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Fractional Calculus Applications in Modeling COVID-19 Dynamics: A Case Study of Algeria

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Abstract

The widespread outbreak of COVID-19 has instigated global panic, prompting extensive exploration of its intricate dynamics through mathematical models. This paper introduces a novel nonlinear fractional order model in the Caputo sense to analyze and simulate the disease's dynamics, with a specific focus on Algeria. Initially, we employ the widely used least squares approach to estimate model parameters based on reported COVID-19 cases in Algeria over a defined timeframe. We establish the existence and uniqueness of the model solution through fixed point theorem. Furthermore, we calculate the basic reproduction numbers and equilibrium points, followed by an investigation into the local and global stability of disease-free and endemic equilibrium points. Finally, numerical findings and graphical simulations are presented to elucidate how various model parameters and fractional orders influence disease dynamics and inform control strategies.

Keywords: Covid-19 , lyapunov function , stability analysis ,.The disease-free equilibrium point (DFE), The endemic equilibrium poin (EE),similation

Résumé

L'épidémie généralisée de COVID-19 a déclenché une panique mondiale, incitant à une exploration approfondie de ses dynamiques complexes à travers des modèles mathématiques. Cet article présente un nouveau modèle non linéaire d'ordre fractionnaire selon Caputo pour analyser et simuler les dynamiques de la maladie, avec un accent spécifique sur l'Algérie. Initialement, nous utilisons l'approche des moindres carrés largement utilisée pour estimer les paramètres du modèle sur la base des cas de COVID-19 signalés en Algérie sur une période définie. Nous établissons l'existence et l'unicité de la solution du modèle par le théorème du point fixe. En outre, nous calculons les nombres de reproduction de base et les points d'équilibre, suivis d'une investigation sur la stabilité locale et globale des points d'équilibre sans maladie et endémique. Enfin, des résultats numériques et des simulations graphiques sont présentés pour élucider comment divers paramètres du modèle et ordres fractionnaires influencent les dynamiques de la maladie et informent sur les stratégies de contrôle.

Mots clés: Covid-19, fonction de Lyapunov, analyse de stabilité, Point d'équilibre sans maladie , Point d'équilibre endémique (EE), simulation

الملخص

يهدف هذا العمل الى دراسة النموذج الوبائي المتعلق بديناميكيات انتشار كوفيد 19 من خلال نمذجة المشكلة الوبائية في مسالة رياضية باستخدام مشتقات كابتو الكسرية.

نناقش الخصائص الاساسية وذلك بإثبات وجود وتفرد الحل النموذجي من خلال نظرية النقطة الثابتة ودراسة الاستقرار.

واخيرا يتم عرض النتائج الرقمية وعمليات المحاكات الرسومية لتوضيح كيفية التأثير لإيجاد استراتيجيات التحكم.

الكلمات المفتاحية : كوفيد-19 ،دالة ليابونوف ، تحليل الاستقرار ، نقطة التوازن الخالية من المرض ،نقطة التوازن المتوطن ،التحاكي .

CONTENTS

LIST OF FIGURES

INTRODUCTION

The COVID-19 pandemic has posed unprecedented challenges to global public health, necessitating rigorous epidemiological analysis and innovative mathematical modeling approaches. Coronaviruses, a family of RNA viruses, include SARS-CoV-2, the causative agent of COVID-19, which has spread globally since its emergence in late 2019. Understanding and predicting the dynamics of such infectious diseases are crucial for informing effective public health interventions.

In mathematical epidemiology, various models have been employed to simulate the transmission dynamics of infectious diseases. Recently, fractional calculus has emerged as a valuable tool for refining these models, offering advantages in capturing complex dynamics that traditional integer-order derivatives may overlook. Fractional calculus allows for the incorporation of memory effects and non-local dependencies into epidemiological models, enhancing their accuracy in reflecting real-world scenarios.

This study aims to explore the dynamics of the COVID-19 pandemic in Algeria using a compartmental model divided into categories, employing Caputo fractional order derivatives. A mathematical model will be presented that divides the infected population into reported and unreported categories, with the analysis and estimation of parameters using real data from Algeria.

The aim of this research is to analyze, study and explain the larticle [1]

This study is structured into three chapters:

Notion about epidemiology systems: This chapter provides an overview of epidemiology, focusing on the transmission dynamics and key parameters involved in modeling infectious diseases like COVID-19. It discusses classical compartmental models such as SIR (Susceptible-Infectious-Recovered) and SEIR (Susceptible-Exposed-Infectious-Recovered),

Notion about fractional operators : Here, fundamental concepts of fractional calculus are introduced, emphasizing definitions and properties of fractional derivatives. The chapter explores how fractional derivatives extend traditional differential equations to better capture long-term memory effects observed in epidemiological data. Specific attention is given to the Caputo fractional derivative, which is utilized in the subsequent modeling approach.

Application (the use of Caputou derivative to study the transmission) of Coronavirus : The final chapter details the application of Caputo fractional derivatives in modeling the dynamics of COVID-19 specifically in Algeria. The research presents a compartmental model that categorizes the infected population into reported and unreported cases, utilizing real data to estimate model parameters. The findings underscore the practical relevance of fractional calculus in epidemiological forecasting and policymaking

NOTATIONS

- ➤ $N(t)$ the total Population.
- ➤ $\mathbb{R}$ Set of real numbers.
- ➤ $\mathbb{C}$ Set of Complex numbers.
- ➤ $\mathbb{N}$ the set of natural integers.
- ➤ $\mathcal{C}([0,T],\mathbb{R})$ set of continuous functions defined on [0, T] into $\mathbb{R}$.
- ➤ Ω Omega set.
- ➤ $\mathcal{R}_0$ the basic reproduction number.
- ➤ V^{-1} the invertible matrix of V.
- ➤ $J\mathcal{F}$ Jacobian matrix of F.
- ➤ FV^{-1} the next generation matrix.
- ➤ $\rho(FV^{-1})$ the matrix spectral radius K
- ➤ DFE Disease-Free equilibrium point.
- ➤ EE point of endemic equilibrium .

CHAPTER 1

NOTION ABOUT EPIDEMIOLOGY SYSTEMS

1.1 INTRODUCTION:

compartmental models like the SIR [25] model serve as foundational frameworks for understanding the dynamics of infectious diseases within populations. By breaking down the population into different compartments based on their epidemiological status (such as susceptible, infected, or recovered), these models allow researchers to simulate the spread of diseases over time. The addition of more compartments, such as those for exposed individuals, those in quarantine, or those who have been vaccinated, enhances the model's ability to capture real-world scenarios and interventions. These models are crucial for informing public health strategies and understanding how different factors influence the transmission and control of infectious diseases.

1.2 DEFINITIONS RELATED TO VARIOUS TERMS USED IN EPIDEMIOLOGY AND COMPARTMENTAL MODELS:

1.2.1 Susceptible individual

These are people who are vulnerable to contracting a particular infectious disease because they lack immunity against it. In other words, they have not been previously exposed to the pathogen or have not been vaccinated against it. Susceptible individuals are at risk of becoming infected if they come into contact with the infectious agent..

1.2.2 Exposed individual

These are individuals who have come into contact with an infectious agent (such as a virus or bacterium) but have not yet developed symptoms of the disease. During this period, the pathogen may be incubating

within the body, and the individual may or may not go on to develop the illness. Exposed individuals may eventually progress to becoming infected and symptomatic, or they may clear the infection without ever developing symptoms.

1.2.3 Infected individual

Individuals who have been infected and can transmit the disease are referred to as infectious.

1.2.4 Transmission coefficient

A central factor in mathematical models of infectious disease is the product of the rate of contact with infected individuals and the probability that this contact leads to transmission to others.

1.2.5 Force of infection

The rate at which susceptible individuals contract the infection per capita.

1.2.6 Incubation period

It refers to the time period between a person's exposure to the disease-causing agent and the first symptoms. This time is important because during this time a person can be infected but not yet show symptoms of the disease. The length of the incubation period varies depending on the disease, and may range from hours to weeks or even months.

1.2.7 Equilibrium points

In the realm of epidemiology, equilibrium points in a differential system typically represent stable states where the spread of a disease remains constant over time. There are two main types of equilibrium points that epidemiologists focus on:

1.2.8 Disease-Free Equilibrium (DFE)

In this state, there are no infected individuals in the population. Mathematically, this equilibrium is reached when all equations representing the transmission and removal of the disease are equal to zero. In epidemiological models, this often corresponds to a situation where the disease has been eradicated or has not yet spread through the population.

1.2.9 Endemic Equilibrium (EE):

This equilibrium represents a stable state where the disease persists within the population at a constant level. In this state, the number of new infections is balanced by the number of recoveries or deaths. Mathematically, this equilibrium is reached when the equations representing the transmission and removal of the

disease reach a steady state, with the number of new infections equaling the number of individuals recovering or dying from the disease.

*** To find these equilibrium points, one typically sets the right-hand side equations of the differential system equal to zero and solves for the values of the variables that satisfy these conditions. These equilibrium points are crucial for understanding the long-term behavior of infectious diseases and for designing effective control strategies.

1.2.10 The basic reproduction number $\mathcal{R}_0$:

is a fundamental epidemiological measure that indicates the average number of secondary infections produced by a typical infectious individual in a completely susceptible population. It's a crucial parameter in understanding the transmission dynamics of infectious diseases.

One common method to calculate $\mathcal{R}_0$ [24, page 91] is through the Next-Generation Matrix approach. This method involves constructing a matrix that captures the transitions from one infection state to another, and $\mathcal{R}_0$ is then determined as the largest positive eigenvalue of this matrix.

Understanding the implications of $\mathcal{R}_0$ is essential in epidemiology:

$\rightarrow$ $If \mathrm{R}_0$ is less than 1 ($\mathcal{R}_0 < 1$), it indicates that, on average, each infected individual infects fewer than one other individual. In this scenario, the disease will die out over time, with the disease-free equilibrium (DFE) being locally asymptotically stable .

$\rightarrow$ $If \mathrm{R}_0$ is greater than 1 ($\mathcal{R}_0 > 1$), it suggests that each infected individual, on average, infects more than one other individual. In such cases, the disease will persist within the population, and an endemic equilibrium point (EE) exists and is stable.

$\rightarrow$ $If \mathrm{R}_0 = 1$, the stability of the disease-free equilibrium (DFE) and the endemic equilibrium point (EE) interchange, resulting in a forward bifurcation phenomenon

1.3 THEORIES IN MATHEMATICAL EPIDEMIOLOGY:

Let us consider a mathematical model of an epidemic defined by the following system of Fractional Ordinary Differential Equations (FODEs) in normal form:

$$\begin{cases} D^{\gamma}X(t) & = f(t, X(t)) \\ X(0) & = X_0, \end{cases} \tag{1.1}$$

Where $X(t) : [0, \infty) \rightarrow \mathbb{R}^n$ denotes the state variables vector, and X_0 denotes the corresponding initial condition. Moreover, $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ defines a continuous vector, and D^{γ} represents a fractional derivative with fractional order γ.

Mathematical Calculation of R_0

[27, 26] There are various methods for calculating the basic reproduction number R_0, depending on the particular context and assumptions of the infectious disease model being employed. One efficient method

for determining R_0 in a deterministic, finite-dimensional case is the Next-Generation Approach.

1.3.1 Next-Generation Approach

In the FODEs epidemic system (1.1), consider that the state variables vector consists of

$$X = (x_1, ..., x_r, x_{r+1}, ..., x_{\text{infected}})^T \tag{1.2}$$

. Next, we separate the state variables and the entering fluxes related to the infectious process from the others, thus we get the following partition of f:

$$f(X) = \mathcal{F}(X) - \mathcal{V}(X),$$

in chich $\mathcal{V}$(X)=($\mathcal{V}_i^+ - \mathcal{V}_i^-$)(X)
and,
$\mathcal{F}_i$: nonnegative function, denoting the flux of newly infected people in compartment i,
$\mathcal{V}_i^+$: nonnegative function, represent other entering fluxes associated with compartment i,
$\mathcal{V}_i^-$: nonnegative function, represent other leaving fluxes associated with compartment i.

