Numerical approximation of some time dependent variational inequalities

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1. Introduction

In this work we shall approximate the parabolic inequality of type one

$$\begin{cases} \text{find } u(t) \in \chi_0 \text{ such that a-e } t \in [0,T] \\ (u\prime + Au - f, v - u) \geq 0\, \forall v \in K \end{cases} \quad (1.1)$$

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2. Problem

in this paper we study this problem:

$$\begin{cases} \text{find } u(t) \in \chi_0 \text{ such that } t \in [0,T] \\ (u\prime + Au - f, v - u) \geq 0\ \forall v \in K \end{cases} \quad (2.1)$$

- Knon- empty convex set of$V$
- $V = H^1(\Omega) H = L^2(\Omega)\ f \in L^2(0,T,V\prime)$
- $V \subset H \subset V\prime (V\prime = dual of V)\ V$ dense in H
- $a(u,v) = (Au, v)$,and $a(.,.)$bilinear from which is continuous and symmetric.
- $W(0,T) = \{v | v \in L^2(0,T,V), v\prime \in L^2(0,T,V\prime)\}$
- $W_0(0,T) = \{v | v \in W(0,T) v(0) = u_0 u_0$ given in $H\}$
- $\chi_0 = \{v | v \in W_0(0,T) v(t) \in K$ a-e $t \in [0,T]\}$

3. Approximation of problem

Approximation of V**and**H
F Hilbert space, $\partial \in L(V,F)\ p_h \in L(V_h, F)\ \ \|p_h\| \leq C$
dim$V_h \leq +\infty$
we say that $V_h$constitutes an exterior approximation of$V$if:

1. $v_h \in V_h\ \ p_h v_h \rightharpoonup \xi in F \Rightarrow \xi \in \partial v$
2. $\forall v \in V\ \ \exists v_h \in V_h$ with $p_h v_h \to \partial v\ \ in\ \ F\ \ and\ \ \|v_h\| \leq C$

q_h a linear operator from $V_h \to H$
we define a second inner product and a second norm onV_h. $(v_h, w_h)_h$and$|v_h|_h$ which approximate the inner product$(,)$ and the norm$||$ on $H$by:

$$\begin{cases} (v_h, w_h)_h = (q_h v_h, q_h w_h) \\ |v_h|_h = |q_h v_h| \end{cases}$$

and we assume that $\forall\, v_h \in V_h$
$|v_h|_h \leq C\|v_h\|_h \quad \|v_h\|_h \leq S(h)|v_h|_h$
Approximation of $a(,)$
we say that the a_h constitute an approximation of $a(,)$if

$$\begin{cases} p_h v_h \rightharpoonup \partial v\, in F\, p_h w_h \to \partial w\, in\, F \\ \text{we have } a_h(v_h, w_h) \longrightarrow a(v,w) \end{cases} \quad (3.1)$$

and if

$$\begin{cases} \text{if } p_h \rightharpoonup \partial v \text{ in } F \text{ then} \\ \liminf a_h(v_h, v_h) \geq a(v,v) \end{cases} \quad (3.2)$$

Approximation of K
$K_h \subset V_h$
K_h constitute an approximation of K if:

1. $v_h \in K_h,\ p_h v_h \rightharpoonup \xi$in $F \Longrightarrow \xi \in \partial K$

and if

$$\begin{cases} \forall v \in K,\ \exists v_h \in K_h\, such that \\ p_h v_h \longrightarrow \partial v \text{ in} F \text{ and} \|v_h\| \leq C \end{cases} \quad (3.3)$$

In view of the approximation assumptions we are led quite naturally to consider the fallowing problem:

$$\begin{cases} (\frac{u_h^{i+1} - u_h^i}{k} + A_h u_h^{i+\theta} - f_h^{i+1}, v_h - u_h^{i+1})_h \geqslant 0 \forall v_h \in K_h \\ u_h^{i+1} \in K_h u_h^0 = u_{0,h} \\ \text{with} u_h^{i+\theta} = u_h^i + \theta(u_h^{i+1} - u_h^i)\theta \in [0,1] \end{cases} \quad (3.4)$$

4. Theorem

Theorem 4.1 *if* $u_{h,k}$*the solution of the problem* $(3,1)$ *then*

$$u_{h,k} \to u\, in\ \ L^2(0,T,V) \quad (4.1)$$

u *the solution of* $(1,1)$
the proof takes place in four stages:

1. **Definition of the weak discrete inequality**:
   for definition of the weak discrete inequality we use this lemma

