KASDI MERBAH UNIVERSITY
OUARGLA

Faculty of Mathematics and Material Sciences

DEPARTMENT OF MATHEMATICS

MASTER

Field: Mathematics and Computer Science

Branch: Mathematics

Specialization: Functional Analysis

By: Hiba Chekli

Positive Solutions of a Nonlinear Three-Point Integral Boundary Value Problem

Publicly defended on: 19/06/2024

Before the jury composed of:

Mr.**BENCHEIKH Abdelkarim**	M. A. Kasdi Merbah University - Ouargla	Chairman
Mr.**HACHIFA Abdelrrazak**	M. A. Kasdi Merbah University - Ouargla	Examiner
Mr.**KOUIDRI Mohammed**	M. A. Kasdi Merbah University - Ouargla	Reporter

Academic Year 2023/2024

Dedication

"And their final prayer is: Praise to Allah, Lord of all the worlds."

All praise is due to Allah, who facilitates beginnings, perfects endings, and enables us to achieve our goals. All praise is due to Allah, whose efforts never cease and whose endeavors are never completed except by His grace and guidance.

I dedicate my success to the steadfast and resilient Gaza, its people, and its brave and steadfast resistance fighters who stand firm against tyrannical oppression.

And to Al-Aqsa Mosque, which has always been the focus of my efforts and the first goal in my life.

To my beloved father, my constant support, and my loving mother, who envelops me with her prayers. They are the source of love and giving, and it is for them that I am here today. These beautiful achievements are the result of your efforts, hard work, and dedication. I will never forget your continuous and limitless support and your genuine love. Thank you for everything. I also thank my brothers for their support and encouragement, as well as my companions and friends who have lived every moment with me and have been a constant source of inspiration.

Special thanks to Dr. Mohammed kouidri for sharing his valuable knowledge and experiences with me .

Finally, all praise is due to Allah, always and forever.

Acknowledgement

Before all consideration, I thank the Great Almighty God who helped me to complete this work. I would first like to thank my supervisor Mr. Kouidri Mohammed for proposing one of the most important themes to me and for its continuity in supporting me and encourage me. I would also like to thank him for his kindness, his availability and his time. devoted to my work. I also thank the members of the Department of Mathematics and Computer Science for allowing me to work in good conditions during the completion of my work. Thank you also to all the teachers who helped me during my studies, without forgetting their valuable advice. I would also like to thank anyone directly or indirectly who contributed to the finalization of this work.

Contents

Notations

$\|.\|$ its norm.

$dist$ the distance associated with this norm.

$\overline{\Omega}$ the closure of Ω and $\partial\Omega$ its boundary.

$B(x_0, r)$ the open ball with center x_0 and radius r:

$u'(t)$ the ordinary derivative with respect to t

$\oplus$ direct sum.

$\langle , \rangle$ inner product. $\mathbb{R}$ the set of real numbers.

$(M.d)$ metric space.

$d(.,.)$: distance function.

$c([a.b])$: the space of continuous functions.

Ω : a bounded open set.

U : a bounded open set.

$\overline{U}$: the closure of U

$\overline{C}^k(.,.)$: the space of functions with values in $\mathbb{R}$, k times differentiable in Ω

deg : topological degree.

max : maximum.

$\overline{B}$: the closed unit ball.

dim : dimension.

K : a cone.

$(E, \|\|)$ a Banach space.

A : operator.

T : operator.

Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential equations was initiated by Il'in and Moiseev [14]. Then Gupta [15] studied three-point boundary value problems for nonlinear second-order ordinary differential equations. Since then, nonlinear second-order three-point boundary value problems have also been studied by several authors. For more details see [16, 17] and the references therein. However, all these papers are concerned with problems with three-point boundary condition restrictions on the slope of the solutions and the solutions themselves, for example,

$$\begin{aligned} u(0) = 0, &\quad \alpha u(\eta) = u(1) \\ u(0) = \beta u(\eta), &\quad \alpha u(\eta) = u(1) \\ u'(0) = 0, &\quad \alpha u(\eta) = u(1) \end{aligned}$$

and so forth.

In our work, we consider the existence of positive solutions to the equation

$$u'' + a(t)f(u) = 0, \quad t \in (0,1) \tag{1}$$

With the following three-point integral boundary condition,

$$u(0) = 0, \quad \alpha \int_0^{\eta} u(s)ds = u(1) \tag{2}$$

where $0 < \eta < 1$. Our aim is to give some results for existence of positive solutions to the system of (1)-(2), in the case when $0 < \alpha < 2/\eta^2$ and f is either superlinear or sublinear.

The organization of this dissertation is as follows: In Chapter 1, we provide a comprehensive overview of fixed-point theorems and their fundamental concepts, including Banach's fixed-point theorem and principles of continuation. We also discuss topological degree theory, covering Brouwer's topological degree and the Leray-Schauder degree. Chapter 2 is dedicated to Guo-Krasnosel'skii's theorem, where we explore its theoretical underpinnings and implications. In Chapter 3, we apply Guo-Krasnosel'skii's theorem to specific problems, illustrating the practical utility of the theoretical results obtained. [2], Graef et Yang [6] [7].

Chapter 1

Reminders and fundamental concepts

This chapter's goal is to study a few fixed-point theories. We will start with the most basic and well-known of them all, the Banach point fixation theory for contracting applications and the fixed-point theorem of Brouwer.

1.1 Fixed-point theorem

The fixed-point theorem is a fundamental principle in mathematics, which states that a continuous function over a given domain may have a point that remains unchanged when the function is repeatedly applied to itself. In other words, there exists an x in the domain such that $f(x) = x$, where f is the function. This point is called the fixed point of the function. [9]

1.1.1 Banach fixed point theorem

Banach's fixed point theorem (also known as the contracting map theorem) is a simple theorem to prove, which guarantees the existence of a unique fixed point for any contracting application, it applies to complete spaces and it has many applications. These applications include the existence theorems of solution for differential equations or integral equations and the study of convergence of some numerical methods.

Definition 1 ***(fixed point)***

Let $F : X \to X$ be a mapping. Any point $x \in X$ such that $F(x) = x$ is called a fixed point.

Theorem 2 *(Banach contraction principle) [3]*
Let (M, d) be a complete metric space and $F : M \to M$ be a contraction mapping, meaning that there exists $0 < k < 1$ such that $d(F(x), F(y)) \leq kd(x, y)$ for all $x, y \in M$. Then F has a fixed point $u \in M$, plus for all x in M we have :$\lim_{n\longrightarrow\infty} F^n(x) = u$ and:

$$d(F^n, u) \leq \frac{k^n}{1-K} d(x, F(x))$$

Proof. First, we prove the unicity .
Assuming the existence of $x, y \in M$ with $x = F(x); y = F(y)$and $d(x, y) = d(F(x), F(y)) \leq kd(x, y)$. Since $0 < k < 1$ hence, the last inequality implies that $d(x, y) = 0 \Longrightarrow x = y$, so $\exists! x \in M$ such that $F(x) = x$.
Now, for the existence. Let $x \in M$.
We'll prove $F^n(x)$ is a Cauchy sequence where $n \in \{0, 1, ...\}$

$$d(F^n(x), F^{n+1}(x)) \leq kd(F^{n-1}(x), F^n(x)) \leq .... \leq k^n d(x, F(x))$$

if $m > n \in \{0, 1, ...\}$

$$\begin{aligned} d(F^n(x), F^m(x)) &\leq d(F^n(x), F^{n+1}(x)) + d(F^{n+1}(x), F^{n+2}(x)) + ... + (F^{m-1}(x), F^m(x)) \\ &\leq k^n d(x, Fx) + k^{n+1} d(x, Fx) + ... + k^{m-1} d(x, F(x)) \\ &\leq k^n d(x, F(x)) \left[1 + k + k^2 + ...\right] \\ &\leq \frac{k^n}{1-k} d(x, F(x)) \end{aligned}$$

for $m > n \in \{0, 1, ...\}$ we have

$$d(F^n(x), F^m(x)) \leq \frac{k^n}{1-k} d(x, F(x)) \tag{1.1}$$

So $F^n(x)$ is a Cauchy sequence in the complete space X, then there exists $u \in X$ with

$$\lim_{n\longrightarrow+\infty} F^n(x) = u$$

moreover by the continuity of F

$$u = \lim_{n\longrightarrow\infty} F^{n+1}(x) = \lim_{n\longrightarrow\infty} F(F^n) = F(u)$$

