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Double-modified Laplace transform to solve a nonlinear modified Boussinesq equation.

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Dedication

With deep affection and sincere gratitude, I dedicate this achievement to the source of tenderness and generosity, my beloved parents, who have been the support and light in my educational journey, and to those who shared the burdens of the days with me, my sisters and brothers, who were both friends and motivators at every step.

I also dedicate my efforts to my honorable teachers, who honed my talents and broadened my horizons with beneficial knowledge, and to my companions on the path, my dear colleagues, who formed a wonderful chapter in the story of this journey.

I extend my thanks to all those in whom I found goodness and kindness, and to all the sincere hearts that prayed for me silently, for with you, I found the strength and motivation to overcome obstacles and achieve this success.

In conclusion, I dedicate this contribution to all those around me, to every soul that believed in me, and every hand that supported me. I am grateful for every moment you stood by my side. My heartfelt thanks and utmost respect to you all.

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Words fall short, but my heart remains grateful for the kindness I have received. Thank you to all the loving and advising souls who have surrounded me. My sincerest thanks and prayers are with you all.

Contents

ملخص

الهدف الرئيسي من هذه المذكرة هو الحصول على نتائج رقمية باستخدام طريقة تحويل لابلاس المزدوجة المعدلة لنماذج بوسينيسک الخطية وغير الخطية مع شروط ابتدائية وحدودية. ندرس تفرد الحل باستخدام طريقة متباينة الطاقة في بعض فضاءات سوبوليف. يتم تقديم بعض الأمثلة لتوضيح فعالية الطريقة في حل المعادلات التفاضلية الجزئية.
الكلمات المفتاحية: التقدير القبلي؛ طريقة تحليل لابلاس المزدوج؛ تحويل لابلاس العكسي؛ معادلة بوسينيسک الخطية المعدلة العامة؛ معادلة بوسينيسک غير الخطية.

Résumé

L'objectif principal de cette mémoire est d'obtenir des résultats numériques grâce à la méthode modifiée de double transformation de Laplace pour les modèles de Boussinesq linéaires et non linéaires avec conditions initiales aux limites. Nous étudions l'unicité de la solution par la méthode d'inégalité d'énergie dans certains espaces de Sobolev. Quelques exemples sont fournis pour illustrer l'efficacité de la méthode dans la résolution des équations aux dérivées partielles.
les mots clés : Estimation a priori ; Méthode de décomposition par double transformée de Laplace ; Transformée de Laplace inverse ; Équation de Boussinesq linéaire modifiée généralisée ; Équation de Boussinesq non linéaire.

Abstract

The main objective of this thesis is to obtain numerical results through the modified double Laplace transform method for linear and nonlinear Boussinesq models with initial boundary conditions. We study the uniqueness of the solution by the energy inequality method in some Sobolev spaces. Some examples are provided to illustrate the effectiveness of the method in solving partial differential equations.
keywords: A priori estimate; Double Laplace decomposition method; Inverse Laplace transform; Generalized modified linear Boussinesq equation; Nonlinear Boussinesq equation.

Introduction

The Boussinesq equations, applicable in both one and higher dimensions, are widely used in coastal and ocean engineering to model tidal oscillations and tsunami waves. Classified as hyperbolic equations similar to nonlinear shallow water equations, they were initially formulated for modeling water waves. These equations describe the irrotational motion of an incompressible fluid in the long wave limit and are derived from the Navier-Stokes equations. Beyond fluid dynamics, Boussinesq equations also appear in contexts such as acoustic, elastic, electromagnetic, and gravitational waves. For further developments in one- and multi-dimensional spaces, refer to the works by Wei et al. [6], Madsen and Schaffer [?], Guido Schneider [?], Nwogu [5], and Kirby [10].

Over the past thirty years, numerous methods have been developed and employed to solve Boussinesq equations. These include the homotopy analysis and homotopy perturbation methods (Francisco and Fernández [19], Gupta and Saha [20], and Dianhen et al. [24]), the analytic method [?], the modified decomposition method (Wazwaz [8], Fang et al. [21], and Basem and Attili [12]), the Laplace-Adomian Decomposition Method (Hardik et al. [22], Zhang et al. [11], Liang et al. [?]), the transformed rational function method (Wang [7], Engui [9]), the integral transform method (Charles et al. [3]), the energy integral method (Joseph [2], Mesloub [15]), and the inverse scattering method (Peter et al. [23]). Additionally, various numerical methods have been employed to explore problems involving Boussinesq equations, as evidenced by the works of Jang [25], Iskandar and Jain [4], Bratsos [14], Dehghan and Salehi [16], Boussinesq [1], and Onorato et al. [13]. For discussions on solution bifurcation and potential applications of Boussinesq equations, references [17, 18, 26] are recommended.

The primary goal of this work is to apply the modified double Laplace decomposition method to solve a singular generalized modified linear Boussinesq equation and a singular nonlinear Boussinesq equation. Additionally, we derive an a priori estimate for the solution and provide examples to validate and demonstrate the effectiveness of the modified double Laplace decomposition method.

This thesis is organized as follows:

Chapter 1 introduces the tools to be used in the subsequent sections.

In Chapter 2, we present the first problem ([11])–([7]), which involves an initial boundary value problem for a singular modified linear Boussinesq equation with a Bessel operator. We establish an a priori bound for the solution of problem ([11])–([7]), from which we deduce the uniqueness of solutions in a weighted Sobolev space.

Chapter 3 discusses the application of the modified double Laplace decomposition method to solve the posed problem ([11])–([7]), with an example provided to illustrate the method. We then address an initial value problem for the one-dimensional singular nonlinear Boussinesq equation. The modified double Laplace decomposition method is again employed to solve this nonlinear problem, and an example is provided to validate the method in the final section.

