Kasdi Merbah University of Ouargla
Faculty of Mathematics and Material Sciences
Department of Mathematics

N° d'ordre :
N° de série :

DEPARTMENT OF MATHEMATICS

MASTER

Domain: Mathematics

Option: Modelisation and Numerical Analysis

By: TAYEBBEY Bakhta

Title:

TWO PARAMETERS IDENTIFICATION PROBLEM IN A THREE DIMENSIONAL SPHERICAL DYNAMO EQUATION

Presented Publicly on: 30/06/2024

Examination Committee:

Dr.	Rezzag Bara Rayhana	M.C.B. Kasdi Merbah University-Ouargla	Chairman
Dr.	Lacheheb Ilyes	M.C.B. Kasdi Merbah University-Ouargla	Examiner
Dr.	Messaoudi Djemaa	M.C.B. Kasdi Merbah University-Ouargla	Supervisor

Dedication

I am forever grateful to my father, "Mohamed", for being an unwavering pillar of support, encouragement, and inspiration.

I also express my gratitude to my mother, "Zouhra", who has been a constant source of light and love in my life.

I also extend my heartfelt appreciation to all my family members espacially , "Fouzia", "Nabila", "Hadjira", "Khaled", "Yassin","Labib",and "Ayoub", for being a part of my life and supporting me in all my endeavors.

To my friends and colleagues, who have been with me every step of the way on my academic journey, I extend my deepest thanks. Their support and encouragement have been instrumental in helping me to achieve this goal.

Finally, I express my gratitude to my teachers and supervisors for their guidance, which have made the challenges of graduation more manageable and rewarding.

Thank you all for being a part of this journey.

Acknowledgement

Firstly, I express my deepest gratitude to Allah Almighty for giving me the strength and health to begin and complete this journey.

I am also deeply grateful to everyone who has supported and helped me throughout my academic journey. Without your contributions, I would not have been able to achieve my goals.

I extend my sincere appreciation to my thesis supervisor "MESSAOUDI Djemaa", whose guidance, expertise, and patience have been invaluable. Her feedback, criticism, and unwavering support have helped me shape this thesis into its final form.

I would also like to express my gratitude to the members of the jury who have kindly agreed to evaluate this work. I hope to meet their expectations and earn their satisfaction.

Contents

Dedication **i**

Acknowledgement **ii**

Acknowledgement **ii**

List of Figures **iv**

Notations and Conventions **vi**

Introduction **1**

1 Generalities **3**
- 1.1 Background and motivation . . . 3
  - 1.1.1 Inverse problems . . . 3
  - 1.1.2 What is an ill-posed problem? . . . 4
- 1.2 Some important tools . . . 5
  - 1.2.1 Tikhonov Regularization . . . 5
  - 1.2.2 Properties . . . 6
  - 1.2.3 Notations . . . 7

2 Parameter identification problem **8**
- 2.1 Setting of the problem . . . 8
- 2.2 Tikhonov regularization method . . . 9
- 2.3 Finite element approximation . . . 10

3 Two Parameters identification problem **15**
3.1 Tikhonov regularization method 15
3.2 Finite element approximation 22

Conclusion **35**

Bibliography **36**

List of Figures

1.1 Inverse problem via direct problem 4
1.2 Techniques for solving ill posed problems 5

Notations

- $\partial_n = n \cdot \nabla$: The derivative according to the normal.
- $\partial\Omega$: The boundary of Ω.
- $\bar{\Omega}$: The closure of Ω.
- $\mathring{\Omega}$: The interior of Ω.
- $C_0^\infty = \mathcal{D}$: The test functions space.
- I : Inverse problem.
- ∇ : The gradient operator.
- C : General constant.
- $\mathrm{div}(u)$: The divergence of u.
- $B(x_0, r) := \{x \in \mathbb{R}^n; |x - x_0| < r\}$.
- $\mathrm{dist}(x, \Omega)$: The distance between x and Ω.
- $(.,.)$: The scalar product.
- $\|f\|^2_{L^2(\Omega)} = \int_\Omega |f(x)|^2 dx$.
- $\|u\|_{0,\Omega} = \|u\|^2_L(\Omega) =:$ the norm of L^2
- $\mathbf{L}^\infty(\Omega)$: The space of essentially bounded functions on Ω.
- $L^2(\Omega)$:The vector space of functions whose square is integrable (in the sense of Lebesgue) on Ω
- $L^2(0, T; B) =: \{u(t) \in B$ for a.e. $t \in (0, T)$ and $\|u\|_{L^2(0,T;B)} < \infty\}$
- $\|u\|_{L^2(0,T;B)} = \left(\int_0^T \|u(t)\|^2_B \mathrm{d}t\right)^{\frac{1}{2}}$
- $H^1(0, T; B) = \{u \in L^2(0, T; B) : u'(t) \in L^2(0, T; B)\}$
- $\|u\|_{H^1(0,T;B)} = \left(\|u(t)\|^2_{L^2(0,T;B)} + \|u'(t)\|^2_{L^2(0,T;B)}\right)^{\frac{1}{2}}$

Introduction

The inverse problem arises in various scientific fields such as astronomy, econometrics, financial mathematics, medical imaging, and quantum physics. For more details, see [2, 3, 6, 10, 15, 17, 18, 27]. Inverse problems are well-studied for their applications to a wide variety of fields. In the past few decades, the development of powerful computers enabled engineers, mathematicians, and scientists to solve inverse problems computationally, leading to significant results in computer vision, medical imaging, physics and many other fields. The scope of applications for the inverse problem has expanded to cover two main problems. These include determining past states or parameters of a physical system, and predicting the outcome of future states or parameters.

Looking into past states and parameters is important in medical imaging. Solving the first type of problem enables us to locate the source of tumors because tumors are generally denser, and therefore resist pulling and pushing more than normal tissue. Studying the second problem is important in computer vision and other physical settings where we are estimating where objects are going to be at a specific time or when we want to steer the environment towards a specific outcome.

It is well known that many astrophysical bodies have intrinsic magnetic fields. But only in the last few decades people begin to understand more about the origin of this field. So far a widely accepted theory is the so-called meanfield dynamo theory . For the numerical simulations and mathematical theory analysis of the direct dynamo problem, one may refer to [2, 3, 7, 10, 11, 12] and the references therein. And for the numerical analysis of some stochastic interface problems, we can refer to [1, 8] and the references therein. While in many applications, the inverse dynamo problems may be more interesting and practically important, where the magnetic property of the physical medium is unknown. But knowing them is indispensable to some research investigations or to a good understanding of the physical medium and how the magnetic field behaves in the physical medium. For example, in [5], the authors make use of the asymmetric time dependence and various statistical properties of polarity reversals of the earth's magnetic field to recover some of parameters of the geodynamo. The objective of this work is studying some of parameter identification problems in a three dimensional spherical dynamo equation. Our work will be divided into three chapters, as outlined below:

In Chapter one, we begin with some background and literature review.

In Chapter two, we focus on recovering the magnetic diffusivity in a three-dimensional (3D) spherical dynamo equation. We will transform The ill-posed problem into a nonlinear minimization problem by using the Tikhonov regularization method. Then we develop a fully discrete scheme based on the finite element method.

In Chapter three, we present a new problem, a two-parameter identification problem. This problem involves recovering the magnetic diffusivity and source strength in a three-dimensional (3D) spherical dynamo equation. Following the same approach as in the second chapter, we study this problem to see what will be different.

Chapter 1

Generalities

1.1 Background and motivation

1.1.1 Inverse problems

Generally, causes and effects are examined in a specific order. The term "inverse problem" derives from physics, when our objective is to recover information by observing the effects and then try to figure out what caused them.
Let X and Y be normed spaces, $K : X \longrightarrow Y$ a (linear or nonlinear) mapping. Then, given the mathematical model

$$K(x) = y,$$

where x is a vector of unknowns and y is a vector of measurements, the direct problem is to find y given x, while the inverse problem is to find x given y. In practice, the unknown could be parameters in our mathematical model or the source term or boundaries or a combination of these.

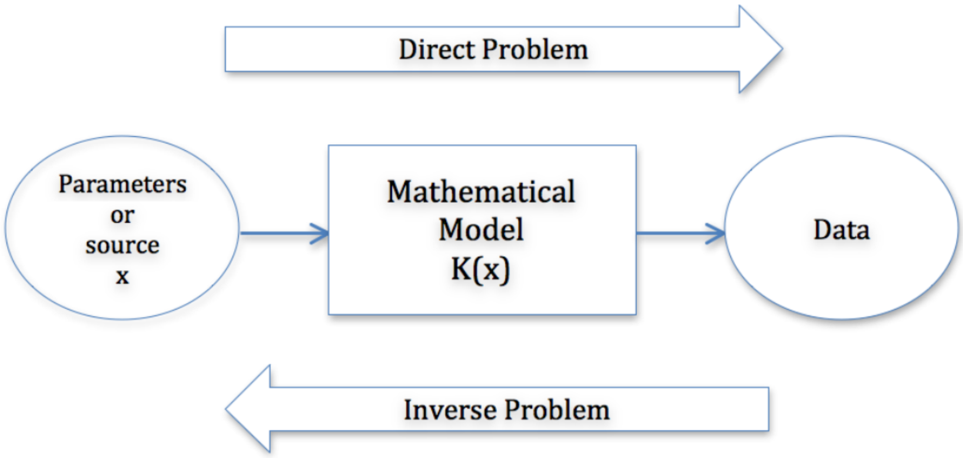

Figure 1.1 – Inverse problem via direct problem

Due to its indirect nature, solving the inverse problem is usually very difficult. In fact, solving such an inverse problem by standard methods numerically is difficult and often yields unstable results, even when the data are exact Therefore, to obtain a stable approximation of the solution, we have to use special techniques. Due to their special properties, most inverse problems are ill-posed.

1.1.2 What is an ill-posed problem?

The French mathematician Jacques Hadamard introduced the concept of a wellposed problem in his paper of 1902 on boundary-value problems for partial differential equations and their physical interpretation [16].

Definition 1.1.1 *Based on Hadamard's definition, a mathematical problem is well-posed if it satisfies the following properties:*
1. Existence: For all (suitable) data, there exists a solution of the problem (in an appropriate sense).
2. Uniqueness: For all (suitable) data, the solution is unique.
3. Stability: The solution depends continuously on the data.

A problem is ill-posed if one of these three conditions is violated.

Remark 1.1.2 *The main concern when studying the inverse problem is the violation of the third condition, that is, the solution does not depend continuously on the data.*

In general to solve an ill-posed problems there are two techniques as in the following schema

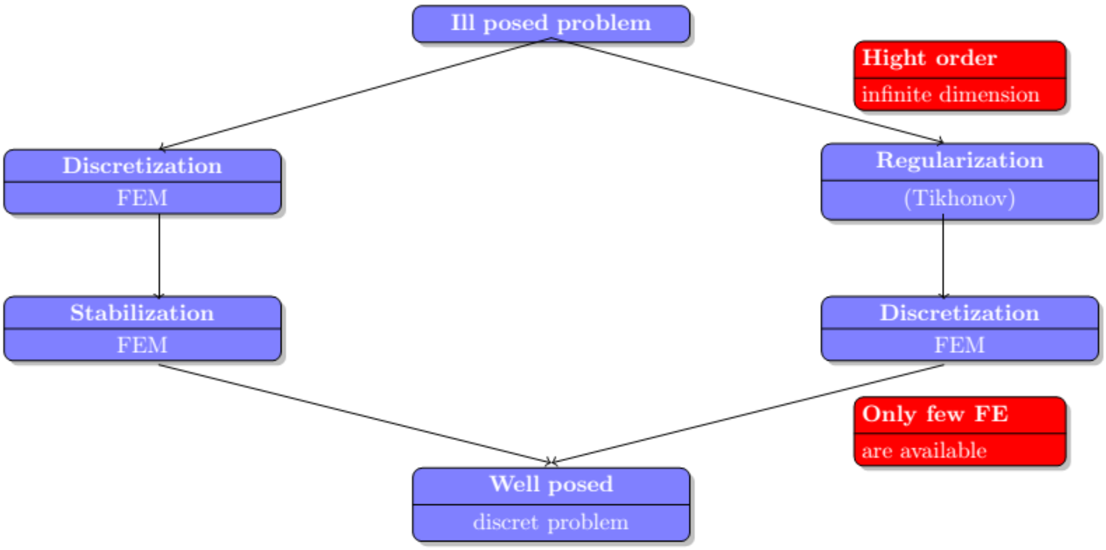

Figure 1.2 – Techniques for solving ill posed problems

1.2 Some important tools

In this section, we will present some important theorems and lemmas for later use.

1.2.1 Tikhonov Regularization

The idea of regularization method is to transform an ill-posed problem into a well-posed one, which can be done by introducing a regularized operator which considers available prior information about the exact solution.
Tikhonov regularization method [1], which is named after the Russian mathematician Andrey Tikhonov and was introduced in the early 20th century, is a commonly employed technique for addressing ill-posed inverse problems. This method entails incorporating a regularization term into the initial problem to ensure the stability of the solution and avoid overfitting.

Theorem 1.2.1 *(Lebesgue's dominated convergence theorem)[5]*
Let (f_n) be a sequence of functions in L^1 that satisfy
(a) $f_n(x) \longrightarrow f(x)$ a.e on Ω,
(b) there is a function $g \in L^1$ such that for all $n, |f_n(x)| \leq g(x)$ a.e on Ω.
Then $f \in L^1$ and $||f_n - f||_1 \longrightarrow 0$.

Lemma 1.2.2 *(Aubin-Lions Lemma, [30], p.189) Let X_0, X be two Banach spaces and X_1 be a Hilbert space with $X_0 \subset X \subset X_1$, the injections being continuous and the injection of X_0 into X being compact. Then the injection of $\mathcal{Y}(0, T; \alpha_0, \alpha_1; X_0, X_1)$ into*

$L^{\alpha_0}(0,T;X)$ *is compact for any finite number* $\alpha_0 > 1$*, where* $\mathcal{Y}(0,T;\alpha_0,\alpha_1;X_0,X_1) = \left\{ v \in L^{\alpha_0}(0,T;X_0);\ v' = \dfrac{dv}{dt} \in L^{\alpha_1}(0,T;X_1) \right\}$.

1.2.2 Properties

Let introduce two important operators for later use. The first one is the so-called modified Scott-Zhang interpolation operator S_h : (see [7] or [29]), which preserves the boundary condition in H_0: for any $\boldsymbol{B} \in H_0$, we have $S_h\boldsymbol{B} \in H_{0h}$ and it has the following properties:

Lemma 1.2.3 *Let* $u \in H^1(\Omega)$*, then there exists a constant* C*, independent of* h*, such that*

$$\|S_h u\|_{1,\Omega} \leq C\|u\|_{1,\Omega}, \quad \|u - S_h u\|_{0,\Omega} \leq Ch\|u\|_{1,\Omega},$$

and

$$\lim_{h\to 0} \|u - S_h u\|_{1,\Omega} = 0.$$

Moreover, if $u \in H^2(\Omega)$*, we have*

$$\|u - S_h u\|_{1,\Omega} \leq Ch\|u\|_{2,\Omega}.$$

The second operator is the L^2 quasi-projection operator $\Pi_h : L^2(\Omega) \to Q_h$, which has the following properties (see [31]):

Lemma 1.2.4 *For* $w \in L^2(\Omega)$*, we have*

$$\|\Pi_h w\|_{0,\Omega} \leq C\|w\|_{0,\Omega}, \qquad \lim_{h\to 0} \|w - \Pi_h w\|_{0,\Omega} \to 0.$$

Moreover, if $w \in H^1(\Omega)$*, we have*

$$\|\Pi_h w\|_{1,\Omega} \leq C\|w\|_{1,\Omega}, \qquad \lim_{h\to 0} \|w - \Pi_h w\|_{1,\Omega} \to 0.$$

Lemma 1.2.5 *(classical approximation result ([32] [33]))*
Let X *be a Banach space. For a given function* $f \in C([0,T];X)$*, we define a step function approximation of* f*:*

$$S_\triangle f(\boldsymbol{x},t) = \sum_{n=1}^{M} \chi_n(t) f(\boldsymbol{x},t_n),$$

where $\chi_n(t)$ *is the characteristic function on the interval* (t_{n-1},t_n)*. Then we have*

$$\lim_{\tau\to 0} \int_0^T \|S_\triangle f(\cdot,t) - f(\cdot,t)\|_X^2 dt = 0. \tag{1.1}$$

1.2.3 Notations

We end this chapter with some useful notations. For $m \in R$, $H^m(\Omega)$ is the usual Sobolev space, and we denote $H^m(\Omega)^3$ and $L^m(\Omega)^3$ by $\boldsymbol{H}^m(\Omega)$ and $\boldsymbol{L}^m(\Omega)$ respectively. We shall use $(\cdot,\cdot)$ and $\|\cdot\|_{m,\Omega}$ to denote the scalar product in $\boldsymbol{L}^2(\Omega)$ or $L^2(\Omega)$ and the norm of $\boldsymbol{H}^m(\Omega)$ or $H^m(\Omega)$ respectively.