Thus, u is a fixed point of F .
ultimately, $m \longrightarrow \infty$ in 1.1, we obtain

$$d(F^n(x), u) \leq \frac{k^n}{1-k} d(x, F(x))$$

■

Example 1 *Consider the application* $T : \mathbb{R} \to \mathbb{R}$ *defined by* $T(x) = \frac{x}{2} + \frac{1}{2}$ *; then* T *is a contraction* $0 < k = \frac{1}{2} < 1$*; and admits as a fixed point* $x = 1$ *moreover* $\lim\{T^n(x)\}_{n=1}^{\infty} = 1$

Remark: To convince oneself of the necessity of the phenomenon, consider the following examples .

Example 2 $T : [0,1] \to \mathbb{R}$, $T(x) = \frac{x}{2} + 1$ *,is contracting but it does not have a fixed point. The issue is that* $T([0,1]) \subsetneq [0,1]$ *and we cannot iterate :* $x_0 = 0$, $x_1 = 1$, $x_2 = 1.5$, *but* x_3 *is not defined !*

Example 3 $T : \mathbb{R} \to \mathbb{R}$ *;* $T(x) = x + \frac{1}{1+e^x}$ *check* $|T(x) - T(y)| < |x - y|$ *for all* $x \neq y$*,but does not have a fixed location. The issue is that* T *is not a contracting map, and for all* $x_0 \in \mathbb{R}$ *we obtain* $x_n \longrightarrow +\infty$

1.1.2 Fixed point theorems for undefined contractions over the whole metric space

Assuming (M,d) to be a complete metric space, it is evident that a function defined just on a subset of M will not inevitably result in a fixed point. There has to be additional requirements in order to guarantee this.

Theorem 3 *Let* $K \subset M$ *be a closed set and* $T : K \longrightarrow M$ *A k-contraction. Assuming the existence of* $x_0 \in K$ *and* $r > 0$ *such that*

$$\overline{B(x_0, r)} \subset K \qquad et \qquad d(x_0, T(x_0)) < (1-k)r$$

then F *has a unique fixed point* $x^* \in B(x_0, r)$.

In certain applications, there are cases where T is Lipschitzian without being a contraction, while a certain power of T is a contraction see[1]. In this case we have the following theorem

Theorem 4 *let (M,d) be a complete metric space* $T : M \longrightarrow M$ *an application such that* $d(T^m(x), T^m(y) \leq kd(x,y), \forall x, y \in M$*, for a certain* $m \geq 1$ *and* $0 \leq k < 1$*. then* T *admits a unique fixed point* $x^* \in M$

Proof. as T^m is a contraction, it follows from the theorem 3 that T^m has a fixed unique point, thus $x^* = T^m x^*$. so $T^m(T(x^*)) = T(T^m(x^*)) = T(x^*)$, i.e. $T(x^*)$ is a fixed point of T^m . But T^m has a unique fixed point $T(x^*) = x^*$, so T has a unique fixed point (x^*), and it is unique because every fixed point of T is also fixed point T^m ■

Example 4 *Consider the metric space* M *given by :*$M = C[a;b]$ *;The space of functions continues with real values defined on the interval* $[a;b]$*.* M *is a Banach space with the norm* $\|u\| = \max_{t\in[a,b]} |u(t)|$, $u \in M$*. We define* $T : M \to M$ *by :*

$$Tu(t) = \int_a^t u(s)ds$$

so,

$$\|T(u) - T(v)\| \leq (b-a)\|u-v\|,$$

so $(b-a)$ *is the Lipchitz constant for* T*. Additionally, we have :*

$$T^2(u)(t) = \int_a^t \left(\int_a^s u(\tau)d\tau \right) ds = \int_a^t (t-s)u(s)ds$$

and by induction

$$T^m u(t) = \frac{1}{(m-1)!} \int_a^t (t-s)^{m-1} u(s)ds,$$

from that we get

$$\|T^m(u) - T^m(v)\| \leq \frac{(b-a)^m}{m!} \|u-v\|,$$

and so T^m *would be a contraction if* $\frac{(b-a)^m}{m!} < 1$

1.1.3 Principles of continuation

Another method of obtaining a fixed point's existence for an infinite dimensional space is obtained through a continuation procedure. This one consists of transforming our application into a simpler one for which we are aware that a fixed point exists. It goes without saying that this must meet certain requirements. [1].

Definition 5 *Let X and Y be two topological spaces. Two continuous applications $f, g : X \longrightarrow Y$ are called homotopes when there exists a continuous application*

$$H : X \times [0, 1] \longrightarrow Y$$

such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. In other words, there exists a family of applications of X in Y, namely $x \longrightarrow H(x, t)$ for $0 \leq t \leq 1$, which varies continuously starting from f and arriving to g. We denote $f \simeq g$.

Example 5 *Let $X = Y = \mathbb{R}^n$, we consider $c : \mathbb{R}^n \to R^n$ constant application $c(x) = 0$, and $i : \mathbb{R}^n \to \mathbb{R}^n$ application $i(x) = x$. show that c et i are homotopic. just take :*

$$H : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$$

$$H(x, t) = tx.$$

so $H(x; 0) = 0 = c(x)$ and $H(x, 1) = x$:

Example 6 *Let $X = Y = \mathbb{R}^n - \{0\}$; we consider this time $p(x) = x/\|x\|$; and $i(x) = x$ again, we see that p and i are homotopice by taking*

$$H : (\mathbb{R}^n - \{0\}) \times [0, 1] \to \mathbb{R}^n - \{0\}$$

$$H(x, t) = (1 - t)x + t\frac{x}{\|x\|}$$

Definition 6 *Let $f : X \longrightarrow Y$ be a continuous application. We say that f is a homotopy equivalence when there exists $g : Y \longrightarrow X$ such that $g \circ f \simeq id_X$ and $f \circ g \simeq id_y$. We then say that X and Y have the same type of homotopy, or sometimes that they are homotopy equivalent, and we note $X \simeq Y$*

Example 7 *Let $X = \mathbb{R}^n - \{0\}$ and $Y = S^{n-1}$, we then take $f : X \to Y$ defined by $f(x) = x/\|x\|$; et $g : Y \to X$ the inclusion. Then $f \circ g = id_y$, and the example 6 to show that $g \circ f \simeq id_x$. so $R^n - \{0\}$ has the same type of homotopy as the sphere S^{n-1}.*

Let (X,d) be a complete space, and U is an open subset of X.

Definition 7 *Let $F : \overline{U} \longrightarrow X$ and $G : \overline{U} \longrightarrow$ two contractions, we say that F and G are homotopic if there exists $H : \overline{U} \times [0,1] \longrightarrow X$ satisfying the following properties:*

(a) $H(.,0) = G$ and $H(.,1) = F$.

(b) $H(x,t) \neq x$ for all $x \in \partial U$ and $t \in [0,1]$

(c) There exists $\alpha \in [0,1)$ such that $d(H(x,t); H(y,t) \leq \alpha d(x,y)$ for all $x, y \in \overline{U}$, and $t \in [0,1]$.

