigma-\int_0^t g(s)ds)\|u'\|^{q/2}_{L^q_\rho(\mathbb{R}^n)}.
\end{aligned}
$$

Our main result reads as follows.

■

Theorem 2.4 : Let $(u_0,u_1)\in\mathcal{H}(\mathbb{R}^n)\times L^q_\rho(\mathbb{R}^n)$ and suppose that $(A1)-(A4)$ hold. Then there exist positive constants $\alpha_0,\alpha_1,\alpha_2,\alpha_3$ such that the energy of solution given by (P) satisfies,

$$
E(t)\leq\alpha_3 H_1^{-1}(\alpha_1 t+\alpha_2),\forall t\geq 0,
$$

where

$$
H_1(t)=\int_t^1\frac{1}{sH_0'(\alpha_0 s)}ds. \tag{2.23}
$$

In order to prove this Theorem, let us define

$$L(t) = \xi_1 E(t) + \psi_1(t) + \xi_2 \psi_2(t), \tag{2.24}$$

for $\xi_1, \xi_2 > 1$ we need the next lemma, which means that there is equivalent between the Lyapunov and energy functions.

Lemma 2.5 . For $\xi_1, \xi_2 > 1$ we have

$$L(t) \sim E(t). \tag{2.25}$$

proof.
From Eq. (2.2), results of Lemma (2.2) and Lemma(2.3), we have

$$\begin{aligned} L'(t) &= \xi_1 E'(t) + \psi_1'(t) + \xi_2 \psi_2'(t) \\ &\leq (\frac{1}{2}\xi_1 - c_\sigma \|\rho\|^q_{L^s(\mathbb{R}^n)}\xi_2)(g' \circ \nabla_x u)^{q/2} + M_0(g \circ \nabla_x u) \\ &- M_1 \|u'\|^q_{L^q_{\rho(\mathbb{R}^n)}} - M_2 \|\nabla_x u\|^2_2, \end{aligned}$$

where

$$\begin{aligned} M_0 &= \left(\frac{4\xi_2 c + (1-l)}{4\sigma} \right), \\ M_1 &= \left(\xi_2 \left(\int_0^{t_1} g(s)ds - \sigma \right) - 1 \right), \\ M_2 &= \left(-\xi_2 \sigma + \frac{1}{2}\xi_1 g(t_1) + (l - \sigma) \right), \end{aligned}$$

and t_1 was introduced in Remark 1.1.
We choose σ so small that $\xi_1 > 2c_\sigma \|\rho\|^q_{L^s(\mathbb{R}^n)}\xi_2$. Whence σ is fixed , we can choose ξ_1, ξ_2 large enough so that $M_1, M_2 > 0$, which yields

$$L'(t) \leq M_0(g \circ \nabla_x u) - cE(t), \forall t \geq t_1. \tag{2.26}$$

On the other hand, by Eq. (2.24)we have

$$\begin{aligned}
|L(t) - \xi_1 E(t)| &\leq |\psi_1(t)| + \xi_2|\psi_2(t)| \\
&\leq \int_{\mathbb{R}^n} |\rho(x)u|u'|^{q-2}u'|dx \\
&+ \int_{\mathbb{R}^n} \left|\rho(x)|u'|^{q-2}u' \int_0^t g(t-s)(u(t)-u(s))ds\right|dx.
\end{aligned}$$

Thanks to Holder and Young's inequalities with exponents $\frac{q}{q-1}, q$, since $\frac{2n}{n+2} \geq q \geq 2$, we have by using Lemma 2.1

$$\begin{aligned}
\int_{\mathbb{R}^n} |\rho(x)u|u'|^{q-2}u'|dx &\leq \left(\int_{\mathbb{R}^n} \rho(x)|u|^q dx\right) + \left(\int_{\mathbb{R}^n} \rho(x)|u'|^q dx\right)^{(q-1)/q} \\
&\leq \frac{1}{q}\left(\int_{\mathbb{R}^n} \rho(x)|u|^q dx\right) + \frac{q-1}{q}\left(\int_{\mathbb{R}^n} \rho(x)|u'|^q dx\right) \\
&\leq c\|u'\|^q_{L^q_{\rho(\mathbb{R}^n)}} + c\|\rho\|^q_{L^s(\mathbb{R}^n)}\|\nabla_x u\|^q_2,
\end{aligned} \tag{2.27}$$

and

$$\int_{\mathbb{R}^n} \left| \left(\rho(x)^{\frac{q-1}{q}} |u'|^{q-2}u' \right) \left(\rho(x)^{\frac{1}{q}} \int_0^t g(t-s)(u(t)-u(s))ds \right) \right| dx$$
$$\leq \left(\int_{\mathbb{R}^n} \rho(x)|u'|^q dx \right)^{(q-1)/q} \times$$
$$\left(\int_{\mathbb{R}^n} \rho(x) | \int_0^t g(t-s)(u(t)-u(s))ds|^q \right)^{1/q}$$
$$\leq \frac{q-1}{q} \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q} \| \int_0^t g(t-s)(u(t)-u(s))ds\|^q_{L^q_\rho(\mathbb{R}^n)}$$
$$\leq \frac{q-1}{q} \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q} \|\rho\|^q_{L^s(\mathbb{R}^n)} (g \circ \nabla_x u)^{q/2}(t)$$

Then, since $q \geq 2$, we have

$$\begin{aligned} |L(t) - \xi_1 E(t)| &\leq c(E(t) + E^{q/2}(t)) \\ &\leq c(E(t) + E(t)E^{(q/2)-1}(t)) \\ &\leq c(E(t) + E(t)E^{(q-2)-1}(0)) \\ &\leq cE(t). \end{aligned}$$

Therefore, we can choose ξ_1 so that

$$L(t) \sim E(t). \tag{2.28}$$

■

proof.

