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Dedication

This thesis is dedicated :

First and foremost, to my parents Lazhar,

who taught me that the best kind of knowledge to have is that which is learned for its own sake,

Aiida Dlili, who taught me that even the largest task can be accomplished if it is done one step at a time.

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To my friends who encourage and support me.

To all the people in my life who touch my heart,

I dedicate this research.

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Notations and conventions

- B° : The adjointe group of a brace B.
- B^+ : The additive group $(B,+)$.
- $RBra$: The category of right braces.
- $LBra$: The category of left brace.
- $\mathrm{Soc}(B)$:The socle of a brace B.
- $\otimes$: Tensor product .
- $QYBE$: The quantum Yang-Baxter equation.
- $T^n(V)$: The tensor product of n folds of a vector space V.
- $End(T^n(V))$: The set of the endomorphisms of $T^n(V)$.
- Δ : Difference function.
- G^Δ : The adjoint group of a bi-group (G,Δ).

Contents

Introduction

Roughly speaking, a *brace* is a half-ring, or better a half radical ring. Recall that a ring R is said to be radical if its underlying set is a group under the circle opertion $x \circ y = x + y + xy$. For a brace, we require the axioms of radical rings except the associativity and the distributivity; this last one is just supposed to be one-sided (right or left following our brace).

Since the introduction of the Jacobson radical by N. Jacobson, a considerable amount of work was devoted to radical rings. One central question in this area is to characterize the groups that can occur as adjoint groups of radical rings. Note that, more recently, this question became central in the realm of braces; that is what are the groups that can occur as adjoint groups of braces? Only very recently, one knows that not every finite p-group is the adjoint group of some brace.

On the one hand, braces are interesting for studying the quantum Yang-Baxter equation; and actually, this equation is behind the discovery of braces by W. Rump in 2007. On the other hand, as braces generalize radical rings, many problems and applications of radical rings extend naturally to braces. Note here that radical rings arise in studying factorized groups (the relevance was first noticed by Y. Sysak), automorphisms of groups (H. Laue), more specifically, automorphisms of finite p-groups, and regular abelian subgroups of affine groups (Caranti et Al), etc.

The abstract notion of brace is treated in the first chapter. The second chapter is devoted to studying the relation between braces and the quantum Yang-Baxter equation. In this chapter we introduce a sort of general Yang-Baxter equations, however no attempt is made to solve them. This family of YB-like equations may shed new lights on the ordinary QYBE, and we think that they underlie some intriguing algebraic structures.

In the last chapter, we introduce the notion of Bi-groups. These may be considered as a far reaching generalization of braces. One curious thing is that braces, radical rings, and general groups represent very particular instances of Bi-groups. Our treatment of these notions is still far from being mature, and the reader will find no deep applications of theme.

Chapter 1

Braces

Braces received wide interest in the last years. They were introduced by Rump in [8] in order to studying set-theoretic solutions of the Yang-Baxter equation.

1.1 Braces, homomorphisms and ideals

Definition 1.1.1 *Let $(B, +)$ be an abelian group together with a right distributive multiplication; that is*

$$(x + y).z = x.z + y.z \text{ for all } x, y, z \in B. \tag{1.1}$$

We say that $(B, +, .)$ is a rigth brace if the circle operation

$$x \circ y := x + y + x.y \tag{1.2}$$

defines a group structure on the set B.

If B is a right brace, the group defined by the circle operation will be denoted by B°, and called the adjoint group of B.

The notion of left brace can be defined similarly; just we require that the multiplication on B is left distributive instead of being right distributive.

There is a canonical way to pass from right braces to left braces and vice versa. If B is a right brace then the opposite brace B^{op} is a left brace, where B^{op} has the same additive group as B and the product $a.b$ in B^{op} is equal to ba in B. It follows similarly that the opposite brace of a left brace is a right brace.

Recall that an associative ring R is said to be *radical* if the operation $x \circ y = x + y + xy$, defines a group structure on the set R. Plainly, R can be seen as a right and a left Brace. Hence, braces can be viewed as generalization of radical rings.

We say that a right brace is trivial if $a.b = a + b$ for every elements a and b of that brace. If B is both a left and a right brace (under the same multiplication), then we say that B is a left-right brace.

We call a brace B abelian if its adjoint group B° is abelian.

Proposition 1.1.2 *Let B be an abelian group. Then B is a left-right brace for a given multiplication if, and only if, B is a radical ring with respect to this multiplication.*

Proof. As we noticed above, every radical ring is a left-right brace. Conversely, assume that B is a left-right brace, then

(i) $a(b + c) = ab + ac$,

(ii) $(a + b)c = ac + bc$,

(iii) $(B, \circ)$ group.

Thus it remains only to prove the associativity of the multiplication. Let $a, b, c \in B$. We have on the one hand,

$$\begin{aligned}(a \circ b) \circ c &= (a + b + ab) \circ c \\ &= a + b + ab + c + (a + b + ab)c \\ &= a + b + ab + c + ac + bc + (ab)c.\end{aligned}$$

On the other hand, from (iii) we have

$$\begin{aligned}(a \circ b) \circ c &= a \circ (b \circ c) \\ &= a + b + c + bc + a(b + c + bc) \\ &= a + b + c + bc + ab + ac + a(bc).\end{aligned}$$

It follows that

$$a(bc) = (ab)c.$$

■

Definition 1.1.3 *A sub-brace of a brace B is an additive subgroup H which is closed under multiplication; i.e., $ab \in H$, whenever $a, b \in H$.*

Given a radical ring A, the right (as well the left) sub-braces of A are exactly the subrings of A. If B is a trivial brace, the sub-braces of B are the additive subgroups of B.

Definition 1.1.4 *A homomorphism between two right (left) braces A and B is a map $f : A \longrightarrow B$ which satisfies $f(a+b) = f(a) + f(b)$ and $f(a.b) = f(a).f(b)$ for all $a, b \in A$.*

Note that every homomophism of right (left) braces induces a group homomorphism between the corresponding adjoint groups.