   Lemma 4.2 *Let* $v_{h,k} = \sum_{i=0}^N v_h^i \chi_k^i\, u_{h,k} = \sum_{i=0}^N u_h^i \chi_k^i$
   then
   $(\frac{v_h^{i+1} - v_h^i}{k}, v_h^{i+1} - u_h^{i+1}) = (\frac{v_h^{i+1} - v_h^i}{k}, v_h^{i+1} - u_h^{i+1}) + \frac{1}{2K}|v_h^{i+1} - u_h^{i+1}|_h^2 - \frac{1}{2K}|v_h^i - u_h^i|^2$
   $+\frac{1}{2K}|v_h^{i+1} - v_h^i - (u_h^{i+1} - u_h^i)|_h^2$ *(4)*
   and if we have

   $$\int_k^T (\partial u_{h,k} + A_h u_{h,k}^\theta - f_{h,k}, v_{h,k} - u_{h,k})_h dt \geq 0 \quad (4.2)$$

   then we also have

   $$\int_k^T (\partial v_{h,k} + A_h u_{h,k}^\theta - f_{h,k}, v_{h,k} - u_{h,k})_h dt \geq \frac{1}{2}|v_{h,k}(T) - u_{h,k}(T)|_h^2 - \frac{1}{2}|v_{h,k}(0) - u_{h,k}(0)|_h^2 \quad (4.3)$$

   By summing $(4)$from $i = 0$ to $N - 1$ we obtain

   $$\int_k^T (\partial u_{h,k}, v_{h,k} - u_{h,k})_h dt \leq \int_k^T (\partial v_{h,k}, v_{h,k} - u_{h,k}) dt - \frac{1}{2}\sum_{i=0}^N |v_h^i - u_h^i|^2 + \frac{1}{2}\sum_{i=0}^{N-1} |v_h^i - u_h^i|^2 \quad (4.4)$$

   and hence we deduce (4,4)by substituting this upper bound in (4,3),and if $v_{h,k}(0) = u_{u,k}(0)$ we deduce from(4,3) that

   $$\int_k^T (\partial v_{h,k} + A_h u_{h,k}^\theta - f_{h,k}, v_{h,k} - u_{h,k})_h dt \geq 0 \quad (4.5)$$

2. **Investigation of the stability** we shall shaw that the solution $u_{h,k} = \sum_{i=0}^N u_h^i \chi_k^i$ as the unique solution of (3,4) remains,with $h$and $k$possibly subject to stability condition
   Poof
   $v = 0$in (3,2) and using relation

   $$(a - b, a) = \frac{1}{2}|a|^2 - \frac{1}{2}|b|^2 + \frac{1}{2}|a - b|^2 \quad (4.6)$$

   we obtain

   $$\frac{1}{2K}[|u_h^{i+1}|_h^2 - |u_h^i|_h^2 + |u_h^{i+1} - u_h^i|_h^2] + a_h(u_h^{i+\theta}, u_h^{i++1}) \leqslant (f_h^{i+1}, u_h^{i+1}) \quad (4.7)$$

   and we use the relation

   $$2(a,b) \leq \frac{1}{\epsilon}|a|^2 + \frac{1}{1}|a - b|^2 \quad (4.8)$$

   for $(f^{i+1}, u^{i+1}), (f^{i+1}, u^i), (f^{i+1}, u^{i+1} - u^i)$ and $a(u^i, u^{i+1} - u^i)$ and substituting the results in $(4,9)$ we obtain

   $$|u^{i+1}|^2 - |u^i|^2 + [1 - (\epsilon + \frac{2}{\alpha}C^2)kS(h)^2(1-\theta)] \times |u^{i+1} - u^i|^2 + (1-\theta)ka(u^i, u^i) + \theta ka(u^{i+1}, u^{i+1}) \leq Ck\|f^{i+1}\|^2 \quad (4.9)$$

   we then introduce the fallowing assumption termed the stability assumption

   $$1 - \frac{2C^2}{\alpha}kS(h)(1-\theta) \geq \beta > 0 \quad (4.10)$$

   $$|u_h^{i+1}|_h^2 - |u_h^i|_h^2 + \beta|u_h^{i+1} - u_h^i|_h^2 + \alpha K[(1-\theta)\|u_h^i\|_h^2 + \theta\|u_h^{i+1}\|_h^2 \leq Ck\|f_h^{i+1}\|_h^2 \quad (4.11)$$

   by summing from $i = 0$ to $n < N - 1$

   $$|u_h^{n+1}|_h^2 + \beta\sum_{i=0}^n |u_h^{i+1} - u_h^i|_h^2 + \alpha\sum_{i=0}^n k[(1-\theta)\|u_h^{i+1}\|_h^2] \leqslant C \quad (4.12)$$

   We hence deduct that,$\forall h, k$ satisfying$(4,10)$

   $$p_h u_{h,k}\ \ remainsinaboundedsetof\ \ L^2(0,T,F) \quad (4.13)$$

   $$q_h u_{h,k}\ \ remainsinaboundedsetof\ \ L^\infty(0,T,H) \quad (4.14)$$

   $$\sum_{i=0}^{N-1} |u_h^{i+1} - u_h^i|_h^2 \leqslant C \quad (4.15)$$

3. **Weak convergence**

   $$\int_k^T (u_{h,k} + A_h u_{h,k}^\theta - f_{h,k}, v_{h,k} - u_{h,k}) dt \geq 0 \quad (4.16)$$

   and in view of lemma(4,2)and taking$v_{h,k}(0) = u_{0,h}$
   we obtain:

   $$\int_k^T (v_{h,k} + A_h u_{h,k}^\theta - f_{h,k}, v_{h,k} - u_{h,k}) dt \geq 0 \quad (4.17)$$

   from (4,13),(4,14),and lemma $(3,1)$ we can extract from $u_{h,k}$a subsequence again denoted by$u_{h,k}$ such that:

   $$\begin{cases} q_h u_{h,k} \rightharpoonup u in L^\infty(0,T,H) \\ p_h u_{h,k} \rightharpoonup \partial u in L^2(0,T,F) \\ u \in K \end{cases} \quad (4.18)$$

   proceeding the limit in (4,17) we obtain:

   $$\int_0^T (v\prime + Au - f, v - u) \geq liminf \int_k^T (A_h u_{h,k}^\theta, u_{h,k} - u_{h,k}^\theta) dt \quad (4.19)$$

   $y_{h,k} = |\int_k^T (A_h u_{h,k}^\theta, u_{h,k} - u_{h,k}^\theta) dt| = |\sum_{i=0}^N K(A_h u_h^{i+\theta}, u_h^{i+1} - u_h^{i+\theta})_h|$
   $= \sum_{i=0}^N k(1-\theta)|A_h u_h^{i+\theta}, u_h^{i+1} - u_h^i)_h|$
   $\leq (1-\theta)\sqrt{K}S\sum_{i=0}^N \sqrt{K}\|u_h^{i+\theta}\|_h |u_h^{i+1} - u_h^i|_h$
   $\leq (1-\theta)S(h)\sqrt{k}(\sum_{i=0}^N k\|u_h^{i+\theta}\|_h^2)^{\frac{1}{2}}(\sum_{i=0}^N |u_h^{i+1} - u_h^i|_h^2)^{\frac{1}{2}}$
   by using(4,13),(4,15) and the convergence assumption(redundant when $\theta = 0$) $(1-\theta)kS(h)^2 \to 0\, if h, k \to 0$
   we obtain $y_{h,k} \to 0$ if $h, k \to 0$
   and hence:
   $\int_0^T (v\prime + Au - f, v - u) dt \geq 0\, \forall v \in \chi_0\, u \in \chi_f$
   and $u$is indeed a weak solution

4. **Strong convergence**
   in fact it my be shown that
   $p_h u_{h,k} \to \partial u$ in$L^2(0,T,F)$
   for this we consider a sequence $u_{h,k}^*$ which approximates$u$and we shall evaluate
   $\chi_{h,k} = \int_k^T (A_h u_{h,k} - A_h u_{h,k}^*, u_{h,k} - u_{h,k}^*) dt$
   and shaw that
   $\chi_{h,k} \to 0$
   from lemma(4,2) and putting $v_{h,k} = u_{h,k}^*$in(4,5) we obtain

   $$\int_k^T (A_h u_{h,k}^\theta, u_{h,k}) dt \leq \int_k^T (\partial u_{h,k}^*, u_{h,k}^* - u_{h,k}) dt + \int_k^T (A_h u_{h,k}^\theta, u_{h,k}^*) dt - \int_k \quad (4.20)$$

   $$\chi_{h,k} = \int_k^T (A_h u_{h,k}, u_{h,k}) dt - \int_k^T (A_h u_{h,k}, u_{h,k}^*) dt + \int_k^T (A_h u_{h,k}^*, u_{h,k}^* - u_{h,k} \quad (4.21)$$

   and hence

   $$\int_k^T (A_h u_{h,k}, u_{h,k}) dt = \int_k^T (A_h u_{h,k}^\theta, u_{h,k}) dt - (1-\theta)\int_k^T (A_h u_{h,k}(t) - A_h u_{h,k} \quad (4.22)$$

   so that (4,20),(4,22) in (4,21) we obtain: $\chi_{h,k} < \int_k^T (\partial u_{h,k}^* + A_h u_{h,k}^* - f_{h,k}, u_{h,k}^* - u_{h,k}) dt + (\theta - 1)\int_k^T (A_h u_{h,k}(t) - A_h u_{h,k}(t-k), u_{h,k}^*(t) - u_{h,k}(t)) dt$
   we shall now shaw that
   $Z_{h,k} = (1-\theta)|\sum_{i=0}^{N-1} k(A_h(u_h^{i+1} - u_h^i), (u_h^{i+1})^* - u_h^{i+1})|$
   $Z_{h,k} \to 0$ consequently $\chi_{h,k} \to 0$ if
   $\int_k^T (\partial u_{h,k}^* + A_h u_{h,k}^* - f_{h,k}, u_{h,k}^* - u_{h,k}) dt \to 0$
   we cane deduce from the uniform coercivity ofA_h in h that
   $p_h(u_{h,k} - u_{h,k}^*) \to 0$
   and thus
   $p_h u_{u,k} \to \partial uinL^2(0,T,F)$

References

[1] .GLowinski,J.L.Lion,R.Trémolières, *Analysis of Variational inequalities* ,Paris(1976).