(d) There exists $M \geq 0$ such that $d(H(x,t), H(x,s) \leq M|t-s|$ for all $x \in \overline{U}$,and $t, s \in [0,1]$.

Theorem 8 *Let $F : \overline{U} \longrightarrow X$ and $G : \overline{U} \longrightarrow X$ be two homotopically contractive applications, and G has a fixed point in U. Then, F has a fixed point in U.*

Proof. Let's consider $Q = \{\lambda \in [0,1] : x = H(x,\lambda)\ ,\ for\ certain\ x \in U\}$, and H a homotopy between F and G as described in the definition (5). Note that Q is not empty because G has a fixed point and that $0 \in Q$. We'll prove that Q is both open and closed in [0,1], and as a result we get that $Q = [0,1]$, so F has a fixed point. First, let's prove that Q is a closed in [0,1]. In fact, Let $\{\lambda_n\}_{n \in N}$ be a sequence in Q such that $\lim_{n \longrightarrow \infty} \lambda_n = \lambda$, so, we must show that $\lambda \in Q$. as $\lambda_n \in Q$ for $n = 1, 2....$, there exists $x_n \in U$ o $x_n = H(x_n, \lambda_n)$. also for $n, m \in \{1, 2, ...\}$ we have

$$\begin{aligned} d(x_n, x_m) &= d(H(x_n, \lambda_n)H(x_m, \lambda_m) \\ &\leq d(H(x_n, \lambda_n)H(x_m, \lambda_m)) + d(H(x_n, \lambda_m), H(x_m, \lambda_m) \\ &\leq M|\lambda_n - \lambda_m| + \alpha d(x_n, x_m) \end{aligned}$$

thus, we have

$$d(x_n, x_m) \leq \frac{M}{1-\alpha}|\lambda_n - \lambda_m|$$

which shows that $\{x_n\}$ is a cauchy sequence in X (because $\{\lambda_n\}$ is too) and since it is complete, there exists $x \in \overline{U}$ such that $\lim_{x \longrightarrow \infty} x_n = x$. by the continuity of H,

$$x = \lim_{n \longrightarrow \infty} x_n = lim_{n \longrightarrow \infty} H(x_n, \lambda n) = H(x, \lambda)$$

thus, $\lambda \in Q$ and Q is closed in $[0, 1]$.

let us show that Q is an open set of $[0, 1]$. Let $\lambda_0 \in Q$, so, there exists $x_0 \in U$ with $x_0 = H(x_0, \lambda_0)$. Since, by hypothesis, $x_0 \in U$, we can find $r > 0$ such that the open ball $B(x_0, r) = \{x \in X : (x, x_0) < r\} \subseteq U$. let's choose $\epsilon > 0$ such that $\epsilon \leq \frac{(1-\alpha)r}{M}$ where $r \leq dist(x_0, \partial U)$, and $dist((x_0, \partial U)) = inf\{(x_0, x) : x \in \partial U\}$. Let's fix $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$. So , for $x_0 \in \overline{B(x_0, r)}$

$$\begin{aligned} d(x_0, H(x, \lambda)) &\leq d(H(x_0, \lambda_0) H(x, \lambda_0)) + d(H(x, \lambda_0), H(x, \lambda) \\ &\leq \alpha d(x_0, x) + M|\lambda, \lambda_0| \\ &\leq \alpha r + (1-\alpha)r = r \end{aligned}$$

then for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$ fix

$$H(., \lambda) : \overline{B(x_0, r)} \longrightarrow \overline{B(x_0, r)}$$

by the theorem (2), (3), we deduce that $H(., \lambda)$ has a fixed point in U. Then, $\lambda \in Q$ for all $\lambda \in (\lambda_0 - \epsilon, \lambda_0 + \epsilon)$.and therefore Q is open in [0, 1] .

■ From the previous term, we deduce the following result.

Theorem 9 *(Nonlinear Leray-Schauder alternative)[1]. Let $U \subset E$ an open set of a Banach space E such that $0 \in U$, and let $F : \overline{U} \longrightarrow E$ a contraction such that $F(\overline{U})$ is bounded. Then one of the following two statements is satisfied:*

(1) F has a fixed point in $(\overline{U})$.

(2) there exists $\lambda \in (0, 1)$ and $x \in \partial U$ such that $x = \lambda F(x)$.

Proof. Suppose that (2) is not verified and that F has no fixed point on ∂U that is to say $x \neq \lambda F(x)$ for all $x \in \partial U$ and $\lambda \in [0,1]$.

Let $H : \overline{U} \times [0,1] \longrightarrow E$ given by $H(x,\lambda) = \lambda F(x)$, and let G be the null application. Note that G has a fixed point in U (such that $(0 = G(0))$ and that F and G are two homotopically contractive applications. by the theorem (8) F also has a fixed point. ■

1.2 Topological degree

In this section, we provide a brief overview of the notion of topological degree, whether in finite or infinite dimensions. The degree, $deg(f, \Omega, y)$ of f in Ω with respect to y gives information about the number of solutions of the equation $f(x) = y$ in an open set $\Omega \subset \mathbb{R}^n$ is continuous, $y \notin f(\partial\Omega)$, and X is a topological space, mostly metric. For more detailed information, see [4], [5], [10], [11].

1.2.1 Brouwer's topological degree

Consider a bounded open set Ω in $\mathbb{R}^n$ with boundary $\partial\Omega$ and closure $\overline{\Omega}$. $\overline{C}^k(\Omega, \mathbb{R}^n)$ denotes the space of k-times differentiable functions in Ω that are continuous on $\overline{\Omega}$. This space is equipped with its usual topology.

Let $x_0 \in \Omega$, if f is differentiable at x_0, we denote by $J_f(x_0) = \det f'(x_0)$ the Jacobian of f at x_0.

Definition 10 *Let f be a C^1 function on Ω. Let $J_f(x_0)$ be the Jacobian of f at a point x_0 in Ω. The point x_0 is called a critical point if $J_f(x_0) = 0$. Otherwise, x_0 is called a regular point.*

We denote by $S_f(\Omega)$ the set of critical points, i.e.,

$$S_f(\Omega) = \{x \in \Omega, J_f(x) = 0\}$$

Definition 11 *An element $y \in \mathbb{R}^n$ is called a regular value of f if $f^{-1}(y) \cap S_f(\Omega) = \emptyset$. Otherwise, y is called a singular value.*

Definition 12 *Let* $f \in \overline{C}^1 \backslash f(\partial\Omega)$ *be a regular value of* f*. The topological degree of* f *in* Ω *with respect to* y *is the integer*

$$deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} SgnJ_f(x)$$

where $SgnJ_f(x)$ *denotes the sign of* $J_f(x)$*, defined by* $sgn(t) = 1$ *if* $t > 0$ *and* $sgn(t) = -1$ *if* $t < 0$

Remark 13 *1) By convention, if* $f^{-1}(y) = \emptyset$*, then* $deg(f, \Omega, y) = 0$*.*

2) $f^{-1}(y)$ *contains a finite number of elements.*

Example 8 *Let* $f : \mathbb{R} \to \mathbb{R}$*,* $x \to x^2$ *then* $deg(f,]0, 2[, 1) = 1$*.*