Chapter 1

Preliminaries

1.1 Function spaces :

Definition 1.1.1. *Let* $L_\rho^2(Q)$ *be the weighted* $L^2(Q)$ *Hilbert space of square integrable functions on* $Q = (0,1) \times (0,T), T < \infty$ *, with scalar product*

$$(\varphi, \psi)_{L_\rho^2(Q)} = \int_Q x\varphi\psi dxdt \quad , \quad \rho = x, \tag{1.1}$$

and with the associated finite norm

$$\|\varphi\|^2_{L_\rho^2(Q)} = \int_Q x\varphi^2 dxdt, \tag{1.2}$$

and let $W_{2,\rho}^{1,1}$ *be the weighted Hilbert space consisting of the elements* φ *of* $L_\rho^2(Q)$ *having first order generalized derivatives square summable on* Q *. the space* $W_{2,\rho}^{1,1}(Q)$ *is equipped with the scalar product*

$$(\varphi, \psi)_{W_{2,\rho}^{1,1}(Q)} = (\varphi, \psi)_{L_\rho^2(Q)} + (\varphi_x, \psi_x)_{L_\rho^2(Q)} + (\varphi_t, \psi_t)_{L_\rho^2(Q)}, \tag{1.3}$$

and the associated norm is

$$\|\varphi\|^2_{W_{2,\rho}^{1,1}(Q)} = \|\varphi\|^2_{L_\rho^2(Q)} + \|\varphi_x\|^2_{L_\rho^2(Q)} + \|\varphi_t\|^2_{L_\rho^2(Q)}. \tag{1.4}$$

We also use the weighted spaces on $(0,1)$, such as $L_\rho^2((0,1))$ and $W_{2,\rho}^{1,1}((0,1))$, whose definitions are analogous to the spaces on Q .

1.2 Laplace transform :

1.2.1 Laplace Transform Definition:

The Laplace transformation is a technique for solving differential equations. It involves transforming a time-domain differential equation into an algebraic equation in the frequency domain. After solving this algebraic equation, the result is transformed back into the time domain to obtain the final solution. Essentially, the Laplace transformation serves as a shortcut for solving differential equations.

Many kinds of transformations already exist but Laplace transforms and Fourier transforms are the most well known. The Laplace transforms is usually used to simplify a differential equation into a simple and solvable algebra problem. Even when the algebra becomes a little complex, it is still easier to solve than solving a differential equation.

To understand the Laplace transform formula: First Let $f(t)$ be the function of t, time for all $t \geq 0$.

Then the Laplace transform of $f(t), F(s)$ can be defined as

$$F(s) = \int_0^\infty f(t) \cdot e^{-st} dt. \tag{1.5}$$

Provided that the integral exists. Where the Laplace Operator, $s = \sigma + iw$; will be real or complex $\mathrm{i}^2 = -1$.

While powerful, the Laplace transform has limitations, such as being applicable only to differential equations with known constants. Without these constants, the method cannot be used, and alternative solutions must be sought.

1.2.2 Method of Laplace Transform:

In control system engineering, the Laplace transform is crucial for analyzing time functions. The inverse Laplace transform is equally important for deriving time-domain functions from their frequency-domain forms, with several properties beneficial for linear systems analysis.

Linearity, Differentiation, integration, multiplication, frequency shifting, time scaling, time shifting, convolution, conjugation, periodic function. There are two very important theorems associated with control systems. These are : Initial value theorem (IVT), and final value theorem (FVT).

The Laplace transform is performed on a number of functions, which are – impulse, unit impulse, step, unit step, shifted unit step, ramp, exponential decay, sine, cosine, hyperbolic sine, hyperbolic cosine, natural logarithm, Bessel function. But the greatest advantage of applying the Laplace transform is solving higher order differential equations easily by converting into algebraic equations.

There are certain steps which need to be followed in order to do a Laplace transform of a time function. In order to transform a given function of time $f(t)$ into its corresponding Laplace transform, we have to follow the following steps:

- First multiply $f(t)$ by e^{-st}, s being a complex number $(s = \sigma + iw)$.

- Integrate this product w.r.t time with limits as zero and infinity. This integration results in Laplace transformation of $f(t)$, which is denoted by $F(s)$. Laplace transform of $f(t) = L[f(t)] = F(s) = \int_0^\infty f(t) \cdot e^{-st} dt$ when $t \geq 0$.

The time function $f(t)$ is obtained back from the Laplace transform by a process called inverse Laplace transformation and denoted by L^{-1} Inverse Laplace transform of $F(s) = L^{-1}[F(s)] = L^{-1}[Lf(t)] = f(t)$. where:

$$L^{-1}\{F\}(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} F(s) e^{st} ds, \tag{1.6}$$

1.2.3 Laplace Transform Properties

The main properties of Laplace Transform can be summarized as follows:

Linearity:

Let $\mathrm{C}_1, \mathrm{C}_2$ be constants. $\mathrm{f(t)}, \mathrm{g(t)}$ be the functions of time, t, then

$$L\left\{C_1 f(t) + C_2 g(t)\right\} = C_1 L\{f(t)\} + C_2 L\{g(t)\}. \tag{1.7}$$

First shifting Theorem:

$$\text{If } L\{f(t)\} = F(s), \quad \text{then } L\left\{e^{at} f(t)\right\} = F(s-a). \tag{1.8}$$

Change of scale property:

If $L\{f(t)\} = F(s)$, then

$$\begin{aligned} L\{f(at)\} &= \frac{1}{a} F\left(\frac{s}{a}\right); \text{ Frequency Scaling },\\ L\left\{f\left(\frac{t}{a}\right)\right\} &= aF(as) \quad ; \text{Time Scaling} . \end{aligned} \tag{1.9}$$

Differentiation:

If $L\{f(t)\} = F(s)$, then

$$L\frac{d^n}{dt^n} f(t) = s^n L\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{n-1}(0). \tag{1.10}$$

For eg. Let n = 1

$$L\left\{\frac{d}{dt} f(t)\right\} = sL\{f(t)\} - f(0). \tag{1.11}$$

Integration:

If $L\{f(t)\} = F(s)$, then

$$L\left[\iiint \int \cdots \int f(t) dt^n\right] = \frac{1}{s^n} L\{f(t)\} + \frac{f^{n-1}(0)}{s^n} + \frac{f^{n-2}(0)}{s^n} + \cdots + \frac{f^n(0)}{s}, \tag{1.12}$$

for eg. Let n = 1

$$L\left\{\int_0^t f(t)\right\} = \frac{1}{s}L\{f(t)\} + \frac{f'(0)}{s}. \tag{1.13}$$

Product:

If $L\{f(t)\} = F(s)$, then the product of two functions, $f_1(t)$ and $f_2(t)$ is

$$L\left\{f_1(t)f_2(t)\right\} = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F_1(w)F_2(w)dw. \tag{1.14}$$

Final Value Theorem:

$$\text{If } L\{f(t)\} = F(s), \text{ then } \lim_{s\to\infty} f(t) = \lim_{s\to 0} sF(s).$$

This theorem is applicable in the analysis and design of feedback control system, as Laplace Transform gives solution at initial conditions

Initial Value Theorem:

$$\text{If } L\{f(t)\} = F(s), \text{ then } f\left(0^+\right) = \lim_{s\to\infty} sF(s).$$

Let us examine the Laplace transformation methods of a simple function $f(t) = e^{at}$ for better understanding the matter.