The kernel of a brace homomorphism $f : A \longrightarrow B$ is defined as usual by

$$\ker f = \{a \in A \,|\, f(a) = 0\}.$$

It follows that $\ker(f)$ is a sub-brace (actually an ideal) of A. Also, $\ker f$ is a normal subgroup of the adjont group A°.

The image of the morphism f is a sub-brace of B.

It is straightforward to see that the right (left) Braces form a category **RBra** (**LBra**, respectively) and the passage from a brace to its adjoint group is a functor from **RBra** (or **LBra**) into the category of groups.

Definition 1.1.5 *Let B a right brace and let I be a subgroup of B^+.*

1. *We say that I is a right ideal of B, if $xb \in I$ for all $b \in B$ and $x \in I$.*
2. *We say that I is a left ideal of B, if $bx \in I$ for all $b \in B$ and $x \in I$.*
3. *I is said to be an ideal, if I is a right and a left ideal of B.*

Proposition 1.1.6 *Every ideal of a brace B is a normal subgroup of the adjoint group B°.*

Proof. Let I be an ideal of B, $a \in I$ and $b \in B$. Denote by b' the inverse of b in B°. We claim that $b' \circ a \circ b \in I$. Indeed,

1. If B is a right brace, then

$$\begin{aligned} b' \circ a \circ b &= (b'a + b' + a) \circ b \\ &= (b'a + b' + a)b + b'a + b' + a + b \\ &= b' + a + b'a + b + b'b + ab + (b'a)b \\ &= a + b'a + ab + (b'a)b \in I. \end{aligned}$$

2. If B is a left brace, then

$$\begin{aligned} b' \circ a \circ b &= b' \circ (a + b + ab) \\ &= b' + a + b + ab + b'(a + b + ab) \\ &= b' + a + b + ab + b'a + b'b + b'(ab) \\ &= a + ab + b'a + b'(ab) \in I. \end{aligned}$$

■

Let I be an ideal of the right (left) brace B. Then B/I has a natural structure of a right (left) brace; the addition is defined as usual, and for the multiplication we set

$$(a + I)(b + I) = ab + I,$$

this multiplication is well defined (see Proposition 3.1.10). Now we can coinsider the adjoint group $(B/I)^\circ$, and on the other hand I° is a normal subgroup of B°, so we can consider the quotient B°/I°. These groups are actually isomorphic.

Proposition 1.1.7 *Under the above notation, we have*

$$(B/I)^\circ \simeq B^\circ/I^\circ.$$

Proof. The canonical map

$$\begin{aligned} B &\longrightarrow B/I \\ b &\longmapsto b + I \end{aligned}$$

is a brace epimorphism; so it induces a group epimorphism

$$B^\circ \longrightarrow (B/I)^\circ.$$

The kernel of this map is exactly $I = I^\circ$, the result follows.

■

1.2 Modules over a braces

Let B be a right brace. A module over B is an abelian group M together with a right operation $M \times B \longrightarrow M$, which satisfies the following conditions, for all $x, y \in M$, and all $a, b \in B$.

1. $(x + y)a = xa + ya$,
2. $x(a \circ b) = (xa)b + xa + xb$,
3. $x0 = 0$.

Note that every right brace B is a module over itself by taking the multiplication in B as operation.

Let R be an associative ring, B be a right brace, and M be a left R-module. Assume also that M is a module over B. We say that M is an (R, B)-bimodule if, for all $x \in M, r \in R$, and $a \in B$, we have

$$(rx)a = r(xa).$$

Proposition 1.2.1 *Let B be a right brace, and*

$$B_0 := \{x \in B \,|\, x(a + b) = xa + xb \text{ for all } a, b \in B\}.$$

Then B_0 is a radical ring.

Proof. Let $x, y \in B_0$ and $a, b \in B$. As the multiplication in B is right distributive, we have

$$\begin{aligned}(x + y)(a + b) &= x(a + b) + y(a + b)\\ &= xa + xb + ya + yb\\ &= xa + ya + xb + yb\\ &= (x + y)a + (x + y)b,\end{aligned}$$

and obviously $0 \in B_0$. Hence, B_0 is an additive subgroup of B. We claim now that B_0 is closed under multiplication. Once this is proved, the first claim in the proposition follows. The associativity of B° shows that

$$\begin{aligned}(xy)(a+b) &= x(y \circ (a+b)) - xy - x(a+b)\\ &= x(y + (a+b) + ya + yb) - xy - x(a+b)\\ &= xy + x(a+b) + x(ya) + x(yb) - xy - x(a+b)\\ &= x(ya) + x(yb).\end{aligned}$$

Now, the expansion of $x \circ (y \circ a) = (x \circ y) \circ a$ yields $x(ya) = (xy)a$; but since b can be replaced by a, we obtain $(xy)(a+b) = (xy)a + (xy)b$, as desired.

■

The above proposition implies that every right brace B can be viewed as a (B_0, B)-bimodule.

For a right brace B, the *socle* is defined as

$$\mathrm{Soc}(B) := \{a \in B \mid ba = 0 \text{ for all } b \in B\}.$$

Proposition 1.2.2 *The socle of a right brace B is an ideal of B.*

Proof. Clearly, $0 \in \mathrm{Soc}(B)$. If $a_1, a_2 \in \mathrm{Soc}(B)$, then

$$b(a_1 + a_2) = ba_1 + ba_2 = 0,$$

for all $b \in B$. Hence $\mathrm{Soc}(B)$ is an additive subgroup of B. Let $a \in \mathrm{Soc}(B)$ and $b_1, \in B$, we claim that $ab_1, b_1a \in \mathrm{Soc}(B)$. We have $b_1a = 0$, so $b_1a \in \mathrm{Soc}(B)$. If $b_2 \in B$. We have

$$b_2(a \circ b_1) = (b_2a)b_1 + b_2a + b_2b_1,$$

the left side of the equation gives

$$\begin{aligned}b_2(a \circ b_1) &= b_2(a + b_1 + ab_1)\\ &= b_2a + b_2b_1 + b_2(ab_1)\\ &= b_2b_1 + b_2(ab_1),\end{aligned}$$

and the right one gives

$$b_2b_1 + b_2(ab_1) = b_2b_1.$$

Thus, $ab_1 \in \mathrm{Soc}(B)$. ■

Similarly, we define the socle of a left brace A as :

$$\mathrm{Soc}(A) = \{a \in A \mid ab = 0 \text{ for all } b \in A\}.$$

Taking in account that the opposite of a left brace is a right brace, one deduces that $\mathrm{Soc}(A)$ is an ideal.

