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Application of topological degree to the study of unilateral problems for nonlinearly elastic plates

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Conclusion

Topological degree theory has been an important tool for the study of nonlinear functional equations. The most important properties of this degree is, of course, the existent and homotopy invariance properties, which forms the basis for the continuation method. In this work we use the degree for Leary-Sauder operators and for operators of type (S_+) to study a general existence results for a nonlinearly elastic plates and apply this results for the inequality variational of von Kármán type.

Dedication

At the first dedicating this work to my lovely

parents

with deepest gratitude whose love and prayers have always been a source of strength for me

I also dedicate this work to my

family

for their interests and encouragement and for my dearest

friends

for their support and cooperation wile conducting the study.

Acknowledgement

I would like to express the deepest appreciation to my supervisor

Dr A. Ghezal

for encouragement support, advises and valuable guidance

Without forget any one from my teachers from the primary to the university,

to every one help me in my studies.

Contents

Notations and conventions

- ▶ $\partial_{\alpha\alpha} = \dfrac{\partial^2}{\partial x_\alpha^2}$: The second partial derivative
- ▶ $\Delta^2 = \Delta\Delta = \partial_{\alpha\alpha}\partial_{\beta\beta}$: The biharmonic operator.
- ▶ $\dfrac{\partial u}{\partial n} = n \cdot \nabla u$: The derivative according to the normal.
- ▶ $[\psi, \xi] = \partial_{11}\psi\partial_{22}\xi + \partial_{22}\psi\partial_{11}\xi - 2\partial_{12}\psi\partial_{12}\xi$: The Monge-Ampere form.
- ▶ $\rightharpoonup$: Denote the weak convergence.
- ▶ (MON) : The class of monotone operators.
- ▶ (QM) : The class of quasi-monotone operators.
- ▶ (S_+) : The class of operators of type S_+.
- ▶ (PM) : The class of pseudo-monotone operators.
- ▶ (LS) : The class of Leary-Schauder operators.
- ▶ ω: Open bounded subset of $\mathbb{R}^2$.
- ▶ $H^s(\omega)$: The usual Sobolev space.
- ▶ $\mathcal{D}(\omega)$: The test functions space.
- ▶ $H_0^s(\omega)$: The closure of $\mathcal{D}(\omega)$ in $H^s(\omega)$.
- ▶ $H^{-s}(\omega)$: The dual space of $H_0^s(\omega)$.

Introduction

The topological degree of mappings is one of the most effective tools for studying the existence of solutions of nonlinear equations. As a measure of the number of solutions of equation $Fx = h$ for a fixed h, the degree has fundamental properties such as existence, normalization, additively, and homotopy invariance.

The topological degree was first introduced by Brouwer [1] in 1912 for continuous functions in R^n. Leary and Schauder [2] generalized in 1934 the degree theory for compact perturbations of identity in infinite-dimensional Banach spaces. References to further applications of the Leray-Schauder continuation theorem to nonlinear elliptic boundary value problems can be found in [3], [4], to nonlinear parabolic boundary value problems in [5] and to ordinary differential equations in [6], [7]. The concept of topological degree has been defined for more and more comprehensive classes of nonlinear single-valued or multi-valued mappings arising in the operator equations. The original definitions of a degree for operators of type (S+) by Skrypnik [8] and Browder [9] were based on Galerkin approximations. For some papers on degree theories and their applications to various problems in nonlinear analysis, we cite Adhikari and Kartsatos [10], Kartsatos and Lin [11], Kartsatos and Quarcoo [12].

The history of the justification and generalization of the classical von Kármán's theory of plates has almost one century. The two-dimensional von Kármán equations for nonlinearly elastic plates were originally proposed in [13] by the American engineer of Hungarian origin Theodore von Kármán (1881-1963). The canonical von Kármán equations, fully justified by Ciarlet [14].

In this memory our interest is how we can apply the topological degree methods to study the existence for the variational inequality of von Kármán, which model unilateral problems for nonlinearly elastic plates. This methods, studied by Goeleven, Nguyen and Theta [15] and Gratie [16].

We shall start in chapter one by giving the definition of the classical topological degree in section one, and the definitions and basic properties and important results of Brouwer degree in section two, Leary and Schauder degree in section three, we first show how a compact map can be approximated by maps with finite dimensional ranges and from here we define the Leray Schauder degree for compact maps. In section four we survey the construction of the degree for mappings of class (S+) and for quasimonotone and pseudo-monotone mappings, that is based on the Leray-Schauder theory. In chapter tow we present the classical von Kármán equations and the reduced von Kármán equations for a nonlinearly elastic plate in section one. In section two we present a relation be twine variational inequality and fixed point problem. In section three and four we use respectively Leary-Schauder degree and the degree for mapping of (S_+) type to study a general existence result of a variational inequality for nonlinearly elastic plate. Finally in section five we use the results in section three and four to study the existence for the variational inequality of von Kármán.

Chapter 1

Topological Degree Theory

1.1 Classical topological degree

Definition 1.1 *A homotopy between tow continuous functions f and g from a topological space X to a space Y is defined to be a continuous function*

$$H : X \times [0,1] \longrightarrow Y$$

from the product of the space X with the unit interval $[0,1]$ to Y such that, if $x \in X$ then

$$H(x,0) = f(x), \quad H(x,1) = g(x).$$

Definition 1.2 *Let X and Y be topological spaces and let O be a class of open subsets G of X. For each $G \in O$, we associate a class F_G of maps from $\overline{G}$ into Y and a class H_G of maps $[0,1] \times \overline{G}$ into Y (admissible homotopies). For any $f \in F_G$; $G \in O$, and for any*

$y \in Y \setminus f(\partial G)$, we associate an integer $d(f, G, y)$.

The integer valued function d is said to be a classical topological degree if the following condition are satisfied:

1. *(**Existence of solution**) If $d(f, G, y) \neq 0$, there exists an $x \in G$ such that $f(x) = y$.*

2. *(**Additivity**) If $D \subset G \in O$ and $f \in F_G$, then the restriction $f|_{\overline{G}} \in F_D$ (the restricted map is usually denoted by the same symbol). Let G_1 and G_2 be a pair of disjoint subsets of G belonging to O and suppose that $y \notin f(\overline{G} \backslash (G_1 \cup G_2))$, then*

$$d(f, G, y) = d(f, G_1, y) + d(f, G_2, y).$$

3. *(**Invariance under homotopy**) If $f_t, 0 \leq t \leq 1$, is a homotopy H_G, then $f_t \in F_G$ for each fixed $t \in [0, 1]$, and if $\{y(t) : t \in [0, 1]\}$ is a continuous curve in Y with $y(t) \notin f_t(\partial G)$ for any $t \in [0, 1]$, then $d(f_t, G, y)$ is constant in $t \in [0, 1]$.*

4. *(**Normalisation**) There exists a map $j : X \longrightarrow Y$ called "normalizing map"such that $j|_{\overline{G}} \in F_G$ for each $G \in O$, and if $y \in j(G)$, then*

$$d(j, G, y) = 1.$$

1.2 Brouwer degree theory

Let $\Omega \subset \mathbb{R}^n$, and let $f : \Omega \longrightarrow \mathbb{R}^n$ be a continuous function. A basic mathematical problem is: Does $f(x) = 0$ have a solution in Ω ? In this section, we will present a number, the topological degree of f with respect to Ω and 0, which is very useful in answering these question.

