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By: Kherfi Razika

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Homotopy Analysis Method for a fractional Order Equation with Dirichlet and Non-Local Integral Conditions

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Before the jury composed of:

Dr. K. Kaliche	M.C.B. Univ Kasdi Merbah-Ouargla	President
Dr. M. Kouidri	M.C.A. Univ Kasdi Merbah-Ouargla	Examiner
Dr. L. Kasmi	M.C.B. Univ Kasdi Merbah-Ouargla	Supervisor

Dedication

My God,the night is not good when i thank you,the day is not good except with your obedience,and the moments are not good when I remamber you,praise and thanks be to you

To the one who conveyed the message,fulfilled the trust,and advised the nation to the prophet of mercy,Light of the worlds,May the blessings and peace of my Lord be upon him

To the one whose name I carry with pride and who taught me to give withot waiting,I ask God to extend your life so that you can see the fruits that have come to be harvested, my beloved father

To my angel in life,the meaning of love and tendrness, the smile of life, and the secret of existence to those whose prayers were the secret of my success, her tenderness a surgical balm, to my beloved women, my dear mother

To those who used to light my way and support me my brothers and all my belloved friends, withot exception,each in his own name

To generous teachers and everyone who inadvertently fell from my pen, I dedicate the fruit of my effort.

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Secondly, I am deeply indebted to my parents and family, whose love, encouragement, and sacrifices have been the bedrock of my success.

I also want to thank my friends and colleagues, who shared my dreams and aspirations and stood by me through both the challenging and joyous times.

Finally, this thesis is not merely an academic document but a collective effort that was made possible through the blessings of God and the cooperation and love of all involved.

Contents

ملخص

قمنا في هذا العمل بدراسة وجود ووحدانية الحل القوي للمعادلات التفاضلية الكسرية ذات الشروط الحدودية التكاملية والكلاسيكية، واستخدمنا طريقة التقدير القبلي في الجزء النظري لدراسة قابلية حل المسألة، ثم استخدمنا طريقة التحليل الهوموتوبي في الجزء التطبيقي، للعثور على النتائج العددية المحصلة لدينا. وقد أرفقنا دراستنا ببعض الأمثلة التي تؤكد فعالية الطرق المستخدمة مع المعادلات التفاضلية ذات الرتبة الكسرية.

الكلمات المفتاحية: المعادلات التفاضلية ذات الرتبة الكسرية؛ الحل المعمم؛ طريقة التقدير المسبق؛ طريقة تحليل الهوموتوبي؛ مشتق كابيتو الكسري.

Résumé

Dans ce travail, nous avons étudié l'existence et l'unicité de la solution fort pour les équations différentielles fractionnaires, avec des conditions aux limites intégrales et classiques, nous utilisons la méthode d'estimation a priori dans la partie théorique pour l'étude de la solvabilité du problème, puis nous avons utilisé l'analyse d'homotopie. méthode en partie pratique, pour retrouver nos résultats numériques. Nous avons joint quelques exemples qui confirment l'efficacité des méthodes utilisées avec les équations différentielles d'ordre fractionnaire.

Mots clés : Équations différentielles d'ordre fractionnaire ; Solution généralisée ; Méthode d'estimation a priori ; Méthode d'analyse d'homotopie ; Dérivé fractionnaire de Caputo.

Abstract

In this work, we studied the existence and uniqueness of the strong solution for fractional differential equations, with integral and classical boundary conditions, we use the a priori estimation method in theory part for study of the solvability of the problem, then we used the Homotopy analysis method in practice part, to find our numerical results. We have attached some examples that confirm the effectiveness of the methods used with fractional order differential equations.

Keywords: Fractional-order differential equations; Generalized solution; Priori estimation method; Homotopy analysis method; Caputo frational derivative.

Notation

${}_0^c\partial_t^\alpha$	The fractional Caputo derivative.
$D_t^{-\alpha}$	The Riemann-Liouvielle integral.
$\mathcal{L}, l$	Linear operators.
$D(k)$	Domain of definition of the operator K.
$\mathcal{L}_p^2(Q)$	Space of square integrable function u with weights function p, defined on Q.
$Im(k)$	Image of the operator$K$.
HAM	Homotopy analysis method.

Introduction

In this thesis, we employ a theoretical method to prove that the non-local initial-boundary value problem for a singular fractional order parabolic equation is well-posed. Additionally, we use the homotopy analysis method, a numerical approach, to investigate approximate solutions for the given problem. For theoretical purposes, we apply the energy inequality method, which relies primarily on a priori estimates and the density of the range of the operator generated by the problem. This method is a vital component of linear and nonlinear functional analysis theory and is crucial for establishing the existence and uniqueness of solutions for a wide variety of local and non-local initial-boundary value problems in partial differential equations.

The model we study is a one-dimensional fractional order diffusion heat equation, which is associated with both classical and non-local integral conditions (see [17, 18]). Over the past few decades, many researchers have studied the existence and uniqueness of fractional order initial-boundary value problems. These problems arise in numerous scientific and engineering fields, including control theory, blood flow, aerodynamics, biology, stochastic transport, viscoelasticity, quantum mechanics, nuclear physics, and many other physical and biological processes (see[5, 13]).

To prove the existence and uniqueness of the solution to the posed problem, we again use the energy inequality method, relying mainly on a priori estimates and the density of the range of the operator generated by the problem. In the literature, there are only a few articles that use the method of energy inequalities to prove the existence and uniqueness of fractional initial-boundary value problems in the fractional case.

For numerical purposes, we employ the homotopy analysis method (HAM), first introduced by Liao [11] to efficiently tackle nonlinear problems. This method provides solutions in the form of a rapidly convergent series, which, in most cases, yields highly accurate results after only a few iterations. Many authors have widely used HAM to successfully solve a broad range of mathematical problems across different disciplines. Recently, it has been utilized to generate reliable approximate solutions for fractional partial differential equations. For instance, HAM has been applied to study approximate solutions of linear and nonlinear fractional diffusion wave equations, systems of nonlinear fractional partial differential equations, time fractional wave-like equations, and nonlinear problems. Numerous authors have analytically and numerically investigated various models of time-fractional differential equations, focusing particularly on the existence and uniqueness of solutions; see, for example, [6, 14].

The homotopy analysis method (HAM)[10, 8], and [12] is a general analytic approach to get series solution of various types of non-linear encoding algebraic equations, ordinary differential equations, partial differential equations, differential-integral equations, and coupled equations of them. Unlike perturbation methods, the HAM is valid even for strongly nonlinear problems. Besides, different from all perturbation and previous non-perturbation methods, the HAM provides us with great freedom to choose proper base functions to approximate a nonlinear problem[9, 7]. More and more researchers have been successfully applying this method to various nonlinear problems in science and engineering,

The HAM is based on homotopy, a fundamental concept in topology and differential geometry[16], Briefly speaking, by means of the HAM, one constructs a continuous mapping of an initial guess approximation to the exact solution of considered equations. An auxiliary linear operator is chosen to construct such kind of continuous mapping, and an auxiliary parameter is used to ensure the convergence of solution series The method enjoys great freedom in choosing initial approximations and auxiliary linear operators, by means of this kind of freedom, a complicated nonlinear problem can be transferred into an infinite number of sampler, linear sub-problem, as shown by Liao and tan[7].

This thesis is divided into three chapters:

Chapter1: in this chapter we present some basic tools(the Gamma function, Mittage-Leffter function,Young enequality with ε...), and some interesting properties.

Chapter2: in this chapter we obtain results on the existence and uniqueness of a generalized solution for a parabolic fractional problem in the sense of Caputo with integral conditions.

Chapter3: the main objective of this chapter is to obtain numerical results via the homotopy analysis method.

Chapter 1

Preliminaries

In this section we recall some function spaces and some basic tools.
The Gumma function:

Definition 1. *The Gumma function, also known as the Eulerian integral of the second kind, is denoted by* Γ*, and defined by:*

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt, \tag{1.1}$$

where z *is any complex number such that* $Re(z) > 0$ *with* $\Gamma(z)$ *is monoton and strictly decreasing function for* $0 < z \leq 1$ *and is continuous on* $]0, +\infty[$.
$\forall z \in \mathbb{R}_+^*$ *we have*

$$\Gamma(z+1) = z\Gamma(z),$$

$\forall n \in \mathbb{N}^*$ *we have:*

$$\Gamma(n) = (n-1)!\,.$$

Mittag-Leffler function:

Definition 2. *The Mittag-Leffler is defined by:*

$$E_\alpha(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + 1)} \quad and \quad E_{\alpha,\mu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\alpha n + \mu)}, \tag{1.2}$$

The fractional Caputo derivative:

Definition 3. *We define the fractional Caputo derivative of order* $0 < \alpha < 1$ *for a differential function by:*

$$ {}_0^c\partial_t^\alpha U(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial}{\partial \tau} U(x,\tau) d\tau, t > 0, \tag{1.3}$$

for more details about the Caputo fracional derivative, we refer the reader to the reference [5].