$$\begin{aligned}
L\left[e^{-\alpha t}\right] &= \int_0^\infty e^{-\alpha t}\cdot e^{-st}dt = \int_0^\infty e^{-(s+\alpha)t}dt,\\
\Rightarrow L\left[e^{-\alpha t}\right] &= \frac{-1}{s+\alpha}\left[e^{-(s+\alpha)t}\right]_0^\infty = \frac{-1}{s+\alpha}\left[e^{-(s+\alpha)\infty} - e^{-(s+\alpha)0}\right],\\
\Rightarrow L\left[e^{-\alpha t}\right] &= \frac{-1}{s+\alpha}[0-1] = \frac{1}{s+\alpha}.
\end{aligned} \tag{1.15}$$

Comparing the above solution, we can write, Laplace transform of

$$L\left[e^{-(-\alpha)t}\right] = \frac{1}{s+(-\alpha)} = \frac{1}{s-\alpha}. \tag{1.16}$$

Similarly, by putting $\alpha = 0$, we get, Laplace transform of $e^{0t} = L\left[e^0\right] = [1] = \frac{1}{s-0} = \frac{1}{s}$,
Hence, Inverse Laplace transform of $\frac{1}{s} = L^{-1}\left[\frac{1}{s}\right] = 1$.

1.2.4 Laplace Transform Table:

There is always a table that is available to the engineer that contains information on the Laplace transforms. An example of Laplace transform table has been made below. We will come to know about the Laplace transform of various common functions from the following table .

$$
\begin{aligned}
L[t] &= \frac{1}{s^2}, \\
L\left[t^2\right] &= \frac{2}{s^3}, \\
L\left[t^3\right] &= \frac{6}{s^4}, \\
L\left[t^n\right] &= \frac{n!}{s^{n+1}}, \\
L\left[e^{-\alpha t}\right] &= \frac{1}{s+\alpha}, \\
L\left[te^{-\alpha t}\right] &= \frac{1}{(s+\alpha)^2}, \\
L\left[te^{\alpha t}\right] &= \frac{1}{(s-\alpha)^2}, \\
L\left[t^n e^{-\alpha t}\right] &= \frac{n!}{(s+\alpha)^{n+1}}, \\
L[\cos wt] &= \frac{s}{s^2+w^2}, \\
L[\sin wt] &= \frac{w}{s^2+w^2}, \\
L\left[e^{-\alpha t}\sin wt\right] &= \frac{w}{(s+\alpha)^2+w^2}, \\
L\left[e^{-\alpha t}\cos wt\right] &= \frac{s+\alpha}{(s+\alpha)^2+w^2}, \\
L[\sinh \alpha t] &= \frac{\alpha}{s^2-\alpha^2}, \\
L[\cosh \alpha t] &= \frac{s}{s^2-\alpha^2}.
\end{aligned}
\tag{1.17}
$$

1.3 Double Laplace transform :

The double Laplace transform $F(p,s)$ of a function $f(x,t)$ is defined by

$$L_x L_t [f(x,t)] = F(p,s) = \int_0^\infty e^{-px} \int_0^\infty e^{st} f(x,t)\, dtdx, \tag{1.18}$$

where $x, t > 0$ and p, s are complex values, and further double Laplace transform of the first order partial derivatives for a function u is given by

$$L_x L_t \left[\frac{\partial u(x,t)}{\partial x}\right] = pU(p,s) - U(0,s,) \tag{1.19}$$

where $U(p,s)$ is the double Laplace transform of $u(x,t)$ Similarly, the double Laplace transform for second partial derivative with respect to x and t are defined by

$$\begin{aligned} L_x L_t\left[\frac{\partial^2 u(x,t)}{\partial^2 x}\right] &= p^2 U(p,s) - pU(0,s) - \frac{\partial U(p,0)}{\partial x}, \\ L_x L_t\left[\frac{\partial^2 u(x,t)}{\partial^2 t}\right] &= s^2 U(p,s) - sU(0,s) - \frac{\partial U(p,0)}{\partial t}. \end{aligned} \tag{1.20}$$

The double Laplace transform of the functions $x\dfrac{\partial^2 \psi}{\partial t^2}$ and $xf(x,t)$ are respectively given by

$$L_x L_t\left(x\frac{\partial^2 \psi}{\partial t^2}\right) = -\frac{d}{dp}\left[s^2\Psi(p,s) - s\Psi(p,0) - \Psi_t(p,0)\right], \tag{1.21}$$

and

$$L_x L_t\left(xf(x,t)\right) = -\frac{dF(p,s)}{dp}. \tag{1.22}$$

The double Laplace transform of the non-constant coefficient second order partial derivative $x^n\frac{\partial^2\psi}{\partial t^2}$ and the function $x^n f(x,t)$ are given by

$$L_x L_t\left(x^n\frac{\partial^2 \psi}{\partial t^2}\right) = (-1)^n\frac{d}{dp}\left[s^2\Psi(p,s) - s\Psi(p,0) - \Psi_t(p,0)\right], \tag{1.23}$$

$$L_x L_t\left(x^n f(x,t)\right) = (-1)^n\frac{d}{dp}\left[L_x L_t f(x,t)\right] = (-1)^n\frac{d^n F(p,s)}{dp^n}, \tag{1.24}$$

where $n = 1, 2, 3, ...$

The inverse double Laplace transform $L_p^{-1}L_s^{-1}\left[F(p,s)\right] = f(x,t)$ is defined by the complex double integral formula

$$L_p^{-1}L_s^{-1}\left[F(p,s)\right] = f(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}dp\frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}ds, \tag{1.25}$$

where $F(p,s)$, must be an analytic function for all p and s in the region defined by the inequalities $\mathrm{Re}p \geq c$ and $\mathrm{Re}s \geq d$, where c and d are real constants to be chosen suitably.

1.4 Young's inequality with ε :

For any $\varepsilon > 0$, we have the inequality

$$ab \leq \frac{1}{p}|\varepsilon a|^p + \frac{p-1}{p}\left|\frac{b}{\varepsilon}\right|^{\frac{p}{p-1}}, \; a,b \in R \;,\; p > 1, \tag{1.26}$$

which is the generalization of *Cauchy inequality with* ε .