Moreover, we introduce some useful Sobolev spaces for the subsequent analysis:

$$\begin{aligned} H(\mathbf{curl}, \mathrm{div}; \Omega) &= \{\boldsymbol{C} \in \boldsymbol{L}^2(\Omega); \mathbf{curl}\boldsymbol{C} \in \boldsymbol{L}^2(\Omega), \mathrm{div}\boldsymbol{C} \in L^2(\Omega)\}, \\ H_0(\mathbf{curl}, \mathrm{div}; \Omega) &= \{\boldsymbol{C} \in H(\mathbf{curl}, \mathrm{div}; \Omega); \boldsymbol{C} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ V &= \{\boldsymbol{C} \in H_0(\mathbf{curl}, \mathrm{div}; \Omega); \mathrm{div}\ \boldsymbol{C} = 0\}, \\ L_0^2(\Omega) &= \{v \in L^2(\Omega); \int_\Omega v d\boldsymbol{x} = 0\}. \end{aligned}$$

As the spaces $H(\mathbf{curl}, \mathrm{div}; \Omega)$ and $H_0(\mathbf{curl}, \mathrm{div}; \Omega)$ will be frequently used, we shall write

$$H = H(\mathbf{curl}, \mathrm{div}; \Omega) \quad \text{and} \quad H_0 = H_0(\mathbf{curl}, \mathrm{div}; \Omega),$$

which are both equipped with the norm

$$\|\boldsymbol{C}\|_H = (\|\boldsymbol{C}\|_{0,\Omega}^2 + \|\nabla \times \boldsymbol{C}\|_{0,\Omega}^2 + \|\nabla \cdot \boldsymbol{C}\|_{0,\Omega}^2)^{\frac{1}{2}}.$$

It has been shown that $\|\cdot\|_H$ is equivalent $\|\cdot\|_{1,\Omega}$ (see, e.g., [14]).

Chapter 2

Parameter identification problem

This chapter focuses on recovering the magnetic diffusivity in a three-dimensional (3D) spherical dynamo equation. The ill-posed problem will be restructured into a nonlinear minimization using the Tikhonov regularization method. The nonlinear optimization problem will be approximated using a fully discretized finite element technique, with its convergence rigorously verified.

2.1 Setting of the problem

Let consider the following nonlinear spherical dynamo equation (see [7]):

$$\begin{cases} \partial_t \boldsymbol{B} + \nabla \times (\beta(\boldsymbol{x}) \nabla \times \boldsymbol{B}) & \\ \quad = R_\alpha \nabla \times \left(\frac{f(x,t)}{1+\sigma|\boldsymbol{B}|^2}\boldsymbol{B}\right) + R_m \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) & \text{in } \Omega \times (0,T), \\ \nabla \cdot \boldsymbol{B} = 0 & \text{in } \Omega \times (0,T), \\ \boldsymbol{B} \cdot \mathbf{n} = 0, \ \nabla \times \boldsymbol{B} \times \mathbf{n} = 0 & \text{on } \partial\Omega \times (0,T), \\ \boldsymbol{B}(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x}) & \text{in } \Omega, \end{cases} \tag{2.1}$$

where $\Omega = B_{r_o}(0) \backslash \overline{B_{r_i}(0)} \subset \mathbb{R}^3$, $0 < r_i < r_o < \infty$ is the physical domain of interest. Here B_{r_o} and B_{r_i} denote two circles with center at 0 and radius r_o and r_i respectively. $\partial\Omega = \Gamma_1 \cup \Gamma_2$ denotes the boundary of Ω, which consists of the inner boundary Γ_1 and outer boundary Γ_2, and $\mathbf{n}$ denotes the unit outer normal vector to the boundary of Ω. The functions $\boldsymbol{B} = \boldsymbol{B}(\boldsymbol{x}, t)$ and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ represent magnetic field and the fluid velocity field respectively, $f(\boldsymbol{x}, t)$ is a model-oriented function, R_α is a dynamo parameter, R_m is the magnetic Reynolds number, σ is a constant and the parameter $\beta(\boldsymbol{x})$ is the magnetic diffusivity.

When $\boldsymbol{u}$, f, σ, β and $\boldsymbol{B}_0$ are given, one can solve the system (2.1) to find the behavior of magnetic field $\boldsymbol{B}$ in Ω. This is usually called a direct dynamo problem. In this problem we shall consider the case when $\boldsymbol{u}$, f, σ and $\boldsymbol{B}_0$ are known, but the magnetic diffusivity $\beta(\boldsymbol{x})$ is unavailable in Ω. In order to recover the magnetic diffusivity $\beta(\boldsymbol{x})$, we need to have some extra measurement data from the magnetic field $\boldsymbol{B}$. We shall assume the measurement data $\boldsymbol{B}$ is available in some small subregion inside Ω over the time interval $(0,T)$, which occurs the following inverse problem.

Inverse Problem I. Let ω be a subregion in Ω. Given the noisy measurement data

$$\boldsymbol{B}(\boldsymbol{x},t) \approx \boldsymbol{z}^\delta(\boldsymbol{x},t), \quad (\boldsymbol{x},t) \in \omega \times (0,T), \tag{2.2}$$

we will reconstruct the magnetic diffusivity $\beta(\boldsymbol{x})$ in the entire domain Ω. Here δ is the noise level.

2.2 Tikhonov regularization method

In this section, we will transform the ill-posed Inverse Problem I presented in Section 1 into a stabilized minimization system. Additionally, we will establish the existence of the solutions and stability with respect to the change in the error of the observation data. Before considering Inverse Problem I, we refer to [7] and recall the equivalent variational problem of system 2.1 and its well-posedness.

Lemma 2.2.1 *The equivalent variational problem of system 2.1: For a.e.* $t \in (0,T)$, *find* $\boldsymbol{B}(\cdot,t) \in H_0,\ p(\cdot,t) \in L_0^2(\Omega)$ *such that* $\boldsymbol{B}(\cdot,0) = \boldsymbol{B}_0(\cdot)$ *and*

$$\begin{cases} (\partial_t \boldsymbol{B}, \boldsymbol{A}) + (\beta\nabla\times\boldsymbol{B}, \nabla\times\boldsymbol{A}) + \gamma(\nabla\cdot\boldsymbol{B}, \nabla\cdot\boldsymbol{A}) + (p, \nabla\cdot\boldsymbol{A}) \\ \quad = R_\alpha\left(\dfrac{f(\boldsymbol{x},t)}{1+\sigma|\boldsymbol{B}|^2}\boldsymbol{B}, \nabla\times\boldsymbol{A}\right) + R_m(\boldsymbol{u}\times\boldsymbol{B}, \nabla\times\boldsymbol{A}) \ \ \forall\ \boldsymbol{A}\in H_0, \\ (\nabla\cdot\boldsymbol{B}, q) = 0 \quad \forall\ q \in L_0^2(\Omega), \end{cases} \tag{2.3}$$

where $p(\boldsymbol{x},t)$ *is a Lagrange multiplier and* γ *is a constant. Moreover, we have the following stability estimate for the solution* $(\boldsymbol{B},p)$ *to system* (2.3)*:*

$$\begin{aligned} & \|\boldsymbol{B}\|_{L^\infty(0,T;V)} + \|\boldsymbol{B}\|_{H^1(0,T;\boldsymbol{L}^2(\Omega))} + \|p\|_{L^2(0,T;L_0^2(\Omega))} \\ \leq\ & C(\|\nabla\times\boldsymbol{B}_0\|_{0,\Omega}^2 + \|\boldsymbol{B}_0\|_{0,\Omega}^2) \max_{0\leq t\leq T}(\|f(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2 + \|\boldsymbol{u}(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2) \\ & \cdot\exp\Big(C\int_0^T \{\|f(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2 + \|f'(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2 + \|\boldsymbol{u}(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2 \\ & + \|\boldsymbol{u}'(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2\}dt\Big), \end{aligned}$$

provided that $\boldsymbol{B}_0 \in V,\ \ f \in H^1(0,T;L^\infty(\Omega))$ *and* $\boldsymbol{u} \in H^1(0,T;\boldsymbol{L}^\infty(\Omega))$.

For convenience, we often write the solutions of the system (2.3) as $(\boldsymbol{B}(\beta), p(\beta))$ to emphasize their dependence on β. In general, Inverse Problem I is mathematically ill-posed, we formulate it into a mathematically stabilized minimization system with Tikhonov regularization:

$$\min_{\beta \in K} \quad J(\beta) = \frac{1}{2} \int_0^T \int_\omega |\boldsymbol{B}(\beta) - \boldsymbol{z}^\delta|^2 d\boldsymbol{x} dt + \frac{\lambda}{2} \int_\Omega |\nabla \beta|^2 d\boldsymbol{x}, \tag{2.4}$$

where the constraint set

$$K = \{\beta(\boldsymbol{x}) \in H^1(\Omega) : 0 < \beta_1 \leq \beta(\boldsymbol{x}) \leq \beta_2\},$$

β_1, β_2 are two positive constants and $\lambda > 0$ is the regularization parameter.

We are now ready to justify the regularizing effects of the nonlinear optimization system (2.4) that it always has solutions and its solutions are stable with respect to the noise error in the observation data $\boldsymbol{z}^\delta$. The first theorem establishes the existence of solutions.

Theorem 2.2.2 *There exists at least a minimizer to the optimization problem (2.4).*

Proof. See [25] ■

The following theorem shows that the minimization system (2.2) is indeed a stabilization of the ill-posed Inverse Problem I with respect to the changes of the observation errors.

Theorem 2.2.3 *Let $\{\boldsymbol{z}_n\}$ be a sequence such that $\boldsymbol{z}_n \to \boldsymbol{z}^\delta$ in $L^2(0,T;\boldsymbol{L}^2(\omega))$ as $n \to \infty$ and $\{\beta_n\}$ be the minimizers of problem (2.4) with $\boldsymbol{z}^\delta$ replaced by $\boldsymbol{z}_n$. Then there exists a subsequence of $\{\beta_n\}$ that converges in $H^1(\Omega)$, and the limit of every such convergent subsequence is a minimizer of (2.4).*

2.3 Finite element approximation

In this section, we shall propose a fully discretized finite element approximation for solving the continuous minimization problem (2.4).

For the space discretization, we consider a shape regular triangulation $\mathcal{T}_h$ of Ω with a mesh size h, consisting of tetrahedral elements. Then we introduce some finite element spaces, which were proposed in [7]:

$$\begin{aligned}
H_h &= \{\boldsymbol{w} \in C(\bar{\Omega})^3 : \ \boldsymbol{w}|_A \in P_2(A)^3, \ \forall \ A \in \mathcal{T}_h\}, \\
H_{0h} &= \{\boldsymbol{w} \in H_h; \ \boldsymbol{w} \cdot \mathbf{n}_F = 0, \ \forall F \in \mathcal{F}_h \cap \partial\Omega\}, \\
Q_h &= \{w \in C(\bar{\Omega}) : \ w|_A \in P_1(A), \ \forall \ A \in \mathcal{T}_h\}, \\
Q_{0h} &= \{w \in Q_h; \ \int_\Omega w d\boldsymbol{x} = 0\}, \\
V_h &= \{w \in H^1(\Omega) : \ w|_A \in P_1(A), \ \forall \ A \in \mathcal{T}_h\},
\end{aligned}$$

where $\mathcal{F}_h$ is the set of all faces of elements in $\mathcal{T}_h$ and $\mathbf{n}_F$ is the unit normal vector of a face $F \in \mathcal{F}_h$, $P_1(A)$ and $P_2(A)$ are the spaces of piecewise linear and quadratic polynomials on A respectively. We will approximate the magnetic field $\boldsymbol{B}$ and Lagrange multiplier p by H_{0h} and Q_{0h} respectively. Moreover, the constrained subset K is approximated by $K_h = K \cap V_h$.

To fully discretize system (2.3)-(2.4), we also need the time discretization. To do so, we divide the time interval $[0,T]$ into M equally spaced subintervals using nodal points

$$0 = t_0 < t_1 < \cdots < t_M = T \tag{2.5}$$

with $t_n = n\tau$, $\tau = \frac{T}{M}$. For a continuous mapping $u : [0,T] \to L^2(\Omega)$, we define $u^n = u(\cdot, t_n)$ for $0 \le n \le M$. For a given sequence $\{u^n\}_{n=0}^M \subset L^2(\Omega)$, we define its first-order backward finite differences and average values as follows:

$$\begin{aligned}
\partial_\tau u^n &= \frac{u^n - u^{n-1}}{\tau}, \; n = 1, 2, \ldots, M, \\
\bar{u}^n &= \frac{1}{\tau}\int_{t_{n-1}}^{t_n} u(\cdot, t)dt, \; n = 1, 2, \ldots, M, \text{ and } \bar{u}^0 = u(\cdot, 0).
\end{aligned}$$

Now we are ready to formulate the finite element approximation of the continuous minimization (2.4) as follows:

$$\min_{\beta_h \in K_h} J_{h,\tau}(\beta_h) = \frac{\tau}{2}\sum_{n=0}^{M} \alpha_n \int_\omega (\boldsymbol{B}_h^n - \boldsymbol{z}^{\delta,n})^2 d\boldsymbol{x} + \frac{\lambda}{2}\int_\Omega |\nabla \beta_h|^2 d\boldsymbol{x}, \tag{2.6}$$

where $(\boldsymbol{B}_h^n, p_h^n) \equiv (\boldsymbol{B}_h^n(\beta_h), p_h^n(\beta_h)) \in H_{0h} \times Q_{0h}$ satisfies $\boldsymbol{B}_h^0 = S_h \boldsymbol{B}_0(\boldsymbol{x})$ and

$$\begin{cases}
\int_\Omega \partial\tau \boldsymbol{B}_h^n \cdot \boldsymbol{A}_h d\boldsymbol{x} + \int_\Omega \beta_h (\nabla \times \boldsymbol{B}_h^n) \cdot (\nabla \times \boldsymbol{A}_h) d\boldsymbol{x} \\
+\gamma \int_\Omega (\nabla \cdot \boldsymbol{B}_h^n)(\nabla \cdot \boldsymbol{A}_h) d\boldsymbol{x} + \int_\Omega p_h^n (\nabla \cdot \boldsymbol{A}_h) d\boldsymbol{x} \\
= R_\alpha \int_\Omega \frac{\bar{f}^n}{1 + \sigma |\boldsymbol{B}_h^n|^2} \boldsymbol{B}_h^n \cdot (\nabla \times \boldsymbol{A}_h) d\boldsymbol{x} \\
+R_m \int_\Omega (\bar{\boldsymbol{u}}^n \times \boldsymbol{B}_h^n) \cdot (\nabla \times \boldsymbol{A}_h) d\boldsymbol{x}, \\
\int_\Omega (\nabla \cdot \boldsymbol{B}_h^n) q_h d\boldsymbol{x} = 0,
\end{cases} \tag{2.7}$$

for all $(\boldsymbol{A}_h,\ q_h) \in H_{0h} \times Q_{0h}$, $n = 1, 2, \cdots, M$. Here $\bar{\boldsymbol{u}}^n \in \boldsymbol{L}^\infty(\Omega)$ and $\bar{f}^n \in L^\infty(\Omega)$, and $\{\alpha_n\}$ are the coefficients of the composite trapezoidal rule, i.e., $\alpha_0 = \alpha_M = \frac{1}{2}$ and $\alpha_n = 1$ for $n \neq 0, M$.

Before analyzing the convergence, we refer to [7] and present the well-posedness and stability estimates for the solutions to the discrete system (2.7).