Definition 1.3 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f \in C^1(\overline{\Omega})$. If $p \notin f(\partial\Omega)$ and $J_f(p) \neq 0$, then we define the Brouwer degree as follow*

$$deg_B(f, \Omega, p) = \sum_{x \in f^{-1}(p)} sgn J_f(x),$$

where $J_f(x) = det(f'(x))$ is the Jacobian of f at a point $x \in \overline{\Omega}$, and $deg_B(f, \Omega, p) = 0$ if $f^{-1}(p) = \emptyset$.

Definition 1.4 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f \in C^2(\overline{\Omega})$. If $p \notin f(\partial\Omega)$, then we define*

$$deg_B(f, \Omega, p) = deg_B(f, \Omega, p'),$$

where p' is any regular value of f that $\mid p' - p \mid < d(p, f(\partial\Omega))$.

Definition 1.5 *Let $\Omega \subset \mathbb{R}^n$ be open and bounded and $f \in C(\overline{\Omega})$. If $p \notin f(\partial\Omega)$. Then we define*

$$deg_B(f, \Omega, p) = deg_B(g, \Omega, p)$$

where $g \in C^2(\overline{\Omega})$ and $\mid g - f \mid < d(p, f(\partial\Omega))$.

Theorem 1.6 *Let $\Omega \subset \mathbb{R}^n$ be open bounded subset, $1 \leq m < n$, let $f : \overline{\Omega} \longrightarrow \mathbb{R}^m$ be a continuous function and let $g = I - f$. If $y \notin (I - f)(\partial\Omega)$, then*

$$deg_B(g, \Omega, y) = deg_B(g_m, \Omega \cap \mathbb{R}^m, y)$$

where g_m is the restriction of g on $\overline{\Omega} \cap \mathbb{R}^m$.

For the proof see [17].

Theorem 1.7 *The Brouwer degree has the following properties*

1. ***(Normality)*** $deg_B(I,\Omega,p) = 1$ *if and only if* $p \in \Omega$*, where* I *denotes the identity mapping.*

2. ***(Solvability)*** *If* $deg_B(f,\Omega,p) \neq 0$*, then* $f(x) = p$ *has a solution in* Ω*.*

3. ***(Homotopy)*** *If* $f_t(x) : [\ 0,1] \ \times \overline{\Omega} \longrightarrow \mathbb{R}^n$ *is a continuous and* $p \notin \cup_{t\in[0,1]} f_t(\partial\Omega)$ *then* $d(f_t,\Omega,p)$ *does not depend on* $t \in [0,1]$*.*

4. ***(Additivity)*** *Suppose that* Ω_1, Ω_2 *are two disjoint open subsets of* Ω *and* $p \notin f(\overline{\Omega} - \Omega_1 \cup \Omega_2)$*. Then*

$$deg_B(f,\Omega,p) = deg_B(f,\Omega_1,p) + deg_B(f,\Omega_2,p).$$

For the proof see [18].

As consequences of Theorem 1.7, we have the following results :

Theorem 1.8 *Let* $f : \overline{B(0,R)} \subset \mathbb{R}^n \longrightarrow \overline{B(0,R)}$ *be a continuous mapping. If* $\mid f(x) \mid \leqslant R$ *for all* $x \in \partial B(0,R)$*, then* f *has a fixed point in* $\overline{B(0,R)}$*.*

Proof. We may assume that $x \neq f(x)$ for all $x \notin \partial B(0,R)$. Put $H(t,x) = x - tf(x)$ for all $(t,x) \in [0,1] \times \overline{B(0,R)}$. Then $0 \neq H(t,x)$ for all $[0,1] \times \partial B(0,R)$. Therefore, we have

$$d_B(I - f, B(0,R), 0) = deg_B(I, B(0,R), 0) = 1$$

Hence f has a fixed point in $\overline{B(0,R)}$. This completes the proof. ■

Theorem 1.9 *Let* $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *be a continuous mapping and* $0 \in \Omega \subset R^N$ *with* Ω *an open bounded subset. If* $(f(x),x) > 0$ *for all* $x \in \partial\Omega$*, then*

$$deg_B(f,\Omega,0) = 1.$$

Proof. Put $H(t,x) = x - tf(x)$ for all $(t,x) \in [0,1] \times \overline{\Omega}$. Then $0 \notin H([0,1] \times \partial\Omega)$, and so we have

$$deg_B(f,\Omega,p) = deg_B(I,\Omega,p) = 1.$$

■

1.3 Leary-Schauder degree theory

Many problems in science lead to the equation $Tx = y$ in infinite dimensional spaces . Hence we again are interested in the questions raised at the beginning of section two. Therefore we construct the Leary Schauder degree such that it is defined for mappings of the form $I - C$, where C is a compact mapping from the closure of an open bounded subset of a Banach space X into X.

Definition 1.10 *Let X be a topological space. A subset $M \subset X$ is called compact if every open covering of M has an finite covering, i.e., if $M \subset \cup_{i\in I} V_i$, where V_i is an open subset of X for all $i \in I$, then there exist $i_j \in I$, $j = 1, 2, ..., k$. such that $M \subset \cup_{j=1}^{k} V_{i_j}$.*
M is called relatively compact if $\overline{M}$ is compact.

Definition 1.11 *Let X be a Banach space. A mapping $T : D(T) \subset X \longrightarrow X$ is called compact if T maps every bounded subset of D(T) to a relatively compact subset in X.*

To construct the Leary-Schauder Degree, we need the following result on the approximation of a compact mapping by finite dimensional mappings.

Lemma 1.12 *Let E be a real Banach space, $\Omega \subset E$ be an open bounded subset and $T : \overline{\Omega} \longrightarrow E$ be a continuous compact mapping. Then, for any $\epsilon > 0$, there exist a finite dimensional space F and a continuous mapping $T_\epsilon : \overline{\Omega} \longrightarrow F$ such that*

$$\|T_\epsilon x - Tx\| < \epsilon \quad for\ all \quad x \in \overline{\Omega}.$$

For the proof see [17].

Lemma 1.13 *Let E be a real Banach space, $B \subset E$ be a closed bounded subset and $T : B \longrightarrow E$ be a continuous compact mapping. Suppose $Tx \neq x$ for all $x \in B$. Then there* *exists* $\epsilon_0 > 0$ *such that* $x \neq tT_{\epsilon_1}x + (1-t)T_{\epsilon_2}$ *for all* $t \in [0,1]$ *and* $x \in B$*, where* $\epsilon_i \in (0, \epsilon_0)$ *and* $T_{\epsilon_i} : B \longrightarrow F_{\epsilon_i}$ *for* $i = 1, 2$ *as in Lemma 1.12.*

For the proof see [17].

