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Theme

Error analysis of obstacle problem for Naghdi shell model

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Thanks

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Contents

Notations and Conventions

Notations

- $a_\alpha(x) = \{a_1(x), a_2(x)\}$
- $H^1_{\gamma_0}(\omega) = \{\mu \in H^1(\omega) \,|\, \mu = 0 \text{ on } \gamma_0\}$.
- $X(\omega) = H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3$,
- $V(\omega) = \left\{(v, s) \in H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3 \;\middle|\; s \cdot a_3 = 0 \text{ in } \omega\right\}$.
- $N_\Phi(\omega) = \{V \in X(\omega) : (v - \frac{e}{2}s) \cdot e_3 \geq \Phi \text{ a.e. in } \omega\}$,
- $H^1_{\gamma_0+}(\omega) = \{\sigma \in H^1(\gamma_0(\omega))\,;\, \sigma \geq 0 \text{ a.e. in } \omega\}$.
- $\Lambda = \left\{\mu \in H^1_{\gamma}{}_0(\omega))\,;\, \forall \sigma \in H^1_{\gamma_0+}(\omega), \langle \sigma, \mu \rangle \geq 0\right\}$.
- $K(\omega) = \{V \in V(\omega) : v - \frac{e(\cdot)}{2s} \cdot e_3 = 0 \text{ a.e. in } \omega\}$.
- $N_{h,\Phi} = \left\{V_h = (v_h, s_h) \in \mathbb{X}_h \mid \mathcal{I}_h\left((v_h - \frac{e}{2}\, s_h) \cdot e_3\right) \geq \Phi_h \text{ a.e. in } \omega\right\}$

Abstracl

In this work, we focus on studying an obstacle problem of Naghdi shell, where we establish the existence and uniqueness of the solution, as well as the continuous finite element approximation , and we also pay attention to estimating the error analysis a priori .

Keywords: obstacle problem of Naghdi shell, the error analysis a priori, finite elements approximation.

Résumé

Dans ce travail, nous nous concentrons sur l'étude d'un probléme d'obstacle de la coque de Naghdi, oú nous établissons l'existence et l'unicité de la solution, ainsi que l'approximation continue par éléments finis, et nous prêtons également attention á l'estimation de l'analyse d'erreur a priori.

Mots-clés : probléme d'obstacle du shell Naghdi, l'analyse d'erreur a priori, approximation par éléments finis.

الملخص:

نركز في هذا العمل على دراسة مشكلة عائق قشرة نقدي، حيث نثبت وجود الحل وتفرده، وكذلك التقريب المستمر للعناصر المحدودة، ونهتم أيضًا بتقدير تحليل الخطأ مسبقًا.

الكلمات الرئيسية: مشكلة معوقات الصدفة النهدي، تقدير تحليل الأخطاء بشكل مسبق، تقريب العناصر المحدودة.

Chapter 1

Introduction

The Naghdi sell model is a two-dimensional shell model. It belongs to the Reissner shell family, which relies on the theory of Cosserat surfaces.The derivation of the model supposes that the distance between a point and the midsurface remains constant throughout the deformation of the shell. Under some mechanical assumptions, this model takes into account membrane deformation and bending of the midsurface. Transverse shear deformations are also taken into account. The unknowns of the problem are the displacement of the points of the shell midsurface and the rotation field of the normal to the midsurface

The formulation of Naghdi's model used here was introduced by Blouza [1] and Blouza and Le Dret [2]. It relies on the idea of using a local basis-free formulation in which the unknowns are described in Cartesian coordinates instead of covariant or contravariant components, as is usually done in shell theory (see, for example, [2]). Such a formulation is capable of handling shells with a $W^{2,\infty}$-midsurface. In particular, midsurface curvature discontinuities are allowed. Furthermore, in view of the discretization and as first proposed in [6], a Lagrange multiplier can be introduced to handle the tangency requirement on the rotation. This leads to a well-posed mixed variational problem.

The literature on finite element approximation of two-dimensional shell models is extensive. Let us mention a few approaches. Concerning conforming methods, the Ganev and Argyris triangles provide interpolation by polynomials of degree 4 and 5, with high-order convergence in ch^4 when the solution is smooth enough. These elements are used, for example, to study the linear Koiter model for C^3-shells in the classical covariant formulation . Such methods are also applied to approximate geometrically exact shell models in [7].

We are interested in two other finite element discretizations relying on the mixed formulation and already studied in [6]. In the second one, a penalty term is added to the mixed formulation as standard for saddle-point problems, which leads to an efficient algorithm for solving the resulting linear system. The convergence of both discretizations is proved in [6], where a complete a priori analysis is performed .

An outline of the paper is as follows.

- In chapter 2, we recall the geometry of the midsurface and Naghdi's equations. Next, we write the mixed formulation and recall its well-posedness.
- in chapter 3 , we intrested by the contact model form and write the mixed formulation and recall its well-posedness .
- in chapter 4, we have the discrete problem and its well-posedness . Next , we recall a priori of the error , with the a posteriori analysis .

Chapter 2

A Naghdi shell Model

2.1 Formulation of the Problem

2.1.1 Geometry of the Midsurface and Notation

Let ω be a bounded connected domain in $\mathbb{R}^2$ with a Lipschitz-continuous boundary $\partial\omega$. We consider a shell whose midsurface is given by $M = \phi(\overline{\omega})$ where ϕ is a one-to-one mapping in $W^{2,\infty}(\omega)^3$ such that the two vectors

$$a_\alpha(x) = (\partial_\alpha \phi)(x)$$

are linearly independent at each point x of $\overline{\omega}$. Thus,

$$a_3(x) = \frac{a_1(x) \wedge a_2(x)}{|a_1(x) \wedge a_2(x)|}.$$

is the unit normal vector on the midsurface at point $\phi(x)$. The vectors $a_i(x)$ define the local covariant basis at point $\phi(x)$. The contravariant basis $a^i(x)$ is defined by the relations $a_i \cdot a_j = \delta_{ji}$ where δ_{ji} is the Kronecker symbol. In particular, $a_3(x)$ coincides with $a^3(x)$. Note that all these vectors belong to $W^{1,\infty}(\omega)^3$.

As standard, Greek indices and exponents take their values in the set $\{1,2\}$ and Latin indices and exponents take their values in the set $\{1,2,3\}$. The first and second fundamental forms of the surface are given in covariant components by

$$a_{\alpha\beta} = a_\alpha \cdot a_\beta \text{ and } b_{\alpha\beta} = a^3 \cdot \partial_\beta a_\alpha.$$

We set $a(x) = |a_1(x) \wedge a_2(x)|^2$ so that $p_a(x)$ is the area element of the midsurface in the chart ϕ. Similarly, the length element on the boundary $\partial\omega$ is given by $p_{a\alpha\beta}\tau^\alpha\tau^\beta$, with the standard summation convention for repeated indices and exponents, where $a_{\alpha\beta} = a_\alpha \cdot a_\beta$ being the contravariant components of the first fundamental form and (τ_1, τ_2) being the covariant coordinates of a unit vector tangent to $\partial\omega$. The thickness of the shell, denoted by e, is a positive continuous function.

The first and second fundamental forms of the surface are given in covariant components by:

$$a_{\alpha\beta} = a_\alpha \cdot a_\beta,$$

and

$$b_{\alpha\beta} = a_3 \cdot \partial_\beta a_\alpha = -a_\alpha \cdot \partial_\beta a_3 \quad (\text{since } a_\alpha \cdot a_3 = 0).$$

Since $W^{1,\infty}$ is a Banach algebra, it follows that $a_{\alpha\beta} \in W^{1,\infty}(\omega)$ and $b_{\alpha\beta} \in L^\infty(\omega)$.

We further introduce the contravariant components of the first fundamental form:

$$a^{\alpha\beta} = a_\alpha \cdot a_\beta,$$

and the mixed (where $b_{\alpha\beta}$ is symmetric) components of the second fundamental form:

$$b^\alpha_\beta = a_{\beta\rho} b^\alpha_\rho.$$

Again, $a_{\beta\rho} \in W^{1,\infty}(\omega)$ and $b^\alpha_\beta \in L^\infty(\omega)$.

Finally, the Christoffel symbols of the midsurface are given by:

$$\Gamma_{\rho\alpha\beta} = \Gamma_{\rho\beta\alpha} = a_\rho \cdot \partial_\beta a_\alpha,$$

and we have $\Gamma_{\rho\alpha\beta} \in L^\infty(\omega)$.

In the case of a homogeneous, isotropic material with Young modulus $E > 0$ and Poisson ratio ν, $0 \leq \nu < \frac{1}{2}$, the contravariant components of the elasticity tensor $a_{\alpha\beta\rho\sigma}$ are given by

$$a^{\alpha\beta\rho\sigma} = \frac{E}{2(1+\nu)} \left(a^{\alpha\rho}a^{\beta\sigma} + a^{\alpha\sigma}a^{\beta\rho}\right) + \frac{E\nu}{1-\nu} \left(\frac{1}{2}a^{\alpha\beta}a^{\rho\sigma}\right) \tag{2.1}$$

We note that each component of the elasticity tensor belongs to $L^{\infty}(\omega)$. Moreover, this tensor satisfies the usual symmetry properties and is uniformly strictly positive.

In this context, the covariant components of the change of metric tensor are given by:

$$\gamma_{\alpha\beta}(u) = \frac{1}{2} \left(\partial_\alpha u \cdot a_\beta + \partial_\beta u \cdot a_\alpha\right) \tag{2.2}$$

And the covariant components of the change of transverse shear tensor are given by:

$$\delta_{\alpha 3}(u, r) = \frac{1}{2} \left(\partial_\alpha u \cdot a_3 + r \cdot a_\alpha\right) \tag{2.3}$$

And the covariant components of the change of curvature tensor are given by:

$$\chi_{\alpha\beta}(u, r) = \frac{1}{2} \left(\partial_\alpha u \cdot \partial_\beta a_3 + \partial_\beta u \cdot \partial_\alpha a_3 + \partial_\alpha r \cdot a_\beta + \partial_\beta r \cdot a_\alpha\right) \tag{2.4}$$

Note that all these quantities make sense for shells with little regularity, and are easily expressed with the Cartesian coordinates of the unknowns and geometrical data.

The objective of the lemma presented below is to provide simplified and intrinsic expressions for the strain tensors, which are more straightforward than those in equations (2,5)-(2,7). Utilizing these expressions facilitates the establishment of existence and uniqueness results for general shells, even with potentially discontinuous curvatures. In this context, rather than representing the displacement and rota

2.1.2 Existence and Uniqueness for the Nagdhi shell Problem

In this section, we delve into the details of the existence and uniqueness theorem, for the linear Nagdhi shell model. We focus specifically on cases where part of the shell's boundary is clamped, accommodating curvature discontinuities along the midsurface. This result represents a significant advancement and simplification compared to prior findings presented . Notably, our analysis extends to scenarios involving C^1-shells, encompassing combinations of planar and cylindrical segments, a departure from the conventional assumption of C^3 continuity in ω-space. Our methodology parallels that of previous works, which tackled similar challenges in the Koiter model.

In the context of elasticity, let $a_{\alpha\beta\rho\sigma} \in L^\infty(\omega)$ denote the elasticity tensor, satisfying the standard symmetries and being uniformly strictly positive. This implies that for any symmetric tensor $\tau_{\alpha\beta}$ and almost every $x \in \omega$, the inequality

$$a_{\alpha\beta\rho\sigma}(x)\tau_{\alpha\beta}\tau_{\rho\sigma} \geq c \sum_{\alpha\beta} |\tau_{\alpha\beta}|^2 \tag{2.5}$$

holds, where c represents a positive constant.

To be more specific, we will concentrate on the case of a homogeneous, isotropic material with Lamu moduli $\mu > 0$ and $\lambda \neq 0$, in which case

$$a_{\alpha\beta\rho\sigma} = 2\mu\left(a_{\alpha\beta}a_{\rho\sigma} + a_{\alpha\sigma}a_{\beta\rho}\right) + \frac{4\lambda\mu}{\lambda + 2\mu}a_{\alpha\beta}a_{\rho\sigma} \tag{2.6}$$

Let $e \in L^\infty(\omega)$ be the thickness of the shell, which we assume to be such that $e(x) \geq c > 0$ almost everywhere in ω.