(oftheorem 2.4)We set $F(t) = L(t) + cE(t)$, which is equivalent to $E(t)$. By Eq.(2.2), and Remark 1.1, we have $\forall t \geq t_1$

$$\int_{\mathbb{R}^n}\int_0^{t_1} g(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2 dsdx \leq -\frac{1}{k}\int_{\mathbb{R}^n}\int_0^{t_1} g'(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2 d$$
$$\leq -cE'(t), \tag{2.29}$$

then by Eq. (2.26), we get

$$\begin{aligned} F'(t) &= L'(t) + cE'(t) \\ &\leq -cE(t) + c\int_{\mathbb{R}^n}\int_{t_1}^{t} g(t-s)|\nabla_x u(t)-\nabla_x u(s)|^2 dsdx, \forall t \geq t_1. \end{aligned} \tag{2.30}$$

At this point, We define

$$I(t) = \int_{t_1}^{t} H_0(-g'(s))(g\circ\nabla_x u)(t)ds. \tag{2.31}$$

Since $\int_0^{+\infty} H_0(-g'(s))(g)ds < +\infty$, we have from Eq.(2.2)

$$\begin{aligned} I(t) &= \int_{t_1}^{t} H_0(-g'(s))\int_{\mathbb{R}^n} g(s)|\nabla_x u(t)-\nabla_x u(t-s)|^2 dxds \\ &\leq 2\int_{t_1}^{t} H_0(-g'(s))g(s)\int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dxds \\ &\leq cE(0)\int_{t_1}^{t} H_0(-g'(s))g(s) \\ &< 1. \end{aligned}$$

And as in [11], there exist β such that

$$0 < \beta \leq I(t) < 1, \forall t \geq t_1 \tag{2.32}$$

Now, we define again a new functional $\lambda(t)$ related with $I(t)$ as

$$\lambda(t) = -\int_{t_1}^{t} H_0(-g'(s))g'(s)\int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \tag{2.33}$$

From,$(A1)-(A3)$ and Remark 1.1, we get

$$H_0(-g'(s))g(s) \leq H_0(H(g(s)))g(s) = D(g(s))g(s) \leq \kappa_0,$$

For some positive constant κ_0. Then,$\forall t \geq t_1$

$$\begin{aligned}
\lambda(t) &\leq -\kappa_0 \int_{t_1}^{t} g'(s)\int_{\mathbb{R}^n} |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&\leq -\kappa_0 \int_{t_1}^{t} g'(s)\int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dxds \\
&\leq -cE(0)\int_{t_1}^{t} g'(s)ds \\
&\leq -cE(0)g(t_1) \\
&< \min\{r, H(r), H_0(r)\}.
\end{aligned} \tag{2.34}$$

Using the properties of H_0(strictly convex in $(0,r], H_0(0)=0$), then for $x \in (0,r], \theta \in [0,1]$

$$H_0(\theta_x) \leq \theta H_0(x).$$

Using Remark 1.1, Eq.(232),Eq(2.34) and Jensen's inequality leads to

$$
\begin{aligned}
\lambda(t) &= \frac{1}{I(t)} \int_{t_1} tI(t)H_0[H_0^{-1}(-g'(s))]H_0(-g'(s))g'(s) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 \\
&\geq \frac{1}{I(t)} \int_{t_1} tH_0[I(t)H_0^{-1}(-g'(s))]H_0(-g'(s))g'(s) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 \\
&\geq H_0\bigg(\frac{1}{I(t)} \int_{t_1} tI(t)H_0^{-1}(-g'(s))H_0(-g'(s))g'(s) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 \\
&\geq H_0\bigg(\int_{t_1} t \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \bigg),
\end{aligned}
$$

which implies

$$
\int_{t_1} t \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \leq H_0^{-1}(\lambda(t)),
$$

then

$$
F'(t) \leq -cE(t) + cH_0^{-1}(\lambda(t)), \forall t \geq t_1.
$$

Now, we will folowing the steps in [11] and using the fact that $E' \leq 0$, $0 < H_0'$, $0 < H_0''$ on $(0, r]$ to define the functional

$$
F_1(t) = H_0'\bigg(\alpha_0 \frac{E(t)}{E(0)}\bigg)F(t) + cE(t), \alpha_0 < r, 0 < c,
$$

where $F_1(t) \sim E(t)$ and

$$
\begin{aligned}
F_1'(t) &= \alpha_0 \frac{E'(t)}{E(0)} H_0''\bigg(\alpha_0 \frac{E(t)}{E(0)}\bigg)F(t) + H_0'\bigg(\alpha_0 \frac{E(t)}{E(0)}\bigg)F'(t) + cE'(t) \\
&\leq -cE(t)H_0'\bigg(\alpha_0 \frac{E(t)}{E(0)}\bigg) + cH_0'\bigg(\alpha_0 \frac{E(t)}{E(0)}\bigg)H_0^{-1}(\lambda(t)) + cE'(t).
\end{aligned}
$$