Taking in mind the definition of the disease-free equilibrium point and (1.2) , the Jacobian matrices of $\mathcal{F}$ and $\mathcal{V}$ at the DFE (X_f) are as follows:

$$F = J_{X_f}\mathcal{F} = \begin{pmatrix} 0 & 0 \\ 0 & M \end{pmatrix}, \quad V = J_{X_f}\mathcal{V} = \begin{pmatrix} 0 & N \\ k_1 & k_2 \end{pmatrix}, \tag{1.3}$$

In which all the off-diagonal components of the matrix N are nonnegative (i.e., N is a Metzler matrix) and M is a positive matrix.

Definition 1.3.1 *Given the condition where the stability modulus of N is strictly negative ($\rho(N) < 0$), the basic reproduction number linked to the DFE of system (1.1) is defined as the dominant eigenvalue of the next generation matrix.*

Stability of Solutions

Let $\mathcal{R}_0 = \rho(FV^{-1})$. Let $\bar{X}$ be an equilibrium point of system (1.1). Then, we define the stability, asymptotic stability, and the unstability of the equilibrium point as follows:

Definition 1.3.2 *(Stability) We state that the equilibrium $\bar{X}(t)$ is stable if, for every $\epsilon > 0$, there exists a $\delta = \delta(\epsilon) > 0$ such that, for any solution $Y(t)$ of (1.1), satisfying:*

$$|\bar{X}(0) - Y(0)| < \delta, \quad |\bar{X}(t) - Y(t)| < \epsilon, \quad \textit{for } t > 0.$$

Definition 1.3.3 *(Asymptotic Stability) We state that the equilibrium $\bar{X}(t)$ is asymptotically stable if:*

1. *$\bar{X}(t)$ is stable.*

2. *There exists a strictly positive constant* s *such that, for any solution* $Y(t)$ *of (1.16) that satisfies* $|\bar{X}(0) - Y(0)| < s$*, then*

$$\lim_{t\to\infty} |\bar{X}(t) - Y(t)| = 0.$$

Definition 1.3.4 *(Unstability) An unstable solution is one that is not stable.*

Theorem 1.3.1 *The equilibrium solution* $X = \bar{X}$ *of the system* (1.1) *is locally asymptotically stable if all the eigenvalues of the Jacobian matrix of f at* $\bar{X}$ *have negative real parts, and in the case where at least one of the eigenvalues has a positive real part, then it is considered unstable.*

CHAPTER 2

NOTION ABOUT FRACTIONAL OPERATORS

2.1 SPECIAL FUNCTIONS :

The following subsection will outline the definitions and properties of some special functions commonly utilized in fractional order calculus.For further detailed information, references [32],[33],[35],and [36] can be consulted.

2.1.1 Gamma function :

The Gamma function $\Gamma(z)$,also know assecond order Euler 1 integral ,is defined by the integral

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, (z \in \mathbb{C}, Re(z) > 0) \tag{2.1}$$

with $\Gamma(1) = 1, \Gamma(0_+) = +\infty, \Gamma(z)$ is a strictly decreasing function for $0 < z \leq 1$. An important property of the Gamma function is the following recurrence relation:

$$\Gamma(z+1) = z\Gamma(z) \tag{2.2}$$

.Which is can be demonstrated using integration by parts. For a detailed proof and additional properties of the Gamma function, refer to [32].

2.1.2 Beta Function :

The Beta function, alternatively know as Euler's first kind integral, is defined as

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx, (p, q \in \mathbb{C}, Re(q) > 0, Re(z) > 0) \tag{2.3}$$

Some important properties of the Beta function are given below:
- the Gamma function is ensured in the following

$$B(p,q) = \frac{\Gamma(q)\Gamma(p)}{\Gamma(q+p)}. \tag{2.4}$$

Proof. see [32] ■

2.1.3 Mittag-Leffler function :

1. The one-parameter Mittag-Leffler function: It is a generalization of the exponential function (e^z) and is defined as

$$E_a(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}, \quad Re(\alpha) > 0. \tag{2.5}$$

2. The two-parameter Mittag-Leffer function: is needed in fractional calculus, particularly in the resolution of fractions. order differential equation. We can define it as

$$E_{\alpha,\beta}(z) = \sum_{k=1}^{\infty} \frac{z^k}{\Gamma(\alpha k+\beta)} \cdot Re(\alpha) > 0, Be(\beta) > 0 \quad \beta, \alpha \in \mathbb{C} \tag{2.6}$$

In the above definition of the two-parameter Mittag-Leffler function for specific values of α and β, we obtain [32]

$$\begin{array}{l} \mathrm{E}_{0,1}(z) = \sum_{k=0}^{\infty} z^k \quad , E_{1,0}(z) = ze^z, \\ \mathrm{E}_{1,1} \quad = \quad e^z \quad , E_{1,2} \quad = \quad \frac{e^z-1}{z}, \end{array}$$

2.2 FRACTIONAL OPERATORS :

2.2.1 Integration and Derivation in Riemann-Liouville sense :

Definition 2.2.1 *.[32] Riemann-Liouville fractional integration of order α of the function $\mathrm{F} : \mathbb{R}_+ \to \mathbb{R}$ is defined as:*

$$I^{\alpha}_{t_0,t}\mathrm{F}(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1}\, \mathrm{F}(s)ds, \alpha > 0, \tag{2.7}$$

where $\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{\alpha-1}dt$, is the Gamma function.

.

Definition 2.2.2 *[32] Riemann-Liouville derivative of order α of the function F, can be defined as:*

$${}^{R}D^{\alpha}_{t_0,t}\mathrm{F}(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int_{t_0}^{t} (t-s)^{n-\alpha-1} F(s)ds, \quad \alpha > 0, \tag{2.8}$$

Where: $\alpha \in [n-1, n\,[, n \in \mathbb{N}^*$

.

2.2.2 Derivation in Caputo sense :

Definition 2.2.3 *[32] The Caputo fractional derivative of order α of a function $\mathrm{F} : \mathbb{R}_+ \to \mathbb{R}$is given by.*

$$ {}^{C}D^{\alpha}_{t_0,t}\mathrm{F}(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-s)^{n-\alpha-1} F^{(n)}(s)ds, \alpha > 0. \tag{2.9} $$

where : n= [α]+1 ,n$\in \mathbb{N}$ with [α] is the integer part of α .

* *Caputo derivative and the Riemann Liouville integral satisfy the following properties*

1. ${}^{C}D^{\alpha}_{0,t}(I^{\alpha}_{0,t}f(t)) = f(t)$
2. ${}^{C}D^{\alpha}_{0,t}(C) = 0$, *where* $C \in \mathbb{R}$
3. $I^{\alpha}_{0,t}({}^{C}D^{\alpha}_{0,t}f(t)) = f(t) - \sum_{k=0}^{n-1} \frac{C^{(k)}}{k!}t^k$
4. *If* α *is such that* $0 < \alpha < 1$, *then* $I^{\alpha}_{0,t}({}^{C}D^{\alpha}_{0,t}f(t)) = f(t) - f(0)$

.

2.3 FIXED POINT THEOREMS

2.3.1 Banach Contraction Principle

[38]

Let (E, d) be a complete metric space and $T : X \to X$ be a contraction mapping with Lipschitz constant $\sigma \in (0, 1)$. Then

1. T has a unique fixed point u in X.
2. For an arbitrary point x_0 in X, the sequence $\{x_n\}$ generated by the Picard iteration process as defined by $x_{n+1} = Tx_n$, $n \in \mathbb{N} \cup \{0\}$, converges to u.
3. $d(x_n, u) \leq \frac{\sigma^n}{1-\sigma} d(x_0, x_1)$ for all $n \in \mathbb{N} \cup \{0\}$.

2.4 CAUCHY PROBLEM FOR FRACTIONAL DIFFERENTIAL EQUATIONS

[] We will study the existence and uniqueness of the solution of a Cauchy problem for fractional differential equations (using the Caputo derivative)[24], and we have the problem in the following form :

$$ \begin{cases} {}^{C}D^{\alpha}y(t) = f(t, y(t)), & t \in [0, T], 0 < \alpha < 1 \\ y(0) = y_0, & y_0 \in \mathbb{R} \end{cases} \tag{2.10} $$

where $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

Lemma 2.4.1 *Suppose* $0 < \alpha < 1$ *and let* $h : [0,T] \to \mathbb{R}$ *be a continuous function. A function* y *is a solution of the Cauchy problem*

$$\begin{cases} {}^{C}D^{\alpha}y(t) = h(t), & t \in [0,T], 0 < \alpha < 1 \\ y(0) = y_0, & y_0 \in \mathbb{R} \end{cases} \tag{2.11}$$

if and only if it is the solution of the integral equation:

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds \tag{2.12}$$

Proof. we apply the operator (2.11) we find

$$\begin{aligned} I^{\alpha C}D^{\alpha}y = I^{\alpha}f(t) &\Rightarrow y(t) + c_0 = I^{\alpha}h(t) \\ &\Rightarrow y(t) = I^{\alpha}h(t) - c_0 \end{aligned}$$

The initial condition gives

$$y(0) = (I^{\alpha}h)(0) - c_0 = -c_0 \Rightarrow c_0 = -y_0$$

so

$$\begin{aligned} y(t) &= I^{\alpha}h(t) - (-y_0) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(t) dx + y_0 \end{aligned}$$

conversely we have

$$y(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(t) dx$$

we apply ${}^{C}D^{\alpha}$ to the integral equation (2.4.1)

$$\begin{aligned} {}^{C}D^{\alpha}y(t) &= {}^{C}D^{\alpha}(I^{\alpha}h)(t) + {}^{C}D^{\alpha}(y_0) \\ &= h(t) \end{aligned}$$

So it remain to verify that $y(0) = y_0$

$$\begin{aligned} y(0) &= I^{\alpha}h(0) + y_0 = 0 + y_0 \\ &= y_0 \end{aligned}$$

then there is a solution to the problem (2.11) ■

Theorem 2.4.1 *Suppose* $0 < \alpha < 1$ *and* $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ *satisfies the Lipschitz condition:*

$$|f(t,y) - f(t,z)| \leq k|y - z|, \quad \forall t \in [0,T], \text{ and } y, z \in \mathbb{R} \tag{2.13}$$

If

$$\frac{kT^{\alpha}}{\Gamma(\alpha+1)} < 1 \tag{2.14}$$

, then there exists a unique solution to the Cauchy problem (2.10).

Proof. . We use the Banach fixed-point theorem

We transform problem (2.10) into a fixed-point problem , considering the operator:

$$F : \mathcal{C}([0,T];\mathbb{R}) \to \mathcal{C}([0,T];\mathbb{R})$$

$$y \mapsto F(y)(t) = y_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y(s)) ds \tag{2.15}$$

where $\mathcal{C}([0,T];\mathbb{R})$ is the Banach space of continuous functions y defined on $[0,T]$ into $\mathbb{R}$, equipped with the norm

$$||y|| = \sup_{t\in[0,T]} |y(t)| \tag{2.16}$$

.

It is clear that the fixed points of the operator F are the solutions of problem (2.10). F is well-defined, indeed: if $y(t) \in \mathcal{C}([0,T];\mathbb{R})$, then $Fy(t) \in \mathcal{C}([0,T];\mathbb{R})$.