Definition 1.14 *Let* E *be a real Banach space,* $\Omega \subset E$ *be an open bounded set and* $T : \overline{\Omega} \longrightarrow E$ *be a continuous compact mapping. Now, suppose that* $0 \notin (I-T)(\partial\Omega)$*. Then, by Lemma 1.13, there exist* $\epsilon_0 > 0$ *such that*

$$x \neq tT_{\epsilon_1}x + (1-t)T_{\epsilon_2} \quad for\ all \quad t \in [0,1] \ and \ x \ \in \partial\Omega,$$

where $\epsilon_i \in (0, \epsilon_0)$ *and* $T_{\epsilon_i} : \overline{\Omega} \longrightarrow F_{\epsilon_i}$ *for* $i = 1, 2$ *as in Lemma 1.12. Hence Brouwer's degree* $deg_B(I - T_\epsilon, \Omega \cap F_\epsilon, 0)$ *is well defined, and so we define*

$$deg_{LS}(I - T, \Omega, 0) = deg_B(I - T_\epsilon, \Omega \cap F_\epsilon, 0),$$

where $\epsilon \in (0, \epsilon_0)$

By the homotopy property of Brouwer degree, we have

$$deg_B(I - T_{\epsilon_1}, \Omega \cap span\{F_{\epsilon_1} \cup F_{\epsilon_2}\}, 0) = deg_B(I - T_{\epsilon_2}, \Omega \cap span\{F_{\epsilon_1} \cup F_{\epsilon_2}\}, 0).$$

But

$$T_{\epsilon_i} : \Omega \cap span\{F_{\epsilon_1} \cup F_{\epsilon_2}\} :\longrightarrow F_{\epsilon_i} \ for \ i = 1, 2,$$

so by Theorem 1.6 we have

$$deg_B(I - T_{\epsilon_1}, \Omega \cap span\{F_{\epsilon_1} \cup F_{\epsilon_2}\}, 0) = deg_B(I - T_{\epsilon_1}, \Omega \cap F_{\epsilon_1}, 0),$$

and

$$deg_B(I - T_{\epsilon_2}, \Omega \cap span\{F_{\epsilon_1} \cup F_{\epsilon_2}\}, 0) = deg_B(I - T_{\epsilon_2}, \Omega \cap F_{\epsilon_2}, 0).$$

Thus we have

$$deg_B(I - T_{\epsilon_1}, \Omega \cap F_{\epsilon_1}, 0) = deg_B(I - T_{\epsilon_2}, \Omega \cap F_{\epsilon_2}, 0),$$

and the degree defined in Definition 1.14 is well defined. For the general case, if $p \notin (I-T)(\partial\Omega)$, *we define*

$$deg_{LS}(I-T,\Omega,p) = deg_{LS}(I-T-p,\Omega,0).$$

Theorem 1.15 *The Leary Schauder degree has te following properties*

1. ***(Normality)*** $deg_{LS}(I,\Omega,0) = 1$ *if and only if* $0 \in \Omega$.
2. ***(Solvability)*** *If* $deg_{LS}(I-T,\Omega,0) \neq 0$, *then* $Tx = x$ *has a solution in* Ω.
3. ***(Homotopy)*** *Let* $T_t : [0,1] \times \overline{\Omega} \longrightarrow E$ *be continuous compact and* $T_t x \neq x$ *for all* $(x,t) \in [0,1] \times \partial\Omega$. *Then* $deg_{LS}(I-T_t,\Omega,0)$ *doesn't depend on* $t \in [0,1]$.
4. ***(Additivity)*** *Let* Ω_1, Ω_2 *be two disjoint open subsets of* Ω *and* $0 \notin (I-T)(\overline{\Omega} - \Omega_1 \cup \Omega_2)$, *then*

$$deg_{LS}(I-T,\Omega,0) = deg_{LS}(I-T,\Omega_1,0) + deg_{LS}(I-T,\Omega_2,0).$$

For the proof see [18].

1.4 Mappings of monotone type

Throughout this section, X is a real reflexive Banach space with norm $\| \, . \, \|$ and X^* denote its dual space. We let $\langle ., . \rangle$ denote the pairing between X^* and X, in the sense that $\langle f, u \rangle = f(u)$, for all $f \in X^*$ and $u \in X$.
We consider here the following classes of mappings of generalized monotone type:

Definition 1.16 *The operator* $A : X \longrightarrow X^*$

(a) is monotone if

$$\langle Au - Av, u - v\rangle \geqslant 0 \quad for\ all \quad u, v \in X,$$

(b) is strictly monotone if

$$\langle Au - Av, u - v\rangle > 0 \quad for\ all \quad u, v \in X,$$

(c) is of type (S_+) if any sequence $\{u_n\} \in X$ that weakly converges to u in X and satisfies

$$\limsup_{n\longrightarrow\infty}\langle Au_n, u_n - u\rangle \leq 0 \tag{1.1}$$

is in fact strongly convergent in X,

(d) is pseudo-monotone if any sequence $\{u_n\}$ weakly converges to u in X and satisfies (1.1) *is such that*

$$\langle Au_n, u_n - u\rangle \longrightarrow 0 \quad as \quad n \longrightarrow \infty$$

and Au_n weakly converges to Au,

(e) is quasi-monotone if any sequence $\{u_n\}$ that weakly converge to u satisfies

$$\limsup_{n\longrightarrow\infty}\langle Au_n - Au, u_n - u\rangle \geq 0.$$

Remark 1.17 *Following [19], we recall the inclusions*

$$(LS) \subset (S_+) \subset (PM) \subset (QM).$$

A basic relation between quasi-monotone operators and mappings of type (S_+), due to Calvert and Webb [20], is given below.

Theorem 1.18

(a) If $A \in (S_+)$ and $B \in (QM)$, then $(A + B) \in (S_+)$.

(b) If $(A + B) \in (S_+)$ for all $A \in (S_+)$, then $B \in (QM)$.

1.5 The degree for mappings of class (S+), (QM) and (PM)

In this section our task is to introduce approximative procedures which extend the LS-degree to further classes of mappings of monotone type.

Definition 1.19 *Let X be a real Banach space a map $T : D(T) \subset X \longrightarrow X^*$ is called demicontinuous if $u_n \longrightarrow u$ then $Tu_n \rightharpoonup Tu$.*

Definition 1.20 *Let X a real Banach space, the duality mapping $J : X \longrightarrow X^*$ is defined as*

$$Jx = \{x^* \in X : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}. \tag{1.2}$$

Definition 1.21 *A Banach space X is said to be locally uniformly convex if for every $\epsilon > 0$ and $x \in X$ with $\|x\| = 1$ there exists $\delta > 0$ such that $\|x - y\| \geqslant \epsilon$ implies*

$$\|\frac{x-y}{2}\| \leqslant 1 - \delta \quad \textit{for all} \quad y \in X \quad \textit{and} \quad \|y\| = 1.$$

Lemma 1.22 *Let X be a real reflexive Banach space such that X and X^* are locally uniformly convex. Then the duality map J defined by (1.2) is strictly monotone and of class (S_+).*

For more details see [21].

Definition 1.23 *(Homotopy of type (S_+)) Let X be a real reflexive Banach space and $G \subset X$ be an open and bounded set and $H : [0, 1] \times \overline{G} \longrightarrow X^*$. Then $H(t, x)$ is said to be a homotopy of type (S_+) if the following condition holds: For every $\{x_n\} \subset \overline{G}$ and $\{t_n\} \subset [0, 1]$ with $x_n \rightharpoonup x_0$ in X and $t_n \rightharpoonup t_0$ in $[0, 1]$ such that*

$$\limsup_{n \longrightarrow \infty} \langle H(t_n, x_n), x_n - x_0 \rangle \leq 0,$$

we have $x_n \longrightarrow x_0$ *and* $H(t_n, x_n) \rightharpoonup H(t_0, x_0)$.