Let $\partial\omega$ be the boundary of the chart domain, divided into two parts: γ_0, where the shell is clamped, and its complement $\gamma_1 = \partial\omega \setminus \gamma_0$, where the shell is subjected to applied tractions and moments. We assume that γ_0 consists of a finite number of connected components and has a strictly positive one-dimensional measure. To incorporate the boundary conditions, we define the function space

$$H^1_{\gamma_0}(\omega) = \left\{\mu \in H^1(\omega) \,|\, \mu = 0 \text{ on } \gamma_0\right\}. \tag{2.7}$$

Consider the function space for shells:

$$V(\omega) = \left\{(v, s) \in H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3 \;\middle|\; s \cdot a_3 = 0 \text{ in } \omega\right\}. \tag{2.8}$$

This space is endowed with the natural Hilbert norm:

$$\|V\|_{V(\omega)} = \left(\|v\|^2_{H^1(\omega)^3} + \|s\|^2_{H^1(\omega)^3}\right)^{\frac{1}{2}} \tag{2.9}$$

Lemma 2.1. The space V is a Hilbert space.

Proof. Clear.

Existence and Uniqueness Theorem

In this subsection, we leverage the function space defined in Lemma 2.1 to establish the existence and uniqueness of solutions for the linear Nagdhi model, even in cases where shells exhibit limited regularity.

Theorem 2.2. Assume that $\phi \in W^{2,\infty}(\omega;\mathbb{R}^3)$. Let $f \in L^2(\omega;\mathbb{R}^3)$ be a given force resultant density, and let $g \in L^2(\gamma_1;\mathbb{R}^3)$ and $m \in L^2(\gamma_1;\mathbb{R}^3)$, with $m \cdot a_3 = 0$, be given traction and moment resultant densities, respectively. Then there exists a unique solution to the variational problem:

$$\begin{cases} \text{Find}(u,r) \in V \text{such that} \\ A(u,r);(v,s) = l(v,s), \quad \text{for all } (v,s) \in V, \end{cases} \tag{2.10}$$

where

$$A(u,r);(v,s) = \int_\omega \left\{ e\, a_{\alpha\beta\rho\sigma} \left[\gamma_{\alpha\beta}(u)\gamma_{\rho\sigma}(v) + \frac{e^2}{12}\chi_{\alpha\beta}(u,r)\chi_{\rho\sigma}(v,s) \right] + 4\mu e\, a_{\alpha\beta}\delta_{\alpha 3}(u,r)\delta_{\beta 3}(v,s) \right\} \sqrt{a}dx \tag{2.11}$$

$$l(v,s) = \int_\omega (f \cdot v)\sqrt{a}\, dx + \int_{\gamma_1} (g \cdot v + m \cdot s)\sqrt{a}\tau_\alpha \tau_\beta \, d\gamma. \tag{2.12}$$

We will prove below that the bilinear form of (2,10) is continuous and V-elliptic by using a contradiction argument, together with Rellich's theorem and the two-dimensional Korn inequality. Then existence and uniqueness follow from the Lax-Milgram lemma applied to problem (2,10).

Lemma 2.3: Let $u \in H^1(\omega;\mathbb{R}^3)$ and $r \in H^1(\omega;\mathbb{R}^3)$ such that $r \cdot a_3 = 0$, where ω represents the domain. Additionally, let $\phi \in W^{2,\infty}(\omega;\mathbb{R}^3)$.

i If u satisfies $\gamma(u) = 0$, then there exists a unique $\psi \in L^2(\omega;\mathbb{R}^3)$ such that

$$\partial_\alpha u = \psi \wedge a_\alpha, \quad \alpha = 1, 2. \tag{2.13}$$

ii Furthermore, if u and r satisfy $\delta_{\alpha 3}(u,r) = 0$, then $\partial_\alpha u \cdot a_3 = -r \cdot a_\alpha$ belong to $H^1(\omega)$. Moreover, $r \cdot a_\alpha = -\varepsilon_{\alpha\beta} a_\beta \cdot \psi$.

iii In addition, under the condition $\chi(u,r)=0$, ψ can be identified with a constant vector in $\mathbb{R}^3$, and for all $x\in\omega$, we have:

$$u(x)=c+\psi\wedge\phi(x),$$

where c is a constant in $\mathbb{R}^3$, and

$$r(x)=-\varepsilon_{\alpha\beta}(x)a_\beta(x)\cdot\psi\, a_\alpha(x).$$

Proof.

(i) for a proof of the existence and uniqueness of the infinitesimal rotation vector ψ such that (2,13) holds true.

(ii) Suppose now that $\delta_{\alpha 3}(u,r)=0$, then

$$\partial_\alpha u\cdot a_3=-r\cdot a_\alpha\in H^1(\omega), \tag{2.14}$$

since $r\in H^1(\omega;\mathbb{R}^3)$ and $a_\alpha\in W^{1,\infty}(\omega;\mathbb{R}^3)$. Therefore, we have $r\cdot a_\alpha=(a_\alpha\wedge a_3)\cdot\psi=-\varepsilon_{\alpha\beta}a_\beta\cdot\psi$.

(iii) Let us first note that under the previous hypotheses, we have.

$$\partial_{\alpha\beta}u\cdot a_3=\partial_\beta(\partial_\alpha u\cdot a_3)-\partial_\alpha u\cdot\partial_\beta a_3=-\partial_\beta(r\cdot a_\alpha)-\partial_\alpha u\cdot\partial_\beta a_3\in L^2(\omega), \tag{2.15}$$

because $\partial_\beta a_3\in L^\infty(\omega;\mathbb{R}^3)$. It follows, by (2.15), that

$$\begin{aligned}\chi_{\alpha\beta}(u,r)&=\frac{1}{2}\left(\partial_\alpha u\cdot\partial_\beta a_3+\partial_\beta u\cdot\partial_\alpha a_3+\partial_\alpha r\cdot a_\beta+\partial_\beta r\cdot a_\alpha\right)\\&=\frac{1}{2}\left(\partial_\alpha u\cdot\partial_\beta a_3+\partial_\beta u\cdot\partial_\alpha a_3+\partial_\alpha(r\cdot a_\beta)+\partial_\beta(r\cdot a_\alpha)-2r\cdot\partial_\alpha a_\beta\right)\\&=\frac{1}{2}\left(-2\partial_{\alpha\beta}u\cdot a_3-2r\cdot\partial_\alpha a_\beta\right)\\&=-(\partial_{\alpha\beta}u\cdot a_3+\varepsilon_{\alpha\beta\rho}r_\rho\cdot a_\rho),\end{aligned}$$

since $\partial_\alpha a_\beta=\varepsilon_{\rho\alpha\beta}a_\rho$. Then, using (2.4), we see that

$$\chi_{\alpha\beta}(u,r)=-(\partial_{\alpha\beta}u-\varepsilon_{\alpha\beta\rho}\partial_\rho u)\cdot a_3. \tag{2.16}$$

At this juncture, we recognize that $(\partial_{\alpha\beta}u - \varepsilon_{\alpha\beta\rho}\partial_\rho u) \cdot a_3 = \Gamma_{\alpha\beta}(u)$ represents the covariant components of the linearized change of curvature tensor in Koiter's model, as discussed . These components belong to the space $L^2(\omega)$ by virtue of (2.16). Assuming now that $\chi(u, r) = 0$, we simply need to leverage the infinitesimal rigid displacement lemma applicable to $W^{2,\infty}$-Koiter shells, as presented , to conclude the proof.

Lemma 2.4. There exists a constant $C > 0$ such that

$$A(v, s); (v, s) \geq C \left(\sum_{\alpha,\beta} \|\gamma(v)_{\alpha,\beta}\|^2_{L^2(\omega)} + \sum_{\alpha,\beta} \|\chi(v, s)_{\alpha,\beta}\|^2_{L^2(\omega)} + \sum_{\alpha} \|\delta_{\alpha,3}(v, s)\|^2_{L^2(\omega)} \right)^{1/2}$$

for all $(v, s) \in H^1(\omega; \mathbb{R}^3)$.

Proof. This is clear in view of inequality (2.5) and the fact that $a_{\alpha\beta}(x)\eta_\alpha\eta_\beta \geq C' \sum_\alpha (\eta_\alpha)^2$ for all $x \in \overline{\omega}$.

Lemma 2.5. The bilinear form of problem (2.11) is V-elliptic.

Proof. Because of Lemma 2.3 and the hypotheses made on the chart ϕ, the elasticity tensor, and the thickness of the shell, it is enough to prove that

$$|||(v, s)||| = \left(\sum_{\alpha,\beta} \left(\|\gamma(v)_{\alpha,\beta}\|^2_{L^2(\omega)} + \|\chi(v, s)_{\alpha,\beta}\|^2_{L^2(\omega)} \right) + \sum_{\alpha} \|\delta_{\alpha,3}(v, s)\|^2_{L^2(\omega)} \right)^{1/2} \tag{2.17}$$

The norm $||| \cdot |||$ on V is bounded from below by a multiple of the natural norm (2.9) of V.

Let's begin by proving that $||| \cdot |||$ is indeed a norm. Suppose $(v, s) \in V$ such that $|||(v, s)||| = 0$. According to Lemma 2.3 of infinitesimal rigid displacement, we have $v(x) = c + \psi \wedge \phi(x)$. By assumption, the displacement v vanishes on γ_0. If $\phi(\gamma_0)$ is not contained in a straight line, it implies $v = 0$ in ω, meaning $\psi = c = 0$. Consequently, $s = 0$ in ω as well. Now, let's assume that $\phi(\gamma_0)$ is contained in a straight line l and $\psi \neq 0$. In this scenario, ψ is parallel to l and thus lies within the planes spanned by $a_\beta(x)$ for all $x \in \gamma_0$. Let's select such an x. Since $s = 0$ on γ_0, it follows that $0 = s \cdot a_\rho = -\varepsilon_{\rho\beta} a_\beta \cdot \psi$ on γ_0. Consequently, $a_\beta \cdot \psi = 0$ on γ_0, and ψ is orthogonal to the plane spanned by $a_\beta(x)$. Therefore, $\psi = 0$, and as before, $v = s = 0$ in ω.

For the second part of the proof, we argue by contradiction. Let us assume that there exists a sequence $(v_n, s_n) \in V$ such that

$$\|(v_n, s_n)\| \in V = 1 \text{ but } \|(v_n, s_n)\|_{\|} \to 0 \text{ when } n \to +\infty. \tag{2.18}$$

By extracting a subsequence, still denoted (v_n, s_n), we may assume that there exists $(v, s) \in V$ such that

$$(v_n, s_n) \rightharpoonup (v, s) \quad \text{weakly in } H^1(\omega; \mathbb{R}^3) \times H^1(\omega; \mathbb{R}^3)$$

and

$$\gamma_{\alpha\beta}(v_n) \rightharpoonup \gamma_{\alpha\beta}(v),$$

$$\chi_{\alpha\beta}(v_n, s_n) \rightharpoonup \chi_{\alpha\beta}(v, s), \text{and } \delta_{\alpha 3}(v_n, s_n) \rightharpoonup \delta_{\alpha 3}(v, s),$$

weakly in $L^2(\omega)$. By hypothesis (2.18), the three tensors tend strongly to zero in $L^2(\omega)$, and using Lemma 2.3 and the discussion above, we infer that $v = s = 0$. Then, Rellich's lemma implies that v_n and s_n both tend to zero strongly in $L^2(\omega; \mathbb{R}^3)$.