To show that F admits a fixed point, it suffices to show that F is a contraction. Indeed, if $y_1, y_2 \in \mathcal{C}([0,T];\mathbb{R})$, $t \in [0,T]$, using the Lipschitz condition we obtain:

$$\begin{aligned}
|Fy_1 - Fy_2| &= \left| \frac{1}{\Gamma(\alpha)} \int_0^t \left(f\left(s, y_1(s)\right) - f\left(s, y_2(s)\right)\right)(t-s)^{\alpha-1} ds \right| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t \left|f\left(s, y_1(s)\right) - f\left(s, y_2(s)\right)\right|(t-s)^{\alpha-1} ds \\
&\leq \frac{k}{\Gamma(\alpha)} \int_0^t |y_1(s) - y_2(s)|(t-s)^{\alpha-1} ds \\
&\leq \frac{k}{\Gamma(\alpha)} \|y_1 - y_2\| \int_0^t (t-s)^{\alpha-1} ds \\
&\leq \frac{kT^\alpha}{\Gamma(\alpha+1)} \|y_1 - y_2\|
\end{aligned}$$

Thus, we can deduce that F is a contraction, and according to the Banach theorem, F admits a unique fixed point which is a solution of problem (2.10). ■

CHAPTER 3

APPLICATION (THE USE OF CAPUTOU DERIVATIVE TO STUDY THE TRANSMISSION) OF CORONAVIRUS

3.1 INTRODUCTION:

The COVID-19 pandemic is a result of the SARS-CoV-2 virus and first appeared in December 2019 in Wuhan, China. The infection is usually transmitted through direct contact, floating, talking, and breathing with an infected person. Common symptoms include fever, cough, and fatigue, and high-risk groups and people with chronic diseases are considered most at risk. There is currently no effective treatment for COVID-19, and prevention occurs through frequent testing, isolation, and precautionary measures. Mathematical models play a vital role in understanding disease dynamics and suggesting control strategies, some of these include fractal transition models that provide a different perspective for analysis.

3.2 MATHEMATICAL MODEL DESCRIPTION

the dynamics of this disease were analyzed with a study of the case of Algeria. This was done by dividing the population into categories and considering these categories as variables in terms of time t
first ,we impose the total population shown by $N(t)$ at time t is classified into variables where :

- S : it represent the number of people susceptible to infect ,
- A : represent asymptom people ,
- I_u, I_r, I_c : As the undetected ,the detected and critical infected people respectively

- R : they are people who have recovered from the virus

- D : the number of dead people by the disease .

the total number of population shown by N(t) at time t given in the form :

$$N(t) = S(t) + A(t) + I_u(t) + I_r(t) + I_c(t) + R(t) + D(t) \tag{3.1}$$

The well-know Caputo derivative whith order $\alpha \in]0,1]$ is utilized to formulate the propped epidemic model that give the dynamic of the virus in different compartments summarized by means of a diagram shown in the (figure (3.1)).Thus we organize the transmission model using the following fractional system:

$$\begin{cases} {}^cD_t^\alpha S = \Delta - \frac{(\nu_1 I_{\mathrm{u}} + \nu_2 I_{\mathrm{r}} + \nu_3 I_c)}{N} S - \mu S, \\ {}^cD_t^\alpha A = \frac{(\nu_1 I_{\mathrm{u}} + \nu_2 I_{\mathrm{r}} + \nu_3 I_c)}{N} S - (\sigma + \mu) A, \\ {}^cD_t^\alpha I_{\mathrm{u}} = \sigma(1-\rho) A - (\mu + \gamma_{I_{\mathrm{u}}} + d_1) I_{\mathrm{u}}, \\ {}^cD_t^\alpha I_r = \sigma\rho A - (\delta_{I_r} + \gamma_{I_{\mathrm{r}}} + \mu + d_2) I_{\mathrm{r}}, \\ {}^cD_t^\alpha I_c = \delta_{I_r} I_r - (\gamma_{I_c} + \mu + d_3) I_c, \\ {}^cD_t^\alpha R = \gamma_{I_{\mathrm{u}}} I_u + \gamma_{I_{\mathrm{r}}} I_r + \gamma_{I_{\mathrm{c}}} I_c - \mu R, \\ {}^cD_t^\alpha D = d_1 I_u + d_2 I_r + d_3 I_c \end{cases} \tag{3.2}$$

in addition the following initial conditions are taken in consideration

$$S(0) = S_0 \geq 0, A(0) = A_0 \geq 0, I_u(0) = I_{u0} \geq 0, I_r(0) = I_{r0} \geq 0, I_c(0) = I_{c0} \geq 0,$$
$$R(0) = R_0 \geq 0, D(0) = D_0 \geq 0.$$

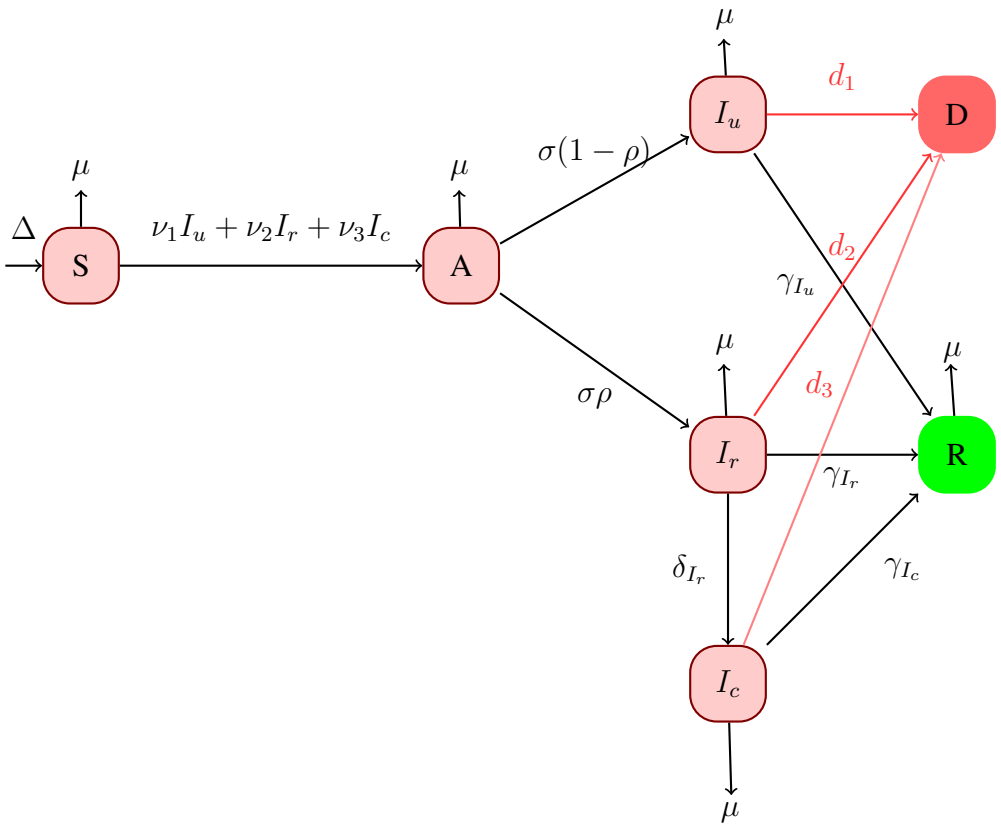


Figure 3.1: scheme of different stages of transmission of a novel coronavirus in different compartments

3.3 INTERACTION BETWEEN COVID-19 COMPARTMENT INDIVIDUALS

1. **Susceptible Population (S):**

   * The individuals susceptible to infection are affected by the recruitment rate Δ..
   * The susceptible population decreases due to interaction with undetected infected, detected infected, or critically infected individuals.
   * Newly infected individuals from compartment (S) become asymptomatic, with transmission rates relevant to undetected, detected, and critically infected people denoted by parameters ν_1, ν_2, and ν_3 respectively.

2. **.Asymptomatic Population (A):**

   * Infected individuals in this compartment progress to undetected infectious (I_u) and detected infectious (I_r) compartments at an average rate of σ.

   * A fraction of asymptomatic individuals move to (I_u) with a fraction $(1-\rho)$ such that $(0 \leq \rho \leq 1)$, while the remainder move to Ir with a fraction ρ.

3. **.Undetected Infectious Population (I_u):**

* Infected individuals who have not been detected through testing transition to this compartment.
* Some individuals recover at a rate of γ_{I_u}, while others move to the dead virus compartment (D) at a rate of d_1.

4. **.Detected Infectious Population (I_r):**

* Infected individuals detected through testing transition to this compartment. Some individuals progress to the infected critical compartment (I_c) at a rate of δ_{I_r}, recover at a rate of γ_{I_r}, or move to the dead virus compartment (D) at a rate of d_2.

5. **Infected Critical Population (I_c):**

* Individuals with critical illness transition to this compartment.
* Some individuals recover at a rate of γ_{I_c}, while others move to the dead virus compartment (D) at a rate of d_3.

.Recovered Population (R) and Dead Virus Population (D):

* Individuals who have recovered from the virus (R) recover at rates γ_{I_u}, γ_{I_r}, and γ_{I_c} for undetected infectious, detected infectious, and critical infectious individuals respectively.
* Death rates are represented by d_1, d_2, and d_3 respectively.

Parameter symbols	Biological meaning	Value	Reference
Δ	Recruitment rate	1534	Estimated [40]
μ	Natural mortality rate	$1/(77.5 \times 365)$	[40]
ν_1	Transmission rate of undetected people	0.5022	Fitted
ν_2	Transmission rate of detected people	0.4666	Fitted
ν_3	Transmission rate of critical people	0.4080	Fitted
d_1	Death rate in I_u class due to infection	0.0575	Fitted
d_2	Death rate in I_r class due to infection	0.0271	Fitted
d_3	Death rate in I_c class due to infection	0.0141	Fitted
σ	Incubation rate	0.6151	Fitted
γ_{I_u}	Recovery rate of undetected infectious people	0.3333	Fitted
γ_{I_r}	Recovery rate of detected infectious people	0.6026	Fitted
γ_{I_c}	Recovery rate of critical infectious people	0.3103	Fitted
δ_{I_r}	Critical rate of detected infectious people	0.4972	Fitted

Table 3.1: Model parameters with biological meaning and respective fitted values

3.4 MATHEMATICAL ANALYSIS OF THE MODEL

adding up the equations given in (3.2) we have as N = S + A + $I_u + I_r + I_c$ + R

$$ {}^{c}D_t^{\alpha} N = \Delta - \mu N - d_1 I_u - d_2 I_r - d_3 I_c \leq \Delta - \mu N \tag{3.3} $$

The above inequality leads to

$$ N \to \frac{\Delta}{\mu} \quad \text{as } t \to \infty $$

we note that D does not appear in the first six equations .Since the population is closed ,we can restrict the system (3.2) as follows :

$$ \begin{cases} {}^{c}D_t^{\alpha} S = \Delta - \frac{(\nu_1 I_{\mathrm{u}} + \nu_2 I_{\mathrm{r}} + \nu_3 I_c)}{N} S - \mu S, \\ {}^{c}D_t^{\alpha} A = \frac{(\nu_1 I_{\mathrm{u}} + \nu_2 I_{\mathrm{r}} + \nu_3 I_c)}{N} S - (\sigma + \mu) A, \\ {}^{c}D_t^{\alpha} I_{\mathrm{u}} = \sigma(1-\rho) A - (\mu + \gamma_{I_{\mathrm{u}}} + d_1) I_{\mathrm{u}}, \\ {}^{c}D_t^{\alpha} I_r = \sigma \rho A - (\delta_{I_r} + \gamma_{I_{\mathrm{r}}} + \mu + d_2) I_{\mathrm{r}}, \\ {}^{c}D_t^{\alpha} I_c = \delta_{I_r} I_r - (\gamma_{I_c} + \mu + d_3) I_c, \\ {}^{c}D_t^{\alpha} R = \gamma_{I_{\mathrm{u}}} I_u + \gamma_{I_{\mathrm{r}}} I_r + \gamma_{I_{\mathrm{c}}} I_c - \mu R, \end{cases} \tag{3.4} $$

consideration

$$ S(0) = S_0 \geq 0, A(0) = A_0 \geq 0, I_u(0) = I_{u0} \geq 0, I_r(0) = I_{r0} \geq 0, I_c(0) = I_{c0} \geq 0, R(0) = R_0 \geq 0. \tag{3.5} $$