1.5 Gronwall's Lemma :

If $f_i(\tau)\,(i=1,2,3)$ are nonnegative functions on $(0,T)$, and $f_1(\tau)\,, f_2(\tau)$ are integrable functions , and $f_3(\tau)$ is non-decreasing on $(0,T)$, then if

$$\Im_\tau f_1 + f_2(\tau) \leq f_3(\tau) + c\Im_\tau f_2,$$

then

$$\Im_\tau f_1 + f_2(\tau) \leq \exp(c\tau)\,.f_3(\tau),$$

where

$$\Im_\tau g(t) = \int_0^\tau g(t)dt.$$

1.6 Poincaré type inequalities :

$$(i)\int_0^1 (\Im_x(\xi u))^2 dx \leq \frac{l^3}{2}\|u(.,t)\|^2_{L^2_\rho(0,l)}\,,$$

$$(ii)\int_0^1 \left(\Im_x^2(\xi u)\right)^2 dx \leq \frac{l^2}{2}\|\Im_x(\xi u)\|^2_{L^2(0,l)}\,,$$

$$(iii)\int_0^1 x(\Im_x(\xi u))^2 dx \leq l\,\|\Im_x(\xi u)\|^2_{L^2(0,l)}\,,$$

where

$$\Im_x(\xi u(\xi,t)) = \int_0^x \xi u(\xi,t)\,d\xi \quad , \quad \Im_x^2(\xi u(\xi,t)) = \int_0^x\int_0^\xi \eta u(\eta,t)\,d\eta d\xi$$

.

Chapter 2

Uniqueness of solution

2.1 Problem Setting for a Singular Generalized Improved Modified Linear Boussinesq Equation

In the rectangle $Q = (0,1) \times (0,T), T < \infty$ we consider an initial boundary value problem for the singular generalized improved modified linear Boussinesq equation with damping and with Bessel operator

$$\begin{aligned} \mathcal{L}_{\psi} &= \frac{\partial^2 \psi}{\partial t^2} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right) - \frac{1}{x}\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial \psi}{\partial x}\right) - \frac{1}{x}\frac{\partial^3}{\partial x \partial t^2}\left(x\frac{\partial \psi}{\partial x}\right) \\ &= f(x,t), \end{aligned} \tag{2.1}$$

$$\psi(x,0) = f_1(x) \ , \ \frac{\partial \psi(x,0)}{\partial x} = f_2(x) \ , \ x \in (0,1), \tag{2.2}$$

$$\left\{ \begin{array}{l} \psi(1,t) = 0 \ , \ t \in (0,T), \\ \psi(0,t) = 0 \ , \ t \in (0,T), \end{array} \right. \tag{2.3}$$

where $f_1(\tau), f_2(\tau)$, and $f_3(\tau)$ are given functions that satisfy certain conditions which will be specified later on. For the solution of problem ([11])–([7]) and use the modified double Laplace decomposition method for solving it.

2.2 A Priori Estimate for the Solution of Problem ([11])–([7])

In this section, we establish an a priori estimate for the solution of problem ([11])–([7]) from which we deduce the uniqueness of the solution.

Theorem 1. *The solution ψ of the initial boundary value problem ([11])–([7]) satisfies the a priori estimate*

$$\sup_{0 \le \tau \le T} \|\psi(.,\tau)\|^2_{W^{1,1}_{2,\rho}(0,l)} \le 2e^{2T}\left(\|f_1\|^2_{W^1_{2,\rho}(0,l)} + \|f_2\|^2_{W^1_{2,\rho}(0,l)} + \|f\|^2_{L^2_\rho(Q)}\right). \tag{2.4}$$

Proof. We consider the scalar product in $L^2(Q^\tau)$ of the operators $\mathcal{L}\psi$ and $M\psi$,where

$$M\psi = x\psi_t$$

With $Q^\tau = (0,l) \times (0,\tau)$, $0 \leq \tau \leq T$, $0 < l < \infty$ we obtain

$$\begin{aligned}(\mathcal{L}\psi, M\psi)_{L^2(Q^\tau)} &= (\psi_{tt}, x\psi_t)_{L^2(Q^\tau)} - ((x\psi_x)_x, \psi_t)_{L^2(Q^\tau)} \\ &\quad - ((x\psi_x)_{xt}, \psi_t)_{L^2(Q^\tau)} - ((x\psi_x)_{xtt}, \psi_t)_{L^2(Q^\tau)}.\end{aligned} \tag{2.5}$$

We obtain the terms of the right side of equation (2.5) as follows

$$\left(\frac{\partial^2\psi}{\partial t^2}, x\psi_t\right)_{L^2(Q^\tau)} = (\psi_{tt}, x\psi_t)_{L^2(Q^\tau)}.$$

$$\begin{aligned}\left(-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right), x\psi_t\right)_{L^2(Q^\tau)} &= -\left(\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right), \psi_t\right)_{L^2(Q^\tau)}, \\ &= -\left(\left(x\frac{\partial\psi}{\partial x}\right)_x, \psi_t\right)_{L^2(Q^\tau)}, \\ &= -((x\psi_x)_x, \psi_t)_{L^2(Q^\tau)}.\end{aligned}$$

$$\left(-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right), x\psi_t\right)_{L^2(Q^\tau)} = -((x\psi_x)_{xt}, \psi_t)_{L^2(Q^\tau)}.$$

$$\left(-\frac{1}{x}\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right), x\psi_t\right)_{L^2(Q^\tau)} = -((x\psi_x)_{xtt}, \psi_t)_{L^2(Q^\tau)}.$$

By using initial and boundary conditions (2.2) and (2.3), terms on the right hand side of (**??**) can be evaluated as follows:

$$\begin{aligned}(\psi_{tt}, x\psi_t)_{L^2(Q^\tau)} &= \int_0^1\int_0^\tau \psi_{tt} \times x\psi_t dxdt, \\ &= \int_0^1 x\left(\int_0^\tau \psi_{tt} \times \psi_t dt\right)dx, \\ &= \int_0^1 x\left[\frac{\psi_t^2(x,t)}{2}\right]_0^\tau dx, \\ &= \int_0^1 x\left[\frac{\psi_t^2(x,\tau)}{2} - \frac{\psi_t^2(x,0)}{2}\right]dx, \\ &= \frac{1}{2}\int_0^1 x\psi_t^2(x,\tau)\,dx - \frac{1}{2}\int_0^1 x\psi_t^2(x,0)\,dx,\end{aligned}$$

$$(\psi_{tt}, x\psi_t)_{L^2(Q^\tau)} = \frac{1}{2}\|\psi_t(.,\tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2}\|f_2\|^2_{L^2_\rho(0,1)}. \tag{2.6}$$