Let us introduce the two-dimensional vector $\mathbf{w}_n = (w_n)_\alpha = v_n \cdot \mathbf{a}_\alpha$. We have, $\mathbf{w}_n \to 0$ in $L^2(\omega; \mathbb{R}^2)$ strongly. Let us define $2e_{\alpha\beta}(w) = \partial_\alpha w'_\beta + \partial_\beta w_\alpha$. It is easy to see that

$$e_{\alpha\beta}(w_n) = \gamma_{\alpha\beta}(v_n) + \frac{1}{2} v_n \cdot (\partial_\beta \mathbf{a}_\alpha + \partial_\alpha \mathbf{a}_\beta) \to 0 \quad \text{strongly in} \quad L^2(\omega). \tag{2.19}$$

Indeed, $\partial_\beta \mathbf{a}_\alpha \in L^\infty(\omega)$. Then, by the two-dimensional Korn inequality, we deduce that

$$\mathbf{w}_n \to 0 \quad \text{strongly in} \quad H^1(\omega; \mathbb{R}^2). \tag{2.20}$$

Next we note that

$$\partial_\rho v_n \cdot \mathbf{a}_\alpha = \partial_\rho (\mathbf{w}_n)_\alpha - v_n \cdot \partial_\rho \mathbf{a}_\alpha \to 0 \quad \text{strongly in} \quad L^2(\omega). \tag{2.21}$$

Moreover, as $s_n \to 0$ strongly in $L^2(\omega; \mathbb{R}^3)$, and $\partial_\rho v_n \cdot \mathbf{a}_3 = 2\delta_{\rho 3}(v_n, s_n) - s_n \cdot \mathbf{a}_\alpha$, we already know that

$$\partial_\rho v_n \cdot \mathbf{a}_3 \to 0 \quad \text{strongly in} \quad L^2(\omega). \tag{2.22}$$

We deduce that

$$\partial_\rho v_n = (\partial_\rho v_n \cdot \mathbf{a}_i)\mathbf{a}_i \to 0 \quad \text{strongly in} \quad L^2(\omega; \mathbb{R}^3), \tag{2.23}$$

by (2.22) and (2.23) and because $\mathbf{a}_i \in L^\infty(\omega; \mathbb{R}^3)$ and $\mathbf{a}_i \in L^\infty(\omega; \mathbb{R}^3)$. It follows that $v_n \to 0$ strongly in $H^1(\omega; \mathbb{R}^3)$.

We use a similar argument to prove that $s_n \to 0$ strongly in $H^1(\omega; \mathbb{R}^3)$. Let $\mathbf{w'}_n = (w_n)_\alpha = s_n \cdot \mathbf{a}_\alpha$. Then we deduce that $\mathbf{w}_n \to 0$ strongly in $L^2(\omega, \mathbb{R}^2)$. On the other hand, we see that

$$2e_{\alpha\beta}(\mathbf{w'}_n) \to 0 \quad \text{strongly in} \quad L^2(\omega). \tag{2.24}$$

Indeed,
$2e_{\alpha\beta}(\mathbf{w'}_n) = 2\chi_{\alpha\beta}(v_n, s_n) - (\partial_\alpha v_n \cdot \partial_\beta \mathbf{a}_3 + \partial_\beta v_n \cdot \partial_\alpha \mathbf{a}_3) + s_n \cdot (\partial_\alpha \mathbf{a}_\beta + \partial_\beta \mathbf{a}_\alpha)$.
Thus, again by the two-dimensional Korn inequality, we conclude that

$$\mathbf{w'}_n \to 0 \quad \text{strongly in} \quad H^1(\omega; \mathbb{R}^2) \tag{2.25}$$

Consequently, since $s_n = (s_n \cdot \mathbf{a}_\alpha)\mathbf{a}_\alpha$, it follows that $s_n \to 0$ strongly in $H^1(\omega; \mathbb{R}^3)$.

Combining now the convergence of v_n and s_n, we see that $\|(v_n, s_n)\|_V \to 0$, which contradicts the hypothesis and proves the lemma.

Proof of Theorem 2.2. The bilinear and linear forms of problem (2.10) are clearly continuous on the space V. We have just shown that the bilinear form is V-elliptic. We use the Lax-Milgram theorem to conclude.

Remark 2.6. Upon considering the discretization process, we notice that the tangency constraint $s \cdot \mathbf{a}_3 = 0$ in the definition of $V(\omega)$ can be addressed by introducing a Lagrange multiplier. We define the relaxed function space as

$$X(\omega) = H^1_{\gamma_0}(\omega)^3 \times H^1_{\gamma_0}(\omega)^3,$$

equipped with the norm denoted by $\|\cdot\|_{X(\omega)}$. Similarly, we set $M(\omega) = H^1_{\gamma_0}(\omega)$. The forms $a(\cdot,\cdot)$ and $L(\cdot)$, defined as in (2.12) and (2.13) respectively, remain continuous on $X(\omega)\times X(\omega)$ and $X(\omega)$.

We pose the variational problem:

$$\begin{aligned} &\text{Find } (U,\psi) \text{ in } X(\omega)\times M(\omega) \text{ such that}\\ &\forall V\in X(\omega),\ a(U,V)+b(V,\psi)=L(V),\\ &\qquad\qquad\forall\chi\in M(\omega),\ b(U,\chi)=0, \end{aligned} \tag{2.26}$$

where the bilinear form $b(\cdot,\cdot)$ is defined by

$$b(V,\chi)=\int_\omega \partial_\alpha(s\cdot\mathbf{a}_3)\partial_\alpha\chi\,\mathrm{d}x. \tag{2.27}$$

Since $\mathbf{a}_3$ belongs to $W^{1,\infty}(\omega)^3$, the form $b(\cdot,\cdot)$ is continuous on $X(\omega)\times M(\omega)$. Moreover, the characterization holds:

$$V(\omega)=\{V=(v,s)\in X(\omega):\forall\chi\in M(\omega),\ b(V,\chi)=0\}.$$

The inf-sup condition on the form $b(\cdot,\cdot)$, derived by considering $V=(0,\chi\mathbf{a}_3)$, ensures the existence of a positive constant c_* such that

$$\forall\chi\in M(\omega),\qquad \sup_{V\in X(\omega)}\frac{b(V,\chi)}{\|V\|_{X(\omega)}}\ge c_*\|\chi\|_{H^1(\omega)}. \tag{2.28}$$

Combining these conditions with the ellipticity property $\|U\|_{V(\omega)}\le c\|L\|$ ensures the well-posedness of problem (2.27). This formulation offers a straightforward approach to handling the contact problem numerically.

Remark 2.7 Problem (2.11) can also be expressed as a system of partial differential equations. Let us define the operator A from $H^1(\omega)^3\times H^1(\omega)^3$ into $H^{-1}(\omega)^3\times H^{-1}(\omega)^3$ by duality:

$$\forall V\in H^1_0(\omega)^3\times H^1_0(\omega)^3,\quad \langle AU,V\rangle=a(U,V),$$

and its associated Neumann operator N defined from the same space into the dual space of $H^{1/2}_{00}(\gamma_1)^3\times H^{1/2}_{00}(\gamma_1)^3$ by:

$$\forall V\in H^1_{\gamma_0}(\omega)^3\times H^1_{\gamma_0}(\omega)^3,\quad \langle NU,V\rangle=a(U,V)-\langle AU,V\rangle$$

(this requires further regularity, which we assume here). Thus, it can be verified that problem (2.2) translates to the following system, in the sense of distributions:

$$\begin{cases} AU = \begin{pmatrix} f\sqrt{a} \\ 0 \end{pmatrix} \text{ in } \omega, \\ r \cdot a_3 = 0 \text{ in } \omega, \\ u = r = 0 \text{ on } \gamma_0, \\ NU = \begin{pmatrix} Nl \\ Ml \end{pmatrix} \text{ on } \gamma_1. \end{cases} \tag{2.29}$$

A similar formulation can also be derived for problem (2.27). An explicit form of the operators A and N is provided .

proof: It can also be noted that the quantity $a(U, V)$ can be written in another form . Indeed, we introduce the contravariant components of the following vectors:

- Stress resultant $n^{\rho\sigma}(u)$:

$$n^{\rho\sigma}(u) = ea^{\alpha\beta\rho\sigma}\gamma_{\alpha\beta}(u), \tag{2.30}$$

- Stress couple $m^{\rho\sigma}(U)$:

$$m^{\rho\sigma}(U) = \frac{e^3}{12}a^{\alpha\beta\rho\sigma}\chi_{\alpha\beta}(U), \tag{2.31}$$

- Transverse shear force $t^{\beta}(U)$:

$$t^{\beta}(U) = e\frac{E}{1+\nu}a^{\alpha\beta}\delta_{\alpha 3}(U). \tag{2.32}$$

We also observe that:

$$\chi_{\rho\sigma}(V) = \theta_{\rho\sigma}(v) + \gamma_{\rho\sigma}(s), \quad \text{with} \quad \theta_{\rho\sigma}(v) = \frac{1}{2}(\partial_\rho v \cdot \partial_\sigma a_3 + \partial_\sigma v \cdot \partial_\rho a_3). \tag{2.33}$$

Thus, $a(U, V)$ is equal to:

$$\begin{aligned} a(U,V) = &\int_\omega \left(n^{\rho\sigma}(u)\gamma_{\rho\sigma}(v) + m^{\rho\sigma}(U)\theta_{\rho\sigma}(v) + t^{\beta}(U)\partial_\beta v \cdot a_3\right)\sqrt{a}\,dx \\ &+ \int_\omega \left(m^{\rho\sigma}(U)\gamma_{\rho\sigma}(s) + t^{\beta}(U)s \cdot a_\beta\right)\sqrt{a}\,dx. \end{aligned} \tag{2.34}$$

and ,$l(v)$ is equal to:

$$l(v) = \int_\omega f \cdot v\sqrt{a}\,dx + \int_\omega \lambda\mathbf{e}_3 \cdot (v - \frac{e(x)}{2}s)\sqrt{a}\,dx + \int_{\gamma_1} (N \cdot v + M \cdot s)\sqrt{a}d\gamma. \tag{2.35}$$

with the bilinear form $b(\cdot, \cdot)$

$$b(V, \psi) = -a_3 \partial_\rho \left(\partial_\sigma \psi\right) \cdot s$$

This form clearly shows how the components v and s of the test function are decoupled, facilitating the implementation of the computations.

Using the "now" form and the symmetry properties $n^{\rho\sigma}(u) = n^{\sigma\rho}(u)$ and $m^{\rho\sigma}(U) = m^{\sigma\rho}(U)$, we can demonstrate that problem (2.6) is equivalent to the following system of partial differential equations.Here, $v = (v_1, v_2)$ represents the unit outward normal vector to w:

$$\gamma_{\rho\sigma}(v) = \frac{1}{2}\left(\partial_\rho v \cdot a_\sigma + \partial_\sigma v \cdot a_\rho\right) = \frac{1}{2}\left(-\partial_\rho a_\sigma - \partial_\sigma a_\rho\right) \cdot v = -\partial_\rho a_\sigma \cdot v$$

and

$$\theta_{\rho\sigma}(v) = \frac{1}{2}(\partial_\rho v \cdot \partial_\sigma a_3 + \partial_\sigma v \cdot \partial_\rho a_3) = \frac{1}{2}(-\partial_\rho \partial_\sigma a_3 - \partial_\sigma \partial_\rho a_3) \cdot v = -\partial_\rho(\partial_\sigma a_3) \cdot v$$

and

$$\partial_\rho v \cdot a_3 = -\partial_\rho a_3 \cdot v$$

In the end we find this form:

$$\begin{cases} -\partial_\rho\left(\left(n^{\rho\sigma}(u)a_\sigma + m^{\rho\sigma}(U)\partial_\sigma a_3 + t^\rho(U)a_3\right)\sqrt{a}\right) = f\sqrt{a} & \text{in } \omega, \\ -\partial_\rho\left(m^{\rho\sigma}(U)a_\sigma\sqrt{a}\right) + t^\beta(U)a_\beta\sqrt{a} - \partial_{\rho\rho}\psi\, a_3 = 0 & \text{in } \omega, \\ r \cdot a_3 = 0 & \text{in } \omega, \\ u = r = 0 & \text{on } \gamma_0, \\ \nu_\rho\left(n^{\rho\sigma}(u)a_\sigma + m^{\rho\sigma}(U)\partial_\sigma a_3 + t^\rho(U)a_3\right)\sqrt{a} = N\ell & \text{on } \gamma_1, \\ \nu_\rho\left(m^{\rho\sigma}(U)a_\sigma\sqrt{a} + \partial_\rho\psi\, a_3\right) = M\ell & \text{on } \gamma_1. \end{cases} \tag{2.36}$$

In conclusion, we observe that the shell is firmly affixed to the obstacle if and only if the function Φ vanishes on γ_0.