Then we reformulate the model (3.4) in the following form:
Let $\mathbb{R}^6_+ = \{X \in \mathbb{R}^6 \quad : X \geq 0\}$

$$ \begin{cases} {}^{C}D_t^{\alpha} X(t) = f(t, X(t)), \quad t \in [0, T], 0 < \alpha \leq 1 \\ X(0) = X_0, \end{cases} \tag{3.6} $$

where the vector $X \in \mathbb{R}^6_+$

$$ X(t) = (S(t), A(t), I_u(t), I_r(t), I_c(t), R(t)) . \tag{3.7} $$

denotes the state variables,
and

$$ X(0) = (S(0), A(0), I_u(0), I_r(0), I_c(0), R(0)) . \tag{3.8} $$

denotes the corresponding initial condition.Moreover $f \in \mathbb{R}^6_+$ defines a continuous vector as follows:

$$f(t,X(t))=\begin{cases} f_1(t,S(t)) = {}^cD_t^\alpha S = \Delta - \frac{(\nu_1 I_{\mathrm{u}}+\nu_2 I_{\mathrm{r}}+\nu_3 I_c)}{N} S - \mu S, \\ f_2(t,A(t)) = {}^cD_t^\alpha A = \frac{(\nu_1 I_{\mathrm{u}}+\nu_2 I_{\mathrm{r}}+\nu_3 I_c)}{N} S - (\sigma+\mu)A, \\ f_3(t,I_u(t)) = {}^cD_t^\alpha I_{\mathrm{u}} = \sigma(1-\rho)A - (\mu+\gamma_{I_{\mathrm{u}}}+d_1)I_{\mathrm{u}}, \\ f_4(t,I_r(t)) = {}^cD_t^\alpha I_r = \sigma\rho A - (\delta_{I_r}+\gamma_{I_{\mathrm{r}}}+\mu+d_2)I_{\mathrm{r}}, \\ f_5(t,I_c(t)) = {}^cD_t^\alpha I_c = \delta_{I_r} I_r - (\gamma_{I_c}+\mu+d_3)I_c, \\ f_6(t,R(t)) = {}^cD_t^\alpha R = \gamma_{I_{\mathrm{u}}} I_u + \gamma_{I_{\mathrm{r}}} I_r + \gamma_{I_{\mathrm{c}}} I_c - \mu R, \end{cases} \tag{3.9}$$

Lemma 3.4.1 *: let $0 < \alpha < 1$,A function X is a solution to a Cauchy problem (3.6) If and only if it is the solution to the integral equation :*

$$X(t) = X(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, X(s)\right) ds. \tag{3.10}$$

Proof. (3.6) $\Rightarrow$ (3.10)
Using initial conditions (3.8) and fractional integral operator (2.7), we have

$$\begin{cases} I_{0,t}^\alpha({}^cD_t^\alpha S) = S(t) - S(0) = I_{0,t}^\alpha \left(\Delta - \frac{(\nu_1 I_u+\nu_2 I_r+\nu_1 I_e)}{N} S - \mu S\right), \\ I_{0,t}^\alpha({}^cD_t^\alpha A) = A(t) - A(0) = I_{0,t}^\alpha \left(\frac{(\nu_1 I_\mu+\nu_2 I_r+\nu_1 I_c)}{N} S - (\sigma+\mu)A\right). \\ I_{0,t}^\alpha({}^cD_t^\alpha I_{\mathrm{u}}) = I_u(t) - I_u(0) = I_{0,t}^\alpha \left(\sigma(1-\rho)A - (\mu+\gamma_{I_u}+d_1)\, I_u\right), \\ I_{0,t}^\alpha({}^cD_t^\alpha I_r) = I_r(t) - I_r(0) = I_{0,t}^\alpha \left(\sigma\rho A - (\delta_{I_r}+\gamma_{I_r}+\mu+d_2)\, I_r\right), \\ I_{0,t}^\alpha({}^cD_t^\alpha I_c) = I_c(t) - I_c(0) = I_{0,t}^\alpha \left(\delta_{I_r} I_r - (\gamma_{I_r}+\mu+d_3)\, I_c\right), \\ I_{0,t}^\alpha({}^cD_t^\alpha R) = R(t) - R(0) = I_{0,t}^\alpha \left(\gamma_{I_u} I_u + \gamma_{I_r} I_r + \gamma_{I_c} I_c - \mu R\right). \end{cases} \tag{3.11}$$

thus ,

$$\begin{cases} S(t) = S(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, S(s)) ds, \\ A(t) = A(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s, A(s)) ds, \\ I_u(t) = I_u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_3(s, I_u(s)) ds, \\ I_r(t) = I_r(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_4(s, I_r(s)) ds, \\ I_c(t) = I_c(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_5(s, I_c(s)) ds, \\ R(t) = R(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_6(s, R(s)) ds, \end{cases} \tag{3.12}$$

This means that

$$X(t) = X(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, X(s)) ds. \tag{3.13}$$

(3.10) $\Rightarrow$ (3.6)
Using fractional Derivative operator (2.9) (the caputo fractional derivative) , we have

$$\begin{cases} {}^cD_t^\alpha S(t) = {}^cD_t^\alpha (S(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_1(s, S(s)) ds), \\ {}^cD_t^\alpha A(t) = {}^cD_t^\alpha (A(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_2(s, A(s)) ds), \\ {}^cD_t^\alpha I_u(t) = {}^cD_t^\alpha (I_u(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_3(s, I_u(s)) ds), \\ {}^cD_t^\alpha I_r(t) = {}^cD_t^\alpha (I_r(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_4(s, I_r(s)) ds), \\ {}^cD_t^\alpha I_c(t) = {}^cD_t^\alpha (I_c(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_5(s, I_c(s)) ds), \\ {}^cD_t^\alpha R(t) = {}^cD_t^\alpha (R(0) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f_6(s, R(s)) ds), \end{cases} \tag{3.14}$$

We know that the Caputo derivative is a linear operator and By applying the mentioned properties in Defi-

nition (2.2.3), we find that

$$\begin{cases} {}^{c}D_{t}^{\alpha}S(t) = f_1(s, S(s)), \\ {}^{c}D_{t}^{\alpha}A(t) = f_2(s, A(s)), \\ {}^{c}D_{t}^{\alpha}I_u(t) = f_3\left(s, I_u(s)\right), \\ {}^{c}D_{t}^{\alpha}I_r(t) = f_4\left(s, I_r(s)\right), \\ {}^{c}D_{t}^{\alpha}I_c(t) = f_5\left(s, I_c(s)\right), \\ {}^{c}D_{t}^{\alpha}R(t) = f_6(s, R(s)), \end{cases} \tag{3.15}$$

and we have :

$$X(0) = X(0) + \frac{1}{\Gamma(\alpha)}0 = X(0)$$

Thus ,function X is a solution to a Cauchy problem (3.6) ■

3.4.1 Existence and Uniqueness :

Naturally, befor analyzing any biological model we ask whether such dynamical problem really exist or not. To answer this question We try to demonstrate the existence and uniqueness of the system (3.4) we use fixed point theory .

Theorem 3.4.1 . *There is a unique solution for the initial value problem given by (3.4)-(3.5), and the solution hold for all $t > 0$ in*

$$\Omega = \left\{ (S, A, I_u, I_r, I_c, R) \in \mathbb{R}^6_+ : 0 \leq S + A + I_u + I_r + I_c + R \leq \frac{\Delta}{\mu} \right\}.$$

to prove this theorem we need Lemma (3.4.2) and Lemma (3.4.3)

Lemma 3.4.2 . *The function $f(t, X(t))$ defined in (3.9) satisfies the Lipschitz condition given by*

$$\|f\,(t, X_1(t)) - f\,(t, X_2(t)\| \leq \Theta \|X_1 - X_2\|, \tag{3.16}$$

where
$\Theta = \max\left((\nu_1 + \nu_2 + \nu_3) + \mu; \sigma + \mu; \mu + \gamma_{I_\mu} + d_1; \delta_{I_r} + \gamma_{I_r} + \mu + d_2; \gamma_{I_c} + \mu + d_3; \mu\right)$ *and the norm* $||.||$ *corresponds to the space* $\mathcal{C}([0, T], \mathbb{R}^6)$

.

Proof. .We proof lemma only for $f_1(t, S(t))$; the other can be obtained in the same manner.
we have

$$f_1(t, S(t)) = \Delta - \frac{(\nu_1 I_u + \nu_2 I_r + \nu_3 I_c)}{N} S - \mu S$$
$$\forall S_1, S_2 \in \mathbb{R},$$

,

$$f_1(t, S_1(t)) - f_1(t, S_2(t)) = (\frac{\nu_1 I_u u + \nu_2 I_r + \nu_3 I_c}{N} - \mu)(S_2 - S_1).$$

Then,

$$\begin{aligned} ||f_1(t, S_1(t)) - f_1(t, S_2(t))|| &= ||(\frac{\nu_1 I_u u + \nu_2 I_r + \nu_3 I_c}{N} - \mu)(S_2 - S_1)|| \\ &\leq |(\nu_1 + \nu_2 + \nu_3) - \mu|||S_2 - S_1|| \quad \leq ((\nu_1 + \nu_2 + \nu_3) + \mu)||S_2 - S_1|| \end{aligned}$$

also we have ;

$$\begin{aligned} &||f_2(t, A_1(t)) - f_2(t, A_2(t)))|| = ||(\sigma + \mu)(A_2 - A_1)|| \leq (\sigma + \mu)||A_2 - A_1|| \\ &||f_3(t, I_{u1}(t)) - f_3(t, I_{u2}(t))|| = ||(\mu + \gamma I_u + d_1)(I_{u2} - I_{u1})|| \quad \leq (\mu + \gamma I_u + d_1)||I_{u2} - I_{u1}||, \\ &|f_4(t, I_{r1}(t)) - f_4(t, I_{r2}(t))| = ||(\delta_{I_r} + \gamma_{I_r} + \mu + d_2)(I_{r2} - I_{r1})|| \\ &\leq (\delta_{I_r} + \gamma_{I_r} + \mu + d_2)||I_{r2} - I_{r1}||, \\ &||f_5(t, I_{c1}(t)) - f_5(t, I_{c2}(t))|| = ||(\gamma_{I_c} + \mu + d_3)(I_{c2} - I_{c1})|| \leq (\gamma_{I_c} + \mu + d_3)||I_{c2} - I_{c1}|| \\ &||f_6(t, R_1(t) - f_6(t, R_2(t)|| = ||\mu(R_2 - R_1)|| \leq \mu||R_2 - R_1||. \end{aligned}$$

■

Lemma 3.4.3 . *Assuming we have (3.16), then there exist a unique solution to the system (3.4)-(3.5) if*

$$\frac{\Theta}{\Gamma(\alpha+1)}T^{\alpha} < 1 \tag{3.17}$$

Proof. .The solution to the system $(3.4)-(3.5)$ is given by:

$$X(t) = F(X(t)),$$

where F is the integral equation defined by F: $\mathcal{C}([0,T],\mathbb{R}) \to \mathcal{C}([0,T],\mathbb{R})$.

$$F(X(t)) = X(0) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\,(s, X(s))\,ds. \tag{3.18}$$

Further, we have:

$$\begin{aligned}
||F(X_1(t)) - F(X_2(t))|| &= ||\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}(f(s,X_1(s)) - f(s,X_2(s)))ds|| \\
&\leq \frac{1}{\Gamma(\alpha)}||(f(s,X_1(s)) - f(s,X_2(s))||\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{\theta}{\Gamma(\alpha)}||X_1(s) - X_2(s)||\int_0^t (t-s)^{\alpha-1}ds \\
&\leq \frac{\theta}{\alpha\Gamma(\alpha)}T^{\alpha}||X_1(s) - X_2(s)||.
\end{aligned}$$

If $\frac{\theta}{\Gamma(\alpha+1)}T^{\alpha} < 1$, then the operator F is a contraction, hence the system $(3.4)-(3.5)$ has a unique solution. ■

3.4.2 Positivity :

An important relevant characteristic of an epidemiological model is the positivity of solution .Therfore ,it is important to prove that all solution are non-negative for all time .to show that the model is mathematically and epidemiologically well-prepared

Lemma 3.4.4 .*(Generalized mean value theorem) Let* $f \in \mathcal{C}([0,t]), {}^{C}D_t^{a} f \in \mathcal{C}([0,T])$ *for* $0 < \alpha \leq 1$, *then we have*

$$f(t) = f(0) + \frac{1}{\Gamma(\alpha)}{}^{c}D_t^{\alpha} f(\xi)t^{\alpha} \tag{3.19}$$

with $0 \leq \xi \leq t$, *for all* $t \in]0,T]$.