The Riemann-Liouvielle integral:

Definition 4. *We define the Riemann-Liouvielle integral of order $0 \leq \alpha \leq 1$ by*

$$D_t^{-\alpha}U(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{U(\tau)}{(t-\tau)^{1-\alpha}} d\tau. \tag{1.4}$$

We denote by $\mathcal{L}_p^2(0,1)$ the Hilbert space of weighted square integrable function with inner product $(U,V)_p = \displaystyle\int_0^1 xUV\,dx$, and by $H_p^1(0,1)$ the wheighted sobolev space with the norme $\|u\|^2_{H_p^1(0,1)} = \|u\|_{\mathcal{L}_p^2(0,1)} + \|u_x\|^2_{\mathcal{L}_p^2(0,1)}$. we also introduce the Hilbert space $\mathcal{L}^2(0,T;H_p^{\alpha,t}(0,1))$ consisting of all abstract strongly measurable function u on [0,T] into$H_p^{\alpha,t}(0,1)$ such that

$$\|u\|^2_{\mathcal{L}^2(0,T;H_p^{\alpha,t}(0,1))} = \int_0^T \|u(.,t)\|^2_{H_p^{\alpha,t}(0,1)} dt = \int_0^T \left(\|u\|^2_{\mathcal{L}_p^2(0,1)} + \|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}_P^2(0,1)} \right) dt < \infty \tag{1.5}$$

$H_p^{\alpha,t}(0,1)$ denotes the weighted sobolev space whose norme is defined by

$$\|u\|^2_{H_p^{\alpha,t}(0,1)} = \|u\|^2_{\mathcal{L}_p^2(0,1)} + \|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}_p^2(0,1)}. \tag{1.6}$$

Lemma 1. *([1]).for any absoltely continous function Z(t) on the interval [0,T],the following inequality holds:*

$$Z(t)\,{}_0^c\partial_t^\alpha Z(t) \geq \frac{1}{2}\,{}_0^c\partial_t^\alpha Z^2(t), 0 < \alpha < 1. \tag{1.7}$$

Lemma 2. *([1]).let a nonnegative absolutely continuous function $\mathcal{J}(s)$satisfy the inequality*

$${}_0^c\partial_t^\alpha \mathcal{J}(t) \leq r_1\mathcal{J}(t) + r_2(t), \quad 0 < \alpha < 1, \tag{1.8}$$

for almost all $t \in [0,T]$, where r_1 is a positive constant and $r_2(t)$ is an integrable non negative function on $[0,T]$, then

$$\mathcal{J}(t) \leq \mathcal{J}(0)E_\alpha(r_1 t^\alpha) + \Gamma(\alpha)E_\alpha(r_1 t^\alpha)D_t^{-\alpha}r_2(t), \tag{1.9}$$

The Cauchy inequality with ε:[3]

$$aW \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}W^2, \quad \varepsilon > 0. \tag{1.10}$$

Young's inequality with ε:[3]

for any $\varepsilon > 0$, we have the inequlity

$$aW \leq \frac{1}{p}|\varepsilon a|^p + \frac{p-1}{p}\left|\frac{W}{\varepsilon}\right|^{\frac{p}{p-1}}, \quad a, W \in \mathbb{R}, p > 1, \tag{1.11}$$

which is the generalization of the Cauchy inequality with ε

A Poincaré type inequality:[15]

$$\|p_x(\zeta U)\|^2_{\mathcal{L}^2_p(0,1)} \leq \frac{l^3}{2}\|U\|^2_{\mathcal{L}^2_p(0,l)}, \tag{1.12}$$

where

$$p_x(V) = \int_0^x V(\zeta,t)d\zeta. \tag{1.13}$$

Energy inequality method:

or what we call "a priori estimates".this method based on the ideas of I.G Pétrovski[4] who used it in the solution of the Cauchy problem for hyperbolic equations.
First, we write the problem posed in the following operatorielle form:

$$Ku = \mathcal{F}, \forall u \in D(K), \tag{1.14}$$

where the operator K is considered from Banach space B in a well-chosen Hilbert space F.
Then, studied the uniqueness of a solution of the previous problem. more precisely, we prove the following energy inequality;

$$\|u\|_B \leq c\|Ku\|_F, \tag{1.15}$$

we obtain this type of a priori estimates by multiplying the equation considered by an integrodifferential operator Mu (container the function u, and its derivatives with a certain weight function) defined on $\Omega^\tau = (0,1) \times (0,\tau)$.
the choice of the Mu operator is very important, it is diracted by the equation and the boundary conditions
Next,we show that the operator K from B into F admits a closure $\overline{K}$, so u is the strong solution of the operatoriel equation

$$\overline{K}u = \mathcal{F}, u \in D(\overline{\mathcal{K}}), \tag{1.16}$$

by going beyond the limit, the estimate(1.14) will be extended to $\overline{K}$ i.e:

$$\|u\|_B \leq c\|\overline{K}u\|_F, \tag{1.17}$$

from wich we deduce the uniqueness of the solution of equation(1.15).
since the image of the operator $\overline{K}$ is closed in F and $Im(\overline{K}) = \overline{Im(K)}$ and the density of $Im(k)$ in F, we guarantee the existence of the solution strength of the problem(15)
for the study of the nonlinear problem, we apply an iterative process based on the results obtained from the linear problem.

Chapter 2

Existence and Uniqueness of solution

2.1 problem setting

We consider a fractional order parabolic equation with a Caputo derivative associated with Dirichlet and non-local conditions of integral type:

$$\begin{cases} \mathcal{L}u = g(x,t), (x,t) \in Q = \Omega \times [0,T], \\ l_1 u = u(x,0) = w(x), x \in \Omega = (0,1), \\ \displaystyle\int_0^1 xu(x,t)dx = 0, u(1,t) = 0, \quad t \in (0,T), \end{cases} \tag{2.1}$$

where $\mathcal{L} =_0^c \partial_t^\alpha - \frac{1}{x}\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} + C$ and the function $g \in \mathcal{L}_p^2(Q)$ and $C \geq 0$.
The solution this problem is equivalent to solution the operator equation $ku = (\mathcal{L}u, l_1 u) = F$, where k is an unbounded opertor wich acts from S to H, with the domain of definition being the set of function $u \in \mathcal{L}_p^2(Q), u_x, u_{xx}, \partial_t^\alpha u \in \mathcal{L}_p^2(Q)$ satisfing the boundary condition, where S is a Banach space of function u assoiated with finite norm:

$$\|u\|_S^2 = \|u\|_{\mathcal{L}^2(0;T,H_p^{\alpha,t}(0,1))} = \int_0^T \left(\|u\|_{\mathcal{L}_p^2(0,1)}^2 + \|_0^c\partial_t^\alpha u\|_{\mathcal{L}_p^2(0,1)}^2 \right) dt, \tag{2.2}$$

such that H is the weighted Hilbert space $\mathcal{L}_p^2(Q) \times H_p^1(0,1)$ constiting of vector valued function $F = (g,w)$ for wich the norm

$$\|F\|_H^2 = \|w\|_{H_p^1(0,1)}^2 + \|g\|_{\mathcal{L}_p^2(Q)}^2 = \|u(x,0)\|_{\mathcal{L}_p^2(0,1)} + \|u_x(x,0)\|_{\mathcal{L}_p^2(0,1)}^2, \tag{2.3}$$

is finite

2.2 Uniqueness of solution

In this section, on the basis of an **a priori estimate**, we establish a uniqueness result for the solution of the given problem and its dependence on the given data of the posed problem.