Conjecture. *The socle of a finite brace is always non-trivial.*

This conjecture is related to the behavior of some solutions of the QYBE.

Chapter 2

Braces and the quantum Yang-Baxter equation

Here we investigate the connection between braces and the set-theoretic solutions of the Yang?Baxter equation.

2.1 The quantum Yang-Baxter equation (QYBE)

The Yang?Baxter equation first appeared in theoretical physics and statistical mechanics in the works of Yang [13] and Baxter [2, 3]. It has applications in many areas of physics, computer sciences and mathematics.

Fix a commutative unital ring K, and let V be a free K-module. We denote by $T^k(V)$ the tensor product of k folds of V. Every linear map

$$R : V \otimes V \longrightarrow V \otimes V,$$

defines a multilinear map

$$\begin{array}{rcl} V \times V \times V & \longrightarrow & V \otimes V \otimes V \\ (x, y, z) & \longmapsto & R(x \otimes y) \otimes z \end{array}$$

which induces a linear map $R \otimes 1 : T^3(V) \to T^3(V)$. We denote this map by R^{12}. Similarly, we define

$$\begin{aligned} R^{23} = 1 \otimes R : V \otimes V \otimes V &\longrightarrow V \otimes V \otimes V \\ x \otimes y \otimes z &\longmapsto x \otimes R(y \otimes z); \end{aligned}$$

and

$$R^{13} = (1 \otimes \tau)(R \otimes 1)(1 \otimes \tau),$$

where τ is the twist map $\tau(x \otimes y) = y \otimes x$. Thus, we have three elements R^{12}, R^{13}, R^{23} in the K-algebra $\mathrm{End}(T^3(V))$.

Definition 2.1.1 *We say that* $R \in \mathrm{End}(T^2(V))$ *is a solution of the quantum Yang-Baxter equation if*

$$R^{12}R^{13}R^{23} = R^{23}R^{13}R^{12} \tag{2.1}$$

holds in $\mathrm{End}(T^3(V))$.

The equation (2.1) is called the quantum Yang-Baxter equation (QYBE, for short).

Example 2.1.2 *The twist map* $\tau : x \otimes y \mapsto y \otimes x$ *is a solution of QYBE. Indeed, for all* $x \otimes y \otimes z \in T^3(V)$, *one has*

$$\begin{aligned} \tau^{12}\tau^{13}\tau^{23}(x \otimes y \otimes z) &= \tau^{12}\tau^{13}(x \otimes z \otimes y) \\ &= \tau^{12}(y \otimes z \otimes x) \\ &= z \otimes y \otimes x. \end{aligned}$$

On the other hand,

$$\begin{aligned} \tau^{23}\tau^{13}\tau^{12}(x \otimes y \otimes z) &= \tau^{23}\tau^{13}(y \otimes x \otimes z) \\ &= \tau^{23}(z \otimes x \otimes y) \\ &= z \otimes y \otimes x. \end{aligned}$$

Note that if R is a solution of the QYBE is olso λR is a solution where $\lambda \in K$. If R is invertible, then R^{-1} is a solution of the QYBE whenever R is.

While the QYBE equation is still far from being solved completely, we shall see how to construct many other examples by means of brace, cycle sets, etc.

2.2 Higher forms of QYBE

The QYBE seems to belong to a larger family of equations, that can be called YB-like equations. Our aim here is to define this family.

First, consider two positive integers n and k such that $k < n$, and define E_{kn} to be the set of all strictly increasing words of length k in $\{1, 2, ..., n\}$. Hence an element $\alpha \in E_{kn}$ is a finite sequence $\alpha = \alpha_1\alpha_2...\alpha_k$, where $\alpha_i \in \{1, 2, ..., n\}$ and $\alpha_i < \alpha_j$ whenever $i < j$. It follows that E_{kn} contains exactly $\binom{n}{k}$ elements.

The set E_{kn} can be ordered lexicographically, that is if $\alpha, \beta \in E_{kn}$, then $\alpha \leq \beta$ if and only if $\alpha = \beta$ or for the smallest index i such that $\alpha_i\beta_i$, we have $\alpha_i \leq \beta_i$. Note that the elements of E_{kn} can be viewed as natural numbers, and for two elements $\alpha, \beta \in E_{kn}$, we have $\alpha \leq \beta$ if $\alpha \leq \beta$ in $\mathbb{N}$.

Let $R \in End(T^kV)$ and let $\alpha = \alpha_1\alpha_2...\alpha_k \in E_{kn}$. Define $R^\alpha \in End(T^nV)$, by

$$R^\alpha(x_1 \otimes x_2... \otimes x_n) = \tilde{\sigma}^{-1}(R \otimes 1)\tilde{\sigma},$$

where $\tilde{\sigma}$ acts on $T^n(V)$ as

$$\tilde{\sigma}(x_1 \otimes x_2... \otimes x_n) = x_{\sigma_{(1)}} \otimes x_{\sigma_{(2)}}... \otimes x_{\sigma_{(n)}}$$

and σ is the permution on $\{1, 2, \ldots, n\}$ defined by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & .. & k & k+1 & .. & n \\ \alpha_1 & \alpha_2 & \alpha_3 & .. & \alpha_k & \beta_{k+1} & .. & \beta_n \end{pmatrix}$$

where $\beta_{k+1} < \cdots < \beta_n$ are the elements of $\{1, 2, \ldots, n\} - \{\alpha_1, ..., \ldots, \alpha_k\}$, listed in an increasing order.

Definition 2.2.1 *Let $\tau \in S_{E_{kn}}$. We say that $R \in End(T^kV)$ is a τ-solution of the YB-like equation if*

$$\Pi_\alpha R^\alpha = \Pi_\alpha R^{\tau(\alpha)}, \tag{2.2}$$

where α runs over E_{kn} in an increasing order.