Remark 3.4.1 . *From lemma (3.4.4) we have:*

1. If ${}^{c}D_{0,t}^{\alpha} f(t) \geq 0$ *then the function* f *is non decreasing for all* $x \in]\,0,T[$.

2. If ${}^{C}D_{0,t}^{\mathrm{a}} f(t) \leq 0$ *then the function* f *is non increasing for all* $x \in]\,0,T[$.

Proof. see [**?**] ■.

3.4.3 Stability analysis of equilibrium points:

We have analyzed the stability of all equilibria for this model, which consist of two equilibrium states:
1.The disease-free equilibrium point (DFE).
2.The endemic equilibrium point (EE).

The disease-free equilibrium point (DFE).

1. **Calculate DEF**.

To obtain the disease-free equilibrium (DEF), let
${}^cD_t^\alpha S = {}^cD_t^\alpha A = {}^cD_t^\alpha I_u = {}^cD_t^\alpha I_r = {}^cD_t^\alpha I_c = {}^cD_t^\alpha R = 0$. Thus, we have:

$$\begin{cases} \Delta - \frac{(\nu_1 I_u + \nu_2 I_r + \nu_3 I_c)}{N} S - \mu S = 0 \\ \frac{(\nu_1 I_u + \nu_2 I_r + \nu_3 I_c)}{N} S - (\sigma + \mu) A = 0 \\ \sigma(1-\rho)A - (\mu + \gamma_{I_u} + d_1)\, I_u = 0 \\ \sigma\rho A - (\delta_{I_r} + \gamma_{I_r} + \mu + d_2)\, I_r = 0 \\ \delta_{I_r} I_r - (\gamma_{I_c} + \mu + d_s)\, I_c = 0 \\ \gamma_{I_u} + \gamma_{I_r} I_r + \gamma_{I_r} I_c - \mu R = 0 \end{cases} \tag{3.20}$$

At the disease-free equilibrium, all infected compartments (I_u, I_r, I_c) and the recovered compartment (R) are zero. Thus, $I_u = I_r = I_c = R = 0$.

From the first equation, we get:

$$\Delta - \mu S = 0 \Rightarrow S = \frac{\Delta}{\mu}$$

From the second equation, we have:

$$\frac{(\nu_1 I_u + \nu_2 I_r + \nu_3 I_c)}{N} S - (\sigma + \mu) A = 0$$

Substituting $I_u = I_r = I_c = 0$:

$$-(\sigma + \mu) A = 0 \Rightarrow A = 0$$

Since $I_u = I_r = I_c = R = 0$, the remaining equations are already satisfied.

Thus, the disease-free equilibrium (DEF) is given by:

$$E_f = \left(\frac{\Delta}{\mu}, 0, 0, 0, 0, 0 \right) \tag{3.21}$$

Basis reproduction number and interpretation

One approach used to evaluate $\mathcal{R}_0$ is the next generation matrix
Let X =$(A(t), I_u(t), I_r(t), I_c(t), R(t), S(t))^t$, the system (3.4) can be written a

$$ {}^cD^{\alpha}_{0,t}(X(t)) = \mathcal{G}(X(t)) - \mathcal{H}(X(t)) \tag{3.22}$$

where ,

$$\mathcal{G}(X) = \begin{pmatrix} \frac{\nu_1 I_u + \nu_2 I_r + \nu_3 I_c}{N} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \mathcal{H}(X) = \begin{pmatrix} (\sigma+\mu)A \\ \mu + \gamma_{I_u} + d_1)I_u - \sigma(1-\rho)A \\ (\delta_{I_r} + \gamma_{I_r} + \mu + d_2)I_r - \sigma\rho A \\ \gamma_{I_c} + \mu + d_3)I_c - \delta_{I_r} \\ \mu R - \gamma_{I_u} I_u - \gamma_{I_r} I_r - \gamma_{I_c} I_c \\ \frac{\nu_1 I_u + \nu_2 I_r + \nu_3 I_c}{N} S - \mu S - \Delta \end{pmatrix}$$

The respective Jacobian of above matrices at the DFE E_f are evaluated as follows:

$$G = J_{E_f}\mathcal{G} = \begin{pmatrix} 0 & \nu_1 & \nu_2 & \nu_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$H = J_{E_f}\mathcal{H} = \begin{pmatrix} \sigma+\mu & 0 & 0 & 0 & 0 & 0 \\ -\sigma(1-\rho) & \mu + \gamma_{I_u} + d_1 & 0 & 0 & 0 & 0 \\ -\sigma\rho & 0 & \delta_{I_r} + \gamma_{I_r} + \mu + d_2 & 0 & 0 & 0 \\ 0 & 0 & -\delta_{I_r} & \gamma_{I_c} + \mu + d_3 & 0 & 0 \\ 0 & -\gamma_{I_u} & -\gamma_{I_r} & -\gamma_{I_c} & \mu & 0 \\ 0 & \nu_1 & \nu_2 & \nu_3 & 0 & \mu \end{pmatrix},$$

Hence, the basic reproduction number $\mathcal{R}_0$ after some calculations is obtain as:

$$\mathcal{R}_0 = \frac{\nu_1\sigma(1-\rho)}{(\mu+\gamma_{I_u}+d_1)(\sigma+\mu)} + \frac{\nu_2\sigma\rho}{(\delta_{I_r}+\gamma_{I_r}+\mu+d_2)(\sigma+\mu)} + \frac{\nu_3\delta_{I_r}\sigma\rho}{(\gamma_{I_c}+d_3+\mu)(\delta_{I_r}+\gamma_{I_r}+d_2+\mu)(\sigma+\mu)} \tag{3.23}$$

we put :

$$\begin{cases} \mathcal{R}_0^{I_u} = \frac{\nu_1\sigma(1-\rho)}{(\mu+\gamma_{I_u}+d_1)(\sigma+\mu)} \\ \mathcal{R}_0^{I_r} = \frac{\nu_2\sigma\rho}{(\delta_{I_r}+\gamma_{I_r}+\mu+d_2)(\sigma+\mu)}, \\ \mathcal{R}_0^{I_c} = \frac{\nu_3\delta_{I_r}\sigma\rho}{(\gamma_{I_c}+d_3+\mu)(\delta_{I_r}+\gamma_{I_r}+d_2+\mu)(\sigma+\mu)} \end{cases}$$

Interpretation of $\mathcal{R}_0$

The basic reproductive number $\mathcal{R}_0$ is indeed often expressed as the sum of three terms, Each of these terms represents the contribution of different pathways by which the infection spreads within a population. Let's break down each of the terms:

(a) $\mathcal{R}_0^{I_u}$**:** This term represents the contribution of new infection cases moving from the asymptomatic compartment A to the undetected Iu compartment. .

(b) $\mathcal{R}_0^{I_r}$**:** This term represents the contribution of new infection cases moved from asymptotic compartment A to the Ir detected compartement .

(c) $\mathcal{R}_0^{I_c}$**:** This term represents the contribution of new infection cases moved from asymptotic compartment A to the Ic critical compartment.

2. **Stability analysis of the DEF**

The following results provide the local and global stability results of the system (3.4) around the DFE $E_{f.}$

Theorem 3.4.2 *The DFE* E_f *of the system (3.4) is locally asymptotically stable when* $\mathcal{R}_0 < 1$.

. **Proof.** .The associated Jacobian Matrix J_{E_f} of (3.4) evaluated at E_f is given by:

$$J_{E_F} = \begin{pmatrix} -\mu & 0 & -\nu_1 & -\nu_2 & -\nu_3 & 0 \\ 0 & -(\sigma+\mu) & \nu_1 & \nu_2 & \nu_3 & 0 \\ 0 & \sigma(1-\rho) & -(\mu+\gamma_{I_u}+d_1) & 0 & 0 & 0 \\ 0 & \sigma\rho & 0 & -(\delta_{I_r}+\gamma_{I_r}+\mu+d_2) & 0 & 0 \\ 0 & 0 & 0 & \delta_{l_r} & -(\gamma_{I_c}+\mu+d_3) & 0 \\ 0 & 0 & \gamma_{I_u} & \gamma_{I_r} & \gamma_{I_c} & -\mu \end{pmatrix},$$

$$= \begin{pmatrix} -\mu & 0 & -\nu_1 & -\nu_2 & -\nu_3 & 0 \\ 0 & -h_1 & \nu_1 & \nu_2 & \nu_3 & 0 \\ 0 & \sigma(1-p) & -h_2 & 0 & 0 & 0 \\ 0 & \sigma\rho & 0 & -h_3 & 0 & 0 \\ 0 & 0 & 0 & \delta_{I_r} & -h_4 & 0 \\ 0 & 0 & \gamma_{I_u} & \gamma_{I_r} & \gamma_{I_c} & -\mu \end{pmatrix}.$$

where

$$h_1 = \sigma+\mu, h_2 = \mu+\gamma_{I_u}+d_1, h_3 = \delta_{I_r}+\gamma_{I_r}+\mu+d_2, h_4 = \gamma_{l_c}+\mu+d_3. \tag{3.24}$$

Computations give the following characteristic polynomial

$$\begin{aligned} &P(\lambda) = \det\left(\lambda I - J_{E_f}\right) \\ &= (\lambda+\mu)(\lambda+h_1)\left(\lambda+h_2\left(1-\mathcal{R}_0^{I_u}\right)\right)\left(\lambda+\frac{h_3\left(1-\mathcal{R}_0^{I_u}-\mathcal{R}_0^{I_r}\right)}{1-\mathcal{R}_0^{I_u}}\right)\left(\lambda+\frac{h_4(1-\mathcal{R}_0)}{\left(1-\mathcal{R}_0^{I_u}-\mathcal{R}_0^{I_r}\right)}\right)(\lambda+\mu). \end{aligned}$$

In the characteristic polynomial of J_{E_f} we have the eigenvalues

$$\begin{cases} \lambda_1 = -\mu, \\ \lambda_2 = -h_1, \\ \lambda_3 = -h_2\left(1-\mathcal{R}_0^{I_u}\right), \\ \lambda_4 = -\frac{h_3\left(1-\mathcal{R}_0^{I_u}-\mathcal{R}_0^{I_r}\right)}{1-\mathcal{R}_0^{I_u}}, \\ \lambda_5 = -\frac{h_4(1-\mathcal{R}_0)}{\left(1-\mathcal{R}_0^{I_u}-\mathcal{R}_0^{I_r}\right)}, \\ \lambda_6 = -\mu. \end{cases}$$