Chapter 3

The contact model

3.1 contact problem

The Naghdi model represents a two-dimensional shell structure. In the preceding section, focus was solely on the shell's mid-surface, denoted as M. This model belongs to the Reissner shell family, which is grounded in the theory of Cosserat surfaces. The contact model's derivation assumes that the distance between a point and the mid-surface remains constant during shell deformation. Additionally, it's presumed that points aligned along a normal line to the mid-surface will stay aligned, though this line will no longer be normal to the deformed mid-surface.

Consequently, the primary unknowns in this two-dimensional model are the displacement (u) of points on the shell mid-surface and the linearized rotation field (r), which describes the normal straight fiber rotations of the mid-surface. Mathematically, this implies that the displacement of a point $\phi(x) + za_3(x)$ is the vector $u(x) + zr(x)$.

It's noteworthy that the rotation field r should be tangential in a linearized theory. This is because the rotation vector associated with the unit normal vector of the displacement field of the mid-surface is tangential to a first-order approximation. Thus, the linearized rotation vector r has a zero component over the vector a_3. For further elaboration.

A shell S with midsurface $M = \phi(\omega)$ and thickness e is given by:

$$S = \left\{ \phi(x) + za_3(x) \, , \, x \in \overline{\omega}, -\frac{e(x)}{2} \leq z \leq \frac{e(x)}{2} \right\} \tag{3.1}$$

In this context, z denotes the distance from a point on the shell to its mid-surface. Let $\{e_1, e_2, e_3\}$ represent the Cartesian basis in $\mathbb{R}^3$. Our focus lies in investigating the contact between the shell and a rigid obstacle situated in the half-space $z \cdot e_3 < 0$, where the boundary of the obstacle extends across the plane $z \cdot e_3 = 0$. Henceforth, we assume, without loss of generality, that the function ϕ satisfies $\phi(x) \cdot e_3 > 0$ for all x in ω. This criterion ensures that contact occurs predominantly on the lower surface of the shell, specifically on the surface given by $\{\phi(x) - \frac{e(x)}{2} a_3 \,|\, x \in \omega\}$.

However, from a numerical point of view, we immediately encounter a problem since the constraint $r \cdot a_3 = 0$ clearly cannot be implemented in a conforming way for a general shell. We thus introduce the convex set in which the unknowns are the displacement u and r, elements of the space $X(\omega)$ without any constraint:

$$N_\Phi(\omega) = \{V \in X(\omega) : (v - \frac{e}{2} s) \cdot e_3 \geq \Phi \text{ a.e. in } \omega\},$$

together with the space $M(\omega) = H^1_{\gamma_0}(\omega)$, and consider the problem:
Find (U, ψ) in $N_\Phi(\omega) \times M(\omega)$ such that

$$a(U, V - U) + b(V - U, \psi) \geq L(V - U), \quad \forall V \in N_\Phi(\omega), \tag{3.2}$$

$$b(U, \chi) = 0, \quad \forall \chi \in M(\omega), \tag{3.3}$$

where the form $b(\cdot, \cdot)$ is now defined by

$$b(V, \chi) = \int_\omega \partial_\alpha (s \cdot a_3) \partial_\alpha \chi \, d\omega. \tag{3.4}$$

It is readily checked that, for any solution (U, ψ) of problem (3.4), its part U is a solution of problem (3.16). Let us check now the well-posedness of this problem. We need a preliminary lemma for that. Let $N](\omega)$ be the following subspace

$$N_{\#}(\omega) = \{V \in X(\omega) : (v - \frac{e}{2}s) \cdot e_3 = 0 \text{ a.e. in } \omega\} \tag{3.5}$$

Lemma 3.1: There exists a constant $\beta > 0$ such that the following inf-sup condition holds:

$$\forall \chi \in M(\omega), \qquad \sup_{V \in N](\omega)} \frac{b(V,\chi)}{\|V\|_{X(\omega)}} \geq \beta \|\chi\|_{M(\omega)}. \tag{3.6}$$

Theorem 3.2: Assume that the function Φ satisfies

$$\Phi(x) \leq 0 \text{ for a.e. } x \text{ in } \omega \text{ and } \Phi(x) = 0 \text{ for a.e. } x \text{ on } \gamma 0. \tag{3.7}$$

Then, for any data $(f, N, M) \in L^2(\omega; \mathbb{R}^3) \times L^2(\gamma 1; \mathbb{R}^3) \times L^2(\gamma 1; \mathbb{R}^3)$, problem (2.16) has a unique solution $(U, \psi) \in X(\omega) \times M(\omega)$.

As is standard for contact models, the contact condition involves three equations or inequalities:

1. The relative positions of the shell and the obstacle,
2. The reaction of the obstacle on the shell, and
3. The location of this reaction.

We will now describe each of these components in detail.

3.1.1 Positions of the Shell and the Obstacle

In accordance with the shell model, the deformed shell S^* has its mid-surface M^* defined as $\phi^*(\omega)$, where ϕ^* equals $\phi + u$. Consequently, we set $\mathbf{a}_3^* = \mathbf{a}_3 + \mathbf{r}$.

Since the shell is assumed to exhibit neither pinching nor dilation, the domain S^* is expressed as:

$$S^* = \{\phi^*(x) + z\mathbf{a}_3^*(x)\,,\, x \in \omega, -\frac{e(x)}{2} \leq z \leq \frac{e(x)}{2}\} \tag{3.8}$$

Here, z still represents the distance from a point of the shell to the mid-surface M^*. Thus, verifying that the shell lies above the obstacle entails:

$$\forall x \in \omega, \forall z \in \left[-\frac{e(x)}{2}, \frac{e(x)}{2}\right], (\phi^*(x) + z\mathbf{a}_3^*(x)) \cdot \mathbf{e}_3 \geq 0,$$

or equivalently:

$$\forall x \in \omega, \left(\phi(x) + u(x) - \frac{e(x)}{2}(\mathbf{a}_3(x) + \mathbf{r}(x))\right) \cdot \mathbf{e}_3 \geq 0.$$

Introducing:

$$\Phi(x) = \left(\frac{e(x)}{2}\mathbf{a}_3(x) - \phi(x)\right) \cdot \mathbf{e}_3, \tag{3.9}$$

the first contact inequality can be expressed as:

$$\left(u - \frac{e(\cdot)}{2}\mathbf{r}\right) \cdot \mathbf{e}_3 \geq \Phi \text{ in } \omega. \tag{3.10}$$

Let ω_c denote the contact zone, i.e., the set of points x in ω such that:

$$\left(u(x) - \frac{e(x)}{2}\mathbf{r}(x)\right) \cdot \mathbf{e}_3 = \Phi(x). \tag{3.11}$$

3.1.2 Obstacle's Reaction

In our considered scenario, the obstacle's response to the presence of the shell is characterized by the scalar function λ, resulting in a reaction of the form $\lambda\mathbf{e}_3$. Consequently, in the equation's right-hand side, the term $\int_\omega f \cdot v\sqrt{a}\, dx$ must be adjusted to:

$$\int_\omega f \cdot v\sqrt{a}\, dx + \int_\omega \lambda\mathbf{e}_3 \cdot (v - \frac{e(x)}{2}s)\sqrt{a}\, dx$$

This modification accounts for the obstacle's resistance against the lower section of the shell, represented by $\left(v - \frac{e(\cdot)}{2}s\right)$.

Furthermore, given the shell's positioning above the obstacle, it follows that

$$\lambda \geq 0. \tag{3.12}$$

3.1.3 Location of the Reaction

Naturally, the obstacle's reaction is confined to the contact zone ω_c as defined by equation (3.15). This condition gives rise to the complementarity equation:

$$\lambda\left(u - \frac{e(\cdot)}{2} r \cdot \mathbf{e}_3 - \Phi\right) = 0 \text{ in } \omega. \tag{3.13}$$

By incorporating these considerations into problem (2.34), we derive the model for the shell's contact. In this model, the unknowns consist of the shell's deformation u, its rotation r, and the reaction coefficient.

$$\begin{cases} AU - \begin{pmatrix} \lambda \mathbf{e}_3 \sqrt{a} \\ -\lambda \mathbf{e}_3 \frac{e(\cdot)}{2} \sqrt{a} \end{pmatrix} = \begin{pmatrix} f\sqrt{a} \\ 0 \end{pmatrix} & \text{in } \omega, \\ r \cdot \mathbf{a}_3 = 0 & \text{in } \omega, \\ u - \frac{e(\cdot)}{2} r \cdot \mathbf{e}_3 \geq \Phi, \quad \lambda \geq 0, \quad \lambda\left(u - \frac{e(\cdot)}{2} r \cdot \mathbf{e}_3 - \Phi\right) = 0 & \text{in } \omega, \\ u = r = 0 & \text{on } \gamma_0, \\ NU = \begin{pmatrix} Nl \\ Ml \end{pmatrix} \end{cases} \tag{3.14}$$

It is noteworthy that the inequalities associated with the contact solely pertain to the third components of the unknowns. Hence, it is crucial to operate within Cartesian coordinates in this context

proof: It can also be noted that the quantity $a(U, V)$ can be written in another form which seems more appropriate for implementation, as it decouples the two components v and s of the test function V. Indeed, we introduce the contravariant components of the following vectors:

- Stress resultant $n^{\rho\sigma}(u)$:

$$n^{\rho\sigma}(u) = e a^{\alpha\beta\rho\sigma} \gamma_{\alpha\beta}(u), \tag{3.15}$$

- Stress couple $m^{\rho\sigma}(U)$:

$$m^{\rho\sigma}(U) = \frac{e^3}{12} a^{\alpha\beta\rho\sigma} \chi_{\alpha\beta}(U), \tag{3.16}$$

- Transverse shear force $t^\beta(U)$:

$$t^\beta(U) = e\frac{E}{1+\nu}a^{\alpha\beta}\delta_{\alpha 3}(U). \tag{3.17}$$

We also observe that:

$$\chi_{\rho\sigma}(V) = \theta_{\rho\sigma}(v) + \gamma_{\rho\sigma}(s), \quad \text{with} \quad \theta_{\rho\sigma}(v) = \frac{1}{2}(\partial_\rho v \cdot \partial_\sigma a_3 + \partial_\sigma v \cdot \partial_\rho a_3). \tag{3.18}$$

Thus, $a(U, V)$ is equal to:

$$\begin{aligned} a(U,V) = \int_\omega \left(n^{\rho\sigma}(u)\gamma_{\rho\sigma}(v) + m^{\rho\sigma}(U)\theta_{\rho\sigma}(v) + t^\beta(U)\partial_\beta v \cdot a_3\right)\sqrt{a}\,dx \\ + \int_\omega \left(m^{\rho\sigma}(U)\gamma_{\rho\sigma}(s) + t^\beta(U)s \cdot a_\beta\right)\sqrt{a}\,dx. \end{aligned} \tag{3.19}$$

and ,$l(v)$ is equal to:

$$l(v) = \int_\omega f \cdot v\sqrt{a}\,dx + \int_\omega \lambda \mathbf{e}_3 \cdot (v - \frac{e(x)}{2}s)\sqrt{a}\,dx + \int_{\gamma_1} (N \cdot v + M \cdot s)\sqrt{a}d\gamma. \tag{3.20}$$

with the bilinear form $b(\cdot,\cdot)$

$$b(V,\psi) = -a_3\partial_\rho\left(\partial_\sigma\psi\right) \cdot s$$

This form clearly shows how the components v and s of the test function are decoupled, facilitating the implementation of the computations.