We say that R is an ordinary solution of YB-like equation if

$$\Pi_\alpha R^\alpha = (\Pi R^\alpha)^{op}. \tag{2.3}$$

Example 2.2.2 *For $E_{12} = \{1, 2\}$, one has two linear transformations R^1, $R^2 : V \otimes V \longrightarrow V \otimes V$, given by $R^1(x \otimes y) = R(x) \otimes y$ and $R^2(x \otimes y) = x \otimes R(y)$. Hence, our equation is*

$$R^1 R^2 = R^2 R^1. \tag{2.4}$$

As $R^1 R^2(x \otimes y) = R(x) \otimes R(y)$, every element of End$(V)$ is a solution of (2.4).

More generally, for E_{1n}, every element $R \in$ End(V) is τ-solution of the YB-like equation:

$$\begin{aligned}\Pi_{i=1}^{i=n} R^i(x_1 \otimes ... \otimes x_n) &= R(x_1) \otimes ... \otimes R(x_n) \\ &= \Pi_{i=1}^{i=n} R^{\tau(i)}(x_1 \otimes ... \otimes x_n).\end{aligned}$$

When $n = 3$ and $k = 2$, we cover the ordinary QYBE. Indeed, we have $E_{23} = \{12, 13, 23\}$, and

$$\Pi_\alpha R^\alpha = R^{12} R^{13} R^{23},$$

and

$$(\Pi_\alpha R^\alpha)^{op} = R^{23} R^{13} R^{12}.$$

So our equation has the form :

$$R^{12} R^{13} R^{23} = R^{23} R^{13} R^{12}.$$

Of course we have to show that our R^{ij}'s coincide with the old ones.

For R^{12}, on has

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} = 1$$

so, $\tilde{\sigma}^{-1}(R \otimes 1)\tilde{\sigma} = R \otimes 1$, which coincides with the ordinary R^{12}.

For R^{23}, one has

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \text{ so } \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}.$$

If $x_1, x_2, x_3 \in V$, then

$$\begin{aligned}\tilde{\sigma}(x_1 \otimes x_2 \otimes x_3) &= x_{\sigma(1)} \otimes x_{\sigma(2)} \otimes x_{\sigma(3)} \\ &= x_2 \otimes x_3 \otimes x_1.\end{aligned}$$

Set

$$R(x_1 \otimes x_2) = \sum_i r_i(x_1, x_2) \otimes s_i(x_1, x_2),$$

with $r_i(x_1, x_2), s_i(x_1, x_2) \in V$, and i runs over a finite set. It follows that

$$\begin{aligned}(R \otimes 1)\tilde{\sigma}((x_1 \otimes x_2 \otimes x_3) &= R(x_2 \otimes x_3) \otimes x_1 \\ &= (\sum_i r_i(x_2, x_3) \otimes s_i(x_2, x_3)) \otimes x_1 \\ &= \sum_i r_i(x_2, x_3) \otimes s_i(x_2, x_3) \otimes x_1.\end{aligned}$$

Therefore,

$$\begin{aligned}\tilde{\sigma}^{-1}(R \otimes 1)\tilde{\sigma}(x_1 \otimes x_2 \otimes x_3) &= \tilde{\sigma}^{-1}(\sum_i (r_i(x_2, x_3) \otimes s_i(x_2, x_3) \otimes x_1) \\ &= \sum_i (\tilde{\sigma}^{-1}(r_i(x_2, x_3) \otimes s_i(x_2, x_3) \otimes x_1)) \\ &= \sum_i (x_1 \otimes r_i(x_2, x_3) \otimes s_i(x_2, x_3)) \\ &= x_1 \otimes \sum_i ((r_i(x_2, x_3) \otimes s_i(x_2, x_3)) \\ &= x_1 \otimes R(x_2 \otimes x_3).\end{aligned}$$

Thus $R^{23} = 1 \otimes R$, as desired.

It remains to show that our R^{13} coincides with the old one. We have $R^{13} = \tilde{\sigma}^{-1}(R \otimes 1)\tilde{\sigma}$, where

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix},$$

on the one hand,

$$\begin{aligned}R^{13}(x_1 \otimes x_2 \otimes x_3) &= \tilde{\sigma}^{-1}(R \otimes 1)\tilde{\sigma}(x_1 \otimes x_2 \otimes x_3) \\ &= \tilde{\sigma}^{-1}(R \otimes 1)(x_1 \otimes x_3 \otimes x_2) \\ &= \tilde{\sigma}^{-1}(R(x_1 \otimes x_3) \otimes x_2) \\ &= \tilde{\sigma}^{-1}(\sum_i (r_i(x_1, x_3) \otimes s_i(x_1, x_3)) \otimes x_2) \\ &= \tilde{\sigma}^{-1}(\sum_i r_i(x_1, x_3) \otimes s_i(x_1, x_3) \otimes x_2) \\ &= \sum_i r_i(x_1, x_3) \otimes x_2 \otimes s_i(x_1, x_3).\end{aligned}$$

On the other hand,

$$\begin{aligned}(1 \otimes \tau)(R \otimes 1)(1 \otimes \tau)(x_1 \otimes x_2 \otimes x_3) &= (1 \otimes \tau)(R \otimes 1)(x_1 \otimes x_3 \otimes x_2) \\ &= (1 \otimes \tau)(\sum_i r_i(x_1, x_3) \otimes s_i(x_1, x_3)) \otimes x_2 \\ &= (1 \otimes \tau)(\sum_i r_i(x_1, x_3) \otimes s_i(x_1, x_3) \otimes x_2)) \\ &= \sum_i r_i(x_1, x_3) \otimes x_2 \otimes s_i(x_1, x_3).\end{aligned}$$

2.3 Set theoretic solutions of the QYBE

For any non-empty set X, we can consider V the free K-module on X, and it follows that the elements $x \otimes y$, with $x, y \in X$, form a basis for T^2V.