Let X is a real reflexive separable Banach space and that X and X^* are locally uniformly convex. In virtue of the embedding theorem by Browder and Ton [22] there exists a separable Hilbert space H and a linear compact injection : $\Psi : H \longrightarrow X$ such that $\Psi(H)$ is dense in X. We define a further map $\hat{\Psi} : X^* \longrightarrow H$ by

$$(\hat{\Psi}(w), v) = (w, \Psi(v)), \quad v \in H, \quad w \in X^*$$

where $(.,.)$ stands for the inner product in H. It is obvious that $\hat{\Psi}$ is also linear compact injection. Let G be an open bounded subset in X. We denote

$$\mathcal{F}_G(S_+) = \{F : \overline{G} \longrightarrow X^* | F \in (S_+), \ \ \textit{bounded and demicontinuous}\}$$

and

$$\mathcal{H}_G(S_+) = \{F_t : \overline{G} \longrightarrow X^*, \ 0 \leqslant t \leqslant 1 | F_t \ \textit{bounded homotopy of class} \ (S_+)\}$$

With each $F \in \mathcal{F}_G(S_+))$ we can now associate a family of mappings $\{F_\epsilon | \epsilon > 0\}$ defined by

$$F_\epsilon(u) = u + \frac{1}{\epsilon}\hat{\Psi}\Psi F(u), \quad u \ \in \overline{G}.$$

For any fixed $\epsilon > 0$, F_ϵ maps $\overline{G}$ into X and has the form $I + C_\epsilon$ where

$$C_\epsilon = \frac{1}{\epsilon}\hat{\Psi}\Psi F$$

is compact. Hence the LS-degree is defined for the triplets (F_ϵ, G, y) whenever $y \notin F_\epsilon(\partial G)$.

We have the following basic

Lemma 1.24 *Let* $F \in \mathcal{F}_G(S_+))$, $A \subset \overline{G}$ *a closed subset and* $0 \notin F(A)$. *Then there exists* $\hat{\epsilon} > 0$ *such that* $0 \notin F_\epsilon(A)$ *for all* $0 < \epsilon < \hat{\epsilon}$. *Moreover, if* $o \notin F(\partial G)$, *there exists* $\epsilon_0 > 0$ *such that* $d_{LS}(F_\epsilon, G, 0)$ *is constant for all* $0 < \epsilon < \epsilon_0$.

For the proof see [23].

Definition 1.25 *In view of Lemma (1.24) it is relevant to define*

$$d_{S_+}(F,G,0) = d_{LS}(F_\epsilon, G, 0) \quad where \quad 0 \leqslant \epsilon \leqslant \epsilon_0.$$

Moreover, for any $y \in X^*$ with $y \notin F(\partial G)$ we can define

$$d_{S_+}(F,G,p) = d_{S_+}(F-p,G,0).$$

To convince ourselves that we have obtained a classical topological degree function d_{S_+} for mappings in $\mathcal{F}_G(S_+)$) the conditions (1) to (4) have to be verified. It is obviously sufficient to deal with the case $y = 0$ or $y(t) \equiv 0$

(a) If $0 \notin F(\overline{G})$ it follows from Lemma 1.24 that $0 \notin F_\epsilon(\overline{G})$ for all $0 < \epsilon < \hat{\epsilon}$. Hence $d_{LS}(F_\epsilon, G, 0) = 0$ for all $0 < \epsilon < \hat{\epsilon}$ implying $d_{S_+}(F,G,0) = 0$. Therefore $d_{S_+}(F,G,0) \neq 0$ implies $0 \in F(G)$.

(b) If G_1 and G_2 are open disjoint subsets of G and $0 \notin F(\overline{G} \backslash (G_1 \cup G_2))$, we can apply again Lemma 1.24 with $A = \overline{G} \backslash (G_1 \cup G_2)$ and use the property (4) for the LS-degree to derive additivity for d_{S_+}.

(c) If $F_t \in \mathcal{H}_G(S_+)$ we can extend Lemma 1.24 for homotopies in the obvious way. The property (c) follows then from the corresponding property for LS-homotopies.

(d) To show that $\mathcal{J}$ plays the role of normalizing map we consider the affine LS-homotopy $(1-t)I + t\mathcal{J}_\epsilon$. Since $\mathcal{J}(u) = 0$ if and only if $u = 0$, and since

$$\langle \mathcal{J}(u), (1-t)u + t\mathcal{J}(u) \rangle = \|u\|^2 + \frac{1}{\epsilon}\|\hat{\Psi}\mathcal{J}(u)\|_H^2 > 0$$

for all $u \neq 0$ and $0 \leqslant t \leqslant 1$, we obtain

$$d_{S_+}(\mathcal{J}, G, 0) = \lim_{\epsilon \longrightarrow 0+} d_{LS}(\mathcal{J}_\epsilon, G, 0) = d_{LS}(I, G, 0) = 1$$

whenever $0 \in \mathcal{J}(G)$.

Theorem 1.26 *For any mapping f of class (S_+) that is one-to-one on the closure of an open set G that contains 0 and $\langle f(u), u\rangle \geqslant 0$ on the boundary of G, then*

$$d(f, G, y_0) = \begin{cases} +1 & if\ y_0 \in f(G) \\ 0 & if\ y_0 \notin f(\overline{G}) \end{cases}$$

Theorem 1.27 *If f_0 and f_1 are two maps of $\overline{G}$ into X^* lying in the class (S_+), then the linear homotopy*

$$f_t = (I - t)f_0 + tf_1, \quad t \in [0, 1],$$

is always a homotopy of class (S_+).

For more details of Theorem 1.26 and Theorem 1.27 see [24].

Remark 1.28 *The (S_+)-degree can be extended for quasimonotone mappings, i.e., to the class $\mathcal{F}_G(QM)$ by using the fact that $F + \epsilon\mathcal{J} \in (S_+)$, whenever $F \in (QM)$ and $\epsilon > 0$.*

Definition 1.29 *If $y \notin \overline{F(\partial G)}$, the QM-degree obtained through approximations*

$$d_{QM}(F, G, y) = \lim_{\epsilon \longrightarrow 0+} d_{S_+}(F + \epsilon\mathcal{J}, G, y).$$

Remark 1.30 *The QM-degree is not a classical degree in the sense of Definition 1.2. For instance we have:*

(á) If $d_{QM}(F, G, y) \neq 0$ then $y \in \overline{F(G)}$.

For more details on weak degree theories we refer to [25].

Remark 1.31 *Since $(S_+) \subset (PM) \subset (QM)$ the QM-degree is defined for all mappings $F \in \mathcal{F}_G(PM)$.*

Theorem 1.32 *Let G be an open bounded subset in X, $T \in \mathcal{F}_G(S_+)$ a reference map and $F \in \mathcal{F}_G(S_+)$. If for a given $y \in X^*$ there exists $w \in T(G)$ such that*

$$tF(u) + (1-t)T(u) \neq ty + (1-t)w \quad for\ all \quad u \in \partial G, 0 \leqslant t \leqslant 1, \tag{1.3}$$

then $d_{S_+}(F, G, y) \neq 0$ and $F(u) = y$ admits a solution u in G.

Theorem 1.33 *Let G be an open bounded subset in X and $F \in \mathcal{F}_G(S_+)$. If there exists $\overline{u}$ $\in G$ such that*

$$\langle F(u) - y, u - \overline{u} \rangle > \|F(u) - y\| \|u - \overline{u}\| \quad for\ all\ u \in \partial G,$$

then $d_{S_+}(F, G, y) = 1$ and $F(u) = y$ admits a solution u in G.

For more details of Theorem 1.32 and Theorem 1.33 see [23].

Chapter 2

Unilateral problems for nonlinearly elastic plates

2.1 von Kármán Equations

2.1.1 The classical von Kármán equation

The canonical von Kármán equations are given by

$$\begin{cases} \Delta^2\xi = [\psi, \xi] + f & \text{in } \omega, \\ \Delta^2\psi = -[\xi, \xi] & \text{in } \omega, \\ \xi = \partial_\nu \xi = 0 & \text{on } \gamma, \\ \psi = \psi_0, \quad \partial_\nu \psi = 0 & \text{on } \gamma, \end{cases}$$

where ω is a domain in $\mathbb{R}^2$ modelling the middle surface of the plate, and f, ψ_0 are given functions. The objective of this boundary-value problem is to find at least one solution $(\xi, \psi) \in H_0^2(\omega) \times H^2(\omega)$, under the assumptions that ω is simply connected and the data have minimal regularities.