Lemma 2.3.1 *There exists a unique solution $(\boldsymbol{B}_h^n, p_h^n)$ to the discrete system (2.7) for each fixed $n\,(1 \le n \le M)$ and the sequence $\{(\boldsymbol{B}_h^n, p_h^n)\}_{n=0}^M$ has the following stability estimates:*

$$\max_{1\le n\le M}\|\boldsymbol{B}_h^n\|_{0,\Omega}^2+\tau\sum_{n=1}^M(\|\nabla\times\boldsymbol{B}_h^n\|_{0,\Omega}^2+\|\nabla\cdot\boldsymbol{B}_h^n\|_{0,\Omega}^2)\le C\|\boldsymbol{B}_h^0\|_{0,\Omega}^2,$$

$$\max_{1\le n\le M}\|\nabla\times\boldsymbol{B}_h^n\|_{0,\Omega}^2+\max_{1\le n\le M}\|\nabla\cdot\boldsymbol{B}_h^n\|_{0,\Omega}^2+\tau\sum_{n=1}^M\|\partial_\tau\boldsymbol{B}_h^n\|_{0,\Omega}^2$$
$$+\tau\sum_{n=1}^M\|p_h^n\|_{0,\Omega}^2+\tau\sum_{n=1}^M\|\partial_\tau\boldsymbol{B}_h^n\|_{(\boldsymbol{H}^1(\Omega))'}^2\le C\|\boldsymbol{B}_h^0\|_{1,\Omega}^2.$$

Theorem 2.3.2 *[26] There exists at least a minimizer to the discrete minimization problem* (2.6)*.*

Proof. See [25] ■

Now we will consider the convergence of the minimizer of the discrete system (2.6) to the minimizer of the continuous problem (2.4). We first define some interpolations based on $\{\boldsymbol{B}_h^n\}$ and $\{p_h^n\}$ as follows: for any $(\boldsymbol{x},t)\in\Omega\times(t_{n-1},t_n)$, let

$$\boldsymbol{B}_{h,\tau}(\boldsymbol{x},t)=\frac{t-t_{n-1}}{\tau}\boldsymbol{B}_h^n(\boldsymbol{x})+\frac{t_n-t}{\tau}\boldsymbol{B}_h^{n-1}(\boldsymbol{x}),\tag{2.8}$$
$$\hat{\boldsymbol{B}}_{h,\tau}(\boldsymbol{x},t)=\sum_{n=1}^M\chi_n(t)\boldsymbol{B}_h^n(\boldsymbol{x})\quad\text{and}\quad\hat{p}_{h,\tau}(\boldsymbol{x},t)=\sum_{n=1}^M\chi_n(t)p_h^n(\boldsymbol{x}).$$

Lemma 2.3.3 *[26] The following results hold:*

$$\|\hat{\boldsymbol{B}}_{h,\tau}\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))}^2=\tau\sum_{n=1}^M\|\boldsymbol{B}_h^n\|_{1,\Omega}^2,$$
$$\|\frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau}\|_{L^2(0,T;(\boldsymbol{H}^1(\Omega))')}^2=\tau\sum_{n=1}^M\|\partial_\tau\boldsymbol{B}_h^n\|_{(\boldsymbol{H}^1(\Omega))'}^2,$$
$$\|\hat{p}_{h,\tau}\|_{L^2(0,T;L^2(\Omega))}^2=\tau\sum_{n=1}^M\|p_h^n\|_{0,\Omega}^2,$$
$$\|\boldsymbol{B}_{h,\tau}\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))}^2\le\tau\sum_{n=0}^M\|\boldsymbol{B}_h^n\|_{1,\Omega}^2.$$

Lemma 2.3.4 *[26] Direct computations give us the following equalities:*

$$\int_0^T\int_\Omega \frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau}(\boldsymbol{x},t)\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt = \tau\sum_{n=1}^M\int_\Omega \partial_\tau \boldsymbol{B}_h^n S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \gamma\nabla\cdot(\hat{\boldsymbol{B}}_{h,\tau})(\nabla\cdot\phi_{h,\tau})d\boldsymbol{x}dt = \tau\sum_{n=1}^M\int_\Omega \gamma(\nabla\cdot\boldsymbol{B}_h^n)(\nabla\cdot S_h\phi(\boldsymbol{x},t_n))d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \beta_h\nabla\times\hat{\boldsymbol{B}}_{h,\tau}(\boldsymbol{x},t)\cdot\nabla\times\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt$$
$$= \tau\sum_{n=1}^M\int_\Omega \beta_h\nabla\times\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \hat{p}_{h,\tau}(\nabla\cdot\phi_{h,\tau})d\boldsymbol{x}dt = \tau\sum_{n=1}^M\int_\Omega p_h^n(\nabla\cdot S_h\phi(\boldsymbol{x},t_n))d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \frac{f}{1+\sigma|\hat{\boldsymbol{B}}_{h,\tau}|^2}\hat{\boldsymbol{B}}_{h,\tau}\cdot\nabla\times\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt$$
$$= \tau\sum_{n=1}^M\int_\Omega \frac{\bar{f}^n}{1+\sigma|\boldsymbol{B}_h^n|^2}\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \boldsymbol{u}\times\hat{\boldsymbol{B}}_{h,\tau}\cdot\nabla\times\phi_{h,\tau}d\boldsymbol{x}dt = \tau\sum_{n=1}^M\int_\Omega \bar{\boldsymbol{u}}^n\times\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x}.$$

Lemma 2.3.5 *For any $\beta_h \in K_h$, $\beta \in K$, $U_{h,\tau}, U \in L^2(0,T;\boldsymbol{H}^1(\Omega))$ and $V_{h,\tau}, V \in L^2(0,T;\boldsymbol{L}^2(\Omega))$, if $\beta_h \to \beta$ in $L^2(\Omega)$ as $h\to 0$, $U_{h,\tau} \rightharpoonup U$ in $L^2(0,T;\boldsymbol{H}^1(\Omega))$ and $V_{h,\tau}\to V$ in $L^2(0,T;\boldsymbol{L}^2(\Omega))$ as $h,\tau\to 0$, we have the following convergence results:*

$$\lim_{h,\tau\to 0}\int_0^T\int_\Omega \beta_h\nabla\times U_{h,\tau}\cdot V_{h,\tau}d\boldsymbol{x}dt = \int_0^T\int_\Omega \beta\nabla\times U\cdot V d\boldsymbol{x}dt, \tag{2.9}$$

$$\lim_{h,\tau\to 0}\int_0^T\int_\Omega \frac{f}{1+\sigma|U_{h,\tau}|^2}U_{h,\tau}\cdot V_{h,\tau}d\boldsymbol{x}dt = \int_0^T\int_\Omega \frac{f}{1+\sigma|U|^2}U\cdot V d\boldsymbol{x}dt, \tag{2.10}$$

$$\lim_{h,\tau\to 0}\int_0^T\int_\Omega \boldsymbol{u}\times U_{h,\tau}\cdot V_{h,\tau}d\boldsymbol{x}dt = \int_0^T\int_\Omega \boldsymbol{u}\times U\cdot V d\boldsymbol{x}dt. \tag{2.11}$$

Lemma 2.3.6 *For the sequence $\{\beta_h\}_{h>0}\subset K_h$, if $\{\beta_h\}_{h>0}$ converges to some $\beta\in K$ in $L^2(\Omega)$ strongly, suppose $z^\delta \in C(0,T;\boldsymbol{L}^2(\omega))$, then there exists a subsequence, also denoted by $\{\beta_h\}_{h>0}$, such that*

$$\lim_{h,\tau\to 0}\tau\sum_{n=0}^M \alpha_n\int_\omega (\boldsymbol{B}_h^n(\beta_h)-\boldsymbol{z}^{\delta,n})^2 d\boldsymbol{x} = \int_0^T\int_\omega |\boldsymbol{B}(\beta)-\boldsymbol{z}^\delta|^2 d\boldsymbol{x}dt.$$

Finally, we are ready to establish the main convergence theorem.

Theorem 2.3.7 *[25] Let* $\{\beta_h^*\}_{h>0}$ *be a sequence of minimizers to the discrete minimization problem* (2.6) *and suppose* $z^\delta \in C(0, T; \boldsymbol{L}^2(\omega))$*, then as* h *and* τ *tend to* 0*, each sequence of* $\{\beta_h^*\}_{h>0}$ *has a subsequence converging in* $L^2(\Omega)$ *to a minimizer of the continuous optimization problem* (3.2)*.*

Chapter 3

Two Parameters identification problem

In this chapter, we shall consider the case when the function f is assumed to take the form

$$f(x,t) = g(\boldsymbol{x})h(\boldsymbol{x},t).$$

We are interested in the recovering of the magnetic diffusivity β and g belonging to the space $L^2(\Omega)$ in a three dimensional (3D) spherical dynamo equation (2.1).

3.1 Tikhonov regularization method

Lemma 3.1.1 *The equivalent variational problem of system 2.1 before the change: For a.e. $t \in (0,T)$, find $\boldsymbol{B}(\cdot,t) \in H_0,\ p(\cdot,t) \in L_0^2(\Omega)$ such that $\boldsymbol{B}(\cdot,0) = \boldsymbol{B}_0(\cdot)$ and*

$$\begin{cases} (\partial_t \boldsymbol{B}, \boldsymbol{A}) + (\beta\nabla\times\boldsymbol{B}, \nabla\times\boldsymbol{A}) + \gamma(\nabla\cdot\boldsymbol{B}, \nabla\cdot\boldsymbol{A}) + (p, \nabla\cdot\boldsymbol{A}) \\ \quad = R_\alpha \left(\dfrac{g(\boldsymbol{x}).h(\boldsymbol{x},t)}{1+\sigma|\boldsymbol{B}|^2}\boldsymbol{B}, \nabla\times\boldsymbol{A}\right) + R_m(\boldsymbol{u}\times\boldsymbol{B}, \nabla\times\boldsymbol{A}) \quad \forall\ \boldsymbol{A}\in H_0, \\ (\nabla\cdot\boldsymbol{B}, q) = 0 \quad \forall\ q \in L_0^2(\Omega), \end{cases} \tag{3.1}$$

where $p(\boldsymbol{x},t)$ is a Lagrange multiplier and γ is a constant. Moreover, we have the following stability estimate for the solution $(\boldsymbol{B},p)$ to system (3.1)*:*

$$\begin{aligned}
&\|\boldsymbol{B}\|_{L^\infty(0,T;V)} + \|\boldsymbol{B}\|_{H^1(0,T;\boldsymbol{L}^2(\Omega))} + \|p\|_{L^2(0,T;L_0^2(\Omega))} \\
\leq\ & C(\|\nabla\times\boldsymbol{B}_0\|_{0,\Omega}^2 + \|\boldsymbol{B}_0\|_{0,\Omega}^2)\max_{0\leq t\leq T}(\|g(x)\|_{L^2}^2 .\|h(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2 + \|\boldsymbol{u}(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2) \\
&\cdot\exp\Big(C\int_0^T\{\|g(x)\|_{L^2}^2 .\|h(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2 + \|g'(x)\|_{L^2}^2 .\|h(x,t)\|_{L^\infty(\Omega)}^2 + \|g(x)\|_{L^2}^2 .\|h'(x,t)\|_{L^\infty(\Omega)}^2 \\
&+\|\boldsymbol{u}(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2 + \|\boldsymbol{u}'(\boldsymbol{x},t)\|_{L^\infty(\Omega)}^2\}dt\Big),
\end{aligned}$$

provided that $\boldsymbol{B}_0\in V,\ \ g\in L^2(\Omega), h\in H^1(0,T;L^\infty(\Omega))$ and $\boldsymbol{u}\in H^1(0,T;\boldsymbol{L}^\infty(\Omega))$.

For convenience, we often write the solutions of the system (3.1) as $(\boldsymbol{B}(\beta,g),p(\beta,g))$ to emphasize their dependence on (β,g). In general, Inverse Problem I is mathematically ill-posed, we formulate it into a mathematically stabilized minimization system with Tikhonov regularization:

$$\min_{(\beta,g)\in K\times L^2(\Omega)} \quad J(\beta,g) = \frac{1}{2}\int_0^T\int_\omega |\boldsymbol{B}(\beta,g)-\boldsymbol{z}^\delta|^2 d\boldsymbol{x}dt + \frac{\lambda_1}{2}\int_\Omega|\nabla\beta|^2 d\boldsymbol{x} + \frac{\lambda_2}{2}\int_\Omega|g|^2 d\boldsymbol{x}, \quad (3.2)$$

where the constraint set

$$K = \{\beta(\boldsymbol{x})\in H^1(\Omega) : 0<\beta_1\leq\beta(\boldsymbol{x})\leq\beta_2\},$$

β_1,β_2 are two positive constants and $\lambda_1,\lambda_2>0$ are the regularization parameters.

We are now ready to justify the regularizing effects of the nonlinear optimization system (3.2) that it always has solutions and its solutions are stable with respect to the noise error in the observation data $\boldsymbol{z}^\delta$. The first theorem establishes the existence of solutions.

Theorem 3.1.2 *There exists at least a minimizer to the optimization problem (3.2).*

Proof. Since $J(\beta,g)\geq 0$ for any $(\beta,g)\in K\times L^2(\Omega)$, there exists a minimizing sequence $\{\beta_n,g_n\}\subset K\times L^2(\Omega)$ such that

$$\lim_{n\to\infty} J(\beta_n,g_n) = \inf_{(\beta,g)\in K\times L^2(\Omega)} J(\beta,gt).$$

Then $|J(\beta_n,g_n)|\leq C$, which implies that $\|\nabla\beta_n\|_{L^2(\Omega)}\leq C$ and $\|g_n\|_{L^2(\Omega)}\leq C$. By the definition of K, $\{\beta_n(\boldsymbol{x}),g_n(\boldsymbol{x})\}$ is bounded in $L^\infty(\Omega)\times L^2(\Omega)$, then in $L^2(\Omega)$. So $\{\beta_n(\boldsymbol{x}),g_n(\boldsymbol{x})\}$ is bounded in $H^1(\Omega)\times L^2(\Omega)$ and there exists a subsequence of $\{\beta_n(\boldsymbol{x}),g_n(\boldsymbol{x})\}$ denoted still by $\{\beta_n(\boldsymbol{x}),g_n(\boldsymbol{x})\}$ and some $(\beta^*,g^*)\in H^1(\Omega)\times L^2(\Omega)$ such that

$$\beta_n\rightharpoonup\beta^* \text{ in } H^1(\Omega), \quad \text{and} \quad \beta_n\to\beta^* \text{ in } L^2(\Omega). \quad (3.3)$$

$$g_n\rightharpoonup g^* \text{ in } L^2(\Omega), \quad (3.4)$$

As K is a closed convex subset of $H^1(\Omega)$, hence K is weakly-closed and we have $\beta^* \in K$.