Theorem 1. *we have the a priori estimate*

$$\|u\|_S^2 = \|u\|^2_{\mathcal{L}^2(0,T;H_p^{\alpha,t}(0,1))} \leq C^{**}\left(\|w\|^2_{H_p^1(0,1)} + \|g\|^2_{\mathcal{L}^2_p(Q)}\right) = C^{**}\|F\|^2_H, \tag{2.4}$$

for all $u \in D(K)$, *where* C^* *and* $C^{**} > 0$

$$C^{**} = max\left\{C^*, \frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\right\}, C^* = \frac{\frac{1}{2C}+2}{min\left(\frac{1}{2}; \frac{C}{2}+\frac{1}{2}; \frac{1}{4}+\frac{3}{4}C\right)} \tag{2.5}$$

Proof. We consider the inner product in $\mathcal{L}^2(0,1)$ of the integro-differential operator $Mu = xu - xp_x(\zeta u) + x {}_0^c\partial_t^\alpha u$ and $\mathcal{L}u$

$$(\mathcal{L}u, Mu)_{\mathcal{L}^2(0,1)} = \left(\partial_t^\alpha u - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + Cu, xu - xp_x(\zeta u) + x {}_0^c\partial_t^\alpha u\right)_{\mathcal{L}^2(0,1)},$$

$$= (g, xu - xp_x(\zeta u) + x {}_0^c\partial_t^\alpha u)_{\mathcal{L}^2(0,1)}, \tag{2.6}$$

where $$p_x(\zeta u) = \int_0^x \zeta u(\zeta, t) d\zeta,$$

using the initial condition we find

$$-({}_0^c\partial_t^\alpha u, xp_x(\zeta u))_{\mathcal{L}^2(0,1)} = -\int_0^1 x {}_0^c\partial_t^\alpha u p_x(\zeta u) dx. \tag{2.7}$$

$$\left(\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right), p_x(\zeta u)\right)_{\mathcal{L}^2(0,1)} = \int_0^1 \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) p_x(\zeta u) dx = \left[x\frac{\partial u}{\partial x}p_x(\zeta u)\right]_0^1 - \int_0^1 x\frac{\partial u}{\partial x} xu dx$$
$$= -\int_0^1 x^2 u \frac{\partial u}{\partial x} dx = -\frac{1}{2}\int_0^1 x^2 \frac{\partial u^2}{\partial x} dx$$

$$= -\frac{1}{2}[u^2x^2]_0^1 + \frac{1}{2}\int_0^1 2xu^2 dx = \int_0^1 xu^2 dx = \|u\|^2_{\mathcal{L}^2_p(0,1)}. \tag{2.8}$$

$$-(Cu, xp_x(\zeta u))_{\mathcal{L}^2(0,1)} = -C\int_0^1 xup_x(\zeta u) dx = -C\int_0^1 \frac{\partial}{\partial x}(p_x(\zeta u))p_x(\zeta u)$$
$$= \frac{-C}{2}\int_0^1 \frac{\partial}{\partial x^2}(p_x(\zeta u))^2$$

$$= \frac{-C}{2}[(p_x(\zeta u))^2]_0^1 = \frac{-c}{2}\left[\left(\int_0^x \zeta u d\zeta\right)^2\right]_0^1 = 0. \tag{2.9}$$

$$({}_0^c\partial_t^\alpha u, xu)_{\mathcal{L}^2(0,1)} = \int_0^1 {}_0^c\partial_t^\alpha u x u dx = \int_0^1 x u {}_0^c\partial_t^\alpha u dx = ({}_0^c\partial_t^\alpha u, u)_{\mathcal{L}_p^2(0,1)}. \tag{2.10}$$

$$-\left(\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right), u\right)_{\mathcal{L}^2(0,1)} = -\int_0^1 \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}u\right)dx = -\left[x\frac{\partial u}{\partial x}u\right]_0^1 + \int_0^1 x\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}dx$$

$$= \int_0^1 x\left(\frac{\partial u}{\partial x}\right)^2 dx = \|u_x\|^2_{\mathcal{L}_p^2(0,1)}. \tag{2.11}$$

$$(Cu, xu)_{\mathcal{L}^2(0,1)} = C\int_0^1 uxudx = C\int_0^1 xu^2dx = C\|u\|^2_{\mathcal{L}_p^2(0,1)}. \tag{2.12}$$

$$({}_0^c\partial_t^\alpha u, x{}_0^c\partial_t^\alpha u)_{\mathcal{L}^2(0,1)} = \int_0^1 x({}_0^c\partial_t^\alpha u)^2 dx = \|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}_p^2(0,1)}. \tag{2.13}$$

$$-\left(\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right), {}_0^c\partial_t^\alpha u\right)_{\mathcal{L}^2(0,1)} = -\int_0^1 \frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right){}_0^c\partial_t^\alpha u dx = -\left[x\frac{\partial u}{\partial x}\,{}_0^c\partial_t^\alpha u\right]_0^1 + \int_0^1 x\frac{\partial u}{\partial x}\frac{\partial}{\partial x}({}_0^c\partial_t^\alpha u)dx$$

$$= \int_0^1 x\frac{\partial u}{\partial x}\,{}_0^c\partial_t^\alpha u_x dx = ({}_0^c\partial_t^\alpha u_x, u_x)_{\mathcal{L}_p^2(0,1)}. \tag{2.14}$$

$$(Cx{}_0^c\partial_t^\alpha u, u)_{\mathcal{L}^2(0,1)} = C\int_0^1 x{}_0^c\partial_t^\alpha u u dx = (C{}_0^c\partial_t^\alpha u, u)_{\mathcal{L}_p^2(0,1)}. \tag{2.15}$$

Substituation of equalities (2.7)-(2.15) into (2.6), gives

$$\|u_x\|^2_{\mathcal{L}_p^2(0,1)} + C\|u\|^2_{\mathcal{L}_p^2(0,1)} + ({}_0^c\partial_t^\alpha u, u)_{\mathcal{L}_p^2(0,1)} + ({}_0^c\partial_t^\alpha u_x, u_x)_{\mathcal{L}_p^2(0,1)} + \|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}_p^2(0,1)} + C({}_0^c\partial_t^\alpha u, u)_{\mathcal{L}_p^2(0,1)} + \|u\|^2_{\mathcal{L}_p^2(0,1)}$$

$$= -(p_x(\zeta u), xg)_{\mathcal{L}^2(0,1)} + (xu, g)_{\mathcal{L}^2(0,1)} + (x{}_0^c\partial_t^\alpha u, g)_{\mathcal{L}^2(0,1)} + ({}_0^c\partial_t^\alpha u, xp_x(\zeta u))_{\mathcal{L}^2(0,1)}$$

$$\|u_x\|^2_{\mathcal{L}_p^2(0,1)} + C\|u\|^2_{\mathcal{L}_p^2(0,1)} + \int_0^1 xu{}_0^c\partial_t^\alpha u dx + \int_0^1 xu_x{}_0^c\partial_t^\alpha u_x dx + \|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}_p^2(0,1)} + C\int_0^1 u{}_0^c\partial_t^\alpha u dx + \|u\|^2_{\mathcal{L}_p^2(0,1)}$$

$$= -\int_0^1 (\sqrt{x}g)(\sqrt{x}p_x(\zeta u))dx + \int_0^1 (\sqrt{x}u)(\sqrt{x}g)dx + \int_0^1 (\sqrt{x}g)(\sqrt{x}{}_0^c\partial_t^\alpha u)dx$$

$$+ \int_0^1 (\sqrt{x}p_x(\zeta u))(\sqrt{x}{}_0^c\partial_t^\alpha u)dx. \tag{2.16}$$

By using **Cauchy ε inequality** for the right hand side of (2.16), and **Lemma(1)** for the left hand side of Equation (2.16), we give:

$$\|u_x\|^2_{\mathcal{L}^2_p(0,1)}+C\|u\|^2_{\mathcal{L}^2_p(0,1)}+\frac{1}{2}\int_0^1 x{}_0^c\partial_t^\alpha u^2dx+\frac{1}{2}\int_0^1 x{}_0^c\partial_t^\alpha u_x^2dx+\|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}^2_p(0,1)}+\frac{C}{2}\int_0^1 x{}_0^c\partial_t^\alpha u^2dx+\|u\|^2_{\mathcal{L}^2_p(0,1)}$$

$$\leq \frac{\varepsilon_1}{2}\int_0^1 x(p_x(\zeta u))^2dx+\frac{1}{2\varepsilon_1}\int_0^1 xg^2dx+\frac{\varepsilon_2}{2}\int_0^1 x({}_0^c\partial_t^\alpha u)^2dx+\frac{1}{2\varepsilon_2}\int_0^1 x(p_x(\zeta u))^2dx$$
$$+\frac{\varepsilon_3}{2}\int_0^1 xu^2dx+\frac{1}{2\varepsilon_3}\int_0^1 (xg)^2dx+\frac{\varepsilon_4}{2}\int_0^1 x({}_0^c\partial_t^\alpha u)^2+\frac{1}{2\varepsilon_4}\int_0^1 xg^2dx$$

$$\|u_x\|^2_{\mathcal{L}^2_p(0,1)}+(1+C)\|u\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{C}{2}+\frac{1}{2}\right){}_0^c\partial_t^\alpha\|u\|^2_{\mathcal{L}^2_p(0,1)}+\frac{1}{2}{}_0^c\partial_t^\alpha\|u_x\|^2_{\mathcal{L}^2_p(0,1)}+\|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}^2_p(0,1)}$$
$$\leq \frac{\varepsilon_1}{2}\|p_x(\zeta u)\|^2_{\mathcal{L}^2_p(0,1)}+\frac{1}{2\varepsilon_1}\|g\|^2_{\mathcal{L}^2_p(0,1)}+\frac{\varepsilon_2}{2}\|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}^2_p(0,1)}+\frac{1}{2\varepsilon_2}\|p_x(\zeta u)\|^2_{\mathcal{L}^2_p(0,1)}$$
$$+\frac{\varepsilon_3}{2}\|u\|^2_{\mathcal{L}^2_p(0,1)}+\frac{1}{2\varepsilon_3}\|g\|^2_{\mathcal{L}^2_p(0,1)}+\frac{\varepsilon_4}{2}\|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}^2_p(0,1)}+\frac{1}{2\varepsilon_4}\|g\|^2_{\mathcal{L}^2_p(0,1)}. \tag{2.17}$$