2.1.2 The reduced von Kármán equation for a nonlinearly elastic plates

The canonical von Kármán equations can be transformed into the reduced von Kármán equation, by means of the following result:
let the bilinear and symmetric operator

$$B : H^2(\omega) \times H^2(\omega) \longrightarrow H_0^2(\omega),$$

be defined as follows: given $(\xi, \psi) \in H^2(\omega) \times H^2(\omega)$, we let $B(\xi, \psi) \in H_0^2(\omega)$ denote the unique solution of the biharmonic equation

$$\Delta^2 B(\xi, \psi) = [\xi, \eta] \text{ in } \omega.$$

Then, defined the operator

$$C : H_0^2(\omega) \longrightarrow H_0^2(\omega),$$

by letting

$$C(\xi) := B(B(\xi, \xi), \xi),$$

which is "cubic", in that

$$C(\alpha\xi) = \alpha^3 C(\xi) \quad for\ all \quad \alpha \in \mathbb{R}.$$

Assuming that $\psi_0 \in H_0^{3/2}(\gamma)$ and $\psi_1 \in H_0^{1/2}(\omega)$, we let θ_0 in $H^2(\omega)$ be the unique solution of the boundary value problem: $\Delta^2\theta_0 = 0 \ in\ \omega,\ \theta_0 = \psi_0$ and $\partial_\nu\theta_0 = \psi_1 \in \gamma$, and we define the linear operator

$$\Lambda : H_0^2(\omega) \longrightarrow H_0^2(\omega),$$

by letting

$$\Lambda(\xi) = B(\theta_0, \xi).$$

Finally, assume that $f \in H^{-2}(\omega)$ and let $F \in H_0^2(\omega)$ be the unique solution of the biharmonic equation

$$\Delta^2 F = f \quad in \quad \omega.$$

Theorem 2.1 *The pair $(\xi, \psi) \in H_0^2(\omega) \times H^2(\omega)$ satisfies the canonical von Kármán equations if and only if $\xi \in H_0^2(\omega)$ satisfies the reduced von Kármán equation*

$$C(\xi) + (I - \Lambda)\xi - F = 0 \quad or\ equivalent \quad \xi - F = B(-B(\xi, \xi) + \theta_0, \xi),$$

where the Airy function is given by

$$\psi = \theta_0 - B(\xi, \xi).$$

For the proof see [26].

2.2 Variational inequality

In what follow, X will be a real Hilbert space, whose scalar product is denoted by $\langle ., . \rangle$, K a non-empty closed convex cone in X, $A : X \longrightarrow X$ an operator denoted on X, and $f \in X$ a fixed element. The problem

$$V.\ I.\ (A; f, K) : \quad \begin{cases} find\ u \in K \quad such\ that \\ \langle Au - f, v - u \rangle \geqslant 0, \quad for\ each\ v \in K, \end{cases} \tag{2.1}$$

is called the variational inequality associated with A, f and K.
If

$$K^* = \{y \in X : \ \langle y, x \rangle \geqslant 0 \quad for\ each\ x \in K\}$$

denoted the dual cone of K, we may define the general complementary problem :

$$C.\ P.\ (A; f, K) : \quad \begin{cases} find\ u \in K \quad such\ that \\ Au - f \in K^*, \ and \ \ \langle Au - f, u \rangle = 0. \end{cases} \tag{2.2}$$

The basic relation between problems $V.I(A; f, K)$ and $C.P(A; f, K)$ is the following:

Proposition 2.2 *Let X be a Hilbert space, K a closed convex cone with vertex at the origin in X, $f \in X$ and $A : K \longrightarrow X$. Then u^* is solution of $V.I(A; f, K)$ if and only if u^* is a solution of $C.P(A; f, K)$.*

Let the set-value mapping

$$P_A : X \longrightarrow 2^K,$$

be defined by

$$P_A(f) := \{u \in K : \quad u\ is\ solution\ of\ V.\ I.\ (A; f, K)\}.$$

It has been shown by A. Szulkin in [27] that, if $A : K \longrightarrow X$ has the following properties:

H1. $A : K \longrightarrow X$ is continuous on finite dimensional subspaces (i.e. the restriction of A to the intersection of k with any finite dimensional subspace of X is weakly continuous),

H2. there exist $\alpha > 0, \ q > 1$ such that

$$\langle Au - Av, u - v \rangle \geqslant \alpha \|u - v\|^q \quad for\ each\ u, v \ \in K,$$

then P_A is single-valued, bounded and continuous.

Let $A, L, T : X \longrightarrow X$ be given, and let g be fixed in X. Let us now suppose that the mapping A is the sum of two operators A_1 and A_2, with A_1 satisfying szulkin's assumptions H_1 and H_2.

It is by now well known in [28] that the complementary problem admits an equivalent fixed point formulation , more precisely, we have:

Proposition 2.3 *Let U be an open bounded set in K, $\lambda \in \mathbb{R}$ and consider the following problem:*

$$V.\ I.\ (A, L, T, g, \lambda, \overline{U}) : \begin{cases} find\ u \in \overline{U},\ \lambda \in \mathbb{R} \ \ such\ that \\ \langle Tu, v-u \rangle \geqslant \langle \lambda Lu - Au + g, v-u \rangle \ \ for\ each\ v \in K. \end{cases} \tag{2.3}$$

If A_1 satisfies assumptions H1 and H2, then $u \in \overline{U}$ is a solution of $V.\ I.\ (A, L, T, g, \lambda, \overline{U})$ if and only if u is a solution of the following fixed point problem:

$$F.\ P.\ (A, L, T, g, \lambda, \overline{U}) : \begin{cases} find\ u \in \overline{U},\ \lambda \in \mathbb{R} \ \ such\ that \\ u = P_{A_1}(-Tu + \lambda Lu - A_2 u + g). \end{cases} \tag{2.4}$$

Remark 2.4 *If $P_{A_1}(-Tu + \lambda Lu - A_2 u + g)$ is compact and if $V.\ I.\ (A, L, T, g, \lambda, \overline{U})$ has no solution on ∂U, then the topological degree of the mapping*

$$\Phi := I - P_{A_1}(-Tu + \lambda Lu - A_2 u + g),$$

with respect to U and 0 is well defined.

Let K be a closed, convex cone in the real Hilbert space X with inner product $< .,. >$ and norm $\|.\|$ and let U be a bounded open subset of K.

Definition 2.5 *The map $F : X \longrightarrow X^*$ is said to be strongly continuous if and only if*

$$u_n \rightharpoonup u \ \ as\ n \longrightarrow \infty$$

implies $Fu_n \longrightarrow Fu \ \ as\ n \longrightarrow \infty$

Proposition 2.6 *Let $A : X \longrightarrow Y$ be an operator where X and Y are real reflexive Banach spaces then the following two assertions are valid:*

(a) A is strongly continuous implies A is compact.

(b) A is linear and compact implies A is strongly continuous

In (b) we do not need that the Banach spaces are reflexive.

For the proof see [18].