For convenience, let $(\boldsymbol{B}^n, p^n) = (\boldsymbol{B}(\beta_n, g_n), p(\beta_n, g_n))$. Due to Lemma 3.1.1, there exists a subsequence, still denoted by $\{\boldsymbol{B}_n, p_n\}$ and some $(\boldsymbol{B}^*, p^*)$ such that

$$\boldsymbol{B}_n \rightharpoonup \boldsymbol{B}^* \text{ in } L^\infty(0,T;\boldsymbol{H}^1(\Omega)), \quad \boldsymbol{B}_n \rightharpoonup \boldsymbol{B}^* \text{ in } H^1(0,T;\boldsymbol{L}^2(\Omega)), \tag{3.5}$$

$$p_n \rightharpoonup p^* \quad \text{in } L^2(0,T;L_0^2(\Omega)). \tag{3.6}$$

Next we shall show that $\boldsymbol{B}^* = \boldsymbol{B}(\beta^*, g^*)$ and $p^* = p(\beta^*, g^*)$. To do so, we multiply both sides of (3.1) ($\boldsymbol{B}$ is replaced by $\boldsymbol{B}^n$, β is replaced by β^n, g is replaced by g^n) by a function $\eta(t) \in C^1[0,T]$ and get

$$\begin{aligned}
&(\partial_t B^n, A)\eta(t) + (\beta^n \nabla \times B^n, \nabla \times A)\eta(t) + \gamma(\nabla \cdot B^n, \nabla \cdot A)\eta(t) + (p^n, \nabla \cdot A)\eta(t) \\
&= R_\alpha\Big(\frac{g(x).h(x,t)}{1+\sigma\left|B^n\right|^2} B^n, \nabla \times A)\eta(t) + R_m(u \times B^n, \nabla \times A)\eta(t) \quad \forall A \in H_0 \\
&(\nabla \cdot B^n, q)\eta(t) = 0 \quad \forall q \in L_0^2(\Omega)
\end{aligned}$$

then

$$\begin{aligned}
&\int_0^T \int_\Omega (\partial_t B^n \cdot A\eta(t) + \beta^n \nabla \times B^n \cdot (\nabla \times A)\eta(t) + \gamma(\nabla \cdot B^n) \cdot (\nabla \cdot A)\eta(t) + p^n(\nabla \cdot A)\eta(t))dxdt \\
&= \int_0^T \int_\Omega R_\alpha \frac{g(x).h(x,t)}{1+\sigma\left|B^n\right|^2} B^n \cdot (\nabla \times A)\eta(t) + R_m(u \times B^n) \cdot (\nabla \times A)\eta(t)dxdt \\
&\int_0^T \int_\Omega \partial_t B^n \cdot A\eta(t)dxdt + \int_0^T \int_\Omega (\beta^n \nabla \times B^n) \cdot (\nabla \times A)\eta(t)dxdt \\
&+ \gamma \int_0^T \int_\Omega (\nabla \cdot B^n) \cdot (\nabla \cdot A)\eta(t)dxdt + \int_0^T \int_\Omega p^n(\nabla \cdot A)\eta(t)dxdt \\
&= R_\alpha \int_0^T \int_\Omega \frac{g(x).h(x,t)}{1+\sigma\left|B^n\right|^2} B^n \cdot (\nabla \times A)\eta(t)dxdt + R_m \int_0^T \int_\Omega (u \times B^n) \cdot (\nabla \times A)\eta(t)dxdt \\
&\int_0^T \int_\Omega \partial_t B^n \cdot A\eta(t)dxdt + \int_0^T \int_\Omega (\beta^n \nabla \times B^n) \cdot (\nabla \times A)\eta(t)dxdt \\
&+ \gamma \int_0^T \int_\Omega (\nabla \cdot B^n) \cdot (\nabla \cdot A)\eta(t)dxdt + \int_0^T \int_\Omega p^n(\nabla \cdot A)\eta(t)dxdt \\
&+ \int_0^T \int_\Omega (\beta - \beta)(\nabla \times B^n) \cdot (\nabla \times A)\eta(t)dxdt \\
&= R_\alpha \int_0^T \int_\Omega \frac{g(x).h(x,t)}{1+\sigma\left|B^n\right|^2} B^n \cdot (\nabla \times A)\eta(t)dxdt + R_m \int_0^T \int_\Omega (u \times B^n) \cdot (\nabla \times A)\eta(t)dxdt
\end{aligned} \tag{3.7}$$

$$\int_0^T \int_\Omega \partial_t B^n \cdot A\eta(t)dxdt + \int_0^T \int_\Omega (\beta\nabla \times B^n) \cdot (\nabla \times A)\eta(t)dxdt$$
$$+ \gamma \int_0^T \int_\Omega (\nabla \cdot B^n) \cdot (\nabla \cdot A)\eta(t)dxdt + \int_0^T \int_\Omega p^n(\nabla \cdot A)\eta(t)dxdt$$
$$= R_\alpha \int_0^T \int_\Omega \frac{g(x).h(x,t)}{1+\sigma\,|B^n|^2} B^n \cdot (\nabla \times A)\eta(t)dxdt + R_m \int_0^T \int_\Omega (u \times B^n) \cdot (\nabla \times A)\eta(t)dxdt$$
$$- \int_0^T \int_\Omega (\beta^n - \beta)(\nabla \times B^n) \cdot (\nabla \times A)\eta(t)dxdt \ \forall \in H_0 \tag{3.8}$$
$$\int_0^T \int_\Omega (\nabla \cdot B^n, q)\eta(t)dxdt = 0 \quad \forall q \in L_0^2(\Omega). \tag{3.9}$$

We first claim that the last term in the right hand side of (3.8) tends to 0 as $n \to \infty$. Indeed, by Cauchy Schwarz inequality and the fact that $\|\boldsymbol{B}^n\|_{L^\infty(0,T;\boldsymbol{H}^1(\Omega))} \le C$, we have

$$\begin{aligned} &\left|\int_0^T\int_\Omega (\beta_n - \beta)(\nabla \times \boldsymbol{B}^n) \cdot (\nabla \times \boldsymbol{A})\eta(t)d\boldsymbol{x}dt\right| \\ \le\ & \left(\int_0^T\int_\Omega |(\nabla \times \boldsymbol{B}^n)|^2 d\boldsymbol{x}dt\right)^{\frac{1}{2}} \left(\int_0^T\int_\Omega |(\beta_n - \beta)(\nabla \times \boldsymbol{A})\eta(t)|^2 d\boldsymbol{x}dt\right)^{\frac{1}{2}} \\ \le\ & C\left(\int_0^T\int_\Omega |(\beta_n - \beta)(\nabla \times \boldsymbol{A})\eta(t)|^2 d\boldsymbol{x}dt\right)^{\frac{1}{2}}, \end{aligned}$$

which converges to zero as $n \to \infty$ by (3.3) and the Lebesgue's dominated convergence theorem.

Then we shall show that

$$\begin{aligned} &\lim_{n\to\infty} R_\alpha \int_0^T\int_\Omega \frac{g(\boldsymbol{x}).h(\boldsymbol{x},t)}{1+\sigma|\boldsymbol{B}^n|^2}\boldsymbol{B}^n \cdot (\nabla \times \boldsymbol{A})\eta(t)d\boldsymbol{x}dt \\ =\ & R_\alpha \int_0^T\int_\Omega \frac{g(\boldsymbol{x}).h(\boldsymbol{x},t)}{1+\sigma|\boldsymbol{B}^*|^2}\boldsymbol{B}^* \cdot (\nabla \times \boldsymbol{A})\eta(t)d\boldsymbol{x}dt. \end{aligned} \tag{3.10}$$

By direct computation, we get

$$
\begin{aligned}
&\left|\int_0^T\int_\Omega \Big(\frac{g(\boldsymbol{x}).h(\boldsymbol{x},t)}{1+\sigma|\boldsymbol{B}^n|^2}\boldsymbol{B}^n\cdot(\nabla\times\boldsymbol{A})\eta(t)-\frac{g(\boldsymbol{x}).h(\boldsymbol{x},t)}{1+\sigma|\boldsymbol{B}^*|^2}\boldsymbol{B}^*\cdot(\nabla\times\boldsymbol{A})\eta(t)\Big)d\boldsymbol{x}dt\right|\\
\le\;& \int_0^T\int_\Omega\left|\frac{g.h\boldsymbol{B}^n(1+\sigma|\boldsymbol{B}^*|^2)-g.h\boldsymbol{B}^*(1+\sigma|\boldsymbol{B}^n|^2)}{(1+\sigma|\boldsymbol{B}^n|^2)(1+\sigma|\boldsymbol{B}^*|^2)}\cdot(\nabla\times\boldsymbol{A})\eta\right|d\boldsymbol{x}dt\\
\le\;& \int_0^T\int_\Omega|\boldsymbol{B}^n-\boldsymbol{B}^*|\cdot|g.h|\cdot|\nabla\times\boldsymbol{A}|\cdot|\eta|d\boldsymbol{x}dt\\
&+\int_0^T\int_\Omega|g.h|\cdot|\nabla\times\boldsymbol{A}|\cdot|\eta|\cdot\left|\frac{\sigma(\boldsymbol{B}^n-\boldsymbol{B}^*)|\boldsymbol{B}^*|^2+\sigma\boldsymbol{B}^*(|\boldsymbol{B}^*|^2-|\boldsymbol{B}^n|^2)}{(1+\sigma|\boldsymbol{B}^n|^2)(1+\sigma|\boldsymbol{B}^*|^2)}\right|d\boldsymbol{x}dt\\
\le\;& 2\int_0^T\int_\Omega|\boldsymbol{B}^n-\boldsymbol{B}^*|\cdot|g.h(\nabla\times\boldsymbol{A})\eta|d\boldsymbol{x}dt\\
&+\int_0^T\int_\Omega|g.h(\nabla\times\boldsymbol{A})\eta|\cdot\left|\frac{\sigma|\boldsymbol{B}^*|(|\boldsymbol{B}^*|+|\boldsymbol{B}^n|)(|\boldsymbol{B}^*|-|\boldsymbol{B}^n|)}{(1+\sigma|\boldsymbol{B}^n|^2)(1+\sigma|\boldsymbol{B}^*|^2)}\right|d\boldsymbol{x}dt\\
\le\;& 2\int_0^T\int_\Omega|\boldsymbol{B}^n-\boldsymbol{B}^*|\cdot|g.h(\nabla\times\boldsymbol{A})\eta|d\boldsymbol{x}dt\\
&+\int_0^T\int_\Omega|g.h(\nabla\times\boldsymbol{A})\eta|\cdot|\boldsymbol{B}^*-\boldsymbol{B}^n|\cdot\frac{\sigma|\boldsymbol{B}^*|^2+\frac{\sigma}{2}(|\boldsymbol{B}^*|^2+|\boldsymbol{B}^n|^2)}{(1+\sigma|\boldsymbol{B}^n|^2)(1+\sigma|\boldsymbol{B}^*|^2)}d\boldsymbol{x}dt\\
\le\;& 4\int_0^T\int_\Omega|\boldsymbol{B}^n-\boldsymbol{B}^*|\cdot|g.h(\nabla\times\boldsymbol{A})\eta|d\boldsymbol{x}dt\\
\le\;& 4\left(\int_0^T\int_\Omega|\boldsymbol{B}^n-\boldsymbol{B}^*|^2d\boldsymbol{x}dt\right)^{\frac12}\left(\int_0^T\int_\Omega|g\times h(\nabla\times\boldsymbol{A})\eta|^2d\boldsymbol{x}dt\right)^{\frac12}\\
\le\;& C\|\boldsymbol{B}^n-\boldsymbol{B}^*\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))},
\end{aligned}
$$

which tends to 0 as $n\to\infty$ if $\|\boldsymbol{B}^n-\boldsymbol{B}^*\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}\longrightarrow 0$ as $n\to\infty$. Now we will prove

$$\boldsymbol{B}^n\to\boldsymbol{B}^*\quad\text{in } L^2(0,T;\boldsymbol{L}^2(\Omega)). \tag{3.11}$$

As $\{\boldsymbol{B}^n\}$ is bounded in $L^2(0,T;\boldsymbol{H}^1(\Omega))$, it suffices to show that $\{\partial_t\boldsymbol{B}^n\}$ is bounded in $L^2(0,T;(\boldsymbol{H}^1(\Omega))')$ by Lemma 1.2.2. For any $\boldsymbol{A}\in H_0$, we have from the variational form (3.1) that for any $t\in(0,T)$

$$
\begin{aligned}
&|(\partial_t\boldsymbol{B}^n,\boldsymbol{A})|\le C(\|\nabla\times\boldsymbol{B}^n\|_{0,\Omega}\|\nabla\times\boldsymbol{A}\|_{0,\Omega}\\
&+\|\nabla\cdot\boldsymbol{B}^n\|_{0,\Omega}\|\nabla\cdot\boldsymbol{A}\|_{0,\Omega}+\|p^n\|_{0,\Omega}\|\nabla\cdot\boldsymbol{A}\|_{0,\Omega})\\
&+R_\alpha\left(\int_\Omega\frac{g(\boldsymbol{x}).h(\boldsymbol{x},t)}{1+\sigma|\boldsymbol{B}^n|^2}|\boldsymbol{B}^n|^2d\boldsymbol{x}\right)^{\frac12}\left(\int_\Omega\frac{g(\boldsymbol{x}).h(\boldsymbol{x},t)}{1+\sigma|\boldsymbol{B}^n|^2}|\nabla\times\boldsymbol{A}|^2d\boldsymbol{x}\right)^{\frac12}\\
&+2R_m\|\boldsymbol{u}\|_{L^\infty(\Omega)}\|\boldsymbol{B}^n\|_{0,\Omega}\|\nabla\times\boldsymbol{A}\|_{0,\Omega}\\
\le\;& C\|\boldsymbol{A}\|_{1,\Omega}(\|\boldsymbol{B}^n\|_{1,\Omega}+\|p^n\|_{0,\Omega})+C\|g\|_{L^2(\Omega)}\|\|h\|_{L^\infty(\Omega)}\|\nabla\times\boldsymbol{A}\|_{0,\Omega}\\
&+C\|\boldsymbol{u}\|_{L^\infty(\Omega)}\|\boldsymbol{B}^n\|_{0,\Omega}\|\nabla\times\boldsymbol{A}\|_{0,\Omega}\\
\le\;& C\|\boldsymbol{A}\|_{1,\Omega}(\|\boldsymbol{B}^n\|_{1,\Omega}+\|p^n\|_{0,\Omega}+\|g\|_{L^2(\Omega)}\|h\|_{L^\infty(\Omega)}+\|\boldsymbol{u}\|_{L^\infty(\Omega)}\|\boldsymbol{B}^n\|_{0,\Omega}).
\end{aligned}
$$

Further, we have

$$\left|\int_0^T (\partial_t \boldsymbol{B}^n, \boldsymbol{A})dt\right| \leq C\|\boldsymbol{A}\|_{L^2(0,T;H^1(\Omega))}$$
$$\cdot(\|\boldsymbol{B}^n\|_{L^2(0,T;\boldsymbol{H}^1(\Omega))} + \|p^n\|_{L^2(0,T;L_0^2(\Omega))} + \|g\|_{L^2(\Omega)}\|h\|_{H^1(0,T;L^\infty(\Omega))}$$
$$+\int_0^T \|\boldsymbol{u}\|^2_{L^\infty(\Omega)}\|\boldsymbol{B}^n\|^2 dt)^{\frac{1}{2}} \leq C\|\boldsymbol{A}\|_{L^2(0,T;H^1(\Omega))},$$

which implies that $\{\partial_t \boldsymbol{B}^n\}$ is bounded.

Our next goal is to show that for any $\boldsymbol{A} \in H_0$,

$$\begin{aligned}&\lim_{n\to\infty} R_m \int_0^T\int_\Omega \boldsymbol{u} \times \boldsymbol{B}^n \cdot (\nabla \times \boldsymbol{A})\eta(t)d\boldsymbol{x}dt \\ =\ & R_m \int_0^T\int_\Omega \boldsymbol{u} \times \boldsymbol{B}^* \cdot (\nabla \times \boldsymbol{A})\eta(t)d\boldsymbol{x}dt. \end{aligned} \tag{3.12}$$

Indeed, by direct computation and (3.5), we have

$$\begin{aligned}&\lim_{n\to\infty}\int_0^T\int_\Omega \boldsymbol{u}\times\boldsymbol{B}^n\cdot(\nabla\times\boldsymbol{A})\eta d\boldsymbol{x}dt = \lim_{n\to\infty}\int_0^T\int_\Omega(\nabla\times\boldsymbol{A})\eta\times\boldsymbol{u}\cdot\boldsymbol{B}^n d\boldsymbol{x}dt\\ =\ &\int_0^T\int_\Omega(\nabla\times\boldsymbol{A})\eta\times\boldsymbol{u}\cdot\boldsymbol{B}^* d\boldsymbol{x}dt = \int_0^T\int_\Omega \boldsymbol{u}\times\boldsymbol{B}^*\cdot(\nabla\times\boldsymbol{A})\eta d\boldsymbol{x}dt.\end{aligned}$$

Finally, passaging to the limit on both sides of (3.8) and (3.9), and making use of (3.5)-(3.6), (3.10) and (3.12), we obtain that

$$\begin{aligned}&\int_0^T\int_\Omega \partial_t\boldsymbol{B}^*\cdot\boldsymbol{A}\eta(t)d\boldsymbol{x}dt + \int_0^T\int_\Omega \beta\nabla\times\boldsymbol{B}^*\cdot(\nabla\times\boldsymbol{A})\eta(t)d\boldsymbol{x}dt\\ &+\gamma\int_0^T\int_\Omega(\nabla\cdot\boldsymbol{B}^*)\cdot(\nabla\cdot\boldsymbol{A})\eta(t)d\boldsymbol{x}dt + \int_0^T\int_\Omega p^*(\nabla\cdot\boldsymbol{A})\eta(t)d\boldsymbol{x}dt\\ =\ & R_\alpha\int_0^T\int_\Omega \frac{g(\boldsymbol{x}).h(\boldsymbol{x},t)}{1+\sigma|\boldsymbol{B}^*|^2}\boldsymbol{B}^*\cdot(\nabla\times\boldsymbol{A})\eta(t)d\boldsymbol{x}dt\\ &+R_m\int_0^T\int_\Omega \boldsymbol{u}\times\boldsymbol{B}^*\cdot(\nabla\times\boldsymbol{A})\eta(t)d\boldsymbol{x}dt \ \ \forall\ \boldsymbol{A}\in H_0,\ \eta\in C^1[0,T],\\ &\int_0^T\int_\Omega(\nabla\cdot\boldsymbol{B}^*)q\eta(t)d\boldsymbol{x}dt = 0,\ \ \forall\ q\in L_0^2(\Omega),\ \eta\in C^1[0,T].\end{aligned}$$