Consider a map

$$r : X \times X \longrightarrow X \times X,$$

and denote its components by r_1 and r_2; so $r(x, y) = (r_1(x, y), r_2(x, y))$, for all $x, y \in X$. Hence, we can define the map

$$x \otimes y \longmapsto r_1(x, y) \otimes r_2(x, y),$$

on the basis elements of $V \otimes V$; and this map extends to a linear map in $End(T^2V)$, which we denote by the same symbol r.

Definition 2.3.1 *Let $r : X^2 \longrightarrow X^2$ be a mapping, and V be the free K-module on X. We say that r is a set theoretic solution of the QYBE if the linear map induced by r in $End(T^2V)$ is a solution of that equation. We say that r is :*

1. *left non-degenerate if the map $x \longmapsto r_1(x, y)$ is bijictive, for all $y \in X$.*
2. *right non-degenerate if the map $y \longmapsto r_2(x, y)$ is bijictive, for all $x \in X$.*
3. *If r is left and right non-degenerate, we say that it is non-degenerate.*

Proposition 2.3.2 *Let $r : X^2 \longrightarrow X^2$, and $r(x, y) = (r_1(x, y), r_2(x, y))$. Then r is a set theoretic solution of the QYBE if, and only if, the following equations hold*

$$r_1(r_1(x, r_2(y, z)), r_1(y, z)) = r_1(r_1(x, y), z) \tag{2.5}$$

$$r_2(r_1(x, r_2(y, z)), r_1(y, z)) = r_1(r_2(x, y), r_2(r_1(x, y), z)) \tag{2.6}$$

$$r_2(x, r_2(y,z)) = r_2(r_2(x,y), r_2(r_1(x,y),z)); \tag{2.7}$$

for all $x, y, z \in X$.

Proof. The map r is a set theoretic solution of the QYBE, iff

$$r^{12}r^{13}r^{23} = r^{23}r^{13}r^{12}.$$

We have

$$\begin{aligned} r^{23}(x \otimes y \otimes z) &= (1 \otimes r)(x \otimes y \otimes z) \\ &= x \otimes r(y \otimes z) \\ &= x \otimes r_1(y,z) \otimes r_2(y,z), \end{aligned}$$

Hence,

$$\begin{aligned} r^{13}(r^{23}(x \otimes y \otimes z)) &= r^{13}(x \otimes r_1(y,z) \otimes r_2(y,z)) \\ &= (1 \otimes \tau)(r \otimes 1)(1 \otimes \tau)(x \otimes r_1(y,z) \otimes r_2(y,z)) \\ &= (1 \otimes \tau)(r \otimes 1)(x \otimes r_2(y,z) \otimes r_1(y,z)) \\ &= (1 \otimes \tau)(r(x, r_2(y,z)) \otimes r_1(y,z)) \\ &= (1 \otimes \tau)(r_1(x, r_2(y,z)) \otimes r_2(x, r_2(y,z) \otimes r_1(y,z) \\ &= r_1(x, r_2(y,z)) \otimes r_1(y,z) \otimes r_2(x, r_2(y,z). \end{aligned}$$

Therefore,

$$r^{12}r^{13}r^{23}(x \otimes y \otimes z) = r_1(r_1(x, r_2(y,z)), r_1(y,z)) \otimes r_2(r_1(x, r_2(y,z)), r_1(y,z)) \otimes r_2(x, r_2(y,z).$$

On the other hand,

$$\begin{aligned} r^{12}(x \otimes y \otimes z) &= r(x \otimes y) \otimes z \\ &= r_1(x,y) \otimes r_2(x,y) \otimes z \end{aligned}$$

and

$$\begin{aligned}
r^{13}r^{12}(x\otimes y\otimes z) &= r^{13}(r_1(x,y)\otimes r_2(x,y)\otimes z)\\
&= (1\otimes\tau)(r\otimes 1)(1\otimes\tau)(r_1(x,y)\otimes r_2(x,y)\otimes z)\\
&= (1\otimes\tau)(r\otimes 1)(r_1(x,y)\otimes z\otimes r_2(x,y))\\
&= (1\otimes\tau)(r_1(r_1(x,y),z)\otimes r_2(r_1(x,y),z)\otimes r_2(x,y))\\
&= r_1(r_1(x,y),z)\otimes r_2(x,y)\otimes r_2(r_1(x,y),z).
\end{aligned}$$

So,

$$\begin{aligned}
r^{23}r^{13}r^{12}(x\otimes y\otimes z) &= r^{23}(r_1(r_1(x,y),z)\otimes r_2(x,y)\otimes r_2(r_1(x,y),z))\\
&= (1\otimes r)(r_1(r_1(x,y),z)\otimes r_2(x,y)\otimes r_2(r_1(x,y),z))\\
&= r_1(r_1(x,y),z)\otimes r(r_2(x,y),r_2(r_1(x,y),z))\\
&= r_1(r_1(x,y),z)\otimes r_1(r_2(x,y),r_2(r_1(x,y),z))\otimes r_2(r_2(x,y),r_2(r_1(x,y),z))
\end{aligned}$$

We conclude that

$$r_1(r_1(x,r_2(y,z)),r_1(y,z)) = r_1(r_1(x,y),z),$$

$$r_2(r_1(x,r_2(y,z)),r_1(y,z)) = r_1(r_2(x,y),r_2(r_1(x,y),z)),$$

and

$$r_2(x,r_2(y,z)) = r_2(r_2(x,y),r_2(r_1(x,y),z)).$$

■

Perhaps, it is better to write the equations in Proposition (2.3.2), in a more dense form. This can be done, for instance, by setting $r_1(x,y)=x^y$ and $r_2(x,y)={}^x y$; from which we obtain :

$$(x^y)^z = (x^{({}^y z)})^{y^z},$$

$${}^{x^{({}^y z)}}(y^z) = ({}^x y)^{{}^{(x^y)}z},$$

$${}^x({}^y z) = {}^{{}^x y}\,({}^{x^y} z).$$

2.4 Cycle sets and solutions of the QYBE

We call a cycle set every set X together with an operation $X^2 \longrightarrow X^2$ such that

1. for all $x \in X$, the map

$$\begin{aligned} \sigma(x) : X &\longrightarrow X \\ y &\longmapsto x.y \end{aligned}$$

   is bijictive.