Using the "now" form and the symmetry properties $n^{\rho\sigma}(u) = n^{\sigma\rho}(u)$ and $m^{\rho\sigma}(U) = m^{\sigma\rho}(U)$, we can demonstrate that problem (2.6) is equivalent to the following system of partial differential equations.Here, $v = (v_1, v_2)$ represents the unit outward normal vector to w:

$$\gamma_{\rho\sigma}(v) = \frac{1}{2}\left(\partial_\rho v \cdot a_\sigma + \partial_\sigma v \cdot a_\rho\right) = \frac{1}{2}\left(-\partial_\rho a_\sigma - \partial_\sigma a_\rho\right) \cdot v = -\partial_\rho a_\sigma \cdot v$$

and

$$\theta_{\rho\sigma}(v) = \frac{1}{2}(\partial_\rho v \cdot \partial_\sigma a_3 + \partial_\sigma v \cdot \partial_\rho a_3) = \frac{1}{2}(-\partial_\rho\partial_\sigma a_3 - \partial_\sigma\partial_\rho a_3) \cdot v = -\partial_\rho(\partial_\sigma a_3) \cdot v$$

and

$$\partial_\rho v \cdot a_3 = -\partial_\rho a_3 \cdot v$$

The problem suggests that the integrals can be expressed separately in terms of v and s, and by utilizing the given symmetry properties, it's evident that the problem can be represented as a system of partial differential equations.

$$
\begin{cases}
-\partial_\rho\left((n^{\rho\sigma}(u)a_\sigma + m^{\rho\sigma}(U)\partial_\sigma a_3 + t^\rho(U)a_3)\sqrt{a}\right) - \lambda\, e_3\sqrt{a} = f\sqrt{a} & \text{in } \omega,\\
-\partial_\rho\left(m^{\rho\sigma}(U)a_\sigma\sqrt{a}\right) + t^\beta(U)a_\beta\sqrt{a} - \partial_{\rho\rho}\psi\, a_3 + \lambda\, e_3\dfrac{e(.)}{2}\sqrt{a} = 0 & \text{in } \omega,\\
r\cdot a_3 = 0 & \text{in } \omega,\\
u - \dfrac{e(\cdot)}{2}r\cdot \mathbf{e}_3 \geq \Phi, \quad \lambda \geq 0, \quad \lambda\left(u - \dfrac{e(\cdot)}{2}r\cdot\mathbf{e}_3 - \Phi\right) = 0 & \text{in } \omega,\\
u = r = 0 & \text{on } \gamma_0,\\
\nu_\rho\left(n^{\rho\sigma}(u)a_\sigma + m^{\rho\sigma}(U)\partial_\sigma a_3 + t^\rho(U)a_3\right)\sqrt{a} = N\ell & \text{on } \gamma_1,\\
\nu_\rho\left(m^{\rho\sigma}(U)a_\sigma\sqrt{a} + \partial_\rho\psi\, a_3\right) = M\ell & \text{on } \gamma_1.
\end{cases}
\tag{3.21}
$$

In conclusion, we observe that the shell is firmly affixed to the obstacle if and only if the function Φ vanishes on γ_0. This assumption aligns with a realistic physical scenario, and its mathematical significance will become apparent in the subsequent section.

3.2 Variational formulation and well-posedness

We introduce the following cones:

$$
H^1_{\gamma_0+}(\omega) = \left\{\sigma \in H^1(\gamma_0(\omega))\,;\, \sigma \geq 0 \text{ a.e. in } \omega\right\}, \tag{3.22}
$$

and

$$
\Lambda = \left\{\mu \in H^1_{\gamma 0}(\omega))\,;\, \forall\sigma \in H^1_{\gamma_0+}(\omega), \langle\sigma,\mu\rangle \geq 0\right\}. \tag{3.23}
$$

Where $H^1_{\gamma_0}(\omega)'$ stands for the dual space of $H^1_{\gamma_0}(\omega)$ and $\langle\cdot,\cdot\rangle$ is the duality pairing between $H^1_{\gamma_0}(\omega)$ and $H^1_{\gamma_0}(\omega)'$. Next, in view of Section 2, we consider the following variational problem:

Find $(U,\psi,\lambda) \in X(\omega)\times M(\omega)\times\Lambda$ such that

$$
\begin{aligned}
&\forall V \in X(\omega), \quad a(U,V) + b(V,\psi) - c(V,\lambda) = L(V),\\
&\forall \chi \in M(\omega), \quad b(U,\chi) = 0,\\
&\forall \mu \in \Lambda, \quad c(U,\mu-\lambda) \geq \langle\Phi\sqrt{a},\mu-\lambda\rangle.
\end{aligned}
\tag{3.24}
$$

where the forms a(., .), b(., .) and L(.) have been introduced in (2.18), (2.34), and (2.19), respectively, while the form c(., .) is given by

$$c(V, \mu) = \langle (v - \frac{e(\cdot)}{2} s) \cdot e_3 \sqrt{a}, \mu \rangle \tag{3.25}$$

In this formulation, the duality pairing can be substituted with an integral whenever μ belongs to $L^2(\Omega)$. However, for general functions μ in Λ, the expression on the right-hand side of the third line is meaningful only if the function Φ lies in $H^1_{\gamma_0}(\omega)$. Prior to assessing the equivalence of this problem with system (3.8), it's crucial to ensure this condition.

Proposition 3.3 If the partition of $\partial\omega$ into γ_0 and γ_1 is suitably smooth such that $D(\omega \cup \gamma_1)$ is dense in $H^1_{\gamma_0}(\omega)$, then any triple (U, ψ, λ) in $X(\omega) \times M(\omega) \times \Lambda$ solves problem (3.11) if and only if it satisfies system (3.8) in the distributional sense.

Proof: Given the preceding assumption, the first equation in (3.17) corresponds to the first and fifth lines in (3.14) (also refer to (2.34) for the Lagrange multiplier ψ), while the second equation in (3.17) aligns with the second line in (3.14). Furthermore, the boundary conditions on u and r on γ_0, as well as the non-negativity of λ, stem from the definitions of the space $H^1_{\gamma_0}(\omega)$ and the cone Λ. Thus, it remains to verify the equivalence of the third line in (3.17) to the appropriate terms.

$$u - \frac{e(\cdot)}{2} r \cdot e_3 \geq \Phi \quad \text{and} \quad \lambda \left(u - \frac{e(\cdot)}{2} r \cdot e_3 - \Phi \right) = 0 \quad \text{in} \quad \omega \tag{3.26}$$

1. If (U, λ) satisfies (3.19), we have for all μ in Λ

   $$c(U, \mu - \lambda) = c(U, \mu) - \langle \Phi\sqrt{a}, \lambda \rangle$$

   whence, due to the nonnegativity of μ,

   $$c(U, \mu - \lambda) \geq \langle \Phi\sqrt{a}, \mu - \lambda \rangle$$

   which is the third line in (3.17).

2. Conversely, if (U, λ) satisfies the third line of (3.17), taking μ equal to $\lambda + \chi_O$ where χ_O is the characteristic function of any measurable subset O of ω, we have

   $$\int_O u - \frac{e(x)}{2} r \cdot e_3 \sqrt{a}\, dx \geq \int_O \Phi\sqrt{a}\, dx$$

which yields the first part of (3.19). On the other hand, taking successively μ equal to 0 and to 2λ leads to

$$\langle (u - \frac{e(\cdot)}{2} r) \cdot e_3 \sqrt{a}, \lambda \rangle = \langle \Phi \sqrt{a}, \lambda \rangle$$

Since both quantities $(u - \frac{e(\cdot)}{2} r \cdot e_3 - \Phi)$ and λ are nonnegative, this gives the second part of (3.19).

The assumption stated in Proposition 3.1 holds true for all geometries under consideration. Now, we delve into the analysis of problem (3.11). While a similar, albeit simpler, system has

$$\begin{cases} \text{Find}(U, \lambda) \ \text{in} V(\omega) \times \Lambda \text{such that} \\ \forall V \in V(\omega), \quad a(U, V) - c(V, \lambda) = L(V), \\ \forall \mu \in \Lambda, \quad c(U, \mu - \lambda) \geq \langle \Phi \sqrt{a}, \mu - \lambda \rangle. \end{cases} \tag{3.27}$$

The next statement is a direct consequence of the inf-sup condition (2.35).

Proposition 3.4 establishes the equivalence between problems (3.17) and (3.19) in the following manner:

i If (U, ψ, λ) is a solution to problem (3.17), then (U, λ) is a solution to problem (3.19).

ii Conversely, if (U, λ) solves problem (3.20), then , there exists a unique ψ in $M(\omega)$ such that (U, ψ, λ) is a solution to problem (3.17).

However, this equivalence alone does not guarantee the existence of a solution for problem (3.17). To address this, we introduce the convex set

$$K_\Phi(\omega) = \{V \in V(\omega) : v - \frac{e(\cdot)}{2s} \cdot e_3 \geq \Phi \text{ almost everywhere in } \omega\} \tag{3.28}$$

From Propositions 3.1 and 3.2, we observe that the solution U of problem (3.19) belongs to $K_\Phi(\omega)$. Consequently, we pose the problem:

Find U in $K_\Phi(\omega)$ such that

$$\forall V \in K_\Phi(\omega), \quad a(U, V - U) \geq L(V - U). \tag{3.29}$$

With this, we are now ready to present the following result.

Proposition 3.5. Assume that the function Φ satisfies

$$\Phi(x) \leq 0 \text{ for almost every } x \text{ in } \omega. \tag{3.30}$$

Then, for any data (f, N, M) in $L^2(\omega)^3 \times L^2(\gamma_1)^3 \times L^2(\gamma_1)^3$, problem (3.22) has a unique solution U in $K_\Phi(\omega)$.

proof: To establish the existence and uniqueness of a solution for problem (3.17), we observe that the set $K_\Phi(\omega)$ is closed, convex, and non-empty due to assumption (3.17). This property, along with the ellipticity property $a(V, V) \geq c^* \|V\|^2_{V(\omega)}$, allows us to directly apply the Lions-Stampacchia theorem.

The condition (3.30) arises from the positivity constraint of $\varphi \cdot e_3$ when the thickness e is moderate. This condition ensures the feasibility of the problem. To further confirm the existence of a solution for problem (3.19), and consequently for problem (3.30), we proceed to examine the properties of the form $c(\cdot, \cdot)$.

Lemma 3.6: The form $c(\cdot, \cdot)$ is continuous on $X(\omega) \times H^1_{\gamma_0}(\omega)_0$ and satisfies the inf-sup condition with a constant $\delta > 0$, expressed as:

$$\forall \mu \in H^1_{\gamma_0}(\omega)_0, \quad \sup_{V \in V(\omega)} \frac{c(V, \mu)}{\|V\|_{X(\omega)}} \geq \delta \|\mu\|_{H^1_{\gamma_0}(\omega)_0} \tag{3.31}$$

Proof: The continuity of $c(\cdot, \cdot)$ stems from its definition and the continuity of the mapping $V \mapsto (v - \frac{e(\cdot)}{2s}) \cdot e_3 \sqrt{a}$ from $X(\omega)$ to $H^1_{\gamma_0}(\omega)$. Additionally, for any μ in $H^1_{\gamma_0}(\omega)_0$, the Lax-Milgram lemma combined with the Poincare-Friedrichs inequality ensures the unique existence of $\sigma \in H^1_{\gamma_0}(\Omega)$ satisfying:

$$\forall \rho \in H^1_{\gamma_0}(\Omega), \quad \int_\Omega \nabla \sigma \cdot \nabla \rho \, dx = \langle \rho, \mu \rangle \tag{3.32}$$

Moreover, the norm definition
$\|\mu\|_{H^1_{\gamma_0}(\omega)_0} \leq |\sigma|_{H^1(\omega)}$ is satisfied.

Furthermore, by considering $V = (v, 0)$ with $v = (0, 0, \sigma/\sqrt{a})$, it follows that
$c(V, \mu) = |\sigma|^2_{H^1(\omega)}$ and $\|V\|_{X(\omega)} \leq c\|\sigma\|_{H^1(\omega)}$.

Additionally, this V belongs to $V(\omega)$. Together with the Poincare-Friedrichs inequality, this establishes the desired condition.