$$\begin{aligned}
-((x\psi_x)_x, \psi_t)_{L^2(Q^\tau)} &= -\int_0^\tau\int_0^1 (x\psi_x)_x\psi_t dxdt,\\
&= -[x\psi_x \times \psi_t]_0^1 + \int_0^\tau\int_0^1 (x\psi_x)\times(\psi_t)_x dxdt,\\
&= \int_0^\tau\int_0^1 x\psi_x \times (\psi_x)_t dxdt,\\
&= \int_0^1 x\int_0^\tau \psi_x(x,t)\times(\psi_x(x,t))_t dtdx,\\
&= \int_0^1 x\left[\frac{\psi_x^2(x,t)}{2}\right]_0^\tau dx,\\
&= \int_0^1 x\left[\frac{\psi_x^2(x,\tau)}{2} - \frac{\psi_x^2(x,0)}{2}\right]dx,
\end{aligned}$$

$$-((x\psi_x)_x, \psi_t)_{L^2(Q^\tau)} = \frac{1}{2}\|\psi_x(.,\tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2}\left\|\frac{\partial f_1}{\partial x}\right\|^2_{L^2_\rho(0,1)}. \tag{2.7}$$

$$-((x\psi_x)_{xt}, \psi_t)_{L^2(Q^\tau)} = \|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)}, \tag{2.8}$$

$$\begin{aligned}
&-((x\psi_x)_{xt}, \psi_t)_{L^2(Q^\tau)} = -\int_0^\tau\int_0^1 (x\psi_x)_{xt}\psi_t dxdt,\\
&= -[(x\psi_x)_t\psi_t]_0^1 + \int_0^\tau\int_0^1 (x\psi_x)_t(\psi_t)_x dxdt,\\
&= \int_0^\tau\int_0^1 x(\psi_{xt})^2 dx,\\
&= \|\psi_{tx}\|^2_{L^2_p(Q^\tau)}.
\end{aligned}$$

$$-((x\psi_x)_{xtt}, \psi_t)_{L^2(Q^\tau)} = \frac{1}{2}\|\psi_{xt}(.,\tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2}\left\|\frac{\partial f_2}{\partial x}\right\|^2_{L^2_\rho(0,1)}, \tag{2.9}$$

$$\begin{aligned}
&-((x,\psi_x)_{xtt}\cdot\psi_t)_{L^2(Q^\tau)} = -\int_0^\tau\int_0^1 (x\psi_x)_{ttx}\psi_t dxdt,\\
&= -[(x\psi_x)_{tt}\psi_t]_0^1 + \int_0^\tau\int_0^1 (x\psi_x)_{tt}\cdot(\psi_t)_x dxdt,
\end{aligned}$$

$$= \int_0^1 x \int_0^\tau (\psi_x)_{tt} \cdot \psi_{xt} dt dx,$$
$$= \int_0^1 x \int_0^\tau (\psi_{xt})_t \cdot \psi_{xt} dt dx,$$
$$= \frac{1}{2} \int_0^1 x \left[\psi_{xt}^2\right]_0^\tau dx,$$

$$= \frac{1}{2} \int_0^1 x\psi_{xt}^2 (x, \tau) dx - \frac{1}{2} \int_0^1 x\psi_{xt}^2 (x, 0) dx,$$
$$= \frac{1}{2} \|\psi_{xt}(., \tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2} \left\|\frac{\partial f_2}{\partial x}\right\|^2_{L^2_\rho(0,1)}.$$

Combination of (2.5–2.9), lead to

$$(\mathcal{L}\psi, x\psi_t)_{L^2(Q^\tau)} = \frac{1}{2} \|\psi_t(., \tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2} \|f_2\|^2_{L^2_\rho(0,1)'} + \frac{1}{2} \|\psi_x(., \tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2} \left\|\frac{\partial f_1}{\partial x}\right\|^2_{L^2_\rho(0,1)} +$$
$$+ \|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)} + \frac{1}{2} \|\psi_{xt}(., \tau)\|^2_{L^2_\rho(0,1)} - \frac{1}{2} \left\|\frac{\partial f_2}{\partial x}\right\|^2_{L^2_\rho(0,1)}.$$
$$= (f, x\psi_t)_{L^2(Q^\tau)}.$$

We take the right side of equation (2.2) and apply Cauchy $-\varepsilon$ inequality to it, and we find

$$= (f, x\psi_t)_{L^2(Q^\tau)} = \int_0^1 \int_0^\tau xf\psi_t dxdt = \int_{Q^\tau} \left(\sqrt{x}f\right)\left(\sqrt{x}\psi_t\right) dxdt,$$
$$\leq \frac{1}{2} \int_{Q^\tau} \left(\sqrt{x}f\right)^2 dxdt + \frac{1}{2} \int_{Q\tau} \left(\sqrt{x}\psi_t\right)^2 dxdt,$$
$$\leq \frac{1}{2} \int_{Q^\tau} xf^2 dxdt + \frac{1}{2} \int_{Q\tau} x\psi_t^2 dxdt,$$
$$\leq \frac{1}{2} \|f\|^2_{L^2_p(Q^\tau)} + \frac{1}{2} \|\psi_t\|^2_{L^2_p(Q^\tau)}.$$

We now consider the elementary inequality

$$\|\psi(., \tau)\|^2_{L^2_\rho(0,1)} \leq \|f_1\|^2_{L^2_\rho(0,1)} + \|\psi_t\|^2_{L^2_\rho(Q^\tau)} + \|\psi\|^2_{L^2_\rho(Q^\tau)}. \tag{2.10}$$

By summing inequalities (2.2),(**??**) and 2.10 side to side, we obtain

$$\begin{aligned} &\|\psi(., \tau)\|^2_{W^{1,1}_{2,\rho}(0,1)} + \|\psi_{xt}(., \tau)\|^2_{L^2_\rho(0,1)} + \|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)} \\ &\leq 2 \left(\|f_2\|^2_{W^1_{2,\rho}(0,1)} + \|f_1\|^2_{W^1_{2,\rho}(0,1)} + \|f\|^2_{L^2_\rho(Q^\tau)} + \|\psi_t\|^2_{L^2_\rho(Q^\tau)} + \|\psi\|^2_{L^2_\rho(Q^\tau)}\right). \end{aligned} \tag{2.11}$$