2. $(x.y).(x.z) = (y.x).(y.z)$.

For all $x, y, z \in X$.

Theoreme 2.4.1 *There is one to one corespondante between the left non-degenerate unitary solutions $r : X^2 \longrightarrow X^2$ of the QYBE, and a cycle sets.*

For a proof see Ramp ([10]).

Definition 2.4.2 *We call a cycle set X non-degenerate if the map*

$$x \longmapsto x.x,$$

is bijective for all $x \in X$.

Proposition 2.4.3 *A cycle set is non-degenerate if the associated solution is non-degenerate.*

Definition 2.4.4 *Let A be an abelian group wich is also a cycle set. We say that A is a linear cycle set if it satisfies the following equations :*

$$a.(b + c) = a.b + a.c \tag{2.8}$$

$$(a + b).c = (a.b).(a.c) \tag{2.9}$$

for all $a, b, c \in A$.

Proposition 2.4.5 *Braces are nothing but linear cycle sets.*

Proof. Write b^a instead for $b \dashrightarrow ab$. substitute b^a by b in (2.9) yields.

$$(a + b^a).c = b.(a.c).$$

$c \longmapsto (c^b)^a$, yields.

$$(b^a + a)(c^b)^a = c.$$

$$(c^b)^a = c^{b^a=a} \Leftrightarrow Eq.(2.9).$$

■

Theoreme 2.4.6 *Every non-degenerate cycle set X induced a brace structure on $Z^{(X)}$ by extending "." to $Z^{(X)}$!*

For a proof see (Ramp.[8]).

A cycle set X is said to be square-free if $x.x = x$ holds for all $x \in X$.

A sub-cycle-set X of a brace B which generates B as an abelian group will be called a cycle base of B.

Chapter 3

Difference functions and Bi-groups

In this chapter we developpe the notion of difference function and bi-groups.

This can be viewed as a far reaching generalization the notion of braces, and we think this last notion become more transparent in the ligth of the language of Bi-groups.

3.1 Definitions and basic facts

Definition 3.1.1 *Let* $(G, .)$ *be a group with identity element* 1. *We call a difference function on* G *every mapping* $\Delta : G \times G \to G$ *such that*

$$x \star y = x.y.\Delta(x, y)$$

defines a group structure on the set G, *with identity element* 1.

It is convenient to call a group endowed with a difference function a *bi-group*. If (G, Δ) is a bi-group, we call the group defined by the law $x \star y = xy\Delta(x, y)$, the *adjoint group* of (G, Δ), and we denote it by G^{Δ}.

Example 3.1.2 *For any group* G, *the commutator* $\Delta(x, y) = [y, x]$ *is a difference function on* G. *The adjoint group in this case is the opposite group of* G.

Example 3.1.3 *Let* R *be a radical ring. The multiplication defines a difference function on* R^+.

Example 3.1.4 *A left skew brace is just a bi-group* (B, Δ), *where the base group* B *is abelian, and* Δ*is left distributive, that is*

$$\Delta(x, yz) = \Delta(x, y)^z \Delta(x, z)$$

for all $x, y, z \in B$. *Similarly, the right brace are the bi-groups with abelian base group and right distributive difference function.*

Example 3.1.5 *Having two group operations xy and $x \star y$ on a set G, with the same identity element, then we can define the function*

$$\Delta(x,y) = x^{-1}y^{-1}(x \star y)$$

where x^{-1} is the inverse of x with respect to the first law. That is obviously a difference function on the first group.

Example 3.1.6 *A generalization of braces to the noncommutative setting was suggested recently by L. Guarnieri and L. Vendramin (see [5]). In the light of bi-groups K, a skew brace is just a bi-group with a difference function Δ satisfying*

$$\Delta(xy,z) = \Delta(x,z)x^{-1}\Delta(y,z)$$

for all $x, y, z \in K$.

Proposition 3.1.7 *Let G be a group and Δ a difference function on G. Then*

1. *$\Delta(x,y)^z\Delta(xy\Delta(x,y),z) = \Delta(y,z)\Delta(x,yz\Delta(y,z))$, for all $x, y, z \in G$;*
2. *$\Delta(x,1) = \Delta(1,x) = 1$, for all $x \in G$;*
3. *for all $x \in G$, there exists $x' \in G$ such that $\Delta(x,x') = (xx')^{-1}$.*

Conversely, if a function $\Delta : G \times G \to G$, satisfies the above conditions, then it defines a difference function on G.

Proof.

1. As "$\star$" assosiative, we have

$$\begin{aligned}\Delta(x,y)^z\Delta(xy\Delta(x,y),z) &= z^{-1}\Delta(x,y)z\Delta(xy\Delta(x,y),z)\\ &= z^{-1}z\Delta(y,z)\Delta(x,yz\Delta(y,z))\\ &= \Delta(y,z)\Delta(x,yz\Delta(y,z)).\end{aligned}$$

2.

$$\begin{aligned}x \star 1 &= x1\Delta(x,1)\\ &= x\\ &= 1x\Delta(1,x)\\ &= 1 \star x.\end{aligned}$$

Then

$$\Delta(x,1) = \Delta(1,x) = 1.$$

3.

$$\begin{aligned} 1 &= x \star \grave{x} \\ &= x\grave{x}\Delta(x, \grave{x}). \end{aligned}$$

Then

$$\Delta(x, \grave{x}) = (x\grave{x})^{-1}.$$

■

Conversely, if a function $\Delta : G \times G \to G$, satisfies the above conditions, then it defines a difference function on G.