Let $H*_0$ given in Remark 1.1 and using Young's inequality (2.12) with $A = H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big), B = H_0^{-1}(\lambda(t))$, to get

$$\begin{aligned} F_1'(t) &\leq -cE(t)H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big) + cH*_0\left(H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big)\right) + c\lambda(t) + cE'(t) \\ &\leq -cE(t)H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big) + c\alpha_0 \frac{E(t)}{E(0)} H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big) - c'E'(t) + cE'(t). \end{aligned}$$

Choosing α_0, c, c', such that $\forall t \geq t_1$

$$\begin{aligned} F_1'(t) &\leq -\kappa \frac{E(t)}{E(0)} H_0'\Big(\alpha_0 \frac{E(t)}{E(0)}\Big) \\ &= -\kappa H_2\Big(\frac{E(t)}{E(0)}\Big), \end{aligned}$$

where $H_2(t) = tH_0'(\alpha_0 t)$. Using the strict convexity of H_0 on $(0, r]$, to find that H_2', H_2 are strict positives on $(0, 1]$,then

$$R(t) = \tau \frac{\kappa_1 F_1(t)}{E(0)} \sim E(t), \tau \in]0, 1[, \tag{2.35}$$

and

$$R'(t) \leq -\tau\kappa_0 H_2(R(t)), \kappa_0 \in]0, +\infty[, \forall t \geq t_1.$$

Then, a simple integration and a suitable choice of τ yield,

$$R(t) \leq H_1^{-1}(\alpha_1 t + \alpha_2), \alpha_1, \alpha_2 \in]0, +\infty[, \forall t \geq t_1,$$

here $H_1(t) = \int_t^1 H_2^{-1}(s)ds$.From Eq. (2.35), for a positive constante α_3, we have

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \alpha_1, \alpha_2 \in]0, +\infty[, \forall t \geq t_1.$$

The fact that H_1 is strictly decreasing function on $(0, 1]$and due to properties of H_2, we have

$$\lim_{t \longrightarrow 0} H_1(t) = +\infty,$$

then

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \forall t \geq 0.$$

This completes the proof of Theorem2.4

■

Chapter 3

General decay of solutions to a Kirchhoff-type damped wave equation with density in R^n

3.1 Introduction

In this chapter we consider the following equation

$$\rho(x)\left(|u'|^{q-2}u'\right)'-M\left(\|\nabla_x u\|_2^2\right)\Delta_x u+\int_0^t g(t-s)\Delta_x u(s)ds=0, x\in\mathbb{R}^n, t>0 \tag{3.1}$$

where $q, n \geq 2$ and M is a positive C^1 function satisfying for $s \geq 0, m_0 > 0, m_1 \geq 0, \gamma \geq 1, M(s) = m_0 + m_1 s^\gamma$ and the scalar function $g(s)$ (so-called relaxation kernel) is assumed to satisfy (A1).

Equation (3.1) is a prototype for PDE of hyperbolic in Kirchhoff type with memory when it is equipped by the following initial data.

$$u(0,x)=u_0(x)\in\mathcal{H}(\mathbb{R}^n), u'(0,x)=u_1(x)\in L^q_\rho(\mathbb{R}^n), \tag{3.2}$$

where the weighted spaces $\mathcal{H}$ is given in Definition 2.2 and the density function satisfies

$$\rho : \mathbb{R}^n \longrightarrow \mathbb{R}^n_+, \rho(x) \in C^{0,\bar{\gamma}}(\mathbb{R}^n) \tag{3.3}$$

with $\bar{\gamma} \in (0,1)$ and $\rho \in L^s(\mathbb{R}^n) \bigcap L^{+\infty}(\mathbb{R}^n)$, where $s = \dfrac{2n}{2n - qn + 2q}$. In this frame works ,(see [10].[9].[16]), it is well known that, for any initial data $u_0 \in \mathcal{H}(\mathbb{R}^n), u_1 \in L^q_\rho(\mathbb{R}^n)$, the problem (3.1)-(3.2) has a unique solution $u \in C([0,T), \mathcal{H}(\mathbb{R}^n), u' \in C([0,T), L^q_\rho(\mathbb{R}^n))$ for T small enough, under hypothesis $(A1) - (A2)$. The energy of u at time t is defined by

$$E(t) = \frac{(q-1)}{q}\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{2}\left(m_0 - \int_0^t g(s)ds\right)\|\nabla_x u\|_2^2 + \frac{1}{2}(g \circ \nabla_x u) + \frac{m_1}{2(\gamma+1)}\|\nabla_x u\|_2^{2(\gamma+1)} \tag{3.4}$$

and the following energy functional law holds :

$$E'(t)\frac{1}{2}(g' \circ \nabla_x u)(t) - \frac{1}{2}g(t)\|\nabla_x u\|_2^2, \forall t \geq 0. \tag{3.5}$$

which means that, our energy is uniformly bounded and decreasing along the trajectories. The following notation will be used throughout this paper

$$(g \circ \nabla_x u)(t) = \int_0^t g(t-\tau)\|\nabla_x u(t) - \nabla_x u(\tau)\|_2^2 d\tau, \tag{3.6}$$

for $u(t) \in \mathcal{H}(\mathbb{R}^n), t \geq 0$.
This kind of systems appears in the models of nonlinear Kirchhoff-type. It is a generalization of a model introduced by Kirchhoff [7] in the case $n = 1$ this type of problem describes a smallamplitude vibration of an elastic string. The original equation is :

$$\rho h u_{tt} + \tau u_t = \left(p_0 + \frac{Eh}{2L}\int_0^L |u_x(x,t)|^2 ds\right) u_{xx} + f, \tag{3.7}$$

where $0 \leq x \leq L$ and $t > 0, u(x,t)$ is the lateral displacement at the space coordinate x and the time t, ρ the mass density, h the croos-section area,L the length, P_0 the initial axial tension, τ the resistance modulus, E the Young modulus and f the external force (for example the action of gravity).