Further, we shall prove $\boldsymbol{B}^*(\boldsymbol{x},0) = \boldsymbol{B}_0(\boldsymbol{x})$, which together with the definition of $(\boldsymbol{B}(\beta^*), p(\beta^*))$ implies that

$$(\boldsymbol{B}^*, p^*) = (\boldsymbol{B}(\beta^*), p(\beta^*)). \tag{3.13}$$

Choosing $\eta(t) \in C^1[0,T]$ with $\eta(T) = 0$, we have by integration by parts with respect to t that

$$\int_0^T\int_\Omega \partial_t\boldsymbol{B}^n\cdot\boldsymbol{A}\eta(t)d\boldsymbol{x}dt = -\int_0^T\int_\Omega \boldsymbol{B}^n\cdot\boldsymbol{A}\eta'(t)d\boldsymbol{x}dt - \int_\Omega \boldsymbol{B}_0(\boldsymbol{x})\cdot\boldsymbol{A}(\boldsymbol{x},0)\eta(0)d\boldsymbol{x}.$$

Letting $n \to \infty$ in the above equality and using (3.5), we have

$$\begin{aligned} \int_0^T \int_\Omega \partial_t \boldsymbol{B}^* \cdot \boldsymbol{A}\eta(t) d\boldsymbol{x}dt &= -\int_0^T \int_\Omega \boldsymbol{B}^* \cdot \boldsymbol{A}\eta'(t) d\boldsymbol{x}dt \\ &\quad - \int_\Omega \boldsymbol{B}_0(\boldsymbol{x}) \cdot \boldsymbol{A}(\boldsymbol{x},0)\eta(0) d\boldsymbol{x}. \end{aligned} \tag{3.14}$$

On the other hand, by integration by parts with respect to t, we also have

$$\int_0^T \int_\Omega \partial_t \boldsymbol{B}^* \cdot \boldsymbol{A}\eta(t) d\boldsymbol{x}dt = -\int_0^T \int_\Omega \boldsymbol{B}^* \cdot \boldsymbol{A}\eta'(t) d\boldsymbol{x}dt - \int_\Omega \boldsymbol{B}^*(\boldsymbol{x},0) \cdot \boldsymbol{A}(\boldsymbol{x},0)\eta(0) d\boldsymbol{x},$$

which together with (3.14) implies $\boldsymbol{B}^*(\boldsymbol{x},0) = \boldsymbol{B}_0(\boldsymbol{x})$.

Therefore, from (3.3), (3.5), (3.13) and the semi-continuity of the norm, we derive

$$J(\beta^*, g^*) \le \liminf_{n\to\infty} J(\beta_n, g_n) = \inf \min_{(\beta,g)\in K\times L^2(\Omega)} J(\beta, g),$$

which implies that (β^*, g^*) is a minimizer to the optimization problem (3.2). ■

The next theorem demonstrates that the minimization system (3.2) is indeed a stabilization of the ill-posed Inverse Problem I with respect to the changes of the observation errors.

Theorem 3.1.3 *Let* $\{\boldsymbol{z}_n\}$ *be a sequence such that* $\boldsymbol{z}_n \to \boldsymbol{z}^\delta$ *in* $L^2(0,T;\boldsymbol{L}^2(\omega))$ *as* $n \to \infty$ *and* $\{\beta_n\, g_n\}$ *be the minimizers of problem* (3.2) *with* $\boldsymbol{z}^\delta$ *replaced by* $\boldsymbol{z}_n$. *Then there exists a subsequence of* $\{\beta_n\, g_n\}$ *that converges in* $H^1(\Omega) \times L^2(\Omega)$, *and the limit of every such convergent subsequence is a minimizer of* (3.2).

Proof. By the definition of $\{\beta_n, g_n\}$, we have

$$\begin{aligned} &\frac{1}{2}\int_0^T \int_\omega |\boldsymbol{B}(\beta_n, g_n) - \boldsymbol{z}_n|^2 d\boldsymbol{x}dt + \frac{1}{2}\lambda_1 \int_\Omega |\nabla \beta_n|^2 d\boldsymbol{x} + \frac{1}{2}\lambda_2 \int_\Omega |g_n|^2 d\boldsymbol{x} \\ \le\ &\frac{1}{2}\int_0^T \int_\omega |\boldsymbol{B}(\beta, g) - \boldsymbol{z}_n|^2 d\boldsymbol{x}dt + \frac{1}{2}\lambda_1 \int_\Omega |\nabla \beta|^2 d\boldsymbol{x} + \frac{1}{2}\lambda_2 \int_\Omega |g|^2 d\boldsymbol{x}, \quad \forall\ (\beta, g) \in K \times L^2(\Omega), \end{aligned}$$

which with $(\beta_n, g_n) \in K \times L^2(\Omega)$ implies that $\{\beta_n\, g_n\}$ is bounded in $H^1(\Omega)\times L^2(\Omega)$. Similar to the proof of Theorem 3.1.2, there exists a subsequence, denoted still by $\{\beta_n\, g_n\}$, and some $(\beta^*, g^*) \in K \times L^2(\Omega)$ such that

$$\begin{aligned} &\beta_n \rightharpoonup \beta^* \quad \text{in } H^1(\Omega), \quad \beta_n \to \beta^* \quad \text{in } L^2(\Omega), \\ &\boldsymbol{B}(\beta_n) \to \boldsymbol{B}(\beta^*) \ \text{ in } L^2(0,T;\boldsymbol{L}^2(\Omega)). \end{aligned} \tag{3.15}$$

Hence we have

$$\lim_{n\to\infty} \int_0^T \int_\omega |\boldsymbol{B}(\beta_n, g_n) - \boldsymbol{z}_n|^2 d\boldsymbol{x}dt = \int_0^T \int_\omega |\boldsymbol{B}(\beta^*, g^*) - \boldsymbol{z}^\delta|^2 d\boldsymbol{x}dt. \tag{3.16}$$

Then, using the lower semi-continuity of a norm, we deduce that

$$\begin{aligned}
J(\beta^*, g^*) &= \frac{1}{2}\int_0^T\int_\omega |\boldsymbol{B}(\beta^*, g^*) - \boldsymbol{z}^\delta|^2 d\boldsymbol{x}dt + \frac{\lambda_1}{2}\int_\Omega |\nabla\beta^*|^2 d\boldsymbol{x} + \frac{\lambda_2}{2}\int_\Omega |\ g^*|^2 d\boldsymbol{x} \\
&\le \liminf_{n\to\infty}\{\frac{1}{2}\int_0^T\int_\omega |\boldsymbol{B}(\beta_n, g_n) - \boldsymbol{z}_n|^2 d\boldsymbol{x}dt + \frac{\lambda_1}{2}\int_\Omega |\nabla\beta_n|^2 d\boldsymbol{x} + \frac{\lambda_2}{2}\int_\Omega |\ g_n|^2 d\boldsymbol{x}\} \\
&\le \limsup_{n\to\infty}\{\frac{1}{2}\int_0^T\int_\omega |\boldsymbol{B}(\beta_n, g_n) - \boldsymbol{z}_n|^2 d\boldsymbol{x}dt + \frac{\lambda_1}{2}\int_\Omega |\nabla\beta_n|^2 d\boldsymbol{x} + \frac{\lambda_2}{2}\int_\Omega |\ g_n|^2 d\boldsymbol{x}\} \qquad (3.17) \\
&= \frac{1}{2}\int_0^T\int_\omega |\boldsymbol{B}(\beta, g) - \boldsymbol{z}^\delta|^2 d\boldsymbol{x}dt + \frac{\lambda_1}{2}\int_\Omega |\nabla\beta|^2 d\boldsymbol{x} + \frac{\lambda_2}{2}\int_\Omega |\ g|^2 d\boldsymbol{x} \quad \forall\ (\beta, g) \in K \times L^2(\Omega) \\
&= J(\beta, g) \quad \forall\ (\beta, g) \in K \times L^2(\Omega).
\end{aligned}$$

This yields that (β^*, g^*) is a minimizer to system (3.2).

Next we shall prove $\nabla\beta_n \to \nabla\beta^*$ in $L^2(\Omega)$, and then $\beta_n \to \beta^*$ in $H^1(\Omega)$. and then $g_n \to g^*$ in $L^2(\Omega)$ Since (3.17) holds for any $(\beta, g) \in K \times L^2(\Omega)$, we take $(\beta, g) = (\beta^*, g^*)$ and obtain that

$$\begin{aligned}
&\lim_{n\to\infty}\{\frac{1}{2}\int_0^T\int_\omega |\boldsymbol{B}(\beta_n, g_n) - \boldsymbol{z}_n|^2 d\boldsymbol{x}dt + \frac{\lambda}{2}\int_\Omega |\nabla\beta_n|^2 d\boldsymbol{x}\} + \frac{\lambda}{2}\int_\Omega |g_n|^2 d\boldsymbol{x}\} \\
= &\ \frac{1}{2}\int_0^T\int_\omega |\boldsymbol{B}(\beta^*, g^*) - \boldsymbol{z}^\delta|^2 d\boldsymbol{x}dt + \frac{\lambda_1}{2}\int_\Omega |\nabla\beta^*|^2 d\boldsymbol{x} + \frac{\lambda_2}{2}\int_\Omega |g^*|^2 d\boldsymbol{x}.
\end{aligned}$$

Combining this with (3.16), we get

$$\lim_{n\to\infty}\int_\Omega |\nabla\beta_n|^2 d\boldsymbol{x} = \int_\Omega |\nabla\beta^*|^2 d\boldsymbol{x},$$

which with $\nabla\beta_n \rightharpoonup \nabla\beta^*$ in $L^2(\Omega)$ by (3.15), we have $\nabla\beta_n \to \nabla\beta^*$ in $L^2(\Omega)$. □

3.2 Finite element approximation

In this section, we shall propose a fully discretized finite element approximation for solving the continuous minimization problem (3.2). The constrained subset K is approximated by $K_h = K \cap V_h$, and g is approximated by $W_h = Q_h \cap L^2(\Omega)$. Using the same finite element discretization and spaces as introduced in the second chapter. We are ready to formulate the finite element approximation of the continuous minimization (3.2) as follows:

$$\min_{(\beta_h, g_h)\in K_h\times W_h} J_{h,\tau}(\beta_h, g_h) = \frac{\tau}{2}\sum_{n=0}^{M}\alpha_n\int_\omega (\boldsymbol{B}_h^n - \boldsymbol{z}^{\delta,n})^2 d\boldsymbol{x} + \frac{\lambda_1}{2}\int_\Omega |\nabla\beta_h|^2 d\boldsymbol{x} + \frac{\lambda_2}{2}\int_\Omega |g_h|^2 d\boldsymbol{x}, \qquad (3.18)$$

where $(\boldsymbol{B}_h^n, p_h^n) \equiv (\boldsymbol{B}_h^n(\beta_h, g_h), p_h^n(\beta_h, g_h)) \in H_{0h} \times Q_{0h}$ satisfies $\boldsymbol{B}_h^0 = S_h \boldsymbol{B}_0(\boldsymbol{x})$ and

$$\begin{cases} \int_\Omega \partial\tau \boldsymbol{B}_h^n \cdot \boldsymbol{A}_h d\boldsymbol{x} + \int_\Omega \beta_h (\nabla \times \boldsymbol{B}_h^n) \cdot (\nabla \times \boldsymbol{A}_h) d\boldsymbol{x} \\ +\gamma \int_\Omega (\nabla \cdot \boldsymbol{B}_h^n)(\nabla \cdot \boldsymbol{A}_h) d\boldsymbol{x} + \int_\Omega p_h^n (\nabla \cdot \boldsymbol{A}_h) d\boldsymbol{x} \\ = R_\alpha \int_\Omega \frac{\bar{g}^n \times \bar{h}^n}{1 + \sigma |\boldsymbol{B}_h^n|^2} \boldsymbol{B}_h^n \cdot (\nabla \times \boldsymbol{A}_h) d\boldsymbol{x} \\ +R_m \int_\Omega (\bar{\boldsymbol{u}}^n \times \boldsymbol{B}_h^n) \cdot (\nabla \times \boldsymbol{A}_h) d\boldsymbol{x}, \\ \int_\Omega (\nabla \cdot \boldsymbol{B}_h^n) q_h d\boldsymbol{x} = 0, \end{cases} \tag{3.19}$$

for all $(\boldsymbol{A}_h,\ q_h) \in H_{0h} \times Q_{0h}$, $n = 1, 2, \cdots, M$, $\bar{\boldsymbol{u}}^n \in \boldsymbol{L}^\infty(\Omega)$ and $\bar{h}^n \in L^\infty(\Omega)$ and $\bar{g}^n \in L^2(\Omega)$. Here $\{\alpha_n\}$ are the coefficients of the composite trapezoidal rule, i.e., $\alpha_0 = \alpha_M = \frac{1}{2}$ and $\alpha_n = 1$ for $n \neq 0, M$.

Lemma 3.2.1 *There exists a unique solution $(\boldsymbol{B}_h^n, p_h^n)$ to the discrete system* (3.19) *for each fixed $n\,(1 \leq n \leq M)$ and the sequence $\{(\boldsymbol{B}_h^n, p_h^n)\}_{n=0}^M$ has the following stability estimates:*

$$\max_{1\leq n\leq M} \|\boldsymbol{B}_h^n\|_{0,\Omega}^2 + \tau \sum_{n=1}^M (\|\nabla \times \boldsymbol{B}_h^n\|_{0,\Omega}^2 + \|\nabla \cdot \boldsymbol{B}_h^n\|_{0,\Omega}^2) \leq C \|\boldsymbol{B}_h^0\|_{0,\Omega}^2,$$

$$\max_{1\leq n\leq M} \|\nabla \times \boldsymbol{B}_h^n\|_{0,\Omega}^2 + \max_{1\leq n\leq M} \|\nabla \cdot \boldsymbol{B}_h^n\|_{0,\Omega}^2 + \tau \sum_{n=1}^M \|\partial_\tau \boldsymbol{B}_h^n\|_{0,\Omega}^2$$
$$+\tau \sum_{n=1}^M \|p_h^n\|_{0,\Omega}^2 + \tau \sum_{n=1}^M \|\partial_\tau \boldsymbol{B}_h^n\|_{(\boldsymbol{H}^1(\Omega))'}^2 \leq C \|\boldsymbol{B}_h^0\|_{1,\Omega}^2.$$

Theorem 3.2.2 *There exists at least a minimizer to the discrete minimization problem* (3.18).

Proof. Due to the stability estimates in Lemma 3.2.1, we could get the existence of the minimizer to (3.18) by the similar technique in the proof of Theorem 3.1.2. ■

Now we will consider the convergence of the minimizer of the discrete system (3.18) to the minimizer of the continuous problem (3.2). For the purpose, we first give the following classical approximation result ([32] [33]).