Proposition 3.7: Assume that the function Φ satisfies (3.23) and

$$\Phi(x) = 0 \quad \text{for almost every} \quad x \quad \text{in}\gamma_0 \tag{3.33}$$

Then, for any data (f, N, M) in $L^2(\omega)^3 \times L^2(\gamma_1)^3 \times L^2(\gamma_1)^3$, problem (3.19) has at least one solution (U, λ) in $V(\omega) \times \Lambda$.

Proof: Let U be the solution of problem (3.22) (see Proposition 3.2). Let $K(\omega)$ denote the kernel of the form $c(\cdot, \cdot)$, namely the set

$$K(\omega) = \{V \in V(\omega) : v - \frac{e(\cdot)}{2s} \cdot e_3 = 0 \text{ a.e. in } \omega\}.$$

For any function W in $K(\omega)$, it is readily checked that the functions $V = U \pm W$ belong to $K_\Phi(\omega)$, so that applying problem (3.22) to these V yields $a(U, W) = L(W)$. Thus, the form $V \mapsto a(U, V) - L(V)$ belongs to the polar set of $K(\omega)$. It then follows from the inf-sup condition (3.24) that there exists a unique λ in $H^1_{\gamma_0}(\omega)_0$ such that

$$\forall V \in V(\omega), \quad c(V, \lambda) = a(U, V) - L(V). \tag{3.34}$$

We now wish to prove that (U, λ) is a solution of problem (3.19).

1) For all σ in $H^+_{\gamma_0}(\omega)$, we take $V = U + W$, with $W = (w, 0)$ and $w = (0, 0, \sqrt{\sigma/a})$. This function belongs to $K_\Phi(\omega)$, and it follows from problem (3.22) that

$$c(W, \lambda) = a(U, V - U) - L(V - U) \geq 0.$$

Since $c(W, \lambda)$ coincides with $\langle \sigma, \lambda \rangle$, we thus derive from definition (3.16) that λ belongs to Λ.

2) By taking $V = (v, 0)$ and $v = (0, 0, \Phi)$, and noting that this V belongs to $K_\Phi(\omega)$, we derive from (3.22) that

$$-c(U, \lambda) \geq -\langle \Phi\sqrt{a}, \lambda \rangle. \tag{3.35}$$

On the other hand, it follows from the definition (3.21) of $K_\Phi(\omega)$, together with assumption (3.26), that, for any μ in Λ,

$$c(U,\mu) \geq \langle \Phi\sqrt{a}, \mu\rangle.$$

By combining these two inequalities, we obtain

$$c(U,\mu-\lambda) \geq \langle \Phi\sqrt{a}, \mu-\lambda\rangle,$$

which is the second line of (3.19).

Thus, the pair (U,λ) belongs to $V(\omega)\times\Lambda$ and satisfies the two lines of problem (3.19).

Remark 3.8: It's conceivable that the variable λ in the solution exhibits a higher degree of regularity than indicated in the preceding proposition. However, to establish this additional regularity seem inapplicable in this context.

Proposition 3.9: Given that the function Φ satisfies conditions (3.23) and (3.26), for any given data (f,N,M) in $L^2(\omega)^3\times L^2(\gamma_1)^3\times L^2(\gamma_1)^3$, problem (3.19) admits at most one solution (U,λ) in $V(\omega)\times\Lambda$. Furthermore, the component U of this solution also satisfies problem (3.22).

Proof: We divide the proof into two steps, beginning with the second part of the statement.

1. Let (U,λ) be any solution of problem (3.19). For any subset O of Ω, with χ_O denoting the characteristic function of O, setting $\mu=\lambda+\chi_O$ in problem (3.19), we obtain
$$\int_O\left(u-\frac{e(x)^2r}{\sqrt{a}}\cdot e_3-\Phi(x)\right)dx\geq 0,$$
which implies that U belongs to $K_\Phi(\omega)$. Now, to prove that, for any V in $K_\Phi(\omega)$, $c(V-U,\lambda)$ is nonnegative, we first observe that inequality (3.26) still holds by setting μ equal to zero in (3.19). Second, using the definition (3.20) of $K_\Phi(\omega)$ and the fact that λ belongs to Λ, we obtain
$$c(V,\lambda)\geq\langle\frac{\Phi}{\sqrt{a}},\lambda\rangle. \tag{3.36}$$
Summing up these two inequalities yields the desired result.

2. Let (U_1, λ_1) and (U_2, λ_2) be two solutions of problem (3.19). It follows from the previous lines that U_1 and U_2 are solutions of problem (3.27). Thus, owing to Proposition 3.3, they are equal. The functions λ_1 and λ_2 satisfy (3.27), and from the inf-sup condition (3.24), they are equal. This yields the uniqueness of the solution of problem (3.19).

By combining Propositions 3.2, 3.5, and 3.7, we derive the main result of this section.

Theorem 3.10: Assume that the function Φ satisfies (3.23) and (3.26). Then, for any data (f, N, M) in $L^2(\omega)^3 \times L^2(\gamma_1)^3 \times L^2(\gamma_1)^3$, problem (3.16) has a unique solution (U, ψ, λ) in $X(\omega) \times M(\omega) \times \Lambda$.

In the subsequent discussion, our primary focus lies in the discretization of problem (3.22). It's noteworthy from Proposition 3.3 that the well-posedness of this problem only necessitates assumption (3.23). However, if assumption (3.26) is not satisfied, there exists no connection between this problem and the contact model.

Chapter 4

Finite element approximation

4.1 The discrete problem

Without constraints, we now consider ω as a polygon. Let $(\mathcal{T}_h)_h$ represent a regular family of triangulations of ω (using triangles), such that for each h:

- $\overline{\omega}$ encompasses all elements of $\mathcal{T}_h$.
- If not empty, the intersection of two distinct elements of $\mathcal{T}_h$ comprises either a vertex or an entire edge of both elements.
- The ratio of the diameter h_K of any element K in $\mathcal{T}_h$ to the diameter of its inscribed circle is less than a constant σ, independent of h.

As usual, h denotes the maximum diameter h_K, where $K \in T_h$. We also assume, without imposing restrictions, that both γ_0 and γ_1 consist of entire edges of elements of $\mathcal{T}_h$.

Given our primary interest in shells with minimal regularity - where classical formulations might not be suitable - pursuing higher-order elements to enhance convergence rates may not be fruitful. In the case of such shells, the underlying system of PDEs exhibits nonsmooth coefficients. Consequently, it remains uncertain whether elliptic regularity can be applied to achieve even H^2-regularity, let alone H^{k+1}-regularity with $k \geq 1$. However, it's worth noting that if the midsurface chart is smooth and we opt to use our formulation for simplicity compared to classical approaches, then elliptic regularity will apply.

Therefore, let us introduce the basic approximation spaces:

$$M_h = \{\mu_h \in H^1(\omega) : \forall K \in \mathcal{T}_h,\ \mu_h|_K \in P_1(K)\} \tag{4.1}$$

$$X^h_{\gamma_0} = \{v_h \in H^1_{\gamma_0}(\omega; \mathbb{R}^3) : \forall K \in \mathcal{T}_h,\ v_h|_K \in P_1(K)\} \tag{4.2}$$

where $P_1(K)$ stands for the space of restrictions to K of affine functions on $\mathbb{R}^2$. The spaces involved in the discrete problem are then:

$$M_h = M_h \cap H^1_{\gamma_0}(\omega), \quad X_h^{\gamma_0} = X^h_{\gamma_0} \times X^h_{\gamma_0}. \tag{4.3}$$

Let $\mathcal{I}_h$ denote the Lagrange interpolation operator with values in $(M_h)^3$. Since ϕ belongs to $W^{2,\infty}(\omega; \mathbb{R}^3)$, the function Φ is continuous on ω. Therefore, we define

$$\Phi_h = \mathcal{I}_h \Phi$$

. and introduce the discrete convex set:

$$N_{h,\Phi} = \left\{ V_h = (v_h, s_h) \in \mathbb{X}_h \mid \mathcal{I}_h \left((v_h - \frac{e}{2}\, s_h) \cdot e_3 \right) \geq \Phi_h \text{ a.e. in } \omega \right\} \tag{4.4}$$

This set's construction is relatively straightforward, as its condition is equivalent to a pointwise requirement: for all vertices a belonging to elements within $\mathcal{T}_h$,

$$\left((\mathbf{v}_h(a) - \frac{e}{2}\mathbf{s}_h(a)) \cdot \mathbf{e}_3 \right) \geq \Phi_h(a) \tag{4.5}$$

The choice of $N_{h,\Phi}$ is, of course, pivotal for constructing our discrete problem. However, given the potentially high variability of Φ, it appears impossible to select it as a subset of N_Φ.

Therefore, we are in a position to construct the discrete problem using the Galerkin method applied to problem (2.27) with a slight modification. It reads:

Find (U_h, ψ_h) in $N_{h,\Phi} \times M_{\gamma 0}^h$ such that

$$a(U_h, V_h - U_h) + b_h(V_h - U_h, \psi_h) \geq L(V_h - U_h), \quad \forall V_h \in N_{h,\Phi}, \tag{4.6}$$

$$b_h(U_h, \chi_h) = 0, \quad \forall \chi_h \in M_h^{\gamma_0}, \tag{4.7}$$

where the form $b_h(\cdot, \cdot)$ is defined on sufficiently smooth functions V and χ by

$$b_h(V, \chi) = \int_\omega \partial_\alpha I_h(s \cdot a_3) \partial_\alpha \chi \, dx. \tag{4.8}$$

remark.4.1 In implementing this problem, approximations of the scalar coefficients $a_{\alpha\beta}$, $a_{\alpha\beta\rho\sigma}$, $\sqrt{a}$, and ` in the space M_h are introduced alongside approximations of the vectors a_k and $\partial_\alpha a_3$ in the space $(M_h)^3$. This results in a modified discrete problem where the forms $a(\cdot, \cdot)$, $b(\cdot, \cdot)$, and L are replaced by $a_h(\cdot, \cdot)$, $b_h(\cdot, \cdot)$, and L_h respectively.

However, it has been verified for the a posteriori analysis that this modification only introduces technical difficulties in the proofs, without any major changes in the final estimates. Hence, we prefer to skip it in what follows.

Proving the well-posedness of problem (4.6) requires several steps. We first introduce the kernel

$$V_h = \{V_h \in X_h \mid \forall \chi_h \in M_h^{\gamma_0}, \ b_h(V_h, \chi_h) = 0\}. \tag{4.9}$$

It is readily checked that, for any solution (U_h, ψ_h) of problem (4.5), its part U_h is a solution of the problem: Find U_h in $N_h, \Phi \cap V_h$ such that:

$$a(U_h, V_h - U_h) \geq L(V_h - U_h), \ \forall V_h \in N_{h,\Phi} \cap V_h. \tag{4.10}$$

The next lemma provides a characterization of V_h.

Lemma 4.2: The space V_h coincides with the set of functions $V_h = (v_h, s_h)$ in X_h, provided that:

$$I_h(s_h \cdot a_3) = 0 \text{ in } \omega. \tag{4.11}$$

Proof: By setting χ_h equal to $I_h(s_h \cdot a_3)$ in the definition of V_h, we find that the gradient of $I_h(s_h \cdot a_3)$ is zero. Thus, the desired property follows from its vanishing on γ_0.

This lack of conformity in discretization complicates the proof of the well-posedness of problem (4.10) slightly. We commence with a technical lemma, and from now on, we assume that a_3 belongs to $W^{2,\infty}(\omega;\mathbb{R}^3)$.

Lemma 4.3: For any function s_h in M_h^3, the following estimate holds:

$$\|s_h \cdot a_3 - I_h(sh \cdot a_3)\|_{H^1(\omega)} \leq c_h \|s_h\|_{H^1(\omega;\mathbb{R}^3)}. \tag{4.12}$$

Proof: Employing the standard properties of the Lagrange interpolation operator with values in $M_{\gamma_0}^h$, we derive for all K in T_h that

$$\|s_h \cdot a_3 - I_h(sh \cdot a_3)\|_{H^1(K)} \leq c_h |sh \cdot a_3|_{H^2(K)}.$$

To evaluate $|sh \cdot a^3|_{H^2(K)}$, note that since $sh|_K$ belongs to $P_1(K)^3$, each partial derivative $\partial_{\alpha\beta}(sh \cdot a^3)$ can be expressed as

$$\partial_{\alpha\beta}(sh \cdot a_3) = \partial_\alpha s_h \cdot \partial_\beta a_3 + \partial_\beta s_h \cdot \partial_\alpha a_3 + s_h \cdot \partial_{\alpha\beta} a_3,$$

yielding the desired result.