The motivation of our work is due to some results regarding viscoelastic wave equations of Kirchhoff type in a bounded domain. The wave equation of the from

$$u'' - M(\|\nabla_x u\|_2^2)\Delta_x u + \int_0^t g(t-s)\Delta_x u(s) ds + h(u') = f(u), x \in \Omega, t > 0 \tag{3.8}$$

The problem (3.1),(3.2) for the case $q = 2, \rho(x) = 1, M \equiv 1$, in a bounded domain $\Omega \subset \mathbb{R}^n, (n \geq 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function was considered in [16], where they established an explicit and general decay rate for relaxation functions satisfying :

$$g'(t) \leq -H(g(t)), t \geq 0, H(0) = 0 \tag{3.9}$$

for a positive function $H \in C^1(\mathbb{R}^+)$ and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$. This improves the conditions considered recently by Alabau-Boussouira and Cannarsa [3] on the relaxation functions

$$g'(t) \leq -\chi(g(t)), \chi(0) = \chi'(0) = 0 \tag{3.10}$$

where χ is a non-negative function, strictly increasing and strictly convex on $(0, \kappa_0], \kappa_0 > 0$.They required that

$$\int_0^{\kappa_0} \frac{dx}{\chi(x)} = +\infty, \int_0^{\kappa_0} \frac{xdx}{\chi(x)} < 1, \lim_{s \longrightarrow 0^+} \inf \frac{\chi(s)/s}{\chi'(s)} > \frac{1}{2} \tag{3.11}$$

and proved a decay result for the energy of (P) with $q = 2, \rho(x) = 1, M \equiv 1$ in a bounded domain.In addition to these assumptions, if

$$\lim_{s \longrightarrow 0^+} \sup \frac{\chi(s)/s}{\chi'(s)} < 1, \tag{3.12}$$

then, in this case,an explicit rate of decay is given.

We omit the space variable x of $u(x,t), u'(x,t)$ and for simplicity reason denote $u(x,t) = u$ and $u'(x,t) = u'$, when no confusion arises .We denote by $|\nabla_x u|^2 = \sum_{i=1}^n \left(\frac{\partial u}{\partial x_i}\right)^2, \Delta_x u = \sum_{i=1}^n \frac{\partial^2 u}{\partial x_i^2}$. The constants c used throughout this paper are positive generic constants which may be different in various occurrences also the functions considered are all real valued, here $u' = du(t)/dt$ and $u'' = d^2u(t)/dt^2$.

The main purpose of this work is to allow a wider class of relaxation functions and obtain a very general decay results to improve earlier results in the literature. The basic mechanism behind the decay rates is the relation between the damping and the energy.

3.2 Statement and Preliminairies

First we recall and make use the following assumptions on the function g as :

(A1)To guarantee the hyperbolicity of the system, we assume that the function g :
$\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is of class C^1 satisfying :

$$m_0 - \bar{g} = l > 0, g(0) = g_0 > 0 \tag{3.13}$$

where $\bar{g} = \int_0^\infty g(t)dt$.
(A2) There exists a positive functions $H \in C^1(\mathbb{R}^+)$such that

$$g'(t) + H(g(t)) \leq 0, t \geq 0, H(0) = 0 \tag{3.14}$$

and H is linear or strictly increasing and strictly convex C^2 function on $(0, r], 1 > r$.

Remark 2.1 [16](A) We can deduce that there exists $t_1 > 0$ large enough such that :

(1)$\forall t \geq t_1$: We have $\lim\limits_{s \longrightarrow +\infty} g(s) = 0$, which implies that $\lim\limits_{s \longrightarrow +\infty} -g'(s)$ cannot be positive, so $\lim\limits_{s \longrightarrow +\infty} -g'(s) = 0$. Then $g(t_1) > 0$ and

$$\max\{g(s), -g'(s) < \min\{r, H(r), H_0(r)\}, \tag{3.15}$$

where $H_0(t) = H(D(t))$ provided that D is a positive C^1 function, with $D(0) = 0$, for which H_0 is strictly increasing and strictly convex