Lemma 3.2.3 *The following results hold:*

$$\|\hat{\boldsymbol{B}}_{h,\tau}\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))} = \tau\sum_{n=1}^{M}\|\boldsymbol{B}_h^n\|^2_{1,\Omega},$$

$$\|\frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau}\|^2_{L^2(0,T;(\boldsymbol{H}^1(\Omega))')} = \tau\sum_{n=1}^{M}\|\partial_\tau\boldsymbol{B}_h^n\|^2_{(\boldsymbol{H}^1(\Omega))'},$$

$$\|\hat{p}_{h,\tau}\|^2_{L^2(0,T;L^2(\Omega))} = \tau\sum_{n=1}^{M}\|p_h^n\|^2_{0,\Omega},$$

$$\|\boldsymbol{B}_{h,\tau}\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))} \le \tau\sum_{n=0}^{M}\|\boldsymbol{B}_h^n\|^2_{1,\Omega}.$$

Proof. We first prove the first three equalities. By direct computation, it is easy to see that

$$\begin{aligned}
&\|\hat{\boldsymbol{B}}_{h,\tau}\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))}\\
=\;& \sum_{n=1}^{M}\left(\int_{t_{n-1}}^{t_n}\int_\Omega|\sum_{n=1}^{M}\chi_n(t)\boldsymbol{B}_h^n(\boldsymbol{x})|^2 d\boldsymbol{x}dt + \int_{t_{n-1}}^{t_n}\int_\Omega|\sum_{n=1}^{M}\chi_n(t)\nabla\boldsymbol{B}_h^n(\boldsymbol{x})|^2\boldsymbol{x}dt\right)\\
=\;& \sum_{n=1}^{M}\left(\int_{t_{n-1}}^{t_n}\int_\Omega|\boldsymbol{B}_h^n(\boldsymbol{x})|^2 d\boldsymbol{x}dt + \int_{t_{n-1}}^{t_n}\int_\Omega|\nabla\boldsymbol{B}_h^n(\boldsymbol{x})|^2\boldsymbol{x}dt\right) = \tau\sum_{n=1}^{M}\|\boldsymbol{B}_h^n\|^2_{1,\Omega};
\end{aligned}$$

$$\begin{aligned}
&\|\frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau}\|^2_{L^2(0,T;(\boldsymbol{H}^1(\Omega))')} = \int_0^T\|\frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau}\|^2_{(\boldsymbol{H}^1(\Omega))')}dt\\
=\;& \int_0^T\|\frac{\partial}{\partial t}\left(\frac{t-t_{n-1}}{\tau}\boldsymbol{B}_h^n(\boldsymbol{x}) + \frac{t_n-t}{\tau}\boldsymbol{B}_h^{n-1}(\boldsymbol{x})\right)\|^2_{(\boldsymbol{H}^1(\Omega))')}dt\\
=\;& \sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\|\frac{\partial}{\partial t}\left(\frac{t-t_{n-1}}{\tau}\boldsymbol{B}_h^n(\boldsymbol{x}) + \frac{t_n-t}{\tau}\boldsymbol{B}_h^{n-1}(\boldsymbol{x})\right)\|^2_{(\boldsymbol{H}^1(\Omega))')}dt\\
=\;& \sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\|\frac{1}{\tau}(\boldsymbol{B}_h^n - \boldsymbol{B}_h^{n-1})\|^2_{(\boldsymbol{H}^1(\Omega))'}dt = \tau\sum_{n=1}^{M}\|\partial_\tau\boldsymbol{B}_h^n\|^2_{(\boldsymbol{H}^1(\Omega))'};
\end{aligned}$$

$$\begin{aligned}
&\|\hat{p}_{h,\tau}\|^2_{L^2(0,T;L^2(\Omega))} = \sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\int_\Omega|\sum_{n=1}^{M}\chi_n(t)p_h^n(\boldsymbol{x})|^2 d\boldsymbol{x}dt\\
=\;& \sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\int_\Omega|p_h^n(\boldsymbol{x})|^2 d\boldsymbol{x}dt = \tau\sum_{n=1}^{M}\|p_h^n\|^2_{0,\Omega}.
\end{aligned}$$

Then we show the last inequality:

$$
\begin{aligned}
&\|\boldsymbol{B}_{h,\tau}\|^2_{L^2(0,T;\boldsymbol{H}^1(\Omega))}\\
= & \sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\Big\{\int_\Omega \Big|\frac{t-t_{n-1}}{\tau}\boldsymbol{B}_h^n(\boldsymbol{x})+\frac{t_n-t}{\tau}\boldsymbol{B}_h^{n-1}(\boldsymbol{x})\Big|^2 d\boldsymbol{x}\\
&\qquad +\int_\Omega \Big|\frac{t-t_{n-1}}{\tau}\nabla\boldsymbol{B}_h^n(\boldsymbol{x})+\frac{t_n-t}{\tau}\nabla\boldsymbol{B}_h^{n-1}(\boldsymbol{x})\Big|^2 d\boldsymbol{x}\Big\}dt\\
= & \sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\int_\Omega\Big[\Big(\frac{t-t_{n-1}}{\tau}\Big)^2|\boldsymbol{B}_h^n|^2+\Big(\frac{t_n-t}{\tau}\Big)^2|\boldsymbol{B}_h^{n-1}|^2\\
&\qquad +2\frac{(t-t_{n-1})(t_n-t)}{\tau^2}\boldsymbol{B}_h^n\cdot\boldsymbol{B}_h^{n-1}\Big]d\boldsymbol{x}dt\\
&+\sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\int_\Omega\Big[\Big(\frac{t-t_{n-1}}{\tau}\Big)^2|\nabla\boldsymbol{B}_h^n|^2+\Big(\frac{t_n-t}{\tau}\Big)^2|\nabla\boldsymbol{B}_h^{n-1}|^2\\
&\qquad +2\frac{(t-t_{n-1})(t_n-t)}{\tau^2}\nabla\boldsymbol{B}_h^n\cdot\nabla\boldsymbol{B}_h^{n-1}\Big]d\boldsymbol{x}dt\\
= & \frac{\tau}{3}\sum_{n=1}^{M}\int_\Omega(|\boldsymbol{B}_h^n|^2+|\boldsymbol{B}_h^{n-1}|^2+\boldsymbol{B}_h^n\cdot\boldsymbol{B}_h^{n-1}\\
&+|\nabla\boldsymbol{B}_h^n|^2+|\nabla\boldsymbol{B}_h^{n-1}|^2+\nabla\boldsymbol{B}_h^n\cdot\nabla\boldsymbol{B}_h^{n-1})d\boldsymbol{x}\\
\le & \tau\sum_{n=0}^{M}\|\boldsymbol{B}_h^n\|^2_{1,\Omega}.
\end{aligned}
$$

■ Next, for any $\varphi(\boldsymbol{x})\in H_0$ and $\psi(t)\in C_0^\infty(0,T)$, let $\phi(\boldsymbol{x},t)=\varphi(\boldsymbol{x})\psi(t)$ and $\phi_{h,\tau}(\boldsymbol{x},t)=\sum_{n=1}^{M}\chi_n(t)S_h\phi(\boldsymbol{x},t_n)$. We have by the triangle inequality, (1.1) and Lemma 1.2.3 that

$$
\begin{aligned}
&\int_0^T\|\phi(\cdot,t)-\phi_{h,\tau}(\cdot,t)\|^2_{1,\Omega}dt\\
\le\ & 2\int_0^T\|\phi(\cdot,t)-S_\Delta\phi(\cdot,t)\|^2_{1,\Omega}dt+2T\max_{0\le t\le T}|\psi(t)|^2\|\varphi(\cdot)-S_h\varphi(\cdot)\|^2_{1,\Omega}\\
\to\ & 0 \quad \text{as} \quad h,\tau\to 0.
\end{aligned}
$$

Lemma 3.2.4 *Direct computations give us the following equalities:*

$$\int_0^T\int_\Omega \frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau}(\boldsymbol{x},t)\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt = \tau\sum_{n=1}^{M}\int_\Omega \partial_\tau \boldsymbol{B}_h^n S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \gamma\nabla\cdot(\hat{\boldsymbol{B}}_{h,\tau})(\nabla\cdot\phi_{h,\tau})d\boldsymbol{x}dt = \tau\sum_{n=1}^{M}\int_\Omega \gamma(\nabla\cdot\boldsymbol{B}_h^n)(\nabla\cdot S_h\phi(\boldsymbol{x},t_n))d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \beta_h\nabla\times\hat{\boldsymbol{B}}_{h,\tau}(\boldsymbol{x},t)\cdot\nabla\times\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt = \tau\sum_{n=1}^{M}\int_\Omega \beta_h\nabla\times\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \hat{p}_{h,\tau}(\nabla\cdot\phi_{h,\tau})d\boldsymbol{x}dt = \tau\sum_{n=1}^{M}\int_\Omega p_h^n(\nabla\cdot S_h\phi(\boldsymbol{x},t_n))d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \frac{f}{1+\sigma|\hat{\boldsymbol{B}}_{h,\tau}|^2}\hat{\boldsymbol{B}}_{h,\tau}\cdot\nabla\times\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt$$
$$= \tau\sum_{n=1}^{M}\int_\Omega \frac{\bar{f}^n}{1+\sigma|\boldsymbol{B}_h^n|^2}\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \boldsymbol{u}\times\hat{\boldsymbol{B}}_{h,\tau}\cdot\nabla\times\phi_{h,\tau}d\boldsymbol{x}dt = \tau\sum_{n=1}^{M}\int_\Omega \bar{\boldsymbol{u}}^n\times\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x}.$$

Proof. By direct computation, we have the following equalities:

$$\int_0^T\int_\Omega \frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau}(\boldsymbol{x},t)\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt = \int_0^T\int_\Omega \partial_\tau\boldsymbol{B}_h^n\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt$$
$$= \sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\int_\Omega \partial_\tau\boldsymbol{B}_h^n\sum_{n=1}^{M}\chi_n(t)S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x}dt = \tau\sum_{n=1}^{M}\int_\Omega \partial_\tau\boldsymbol{B}_h^n S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x};$$

$$\int_0^T\int_\Omega \gamma\nabla\cdot(\hat{\boldsymbol{B}}_{h,\tau})(\nabla\cdot\phi_{h,\tau})d\boldsymbol{x}dt$$
$$=\sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\int_\Omega \gamma\nabla\cdot(\sum_{n=1}^{M}\chi_n(t)\boldsymbol{B}_h^n(\boldsymbol{x}))(\nabla\cdot\sum_{n=1}^{M}\chi_n(t)S_h\phi(\boldsymbol{x},t_n))d\boldsymbol{x}dt$$
$$=\sum_{n=1}^{M}\int_{t_{n-1}}^{t_n}\int_\Omega \gamma(\nabla\cdot\boldsymbol{B}_h^n)(\nabla\cdot S_h\phi(\boldsymbol{x},t_n))d\boldsymbol{x}dt$$
$$=\tau\sum_{n=1}^{M}\int_\Omega \gamma(\nabla\cdot\boldsymbol{B}_h^n)(\nabla\cdot S_h\phi(\boldsymbol{x},t_n))d\boldsymbol{x};$$

$$
\begin{aligned}
&\int_0^T\int_\Omega \beta_h \nabla \times \hat{\boldsymbol{B}}_{h,\tau}(\boldsymbol{x},t)\cdot\nabla\times\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt\\
&=\int_0^T\int_\Omega \beta_h\nabla\times\sum_{n=1}^M\chi_n(t)\boldsymbol{B}_h^n(\boldsymbol{x})\cdot\nabla\times\sum_{n=1}^M\chi_n(t)S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x}dt\\
&=\sum_{n=1}^M\int_{t_{n-1}}^{t_n}\int_\Omega\beta_h\nabla\times\boldsymbol{B}_h^n(\boldsymbol{x})\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x}dt\\
&=\tau\sum_{n=1}^M\int_\Omega\beta_h\nabla\times\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x};
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T\int_\Omega \hat{p}_{h,\tau}(\nabla\cdot\phi_{h,\tau})d\boldsymbol{x}dt\\
&=\sum_{n=1}^M\int_{t_{n-1}}^{t_n}\int_\Omega(\sum_{n=1}^M\chi_n(t)p_h^n(\boldsymbol{x}))(\nabla\cdot\sum_{n=1}^M\chi_n(t)S_h\phi(\boldsymbol{x},t_n))d\boldsymbol{x}dt\\
&=\tau\sum_{n=1}^M\int_\Omega p_h^n(\boldsymbol{x})(\nabla\cdot S_h\phi(\boldsymbol{x},t_n))d\boldsymbol{x};
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T\int_\Omega\frac{g.h}{1+\sigma|\hat{\boldsymbol{B}}_{h,\tau}|^2}\hat{\boldsymbol{B}}_{h,\tau}\cdot\nabla\times\phi_{h,\tau}(\boldsymbol{x},t)d\boldsymbol{x}dt\\
&=\sum_{n=1}^M\int_{t_{n-1}}^{t_n}\int_\Omega\frac{g.h}{1+\sigma|\boldsymbol{B}_h^n(\boldsymbol{x})|^2}\boldsymbol{B}_h^n(\boldsymbol{x})\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x}dt\\
&=\sum_{n=1}^M\int_\Omega\frac{\tau\bar{g}^n.\bar{h}^n}{1+\sigma|\boldsymbol{B}_h^n|^2}\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x},\\
&=\tau\sum_{n=1}^M\int_\Omega\frac{\bar{g}^n.\bar{h}^n}{1+\sigma|\boldsymbol{B}_h^n|^2}\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x};
\end{aligned}
$$

$$
\begin{aligned}
&\int_0^T\int_\Omega\boldsymbol{u}\times\hat{\boldsymbol{B}}_{h,\tau}\cdot\nabla\times\phi_{h,\tau}d\boldsymbol{x}dt\\
&=\sum_{n=1}^M\int_{t_{n-1}}^{t_n}\int_\Omega\boldsymbol{u}\times\sum_{n=1}^M\chi_n(t)\boldsymbol{B}_h^n(\boldsymbol{x})\cdot\nabla\times\sum_{n=1}^M\chi_n(t)S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x}dt\\
&=\sum_{n=1}^M\int_\Omega\int_{t_{n-1}}^{t_n}\boldsymbol{u}dt\times\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x}\\
&=\tau\sum_{n=1}^M\int_\Omega\bar{\boldsymbol{u}}^n\times\boldsymbol{B}_h^n\cdot\nabla\times S_h\phi(\boldsymbol{x},t_n)d\boldsymbol{x}.
\end{aligned}
$$

We then derive some important convergence results.

Lemma 3.2.5 *For any* $(\beta_h, g_h) \in K_h \times L^2(\Omega)$, $(\beta, g) \in K \times L^2(\Omega)$, $U_{h,\tau}$, $U \in L^2(0,T;\boldsymbol{H}^1(\Omega))$ *and* $V_{h,\tau}$, $V \in L^2(0,T;\boldsymbol{L}^2(\Omega))$, *if* $\beta_h \to \beta$ *in* $L^2(\Omega)$ *as* $h \to 0$, $U_{h,\tau} \rightharpoonup U$ *in* $L^2(0,T;\boldsymbol{H}^1(\Omega))$ *and* $V_{h,\tau} \to V$ *in* $L^2(0,T;\boldsymbol{L}^2(\Omega))$ *as* $h,\tau \to 0$, *we have the following convergence results:*

$$\lim_{h,\tau\to 0}\int_0^T\int_\Omega \beta_h \nabla\times U_{h,\tau}\cdot V_{h,\tau} d\boldsymbol{x}dt = \int_0^T\int_\Omega \beta\nabla\times U\cdot V d\boldsymbol{x}dt, \tag{3.20}$$

$$\lim_{h,\tau\to 0}\int_0^T\int_\Omega \frac{g.h}{1+\sigma|U_{h,\tau}|^2} U_{h,\tau}\cdot V_{h,\tau} d\boldsymbol{x}dt = \int_0^T\int_\Omega \frac{g.h}{1+\sigma|U|^2} U\cdot V d\boldsymbol{x}dt, \tag{3.21}$$

$$\lim_{h,\tau\to 0}\int_0^T\int_\Omega \boldsymbol{u}\times U_{h,\tau}\cdot V_{h,\tau} d\boldsymbol{x}dt = \int_0^T\int_\Omega \boldsymbol{u}\times U\cdot V d\boldsymbol{x}dt. \tag{3.22}$$

Proof. We first prove (3.20). By the triangle inequality, we have

$$\begin{aligned}
& |\int_0^T\int_\Omega (\beta_h\nabla\times U_{h,\tau}\cdot V_{h,\tau} - \beta\nabla\times U\cdot V)\ d\boldsymbol{x}dt| \\
\le\ & |\int_0^T\int_\Omega (\beta_h - \beta)\nabla\times U_{h,\tau}\cdot (V_{h,\tau} - V)\ d\boldsymbol{x}dt| \\
& + |\int_0^T\int_\Omega (\beta_h - \beta)\nabla\times U_{h,\tau}\cdot V\ d\boldsymbol{x}dt| \\
& + |\int_0^T\int_\Omega \beta(\nabla\times U_{h,\tau}\cdot V_{h,\tau} - \nabla\times U\cdot V)\ d\boldsymbol{x}dt| \\
\doteq\ & I + II + III.
\end{aligned}$$

To estimate I, it is readily to see that

$$\begin{aligned}
I &= |\int_0^T\int_\Omega (\beta_h - \beta)\nabla\times U_{h,\tau}\cdot (V_{h,\tau} - V)\ d\boldsymbol{x}dt| \\
&\le 2\beta_2 ||\nabla\times U_{h,\tau}||_{L^2(0,T;\boldsymbol{L}^2(\Omega))} ||V_{h,\tau} - V||_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \longrightarrow 0,
\end{aligned}$$

as $h,\tau\to 0$ due to the fact that $V_{h,\tau}\to V$ in $L^2(0,T;\boldsymbol{L}^2(\Omega))$.