Taking $\varepsilon_1 = C, \varepsilon_2 = \frac{1}{2}, \varepsilon_3 = \frac{1}{2}$ and $\varepsilon_4 = \frac{1}{2}$, we obtain

$$\|u_x\|^2_{\mathcal{L}^2_p(0,1)}+(1+C)\|u\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{C}{2}+\frac{1}{2}\right)\left({}_0^c\partial_t^\alpha\|u\|^2_{\mathcal{L}^2_p(0,1)}\right)+\frac{1}{2}{}_0^c\partial_t^\alpha\|u_x\|^2_{\mathcal{L}^2_p(0,1)}+\|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}^2_p(0,1)}$$

$$\leq\left(\frac{C}{2}+1\right)\|p_x(\zeta u)\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{1}{2C}+2\right)\|g\|^2_{\mathcal{L}^2_p(0,1)}+\frac{1}{4}\|u\|^2_{\mathcal{L}^2_p(0,1)}+\frac{1}{2}\|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}^2_p(0,1)}, \tag{2.18}$$

We move the third and fourth terms from the right side to the left side of the inequality (2.18)

$$\|u_x\|^2_{\mathcal{L}^2_p(0,1)}+(1+C)\|u\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{C}{2}+\frac{1}{2}\right)\left({}_0^c\partial_t^\alpha\|u\|^2_{\mathcal{L}^2_p(0,1)}\right)+\frac{1}{2}{}_0^c\partial_t^\alpha\|u_x\|^2_{\mathcal{L}^2_p(0,1)}+\|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}^2_p(0,1)}$$

$$-\frac{1}{4}\|u\|^2_{\mathcal{L}^2_p(0,1)}-\frac{1}{2}\|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}^2_p(0,1)}\leq\left(\frac{C}{2}+1\right)\|p_x(\zeta u)\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{1}{2C}+2\right)\|g\|^2_{\mathcal{L}^2_p(0,1)}. \tag{2.19}$$

Applying Poincaré's theorem to the first term on the right-hand side of (2.19), leads to:

$$\|u_x\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{3}{4}+C\right)\|u\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{C}{2}+\frac{1}{2}\right)\left({}_0^c\partial_t^\alpha\|u\|^2_{\mathcal{L}^2_p(0,1)}\right)+\frac{1}{2}{}_0^c\partial_t^\alpha\|u_x\|^2_{\mathcal{L}^2_p(0,1)}$$

$$+\frac{1}{2}\|{}_0^c\partial_t^\alpha u\|^2_{\mathcal{L}^2_p(0,1)}\leq\left(\frac{C}{4}+\frac{1}{2}\right)\|u\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{1}{2C}+2\right)\|g\|^2_{\mathcal{L}^2_p(0,1)}. \tag{2.20}$$

So

$$\|u_x\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{1}{4}+\frac{3}{4}C\right)\|u\|^2_{\mathcal{L}^2_p(0,1)}+\left(\frac{C}{2}+\frac{1}{2}\right)\left({}^c_0\partial^\alpha_t\|u\|^2_{\mathcal{L}^2_p(0,1)}\right)+\frac{1}{2}\,{}^c_0\partial^\alpha_t\|u_x\|^2_{\mathcal{L}^2_p(0,1)}$$

$$+\frac{1}{2}\|{}^c_0\partial^\alpha_t u\|^2_{\mathcal{L}^2_p(0,1)}\leq\left(\frac{1}{2C}+2\right)\|g\|^2_{\mathcal{L}^2_p(0,1)}. \tag{2.21}$$

We eliminate the first term on the left hand side of the inequality(2.21):

$$min\left(\frac{1}{2};\frac{C}{2}+\frac{1}{2};\frac{1}{4}+\frac{3}{4}C\right)\times\left[\|u\|^2_{\mathcal{L}^2_p(0,1)}+\|\partial^\alpha_t u\|^2_{\mathcal{L}^2_p(0,1)}+{}^c_0\partial^\alpha_t\left(\|u\|^2_{\mathcal{L}^2_p(0,1)}+\|u_x\|^2_{\mathcal{L}^2_p(0,1)}\right)\right]$$

$$\leq\left(\frac{1}{2C}+2\right)\|g\|^2_{\mathcal{L}^2_p(0,1)}. \tag{2.22}$$

There fore:

$$\|u\|^2_{\mathcal{L}^2_p(0,1)}+\|{}^c_0\partial^\alpha_t u\|^2_{\mathcal{L}^2_p(0,1)}+{}^c_0\partial^\alpha_t\left(\|u\|^2_{\mathcal{L}^2_p(0,1)}+\|u_x\|^2_{\mathcal{L}^2_p(0,1)}\right)\leq\frac{\frac{1}{2C}+2}{min\left(\frac{1}{2};\frac{C}{2}+\frac{1}{2};\frac{1}{4}+\frac{3}{4}C\right)}\|g\|^2_{\mathcal{L}^2_p(0,1)}. \tag{2.23}$$

So

$$\|u\|^2_{H^{\alpha,t}_p(0,1)}+{}^c_0\partial^\alpha_t\|u\|^2_{H^1_p(0,1)}\leq C^*\|g\|^2_{\mathcal{L}^2_p(0,1)}. \tag{2.24}$$

Where

$$C^*=\frac{\frac{1}{2C}+2}{min\left(\frac{1}{2};\frac{C}{2}+\frac{1}{2};\frac{1}{4}+\frac{3}{4}C\right)}. \tag{2.25}$$

Integration both sides of equation (2.24) over $(0.t)$ gives

$$\int_0^t\|u(x,v)\|^2_{H^{\alpha,t}_p(0,1)}dv+D^{\alpha-1}\|u\|^2_{H^1_p(0,1)}-\frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\|u(x,0)\|^2_{H^1_p(0,1)}\leq C^*\int_0^t\|g(x,v)\|^2_{\mathcal{L}^2_p(0,1)}dv,$$

$$\int_0^t\|u(x,v)\|^2_{H^{\alpha,t}_p(0,1)}dv+D^{\alpha-1}\|u\|^2_{H^1_p(0,1)}\leq C^*\int_0^t\|g(x,v)\|^2_{\mathcal{L}^2_p(0,1)}dv+\frac{\Gamma^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\|w\|^2_{H^1_p(0,1)},$$

$$\leq C^{**}\left(\int_0^t\|g(x,v)\|^2_{\mathcal{L}^2_p(0,1)}dv+\|w\|^2_{H^1_p(0,1)}\right), \tag{2.26}$$

where

$$C^{**}=max\left\{C^*,\frac{T^{1-\alpha}}{(1-\alpha)\Gamma(1-\alpha)}\right\}. \tag{2.27}$$

If we discard the second term on the left hand side of equation(2.26) and replace t by T, we obtain the desired inequality:

$$\|u\|^2_{\mathcal{L}^2(0,T;H_p^{\alpha,t}(0,1))} \leq C^{**}\left(\|w\|^2_{H_p^1(0,1)} + \|g\|^2_{\mathcal{L}_p^2(Q)}\right), \tag{2.28}$$

$$\|u\|^2_S \leq C^{**}\|F\|^2_H. \tag{2.29}$$

□

Corollary 1. *the solution of problem(2.1) is unique*

Proof. Let u_1 and u_2 be two solution of problem (2.1)

$$\begin{cases} Ku_1 = F \\ Ku_2 = F \end{cases} \Rightarrow Ku_1 - Ku_2 = 0, \tag{2.30}$$

and as K is linear we obtain $K(u_1 - u_2) = 0$,
according to (2.4)

$$\|u_1 - u_2\|^2_S \leq 0 \Rightarrow \|u_1 - u_2\|^2_S = 0 \Rightarrow u_1 - u_2 = 0 \Rightarrow u_1 = u_2. \tag{2.31}$$

Hence, we deduce the uniqueness of solution. □

2.3 Existence of solution

In this section, we prove a result concerning the existence of the solution of the given problem it follows from Inequality (2.4) that the operator K admit an inverse K^{-1} : $Im(K) \to S$ since $Im(K) \subset H$, we then can construct its clocure $\overline{K}$ such that Inequality (2.4) holds for $\overline{K}$ and $Im(\overline{K}) = H$.

Proving the existence of the solution is based on the following three steps
1. The operator $K : S \to H$ is closed,
2. $\overline{K}u = H$ and $Im(\overline{K}) = H$ is a closed subset in H.
and $Im(K) = Im(\overline{K})$ and $\overline{K}^{-1} = \overline{K^{-1}}$,
3. $Im(K)$ is dense in H.