Example 9 *Let* $0 < \epsilon < 1$ *and consider the function* $f(x, y) = (x^2 - y^2 - \epsilon, 2xy)$*, and* $f^{-1}(0, 0) = \{(x, y) \in \mathbb{R}^2; f(x, y) = (0, 0)\}$ *then, we have*

$$x^2 - y^2 - \epsilon = 0 \tag{1.2}$$

and

$$2xy = 0 \tag{1.3}$$

According to (1.3), we find $x = 0$ or $y = 0$.
If $x = 0$ then: $-y^2 - \epsilon = 0 \Longrightarrow y^2 = -\epsilon$, which is a contradiction.
If $y = 0$ then: $x^2 - \epsilon = 0 \Longleftrightarrow x = \sqrt{\epsilon}$ or $x = -\sqrt{\epsilon}$, thus

$$f^{-1}(0, 0) = \{(-\sqrt{\epsilon}, 0); (\sqrt{\epsilon}, 0)\}$$

If $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$ then $f^{-1}(0) \cap \partial\Omega = \emptyset$. Moreover, as

$$J_f(x, y) = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}$$

and $det(J_f((x, y))) = 4(x^2 + y^2)$ and since $deg(f, \Omega, y) = \sum_{x \in f^{-1}(y)} SgnJ_f(x)$ then

$$SgndetJ_f(\sqrt{\epsilon}, 0) = Sgn4\epsilon = 1$$
$$SgndetJ_f(-\sqrt{\epsilon}, 0) = Sgn4\epsilon = 1$$
$$\Longrightarrow deg(f, \Omega, 0) = 1 + 1 = 2$$

Remark 14 *In the case where $f^{-1}(y) \cap S_f(\Omega) \neq 0$, we have the following lemma.*

Lemme 1 (**Sard′s Lemma**) *Let $f \in C^1(\overline{\Omega}, \mathbb{R}^n)$. Then the set $f(S_f)$ of critical values of f has measure zero.*

We will now see that we can extend the notion of degree to the case where the function f is continuous

Definition 15 *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $f \in C(\overline{\Omega}, \mathbb{R}^n)$, and $y \in \mathbb{R}^n$ such that $y \notin f(\partial\Omega)$. We define the topological degree of f in Ω with respect to y as*

$$deg(f, \Omega, y) = \lim_{n \to \infty} deg(f_n, \Omega, y)$$

where $\{f_n\}_{n \in N^*}$ is a sequence of functions $C^1(\overline{\Omega}, \mathbb{R}^n)$ that converges uniformly to f in $\overline{\Omega}$.

1.2.2 Topological degree of Leray-Schauder

In the previous section, we saw that in finite dimensions, $C(\bar{\Omega}, X)$ is a suitable class of functions for which there exists a unique degree function, the Brouwer degree, satisfying properties 1, 2, and 3 of the theorem. Unfortunately, in infinite dimensions, $C(\bar{\Omega}, X)$ doesn't have it. Indeed, an example by Leray shows that we need to restrict the class of functions for which there is existence and uniqueness of a Leray-Schauder degree function, to a set strictly contained in $C(\bar{\Omega}, X)$.

Definition 16 *[10] Let X be a Banach space and Ω a closed subset of X. If $T : \Omega \to X$ is a continuous operator, we say that T is compact if for every bounded subset B of Ω, $T(B)$ is relatively compact in X.*

Definition 17 *Let X be a Banach space and Ω a subset of X. We say that the mapping $T : \Omega \to X$ has finite rank if $dim(Im(T)) < \infty$, in other words, if $Im(T)$ is a finite-dimensional subspace of X.*

Lemme 2 *Let X be a Banach space, $\Omega \subset X$ an open bounded set, and $T : \bar{\Omega} \to X$ a compact mapping. Then, for any $\epsilon > 0$, there exists a finite-dimensional space denoted F and a continuous mapping $T_\epsilon : \bar{\Omega} \to F$ such that*

$$\|T_\epsilon x - Tx\| < \epsilon \quad \text{for all } x \in \bar{\Omega}.$$

Definition 18 *Let X be a Banach space, $\Omega \subset X$ an open bounded set, and $T : \bar{\Omega} \to X$ a compact mapping. Suppose now that $0 \notin (I - T)(\partial\Omega)$. There exists $\epsilon_0 > 0$ such that for $\epsilon \in (0, \epsilon_0)$, the Brouwer degree $deg(I - T_\epsilon, \Omega \cap F_\epsilon, 0)$ is well-defined as in Lemma 2. Therefore, we define the Leray-Schauder degree by*

$$deg(I - T, \Omega, 0) = deg(I - T_\epsilon, \Omega \cap F_\epsilon, 0).$$

Remark 19 *This definition depends only on T and Ω. If $y \in X$ such that $y \notin (I-T)(\partial\Omega)$, then the degree of $I - T$ in Ω with respect to y is defined as*

$$deg(I - T, \Omega, y) = deg(I - T - y, \Omega, 0).$$

Theorem 20 *[4] Let X be a Banach space and*
$A = \left\{(I - T, \Omega, 0), \Omega \text{ a bounded open subset of } X, T : \overline{\Omega} \to X \text{ compact}, 0 \notin (I - T)(\partial\Omega)\right\}$
then, there exists a unique application $deg(f, \Omega, y) : A \to \mathbb{Z}$ called the Leray-Schauder topological degree such that :

1. *(Normality) If $0 \in \Omega$ then $deg(I, \Omega, 0) = 1$;*
2. *(Solvability) If $deg(I - T, \Omega, 0) \neq 0$, then there exists $x \in \Omega$ such that $(I - T)x = 0$;*
3. *(Invariance by homotopy) Let $H : [0, 1] \times \overline{\Omega}$ be a compact homotopy, such that $0 \notin (I - H(t, .))(\partial\Omega)$. Then $deg(I - H(t, .), \Omega, 0)$ does not depend on $t \in [0, 1]$;*
4. *(Additivity) Let Ω_1 and Ω_2 be two disjoint open subsets of Ω and*

$$0 \notin (I - T)(\bar{\Omega} \setminus (\Omega_1 \cup \Omega_2)).$$

Then,

$$deg(I - T, \Omega, 0) = deg(I - T, \Omega_1, 0) + deg(I - T, \Omega_2, 0).$$

The Leray-Schauder degree preserves all basic properties of the Brouwer degree. As a consequence of this notion of degree, we will prove some topological fixed point theorems, in particular the Leray-Schauder nonlinear alternative.

Theorem 21 *(Brouwer) Let $\bar{B}$ be the closed unit ball of $\mathbb{R}^n$ and $f : \bar{B} \to \bar{B}$ continuous. Then f has a fixed point: there exists $x \in \bar{B}$ such that $f(x) = x$.*

Proof. If there exists $x \in \partial B$, then there is nothing to prove. Otherwise, consider the function $h(t,x) = x - tf(x)$. We have that h is continuous, $h(0,x) = x$, and $h(1,x) = x - f(x)$. Moreover, if we assume that $h(t,x_0) = 0$ for some $x_0 \in \partial B$, then we obtain $x_0 = tf(x_0)$, which implies, since $0 \leqslant t \leqslant 1$, that $f(x_0) \in \partial B$, contradiction. Since h is a suitable homotopy between $I - f$ and I, then

$$deg(I - f, \Omega, 0) = deg(I, \Omega, 0) = 1.$$

In conclusion, there exists an $x \in B$, such that $x - tf(x) = 0$, i.e., $f(x) = x$. ■

Theorem 22 *(Schauder) Let $\overline{B}$ be the closed unit ball of a Banach space E and $f : \overline{B} \to \overline{B}$ compact. Then f has a fixed point: there exists $x \in \overline{B}$ such that $f(x) = x$.*

Proof. Let $h(t,x) = tf(x)$ be a compact function on $[0,1] \times \overline{B}$. If, for some $t \in [0,1]$ and $x \in \partial B$, we have $x - h(t,x) = 0$, then $tf(x) = x$; since $|x| = 1$ and $|f(x)| \leq 1$, this implies $t = 1$ and $x = f(x)$, thus a fixed point on ∂B, a situation that we have excluded. Therefore, we can apply the properties of normality and invariance by homotopy of the degree, which gives

$$1 = deg(I, B, 0) = deg(I - f, B, 0)$$

since $h(0,.) = 0$ and $h(1,0) = f$, hence the existence of a fixed point. ■

Theorem 23 *[4] (Leray-Schauder Nonlinear Alternative). Let $\Omega \subset X$ be a non-empty bounded open subset of a Banach space X such that $0 \in \Omega$, and let $T : \overline{\Omega} \to X$ be a compact operator. Then one of the following statements holds:*

(1) T has a fixed point in Ω;

(2) there exist $\lambda > 1$ and $x \in \partial\Omega$ such that $Tx = \lambda x$.