Application of Gronwall's lemma to inequality (2.11) with

$$\begin{cases} \Im_\tau F_1 = \|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)}, \\ F_2(\tau) = \|\psi_t(., \tau)\|^2_{L^2_\rho(0,1)} + \|\psi(., \tau)\|^2_{L^2_\rho(0,1)}, \\ F_3(\tau) = \|f_2\|^2_{W^1_{2,\rho}(0,1)} + \|f_1\|^2_{W^1_{2,\rho}(0,1)} + \|f\|^2_{L^2_\rho(Q^\tau)}, \end{cases} \tag{2.12}$$

gives

$$
\begin{aligned}
&\|\psi(.,\tau)\|^2_{W^{1,1}_{2,\rho}(0,1)} + \|\psi_{xt}(.,\tau)\|^2_{L^2_\rho(0,1)} + \|\psi_{xt}\|^2_{L^2_\rho(Q^\tau)} \\
\leq\ & 2e^{2\tau}\left(\|f_1\|^2_{W^1_{2,\rho}(0,l)} + \|f_2\|^2_{W^1_{2,\rho}(0,l)} + \|f\|^2_{L^2_\rho(0,1)}\right).
\end{aligned}
\tag{2.13}
$$

By discarding the last two terms in the left-hand side of (2.13) and then taking the upper bound for both sides with respect to τ over $[0, T]$ of the obtained inequality, we obtain the following a priori estimate for the solution of the posed problem ([11])–([7])

$$
\begin{aligned}
&\sup_{0\leq\tau\leq T}\|\psi(.,\tau)\|^2_{W^{1,1}_{2,\rho}(0,1)} \\
\leq\ & 2e^{2T}\left(\|f_1\|^2_{W^1_{2,\rho}(0,1)} + \|f_2\|^2_{W^1_{2,\rho}(0,1)} + \|f\|^2_{L^\rho_2(0,1)}\right).
\end{aligned}
\tag{2.14}
$$

□

Corollary 1. *Let ψ_1 and ψ_2 be two solutions of problem* (2.1) − (2.3)*:*

$$
\begin{cases} L\psi_1 = f \\ L\psi_2 = f \end{cases} \implies L(\psi_1 - \psi_2) = 0
$$

and as L is linear, we obtain

$$
L(\psi_1 - \psi_2) \geq 0,
$$

according to (2.14)

$$
\sup_{0\leq t\leq T}\|\psi_1 - \psi_2\| \leq 0 \implies \sup_{0\leq t\leq T}\|\psi_1 - \psi_2\| = 0
$$

$$
\implies \psi_1 - \psi_2 = 0 \implies \psi_1 = \psi_2.
$$

Hence, we deduce the uniqueness of the solution.

Chapter 3

Double Laplace Decomposition Method

3.1 The Modified Double Laplace Decomposition Method

The main aim of this section is to discuss the use of the modified double Laplace decomposition method for solving the linear initial value problem (2.1) and (2.2).

By using (1.19)–(1.22), we obtain

$$\begin{aligned}\frac{d\Psi}{dp} &= \frac{dF_1(p)}{sdp} + \frac{dF_2(p)}{s^2dp} - \frac{1}{s^2}L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right)\right] \\ &- \frac{1}{s^2}L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right)\right] + \frac{1}{s^2}\frac{dF(p,s)}{dp}.\end{aligned} \tag{3.1}$$

Integration of both sides of Equation (3.1) from 0 to p with respect to p, yields

$$\begin{aligned}\Psi(p,s) &= \frac{F_1(p)}{s} + \frac{F_2(p)}{s^2} - \frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp \\ &- \frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp + \frac{F(p,s)}{s^2},\end{aligned} \tag{3.2}$$

where $F(p,s)$, $F_1(p)$ and $F_2(p)$ are Laplace transform of the functions $f(x,t)$, $f_1(x)$ and $f_2(x)$ respectively and the double Laplace transform with respect to x, t is defined by L_xL_t . Operating with the double Laplace inverse on both sides of Equation (2.13), we obtain

$$\begin{aligned}\psi(x,t) &= f_1(x) + tf_2(x) - L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right) + \frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp - \frac{F(p,s)}{s^2}\right].\end{aligned} \tag{3.3}$$

The modified double Laplace decomposition method (MDLDM) defines the solutions $\psi(x,t)$ by the infinite series

$$\psi(x,t) = \sum_{n=0}^{\infty}\psi_n(x,t.) \tag{3.4}$$

Upon substitution of Equation (3.4) into (3.3), we get

$$\begin{aligned}\sum_{n=0}^{\infty}\psi_n(x,t) &= f_1(x)+tf_2(x)-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right)\right]dp\right]\\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right)\right]dp\right]\\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n(x,t)\right)\right)\right]dp\right]\\ &+ L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s^2}\right].\end{aligned} \tag{3.5}$$

On comparing both sides of (3.5), we get

$$\psi_0(x,t)=f_1(x)+tf_2(x)+L_p^{-1}L_s^{-1}\left[\frac{F(p,s)}{s^2}\right]. \tag{3.6}$$

In general, the recursive relation is given by

$$\begin{aligned}\psi_{n+1}(x,t) &= -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}(\psi_n(x,t))\right)\right]dp\right]\\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}(\psi_n(x,t))\right)\right]dp\right]\\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial}{\partial x}(\psi_n(x,t))\right)\right]dp\right],\end{aligned} \tag{3.7}$$

where $L_p^{-1}L_s^{-1}$ is the double inverse Laplace transform with respect to p, s. Here we assume that the double inverse Laplace transform with respect to p and s exists for each term in the right hand side of Equations (3.6) and (3.7). To illustrate this method, we consider the following example.

Example: Consider the following singular generalized modified linear Boussinesq equation with Bessel operator:

$$\begin{aligned}&\frac{\partial^2\psi}{\partial t^2}-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)-\frac{1}{x}\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial\psi}{\partial x}\right)-\frac{1}{x}\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial\psi}{\partial x}\right)\\ &=-x^2\sin t-4\cos t,\end{aligned} \tag{3.8}$$

subject to the initial conditions

$$\psi(x,0)=0\ ,\ \frac{\partial\psi(x,0)}{\partial t}=x^2. \tag{3.9}$$

By multiplying Equation (3.8) by x and using the definition of partial derivatives of the double Laplace transform and single Laplace transform for Equations (3.8) and (3.9), we obtain

$$\begin{aligned}\frac{d\Psi}{dp} &= -\frac{1}{s^2} L_x L_t \left[\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right) + \frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial \psi}{\partial x}\right) + \frac{\partial^3}{\partial x \partial t^2}\left(x\frac{\partial \psi}{\partial x}\right)\right] \\ &+ \frac{6}{p^4 s^2 \left(s^2+1\right)} + \frac{4}{p^2 s \left(s^2+1\right)} - \frac{6}{p^4 s^2}.\end{aligned} \tag{3.10}$$