C^2 function on $(0, r]$ and

$$\int_0^{+\infty} g(s)H_0(-g'(s))ds < +\infty.$$

(2)$\forall_t \in [0, t_1]$: As g is nonincreasing, $g(0) > 0$ and $g(t_1) > 0$ then $g(t) > 0$ and

$$g(0) \geq g(t) \geq g(t_1) > 0.$$

There fore, since H is a positive continuous function, then

$$a \leq H(g(t)) \leq b$$

for some positive constants a and b. Consequently,

$$g'(t) \leq -H(g(t)) \leq -kg(t), k > 0$$

which gives

$$g'(t) \leq -kg(t), k > 0 \tag{3.16}$$

(B) Let H_0^* be the convex conjugate of H_0 in the sense of Young (see [23], pages 61-64). Then

$$H_0^*(s) = s(H_0^{'})^{-1}(s) - H_0[(H_0^{'})^{-1}(s)], s \in (0, H_0^{'}(r))$$

and satisfies the following Young's inequality

$$AB \leq H_0^*(A) + H_0(B), A \in (0, H_0^{'}(r)), B \in (0, r]. \tag{3.17}$$

Definition 2.2 [10],[19] We define the function spaces of our problem and its norm as follows :

$$\mathcal{H}(\mathbb{R}^n) = \{f \in L^{2n/(n-2)}(\mathbb{R}^n) : \nabla_x f \in (L^2(\mathbb{R}^n))\} \tag{3.18}$$

and the space $L^2_\rho(\mathbb{R}^n)$ to be the closure of $C_0^\infty(\mathbb{R}^n)$ functions with respect to the inner product

$$(f,h)_{L^2_\rho(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \rho f h dx.$$

For $1 < q < \infty$, if f is a measurable function on $\mathbb{R}^n$, we define

$$\|f\|_{L^q_\rho(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} \rho |f|^q dx \right)^{1/q}. \tag{3.19}$$

The space $L^2_\rho(\mathbb{R}^n)$ is a separable Hilbert space.
The following technical lemma will play an important role in the sequel.

Lemma 2.3: [14],Lemma 1.1]For any two functions $g, v \in C^1(\mathbb{R})$ and $\theta \in [0,1]$ we have

$$\begin{aligned} v'(t) \int_0^t g(t-s)v(s)ds = & -\frac{1}{2}\frac{d}{dt}\int_0^t g(t-s)|v(t)-v(s)|^2 ds \\ & + \frac{1}{2}\frac{d}{dt}\left(\int_0^t g(s)ds\right)|v(t)|^2 \\ & + \frac{1}{2}\int_0^t g'(t-s)|v(t)-v(s)|^2 ds - \frac{1}{2}g(t)|v(t)|^2. \end{aligned} \tag{3.20}$$

and

$$\begin{aligned} & |\int_0^t g(t-s)(v(t)-v(s))ds|^2 \\ & \qquad \leq \left(|g(s)|^{2(1-\theta)} ds \right) \left(\int_0^t |g(t-s)|^{2\theta} |v(t)-v(s)|^2 ds \right) \end{aligned} \tag{3.21}$$

We are now ready to state and prove our main results.

3.3 Main result

The next Lemma can be rasily shown (see [17],Lemma2.1]).

Lemma3.1: Let ρ satisfies (3.3),then for any $u \in \mathcal{H}(\mathbb{R}^n)$

$$\|u\|_{L^q_\rho(\mathbb{R}^n)} \leq \|\rho\|_{L^s(\mathbb{R}^n)} \|\nabla_x u\|_{L^2(\mathbb{R}^n)} \tag{3.22}$$

with $s = \dfrac{2n}{2n - qn + 2q}, 2 \leq q \leq \dfrac{2n}{n-2}$.

For the purpose of constrcting a Lyapunov functional L equivalent to E, we introduce the next functionals

$$\psi_1(t) = \int_{\mathbb{R}^n} \rho(x) u |u'|^{q-2} u' dx, \tag{3.23}$$

$$\psi_2(t) = -\int_{\mathbb{R}^n} \rho(x) |u'|^{q-2} u' \int_0^t g(t-s)(u(t) - u(s)) ds dx. \tag{3.24}$$

Lemma3.2: Under the assumptions (A1) and (A2), the functional ψ_1 satisfies, along the solution of (3.1),(3.2)

$$\psi_1'(t) \leq \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c_1 m_1 \|\nabla_x u\|_2^{2(\gamma+1)} + (\sigma - l)\|\nabla_x u\|_2^2 + \frac{(1-l)}{4\sigma}(g \circ \nabla_x u). \tag{3.25}$$

proof.

From (3.23), integrate over $\mathbb{R}^n$, we have

$$\begin{aligned}
\psi_1'(t) &= \int_{\mathbb{R}^n} \rho(x)|u'|^q dx + \int_{\mathbb{R}^n} \rho(x) u \big(|u'|^{q-2}u'\big)' dx \\
&= \int_{\mathbb{R}^n} \big(\rho(x)|u'|^q + M(\|\nabla_x u\|_2^2) u \Delta_x u - u \int_0^t g(t-s)\Delta_x u(s,x) ds\big) dx \\
&\leq \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + m_1 \|\nabla_x u\|_2^{2(\gamma+1)} - l\|\nabla_x u\|_2^2 \\
&+ \int_{\mathbb{R}^n} \nabla_x u \int_0^t g(t-s)(\nabla_x u(s) - \nabla_x u(t)) ds dx.
\end{aligned} \tag{3.26}$$