Proposition 1. *(spesial case of density) If for all $u \in D_0(k)$ such that $l_1 u = 0$ and for some function $\phi \in \mathcal{L}^2(Q)$ we have:*

$$\int_0^T (\mathcal{L}u, \phi)_{\mathcal{L}^2_p(0,1)} dt = 0, \tag{2.32}$$

then ϕ is zero a.e in Q.

Proof.

$$\int_0^T (\mathcal{L}u, \phi)_{\mathcal{L}^2_p(0,1)} dt = 0,$$

are equivalent

$$\int_0^T \left(\partial_t^\alpha u - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right) + Cu, \phi \right)_{\mathcal{L}^2_p(0,1)} dt = 0. \tag{2.33}$$

Suppose that a function $\Gamma(x,t)$ satisfies boundary and initial conditions of equation (2.1) and such that $\Gamma, \Gamma_x, \frac{\partial}{\partial x}(x\mathcal{I}_t(\Gamma(x,s)),$
and $\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)) \in \mathcal{L}^2(Q_t)$ we then let:

$$u(x,t) = \mathcal{I}_t(\Gamma(x,s)) = \int_0^t \Gamma(x,s)ds. \tag{2.34}$$

Equation (2.33) takes the form

$$\int_0^T \left({}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)) - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s)))\right) + C(\mathcal{I}_t(\Gamma(x,s))), \phi \right)_{\mathcal{L}^2_p(0,1)} dt = 0. \tag{2.35}$$

We now consider the function

$$\phi(x,t) = \mathcal{I}_t(\Gamma(x,s)) + p_x(\zeta\mathcal{I}_t(\Gamma(x,s))) +_0^c \partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)), \tag{2.36}$$

we replace ϕ with its value

$$\int_0^T ({}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)), x\mathcal{I}_t(\Gamma(x,s)) + xp_x(\zeta\mathcal{I}_t(\Gamma(x,s))) + x{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)))_{\mathcal{L}^2(0,1)}dt$$

$$-\int_0^T \left(\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s)))\right), \mathcal{I}_t(\Gamma(x,s)) + p_x(\zeta\mathcal{I}_t(\Gamma(x,s))) +_0^c \partial_t^\alpha \mathcal{I}_t(\Gamma(x,s))\right)_{\mathcal{L}^2(0,1)})dt$$

$$+\int_0^T (C\mathcal{I}_t(\Gamma(x,s)), x\mathcal{I}_t(\Gamma(x,s)) + xp_x(\zeta\mathcal{I}_t(\Gamma(x,s))) + x{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)))_{\mathcal{L}^2(0,1)})dt = 0. \tag{2.37}$$

Since Γ satisfies boundary codition in equation (2.1), and **Lemma 1**, then we have

$$\begin{aligned}({}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)), x\mathcal{I}_t(\Gamma(x,s)))_{\mathcal{L}^2(0,1)} &= \int_0^1 x{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s))\mathcal{I}_t(\Gamma(x,s))dx \\ &\geq \frac{1}{2}\int_0^1 x{}_0^c\partial_t^\alpha (\mathcal{I}_t(\Gamma(x,s)))^2 \\ &= \frac{1}{2}\, {}_0^c\partial_t^\alpha \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}_p^2(0,1)},\end{aligned} \tag{2.38}$$

by using **the Cauchy inequality with** ε we find

$$\begin{aligned}-(\,{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)), xp_x(\zeta\mathcal{I}_t(\Gamma(x,s))))_{\mathcal{L}^2(0,1)} &= \int_0^1 \sqrt{x}(-{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)))\sqrt{x}p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))dx \\ &\leq \frac{\varepsilon_1}{2}\int_0^1 x(\,{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)))^2dx \\ &+\frac{1}{2\varepsilon_1}\int_0^1 x(p_x(\zeta\mathcal{I}_t(\Gamma(x,s))))^2dx, \\ &= \frac{\varepsilon_1}{2}\|{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}_p^2(0,1)} \\ &+\frac{1}{2\varepsilon_1}\|p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))\|^2_{\mathcal{L}_p^2(0,1)}.\end{aligned} \tag{2.39}$$

$$\begin{aligned}({}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)), x{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)))_{\mathcal{L}^2(0,1)} &= \int_0^1 x{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s)){}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s))dx \\ &= \|{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}_p^2(0,1)}.\end{aligned} \tag{2.40}$$

$$
\begin{aligned}
-\left(\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s)))\right),\mathcal{I}_t(\Gamma(x,s))\right)_{\mathcal{L}^2(0,1)} &= -\int_0^1 \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s)))\right)\mathcal{I}_t(\Gamma(x,s))dx,\\
&= -\left[x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s)))\mathcal{I}_t(\Gamma(x,s)))\mathcal{I}_t(\Gamma(x,s))\right]_0^1\\
&+\int_0^1 x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s)))\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s)))dx\\
&= \left\|\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s)))\right\|^2_{\mathcal{L}^2_p(0,1)}\\
&= \|\mathcal{I}_t(\Gamma_x(x,s))\|^2_{\mathcal{L}^2_p(0,1)}. \qquad (2.41)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))\right),p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))\right)_{\mathcal{L}^2(0,1)}\\
&= -\int_0^1 \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))\right)p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))dx\\
&= -\left[x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))\right]_0^1+\int_0^1 x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))\frac{\partial}{\partial x}(p_x(\zeta\mathcal{I}_t(\Gamma(x,s))))dx\\
&= \int_0^1 x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))x\mathcal{I}_t(\Gamma(x,s))dx = \int_0^1 x^2\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))\mathcal{I}_t(\Gamma(x,s))dx\\
&= \int_0^1 \frac{x^2}{2}\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))^2dx = \frac{1}{2}[x^2(\mathcal{I}_t(\Gamma(x,s)))^2]_0^1 - \int_0^1 x(\mathcal{I}_t(\Gamma(x,s)))^2dx\\
&= \frac{1}{2}(\mathcal{I}_t(\Gamma(1,s)))^2 - \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)}. \qquad (2.42)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))\right),{}^c_0\partial_t^\alpha\mathcal{I}_t(\Gamma(x,s))\right)_{\mathcal{L}^2(0,1)}\\
&= -\int_0^1 \frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))\right)\,{}^c_0\partial_t^\alpha\mathcal{I}_t(\Gamma(x,s))dx,\\
&= -\left[x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))^\partial_t\alpha\mathcal{I}_t(\Gamma(x,s))\right]_0^1+\int_0^1 x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))\frac{\partial}{\partial x}(\,{}^c_0\partial_t^\alpha\mathcal{I}_t(\Gamma(x,s)))dx,\\
&= \int_0^1 x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s)))\frac{\partial}{\partial x}(\,{}^c_0\partial_t^\alpha\mathcal{I}_t(\Gamma(x,s)))dx = \int_0^1 x\mathcal{I}_t(\Gamma_x(x,s))\,{}^c_0\partial_t^\alpha\mathcal{I}_t(\Gamma_x(x,s))dx.
\end{aligned}
$$

By using **Lemma 1** we find

$$-\left(\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\mathcal{I}_t(\Gamma(x,s))\right),{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s))\right)_{\mathcal{L}^2(0,1)} \geq \frac{1}{2}\int_0^1 x{}_0^c\partial_t^\alpha(\mathcal{I}_t(\Gamma_x(x,s)))^2 dx$$

$$= \frac{1}{2}\,{}_0^c\partial_t^\alpha \|\mathcal{I}_t(\Gamma_x(x,s))\|^2_{\mathcal{L}^2_p(0,1)}. \quad (2.43)$$

$$(C\mathcal{I}_t(\Gamma(x,s)), x\mathcal{I}_t(\Gamma(x,s)))_{\mathcal{L}^2(0,1)} = C\int_0^1 x\mathcal{I}_t(\Gamma(x,s))\mathcal{I}_t(\Gamma(x,s))dx$$

$$= C\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)}. \quad (2.44)$$

By using **the cauchy inequality with** ε we find

$$-(C\mathcal{I}_t(\Gamma(x,s)), xp_x(\zeta\mathcal{I}_t(\Gamma(x,s))))_{\mathcal{L}^2(0,1)} = C\int_0^1 \sqrt{x}(-\mathcal{I}_t(\Gamma(x,s)))\sqrt{x}p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))dx,$$

$$\leq \frac{C\varepsilon_2}{2}\int_0^1 x(\mathcal{I}_t(\Gamma(x,s)))^2 dx$$

$$+\frac{C}{2\varepsilon_2}\int_0^1 x(p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))^2 dx,$$

$$= \frac{C\varepsilon_2}{2}\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)}$$

$$+\frac{C}{2\varepsilon_2}\|p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))\|^2_{\mathcal{L}^2_p(0,1)}. \quad (2.45)$$

$$(C\mathcal{I}_t(\Gamma(x,s)), x{}_0^c\partial_t^\alpha\mathcal{I}_t(\Gamma(x,s)))_{\mathcal{L}^2(0,1)} = C\int_0^1 x\mathcal{I}_t(\Gamma(x,s)){}_0^c\partial_t^\alpha\mathcal{I}_t(\Gamma(x,s))dx,$$

by using **Lemma 1** we find

$$(C\mathcal{I}_t(\Gamma(x,s)), x{}_0^c\partial_t^\alpha\mathcal{I}_t(\Gamma(x,s)))_{\mathcal{L}^2(0,1)} \geq \frac{C}{2}\,{}_0^c\partial_t^\alpha\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)}. \quad (2.46)$$