By integrating both sides of (3.10) from 0 to p with respect to p, we obtain

$$\begin{aligned}\Psi(p,s) &= -\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right) + \frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial \psi}{\partial x}\right) + \frac{\partial^3}{\partial x \partial t^2}\left(x\frac{\partial \psi}{\partial x}\right)\right] dp \\ &- \frac{2}{p^3 s^2 \left(s^2+1\right)} - \frac{4}{p s \left(s^2+1\right)} + \frac{2}{p^3 s^2}.\end{aligned} \tag{3.11}$$

Application of the inverse double Laplace transform to (3.11), yields

$$\begin{aligned}\psi(x,t) &= -L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial}{\partial x}\left(x\frac{\partial \psi}{\partial x}\right) + \frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial \psi}{\partial x}\right) + \frac{\partial^3}{\partial x \partial t^2}\left(x\frac{\partial \psi}{\partial x}\right)\right] dp\right] \\ &+ x^2 \sin t + 4\cos t - 4.\end{aligned} \tag{3.12}$$

Putting (3.4) into (3.12) to have

$$\begin{aligned}\sum_{n=0}^{\infty} \psi_n(x,t) &= -L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} \sum_{n=0}^{\infty} \psi_n(x,t)\right)\right] dp\right] \\ &- L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial}{\partial x} \sum_{n=0}^{\infty} \psi_n(x,t)\right)\right] dp\right] \\ &- L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial^3}{\partial x \partial t^2}\left(x\frac{\partial}{\partial x} \sum_{n=0}^{\infty} \psi_n(x,t)\right)\right] dp\right] \\ &+ x^2 \sin t + 4\cos t - 4.\end{aligned} \tag{3.13}$$

By modified Laplace decomposition method, we have

$$\psi_0 = x^2 \sin t + 4\cos t - 4, \tag{3.14}$$

and

$$\begin{aligned}\psi_{n+1} = &-L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x} \psi_n(x,t)\right)\right] dp\right] \\ &- L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial^2}{\partial x \partial t}\left(x\frac{\partial}{\partial x} \psi_n(x,t)\right)\right] dp\right] \\ &- L_p^{-1} L_s^{-1} \left[\frac{1}{s^2} \int_0^p L_x L_t \left[\frac{\partial^3}{\partial x \partial t^2}\left(x\frac{\partial}{\partial x} \psi_n(x,t)\right)\right] dp\right].\end{aligned}$$

Now the components of the series solution are

$$\psi_1 = -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_0(x,t)\right)\right]dp\right]$$
$$-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^2}{\partial x\partial t}\left(x\frac{\partial}{\partial x}\psi_0(x,t)\right)\right]dp\right]$$
$$-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[\frac{\partial^3}{\partial x\partial t^2}\left(x\frac{\partial}{\partial x}\psi_0(x,t)\right)\right]dp\right].$$
$$\psi_1 = -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[4x\cos t\right]dp\right] = -L_p^{-1}L_s^{-1}\left[\frac{-4}{ps\left(s^2+1\right)}\right]$$
$$= 4 - 4\cos t,$$

and

$$\psi_2 = -L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_xL_t\left[0\right]dp\right] = 0.$$

Eventually, the approximate solution of the unknown functions is given by

$$\sum_{n=0}^{\infty}\psi_n(x,t) = \psi_0 + \psi_1 + \psi_2 + \cdots$$
$$= x^2\sin t + 4\cos t - 4 + 4 - 4\cos t + 0.$$

Hence, the exact solution is given by

$$\psi(x,t) = x^2\sin t.$$

3.2 A Nonlinear Singular Boussinesq Equation with Bessel Operator

In this section, we consider the following nonlinear singular one dimensional Boussinesq equation

$$\psi_{tt} - \frac{1}{x}\frac{\partial}{\partial x}\left(x\psi_x\right) + a(x)\psi_{xxxx} - b(x)\psi_{xxtt} + c(x)\psi_t\psi_{xx} + d(x)\psi_x\psi_{xt} = f(x,t), \tag{3.15}$$

subject to the initial conditions

$$\psi(x,0) = g_1(x)\ ,\ \frac{\partial\psi(x,0)}{\partial t} = g_2(x), \tag{3.16}$$

where $a(x), b(x), c(x)$ and $d(x)$ are given functions.

Multiplication of Equation (50) by x and application of double Laplace transform, give

$$L_xL_t\left[x\psi_{tt}-\frac{\partial}{\partial x}\left(x\psi_x\right)+xa\left(x\right)\psi_{xxxx}-xb\left(x\right)\psi_{xxtt}+xc\left(x\right)\psi_t\psi_{xx}+xd\left(x\right)\psi_x\psi_{xt}\right]$$
$$=L_xL_t\left[xf\left(x,t\right)\right]. \tag{3.17}$$

On using the differentiation property of double Laplace transform and initial conditions (51) , we get

$$\begin{aligned}\frac{d\Psi}{dp}&=\frac{dG_1\left(p\right)}{sdp}+\frac{dG_2\left(p\right)}{s^2dp}-\frac{1}{s^2}L_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)-xa\left(x\right)\psi_{xxxx}+xb\left(x\right)\psi_{xxtt}\right]\\&\quad+\frac{1}{s^2}L_xL_t\left[c\left(x\right)\psi_t\psi_{xx}+d\left(x\right)\psi_x\psi_{xt}\right]+\frac{1}{s^2}\frac{dF\left(p,s\right)}{dp}.\end{aligned} \tag{3.18}$$

By integrating both sides of (53) from 0 to p with respect to p, we have

$$\begin{aligned}\Psi\left(p,s\right)=&\frac{G_1\left(p\right)}{s}+\frac{G_2\left(p\right)}{s^2}+\frac{1}{s^2}\int_0^p\frac{dF\left(p,s\right)}{dp}d_P\\&-\frac{1}{s^2}\int_0^pL_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp\\&+\frac{1}{s^2}\int_0^pL_xL_t\left(xa\left(x\right)\psi_{xxxx}-xb\left(x\right)\psi_{xxtt}\right)dp\\&-\frac{1}{s^2}\int_0^pL_xL_t\left(xc\left(x\right)\psi_t\psi_{xx}+xd\left(x\right)\psi_x\psi_{xt}\right)dp.\end{aligned} \tag{3.19}$$

Using double inverse Laplace transform, it follows from (54) that

$$\begin{aligned}\psi\left(x,t\right)=&f_1\left(x\right)+tf_2\left(x\right)+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p\frac{dF\left(p,s\right)}{dp}dp\right]\\&-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^pL_xL_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial\psi}{\partial x}\right)\right]dp\right]\\&+L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^pL_xL_t\left[xa\left(x\right)\psi_{xxxx}-xb\left(x\right)\psi_{xxtt}\right]dp\right]\\&-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^pL_xL_t\left[xc\left(x\right)\psi_t\psi_{xx}+xd\left(x\right)\psi_x\psi_{xt}\right]dp\right].\end{aligned} \tag{3.20}$$