Using Young's inequality and Lemma 2.3 for $\theta = 1/2$, we obtain

$$\begin{aligned}\psi_1'(t) &\le \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + m_1\|\nabla_x u\|_2^{2(\gamma+1)} - l\|\nabla_x u\|_2^2 \\ &\quad + \sigma\|\nabla_x u\|_2^2 + \frac{1}{4\sigma}\int_{\mathbb{R}^n}\left(\int_0^t g(t-s)|\nabla_x u(s) - \nabla_x u(t)|ds\right)^2 dx \\ &\le \|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + m_1\|\nabla_x u\|_2^{2(\gamma+1)} + (\sigma - l)\|\nabla_x u\|_2^2 + \frac{(1-l)}{4\sigma}(g\circ\nabla_x u).\end{aligned} \tag{3.27}$$

■

Lemma3.3: Under the assumptions (A1) and (A2), the functional ψ_2 satisfies, along the solution of (3.1) (3.2), for any $\sigma \in (0, m_0)$

$$\begin{aligned}\psi_2'(t) &\le \left(\sigma - \int_0^t g(s)ds\right)\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + m_1\|\nabla_x u\|_2^{2(\gamma+1)} \\ &\quad + \sigma\|\nabla_x u\|_2^2 + \frac{c}{\sigma}(g\circ\nabla_x u) - c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}(g'\circ\nabla_x u)^{q/2}\end{aligned} \tag{3.28}$$

proof. Exploiting Eqs. (3.1),(3.25) to get

$$\begin{aligned}\psi_2'(t) &= -\int_{\mathbb{R}^n}\rho(x)\big(|u'|^{q-2}u'\big)'\int_0^t g(t-s)(u(t)-u(s))dsdx \\ &\quad - \int_{\mathbb{R}^n}\rho(x)|u'|^{q-2}u'\int_0^t g'(t-s)(u(t)-u(s))dsdx \\ &\quad - \int_0^t g(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)}\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} M(\|\nabla u\|_2^2)\nabla_x u \int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx \\
&- \int_{\mathbb{R}^n} \Big(\int_0^t g(t-s)\nabla_x u(s,x)ds \Big)\Big(\int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx \Big)dx \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} \\
&= \big(m_0 - \int_0^t g(s)ds\big) \int_{\mathbb{R}^n} \nabla_x u \int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))dsdx \\
&+ \int_{\mathbb{R}^n} \big(\int_0^t g(t-s)(\nabla_x u(t) - \nabla_x u(s))ds\big)^2 dx + c_1 m_1 \|\nabla_x u\|_2^{2(\gamma+1)} \\
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&- \int_0^t g(s)ds\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c(g\circ\nabla_x u)(t).
\end{aligned}
$$

By Holder's and Young's inequalities and Lemma 3.1, we estimate

$$
\begin{aligned}
&- \int_{\mathbb{R}^n} \rho(x)|u'|^{q-2}u' \int_0^t g'(t-s)(u(t)-u(s))dsdx \\
&- \leq \big(\int_{\mathbb{R}^n} \rho(x)|u'|^q dx\big)^{(q-1)/q} \\
&\times \big(\int_{\mathbb{R}^n} \rho(x)| \int_0^t g'(t-s)(u(t)-u(s))ds|^q\big)^{1/q} \\
&\leq \sigma\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}(g'\circ\nabla_x u)^{q/2}(t).
\end{aligned}
$$

Using Young's and Poincare's inequalities and Lemma 2.3 for $\theta = 1/2$,we obtain

$$\psi_2'(t) \leq \sigma\|\nabla_x u\|_2^2 + \frac{c}{\sigma}(g \circ \nabla_x u) - c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}(g' \circ \nabla_x u)^{q/2} + \left(\sigma - \int_0^t g(s)ds\right)\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + m_1\|\nabla_x u\|_2^{2(\gamma+1)}$$

■

Our main result reads as follows,

Theorem 3.4: Let $(u_0, u_1) \in \mathcal{H}(\mathbb{R}^n) \times L^q_\rho(\mathbb{R}^n)$ and suppose that (A1)-(A2) hold .
Then there exist positive constants $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ such that the energy of solution given by (3.1),(3.2) satisfies .

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \forall t \geq 0,$$

where

$$H_1(t) = \int_t^1 \frac{1}{sH_0'(\alpha_0 s)} ds \tag{3.29}$$

In order to prove this theorem, let us define

$$L(t) = \xi_1 E(t) + \psi_1(t) + \xi_2\psi_2(t) \tag{3.30}$$

For $\xi_1, \xi_2 > 1$. We need the next lemma, which means that there is equivalent between the Lyapunov and energy functions, that is for $\xi_1, \xi_2 > 1$, we have

$$\beta_1 L(t) \leq E(t) \leq \beta_2 L(t) \tag{3.31}$$

holds for two positive constants β_1 and β_2.

Lemma 3.5: For $\xi_1, \xi_2 > 1$, we have

$$L(t) \sim E(t) \tag{3.32}$$

proof. By (3.30) we have

$$\begin{aligned}
|L(t) - \xi_1 E(t)| &\leq |\psi_1(t)| + \xi_2|\psi_2(t)| \\
&\leq \int_{\mathbb{R}^n} |\rho(x)u|u'|^{q-2}u'|dx \\
&\quad + \xi_2 \int_{\mathbb{R}^n} |\rho(x)|u'|^{q-2}u' \int_0^t g(t-s)(u(t)-u(s))ds|dx.
\end{aligned}$$