Definition 3.1.8 *Let (G, Δ) be a bi-group, and H a subgroup of G.*

1. *We say that H is a left ideal of G, if $\Delta(x, h) \in H$, for all $x \in G$ and $h \in H$. The notion of right ideal can be defined similarly.*
2. *We say that H is an ideal of G, if it is a left and a right ideal.*
3. *If H is a normal subgroup and an ideal of G, we say that H is a normal ideal.*

Lemma 3.1.9 *If Δ is the commutator function, then H is a left ideal if and only if H is normal in G.*

Proof. $(\Rightarrow)$:
H is a left ideal then $\Delta(x, h) \in H$ for all $x \in G$ and $h \in H$, we can write $\Delta(x, h) = \grave{h}$ for some $\grave{h} \in G$. Thus $[h, x] = \grave{h}$ then $h^{-1}x^{-1}hx = \grave{h}$ so $x^{-1}hx = h\grave{h}$. It follows that $x^{-1}hx \in H$, this implies that H is normal in G.
$(\Leftarrow)$:
H is normal subgroup in G then $x^{-1}hx = \grave{h}$ for all $x \in G$ and $h, \grave{h} \in H$. Thus $h^{-1}[h, x] = \grave{h}$ so $[h, x] = h\grave{h} = \Delta(x, h)$ it follows that $\Delta(x, h) \in H$. This implies that H is a left ideal of G. ■

Proposition 3.1.10 *Let (G, Δ) be a bi-group, and H be a normal ideal of G. Then the following two conditions are equivalent*

(1) $\Delta(xh_1, yh_2) = \Delta(x, y) \mod H$, *for all $x, y \in G$ and $h_1, h_2 \in H$. (We mean by $x = y \mod H$, that $xH = yH$).*

(2) $x \star H = xH$, *for all $x \in G$.*

Proof.

Assume that *(1)* holds. Let $x \in G$, and x' be the inverse of x in the group $(G, \star)$. For any $h \in H$, we have

$$x' \star (xh) = x'xh\Delta(x', xh) = x'x\Delta(x', x) \mod H.$$

Since $x'x\Delta(x', x) = 1$, it follows that $x' \star (xh) = h_1$, for some $h_1 \in H$. Therefore $xh = x \star h_1$. This shows that $xH \subseteq x \star H$, and clearly $x \star H \subset xH$, so $x \star H = xH$.

Conversely, assume that *(2)* is true. First, we claim that H is a normal subgroup of G^{Δ}. Let $x \in G$. We have $H \star x \subseteq Hx = xH = x \star H$. It follows that $x' \star H \star x \subseteq H$. This prove the claim. Now let $x, y \in G$ and $h_1 \in H$, by assumption there exists $h \in H$ such that $xh_1 = x \star h$. We have

$$x \star h \star y = xhy\Delta(x, h)^y\Delta(x \star h, y) = xy\Delta(xh_1, y) \mod H.$$

As H is normal in $(G, \star)$, we can write $h \star y = y \star h_2$, for some $h_2 \in H$. Thus

$$x \star h \star y = x \star y \star h_2 = xy\Delta(x, y)^{h_2}\Delta(x \star y, h_2) = xy\Delta(x, y) \mod H.$$

It follows that $\Delta(xh_1, y) = \Delta(x, y) \mod H$.

Let $h_2 \in H$. We can write $yh_2 = y \star h$, for some $h \in H$. On the one hand, we have

$$x \star (y \star h) = xyh\Delta(y, h)\Delta(x, y \star h) = xy\Delta(x, yh_2) \mod H,$$

on the other hand,

$$(x \star y) \star h = xyh\Delta(x, y)^h\Delta(x \star y, h) = xy\Delta(x, y) \mod H.$$

This implies that

$$\Delta(x, yh_2) = \Delta(x, y) \mod H.$$

■

Definition 3.1.11 *Let (G, Δ) be a bi-group. We call a bi-subgroup of G every normal ideal H of G which satisfies one of the condition of Proposition 3.1.10.*

Remark 3.1.12 *Suppose that H is a bi-subgroup of (G, Δ). Then H can be seen as a subgroup of G^{Δ}, and we may denote it by H^{Δ}. As we seen in the previous proof, H^{Δ} is a normal subgroup of G^{Δ}; hence we can consider the quotient subgroup G^{Δ}/H^{Δ}. On the other hand, the relation $\Delta(xH, yH) = \Delta(x, y)H$, defines a difference function on G/H, and the adjoint group $(G/H)^{\Delta}$ coincides with G^{Δ}/H^{Δ}.*

Definition 3.1.13 *A map $f : G \to G'$, between the two bi-groups (G, Δ) and (G', Δ') is a morphism of bi-group if f is a group homomorphism from G to G', that satisfies*

$$f(\Delta(x, y)) = \Delta'(f(x), f(y)),$$

for all $x, y \in G$. In this case, f determines a group homomorphism from G^{Δ} to $G'^{\Delta'}$. An obvious example is the canonical map $\pi : G \longrightarrow G/H$, when H is a bi-subgroup of (G, Δ).

The composition of two morphisms of bi-groups is again a morphism of bi-group.

Remark 3.1.14 *In the category of bi-groups, a morphism is an isomorphism if and only if it is bijective.*

The kernel of a morphism $f : (G, \Delta) \to (G', \Delta')$, is a bi-subgroup of (G, Δ).

Proposition 3.1.15 *Let f be a morphism of bi-group, between the two bi-groups (G, Δ) and (G', Δ') ; if H' is a bi-subgroup of G' then $f^{-1}(H')$ is a bi-subgroupe of G.*

Proof. Let $x \in G$ and $h \in f^{-1}(H')$.

We have

$$f(\Delta(x, h)) = \Delta'(f(x), f(h)) \in H',$$

and

$$f(\Delta(h, x)) = \Delta'(f(h), f(x)) \in H'.$$

Therefore $\Delta(x, h)$ et $\Delta(h, x) \in f^{-1}(H')$. On the other hand, we have

$$f(x^{-1}hx) = f(x^{-1})f(h)f(x) \in H',$$

then $x^{-1}hx \in f^{-1}(H')$. It follows that $f^{-1}(H')$ is a normal ideal of G.

We have,

$$x \star h = xh\Delta(x, h) \in xf^{-1}(H').$$

$f(xh) = f(x) \star h' \in f(x) \star H'$, for some $h' \in H'$.

Let x be the inverse of x in the group $(G; \star)$. we can rite

$$f(x)' \star f(xh) \in H'.$$

Thus

$$f(x' \star xh) \in H',$$

then $(x' \star xh) \in f^{-1}(H')$, this implies that $x \star f^{-1}(H') = xf^{-1}(H')$.