Proof. If (2) is true, then there is nothing to prove. Otherwise, we define the homotopy

$$H(t,x) = tTx \quad \text{for } t \in [0,1].$$

Thus defined, $H(t,x)$ is compact, $H(0,x) = 0$ and $H(1,x) = Tx$. Suppose that $H(t,x_0) = x_0$ for some $t \in [0,1]$ and $x_0 \in \partial\Omega$. Then we have $tTx_0 = x_0$. If $t = 0$ or $t = 1$ we have (1); Otherwise

$$Tx_0 = \frac{1}{t}x_0 \quad \text{for some } t \in (0,1),$$

and then we have (2). Otherwise, we have $deg(I-T,\Omega,0)=deg(I,\Omega,0)=1$ and then T has a fixed point in Ω. ■

Theorem 24 ***(Brouwer)*** *Let M be a convex, compact, non-empty subset of a finite-dimensional normed space $(X,\|.\|)$ and let $A:M\to M$ be a continuous function, then A has a fixed point.*

Theorem 25 ***(Schauder)*** *Let M be a bounded, closed, convex, non-empty subset of a Banach space X and let $A:M\to M$ be a compact function, then A has a fixed point.*

Theorem 26 ***(Schaeffer)*** *Let X be a Banach space and let $A:X\to X$ be a compact operator, then*

i. Either the equation $x=\lambda Ax$ has a solution for $\lambda=1$,

ii. Or, the set $\epsilon=\{x\in X, x=\lambda Ax, \lambda\in(0,1)\}$ is unbounded.

Theorem 27 ***(Krasnoselskii)*** *Let M be a closed and non-empty convex subset of a Banach space $(X,\|.\|)$. Suppose that A and B are two mappings from M to X such that*

i. $Ax+By\in M, \forall x,y\in M$.

ii. A is continuous and AM is contained in a compact set.

iii. B is a contraction with constant $\alpha<1$.
Then, there exists $x\in M$, such that $Ax+Bx=x$.

Note that if $A=0$, the theorem reduces to the **Banach fixed-point theorem**. If $B=0$, then the theorem is nothing but the **Schauder fixed-point theorem**.

Theorem 28 *[8]**Krasnoselskii's Compression-Expansion Theorem for a Cone*** *Let Ω_1 and Ω_2 be two bounded open sets in a Banach space E such that $0\in\Omega_1, \bar{\Omega_1}\subset\Omega_2$, and K a cone of E. Let $\mathcal{A}:K\cap(\bar{\Omega_2}\setminus\Omega_1)\to K$ be a completely continuous operator, such that one of the following conditions is satisfied,*

(i) $\|\mathcal{A}u\|\leqslant\|u\|$, $u\in K\cap\partial\Omega_1$, and $\|\mathcal{A}u\|\geqslant\|u\|, \quad u\in K\cap\partial\Omega_2$.

(ii) $\|\mathcal{A}u\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_1$, $\|\mathcal{A}u\| \leqslant \|u\|, u \in K \cap \partial\Omega_2$.

Then the operator $\mathcal{A}$ *has at least one fixed point in* $K \cap (\bar{\Omega_2} \setminus \Omega_1)$.

. Now let's recall the Ascoli-Arzela theorem and the dominated convergence theorem of Lebesgue

Theorem 29 ***(Ascoli-Arzela)*** *Consider* $X = C([a,b])$ *equipped with the norm* $\|u\| = \max\limits_{a\leqslant t\leqslant b}|u(t)|$, *where* $-\infty < a < b < +\infty$. *If* M *is a subset of* X *such that*

i. M *is bounded, i.e.,* $\|u\| \leqslant r, \quad \forall u \in M$ *and* $r > 0$ *is a fixed number,*

ii. M *is equicontinuous, i.e.,*

$$\forall \epsilon > 0, \exists \delta > 0, \text{ such that } |t_1 - t_2| < \delta \text{ and } \forall u \in M \Rightarrow |u(t_1) - u(t_2)| < \epsilon.$$

Then, M *is relatively compact.*

Theorem 30 *Let* Ω *be an open set in* $\mathbb{R}^n$ *and* $(f_n)_{n\in\mathbb{N}}$ *a sequence in* $L^p(\Omega)$ *such that*

i. $f_n(x) \to f(x)$ *almost everywhere on* Ω.

ii. $|f_n(x)| \leqslant g(x)$ *almost everywhere on* Ω, $\forall n$ *with* $g \in L^p(\Omega)$. *Then,*

$$f \in L^p(\Omega) \text{ and } \|f_n - f\|_{L^p} \to 0.$$

Chapter 2

Guo-Krasnoselskii's theorem

Definition 31 *(a Cone)*

Let E be a Banach space. A subset $K \subset E$ is called a cone if the following conditions are satisfied:

(i) K is convex: for all $x, y \in K$ and for all $0 \leq \lambda \leq 1$, we have $\lambda x + (1 - \lambda)y \in K$.

(ii) K is closed: K contains all its limits of convergent sequences.

(iii) For all $x \in K$ and for all $\alpha \geq 0$, we have $\alpha x \in K$.

(iv) $K \cap (-K) = \{0\}$: If $x \in K$ and $-x \in K$, then $x = 0$.

In other words, a cone is a convex and closed subset of a Banach space that is stable under multiplication by positive scalars and does not contain any nontrivial linear subspaces.

Definition 32 *(Completely Continuous Operator)*

Let E be a Banach space. An operator $A : E \to E$ is called completely continuous if the following two conditions are satisfied:

(i) A is continuous, that is, for any sequence (x_n) in E such that $x_n \to x$ in E, we have $Ax_n \to Ax$.

(ii) A maps bounded sets in E to relatively compact sets, that is, for any bounded set $B \subset E$, the image $A(B)$ is relatively compact in E (i.e., the closure of $A(B)$ is compact).

In other words, a completely continuous operator is a linear and continuous operator that transforms bounded sets into sets whose closures are compact.

Theorem 33 *Let Ω_1 and Ω_2 be two bounded open sets in a Banach space E such that $0 \in \Omega_1$, $\Omega_1 \subset \Omega_2$, and let K be a cone in E. Let $A : K \cap (\Omega_2 \setminus \Omega_1) \to K$ be a completely continuous operator such that one of the following conditions is satisfied:*

(i) $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_2$,

(ii) $\|Au\| \geq \|u\|$ *for* $u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leq \|u\|$ *for* $u \in K \cap \partial\Omega_2$.

Then the operator A has at least one fixed point in $K \cap (\Omega_2 \setminus \Omega_1)$.

Proof.
We will use the Krasnoselskii fixed point theorem.
Step 1: Construction of the Homotopy
Define a homotopy $H : K \cap (\Omega_2 \setminus \Omega_1) \times [0, 1] \to K$ by:

$$H(u, t) = (1 - t)u + tAu.$$

We will show that $H(u, t) \neq 0$ for all $u \in K \cap (\Omega_2 \setminus \Omega_1)$ and for all $t \in [0, 1]$.
Step 2: Boundary Properties

- If $u \in K \cap \partial\Omega_1$, then by (i) $\|Au\| \leq \|u\|$ and thus:

$$\|H(u, t)\| = \|(1 - t)u + tAu\| \leq (1 - t)\|u\| + t\|Au\| \leq \|u\|.$$

  By the definition of $\partial\Omega_1$, $\|u\|$ is constant for $u \in \partial\Omega_1$. Thus, $\|H(u, t)\| < \|u\|$ or $H(u, t) \neq 0$.