Thanks to Holder and Young's inequalities with exponents $\frac{q}{q-1}, q$, since $\frac{2n}{n+2} \geq q \geq 2$, we have by using Lemma3.1

$$\begin{aligned}
\int_{\mathbb{R}^n} |\rho(x)u|u'|^{q-2}u'|dx &\leq \left(\int_{\mathbb{R}^n} \rho(x)|u|^q dx\right)^{1/q} \left(\int_{\mathbb{R}^n} \rho(x)|u|^q dx\right)^{(q-1)/q} \\
&\leq \frac{1}{q}\left(\int_{\mathbb{R}^n} \rho(x)|u|^q dx\right) + \frac{q-1}{q}\left(\int_{\mathbb{R}^n} \rho(x)|u'|^q dx\right) \\
&\leq c\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + c\|\rho\|^q_{L^s(\mathbb{R}^n)}\|\nabla_x u\|^q_2
\end{aligned} \tag{3.33}$$

and

$$\int_{\mathbb{R}^n} |\left(\rho(x)^{\frac{q-1}{q}} |u'|^{q-2}u'\right)\left(\rho(x)^{\frac{1}{q}} \int_0^t g(t-s)(u(t)-u(s))ds\right)|dx$$
$$\leq \left(\int_{\mathbb{R}^n} \rho(x)|u'|^q dx\right)^{(q-1)/q}$$
$$\times \left(\int_{\mathbb{R}^n} \rho(x)|\int_0^t g(t-s)(u(t)-u(s))ds|^q dx\right)^{1/q}$$
$$\leq \frac{q-1}{q}\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q}\|\int_0^t g(t-s)(u(t)-u(s))ds\|^q_{L^q_\rho(\mathbb{R}^n)}$$
$$\leq \frac{q-1}{q}\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} + \frac{1}{q}\|\rho\|^q_{L^s(\mathbb{R}^n)}(g\circ\nabla_x u)^{q/2}(t).$$

Then, since $q \geq 2$, we have

$$\begin{aligned}|L(t)-\xi_1 E(t)| &\leq c(E(t)+E^{q/2}(t))\\ &\leq c(E(t)+E(t)E^{(q/2)-1}(t))\\ &\leq c(E(t)+E(t)E^{(q/2)-1}(0))\\ &\leq cE(t).\end{aligned}$$

Therefore, we can choose ξ_1 so that

$$L(t) \sim E(t) \tag{3.34}$$

■

proof. of theorem 3.4 from(3.5), results of Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned}L'(t) &= \xi_1 E'(t) + \psi_1'(t) + \xi_2\psi_2'(t)\\ &\leq \left(\frac{1}{2}\xi_1 - c_\sigma\|\rho\|^q_{L^s(\mathbb{R}^n)}\xi_2\right)(g'\circ\nabla_x u)^{q/2} + M_0(g\circ\nabla_x u)\\ &- M_1\|u'\|^q_{L^q_\rho(\mathbb{R}^n)} - M_2\|\nabla_x u\|_2^2 + c_1 m_1\|\nabla_x u\|_2^{2(\gamma+1)}\end{aligned}$$

where

$$\begin{aligned} M_0 &= \Big(\frac{4\xi_2 c + (1-l)}{4\sigma}\Big), \\ M_1 &= \Big(\xi_2\Big(\int_0^{t_1} g(s)ds - \sigma\Big) - 1\Big), \\ M_2 &= \Big(-\xi_2\sigma + \frac{1}{2}\xi_1 g(t_1) + (l-\sigma)\Big), \end{aligned} \tag{3.35}$$

and t_1 was introduced in Remark2.1.
We choose σ so small that $\xi_1 > 2c_\sigma \|\rho\|^q_{L^s(\mathbb{R}^n)}\xi_2$ .Whence σ is fixed, we can choose ξ_1, ξ_2 large enough so that $M_1, M_2 > 0$, which yields

$$L'(t) \le M_0(g \circ \nabla_x u) + c_1 m_1 \|\nabla_x u\|_2^{2(\gamma+1)} - cE(t), \forall t \geqslant t_1. \tag{3.36}$$

Now we set $F(t) = L(t) + cE(t)$, which is equivalent to $E(t)$. Then by (3.36), we get for some $c > 2c_1(\gamma+1)$

$$\begin{aligned} F'(t) &= L'(t) + cE'(t) \\ &\le -cE(t) + c\int_{\mathbb{R}^n}\int_{t_1}^t g(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 dsdx, \forall t \ge t_1. \end{aligned} \tag{3.37}$$

By (3.5) and Remark 2.1, we have for all $t \ge t_1$

$$\begin{aligned} &\int_{\mathbb{R}^n}\int_0^t g(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 dsdx \\ &\le -\frac{1}{k}\int_{\mathbb{R}^n}\int_0^t g(t-s)|\nabla_x u(t) - \nabla_x u(s)|^2 dsdx \\ &\le -cE'(t). \end{aligned}$$

At this point, we define

$$I(t) = \int_{t_1}^{t} H_0(-g'(s))(g \circ \nabla_x u)(t)ds. \tag{3.38}$$

Since $\int_0^{+\infty} H_0(-g'(s))g(s)ds < +\infty$, from (3.5) we have

$$\begin{aligned}
I(t) &= \int_{t_1}^{t} H_0(-g'(s)) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&\leq 2\int_{t_1}^{t} H_0(-g'(s))g(s) \int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dxds \\
&\leq cE(0) \int_{t_1}^{t} H_0(-g'(s))g(s)ds < 1.
\end{aligned} \tag{3.39}$$