2.3 Topological degree for Leary-Schauder operators

Consider the following nonlinear variational eigenvalue problem:

$$\text{Fined } (u,\lambda) \in \overline{U} \times \mathbb{R}_+ \text{ such that } < Au - \lambda Lu + Cu - f, v - u > \geqslant 0, \text{ for all } v \in K, \quad (2.5)$$

where f is given in X, λ is a positive parameter, and A, L, C are operators satisfying the following assumptions:

1. $A : X \longrightarrow X$ is such that $A = A_1 + A_2$, where
   (a) $A_1 : X \longrightarrow X$ is bounded, linear and α-coercive, ($i.e.\langle A_1 u, u\rangle \geqslant \alpha \|u\|^2$, for each $u \in X$);
   (b) $A_2 : X \longrightarrow X$ is strongly continuous, positively homogeneous of order one ($i.e. A_2(tu) = tA_2 u \ \ for\ each\ u \in X,\ t > 0$).
2. $L : K \longrightarrow X$ is strongly continuous and positively homogeneous of order one.
3. $C : K \longrightarrow X$ is strongly continuous and positively homogeneous of order $p > 1$.
4. $\langle Cu, u\rangle > 0$, for each $u \in K\backslash 0$.

Remark 2.7 *It should be observed that Szulkin's assumptions are fulfilled for* A_1 *and therefore* P_{A_1} *is a single-value, bounded and continuous.*

Lemma 2.8 *Assume that hypotheses (1) to (4) hold. Then there exists* $r_0 > 0$ *depending on* $\lambda \in \mathbb{R}$ *and* $g \in X$ *such that, for each* $r \geqslant r_0$

$$d_{LS}(u - P_{A_1}(-Cu + \lambda Lu - A_2 u + g), K_r, 0) = 1.$$

Proof. The map

$$u \longrightarrow -Cu + \lambda Lu - A_2u + g$$

is compact and since P_{A_1} is continuous, the map

$$u \longrightarrow P_{A_1}(-Cu + \lambda Lu - A_2u + g)$$

is compact.
Let U be a bounded open set in X such that $0 \notin \Phi(\partial U)$ where $\Phi : \overline{U} \longrightarrow X$ is given by

$$x \longrightarrow x - P_{A_1}(-Cx + \lambda Lx - A_2x + g).$$

Since the topological degree of Φ with respect to U and 0 is clearly defined, we may define the homotopy

$$H_\lambda(t, u) = P_{A_1}(-Cu + t(\lambda Lu - A_2u + g)).$$

We claim that there exists $r_0 > 0$ such that for each $r \geqslant r_0$, $(I - H_\lambda(t, u))(\partial K_r) \neq 0$ for each $t \in [0, 1]$. Indeed, suppose on the contrary, we may find sequences $\{u_n; n \in \mathbb{N}\}$ and $\{t_n; n \in \mathbb{N}\}$ such that $u_n \in k$, $t_n \in [0, 1]$, $\lim\limits_{n \longrightarrow \infty} \|u_n\| = \infty$ and

$$\langle Cu_n + A_1u_n, v - u_n\rangle \geqslant t_n\langle \lambda Lu - A_2u_n + g, v - u_n\rangle, \quad for\ each\ v \in K. \tag{2.6}$$

In particular , for $v = 0$ we obtain

$$\langle Cu_n + A_1u_n, u_n\rangle \leqslant t_n\langle \lambda Lu - A_2u_n + g, u_n\rangle. \tag{2.7}$$

We claim that there exists some $\tau > 0$ such that

$$\langle Cu_n, u_n\rangle \geqslant \tau\|u_n\|^{p+1}, \quad for\ all\ \in \mathbb{N}.$$

Otherwise, on relabelling if necessary and setting $v_n := u_n/\|u_n\|$ we would obtain

$$\lim_{n \longrightarrow \infty} \langle Cv_n, v_n\rangle = 0.$$

Since we may assume that $v_n \rightharpoonup v_0, v_0 \in K$, by strong continuity of C we would obtain $\langle Cv_0, v_0\rangle = 0$, and therefore by assumption (4), $v_0 = 0$.

Using (2.7) and assumption (2.1) and (4) we have

$$t_n\lambda\langle Lu_n, u_n\rangle \geqslant \langle A_1u_n, u_n\rangle + t_n\langle A_2u_n, u_n\rangle - t_n\langle g, u_n\rangle,$$

and therefore

$$t_n\lambda\langle Lv_n, v_n\rangle \geqslant \alpha + t_n\langle A_2v_n, v_n\rangle - t_n\langle g, u_n\rangle/\|u_n\|,$$

Hence by passing to a subsequence, if necessary (this is possible since $t_n \in [0,1]$), we may assume that $\lim_{n\longrightarrow\infty} t_n = t^*$ and we get

$$t^*\lambda\langle Lv_0, v_0\rangle \geqslant \alpha + t^*\langle A_2v_0, v_0\rangle,$$

and $\alpha \leqslant 0$, contradiction.

By applying again (2.7), Assumption (1.2) and the previous claim we have

$$\begin{aligned}\alpha\|u_n\|^2 + \tau\|u_n\|^{p+1} &\leqslant \langle Cu_n + A_1u_n, u_n\rangle \\ &\leqslant |\lambda|\|Lu_n\|\|u_n\| + \|A_2u_n\|\|u_n\| + \|g\|\|u_n\|.\end{aligned}$$

In particular, dividing the last inequality by $\|u_n\|^{p+1}$ we obtain:

$$\alpha\|u_n\|^{1-p}\| + \tau \leqslant |\lambda|\|Lu_n\|/\|u_n\|^p + \|A_2u_n\|/\|u_n\|^p + \|g\|/\|u_n\|^p.$$

Since A_2 and L are continuous positively homogeneous of order one, there exits $\Gamma_{A_2}, \Gamma_L > 0$ such that

$$\|A_2x\| \leqslant \Gamma_{A_2}\|x\|, \quad for\ each\ x \in K$$

and

$$\|Lx\| \leqslant \Gamma_L\|x\|, \quad for\ each\ x \in K.$$

This yields

$$\alpha\|u_n\|^{1-p} + \tau \leqslant |\lambda|\Gamma_L\|u_n\|^{1-p} + \Gamma_{A_2}\|u_n\|^{1-p} + \|g\|\|u_n\|^{-p},$$

and therefore by taking the limit as n tends to $+\infty$ we obtain $\tau \leqslant 0$, a contradiction.

Using now property (3) of the Leary-Schauder degree

$$\begin{aligned}d_{LS}(\Phi, K_r, 0) &= d_{LS}(I - H_\lambda(1,.), K_r, 0) \\ &= d_{LS}(I - H_\lambda(0,.), K_r, 0) \\ &= d_{LS}(I - P_{A_1}(-Cu), K_r, 0).\end{aligned}$$

We now define the homotopy

$$G_\lambda(t,u) := P_{A_1}(-tCu),$$

we claim that for each $r > 0$,

$$I - G_\lambda(t,.)(\partial K_r) \neq 0 \quad for\ each\ t \in [0,1].$$

Indeed, suppose, on the contrary, that there exist $r > 0$, $t^* \in [0,1]$ and $u^* \in K$ with $\|u^*\| = r$ such that

$$u^* = P_{A_1}(-tCu^*),$$

or equivalently,

$$\langle A_1 u^* + t^* C u^*, v - u^* \rangle \geqslant 0, \quad for\ each\ v \in K.$$

For $v = 0$, we get

$$\langle A_1 u^* + t^* C u^*, u^* \rangle \leqslant 0,$$

from which, by assumption (4) and properties of A_1 we derive $\alpha \|u^*\|^2 \leqslant$. This yields $u^* = 0$, a contradiction. Thus

$$\begin{aligned} d_{LS}(\Phi, K_r, 0) &= d(I - P_{A_1}(-Cu), K_r, 0) \\ &= d(I - G_\lambda(1,.), K_r, 0) \\ &= d(I - G_\lambda(0,.), K_r, 0) \\ &= d(I - P_{A_1}(0), K_r, 0). \end{aligned}$$