Then we start to analyze II:

$$\begin{aligned}
& |\int_\Omega (\beta_h - \beta)\nabla\times U_{h,\tau}\cdot V\ d\boldsymbol{x}dt| \\
\le\ & (\int_\Omega |\beta_h - \beta||\nabla\times U_{h,\tau}|^2 d\boldsymbol{x}dt)^{\frac{1}{2}} (\int_\Omega |\beta_h - \beta||V|^2 d\boldsymbol{x}dt)^{\frac{1}{2}} \\
\le\ & C(\int_\Omega |\beta_h - \beta||V|^2 d\boldsymbol{x}dt)^{\frac{1}{2}} \to 0
\end{aligned}$$

as $h, \tau \to 0$ by Lebesgue dominated convergence theorem.

For III, we have

$$
\begin{aligned}
III &= |\int_0^T\int_\Omega \beta\nabla\times U_{h,\tau}\cdot(V_{h,\tau}-V)+\beta\nabla\times(U_{h,\tau}-U)\cdot V\,d\boldsymbol{x}dt| \\
&\leq \beta_2||\nabla\times U_{h,\tau}||_{L^2(0,T;\boldsymbol{L}^2(\Omega))}||V_{h,\tau}-V||_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&\quad +|\int_0^T\int_\Omega \nabla\times(U_{h,\tau}-U)\cdot\beta V\,d\boldsymbol{x}dt| \\
&\to 0 \quad \text{as } h,\tau\to 0.
\end{aligned}
$$

Next, we shall show (3.21).

$$
\begin{aligned}
&|\int_0^T\int_\Omega \frac{g.h}{1+\sigma|U_{h,\tau}|^2}U_{h,\tau}\cdot V_{h,\tau}d\boldsymbol{x}dt-\int_0^T\int_\Omega\frac{g.h}{1+\sigma|U|^2}U\cdot Vd\boldsymbol{x}dt| \\
\leq\;& |\int_0^T\int_\Omega \frac{g.h}{1+\sigma|U_{h,\tau}|^2}U_{h,\tau}\cdot(V_{h,\tau}-V)d\boldsymbol{x}dt| \\
&+|\int_0^T\int_\Omega g.h\frac{(1+\sigma|U|^2)(U_{h,\tau}-U)+\sigma(|U|^2-|U_{h,\tau}|^2)U}{(1+\sigma|U_{h,\tau}|^2)(1+\sigma|U|^2)}\cdot Vd\boldsymbol{x}dt| \\
\leq\;& C\|g\|_{L^2(\Omega)}.\|h\|_{H^1(0,T;L^\infty(\Omega))}\|U_{h,\tau}\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}||V_{h,\tau}-V||_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&+C\|g\|_{L^2(\Omega)}.\|h\|_{H^1(0,T;L^\infty(\Omega))}\|U_{h,\tau}-U\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}||V||_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
&+|\int_0^T\int_\Omega g.h\frac{\sigma(|U|+|U_{h,\tau}|)(|U|-|U_{h,\tau}|)U}{(1+\sigma|U_{h,\tau}|^2)(1+\sigma|U|^2)}\cdot Vd\boldsymbol{x}dt| \\
\leq\;& C(||V_{h,\tau}-V||_{L^2(0,T;\boldsymbol{L}^2(\Omega))}+\|U_{h,\tau}-U\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}) \\
&+\int_0^T\int_\Omega |g.h||U-U_{h,\tau}|\frac{\sigma|U|^2+\frac{\sigma}{2}|U_{h,\tau}|^2+\frac{\sigma}{2}|U|^2}{(1+\sigma|U_{h,\tau}|^2)(1+\sigma|U|^2)}|V|d\boldsymbol{x}dt \\
\leq\;& C(||V_{h,\tau}-V||_{L^2(0,T;\boldsymbol{L}^2(\Omega))}+\|U_{h,\tau}-U\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))}) \\
&+C\|g\|_{L^2(\Omega)}.\|h\|_{H^1(0,T;L^\infty(\Omega))}||U-U_{h,\tau}||_{L^2(0,T;\boldsymbol{L}^2(\Omega))}||V||_{L^2(0,T;\boldsymbol{L}^2(\Omega))} \\
\to\;& 0 \quad \text{as } h,\tau\to 0.
\end{aligned}
$$

Finally, we shall prove the last equation (3.22).

$$
\begin{aligned}
&|\int_0^T\int_\Omega \boldsymbol{u}\times U_{h,\tau}\cdot V_{h,\tau}d\boldsymbol{x}dt-\int_0^T\int_\Omega\boldsymbol{u}\times U\cdot Vd\boldsymbol{x}dt| \\
\leq\;& |\int_0^T\int_\Omega \boldsymbol{u}\times U_{h,\tau}\cdot(V_{h,\tau}-V)d\boldsymbol{x}dt|+|\int_0^T\int_\Omega\boldsymbol{u}\times(U_{h,\tau}-U)\cdot Vd\boldsymbol{x}dt| \\
\leq\;& C(||V_{h,\tau}-V||_{L^2(0,T;\boldsymbol{L}^2(\Omega))}+\|U_{h,\tau}-U\|_{L^2(0,T;\boldsymbol{L}^2(\Omega))})\to 0 \;\text{ as } h,\tau\to 0.
\end{aligned}
$$

■

In the following, we prove a crucial lemma for our purpose

Lemma 3.2.6 *For the sequence* $\{\beta_h, g_h\}_{h>0} \subset K_h \times L^2(\omega)$ *, if* $\{\beta_h, g_h\}_{h>0}$ *converges to some* $(\beta, g) \in K \times L^2(\omega)$ *in* $L^2(\Omega)$ *strongly, suppose* $z^\delta \in C(0,T;\boldsymbol{L}^2(\omega))$*, then there exists a subsequence, also denoted by* $\{\beta_h, g_h\}_{h>0}$*, such that*

$$\lim_{h,\tau\to 0} \tau \sum_{n=0}^{M} \alpha_n \int_\omega (\boldsymbol{B}_h^n(\beta_h) - \boldsymbol{z}^{\delta,n})^2 d\boldsymbol{x} = \int_0^T \int_\omega |\boldsymbol{B}(\beta) - \boldsymbol{z}^\delta|^2 d\boldsymbol{x} dt.$$

Proof. For $1 \le n \le M$, we denote by $\boldsymbol{B}_h^n = \boldsymbol{B}_h^n(\beta_h, g_h)$, $\boldsymbol{B}^n = \boldsymbol{B}(\beta, g)(\cdot, t_n)$. Making use of (1.1), we find that

$$\lim_{\tau\to 0} \tau \sum_{n=0}^{M} \alpha_n \int_\omega (\boldsymbol{B}^n - \boldsymbol{z}^{\delta,n})^2 d\boldsymbol{x} = \int_0^T\int_\omega (\boldsymbol{B}(\beta, g) - \boldsymbol{z}^\delta)^2 d\boldsymbol{x} dt.$$

So it suffices to show that

$$\lim_{h,\tau\to 0} \tau \sum_{n=0}^{M} \alpha_n \int_\omega (\boldsymbol{B}_h^n - \boldsymbol{B}^n)^2 d\boldsymbol{x} = 0. \tag{3.23}$$

From Lemma 3.2.3 and Lemma 3.2.1, we conclude that $\{\boldsymbol{B}_{h,\tau}\}$ and $\{\hat{\boldsymbol{B}}_{h,\tau}\}$ are bounded in $L^2(0,T;\boldsymbol{H}^1(\Omega))$, $\{\frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau}\}$ is bounded in $L^2(0,T;(\boldsymbol{H}^1(\Omega))')$ and $\{\hat{p}_{h,\tau}\}$ is bounded in $L^2(0,T;L^2(\Omega))$. Hence there exists a subsequence of $\{\boldsymbol{B}_{h,\tau}\}$ such that

$$\boldsymbol{B}_{h,\tau} \rightharpoonup \boldsymbol{B}^* \text{ in } L^2(0,T;\boldsymbol{H}^1(\Omega)), \tag{3.24}$$
$$\boldsymbol{B}_{h,\tau} \to \boldsymbol{B}^* \text{ in } L^2(0,T;\boldsymbol{L}^2(\Omega)), \tag{3.25}$$
$$\frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau} \rightharpoonup \boldsymbol{C}^* \text{ in } L^2(0,T;(\boldsymbol{H}^1(\Omega))') \tag{3.26}$$

and a subsequence of $\{\hat{\boldsymbol{B}}_{h,\tau}\}$ and a subsequence of $\{\hat{p}_{h,\tau}\}$ such that

$$\hat{\boldsymbol{B}}_{h,\tau} \rightharpoonup \boldsymbol{B}^{**} \text{ in } L^2(0,T;\boldsymbol{H}^1(\Omega)), \tag{3.27}$$
$$\hat{p}_{h,\tau} \rightharpoonup p^* \text{ in } L^2(0,T;L^2(\Omega)), \tag{3.28}$$

for some $\boldsymbol{B}^*, \boldsymbol{B}^{**} \in L^2(0,T;\boldsymbol{H}^1(\Omega))$, $\boldsymbol{C}^* \in L^2(0,T;(\boldsymbol{H}^1(\Omega))')$ and $p^* \in L^2(0,T;L^2(\Omega))$.

Next, we show $\boldsymbol{B}^* = \boldsymbol{B}^{**}$ and $\boldsymbol{C}^*(\boldsymbol{x},t) = \frac{\partial}{\partial t}\boldsymbol{B}^*(\boldsymbol{x},t)$. Firstly, by (3.26) we have for any $\varphi(\boldsymbol{x}) \in \boldsymbol{H}^1(\Omega)$ and $\psi(t) \in C_0^\infty(0,T)$ that

$$\lim_{h,\tau\to 0} \int_0^T\int_\Omega \frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau} \cdot \varphi(\boldsymbol{x})\psi(t) d\boldsymbol{x} dt = \int_0^T\int_\Omega C^*(\boldsymbol{x},t) \cdot \varphi(\boldsymbol{x})\psi(t) d\boldsymbol{x} dt. \tag{3.29}$$

On the other hand, by integration by parts with respect to t and using (3.24), we get

$$\lim_{h,\tau\to 0} \int_0^T\int_\Omega \frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau} \cdot \varphi(\boldsymbol{x})\psi(t) d\boldsymbol{x} dt = \lim_{h,\tau\to 0} - \int_0^T\int_\Omega \boldsymbol{B}_{h,\tau} \cdot \varphi(\boldsymbol{x})\psi'(t) d\boldsymbol{x} dt$$
$$= - \int_0^T\int_\Omega \boldsymbol{B}^* \cdot \varphi(\boldsymbol{x})\psi'(t) d\boldsymbol{x} dt,$$

which together with (3.29) gives

$$\boldsymbol{C}^*(\boldsymbol{x}, t) = \frac{\partial}{\partial t}\boldsymbol{B}^*(\boldsymbol{x}, t). \tag{3.30}$$

Then taking any $\varphi(\boldsymbol{x}) \in \boldsymbol{H}^1(\Omega)$ and $\psi(t) \in C^1[0, T]$ with $\psi(T) = 0$, integrating by parts with respect to t to both sides of (3.29) and using (3.30) and (3.24), we have

$$\begin{aligned} & \lim_{h,\tau\to 0}\left\{-\int_\Omega S_h\boldsymbol{B}_0(\boldsymbol{x})\cdot\varphi(\boldsymbol{x})\psi(0)d\boldsymbol{x} - \int_0^T\int_\Omega \boldsymbol{B}_{h,\tau}\cdot\varphi(\boldsymbol{x})\psi'(t)d\boldsymbol{x}dt\right\} \\ = & -\int_\Omega \boldsymbol{B}^*(\boldsymbol{x}, 0)\cdot\varphi(\boldsymbol{x})\psi(0)d\boldsymbol{x} - \int_0^T\int_\Omega \boldsymbol{B}^*(\boldsymbol{x}, t)\cdot\varphi(\boldsymbol{x})\psi'(t)d\boldsymbol{x}dt. \end{aligned}$$

By (3.24) and Lemma 1.2.3 we derive that

$$\boldsymbol{B}^*(\boldsymbol{x}, 0) = \boldsymbol{B}_0(\boldsymbol{x}).$$

Now we will show that $\boldsymbol{B}^*(\boldsymbol{x}, t) = \boldsymbol{B}^{**}(\boldsymbol{x}, t)$. By direct computation and using Lemma 1.2.3, we have

$$\int_0^T \|\boldsymbol{B}_{h,\tau}(\cdot, t) - \hat{\boldsymbol{B}}_{h,\tau}(\cdot, t)\|_{0,\Omega}^2 dt = \frac{\tau^3}{3}\sum_{n=1}^{M}\|\partial_\tau \boldsymbol{B}_h^n\|_{0,\Omega}^2 \le C\tau^2 \to 0 \ \text{ as } h, \tau \to 0,$$

which, together with (3.25) implies

$$\hat{\boldsymbol{B}}_{h,\tau} \to \boldsymbol{B}^* \text{ in } L^2(0, T; \boldsymbol{L}^2(\Omega)).$$

Then from (3.27) and the uniqueness of the limits, we get $\boldsymbol{B}^* = \boldsymbol{B}^{**}$.

It is time to show that $\boldsymbol{B}^* = \boldsymbol{B}(\beta),\ p^* = p(\beta)$. Using Lemma 3.2.4 and system (2.7), we get

$$\begin{aligned} & \int_0^T\int_\Omega \frac{\partial}{\partial t}\boldsymbol{B}_{h,\tau}(\boldsymbol{x}, t)\phi_{h,\tau}(\boldsymbol{x}, t)d\boldsymbol{x}dt + \int_0^T\int_\Omega \beta_h\nabla\times\hat{\boldsymbol{B}}_{h,\tau}\cdot\nabla\times\phi_{h,\tau}d\boldsymbol{x}dt \\ & +\gamma\int_0^T\int_\Omega \nabla\cdot(\hat{\boldsymbol{B}}_{h,\tau})(\nabla\cdot\phi_{h,\tau})d\boldsymbol{x}dt + \int_0^T\int_\Omega \hat{p}_{h,\tau}(\nabla\cdot\phi_{h,\tau})d\boldsymbol{x}dt \\ = & \ R_\alpha\int_0^T\int_\Omega \frac{g\times h}{1+\sigma|\hat{\boldsymbol{B}}_{h,\tau}|^2}\hat{\boldsymbol{B}}_{h,\tau}\cdot\nabla\times\phi_{h,\tau}(\boldsymbol{x}, t)d\boldsymbol{x}dt \\ & +R_m\int_0^T\int_\Omega \boldsymbol{u}\times\hat{\boldsymbol{B}}_{h,\tau}\cdot\nabla\times\phi_{h,\tau}d\boldsymbol{x}dt. \end{aligned} \tag{3.31}$$

Letting $h, \tau \to 0$ in the above equation and making use of (3.24)-(3.28) and (3.20)-(3.22), we have

$$\begin{aligned}
& \int_0^T\int_\Omega \frac{\partial}{\partial t}\boldsymbol{B}^*\cdot\varphi(\boldsymbol{x})\psi(t)d\boldsymbol{x}dt + \int_0^T\int_\Omega \beta(\nabla\times\boldsymbol{B}^*)\cdot(\nabla\times\varphi)\psi(t)d\boldsymbol{x}dt \\
& +\gamma\int_0^T\int_\Omega(\nabla\cdot\boldsymbol{B}^*)(\nabla\cdot\varphi)\psi(t)d\boldsymbol{x}dt + \int_0^T\int_\Omega p^*(\nabla\cdot\varphi)\psi(t)d\boldsymbol{x}dt \\
= \ & R_\alpha\int_0^T\int_\Omega \frac{g.h}{1+\sigma|\boldsymbol{B}^*|^2}\boldsymbol{B}^*\cdot(\nabla\times\varphi)\psi(t)d\boldsymbol{x}dt \\
& +R_m\int_0^T\int_\Omega \boldsymbol{u}\times\boldsymbol{B}^*\cdot(\nabla\times\varphi)\psi(t)d\boldsymbol{x}dt. \qquad (3.32)
\end{aligned}$$

Further, we shall prove

$$\int_0^T\int_\Omega(\nabla\cdot\boldsymbol{B}^*)\varphi(\boldsymbol{x})\psi(t)d\boldsymbol{x}dt = 0, \quad \forall\ \varphi\in(L_0^2(\Omega))^3,\ \psi\in C_0^\infty(0,T), \qquad (3.33)$$

which together with (3.32) and the definitions of $\boldsymbol{B}(\beta,g)$ and $p(\beta,g)$ in (3.1) yields that

$$\boldsymbol{B}^* = \boldsymbol{B}(\beta,g) \quad \text{and} \quad p^* = p(\beta,g). \qquad (3.34)$$

Indeed, for any $\varphi\in L_0^2(\Omega)^3$ and $\psi\in C_0^\infty(0,T)$, let $\tilde{q}_h = \Pi_h\varphi - \frac{1}{|\Omega|}\int_\Omega\Pi_h\varphi d\boldsymbol{x}$. Then $\tilde{q}_h\in Q_{0h}$ and we get by (2.7) and the divergence theorem that

$$\int_\Omega(\nabla\cdot\boldsymbol{B}_h^n)\Pi_h\varphi d\boldsymbol{x} = \int_\Omega(\nabla\cdot\boldsymbol{B}_h^n)\tilde{q}_h d\boldsymbol{x} + \frac{1}{|\Omega|}\int_\Omega\Pi_h\varphi d\boldsymbol{x}\int_\Omega\nabla\cdot\boldsymbol{B}_h^n d\boldsymbol{x} = 0.$$

We can also derive

$$\int_0^T\int_\Omega(\nabla\cdot\hat{\boldsymbol{B}}_{h,\tau})\Pi_h\varphi\psi(t)d\boldsymbol{x}dt = \sum_{n=1}^M\int_{t_{n-1}}^{t_n}\left(\int_\Omega(\nabla\cdot\boldsymbol{B}_h^n)\Pi_h\varphi d\boldsymbol{x}\right)\psi(t)dt = 0.$$

Hence (3.33) immediately holds by taking $h,\tau\to 0$ in the above equation and making use of Lemma 1.2.4 and (3.27).