Now , we define again a new functional $\lambda(t)$ related with $I(t)$ as

$$\lambda(t) = -\int_{t_1}^{t} H_0(-g'(s))g'(s) \int_{\mathbb{R}^n} g(s)|\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds. \tag{3.40}$$

From (A1) - (A2) and Remark2.1 we get

$$H_0(-g'(s))g(s) \leq H_0(H(g(s)))g(s) = D(g(s))g(s) \leq \kappa_0.$$

for some positive constant κ_0. Then, for all $t \geq t_1$

$$\begin{aligned}
\lambda(t) &\leq -\kappa_0 \int_{t_1}^{t} g'(s) \int_{\mathbb{R}^n} |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \\
&\leq -\kappa_0 \int_{t_1}^{t} g'(s) \int_{\mathbb{R}^n} |\nabla_x u(t)|^2 + |\nabla_x u(t-s)|^2 dxds \\
&\leq -cE(0) \int_{t_1}^{t} g'(s) ds \\
&\leq cE(0) g(t_1) \\
&< \min\{r, H(r), H_0(r)\}.
\end{aligned} \tag{3.41}$$

Using the properties of H_0 (strictly convex in $(0, r], H_0(0) = 0$), then for $x \in (0, r], \theta \in [0, 1]$

$$H_0(\theta x) \leq \theta H_0(x).$$

Using Remark 2.1, (3.39),(3.41) and jensen's inequality leads to

$$\begin{aligned}
\lambda(t) &= \frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0[H_0^{-1}(-g'(s))] H_0(-g'(s)) g'(s) \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 d \\
&\geq \frac{1}{I(t)} \int_{t_1}^{t} H_0[I(t) H_0^{-1}(-g'(s))] H_0(-g'(s)) g'(s) int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 \\
&\geq H_0\bigg(\frac{1}{I(t)} \int_{t_1}^{t} I(t) H_0^{-1}(-g'(s)) H_0(-g'(s)) g'(s) \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 d \\
&\geq H_0\bigg(\int_{t_1}^{t} \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \bigg)
\end{aligned}$$

which implies

$$\int_{t_1}^{t} \int_{\mathbb{R}^n} g(s) |\nabla_x u(t) - \nabla_x u(t-s)|^2 dxds \leq H_0^{-1}(\lambda(t)).$$

Then

$$F'(t) \leq -cE(t) + cH_0^{-1}(\lambda(t)) \forall t \geq t_1.$$

Now, we will following the steps in [16] and using the fact that $E' \leq 0, 0 < H_0', 0 < H_0''$ on $(0, r]$ to define the functional

$$F_1(t) = H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right)F(t) + cE(t), \alpha_0 < r, 0 < c,$$

where $F_1(t) \sim E(t)$ and

$$\begin{aligned} F_1'(t) &= \alpha_0 \frac{E'(t)}{E(0)} H_0''\left(\alpha_0 \frac{E(t)}{E(0)}\right)F(t) + H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right)F'(t) + cE'(t) \\ &\leq -cE(t)H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) + cH_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right)H_0^{-1}(\lambda(t)) + cE'(t). \end{aligned}$$

Let $H*_0$ given in Remark 2.1 and using Young's inequality (3.17) with $A = H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right), B = H_0^{-1}(\lambda(t))$, to get

$$\begin{aligned} F_1'(t) &\leq -cE(t)H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) + cH *_0 \left(H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right)\right)c(\lambda(t)) + cE'(t) \\ &\leq -cE(t)H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) + c\alpha_0 \frac{E(t)}{E(0)} H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) - c'E'(t) + cE'(t). \end{aligned}$$

Choosing α_0, c, c', such that for all $t \geq t_1$ we have

$$\begin{aligned} F_1'(t) &\leq -\kappa \frac{E(t)}{E(0)} H_0'\left(\alpha_0 \frac{E(t)}{E(0)}\right) \\ &= -\kappa H_2\left(\frac{E(t)}{E(0)}\right), \end{aligned}$$

where $H_2(t) = tH_0'(\alpha_0 t)$. Using the strict convexity of H_0 on $(0, r]$, to find that H_2', H_2 are strict positives on $(0; 1]$, then

$$R(t) = \tau \frac{\kappa_1 F_1(t)}{E(0)} \sim E(t), \tau \in (0, 1) \tag{3.42}$$

and

$$R'(t) \leq -\tau \kappa_0 H_2(R(t)), \kappa_0 \in (0, +\infty), t \geq t_1.$$

Then, a simple integration and a suitable choice of τ yield,

$$R(t) \leq H_1^{-1}(\alpha_1 t + \alpha_2), \alpha_1, \alpha_2 \in (0, +\infty), t \geq t_1.$$

here $H_1(t) = \int_t^1 H_2^{-1}(s)ds$ Form(3.42), for a positive constant α_3, we have

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2)\alpha_1, \alpha_2 \in (0, +\infty), t \geq t_1.$$

The fact that H_1 is strictly decreasing function on $(0, 1]$ and due to properties of H_2. we have

$$\lim_{t \longrightarrow 0} H_1(t) = +\infty,$$

Then

$$E(t) \leq \alpha_3 H_1^{-1}(\alpha_1 t + \alpha_2), \forall t \geq 0.$$

■ This completes the proof of Theorem 3.4.

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