A combination of equation (2.37)-(2.46) gives the inequality

$$(\mathcal{I}_t(\Gamma(1,s)))^2 +{}_0^c\partial_t^\alpha\|\mathcal{I}_t(\Gamma_x(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + 2\|\mathcal{I}_t(\Gamma_x(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + 2\|{}_0^c\partial_t^\alpha\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)}$$

$$+(1+C)\,{}_0^c\partial_t^\alpha\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + 2C\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)}$$

$$\leq 2\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + \varepsilon_1\|{}_0^c\partial_t^\alpha\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + \frac{1}{\varepsilon_1}\|p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))\|^2_{\mathcal{L}^2_p(0,1)}$$

$$+C\varepsilon_2\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + \frac{C}{\varepsilon_2}\|p_x(\zeta\mathcal{I}_t(\Gamma(x,s)))\|^2_{\mathcal{L}^2_p(0,1)}.$$

By applying the **Poincaré inequality** for the third and fifth term on the right-hand

$$(\mathcal{I}_t(\Gamma(1,s)))^2 +_0^c \partial_t^\alpha \|\mathcal{I}_t(\Gamma_x(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + 2\|\mathcal{I}_t(\Gamma_x(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + 2\|{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)}$$

$$+(1+C)\ {}_0^c\partial_t^\alpha \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + 2C\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)}$$

$$\leq 2\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + \varepsilon_1 \|{}_0^c\partial_t^\alpha \mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + \frac{1}{2\varepsilon_1}\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)}$$

$$+C\varepsilon_2 \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} + \frac{C}{2\varepsilon_2}\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)}. \tag{2.47}$$

Put $\varepsilon_1 = 2$ and $\varepsilon_2 = 2$ and ingnore the first three terms on the left hand side of equation (2.47) it follows that

$$ {}_0^c\partial_t^\alpha \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} \leq \frac{C+9}{4+4C}\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)}. \tag{2.48}$$

Integration over (0,t) in equation (2.48) leads to

$$D^{\alpha-1}\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_p(0,1)} \leq C^* \int_0^t \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)} d\tau, \tag{2.49}$$

where

$$C^* = \frac{C+9}{4C+4}$$

. Applying **Lemma 2** to equation (2.49), after putting:

$$\mathcal{J}(t) = \int_0^t \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)} d\tau,$$

$$\mathcal{J}(0) = 0, r_1 = C^*, r_2 = 0.$$

and

$$ {}_0^c\partial_t^\alpha \mathcal{J}(t) =_0^c \partial_t^\alpha \left(\int_0^t \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)} d\tau \right) = D_t^{\alpha-1}\|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)},$$

then

$$\int_0^t \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)} d\tau \leq \mathcal{J}(0)E_\alpha(C^* t^\alpha) + \Gamma(\alpha)E_{\alpha\alpha}(C^* t^\alpha)D_t^{-\alpha}(0) = 0. \tag{2.50}$$

Realacing t by T, it follows then from Equation (2.50)that

$$\int_0^T \|\mathcal{I}_t(\Gamma(x,s))\|^2_{\mathcal{L}^2_P(0,1)} d\tau \leq 0. \tag{2.51}$$

Hence $\phi = 0$ a.e in Q.

We now complete the proof,

we suppose that for $(\psi, \xi_1) \in Im(K)^\perp$ we have

$$\int_0^T (\mathcal{L}u, \psi)_{\mathcal{L}^2_p(0,1)} dt + (l_1, \xi_1)_{H^1_p(0,1)} = 0, \tag{2.52}$$

then we should show that $\psi = 0$, take $u \in D(k)$ such that $l_1 u = 0$ in (1.53), then we have

$$\int_0^T (\mathcal{L}u, \psi)_{\mathcal{L}_p^2(0,1)} dt = 0, \forall u \in D(K), \tag{2.53}$$

it follow from equation(2.32)and equation(2.53), that $\psi = 0$ a.e in Q. Hence equation (2.52) takes the forms

$$(l_1 u, \xi_1)_{H_p^1(0,1)} = 0, \forall u \in D(k). \tag{2.54}$$

Since $Im(l_1)$ is dence in $H_p^1(0,1)$ (because it is compact operator), we deduce from equation (2.54) that $\xi_1 = 0$.
From Proposition 1, we find $Im(K_0) = 0, ie; \overline{Im(K_0)} = H$.
Now consider the general case. From the fact that $Im(K)$ is dense in H, we conclude that we can prove that $Im(K)$ is dense in H by means of the continuation method along the parameter (see [2]). This is what leads to Proposition 2.

Proposition 2. *$Im(K)$ is dense in H, ie $\overline{Im(K)} = H$.*

Theorem 2. *Assume that conditions of Theorem (1) hold. Then for all $F = (g, w) \in H$, there exists a unique strong solution $u = \overline{K}^{-1} F = \overline{K^{-1}} F$ of problem (2.1).*

□

Chapter 3

The Homotopy analysis method

In this chapter we test the efficiency **the homotopy analysis method** for solving the fractional nonlocal mixed problem with the bessel operator and we provide some examlpes then search for the numerical solutions by using this method.

3.1 Application of the method and numerical results

To test the efficiency of the HAM for solving the fractional non-local mixed problem with the Bessel operator, we consider the equivalent initial-boundary value problem

$$\begin{cases} {}_0^c\partial_t^\alpha u - \frac{1}{x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + Cu(x,t) = g(x,t) \quad 0 < x, \alpha < 1, 0 < t < T, \\ u(x,0) = w(x), x \in (0,1), \\ u_x(1,t) = 0, u(1,t) = d(t), \end{cases} \tag{3.1}$$

where g, w and d are some given functions and $C \geq 0$.
We consider the initial approximation

$$u_0(x,t) = u(x,0),$$

and the linear operator with the non-integer order

$$\mathcal{L}; [\phi(x,t;q)] =_0^c \partial_t^\alpha \phi(x,t;q), \quad 0 < \alpha < 1,$$

such that $\mathcal{L}(k) = 0$ where k represents an integral constant.
We consider the fractional partial differential operator

$$\mathcal{F}[\phi(x,t;q)] =_0^c \partial_t^\alpha \phi(x,t;q)) - \frac{1}{x}\frac{\partial \phi}{\partial x} - \frac{\partial^2 \phi}{\partial x^2} + C\phi(x,t) - g(x,t),$$

hence the zeroth-order deformation equation is given by

$$(1-q)\mathcal{L}[\phi(x,t;q) - u_0(x,t)] = qh\mathcal{F}[\phi(x,t;q)],$$

then, at $q = 0$ and $q = 1$, we have

$$\phi(x, t; 0) = u_0(x, t) = u(x, 0), and \quad \phi(x, t; 1) = u(x, t),$$

respectively.
On the other hand, the mth-order deformation equation is given by

$$\mathcal{L}[u_m(x, t) - \chi_m u_{m-1}(x, t)] = h\mathcal{R}_m(\vec{u}_{m-1}), \tag{3.2}$$

where

$$\mathcal{R}_m(\vec{u}_{m-1}) = {}_0^c\partial_t^\alpha u_{m-1} - \frac{1}{x}\frac{\partial u_{m-1}}{\partial x} - \frac{\partial^2 u_{m-1}}{\partial x^2} + Cu_{m-1} - (1 - \chi_m)g(x, t), \tag{3.3}$$

and

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{3.4}$$

For $m \geq 1$, the solution of the mth-order deformation equation (3.2) can be obtained recessively through the iterative scheme:

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + h\mathcal{L}^{-1}[\mathcal{R}_m(\vec{u}_{m-1}(x, t))], \tag{3.5}$$

or

$$u_m(x, t) = \chi_m u_{m-1}(x, t) + h\partial_t^{-\alpha}[\mathcal{R}_m(\vec{u}_{m-1}(x, t))]. \tag{3.6}$$

Now, we apply the HAM to the following test examples, to illustrat the efficiency of this method in solving fractional partial differntial equations in the from of equation (3.2)
Example 1. Consider the fractional homogeous initial/boundary value problem

$$\begin{cases} {}_0^c\partial_t^\alpha u - \frac{1}{x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + Cu = cos(t) + \frac{1}{2x}, & 0 < t < T, 0 < \alpha < 1, \\ u(x, 0) = \frac{1}{2}ln(x) - \frac{1}{2}x + \partial_t^{-\alpha}(1), & x \in (0, 1), \\ u(1, t) = -\frac{1}{2} + \partial_t^{-\alpha}(cos(t)), u_x(1, t) = 0, & \forall t \in (0, T). \end{cases} \tag{3.7}$$