Now we will prove (3.18). Indeed, noting that $\boldsymbol{B}_{h,\tau}(\cdot, t_n) = \boldsymbol{B}_h^n$ by the definition of $\boldsymbol{B}_{h,\tau}$ in (2.8), we have

$$\begin{aligned}
& \tau \sum_{n=1}^{M} \int_\Omega (\boldsymbol{B}_h^n - \boldsymbol{B}^n)^2 d\boldsymbol{x} - \int_0^T \|\boldsymbol{B}_{h,\tau}(\cdot,t) - \boldsymbol{B}(\cdot,t)\|_\Omega^2 dt \\
= & \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \int_\Omega \{(\boldsymbol{B}_{h,\tau}(\cdot,t_n) - \boldsymbol{B}^n)^2 - (\boldsymbol{B}_{h,\tau}(\cdot,t) - \boldsymbol{B}(\cdot,t))^2\} d\boldsymbol{x} dt \\
\leq & C \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|(\boldsymbol{B}_{h,\tau}(\cdot,t_n) - \boldsymbol{B}_{h,\tau}(\cdot,t)) + (\boldsymbol{B} - \boldsymbol{B}^n)\| dt \right\}^{1/2} \\
= & C \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|\boldsymbol{B} - \boldsymbol{B}^n + (t_n - t)\partial_\tau \boldsymbol{B}_h^n\|_\Omega^2 dt \right\}^{1/2} \\
\leq & C \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|\boldsymbol{B} - \boldsymbol{B}^n\|_\Omega^2 dt \right\}^{1/2} + C \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} \|(t_n - t)\partial_\tau \boldsymbol{B}_h^n\|_\Omega^2 dt \right\}^{1/2} \\
\leq & C \left\{ \int_0^T \|\boldsymbol{B} - \boldsymbol{B}^n\|_\Omega^2 dt \right\}^{1/2} + C \left\{ \sum_{n=1}^{M} \int_{t_{n-1}}^{t_n} (t_n - t)^2 \|\partial_\tau \boldsymbol{B}_h^n\|_\Omega^2 dt \right\}^{1/2} \\
\leq & C \left\{ \int_0^T \|\boldsymbol{B} - \boldsymbol{B}^n\|_\Omega^2 dt \right\}^{1/2} + C\tau^{3/2} \left(\sum_{n=1}^{M} \|\partial_\tau \boldsymbol{B}_h^n\|_\Omega^2 \right)^{1/2} .
\end{aligned}$$

This together with Lemma 3.2.1, Lemma 1.2.5 and (3.25) implies that

$$\begin{aligned}
& \tau \sum_{n=1}^{M} \int_\Omega (\boldsymbol{B}_h^n - \boldsymbol{B}^n)^2 d\boldsymbol{x} \leq \int_0^T \|\boldsymbol{B}_{h,\tau}(\cdot,t) - \boldsymbol{B}(\cdot,t)\|_\Omega^2 dt \\
+ \quad & C \left\{ \int_0^T \|\boldsymbol{B} - \boldsymbol{B}^n\|_\Omega^2 dt \right\}^{1/2} + C\tau \left(\tau \sum_{n=1}^{M} \|\partial_\tau \boldsymbol{B}_h^n\|_\Omega^2 \right)^{1/2} \\
\longrightarrow \quad & 0 \quad \text{as} \quad h, \tau \to 0,
\end{aligned}$$

which completes the proof. ■

Finally, we are ready to establish the main convergence theorem.

Theorem 3.2.7 *Let $\{\beta_h^*, g_h^*\}_{h>0}$ be a sequence of minimizers to the discrete minimization problem* (2.6) *and suppose $z^\delta \in C(0,T; \boldsymbol{L}^2(\omega))$, then as h and τ tend to* 0*, each sequence of $\{\beta_h^*, g_h^*\}_{h>0}$ has a subsequence converging in $L^2(\Omega)$ to a minimizer of the continuous optimization problem* (3.2)*.*

Proof. The uniform boundedness of the sequence $\{\beta_h^*, g_h^*\}_{h>0}$ in $K_h \times L^2(\Omega)$ implies that there exists a subsequence, still denoted by $\{\beta_h^*, g_h^*\}_{h>0}$, and some element $(\beta^*, g^*) \in K \times L^2(\Omega)$ such that

$g_h^* \rightharpoonup g^*$ in $L^2(\Omega)$ $\beta_h^* \rightharpoonup \beta^*$ in $H^1(\Omega)$ and $\beta_h^* \to \beta^*$ in $L^2(\Omega)$ as $h,\ \tau \to 0$, Next we will show (β^*, g^*) is a minimizer of the continuous optimization problem (3.2). To do so, for any $(\beta, g) \in K \times L^2(\Omega)$, we define $(\beta_h, g_h) = (\Pi_h \beta, \Pi_h g)$, then we know (cf. [31]) that $(\beta_h, g_h) \in K \times L^2(\Omega)$ and

$$\beta_h \to \beta \quad \text{in} \quad H^1(\Omega) \quad \text{as} \quad h,\ \tau \to 0, g_h \to g \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad h,\ \tau \to 0.$$

Therefore we can deduce by Lemma 3.2.6 and the lower semi-continuity of a norm that

$$\begin{aligned}
J(\beta^*, g^*) &= \frac{1}{2}\int_0^T\int_\omega |\boldsymbol{B}(\beta^*, g^*) - \boldsymbol{z}^\delta|^2 d\boldsymbol{x}dt + \frac{\lambda_1}{2}\int_\Omega |\nabla\beta^*|^2 d\boldsymbol{x} + \frac{\lambda_2}{2}\int_\Omega |g^*|^2 d\boldsymbol{x} \\
&\le \lim_{h,\tau\to 0} \frac{\tau}{2}\sum_{n=0}^{M} \alpha_n \int_\omega |\boldsymbol{B}_h^n(\beta_h^*, g_h^*) - \boldsymbol{z}^{\delta,n}|^2 d\boldsymbol{x} + \liminf_{h\to 0} \frac{\lambda_1}{2}\int_\Omega |\nabla\beta_h^*|^2 d\boldsymbol{x} \\
&\quad + \liminf_{h\to 0} \frac{\lambda_2}{2}\int_\Omega |g_h^*|^2 d\boldsymbol{x} \\
&\le \liminf_{h,\tau\to 0} J_{h,\tau}(\beta_h^*, g_h^*) \\
&\le \liminf_{h,\tau\to 0} J_{h,\tau}(\beta_h, g_h) \\
&= \liminf_{h,\tau\to 0} \left\{ \frac{\tau}{2}\sum_{n=0}^{M} \alpha_n \int_\omega |\boldsymbol{B}_h^n(\beta_h, g_h) - \boldsymbol{z}^{\delta,n}|^2 d\boldsymbol{x} + \frac{\lambda_1}{2}\int_\Omega |\nabla\beta_h|^2 d\boldsymbol{x} + + \frac{\lambda_2}{2}\int_\Omega |g_h|^2 d\boldsymbol{x} \right\} \\
&= \frac{1}{2}\int_0^T\int_\omega |\boldsymbol{B}(\beta, g) - \boldsymbol{z}^\delta|^2 d\boldsymbol{x}dt + \frac{\lambda_1}{2}\int_\Omega |\nabla\beta|^2 d\boldsymbol{x} + \frac{\lambda_2}{2}\int_\Omega |g|^2 d\boldsymbol{x} \\
&= J(\beta, g).
\end{aligned}$$

This yields that (β^*, g^*) is a minimizer of the continuous problem (3.2). ■

Conclusion

In this work, we have discussed some parameter identification problems in a three-dimensional (3D) spherical dynamo equation. We have considered the inverse problem of recovering the magnetic diffusivity for a 3D spherical dynamo equation. The highly ill-posed inverse problem has been transformed into a stable minimization problem by using Tikhonov regularization and the existence and stability of the minimizers to the minimization problem has also been verified. Then the finite element approximation and its convergence have investigated. Then by using the same way, we have introduced and studied the case when we have two parameters unknown.

BIBLIOGRAPHY

[1] A.ALDOGHAITHER. *Methods and Algorithms for Solving Inverse Problems for Fractional Advection-Dispersion Equations*. A PhD thesis, King Abdullah University of Science and Technology, Thuwal, Kingdom of Saudi Arabia.

[2] P. ALQUIER, E. GAUTIER AND G. STOLTZ (EDS). *Inverse problems and high-dimensional estimation, Summer School, August 31-September 4, 2009*, vol. 203, Springer Science and Business Media, 2011.

[3] R. C. ASTER, B. BORCHERS AND C. H. THURBER. *Parameter estimation and inverse problems*, Elsevier, 2018.

[4] H.BREZIS, *Inverse problem in mathematical physics*, Vol., 1987.

[5] H.BREZIS, HAIM BRÉZIS. *Functional analysis, Sobolev spaces and partial differential equations*. Vol. 2. No. 3. New York: Springer, 2011.

[6] L. BEILINA AND Y.G. SMIRNOV (EDS). *Nonlinear and inverse problems in electromagnetics: PIERS 2017, St. Petersburg, Russia, May 22-25*, vol. 243, Springer, 2018.

[7] K. H. CHAN, K. ZHANG AND J. ZOU, *Spherical interface dynamos: mathematical theory, finite element approximation, and application*, SIAM J. Numer. Anal. 44 (2006), No. 5, pp. 1877-1902.

[8] P. CARDIN, P. OLSON, *An experimental approach to thermochemical convection in the Earth's core*, Geophys. Res. Lett., 19 (1992), pp. 1995–1998.

[9] JAMES CLERK MAXWELL, *A Treatise On Electricity And Magnotism* , 1873.

[10] P.G. DANILAEV. *Coefficient inverse problems for parabolic type equations And their applications*, VSP, Utrecht, 2001.

[11] ENGL ,H.W.,HANKE,M.,NEUBAUER,A., *Regularisation Of Inverse Problem*, Springer-verlag., (1996).

[12] M. Fischer, G. Gerbeth, A. Giesecke, F. Stefani, *Inferring basic parameters of the geodynamo from sequences of polarity reversals*, Inverse Problems 25 (2009), no. 6, 065011 (12 pages).

[13] G. A. Glatzmaier, P. H. Roberts, *A three-dimensional convective dynamo solution with rotating and finitely conducting inner core and mantle*, Phys. Earth Planet. Inter. 91, 63 (1995).

[14] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations* SpringerVerlag Berlin Heidelberg, 1986.

[15] W.P. Hawkes. *Advances in imaging and electron physics*, Academic press, 1998.

[16] J. Hadamard. *Sur les problemes aux derivees partielles et leur signification physique*, Bull. Univ. Princeton, 13, 1902.

[17] V. Isakov. *Inverse problems in partial differential equations*, Springer, New York, 1998.

[18] S.I. Kabanikhin. *Inverse and ill-posed problems. Theory and applications*, De Gruyter, Berlin, 2012.

[19] V. Kolehmainen,Tarvainen,T.,vauhkonen,M., *Computational Methods For inverse Problems* ,inverse problem and imaging ,6(3),373-407

[20] W. Kuang, J. Bloxham, *Numerical Modeling of Magnetohydrodynamic Convection in a Rapidly Rotating Spherical Shell: Weak and Strong Field Dynamo Action*, J. Comput. Phys., 153 (1999), pp. 51–81.

[21] J.Larmor, *How Could a Rotating Body such as the sun Become a Magnet?*, Reports of The British Association For The Advencement Of Science., 88th Meeting (1919), pp. 159-160.

[22] A.Londero, Afonso. *A Cut-Cell Implementation of the Finite Element Method in deal.* ii. 2015.

[23] D. Liu, W. Kuang, A. Tangborn, *High-order compact implicit difference methods for parabolic equations in geodynamo simulation*, Adv. Math. Phys. 2009, Art. ID 568296 (23 pages).

[24] H.K, Moffatt., *Magnetic Field Generation in Electrically Conducting Fluids* , 1978, Cambridge university Press

[25] D. Messaoudi,O. Said Ahmed ,K. Souley Agbodjan, Ting Cheng and Daijun Jiang , *numerical recovery of magnotic diffusivity in a three dimensional spherical dynamo equation*

[26] D. Messaoudi. *Parameter identification problems in some three dimensional partial differential equations.* A PhD thesis.

[27] F. D. M. Neto and A. J. da Silva Neto. *An introduction to inverse problems with applications*, Springer Science and Business Media, 2012.

[28] M. Schrinner, K.-H. Rädler, D. Schmitt, M. Rheinhardt, U. R. Christensen, *Mean-field concept and direct numerical simulations of rotating magnetoconvection and the geodynamo*, Geophys. Astrophys. Fluid Dyn. 101 (2007), no. 2, 81–116.

[29] L. R. Scott, S. Zhang, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Math. Comp., 54 (1990), pp. 483–493

[30] R. Temam , *Sur l'approximation de la solution des équations de Navier-Stokes par la methode des pas fractionnaires II* , Arch. Rat. Mech. Anal., 33 (1969), 377-385

[31] J. Xu, *Theory of multilevel methods*, PhD Thesis, Cornell University, 1989.

[32] J. L. Xie and J. Zou, *Numrecal reconstruction of heat fluxes*, SIAM J. Numer. Anal., 43 (2005), 1504-1535.

[33] E. Zeidler, *Nonlinear Functional Analysis and Its Applications*, Vol. 2, Springer-Verlag, New York, 1985.

ملخص

في هذا العمل، نقدم مسألة تحديد ذات معلمتين. تتضمن استعادة توزيع التوصيل المغناطيسي وقوة المصدر في معادلة دينامو كروية ثلاثية الأبعاد نحول المسألة التي تعاني من سوء التصيّد إلى مسألة تقليص مستقرة باستخدام التنظيم بواسطة Tikhonov، ثم نثبت الصحة الجيدة. نقوم بتطوير مخطط كامل استنادًا إلى طريقة العناصر المحددة.

الكلمات المفتاحية مسألة تحديد المعلمات،مسألة سوء التصيّد، تنظيم Tikhonov، طريقة العناصر المحددة: .

Résumé

Dans ce travail, nous présentons un problème d'identification à deux paramètres, impliquant la récupération de la diffusivité magnétique et de la force de la source dans une équation de dynamo sphérique tridimensionnelle (3D). Nous transformons le problème mal posé en un problème de minimisation stable en utilisant la régularisation de Tikhonov, puis établissons la bien-poséité. Nous développons un schéma entièrement discret basé sur la méthode des éléments finis.

Mots clés : problème d'identification à paramètres, problème mal posé, régularisation de Tikhonov, méthode des éléments finis

Abstract

In this work, we present a two-parameter identification problem. Which involves recovering the magnetic diffusivity and source strength in a three-dimensional (3D) spherical dynamo equation. We transform the ill posed problem into a stable minimization problem by using Tikhonov regularization, We then establish the well-posedness. We develop a fully discrete scheme based on the finite element method.

. **Key words:** parameter identification problem ,ill-posed problem , Tikhonov regularization, finite element method.