Since A_1 is coercive, necessarily $P_{A_1}(0) = 0$, and therefore by virtue of property (1) of the Leary-Schauder degree we obtain

$$d_{LS}(\Phi, K_r, 0) = 1,$$

and the desired result. ■

Theorem 2.9 *Assume that hypotheses (1)-(4) hold. Let* $g \in X$ *be fixed. If there exists* $u_0 \in K$ *such that* $\langle g, u_0 \rangle > 0$*, then for each* $\lambda \in \mathbb{R}$*, there exists* $u(\lambda) \in K$ *such that* $u(\lambda) \neq 0$ *and*

$$\langle Au(\lambda) - \lambda Lu(\lambda) + Cu(\lambda), v - u(\lambda) \rangle \geqslant \langle g, v - u(\lambda) \rangle, \quad for\ each\ v \in K.$$

Proof. The existence of $u(\lambda)$, solution of $V.\ I.\ (A, L, C, g, \lambda, K)$ follows from Lemma 2.8 and property (2) of the Leary-Schauder degree. For zero to be a solution, it is necessary that $\langle g, v \rangle \leqslant 0$, for each $v \in K$, and thus $u(\lambda) \neq 0$. ■

2.4 Topological degree for (S_+) operators

Consider the same nonlinear variational eigenvalue problem 2.5 such that A, L, C satisfying the following assumptions:

1. $A : X \longrightarrow X$ is linear, continuous, and α-coercive.

2. $L : K \longrightarrow X$ is continuous and positively homogeneous of order one.

3. $C : K \longrightarrow X$ is continuous and positively homogeneous of order three. and satisfies

$$< Cu, u >> 0 \quad \text{for all } u \in K \backslash \{0\}.$$

If $f = 0$, then (2.5) has the trivial solution $u = 0$, which corresponds to a state of plate without buckling. When $f \neq 0$ and λ increases from zero onward, buckling occurs and we are interested in the modelling of this phenomenon.

Let

$$F_\lambda = Au - \lambda Lu + Cu - f,$$

where the sum $(-\lambda L + C)$ is quasi monotone operator. Since A is an operator of type (S_+), so is F_λ (Theorem 1.18).

Let

$$K_r = \{x \in K; \|x\| < r\},$$

then, the topological degree $d_{S_+}(F_\lambda, K_r, 0)$ well defined.

Theorem 2.10 *Under hypotheses (1), (2) and (3), there exists* $r_0 = r_0(\lambda, f) > 0$ *such that, for each* $r \geqslant r_0$

$$d_{S_+}(F_\lambda, K_r, 0) = 1.$$

Proof. Let U be a bounded open set in X such that (2.5) has no solutions on ∂U. Since the (S_+)-degree of F_λ at 0 relative to U is well defined, we may consider the homotopy of type (S_+)

$$H_\lambda(t, u) = Au + Cu - t(\lambda Lu + f).$$

We claim that there exists $r_0 > 0$ such that the problem

$$\text{Fined } (u,\lambda) \in \overline{U} \times \mathbb{R}_+ \text{ such that } < H_\lambda(t,u), v-u >\geqslant 0, \text{ for all } v \in K,$$

has no solutions on ∂K_r for $r \geqslant r_0$ and $t \in [0,1]$, where

$$K_r = \{x \in K; \|x\| < r\}.$$

Indeed, suppose the contrary. Then, we can find sequences $\{u_n\}$ and $\{t_n\}$ such that $\|u_n\| \longrightarrow \infty$ and

$$\langle Au_n + Cu_n, v - u_n\rangle \leqslant t_n \langle \lambda L u_n + f, v - u_n\rangle \quad \forall v \in k.$$

Taking v $= 0$, we obtain

$$\langle Au_n + Cu_n, u_n\rangle \leqslant t_n \langle \lambda L u_n + f, u_n\rangle. \tag{2.8}$$

We prove that there exists $\epsilon > 0$ such that

$$\langle Cu_n, u_n\rangle \geqslant \epsilon \|u_n\|^4 \quad for\ all\ n \in N.$$

Otherwise, setting $v_n = \frac{u_n}{\|u_n\|}$, we would obtain $\langle Cv_n, v_n\rangle \longrightarrow 0$ as $n \longrightarrow \infty$. Since we may assume that $v_n \longrightarrow v_0 \in K$, we have $\langle Cv_0, v_0\rangle = 0$ by the strong continuity of C, and therefore $v_0 = 0$ by virtue of (3).
Using (2.8), (1) and (3), we get

$$\lambda t_n \langle L u_n, u_n\rangle \geqslant \alpha \|u_n\|^2 - t_n \langle f, u_n\rangle.$$

We may assume that $t_n \longrightarrow t_0 \in [0,1]$. Dividing by $\lambda\|u_n\|^2$ and letting $n \longrightarrow \infty$ we obtain

$$t_0 \langle L v_0, v_0\rangle \geqslant \frac{\alpha}{\lambda} > 0,$$

which is a contradiction for λ small enough, because $\langle Lv_0, v_0\rangle)$ is a constant. Using again (2.8), hypotheses (1), (2), (3) and the previous estimate, we get

$$\begin{aligned} \epsilon\|u_n\|^4 + \alpha\|u_n\|^2 &\leqslant \langle Cu_n + Au_n, u_n\rangle \\ &\leqslant |\lambda| \|Lu_n\| \|u_n\| - \|f\| \|u_n\|, \end{aligned}$$

and dividing by $\|u_n\|^4$, we infer that

$$\epsilon + \|u_n\|^{-2}(\alpha - \lambda\|L\|) - \|f\|\|u_n\|^{-4} \leqslant 0.$$

Taking the limit as $n \longrightarrow \infty$, we obtain $\epsilon \leqslant 0$, which is a contradiction. Using now the homotopy invariance property of the (S_+)-degree, we get

$$\begin{aligned} d_{S_+}(F_\lambda, K_r, 0) &= d_{S_+}(H_\lambda(1,.), K_r, 0) \\ &= d_{S_+}(H_\lambda(0,.), K_r, 0) \\ &= d_{S_+}(A + C, K_r 0). \end{aligned}$$

Define another homotopy

$$G(t,u) = Au + tCu.$$

We claim that, for each $r > 0$, the problem

$$Find\ u \in \overline{U} \ \ such\ that\ \ \langle G(t,u), v-u\rangle \geqslant 0 \ \ for\ all\ v \in K,$$

has no solution on ∂K_r for $t \in [0,1]$ Indeed, suppose the contrary. Then there exist $r > 0, s \in [0,1]$ and $y \in K$ with $\|y\| = r$ such that

$$\langle Ay + sCy, v - y\rangle \geqslant 0 \ \ forall \ \ v \in K.$$

For $v = 0$, we get

$$\langle Ay + sCy, y\rangle \leqslant 0,$$

and by hypotheses (1) and (3), it follows that $\alpha\|y\|^2 \leqslant 0$. This yields $y = 0$, a contradiction. Therefore

$$\begin{aligned} d_{S_+}(F_\lambda, K_r, 0) &= d_{S_+}(A + C, K_r, 0) \\ &= d_{S_+}(G_\lambda(1,.), K_r, 0) \\ &= d_{S_+}(G_\lambda(0,.), K_r, 0) \\ &= d_{S_+}(A, K_r 0). \end{aligned}$$

Since A is coercive, $Au = 0$ has a solution and thus

$$d_{S_+}(A, K_r 0) = 1.$$

Hence the proof is complete. ■

We are now in a position to prove a general existence result for nontrivial solutions.