Taking $C = 0, g(x, t) = cos(t) + \frac{1}{2x}, d(t) = u(1, t) = -\frac{1}{2} + \partial_t^{-\alpha}(cos(t))$ and
$u_0(x, t) = u(x, 0) = \frac{1}{2}ln(x) - \frac{1}{2}x + \partial_t^{-\alpha}(1)$ then:
put $m = 1$ in eq(3.6), we get

$$\begin{aligned} u_1(x, t) &= \chi_1 u_0(x, t) + h\partial_t^{-\alpha}[\mathcal{R}_1(\vec{u}_0(x, t))], \\ &= h\partial_t^{-\alpha}\left[{}_0^c\partial_t^\alpha u_0 - \frac{1}{x}\frac{\partial u_0}{\partial x} - \frac{\partial^2 u_0}{\partial x^2} - cos(t) - \frac{1}{2x}\right], \\ &= h\partial_t^{-\alpha}(-cos(t) + 1), \end{aligned} \tag{3.8}$$

then, put $m = 2$ in eq(2.6), we get

$$u_2(x,t) = \chi_2 u_1(x,t) + h\partial_t^{-\alpha}[\mathcal{R}_2(\vec{u}_1(x,t))],$$

$$= u_1(x,t) + h\partial_t^{-\alpha}\left[\ {}_0^c\partial_t^{\alpha} u_1 - \frac{1}{x}\frac{\partial u_1}{\partial x} - \frac{\partial^2 u_1}{\partial x^2}\right],$$

$$= h\partial_t^{-\alpha}(-cos(t)+1)+$$

$$h\partial_t^{-\alpha}\left[\ {}_0^c\partial_t^{\alpha}(h\partial_t^{-\alpha}(-cos(t)+1)) - \frac{1}{x}\frac{\partial}{\partial x}(h\partial_t^{-\alpha}(-cos(t)+1)) - \frac{\partial^2}{\partial x^2}(h\partial_t^{-\alpha}(-cos(t)+1))\right],$$

$$= h\partial_t^{-\alpha}(-cos(t)+1) + h\partial_t^{-\alpha}[\partial_t^{\alpha}(h\partial_t^{-\alpha}(-cos(t)+1))],$$

$$= h\partial_t^{-\alpha}(-cos(t)+1) + h^2\partial_t^{-\alpha}(-cos(t+1)),$$

$$= h(1+h)\partial_t^{-\alpha}(-cos(t)+1) \tag{3.9}$$

take $m = 3$, then

$$u_3(x,t) = \chi_3 u_2(x,t) + h\partial_t^{-\alpha}[\mathcal{R}_3(\vec{u}_2(x,t))],$$

$$= u_2(x,t) + h\partial_t^{-\alpha}\left[\ {}_0^c\partial_t^{\alpha} u_2 - \frac{1}{x}\frac{\partial u_2}{\partial x} - \frac{\partial^2 u_2}{\partial x^2}\right]$$

$$= h(1+h)\partial_t^{-\alpha}(-cos(t)+1) + h^2(1+h)\partial_t^{-\alpha}(\ {}_0^c\partial_t^{\alpha}(\partial_t^{-\alpha}(-cos(t)+1)))$$

$$= h(1+h)\partial_t^{-\alpha}(-cos(t)+1) + h^2(1+h)\partial_t^{-\alpha}(-cos(t)+1)$$

$$= h(1+h)^2\partial_t^{-\alpha}(-cos(t)+1). \tag{3.10}$$

$$u_4(x,t) = \chi_4 u_3(x,t) + h\partial_t^{-\alpha}[\mathcal{R}_4(\vec{u}_3(x,t))]$$

$$= u_3(x,t) + h\partial_t^{-\alpha}\left[\ {}_0^c\partial_t^{\alpha} u_3 - \frac{1}{x}\frac{\partial u_3}{\partial x} - \frac{\partial^2 u_3}{\partial x^2}\right]$$

$$= h(1+h)^2\partial_t^{-\alpha}(-cos(t)+1) + h^2(1+h)^2\partial_t^{-\alpha}(-cos(t)+1),$$

$$= h(1+h)^3\partial_t^{-\alpha}(-cos(t)+1). \tag{3.11}$$

and so on, Thus, the series solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...,$$

$$= \frac{1}{2}ln(x) - \frac{1}{2}x + \partial_t^{-\alpha}(1)+$$

$$h\partial_t^{-\alpha}(-cos(t)+1) + h(1+h)\partial_t^{-\alpha}(-cos(t)+1) + h(1+h)^2\partial_t^{-\alpha}(-cos(t)+1) + ...,$$

$$= \frac{1}{2}ln(x) - \frac{1}{2}x + \partial_t^{-\alpha}(1) + h\partial_t^{-\alpha}(-cos(t)+1)(1+(1+h)+(1+h)^2+...),$$

$$= \frac{1}{2}ln(x) - \frac{1}{2}x + \partial_t^{-\alpha}(1) + h\partial_t^{-\alpha}(-cos(t) + 1)\sum_{j=0}^{\infty}(1+h)^j,$$

$$= \frac{1}{2}x\ln(x) - \frac{1}{2}x + \partial_t^{-\alpha}(1) + h\partial_t^{-\alpha}(-cos(t) + 1)\left(\frac{1-(1+h)^{j+1}}{-h}\right). \tag{3.12}$$

If the auxilary parametre h is selected so that $|1 + h| < 1$, then the last power series converges and gives

$$u(x,t) = \frac{1}{2}ln(x) - \frac{1}{2}x + \partial_t^{-\alpha}(1) - \partial_t^{-\alpha}(-cos(t) + 1),$$

$$= \frac{1}{2}ln(x) - \frac{1}{2}x + \partial_t^{-\alpha}(cos(t)). \tag{3.13}$$

Which is the exact solution for $0 < \alpha < 1$,
for $\alpha = 1$ setting $u_0(x,t) = \frac{1}{2}ln(x) - \frac{1}{2}x + \partial_t^{-1}(1) = \frac{1}{2}ln(x) - \frac{1}{2}x + t$,

$$u_1(x,t) = h\partial_t^{-1}(-cos(t) + 1) = h(\partial_t^{-1}(-cos(t)) + \partial_t^{-1}(1)),$$

$$= h\left(\frac{-1}{\Gamma(1)}\int_0^t cos(\tau)d\tau + t\right) = h(-sin(t) + t). \tag{3.14}$$

then:

$$u_2(x,t) = h(1+h)(-sin(t) + t). \tag{3.15}$$

$$u_3(x,t) = h(1+h)^2(-sin(t) + t). \tag{3.16}$$

$$u_4(x,t) = h(1+h)^3(-sin(t) + t). \tag{3.17}$$

Hence the series solution becomes

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...,$$

$$= \frac{1}{2}ln(x) - \frac{1}{2}x + t + h(-sin(t) + t) + h(1+h)(-sin(t) + t) + h(1+h)^2(-sin(t) + t) + ...,$$

$$= \frac{1}{2}ln(x) - \frac{1}{2}x + t + h(-sin(t) + t)(1 + (1+h) + (1+h)^2 + ...),$$

$$= \frac{1}{2}ln(x) - \frac{1}{2}x + t + h(-sin(t) + t)\sum_{j=0}^{\infty}(1+h)^j,$$

$$= \frac{1}{2}ln(x) - \frac{1}{2}x + sin(t), \tag{3.18}$$

provided that $|1 + h| < 1$.