- If $u \in K \cap \partial\Omega_2$, then by (i) $\|Au\| \geq \|u\|$ and thus:

$$\|H(u, t)\| = \|(1 - t)u + tAu\| \geq (1 - t)\|u\| + t\|Au\| \geq \|u\|.$$

  By the definition of $\partial\Omega_2$, $\|u\|$ is constant for $u \in \partial\Omega_2$. Thus, $\|H(u, t)\| > \|u\|$ or $H(u, t) \neq 0$.

Step 3: Conclusion by Homotopy

Since $H(u,t) \neq 0$ on the boundaries $\partial\Omega_1$ and $\partial\Omega_2$, we can apply the Krasnoselskii fixed point theorem which guarantees that there exists a fixed point $u \in K \cap (\Omega_2 \setminus \Omega_1)$ such that $Au = u$.

■

Chapter 3

Application to Guo-Krasnoselskii's theorem

In this chapter, we consider the existence of positive solutions to the equation [13]

$$u'' + a(t)f(u) = 0, \quad t \in (0, 1) \tag{3.1}$$

with the three-point integral boundary condition

$$u(0) = 0, \quad \alpha \int_0^\eta u(s)ds = u(1) \tag{3.2}$$

where $0 < \eta < 1$. We note that the new three-point boundary conditions are related to the area under the curve of solutions $u(t)$ from $t = 0$ to $t = \eta$.

The aim of this chapter is to give some results for existence of positive solutions to 3.1, 3.2, assuming that $0 < \alpha < 2/\eta^2$ and f is either superlinear or sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \quad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case. By the positive solution of 3.1-3.2 we mean that a function $u(t)$ is positive on $0 < t < 1$ and satisfies the problem 3.1-3.2.

Throughout this chapter, we suppose the following conditions hold:

- $(H1)$: $f \in C([0, \infty), [0, \infty))$;
- $(H2)$: $a \in C([0, 1], [0, \infty))$ and there exists $t_0 \in [\eta, 1]$ such that $a(t_0) > 0$.

The proof of the main theorem is based upon an application of Krasnoselskii's fixed point theorem in a cone which we proved in the previous chapter:

Theorem 34 *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let ...*

$$A : K \cap \left(\bar{\Omega}_1 \backslash \Omega_2\right) \longrightarrow K \tag{3.3}$$

be a completely continuous operator such that

(i) $\|Au\| \leqslant \|u\|, u \in K \cap \partial\Omega_1$, and $\|Au\| \geqslant \|u\|, u \in K \cap \partial\Omega_2$; or

(ii) $\|Au\| \geqslant \|u\|, u \in K \cap \partial\Omega_1$, and $\|Au\| \geqslant \|u\|, u \in K \cap \partial\Omega_2$. Then A has a fixed point in $K \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$.

3.1 Preliminaries

In this section we state and prove a number of lemmas.

Lemme 3 *Let $\alpha\eta^2 \neq 2$. Then for $y \in C[0,1]$, the problem*

$$u'' + y(t) = 0, \quad t \in (0,1) \tag{3.4}$$

$$u(0) = 0, \quad \alpha \int_0^\eta u(s)ds = u(1) \tag{3.5}$$

has a unique solution

$$u(t) = \frac{2t}{2 - \alpha\eta^2} \int_0^1 (1-s)y(s)ds - \frac{\alpha t}{2 - \alpha\eta^2} \int_0^\eta (\eta - s)^2 y(s)ds - \int_0^t (t-s)y(s)ds$$

Proof.

From 3.4, we have

$$u''(t) = -y(t)$$

For $t \in [0,1)$, integration from 0 to t, gives

$$u'(t) = u'(0) - \int_0^t y(s)ds \tag{3.6}$$

For $t \in [0,1]$, integration from 0 to t yields that

$$u(t) = u'(0)t - \int_0^t \left(\int_0^x y(s)ds\right) dx \tag{3.7}$$

that is,

$$u(t) = u'(0)t - \int_0^t (t-s)y(s)ds \tag{3.8}$$

So,

$$u(1) = u'(0) - \int_0^1 (1-s)y(s)ds \tag{3.9}$$

Integrating 3.8 from 0 to η, where $\eta \in (0,1)$, we have

$$\begin{aligned}\int_0^\eta u(s)ds &= u'(0)\frac{\eta^2}{2} - \int_0^\eta \left(\int_0^x (x-s)y(s)ds\right)dx \\ &= u'(0)\frac{\eta^2}{2} - \frac{1}{2}\int_0^\eta (\eta-s)^2 y(s)ds\end{aligned} \tag{3.10}$$

From 3.5, we obtain that

$$u'(0) - \int_0^1 (1-s)y(s)ds = u'(0)\frac{\alpha\eta^2}{2} - \frac{\alpha}{2}\int_0^\eta (\eta-s)^2 y(s)ds \tag{3.11}$$

Thus,

$$u'(0) = \frac{2}{2-\alpha\eta^2}\int_0^1 (1-s)y(s)ds - \frac{\alpha}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 y(s)ds \tag{3.12}$$

Therefore, 3.4, 3.5 has a unique solution

$$u(t) = \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)y(s)ds - \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 y(s)ds - \int_0^t (t-s)y(s)ds \tag{3.13}$$

■

Lemme 4 *Let $0 < \alpha < \frac{2}{\eta^2}$. If $y \in C(0,1)$ and $y(t) \geqslant 0$ on $(0,1)$, then the unique solution u of 3.4, 3.5 satisfies $u \geqslant 0$ for $t \in [0,1]$.*

Proof. If $u(1) \geqslant 0$, then, by the concavity of u and the fact that $u(0) = 0$, we have $u(t) \geqslant 0$ for $t \in [0,1]$.

Moreover, since the graph of $u(t)$ is concave down on $(0,1)$, we have

$$\int_0^\eta u(s)ds \geqslant \frac{1}{2}\eta u(\eta) \tag{3.14}$$

where $\frac{1}{2}\eta u(\eta)$ is the area of the triangle under the curve $u(t)$ from $t = 0$ to $t = \eta$ for $\eta \in (0,1)$.

Assume that $u(1) < 0$. From 3.5, we have

$$\int_0^\eta u(s)ds < 0 \tag{3.15}$$

By concavity of u and $\int_0^\eta u(s)ds < 0$, it implies that $u(\eta) < 0$.

Hence,

$$u(1) = \alpha \int_0^\eta u(s)ds \geqslant \frac{\alpha\eta}{2} u(\eta) > \frac{u(\eta)}{\eta} \tag{3.16}$$

which contradicts the concavity of u. ■

Lemme 5 *Let $\alpha\eta^2 > 2$. If $y \in C(0,1)$ and $y(t) \geqslant 0$ for $t \in (0,1)$, then 3.4, 3.5 has no positive solution.*

Proof. Assume 3.4, 3.5 has a positive solution u.

If $u(1) > 0$, then $\int_0^\eta u(s)ds > 0$, it implies that $u(\eta) > 0$ and

$$\frac{u(1)}{1} = \alpha \int_0^\eta u(s)ds \geqslant \frac{\alpha\eta}{2} u(\eta) = \frac{\alpha\eta^2}{2}\frac{u(\eta)}{\eta} > \frac{u(\eta)}{\eta} \tag{3.17}$$

which contradicts the concavity of u.