Moreover, the nonlinear terms $N_1=\psi_t\psi_{xx}$ and $N_2=\psi_x\psi_{xt}$ are defined by

$$N_1=\psi_t\psi_{xx}=\sum_{n=0}^{\infty}A_n\quad,\quad N_2=\psi_x\psi_{xt}=\sum_{n=0}^{\infty}B_n, \tag{3.21}$$

where the Adomian polynomials for A_n and B_n are defined by

$$A_n=\frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_1\left(\sum_{j=0}^{n}\left(\lambda^j\psi_j\right)\right)\right]\right)_{\lambda=0}\quad,\quad n=0,1,2,..., \tag{3.22}$$

and

$$B_n = \frac{1}{n!}\left(\frac{d^n}{d\lambda^n}\left[N_2 \sum_{j=0}^{n}\left(\lambda^j \psi_j\right)\right]\right)_{\lambda=0} \quad , \quad n = 0, 1, 2, ... \tag{3.23}$$

By substitution of Equations (57)–(59) into (56), we obtain:

$$\begin{aligned}
\sum_{n=0}^{\infty} \psi_n\left(x,t\right) = f_1\left(x\right) + t f_2\left(x\right) + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p \frac{dF\left(p,s\right) d_P}{dp}\right] \\
- L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty}\psi_n\right)\right)\right] dp\right] \\
+ L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xa\left(x\right)\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxxx} - xb\left(x\right)\left(\sum_{n=0}^{\infty}\psi_n\right)_{xxtt}\right] dp\right] \\
- L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xc\left(x\right)\sum_{n=0}^{\infty} A_n + xd\left(x\right)\sum_{n=0}^{\infty} B_n\right] dp\right],
\end{aligned} \tag{3.24}$$

where some few terms of A_n and B_n for $n = 0, 1, 2, 3$ are given by

$$\begin{aligned}
A_0 &= \psi_{0t}\psi_{0xx} \\
A_1 &= \psi_{0t}\psi_{1xx} + \psi_{1t}\psi_{0xx} \\
A_2 &= \psi_{0t}\psi_{2xx} + \psi_{1t}\psi_{1xx} + \psi_{2t}\psi_{0xx} \\
A_3 &= \psi_{0t}\psi_{3xx} + \psi_{1t}\psi_{2xx} + \psi_{2t}\psi_{1xx} + \psi_{3t}\psi_{0xx},
\end{aligned} \tag{3.25}$$

and

$$\begin{aligned}
B_0 &= \psi_{0x}\psi_{0xt} \\
B_1 &= \psi_{0x}\psi_{1xt} + \psi_{1x}\psi_{0xt} \\
B_2 &= \psi_{0x}\psi_{2xt} + \psi_{1x}\psi_{1xt} + \psi_{2x}\psi_{0xt} \\
B_3 &= \psi_{0x}\psi_{3xt} + \psi_{1x}\psi_{2xt} + \psi_{2x}\psi_{1xt} + \psi_{3x}\psi_{0xt}.
\end{aligned} \tag{3.26}$$

Therefore, from (60) above, it follows that

$$\psi_0\left(x,t\right) = f_1\left(x\right) + t f_2\left(x\right) + L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p \frac{dF\left(p,s\right)}{dp}\right],$$

and

$$\begin{aligned}
\psi_{n+1}\left(x,t\right) = -L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_n\right)\right] dp\right] \\
+ L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xa\left(x\right)\left(\psi_n\right)_{xxxx} - xb\left(x\right)\left(\psi_n\right)_{xxtt}\right] dp\right] \\
- L_p^{-1} L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[xc\left(x\right) A_n + xd\left(x\right) B_n\right] dp\right].
\end{aligned} \tag{3.27}$$

To illustrate the used method, we consider the following example, where we let that $a(x) = b(x) = 1, c(x) = -4, d(x) = 2$ and $f(x,t) = -4t$ in Equation (50).

Example: We consider the nonlinear Boussinesq equation with Bessel operator

$$\psi_{tt} - \frac{1}{x}\frac{\partial}{\partial x}(x\psi_x) + \psi_{xxxx} - \psi_{xxtt} - 4\psi_t\psi_{xx} + 2\psi_x\psi_{xt} = -4t, \tag{3.28}$$

subject to the initial conditions

$$\psi(x,0) = 0 \quad , \quad \psi_t(x,0) = x^2. \tag{3.29}$$

The double Laplace decomposition method leads to the following:

$$\psi_0(x,t) = x^2 t - \frac{2}{3}t^3,$$

and

$$\begin{aligned} \psi_{n+1}(x,t) = &-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_n\right)\right]dp\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[x(\psi_n)_{xxxx} - x(\psi_n)_{xxtt}\right]dp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[4xA_n - 2xB_n\right]dp\right]. \end{aligned} \tag{3.30}$$

The first iteration is given by

$$\begin{aligned} \psi_1(x,t) = &-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_0\right)\right]dp\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[x(\psi_0)_{xxxx} - x(\psi_0)_{xxtt}\right]dp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[4xA_0 - 2xB_0\right]dp\right]. \end{aligned} \tag{3.31}$$

$$\psi_1(x,t) = \frac{2}{3}t^3 - \frac{4}{5}t^5. \tag{3.32}$$

The subsequent terms are given by

$$\begin{aligned} \psi_2(x,t) = &-L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}\psi_1\right)\right]dp\right] \\ &- L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[x(\psi_1)_{xxxx} - x(\psi_1)_{xxtt}\right]dp\right] \\ &+ L_p^{-1}L_s^{-1}\left[\frac{1}{s^2}\int_0^p L_x L_t\left[4xA_1 - 2xB_1\right]dp\right] \\ = &\ 0. \end{aligned} \tag{3.33}$$

and the rest terms are all zeros. Hence

$$\psi(x,t) = x^2 t. \tag{3.34}$$

Conclusion

In this work, we introduce a modified double Laplace decomposition method to analyze both a singular generalized modified linear Boussinesq equation and a singular nonlinear Boussinesq equation. To demonstrate the method's effectiveness, we provide several examples that confirm its validity, efficiency, and accuracy. Our findings indicate that the modified double Laplace decomposition method is not only highly efficient but also straightforward to apply to both linear and nonlinear Boussinesq models. This makes it a valuable tool for solving complex problems in the study of Boussinesq equations. The examples included highlight the practical applications and benefits of using this method in various scenarios, showcasing its robustness and reliability in producing accurate solutions.

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