Which show negative real parts if $\mathcal{R}_0 = \mathcal{R}_0^{I_u} + \mathcal{R}_0^{I_r} + \mathcal{R}_0^{I_c} < 1$. ■

Remark 3.4.2 . *With the notation in (3.24) system (3.4) becomes*

$$\begin{cases} {}^cD_t^\alpha S = \Delta - \frac{(\nu_1 I_u + \nu_2 I_r + \nu_3 I_c)}{N} S - \mu S, \\ {}^cD_t^\alpha A = \frac{(\nu_1 I_u + \nu_2 I_r + \nu_3 I_c)}{N} S - h_1 A, \\ {}^cD_t^\alpha I_u = \sigma(1-\rho)A - h_2 I_u, \\ {}^cD_t^\alpha I_r = \sigma\rho A - h_3 I_r, \\ {}^cD_t^\alpha I_c = \delta_{I_r} I_r - h_4 I_c, \\ {}^cD_t^\alpha R = \gamma_{I_u} I_u + \gamma_{I_r} I_r + \gamma_{I_c} I_c - \mu R, \end{cases} \tag{3.25}$$

and the number $\mathcal{R}_0$ become

$$\mathcal{R}_0 = \frac{\nu_1\sigma(1-\rho)}{h_2 h_1} + \frac{\nu_2\sigma\rho}{h_3 h_1} + \frac{\nu_3\delta_{I_r}\sigma\rho}{h_4 h_3 h_1}. \tag{3.26}$$

We establish result of global stability for fractional differential system (3.4). For the proof of the theorem (3.4.3) we need this lemma:

Lemma 3.4.5 . *[lemma 2.4 [41]] Let $x(t) \in \mathbb{R}$ denotes a continuous and derivable function then, for any time instant $t \geq t_0$.*

$$\frac{1}{2}{}_{f_0}^{C}D_t^{\mathrm{a}} x^2(t) \leq x(t)\,{}_{t_0}^{C}D_t^{\mathrm{a}} x(t), \forall \alpha \in (0,1). \tag{3.27}$$

Theorem 3.4.3 . *The DFE E_f for the system (3.4) is globally asymptotically stable, if $\mathcal{R}_0 \leq 1$*

Proof. :see [1] ■

Endemic equilibrium point (EE)

1. **Calculat EE**

to calculate the endemic equilibrium point (EE) E^* of the system (3.4) .first since R(t) does not appear in the sirst five equation of the system (3.4) then we consider X(t)=(S(t),A(t),$I_u(t), I_r(t), I_c(t)$) and the folowing system

$$\begin{cases} {}^{C}D_t^a S = \Delta - \frac{(\nu_1 I_u + \nu_2 I_r + \nu_3 I_c)}{N} S - \mu S, \\ {}^{c}D_t^a A = \frac{(\nu_1 I_u + \nu_2 I_r + \nu_3 I_c)}{N} S - h_1 A, \\ {}^{C}D_t^{\mathrm{a}} I_{\mathrm{u}} = \sigma(1-\rho)A - h_2 I_{\mathrm{u}}. \\ {}^{C}D_t^a I_r = \sigma\rho A - h_5 I_r, \\ {}^{c}D_t^a I_c = \delta_{I_r} I_r - h_4 I_c. \end{cases} \tag{3.28}$$

For E^* we have ${}^{c}D_t^{\alpha}(E^*) = 0$, so we have

$$\Delta - \frac{(\nu_1 I_u^* + \nu_2 I_r^* + \nu_3 I_c^*)}{N^*} S^* - \mu S^* = 0, \tag{3.29}$$

$$\frac{(\nu_1 I_u^* + \nu_2 I_r^* + \nu_3 I_c^*)}{N^*} S^* - h_1 A^* = 0, \tag{3.30}$$

$$\sigma(1-\rho)A^* - h_2 I_u^* = 0, \tag{3.31}$$

$$\sigma\rho A^* - h_3 I_r^* = 0, \tag{3.32}$$

$$\delta_{I_r} I_r^* - h_4 I_c^* = 0. \tag{3.33}$$

1.Calculate S* :
we have :$(\nu_1 I_u^* + \nu_2 I_r^* + \nu_3 I_c^*) = h_1 \mathcal{R}_0 A^*$
from (3.30) we get .

$$\frac{h_1 \mathcal{R}_0}{N^*} A^* S^* - h_1 A^* = 0$$

$$S^* = \frac{N^*}{\mathcal{R}_0} \tag{3.34}$$

2.Calculate A^* :
From (3.29) we get .

$$\Delta - \frac{h_1 \mathcal{R}_0}{N^*} A^* S^* - \mu S^* = 0 \tag{3.35}$$

substituting (3.34) in (3.35) we get

$$A^* = \frac{1}{h_1}\left(\Delta - \mu \frac{N^*}{\mathcal{R}_0}\right). \tag{3.36}$$

3.Calculate I_u^* :
From (3.31) we get.

$$I_{\mathrm{u}}^* = \frac{\sigma(1-\rho)}{h_2} A^* = \frac{h_1}{\nu_1} \mathcal{R}_0^{I_{\mathrm{u}}} A^* \tag{3.37}$$

4.Calculate I_r^* :
from (3.32) we have

$$I_r^* = \frac{\sigma\rho}{h_3} A^* = \frac{h_1}{\nu_2} \mathcal{R}_0^{I_r} A^* \tag{3.38}$$

5.Calculate I_c^* :
Inserting (3.38) in (3.33) gives

$$I_c^* = \frac{\delta_{I_r}\sigma\rho}{h_4 h_3} A^* = \frac{h_1}{\nu_3} \mathcal{R}_0^{I_c} A^*. \tag{3.39}$$

- in the other hand we have

$$N^* = S^* + A^* + I_u^* + I_r^* + I_{c.}^* \tag{3.40}$$

$$= \frac{N^*}{\mathcal{R}_0} + \left(1 + \frac{h_1}{\nu_1}\mathcal{R}_0^{I_u} + \frac{h_1}{\nu_2}\mathcal{R}_0^{I_r} + \frac{h_1}{\nu_3}\mathcal{R}_0^{l_c}\right) A^*. \tag{3.41}$$

Then

$$\left(1 - \frac{1}{\mathcal{R}_0}\right) N^* = \left(1 + \frac{h_1}{\nu_1}\mathcal{R}_0^{I_u} + \frac{h_1}{\nu_2}\mathcal{R}_0^{l_r} + \frac{h_1}{\nu_3}\mathcal{R}_0^{I_c}\right) A^*, \tag{3.42}$$

and we get

$$N^* = \frac{\mathcal{R}_0\left(1 + \frac{h_1}{\nu_1}\mathcal{R}_0^{I_u} + \frac{h_1}{\nu_2}\mathcal{R}_0^{I_r} + \frac{h_1}{\nu_3}\mathcal{R}_0^{I_c}\right)}{1 - \mathcal{R}_0} A^* \tag{3.43}$$

Proposition 3.4.1 *:Suppose $\mathcal{R}_0 > 1$, then the system (3.28) has a unique EE denoted by $E^* = (S^*, A^*, I_u^*, I_r^*, I_e^*)$, with*

$$\begin{cases} A^* = \frac{\Delta(\mathcal{R}_0 - 1)}{h_1(\mathcal{R}_0 - 1) + \mu\left(1 + \frac{h_1}{\nu_1}\mathcal{R}_0^{I_u} + \frac{h_1}{\nu_2}\mathcal{R}_0^{I_r} + \frac{h_1}{\nu_3}\mathcal{R}_0^{I_c}\right)} \\ S^* = \frac{\left(1 + \frac{h_1}{\nu_1}\mathcal{R}_0^{I_u} + \frac{h_1}{\nu_2}\mathcal{R}_0^{I_r} + \frac{h_1}{\nu_3}\mathcal{R}_0^{I_c}\right)}{\mathcal{R}_0 - 1} A^* \\ I_u^* = \frac{h_1}{\nu_1}\mathcal{R}_0^{I_u} A^* \\ I_r^* = \frac{h_1}{\nu_2}\mathcal{R}_0^{I_r} A^* \\ I_c^* = \frac{h_1}{\nu_3}\mathcal{R}_0^{I_r} A^* \end{cases} \tag{3.44}$$

2. **Stability analysis of the EE:**

The following results provide the local and global stability results of the system (3.4) around the EE E^*

Theorem 3.4.4 *If $\mathcal{R}_0 > 1$, then the EE denoted by E^* of the system (3.28) is locally symptomatically stable when*

$$\frac{h_1\mu}{\Delta}A^* > \frac{1}{\mathcal{R}_0}\max(\nu_1,\nu_2,\nu_3)$$

.

Proof. .see [1] . ■

Lemma 3.4.6 *(lemma 3.1 [41]). Let $x(t) \in \mathbb{R}^+$ be a continuous and derivable function. Then for any time instant $t \geq t_0$*

$$ {}_{t_0}^{C}D_t^{\alpha}\left[x(t) - x^* - x^*\ln\left(\frac{x(t)}{x^*}\right)\right] \leq \left(1 - \frac{x^*}{x(t)}\right){}_{t_0}^{C}D_t^{\alpha}x(t), \quad x^* \in \mathbb{R}_+, \forall\alpha \in (0,1). \tag{3.45}$$

Theorem 3.4.5 . *If $\mathcal{R}_0 > 1$, then the E E E^* of the system (3.28) is globally symptomatically stable.*

Proof. . First all considering the following simpler model obtained by normalizing $N(t)$ in (3.28) to be 1

$$\begin{cases} {}^{c}D_t^{a}S = \Delta - (\nu_1 I_u + \nu_2 I_r + \nu_3 I_c)\,S - \mu S, \\ {}^{C}D_t^{\alpha}A = (\nu_1 I_u + \nu_2 I_r + \nu_s I_c)\,S - h_1 A, \\ {}^{C}D_t^{a}I_u = \sigma(1-\rho)A - h_2 I_{u+} \\ {}^{c}D_t^{\alpha}I_r = \sigma\rho A - h_3 I_{r+} \\ {}^{c}D_t^{a}I_c = \delta_{I_r} I_r - h_4 I_{c.}. \end{cases} \tag{3.46}$$

then the EE E^* satisfy the following equations

$$\Delta - (\nu_1 I_u^* + \nu_2 I_r^* + \nu_3 I_c^*)\,S^* - \mu S^* = 0, \tag{3.47}$$
$$(\nu_1 I_u^* + \nu_2 I_r^* + \nu_3 I_c^*)\,S^* - h_1 A^* = 0, \tag{3.48}$$
$$\sigma(1-\rho)A^* - h_2 I_u^* = 0, \tag{3.49}$$
$$\sigma\rho A^* - h_3 I_r^* = 0, \tag{3.50}$$
$$\delta_{I_r} I_r^* - h_4 I_c^* = 0. \tag{3.51}$$

Let $X(t) = (S(t), A(t), I_u(t), I_r(t), I_c(t))^T \in \mathbb{R}_+$, and the following Lyapunov function is define for the required result

$$\mathcal{L}(X(t)) = \left(S - S^* - S^*\ln\frac{S}{S^*}\right) + \left(A - A^* - A^*\ln\frac{A}{A^*}\right) + \frac{\nu_1 S^*}{h_2}\left(I_{\mathrm{ut}} - I_{\mathrm{u}}^* - I_{\mathrm{u}}^*\ln\frac{I_u}{I_u^*}\right)$$
$$+\frac{\nu_2 S^*}{h_3}\left(I_r - I_r^* - I_r^*\ln\frac{I_r}{I_r^*}\right) + \frac{\nu_3 S^*}{h_4}\left(I_c - I_c^* - I_c^*\ln\frac{I_c}{I_c^*}\right)$$