Example 2. Consider the fractional homogeous initial/boundary value problem

$$\begin{cases} {}_0^c\partial_t^\alpha u - \frac{1}{x}\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + Cu = e^t + \frac{1}{x}, \quad 0 < t < T, 0 < \alpha < 1, \\ u(x,0) = -x + ln(x) + \partial_t^{-\alpha}(1), \quad x \in (0,1), \\ u(1,t) = -1 + \partial_t^{-\alpha}(e^t), u_x(1,t) = 0, \quad \forall t \in (0,T). \end{cases} \tag{3.19}$$

Taking $C = 0, g(x,t) = e^t + \frac{1}{x}, d(t) = u(1,t) = -1 + \partial_t^{-\alpha}(e^t)$ and
$u_0(x,t) = u(x,0) = -x + ln(x) + \partial_t^{-\alpha}(1)$
then: put $m = 1$ in (eq)(2.3),we get

$$\begin{aligned} u_1(x,t) &= \chi_1 u_0(x,t) + h\partial_t^{-\alpha}[\mathcal{R}_1(\vec{u}_0(x,t))], \\ &= h\partial_t^{-\alpha}\left[{}_0^c\partial_t^\alpha u_0 - \frac{1}{x}\frac{\partial u_0}{\partial x} - \frac{\partial^2 u_0}{\partial x^2} - e^t - \frac{1}{x}\right], \\ &= h\partial_t^{-\alpha}(1 - e^t), \end{aligned} \tag{3.20}$$

then:

$$\begin{aligned} u_2(x,t) &= \chi_2 u_1(x,t) + h\partial_t^{-\alpha}[\mathcal{R}_2(\vec{u}_1(x,t))], \\ &= u_1(x,t) + h\partial_t^{-\alpha}\left[{}_0^c\partial_t^\alpha u_1 - \frac{1}{x}\frac{\partial u_1}{\partial x} - \frac{\partial^2 u_1}{\partial x^2}\right], \\ &= h\partial_t^{-\alpha}(1 - e^t) + h\partial_t^{-\alpha}\left[{}_0^c\partial_t^\alpha(h\partial_t^{-\alpha}(1 - e^t)) - \frac{1}{x}{}_0^c\frac{\partial}{\partial x}(h\partial_t^{-\alpha}(1 - e^t)) - \frac{\partial^2}{\partial x^2}(h\partial_t^{-\alpha}(1 - e^t))\right], \\ &= h\partial_t^{-\alpha}(1 - e^t) + h\partial_t^{-\alpha}h(1 - e^t), \\ &= \partial_t^{-\alpha}(1 - e^t) + h^2\partial_t^{-\alpha}(1 - e^t) \\ &= h(1 + h)\partial_t^{-\alpha}(1 - e^t), \end{aligned} \tag{3.21}$$

take $m = 3$, then

$$\begin{aligned} u_3 &= \chi_3 u_2(x,t) + h\partial_t^{-\alpha}[\mathcal{R}_3(\vec{u}_2(x,t))], \\ &= u_2(x,t) + h\partial_t^{-\alpha}\left[{}_0^c\partial_t^\alpha u_2 - \frac{1}{x}\frac{\partial u_2}{\partial x} - \frac{\partial^2 u_2}{\partial x^2}\right], \\ &= h(1 + h)\partial_t^{-\alpha}(1 - e^t) + h\partial_t^{-\alpha}[{}_0^c\partial_t^\alpha(h(1 + h)\partial_t^{-\alpha}(1 - e^t))], \\ &= h(1 + h)\partial_t^{-\alpha}(1 - e^t) + h^2(1 + h)\partial_t^{-\alpha}(1 - e^t), \\ &= h(1 + h)^2\partial_t^{-\alpha}(1 - e^t), \end{aligned} \tag{3.22}$$

$$u_4(x,t) = \chi_4 u_3(x,t) + h\partial_t^{-\alpha}[\mathcal{R}_4(\vec{u}_3(x,t))],$$
$$= u_3(x,t) + h\partial_t^{-\alpha}\left[\ {}_0^c\partial_t^\alpha u_3 - \frac{1}{x}\frac{\partial u_3}{\partial x} - \frac{\partial^2 u_3}{\partial x^2}\right],$$
$$= h(1+h)^2\partial_t^{-\alpha}(1-e^t) + h^2(1+h)^2\partial_t^{-\alpha}(\ {}_0^c\partial_t^\alpha(\partial_t^{-\alpha}(1-e^t))),$$
$$= h(1+h)^2\partial_t^{-\alpha}(1-e^t) + h^2(1+h)^2\partial_t^{-\alpha}(1-e^t),$$
$$= h(1+h)^2(1+h)\partial_t^{-\alpha}(1-e^t),$$
$$= h(1+h)^3\partial_t^{-\alpha}(1-e^t), \tag{3.23}$$
$$u_5(x,t) = \chi_5 u_4(x,t) + h\partial_t^{-\alpha}[\mathcal{R}_5(\vec{u}_4(x,t))],$$
$$= u_4(x,t) + h\partial_t^{-\alpha}\left[\ {}_0^c\partial_t^\alpha u_4 - \frac{1}{x}\frac{\partial u_4}{\partial x} - \frac{\partial^2 u_4}{\partial x^2}\right],$$
$$= h(1+h)^3\partial_t^{-\alpha}(1-e^t) + h^2(1+h)^3\partial_t^{-\alpha}(1-e^t),$$
$$= h(1+h)^4\partial_t^{-\alpha}(1-e^t), \tag{3.24}$$

and so on. Thus, the series solution is:

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ...,$$
$$= -x + ln(x) + \partial_t^{-\alpha}(1) + h\partial_t^{-\alpha}(1-e^t) + h(1+h)\partial_t^{-\alpha}(1-e^t) + h(1+h)^2\partial_t^{-\alpha}(1-e^t) + ...,$$
$$= -x + ln(x) + \partial_t^{-\alpha}(1) + h\partial_t^{-\alpha}(1-e^t)(1 + (1+h) + (1+h)^2 + ...),$$
$$= -x + ln(x) + \partial_t^{-\alpha}(1) + h\partial_t^{-\alpha}(1-e^t)\sum_{j=0}^{+\infty}(1+h)^j,$$
$$= -x + ln(x) + \partial_t^{-\alpha}(1) + h\partial_t^{-\alpha}(1-e^t)\left(\frac{1-(1+h)^{j+1}}{-h}\right). \tag{3.25}$$

If the auxilary parametre h is selected so that $|1+h| < 1$, then the last power series converges, and gives

$$u(x,t) = -x + ln(x) + \partial_t^{-\alpha}(1) - \partial_t^{-\alpha}(1-e^t),$$
$$= -x + ln(x) + \partial_t^{-\alpha}(1) - \partial_t^{-\alpha}(1) + \partial_t^{-\alpha}(e^t) = -x + ln(x) + \partial_t^{-\alpha}(e^t). \tag{3.26}$$

wich is the exact solution for $0 < \alpha < 1$.

for $\alpha = 1$ setting $u_0(x,t) = -x + ln(x) + \partial_t^{-1}(1) = -x + ln(x) + \frac{1}{\Gamma(1)}\int_0^t 1d\tau = -x + ln(x) + t$.

then:

$$u_1(x,t) = h\partial_t^{-1}(1-e^t) = h(\partial_t^{-1}(1) - \partial_t^{-1}(e^t)),$$

$$= h\left(t - \frac{1}{\Gamma(1)}\int_0^t e^{\tau} d\tau\right) = h(t - e^t + 1). \tag{3.27}$$

$$u_2(x,t) = h(1+h)\partial_t^{-1}(1 - e^t) = h(1+h)(t - e^t + 1).$$

$$u_3(x,t) = h(1+h)^2(t - e^t + 1). \tag{3.28}$$

$$u_4(x,t) = h(1+h)^3(t - e^t + 1). \tag{3.29}$$

Hence the series solution becomes

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + ..., \\ &= -x + ln(x) + t + h(t - e^t + 1) + h(1+h)(t - e^t + 1) + h(1+h)^2(t + e^t - 1) + ..., \\ &= -x + ln(x) + t + h(t - e^t + 1)(1 + (1+h) + (1+h)^2 + (1+h)^2) + ..., \\ &= -x + ln(x) + t + h(t - e^t + 1)\sum_{j=0}^{\infty}(1+h)^j, \\ &= -x + ln(x) + e^t - 1. \end{aligned} \tag{3.30}$$

The tables

m	x	$u^{(m)}$	u_{exact}	$\|u^{(m)} - u_{exact}\|$
2	0.5	0.244991	0.230497	0.014494
4		0.23412		0.003623
6		0.231402		0.000905

Table 3.1: Estimating error of problem (3.7) in cases: $m = 2, m = 4, m = 6$, $\alpha = 0.2$ and $t = 0.3$.

m	x	$u^{(m)}$	u_{exact}	$\|u^{(m)} - u_{exact}\|$
2	0.2	-1.344	-1.32732	0.01668
4		-1.331		0.00368
6		-1.327		0.00032

Table 3.2: Estimating error of problem (3.19) in cases: $m = 2, m = 4, m = 6$, $\alpha = 0.4$ and $t = 0.1$.

from the result recorded in table 1 and table 2, through the Wolfram Mathematica program, using the **homotopy analysis method**, we note that whenever the larger the m the smaller the error (the approximate solution is close to the exact solution), wich proves to us the effectiveness of the method.

Conclusion

In this work we are interested in the numerical resolution for diffusion equations of fractional orders, via the homotopy analysis method, with integral type boundary conditions. The numerical study was carried out, this was done with a test the efficiency to the Homotopy analysis method for solving the fractional non-local mixed problem with the Bessel operator, then we apply it to some test examples, to demonstrate the effectiveness of this method in solving fractional partial differential equations, following the study on the existene and uniqeness of the generalized solution, which was based on the method of a priori estimation.

We plan to continue to employ the methods reviewed in this thesis for a different field of equations in the future, and we also hope to develop additional computational techniques with higher accuracy for solving differential equations with fractional derivatives than those presented in this study.

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