If $u(1) = 0$, then $\int_0^\eta u(s)ds = 0$, this means $u(t) \equiv 0$ for all $t \in [0, \eta]$. If there exists $\tau \in (\eta, 1)$ such that $u(\tau) > 0$, then $u(0) = u(\eta) < u(\tau)$, which contradicts the concavity of u. Therefore, no positive solutions exist. ■

In the rest of the chapter, we assume that $0 < \alpha\eta^2 < 2$. Moreover, we will work in the Banach space $C[0,1]$, and only the sup norm is used.

Lemme 6 *Let $0 < \alpha < \frac{2}{\eta^2}$. If $y \in C[0,1]$ and $y \geqslant 0$, then the unique solution u of the problem 3.4, 3.5 satisfies*

$$\inf_{t\in[\eta,1]} u(t) \geqslant r\|u\| \tag{3.18}$$

where

$$r := \min\left\{\eta, \frac{\alpha\eta^2}{2}, \frac{\alpha\eta(1-\eta)}{2-\alpha\eta^2}\right\} \tag{3.19}$$

Proof. Set $u(\tau) = \|u\|$. We divide the proof into three cases.

Case 1. If $\eta \leqslant \tau \leqslant 1$ and $\inf_{t\in[\eta,1]} u(t) = u(\eta)$, then the concavity of u implies that

$$\frac{u(\eta)}{\eta} \geqslant \frac{u(\tau)}{\tau} \geqslant u(\tau) \tag{3.20}$$

Thus,

$$\inf_{t\in[\eta,1]} u(t) \geqslant \eta\|u\| \tag{3.21}$$

Case 2. If $\eta \leqslant \tau \leqslant 1$ and $\inf_{t\in[\eta,1]} u(t) = u(1)$, then $3.5, 3.14$, and the concavity of u imply

$$u(1) = \alpha \int_0^\eta u(s)ds \geqslant \frac{\alpha\eta^2}{2}\left(\frac{u(\eta)}{\eta}\right) \geqslant \frac{\alpha\eta^2}{2}\frac{u(\tau)}{\tau} \geqslant \frac{\alpha\eta^2}{2}u(\tau) \tag{3.22}$$

Therefore,

$$\inf_{t\in[\eta,1]} u(t) \geqslant \frac{\alpha\eta^2}{2}\|u\| \tag{3.23}$$

Case 3. If $\tau \leqslant \eta < 1$, then $\inf_{t\in[\eta,1]} u(t) = u(1)$. Using the concavity of u and $3.5, 3.14$, we have

$$\begin{aligned} u(\sigma) &\leqslant u(1) + \frac{u(1) - u(\eta)}{1-\eta}(0-1) \\ &\leqslant u(1)\left(1 - \frac{1 - 2/\alpha\eta}{1-\eta}\right) \\ &= u(1)\frac{2-\alpha\eta^2}{\alpha\eta(1-\eta)} \end{aligned} \tag{3.24}$$

This implies that

$$\inf_{t\in[\eta,1]} u(t) \geqslant \frac{\alpha\eta(1-\eta)}{2-\alpha\eta^2}\|u\| \tag{3.25}$$

This completes the proof. ■

3.2 Main Results

Now we are in the position to establish the main result.

Theorem 35 *Assume (H1) and (H2) hold. Then the problem 3.1- 3.2 has at least one positive solution in the case*

(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear), or*

(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. It is known that $0 < \alpha < \frac{2}{\eta^2}$. From 3, u is a solution to the boundary value problem 3.1- 3.2 if and only if u is a fixed point of operator A, where A is defined by

$$\begin{aligned} Au(t) =& \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))\,ds \\ &- \frac{\alpha t}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 a(s)f(u(s))\,ds \\ &- \int_0^t (t-s)a(s)f(u(s))\,ds. \end{aligned} \tag{3.26}$$

Denote that

$$K = \left\{ u \mid u \in C[0,1], u \geqslant 0, \inf_{\eta\leqslant t\leqslant 1} u(t) \geqslant \gamma\|u\| \right\} \tag{3.27}$$

where γ is defined in 3.19.

It is obvious that K is a cone in $C[0,1]$. Moreover, by 4 and 6, $AK \subset K$. It is also easy to check that $A : K \to K$ is completely continuous.

Superlinear Case ($f_0 = 0$ and $f_\infty = \infty$).

Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leqslant \epsilon u$, for $0 < u \leqslant H_1$, where $\epsilon > 0$ satisfies

$$\frac{2\epsilon}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)\,ds \leqslant 1 \tag{3.28}$$

Thus, if we let

$$\Omega_1 = \{u \in C[0,1] \mid \|u\| < H_1\} \tag{3.29}$$

then, for $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned} Au(t) &\leqslant \frac{2t}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))\,ds \\ &\leqslant \frac{2t\epsilon}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)u(s)\,ds \\ &\leqslant \frac{2\epsilon}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)\,ds\|u\| \\ &\leqslant \|u\| \end{aligned} \tag{3.30}$$

Thus $\|Au\| \leqslant \|u\|$ for $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \geqslant \rho u$, for $u \geqslant \widehat{H}_2$, where $\rho > 0$ is chosen so that

$$\rho\gamma\frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)\,ds \geqslant 1$$

Let $H_2 = \max\left\{2H_1, \widehat{H}_2/\gamma\right\}$ and $\Omega_2 = \{u \in C[0,1] \mid \|u\| < H_2\}$. Then $u \in K \cap \partial\Omega_2$ implies that

$$\inf_{\eta\leqslant t\leqslant 1} u(t) \geqslant \gamma\|u\| = \gamma H_2 \geqslant \widehat{H}_2$$

and so

$$\begin{aligned}
Au(\eta) =& \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))\,ds - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 a(s)f(u(s))\,ds \\
&- \int_0^\eta (\eta-s)a(s)f(u(s))\,ds \\
=& \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))\,ds - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta \left(\eta^2-2\eta s+s^2\right) a(s)f(u(s))\,ds \\
&- \frac{1}{2-\alpha\eta^2}\int_0^\eta \left(2-\alpha\eta^2\right)(\eta-s)a(s)f(u(s))\,ds \\
=& \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))\,ds + \frac{\alpha\eta^2}{2-\alpha\eta^2}\int_0^\eta sa(s)f(u(s))\,ds \\
&- \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta s^2 a(s)f(u(s))\,ds - \frac{2\eta}{2-\alpha\eta^2}\int_0^\eta a(s)f(u(s))\,ds \\
&+ \frac{2}{2-\alpha\eta^2}\int_0^\eta sa(s)f(u(s))\,ds \\
=& \frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)f(u(s))\,ds + \frac{2(1-\eta)}{2-\alpha\eta^2}\int_0^\eta sa(s)f(u(s))\,ds \\
&+ \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta s(\eta-s)a(s)f(u(s))\,ds \\
\geqslant& \frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)f(u(s))\,ds \\
\geqslant& \frac{2\eta\rho}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)u(s)\,ds \\
\geqslant& \frac{2\eta\rho\gamma}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)\,ds\|u\| \\
\geqslant& \|u\|
\end{aligned} \tag{3.8}$$

Hence, $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$. By the first part of 33, A has a fixed point in $K \cap \left(\bar{\Omega}_2 \backslash \Omega_1\right)$ such that $H_1 \leqslant \|u\| \leqslant H_2$.

Sublinear Case ($f_0 = \infty$ **and** $f_\infty = 0$).

Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(u) \geqslant Mu$ for $0 < u \leqslant H_3$, where $M > 0$ satisfies

$$\frac{2\eta\gamma M}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)\,ds \geqslant 1$$

Let

$$\Omega_3 = \{u \in C[0,1] \mid \|u\| < H_3\}$$

then for $u \in K \cap \partial\Omega_3$, we get

$$\begin{aligned}
Au(\eta) =& \frac{2\eta}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)f(u(s))\,ds - \frac{\alpha\eta}{2-\alpha\eta^2}\int_0^\eta (\eta-s)^2 a(s)f(u(s))\,ds \\
&- \int_0^\eta (\eta-s)a(s)f(u(s))\,ds \\
\geqslant& \frac{2\eta}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)f(u(s))\,ds \\
\geqslant& \frac{2\eta\gamma M}{2-\alpha\eta^2}\int_\eta^1 (1-s)a(s)\,ds\|u\| \\
\geqslant& \|u\|
\end{aligned} \tag{3.11}$$

Thus, $\|Au\| \geqslant \|u\|, u \in K \cap \partial\Omega_3$. Now, since $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ so that $f(u) \leqslant \lambda u$ for $u \geqslant \widehat{H}_4$, where $\lambda > 0$ satisfies

$$\frac{2\lambda}{2-\alpha\eta^2}\int_0^1 (1-s)a(s)\,ds \leqslant 1$$

Choose $H_4 = \max\left\{2H_3, \widehat{H}_4/\gamma\right\}$. Let

$$\Omega_4 = \{u \in C[0,1] \mid \|u\| < H_4\}$$

then $u \in K \cap \partial\Omega_4$ implies that

$$\inf_{\eta \leqslant t \leqslant 1} u(t) \geqslant \gamma \|u\| = \gamma H_4 \geqslant \widehat{H}_4$$

Therefore,

$$\begin{aligned}
Au(t) =& \frac{2t}{2-\alpha\eta^2} \int_0^1 (1-s)a(s)f(u(s))\,ds - \frac{\alpha t}{2-\alpha\eta^2} \int_0^\eta (\eta-s)^2 a(s)f(u(s))\,ds \\
& - \int_0^t (t-s)a(s)f(u(s))\,ds \\
\leqslant & \frac{2t}{2-\alpha\eta^2} \int_0^1 (1-s)a(s)f(u(s))\,ds \\
\leqslant & \frac{2\lambda\|u\|}{2-\alpha\eta^2} \int_0^1 (1-s)a(s)\,ds \\
\leqslant & \|u\|
\end{aligned} \tag{3.15}$$

Thus $\|Au\| \leqslant \|u\|, u \in K \cap \partial\Omega_4$. By the second part of Theorem 33, A has a fixed point u in $K \cap \left(\bar{\Omega}_4 \backslash \Omega_3\right)$, such that $H_3 \leqslant \|u\| \leqslant H_4$. This completes the sublinear part of the theorem . Therefore, the problem 3.13.2 has at least one positive solution. ■

Conclusion

The fixed-point principle has many applications. It is particularly involved in solving several nonlinear differential equations, especially in the study of existence and uniqueness.

In this dissertation, we address different applications of this principle as well as some of its extensions and generalizations that are involved in solving non-local boundary value problems.

We demonstrate the existence of solutions using Banach's contraction principle and the nonlinear Leray-Schauder alternative. We study the positivity of the solution via the Guo-Krasnosel'skii fixed-point theorem on a cone.

Bibliography

[1] R. P. Agarwal, M. Meehan, D. O'Regan, Fixed Point Theory and Applications, Cambridge Tracts in Mathematics, Cambridge University Press, 141, 2001.

[2] D. R. Anderson, J. M. Davis, Multiple solutions and eigenvalues for third-order right-focal boundary value problems, J. Math. Anal. Appl., 267 (2002), 135-157.

[3] V. IS. Banach, Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales, Fundamenta Math., 3 (1922), pp. 133-181.

[4] K. Deimling, Nonlinear Functional Analysis, Springer, Berlin, 1985.

[5] G. Dinca, J. Mawhin, Brouwer Degree and Applications, January 17, 2009.

[6] J. R. Graef and Bo Yang, Positive solutions for a third order nonlocal boundary value problem, Discrete Contin. Dyn. Syst., Ser. S., 1 (2008), 89-97.

[7] J. R. Graef, B. Yang, Positive solutions of a third order eigenvalue problem, Dynam. Systems Appl. 15 (2006), 97-110.

[8] D. Guo, V. Lakshmikantham, Nonlinear problems in abstract cones. Academic Press, San Diego. 1988.

[9] V. I. Istratescu, Fixed Point Theory : An Introduction, Springer, 30 nov. 2001 - 488 pages

[10] O. Kavian. Introduction á la théorie des points critiques et applications au problemes elliptiques. Vol. 13. Math ematiques Applications. Springer-Verlag, 1993. Hiroshima Math. J. 33 (2003), 217-227.

[11] D. O'Regan, Y. Je Cho and Y.Q. Chen, Topological Degree Theory and Applications, Series in Mathematical Analysis and Applications, vol. 10, Chapman Hall/CRC, (2006).

[12] J. R. Graef, B. Yang, Positive solutions of a third order eigenvalue problem, Dynam. Systems Appl. 15 (2006), 97-110.

[13] J.Tariboon, T.Sitthiwirattham, Positive Solutions of a Nonlinear Three-Point Integral Boundary Value Problem, Hindawi Publishing Corporation Boundary Value Problems Volume 2010, Article ID 519210, 11 pages

[14] V. A. Il'in and E. I. Moiseev, Nonlocal boundary-value problem of the first kind for a Sturm-Liouville operator in its differential and finite difference aspects, Differential Equations, vol. 23, pp. 803–810,1987.

[15] C. P. Gupta, Solvability of a three-point nonlinear boundary value problem for a second order ordinary differential equations, Journal of Mathematical Analysis and Applications, vol. 168, no. 2, pp.540–551, 1992.

[16] Chengbo, Positive solutions for semi-positone three-point boundary value problems, Journal of Computational and Applied Mathematics, vol. 228, no. 1, pp. 279–286, 2009.

[17] X. Xu, Multiplicity results for positive solutions of some semi-positone three-point boundary value problems, Journal of Mathematical Analysis and Applications, vol. 291, no. 2, pp. 673–689, 2004.

Abstract

We investigate the existence of positive solutions for a three-point integral boundary value problem of the form $u'' + a(t)f(u) = 0$ with $t \in [0,1]$, $u(0) = 0$ and $\alpha \int_0^\eta u(s)ds = u(1)$, where $0 < \eta < 1$ and $0 < \alpha < \frac{2}{\eta^2}$. By applying the fixed-point theorem in cones, we demonstrate that there exists at least one positive solution when f is either superlinear or sublinear.

Keywords: positive solution, superlinear, sublinear, fixed-point theorem, cones.

Résumé

Nous étudions l'existence de solutions positives pour le problème aux limites intégral à trois points de la forme $u'' + a(t)f(u) = 0$ avec $t \in [0,1]$, $u(0) = 0$ et $\alpha \int_0^\eta u(s)ds = u(1)$, où $0 < \eta < 1$ and $0 < \alpha < \frac{2}{\eta^2}$. En appliquant le théorème du point fixe dans les cônes, nous montrons qu'il existe au moins une solution positive lorsque f est soit superlinéaire, soit sous-linéaire.

Mots-clés: solution positive, superlinéaire, sous-linéaire, théorème du point fixe,

Cones

ملخص

ندرس وجود الحلول الإيجابية لمسألة القيمة الحدية التكاملية ثلاثية النقط من الشكل $u'' + a(t)f(u) = 0$ حيث $t \in [0,1]$ ، $u(0) = 0$ و $\alpha \int_0^\eta u(s)ds = u(1)$ أين $0 < \eta < 1$ و $0 < \alpha < \frac{2}{\eta^2}$. من خلال تطبيق نظرية النقطة الثابتة في المخاريط نثبت وجود حل إيجابي واحد على الأقل عندما تكون f إما فوق خطية أو تحت خطية.

الكلمات المفتاحية: حل إيجابي، فوق خطي، تحت خطي، نظرية النقطة الثابتة، مخاريط.