Using linearity propriety of Caputo derivatives and result in lemma 4.9, and from model (3.46) we have

$$
\begin{aligned}
{}^{C}D_t^{\alpha}\left(\mathcal{L}(X(t))\right) \leq \left(1-\frac{S^*}{S}\right){}^{C}D_t^{\alpha}S + \left(1-\frac{A^*}{A}\right){}^{C}D_t^{\alpha}A + \left(1-\frac{I_u^*}{I_u}\right){}^{C}D_t^{\alpha}I_u + \left(1-\frac{I_r^*}{I_r}\right){}^{C}D_tI_r \\
+\left(1-\frac{I_c^*}{I_c}\right){}^{C}D_t^{\alpha}I_c \\
\leq \left(1-\frac{S^*}{S}\right)\left(\Delta-\left(\nu_1 I_u+\nu_2 I_r+\nu_3 I_c\right)S-\mu S\right)+\left(1-\frac{A^*}{A}\right)\left(\nu_1 I_u+\nu_2 I_r+\nu_3 I_c\right)S-h_1A\Big) \\
+\frac{\nu_1 S^*}{h_2}\left(1-\frac{I_u^*}{I_u}\right)\left(\sigma(1-\rho)A-h_2I_u\right)+\frac{\nu_2 S^*}{h_3}\left(1-\frac{I_r^*}{I_r}\right)\left(\sigma\rho A-h_3I_r\right) \\
+\frac{\nu_3 S^*}{h_4}\left(1-\frac{I_c^*}{I_c}\right)\left(\delta_{I_r}I_r-h_4I_c\right).
\end{aligned}
$$

Using direct calculation, and formulas eqs. (3.47) to (3.51), we obtain

$$
\begin{cases}
\left(1-\frac{S^*}{S}\right)^{C}D_t^{\alpha}S(t) = \left(1-\frac{S^*}{S}\right)\left(\nu_1 I_u^*+\nu_2 I_r^*+\nu_3 I_c^*\right)S^*+\mu S^*-\left(\nu_1 I_u+\nu_2 I_r+\nu_3 I_c\right)S-\mu S\Big) \\
= \mu S^*\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right)+\left(\nu_1 I_u^*+\nu_2 I_r^*+\nu_3 I_c^*\right)S^*-\left(\nu_3 I_u+\nu_2 I_r+\nu_3 I_c\right)S \\
+\left(\nu_1 I_u+\nu_2 I_r+\nu_3 I_c\right)S^*-\left(\nu_1 I_u^*+\nu_2 I_r^*+\nu_3 I_c^*\right)\frac{S^{*2}}{S}.
\end{cases}
\tag{3.52}
$$

$$
\begin{cases}
\left(1-\frac{A^*}{A}\right)^{c}D_t^{\mathrm{a}}A(t) = \left(1-\frac{A^*}{A}\right)\left(\left(\nu_1 I_u+\nu_2 I_r+\nu_3 I_c\right)S-h_1A\right) \\
= \left(\nu_1 I_u+\nu_2 I_r+\nu_3 I_c\right)S-\left(\nu_1 I_u+\nu_2 I_r+\nu_3 I_c\right)S\frac{A^*}{A}-h_1A+h_1A^* \\
= \left(\nu_1 I_n+\nu_2 I_r+\nu_\beta I_c\right)S-\left(\nu_1 I_{\mathrm{u}}+\nu_2 I_r+\nu_\beta I_c\right)S\frac{A^*}{A}-h_1A+\left(\nu_1 I_{\mathrm{u}}^*+\nu_2 I_r^*+\nu_3 I_c^*\right)S^*.
\end{cases}
\tag{3.53}
$$

$$
\begin{cases}
\frac{\nu_1 S^*}{h_2}\left(1-\frac{I_u^*}{I_u}\right)^{C}D_t^{a}I_u(t) = \frac{\nu_1 S^*}{h_2}\left(1-\frac{I_u^*}{I_u}\right)\left(\sigma(1-\rho)A-h_2I_u\right) \\
= \frac{\nu_1 S^*}{h_2}\sigma(1-\rho)A-\frac{\nu_1 S^*}{h_2}\sigma(1-\rho)A\frac{I_u^*}{I_u}-\nu_1 I_uS^*+\nu_1 I_u^*S^* \\
= \nu_1 I_u^*S^*\frac{A}{A^*}-\nu_1 I_u^*S^*\frac{A}{A^*}\frac{I_u^*}{I_u}-\nu_1 I_uS^*+\nu_1 I_u^*S^*.
\end{cases}
\tag{3.54}
$$

$$
\begin{cases}
\frac{\nu_2 S^*}{h_3}\left(1-\frac{I_r^*}{I_r}\right)^{C}D_t^{n}I_r(t) = \frac{\nu_2 S^*}{h_3}\left(1-\frac{I_r^*(\sigma\rho A-h_3I_r)}{I_r}\right) \\
= \frac{\nu_2 S^*}{h_3}\sigma\rho A-\frac{\nu_2 S^*}{h_3}\sigma\rho A\frac{I_r^*}{I_r}-\nu_2 I_rS^*+\nu_2 I_r^*S^* \\
= \nu_2 I_r^*S^*\frac{A}{A^*}-\nu_2 I_r^*S^*\frac{A}{A^*}\frac{I_r^*}{I_r}-\nu_2 I_rS^*+\nu_2 I_r^*S^*.
\end{cases}
\tag{3.55}
$$

$$\begin{cases} \frac{\nu_3 S^*}{h_4}\left(1-\frac{I_c^*}{I_e}\right)^C D_t^a I_c(t) = \frac{\nu_s S^*}{h_4}\left(1-\frac{I_c^*}{I_c}\right)(\delta_{t_r} I_r - h_4 I_c) \\ = \frac{\nu_3 S^*}{h_4}\delta_{r_r} I_r - \frac{\nu_3 S^*}{h_4}\delta_{t_r} I_r \frac{I_c^*}{I_c} - \nu_3 I_c S^* + \nu_3 I_c^* S^* \\ = \nu_3 I_c^* S^* \frac{I_r}{I_r^*} - \nu_3 I_c^* S^* \frac{I_r}{I_r^*}\frac{I_c^*}{I_e} - \nu_3 I_c S^* + \nu_3 I_c^* S^* \end{cases} \tag{3.56}$$

Adding up equations eqs. (3.51) to (3.52) we get.

$$\begin{aligned} {}^cD_t^\alpha(\mathcal{L}(X(t)) &\leq \mu S^*(2-\frac{S^*}{S}-\frac{S}{S^*}) + 3\left(\nu_1 I_u^* + \nu_2 I_r^* + \nu_s I_c^*\right) S^* - \left(\nu_1 I_u^* + \nu_2 I_r^* + \nu_s I_c^*\right)\frac{S^{*2}}{S} \\ &- \left(\nu_1 I_u + \nu_2 I_r + \nu_3 I_c\right) S\frac{A^*}{A} - \nu_s I_c^*\frac{S^*}{A^*} + \nu_3 I_c^* S^*\frac{I_r}{I_r^*} - \nu_1 I_u^* S^*\frac{A}{A^*}\frac{I_u^*}{I_u} - \nu_2 I_r^* S^*\frac{A}{A^*}\frac{I_r^*}{I_r} - \nu_s I_c^* S^*\frac{I_r}{I_r^*}\frac{I_c^*}{I_c} \\ &\leq \mu S^*(2-\frac{S^*}{S}-\frac{S}{S^*}) + \nu_1 I_u^* S^*\left(3-\frac{S^*}{S}-\frac{AI_u^*}{A^*I_u}-\frac{SI_uA^*}{S^*I_u^*A}\right) + \nu_2 I_r^* S^*\left(3-\frac{S^*}{S}-\frac{AI_r^*}{A^*I_r}-\frac{SI_rA^*}{S^*I_r^*A}\right) \\ &+\nu_3 I_c^* S^*\left(3-\frac{S^*}{S}-\frac{I_r^*I_c^*}{I_r^*I_c}-\frac{SI_cA^*}{S^*I_c^*A}-\frac{1}{A^*}+\frac{I_r}{I_r^*}\right). \end{aligned}$$

Finally by the arithmetic-geometric means inequality, it follows that

$$\left(2-\frac{S^*}{S}-\frac{S}{S^*}\right) \leq 0$$
$$\left(3-\frac{S^*}{S}-\frac{AI_u^*}{A^*I_u}-\frac{SI_uA^*}{S^*I_u^*A}\right) \leq 0,$$
$$\left(3-\frac{S^*}{S}-\frac{AI_r^*}{A^*I_r}-\frac{SI_rA^*}{S^*I_r^*A}\right) \leq 0,$$

and if in addition

$$\left(3-\frac{S^*}{S}-\frac{I_rI_c^*}{I_r^*I_c}-\frac{SI_cA^*}{S^*I_c^*A}-\frac{1}{A^*}+\frac{I_r}{I_r^*}\right) \leq 0,$$

then, we have ${}^cD_1^\alpha\left(\mathcal{L}(X(t))\right) \leq 0$.. In addition we have ${}^cD_t^\alpha\left(\mathcal{L}\left(E^*\right)=0\right.$ if and only if $(S(t), A(t), I_w(t), I_c(t), I_r(t), R(t)) = E^*$, hence the maximum invariant set for

$$\left\{(S(t), A(t), I_n(t), I_c(t), I_r(t), R(t)) \in \mathbb{R}^6, {}^cD_t^\circ \mathcal{L}\left(E^*\right) = 0\right\}$$

in the singleton set E^* and according to the LASalle inviance principle the EE E^* for the system (3.28) is globally asymptotically stable ,whenever $\mathcal{R}_0 > 1$ ■

3.5 NUMERICAL SIMULATION OF THE MODEL :

The numerical simulation section of the paper focuses on analyzing the dynamics of a COVID-19 model based on fractional calculus, particularly employing the Caputo sense. Here's a breakdown of the key points mentioned:

Model Solution Method: The COVID-19 model is solved using a fractional Adams-Molten type iterative scheme.

Parameter Values: Parameters for the model are obtained from reported infected cases in Algeria and are listed in Table (3.1).
In this simulation, our main purpose is to discuss:

1. **Impact of Memory Index (α):**

   In **Figure** (3.2), we can observe the graphical representation of the impact of arbitrary fractional order α, which represents the memory index. Here's a breakdown of the findings:

   (a) **.Susceptible Population dynamics (Figure** (3.2(a))**)** We observe that the density of individuals in the susceptible subgroup decreases until it stabilizes at a specific value. The rate of decay is faster for larger fractional order values compared to smaller ones.

   (b) **Asymptomatic population Dynamics (Figure** (3.2(b)) In this class, we see that the number of asymptomatic cases increases until reaching a certain peak, then decreases thereafter. For varying values of α, the dynamics show a slight deviation from approaching the peaks significantly, and the category exhibits a slight decrease in the peaks of infected curves and continues for a longer period for values lower than α.

   (c) **Infected populations Dynamics (Figures** (3.2(c)) **to** (3.2(e))**)** Like the asymptomatic class, the dynamics of the remaining infected groups (I_u,I_c,I_r) exhibit the same behavior, with a slight decrease in the peaks of infected curves and a longer duration for values lower than α.

   (d) **Recovered Population Dynamics (Figure** (3.2(f)) The dynamics of the recovered population under different α values are analyzed. Initially, we observed no increase in the number of recoveries for some time, then it began to increase rapidly until stabilizing at certain values of α.We notice that the growth rate is faster for larger values of α compared to smaller ones.