Remark 4.4: Remark 4.2 highlights the possibility of employing a weaker assumption for a_3 by transitioning to the reference triangle $\hat{K}$ and utilizing the embedding of $H^1(\hat{K})$ into all $L^p(\hat{K})$, where $1 \leq p < +\infty$, to establish an analogous result to the estimate (4.12). However, for the sake of simplicity, we opted not to pursue this modification.

Hence, we are in a position to deduce the following ellipticity property.

Lemma 4.5: There exists a real number $h_0 > 0$ and a constant $\alpha_* > 0$ such that, for all $h \leq h_0$,

$$a(V_h, V_h)_\Omega \geq \alpha_* \|V_h\|^2_{X(\Omega)}, \quad \forall V_h \in V_h. \tag{4.13}$$

proof: For any $V_h = (v_h, s_h)$ in V_h, the function $V_h - W$, where $W = (0, (s_h \cdot a_3)a_3)$, belongs to $V(\Omega)$. Hence, applying the ellipticity property $\{a(V,V)_\Omega \geq \alpha \|V\|^2_X\}$, we obtain:

$$a(V_h - W, V_h - W) \geq \alpha \|V_h - W\|^2_{X(\Omega)}.$$

Denoting the norm of $a(\cdot,\cdot)$ on $X(\Omega)$ by c, we derive:

$$a(V_h, V_h) \geq \alpha \|V_h\|^2_{X(\Omega)} - 2\alpha \|V_h\|_{X(\Omega)} \|W\|_{X(\Omega)} - 2c \|V_h\|_{X(\Omega)} \|W\|_{X(\Omega)} - c\|W\|^2_{X(\Omega)}.$$

Since V_h belongs to $\mathcal{V}_h$, it follows from Lemma 4.1 that the quantity W equals:

$$(0, (s_h \cdot a_3 - I_h(s_h \cdot a_3))a_3).$$

Therefore, the desired ellipticity property is a direct consequence of Lemma 4.2.

This property implies that, for $h \le h_0$, problem (4.10) is well-posed. The arguments for proving that problem (4.7) has a unique solution are exactly the same as in Section 2; we only present a condensed version of the proofs where necessary. First, note that the formula:

$$b_h(V, \chi) = b(V, \chi) - \int_\Omega \partial_\alpha(s \cdot a_3 - I_h(s \cdot a_3))\partial_\alpha \chi \, dx,$$

and Lemma 4.2 yield the continuity of $b_h(\cdot, \cdot)$ on $X_h \times M_h$; furthermore, its norm is bounded independently of h. We also define the set:

$$N_{h,\#} = \{V_h \in X_h; I_h(v_h - \frac{e}{2}s_h) \cdot e_3 = 0 \text{ almost everywhere in } \omega\}. \tag{4.14}$$

Lemma 4.6: There exists a real number $h^* > 0$ and a constant $\beta^* > 0$ such that the following inf-sup condition holds for all $h \le h^*$:

$$\forall \chi_h \in M_h, \quad \sup_{V_h \in N_{h,\#}} \frac{b_h(V_h, \chi_h)}{\|V_h\|_{X(\omega)}} \ge \beta^* \|\chi_h\|_{M(\omega)}. \tag{4.15}$$

Proof: For any function χ_h in M_h, the pair $V_h = (v_h, s_h)$ with $s_h = I_h(\chi_h a_3)$ and $v_h = \frac{e}{2} s_h$ belongs to $N_{h,\#}$. We have:

$$\begin{aligned} b_h(V_h, \chi_h) \ge \|\chi_h\|^2_{M(\omega)} - \|s_h \cdot a_3 - I_h(s_h \cdot a^3)\|_{H^1(\omega)} \|\chi_h\|_{M(\omega)} \\ - \|\chi_h a_3 - I_h(\chi_h a_3)\|_{H^1(\omega;\mathbb{R}^3)} \|\chi_h\|_{M(\omega)}. \end{aligned}$$

From Lemma 4.2 and its extension, this quantity is greater than $c\|\chi_h\|^2_{M(\omega)}$ for sufficiently small h. Lemma 4.2 also implies that:

$$\|V_h\|_{X(\omega)} \le c\|\chi_h\|_{M(\omega)}.$$

Thus, the inf-sup condition is satisfied as required.

Theorem 4.7 Let Φ be a function satisfying condition (3.29). Then, for any data (f, N, M) in $L^2(\omega)^3 \times L^2(\gamma_1)^3 \times L^2(\gamma_1)^3$, problem (4.6) admits a unique solution (U_h, ψ_h) in the function spaces $X_h \times M_h$.

Proof. The proof follows the same arguments as for Theorem 3.10, incorporating the newly introduced ellipticity and inf-sup properties from Lemmas 4.3 and 4.4 .

4.1.1 Different way:

Let $\mathcal{T}_h$ be a regular affine triangulation that covers the domain ω. Here, h is the mesh size, or more precisely:

$$h = \max_{T \in \mathcal{T}_h} \operatorname{diam}(T)$$

where $\operatorname{diam}(T)$ is the diameter of the triangle T.

For a positive integer k, $P_k(T)$ stands for the set of functions on T which are the restrictions to T of polynomials of degree less than or equal to k.

For $T \in \mathcal{T}_h$, b_T denotes the bubble function defined by:

$$b_T = \frac{\lambda_1 \lambda_2 \lambda_3}{27}$$

where λ_i, $i = 1, 2, 3$ are the barycentric coordinates of T. Note that $b_T \in H_0^1(T) \cap P_3(T)$ has a maximum value of one.

We further define:

$$B_3(T) = \{v \in H_0^1(T); v = b_T w,\ w \in P_0(T)\}$$

Let us define the finite dimensional spaces:

$$M_h := \{\chi_h \in H^1_{\gamma_0}(\omega) \mid \chi_h|_T \in P_1(T) \oplus B_3(T),\ \forall T \in \mathcal{T}_h\}$$

$$Q_h := \{\mu_h \in L^2(\omega) \mid \mu_h|_T \in P_0(T),\ \forall T \in \mathcal{T}_h\}$$

$$X_h := (M_h)^3 \times (M_h)^3$$

$$W_h := \{(v_h, s_h) \in X_h; (v_h - \frac{\epsilon}{2} s_h) \cdot e_3 = 0\}$$

Then, we introduce the discrete convex cone:

$$N_h := \{(v_h, s_h) \in X_h; (v_h - \frac{e}{2} s_h) \cdot e_3 \geq \Phi_h\}$$

where $\Phi_h := \mathcal{I}_h\Phi$, with $\mathcal{I}_h$ being the standard Lagrange interpolant operator. Specifically, $(\mathcal{I}_h\Phi)_T \in P_1(T)$ and $(\mathcal{I}_h\Phi)_T(x) = \Phi(x)$ for all vertices x of T.

Clearly, we have $X_h \subset X$, $M_h \subset M$, and $W_h \subset N_h$, but N_h is not necessarily contained in N_Φ.

$$(P1)\begin{cases}\text{Find}\,(U_h,\psi_h)\in N_h\times M_h \text{ such that:}\\ \forall V_h\in N_h, \quad a_\rho(U_h,V_h-U_h)+b(V_h-U_h,\psi_h)\geq \mathcal{L}(V_h-U_h),\\ \forall\chi_h\in M_h, \quad b(U_h,\chi_h)=0.\end{cases} \tag{4.16}$$

where for any real parameter $\rho > 0$, we set

$$a_\rho(U,V) = a(U_h,V_h) + \rho\int_\omega \partial_\alpha(r_h\cdot a_3)\partial_\alpha(s_h\cdot a_3)\,dx, \quad \forall U_h=(u_h,r_h),\, V_h=(v_h,s_h)\in X_h.$$

$$(P2)\begin{cases}\text{Find}\,(U_h,\psi_h)\in N_h\times M_h \text{ such that:}\\ \mathcal{A}_\rho((U_h,\psi_h);(V_h-U_h,\chi_h))\geq \mathcal{L}(V_h-U_h), \quad \forall (V_h,\chi_h)\in N_h\times M_h.\end{cases} \tag{4.17}$$

We define the bilinear form $\mathcal{A}_\rho : K\times K\to\mathbb{R}$ through

$$\mathcal{A}_\rho((W_h,\mu_h),(V_h,\chi_h)) = a_\rho(W_h,V_h)+b(V_h,\mu_h)+b(W_h,\chi_h)$$

Lemma 4.8 asserts that for a sufficiently small mesh size h, there exists a positive constant C_b such that the following inequality holds uniformly for all $\chi_h \in M_h$ and $V_h \in W_h$:

$$\inf_{\chi_h\in M_h}\sup_{V_h\in W_h}\frac{b(V_h,\chi_h)}{\|\chi_h\|_M\|V_h\|_X}\geq C_b$$

Theorem 4.9. If the mesh size ` is sufficiently small, then $(P1)$ admits a unique solution

$$(P3)\begin{cases}\text{find}\,(U_h,\psi_h,\lambda_h)\in X_h\times M_h\times\Lambda_h \text{ such that:}\\ \forall V_h\in X_h, \quad a(U_h,V_h)+b(V_h,\psi_h)-c(V_h,\lambda_h)=\langle\varphi,V_h\rangle,\\ \forall\chi_h\in M_h, \quad b(U_h,\chi_h)=0,\\ \forall\mu_h\in\Lambda_h, \quad c(U_h,\mu_h-\lambda_h)\geq\langle\theta_h,\mu_h-\lambda_h\rangle,\end{cases} \tag{4.18}$$

The h-dependent norm $\|\chi_h\|_{2,h}$ is defined as follows:

$$\|\chi_h\|_{2,h}^2 = \sum_{T\in\mathcal{T}_h} h_T^2 \|\chi_h\|_T^2, \quad \forall \chi_h \in Q_h.$$

Lemma 4.10 states that there exist two positive constants C_1 and C_2 (independent of h) such that for all $\chi_h \in Q_h$:

$$\sup_{V_h\in X_h\cap\ker b} \frac{c(V_h,\chi_h)}{\|V_h\|_{X_h}} \geq C_1\|\chi_h\|_{M'} - C_2\|\chi_h\|_{2,h}$$

Lemma 4.11 asserts the following inf-sup condition for the mesh-dependent norm $\|\cdot\|_h$: there exists a positive constant C_3 (independent of h) such that for all $\chi_h \in Q_h$:

$$\sup_{V_h\in X_h\cap\ker b} \frac{c(V_h,\chi_h)}{\|V_h\|_X} \geq C_3\|\chi_h\|_h$$

Lemma 4.12 states that there exists a positive constant C such that for all $\chi_h \in Q_h$:

$$\sup_{V_h\in X\cap\ker b} \frac{c(V_h,\chi_h)}{\|V_h\|_X} \gtrsim \|\chi_h\|_{M'}$$

Proposition 4.13. The full problem (P3) and the reduced problem (P1) are equivalent, in the following sense: If (U_h,ψ_h,λ_h) is a solution of the full problem, then (U_h,ψ_h) is a solution of the reduced problem. If (U_h,ψ_h) is a solution of the reduced problem, then there exists $\lambda_h \in \Lambda_h$ such that (U_h,ψ_h,λ_h) is a solution of the full problem.

4.2 A PRIORI ERROR ANALYSIS

Our problem is expressed as variational inequalities in H^1. To demonstrate the convergence of non-conforming finite element approximations, we adhere to the standard approach for variational inequalities in mixed problems. Consequently, we initially present the subsequent version of Strang's lemma, which isn't entirely evident and entails the complete problem (3.11).