Theorem 2.11 *Assume that assumptions (1),(2), and (3) hold and let* $f \in X$ *be fixed. If there exists* $u_0 \in K$ *such that* $< f, u_0 >> 0$*, then for each* $\lambda \in \mathbb{R}_+$*, there exists a nontrivial solution* $u(\lambda) \in K$ *of the problem (2.5)*

Proof. The existence of a solution for (2.5) follows from Theorem 2.10 and from the existence property of the S_+-degree. For zero to be a solution, it is necessary that $\langle f, v \rangle \leqslant 0 \;\; for \; all \;\; v \in K$, and thus $u(\lambda) \neq 0$. ■

2.5 Application to variational inequalities of von Kármán type

Let there be given a thin plate, identified with the closure of a bounded, open subset ω of $\mathbb{R}^2$, with a boundary $\partial\omega$ of class C^1. Assume that the plate is clamped on a part Γ_0 of its boundary $\partial\omega$ and simply supported on the remaining part of the boundary. Define the space

$$X := \{u \in H^2(\omega) : u = 0 \text{ on } \Gamma, \frac{\partial u}{\partial n} = 0 \text{ a.e. on } \Gamma_0\},$$

and let the set K of admissible displacements be the closed convex cone of X defined by

$$K := \{u \in X : \; u \geqslant 0 \text{ a.e. on } \Gamma_0\}.$$

The equilibrium of a non linearly elastic plate subjected to unilateral conditions is governed by the following variational inequalities:

$$\text{Fined } u \in K \text{ and } \lambda \in \mathbb{R} \text{ such that } < u - \lambda Lu + Cu - f, v - u > \geqslant 0, \text{ for all } v \in K, \quad (2.9)$$

where L is a linear operator describing the lateral loading in the the plane of the plate, C is a “cubic ”nonlinear operator generalizing that introduced in the Von Karman nonlinear theory of plates (see[26]), f is the density of the vertical force, λ is a parameter measuring the magnitude of the lateral loading, and u is the unknown transverse displacement.

Remark 2.12 *For a nonlinearly elastic plate with unilateral conditions, subjected to a body force of density f, the equilibrium of the plate is governed by a variational inequality of type*

(2.5), and if there exists $u_0 \in K$ such that $\langle f, u_0 \rangle > 0$, then we may apply Theorem 2.9 and Theorem 2.11 to get the existence of an equilibrium for any $\lambda \in \mathbb{R}_+$.

Remark 2.13 *Applying Theorem (2.9) with $A_1 = I$ and $A_2 = 0$, we obtain the existence of solutions for (2.9).*

Remark 2.14 *Applying Theorem 2.11 with $A = I$, we obtain the existence of solutions for (2.9).*

Bibliography

[1] L. E. J. Brouwer, Über abbildungen von mannichfaltigkeiten, Math. Ann., **71**, 1912, 97-115.

[2] J. Leray, J. Schauder, Topologie et equationes functionelles, Ann. Sci. Ecole Norm. Sup., **51**, 1934, 45-78.

[3] J. Cronin, Fixed points and topological degree in nonlinear analysis, Math. Surveys and Monogr, Amer. Math. Soc., Providence, **11**, 1964.

[4] S. Fučik, Solvability of nonlinear equations and boundary value problems, Reidel, Dordrecht, 1980.

[5] O. A. Ladyzenskaya, V. A. Solonnikov and N. N. Ural'ceva, Linear and quasilinear equations of parabolic type, Nauka, Moscow, 1967, English transl. in Amer. Math.

Soc., Providence, 1968.

[6] S. Fučik and A. Kufner, Nonlinear differential equations, Elsevier, Amsterdam, 1980.

[7] R. E. Gaines and J. Mawhin, Coincidence degree and nonlinear differential equations, vol. 568, Lecture Notes in Math., Springer-Verlag, Berlin, 1977.

[8] I. V. Skrypnik, Nonlinear elliptic equations of higher order, Naukova Dumka, Kiev, Russian, 1973.

[9] F. E. Browder, Degree of mapping for nonlinear mappings of monotone type, Proc. Nat. Acad. Sci. USA., **80**, 1983, 1771-1773.

[10] D.R. Adhikari and A.G. Kartsatos, Topological degree theories and nonlinear operator equations in Banach spaces, Nonl. Anal., **69**, 2008, 1235-1255.

[11] A.G. Kartsatos and J. Lin, Homotopy invariance of parameter-dependent domains and perturbation theory for maximal monotone and m-accretive operators in Banach spaces, Adv. Differential equations, **8**, 2003, 129-160.

[12] A.G. Kartsatos and J. Quarcoo, A new topological degree theory for densely defined $(S_+)_L$-perturbations of multivalued maximal monotone operators in reflexive separable Banach spaces, Nonlinear Anal., **69**, 2000, 2339-2354.

[13] T. von Kármán, Festigkeitsprobleme in maschinenbau, in encyclopadie der mathematischen wissenschaften, Leipzig, vol. 4, 1910, 311-385.

[14] P. G. Ciarlet, A justification of the von Kármán equations, Rational Mech. Anal., **73**, 1980, 349-389.

[15] D. Goeleven, V. H. Nguyen and M. Thera, Nonlinear eigenvalue problems governed by a variational inequality of von Kármán's type: A Degree theoretic approach, Topological methods in nonlinear analysis, vol. 2. 1993, 253-262.

[16] L. Gratie, Unilateral problems for nonlinearly elastic plates and shallow shells, Math. Mech. Solids, vol. 6, 2001, 343-349.

[17] D. O'Regan, Y. Je Cho and Y. Chen, Topological degree theory and application, mathematical analysis and applications, Taylor and Francis Group, LLC, New York, vol. 10, 2006, 1-30.

[18] S. Fučik, J. Nečas, J. Souček and V. Souček, Spectral analysis of nonlinear operator, Lecture Notes in Math., **346**, 1973.

[19] D. Pascali, Variational and topological methods in nonlinear analysis ,Topological degree for monotone mappings [M], Lecture Notes, New York University, Courant Institute, 1997.

[20] B. Calvert and J. R. L. Webb, An existence theorem for quasimonotone operators [J], Atii. Acad. Nat. Lincei, **50**, 1972, 362-368.

[21] K. Deimling, Nonlinear functional analysis, Springer-Verlag, Berlin, 1985.

[22] F. E. Brouwder and B. A. Ton, Nonlinear functional equations in Banach spaces and elliptic super-regularization, Math. Z., **105**, 1968, 177-195.

[23] V. Mustonen, Mappings of monotone type: Theory and applications, Spring School held in Roudnice nad Labem, Vol. 4, 1990, 104-126.

[24] F. E. Brouwder, Degree of mapping for nonlinear mappings of monotone type, Proc. Natl Acad. Sci. USA, Chicago, Illinois 60637, Vol. 80, 1983, 1771-1773.

[25] J. Berkovits, On the degree theory for nonlinear mappings of monotone type, Ann Acad. Sci. Fenn. Ser. Al, Dissertationes **58**, 1986.

[26] P. G. Ciarlet, Mathematical elasticity, Theory of plates, vol. 2, North-Holland, Amsterdam, 1997.

[27] A. Suzulkin, Popsitive solutions of variational inequalities: a degree-theoric approach, Jour of differential equations **57**, 1985, 90-111.

[28] R. W. Cottle, F. Giannessi and J. L. Lions, Vriational inequalities and complementarity problems, Johon Wiley and Sons, New York, 1980.

[29] L. Gratie, Unilateral eigenvalue problems for nonlinearly elastic plates and pseudo-monotone operators, C. R. Acad. Sci. Paris, **327**(1), 1998, 959-964.