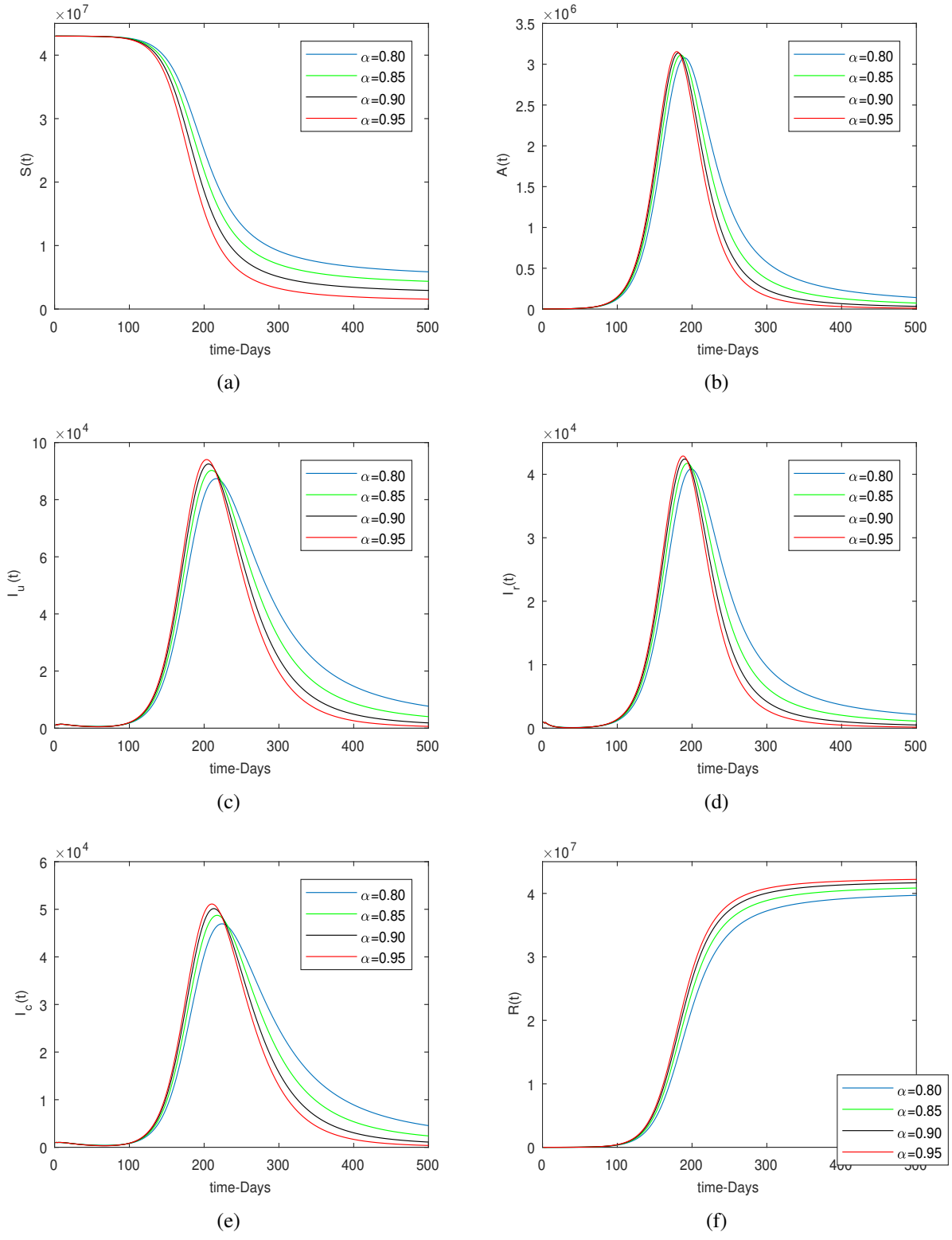


Figure 3.2: Dynamics of S(t),A(t),Iu(t),Ir(t),Ic(t) individuals for various values of α

2. **Impact of Disease Transmission Rates (ν):**

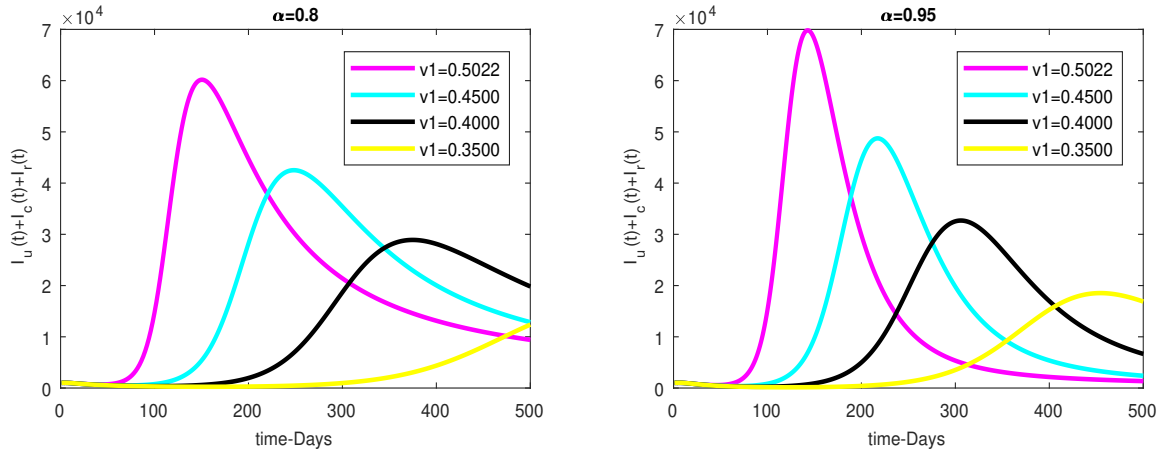
In Figures (3.3)–(3.5) , we examine the dynamic behavior of population groups infected with cumulative infection ($Iu + Ir + Ic$) for different values of disease transmission rates (ν_1, ν_2, ν_3) corresponding to population groups in compartments Iu, Ir, and Ic, respectively.

Figure (3.3) illustrates the effect of ν_1 by gradually reducing it to its baseline value and observing its impact on the total infected individuals ($Iu + Ir + Ic$). This analysis is performed across two values of the memory index α. A significant decrease in the peaks of the infected curves is observed as the disease transmission coefficient ν_1 approaches its estimated value, as shown in Table (3.1). This effect becomes more pronounced for smaller α values, as depicted in (3.4(b))–(3.4(a)).

Figure (3.4) we explore the impact of the disease transmission rate ν_2 and the fractional order α on the total infected individuals. Simulation processes involve varying ν_2 at different rates relative to the baseline across two α values such that $\alpha \in (0, 1]$. Results indicate a noticeable flattening in the peaks of the infected curves with decreasing ν_2.

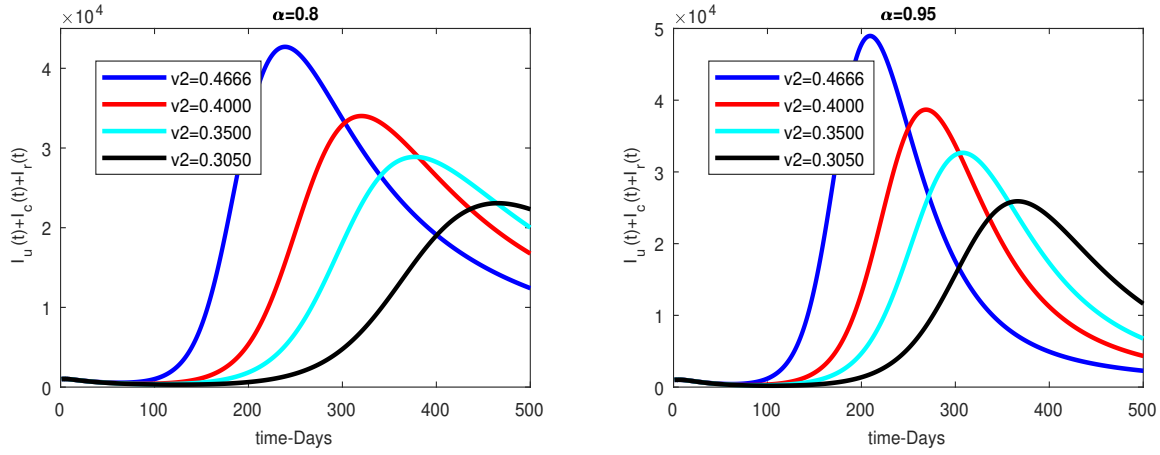
Figure (3.5) extends the analysis to study the effect of the transmission rate ν_3 on the total number of infected populations. Similar to previous analyses, simulation processes consider different rates for ν_3 relative to its baseline, along with two α values. It is worth noting that a reasonable decrease in the number of infected individuals was observed with the decrease in the parameter ν_3.

The speed of growth and decay in the three curves is slower for α values of 0.8 (small values) compared to their large values $(\alpha = 0.95)$.



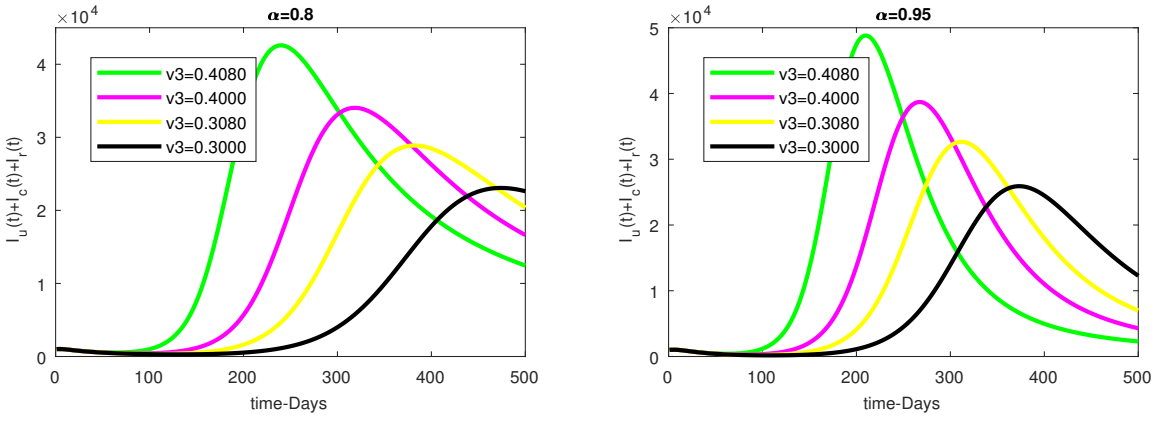
(a) (b)

Figure 3.3: Inflence of ν_1 Time (days) the cumulative infected individuals



(a) (b)

Figure 3.4: Inflence of ν_2 Time (days) the cumulative infected individuals



(a) (b)

Figure 3.5: Influenc of ν_3 Time (days) the cumulative infected individuals

CONCLUSION

In this research we presented a new mathematical model for understanding the dynamics of COVID-19 transmission. Individuals infected are divided into two categories: detected and undetected, and the Caputo derivative is used to explore the disease dynamics more effectively. The stability of disease-free and endemic states is studied, with some essential mathematical results of the fractional model being presented. Additionally, some model parameters are estimated using data from reported cases in Algeria, while others are inferred from the literature. The fractional model is numerically solved, and detailed simulation results for various estimated parameters are provided, along with an analysis of the impact of the memory parameter on disease dynamics. It is believed that this study will contribute to mitigating the COVID-19 pandemic, and the current model can be expanded to include more sophisticated mathematical models

. The numerical results demonstrate the crucial role of the memory index α in shaping the dynamics of COVID-19 spread. By altering the value of α, researchers can understand how memory affects virus transmission and infection growth. This implies that studying and comprehending the role of memory in infectious disease models can contribute to improving preventive and therapeutic measures to combat the pandemic and mitigate its impact on society.

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