Lemma 4.14 Suppose that the component λ of the solution to problem (3.27) belongs to $L^2(\omega)$. Then, the following inequality holds between the component U of the solution to this problem and the component U_h of the solution to problem (4.6):

$$\begin{aligned}\|U - U_h\|_{X(\omega)} \leq c(&\inf_{V_h \in N_{h,\Phi} \cap V_h} (\|U - V_h\|_{X(\omega)} + \|U - V_h\|^2_{\frac{1}{2}L^2(\omega;\mathbb{R}^3)}) \\ &+ \inf_{\chi_h \in M_h} \|\psi - \chi_h\|_{M(\omega)} + \|\Phi - \Phi_h\|_{\frac{1}{2}L^2(\omega;\mathbb{R}^3)})\end{aligned} \tag{4.19}$$

where c is a constant dependent on $\|\lambda\|_{L^2(\omega)}$.

Proof. Let U_h be an approximation of U in $N_{h,\Phi} \cap V_h$. We start by considering problem (4.10), which yields:

$$a(U_h - V_h, U_h - V_h) \leq L(U_h - V_h) - a(V_h, U_h - V_h).$$

Applying problem (3.11) gives:

$$a(U_h - V_h, U_h - V_h) \leq a(U - V_h, U_h - V_h) + b(U_h - V_h, \psi) - c(U_h - V_h, \lambda). \tag{4.20}$$

Now, evaluating $b(U_h - V_h, \psi)$ and $c(U_h - V_h, \lambda)$ poses a challenge. By Lemma 4.3, we find:

$$b(U_h - V_h, \psi) = b(U_h - V_h, \psi - \chi_h), \tag{4.21}$$

where χ_h represents any function in M_h.

Considering the non-negativity of λ, we deduce:

$$-c(U_h, \lambda) \leq -h\Phi_h \sqrt{a, \lambda}.$$

Furthermore, applying problem (3.11) with $\mu = 0$, we get:

$$-c(U_h - V_h, \lambda) \leq -c(U - V_h, \lambda) + h(\Phi - \Phi_h)\sqrt{a, \lambda}. \tag{4.22}$$

Combining the ellipticity property of the form $a(\cdot, \cdot)$ (Lemma 4.3) with the continuity of the involved forms, considering $\lambda \in L^2(\omega)$, yields:

$$\begin{aligned}\alpha^* \|U_h - V_h\|^2_{X(\omega)} \leq c \big(&\|U - V_h\|_{X(\omega)} + \|\psi - \chi_h\|_{M(\omega)} \|U_h - V_h\|_{X(\omega)} \\ &+ \|U - V_h\|^2_{L^2(\omega;\mathbb{R}^3)} + \|\Phi - \Phi_h\|_{L^2(\omega;\mathbb{R}^3)} \Big).\end{aligned}$$

The desired estimate then follows from a triangle inequality.

To proceed, we must now devise approximations V_h of U and χ_h of ψ that fulfill the established criteria. Given the continuity of ψ, we select χ_h to be the interpolation $I_h \psi$.

Estimating the term $|\psi - \chi_h|_{M(\omega)}$ is straightforward. However, for the approximation of U, we require a lemma.

lemma4.15 For any function U belonging to $K_\Phi(\omega) \cap H^{s+1}(\omega; \mathbb{R}^3)$, where $0 < s \leq 1$, there exists a function V_h in $N_{h,\Phi} \cap V_h$ such that:

$$\|U - V_h\|_{X(\omega)} \leq ch^s \|U\|_{H^{s+1}(\omega;\mathbb{R}^3)}, \tag{4.23}$$

$$\|U - V_h\|_{L^2(\omega)} \leq ch^{s+1} \|U\|_{H^{s+1}(\omega;\mathbb{R}^3)}. \tag{4.24}$$

proof: The function $V_h = I_h U$ meets the required approximation criteria. Additionally, as per the definition of $N_{h,\Phi}$ (see (4.5)), it falls within this set. Furthermore, since $r \cdot a^3$ vanishes universally, when $V_h = (v_h, s_h)$ is set, it's observed that $s_h \cdot a^3$ nullifies at all vertices of elements in T_h. Consequently, $I_h(s_h \cdot a^3)$ equals zero. Given that this operator solely appears in the definition (4.8) of the form $b_h(\cdot, \cdot)$, it implies that V_h also lies within V_h. These considerations substantiate the desired estimates.

Theorem 4.16 Assume that the solution (U, ψ, λ) of problem (3.27) belongs to $H^2(\omega; \mathbb{R}^3) \times H^2(\omega) \times L^2(\omega)$ and that the vector a^3 belongs to $W^{2,p}(\omega; \mathbb{R}^3)$, $p > 2$. Then, the following a priori error estimate holds between the solution (U, ψ) of problem (3.24) and the solution (U_h, ψ_h) of problem (4.6):

$$\|U - U_h\|_{X(\omega)} + \|\psi - \psi_h\|_{M(\omega)} \leq ch, \tag{4.25}$$

for a constant c depending on Φ and (U, ψ, λ).

Proof. The estimate for $\|U - U_h\|_{X(\omega)}$ follows directly from Lemmas 4.5 and 4.6, since Φ_h is taken as $I_h \Phi$. To evaluate $\|\psi - \psi_h\|_{M(\omega)}$, we use the inf-sup condition (4.15), which implies that for any $\chi_h \in M_h$,

$$\|\psi_h - \chi_h\|_{M(\omega)} \leq \beta^{-1} \sup_{W_h \in N_{h,\#}} \frac{b_h(W_h, \psi_h - \chi_h)}{\|W_h\|_{X(\omega)}}.$$

Using problem (4.5) with $V_h = U_h \pm W_h$ results in

$$b_h(W_h, \psi_h - \chi_h) = L(W_h) - a(U_h, W_h) - b_h(W_h, \chi_h).$$

From problem (3.10), we get

$$b_h(W_h, \psi_h - \chi_h) = a(U - U_h, W_h) + b(W_h, \psi) - b_h(W_h, \chi_h) - c(W_h, \lambda),$$

and the definition (4.14) of $N_{h,\#}$ ensures that the last term is zero. Therefore, using the notation $W_h = (w_h, t_h)$, we obtain

$$\|\psi_h - \chi_h\|_{M(\omega)} \leq c\left(\|U - U_h\|_{X(\omega)} + \|\psi - \chi_h\|_{M(\omega)} + \sup_{t_h \in M_h^3} \frac{\|t_h \cdot a_3 - I_h(t_h \cdot a_3)\|_{M(\omega)}}{\|t_h\|_{H^1(\omega;\mathbb{R}^3)}}\right).$$

Choosing $\chi_h = I_h\psi$, and observing that for each K in T_h,

$$\|t_h \cdot a_3 - I_h(t_h \cdot a_3)\|_{H^1(K)} \leq ch|t_h \cdot a_3|_{H^2(K)},$$

we conclude, using the regularity of a_3, that the desired estimate holds.

4.2.1 Different way:

We recall that it consists in finding $(U, \psi, \lambda) \in X \times M \times \Lambda$ such that

$$(P4)\begin{cases} \forall V \in X, \quad \alpha(U,V) + \beta(V,\psi) - \gamma(V,\lambda) = \phi(V), \\ \forall \chi \in M, \quad \beta(U,\chi) = 0, \\ \forall \mu \in \Lambda, \quad \gamma(U, \mu - \lambda) \geq \langle \Psi, \mu - \lambda \rangle. \end{cases} \tag{4.26}$$

In this section we derive a priori error analysis for the discrete approximation (P3) consists in finding $(U_h, \psi_h, \lambda_h) \in X_h \times M_h \times \Lambda_h$ such that

$$(P5)\begin{cases} \forall V_h \in X_h, \quad a(U_h,V_h) + b(V_h,\psi_h) - c(V_h,\lambda_h) = \langle \varphi, V_h \rangle, \\ \forall \chi_h \in M_h, \quad b(U_h,\chi_h) = 0, \\ \forall \mu_h \in \Lambda_h, \quad c(U_h, \mu_h - \lambda_h) \geq \langle \theta_h, \mu_h - \lambda_h \rangle, \end{cases} \tag{4.27}$$

we con written the problem (P5) in a compact way as follows :

$$\begin{cases} \text{Find}(\mathrm{U}_h, \psi_h, \lambda_h) \in X_h \times M_h \times \Lambda_h \text{ such that:} \\ \mathcal{B}(U_h, \psi_h, \lambda_h; V_h, \chi_h, \mu_h - \lambda_h) \geq \mathcal{L}_h(V_h, \mu_h - \lambda_h), \quad \forall (V_h, \chi_h, \mu_h) \in X_h \times M_h \times \Lambda_h, \end{cases} \tag{4.28}$$

where

$$\mathcal{B}(U_h, \psi_h, \lambda_h; V_h, \chi_h, \mu_h) := \alpha(U_h, V_h) + \beta(V_h, \psi_h) + \beta(U_h, \chi_h) - \gamma(V_h, \lambda_h) + \gamma(U_h, \mu_h),$$

$$\mathcal{L}_h(V_h, \chi_h, \mu_h) := \phi(V_h) + \langle \Psi_h, \mu_h \rangle.$$

Lemma 4.17. There exists a constant $\beta > 0$ such that:

$$\inf_{(\chi_h,\mu_h)\in M_h\times Q_h} \sup_{Z_h=(z_h,t_h)\in X_h}, \frac{\gamma(z_h,\mu_h)-\beta(z_h,\chi_h)}{\|(\chi_h,\mu_h)\|_{M_h\times M'}\|z_h\|_X} \geq \beta^{\cdot} \tag{4.29}$$

Lemma 4.18. For any $(W_h, \chi_h, \mu_h) \in X_h \times M_h \times Q_h$ there exists $Y_h \in X_h$ such that:

$$\mathcal{B}(W_h, \chi_h, \mu_h; Y_h, -\chi_h, \mu_h) \gtrsim (\|W_h\|_X + \|\chi_h\|_M + \|\mu_h\|_{M'})^2, \tag{4.30}$$

$$\|Y_h\|_X + \|\chi_h\|_M + \|\mu_h\|_{M'} \lesssim \|W_h\|_X + \|\chi_h\|_M + \|\mu_h\|_{M'}. \tag{4.31}$$

Theorem 4.19. Let (U, ψ, λ) and (U_h, ψ_h, λ_h) be the solution of Problem (P4) and Problem (P3) respectively. Then

$$\|U - U_h\|_X + \|\psi - \psi_h\|_M + \|\lambda - \lambda_h\|_{M'} \lesssim \inf_{V_h\in N_h} \|U - V_h\|_X + \inf_{\chi_h\in M_h} \|\psi - \chi_h\|_M$$
$$+ \inf_{\mu_h\in\Lambda_h} \left(\|\mu_h - \lambda\|_{M'} + \sqrt{\gamma(U, \mu_h - \lambda) - \langle\Psi, \mu_h - \lambda\rangle}\right) + \|\Psi - \Psi_h\|_M.$$

Corollary 4.20. Assume that the solution (U, ψ, λ) of Problem (P4) belongs to $(H^2(\Omega, \mathbb{R}^3))^2 \times H^2(\Omega) \times L^2(\Omega)$ and the function Ψ belongs to $H^2(\Omega) \cap H^1_{\gamma_0}(\Omega)$. Let (U_h, ψ_h, λ_h) be the solution of Problem (P3). Then

$$\|U - U_h\|_X + \|\psi - \psi_h\|_M + \|\lambda - \lambda_h\|_{M'} \lesssim \sqrt{h\,[\|U\|^2_\Omega + \|\psi\|^2_\Omega + \|\Psi\|^2_\Omega + \|\lambda\|_\Omega]}.$$

Remark 4.21: "Proof of theories and problem transformation can be understood by referring to the literature (see [5])."

Conclusion and Perspective

In short, this memorandum discussed the main points that were formed around our topic, through previous studies and current research. Where we reached the existence and uniqueness of the solution to the Naghdi shell model using the Lax-Miligram theorem .as we touched on the contact problem and its well-posedness , then the discrete problem in tow ways in addition to a priori error analysis .

Due to time constraints , we could not study The a posteriori error analysis and the test numerical , We promise to study it in other projects.

In the end, we hope that this memorandum has provided a comprehensive and useful analysis on the topic, and that it has a positive impact on understanding the problem and proposing appropriate solutions.

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