■

Definition 3.1.16 *Let* $(G;\Delta)$ *be a bi-group*

$$Z(G,\Delta) = \{x \in G \mid \Delta(x,y) = \Delta(y,x) = 1, \forall y \in G\}.$$

Example 3.1.17 *Let* $G = \{0,1,2,3\}$ *with the usual addition. We can say that* $G \simeq \mathbb{Z}_4$*, and define the map* $\mu : \mathbb{Z}_2 \times \mathbb{Z}_2 \longrightarrow G$ *such that*

$$\begin{array}{c} \mu(0,0) \longleftrightarrow 0 \\ \mu(1,0) \longleftrightarrow 1 \\ \mu(0,1) \longleftrightarrow 2 \\ \mu(1,1) \longleftrightarrow 3. \end{array}$$

So $(G,+,.)$ *depicted by Table :*

.	0	1	2	3
0	0	1	2	3
1	1	0	3	2
2	2	3	0	1
3	3	2	1	0

Then

$$\begin{array}{rcl} \Delta : G \times G & \longrightarrow & G \\ (x,y) & \longmapsto & (xy)^{-1}(x \star y) \end{array}$$

is a difference function on G*, we have :*

$$\Delta(0,0) = \Delta(0,1) = \Delta(0,2) = \Delta(0,3) = \Delta(1,2) = \Delta(2,2) = \Delta(2,3) = 0,$$
$$\Delta(1,1) = \Delta(1,3) = \Delta(3,3) = 2.$$

We conclude that $Z(G,\Delta) = \{0,2\}$ *; then*

$$G/Z(G,\Delta) = \{\bar{0},\bar{1}\} \simeq \mathbb{Z}_2.$$

Can be define

$$\begin{array}{rcl} \bar{\Delta} : G/Z(G,\Delta) \times G/Z(G,\Delta) & \longrightarrow & G/Z(G,\Delta) \\ (\bar{x},\bar{y}) & \longmapsto & \overline{\Delta(x,y)} \end{array}$$

3.2 Composition of difference function

Let (G,Δ_1) and (G,Δ_2) be two bi-groups. We have $\Delta_3 = \Delta_2 \circ \Delta_1$.

- $x \star y = xy\Delta_1(x,y)$.
- $x * y = x \star y \star \Delta_2(x,y)$.

Then

$$\begin{aligned} x * y &= (xy\Delta_1(x,y)) \star \Delta_2(x,y) \\ &= xy\Delta_1(x,y)\Delta_2(x,y)\Delta_2(xy\Delta_1(x,y), \Delta_2(x,y)). \end{aligned}$$

So,

$$\Delta_3(x,y) = \Delta_1(x,y)\Delta_2(x,y)\Delta_2(xy\Delta_1(x,y), \Delta_2(x,y)),$$

is a difference function.

It is worth noting that we have an obvious category of bi-groups with their homomorphisms. On another side, one can take a set G with a distinguished element 1, and consider the category $\mathcal{C}$ defined by :

- The objects of $\mathcal{C}$ are the group operations on G having 1 as an identity element.
- If a and b are two objects of $\mathcal{C}$, then there is only one morphism from a to b which is the difference function between a and b.

Proposition 3.2.1 *The category $\mathcal{C}$ defined above is a groupoid.*

Our main purpose in our thesis is to investigate the properties of this groupoid. For instance, knowing the order of G, what can one say about the number of objects of our groupoid.

3.3 The bi-subgroup generated by a subset

Let X be a subset of the bi-group G.

Lemma 3.3.1 *The intersection of a familly of bi-subgroups is a bi-subgroup.*

Proof. Let H_i be a familly of bi-subgroups of G, then $H_i \lhd G$ so $\bigcap H_i$ is a normal subgroups of G;
assume that $x \in G$,

$$h \in H_i \Rightarrow h \in H_i, \forall i \Rightarrow \Delta(x,h) \in H_i \forall i$$

$$\Delta(x,h) \text{ and } \Delta(h,x) \text{ lies in } H_i \text{ for all } i.$$

Thus H is a normal ideal of (G,Δ).

Finally, let be $h_1, h_2 \in \bigcap H_i$ and $x, y \in G$

$$\Delta(xh_1, yh_2)\Delta(x,y)^{-1} \in H_i, \forall i.$$

Thus

$$\Delta(xh_1, yh_2)\Delta(x,y)^{-1} \in \bigcap H_i.$$

■

Now we define the bi-subgroup generated by X to be the intersection of all bi-subgroup of (G,Δ) containing X, let us denoted by $< X, \Delta >$.

We can see $< X >$ and $< X >^{\Delta}$........... subgroup of G.

Question : Consider $X = \{\Delta(x,y), x, y \in G\}$. Then we can define :

$$\Delta(G,G) \text{ the bi-subgroupe generated by } \Delta(x,y) \text{ for all } x, y.$$

Note that in $G/\Delta(G,G)$

$$(G/\Delta(G,G))^{\Delta} = G/\Delta(G,G).$$

More generaly, given $X, Y \subseteq G$, define

$$\Delta(X,Y) = \text{the bi-subgroup generated by} \Delta(x,y); (x,y) \in X \times Y.$$

We define $\Delta_1(G) = \Delta(G,G)$ and by induction $\Delta_{n+1}(G) = \Delta(G, \Delta_n(G))$.

Remark 3.3.2

$$... \leq \Delta_2(G) \leq \Delta_1(G) \leq G$$

we say that (G,Δ) is right convidable if $\Delta_n(G) = 1$ for some n.
Idem $\acute{\Delta}_1(G) = \Delta(G,G)$, $\acute{\Delta}_{n+1}(G) = \Delta(\acute{\Delta}_n(G), G)$; we say that (G,Δ) is leftt convidable if $\acute{\Delta}_n(G) = 1$ for some n.

If (G,Δ) is a bi-group then we can define the opposite difference function $\acute{\Delta} : G \times G \longrightarrow G$ by $xy = x \star y \star \Delta^0(x,y)$.So we have the opposite bigroup (G,Δ^0).

Proposition 3.3.3 *If (G,Δ) is nilpotent, then (G,Δ^0) is nilpotent ?*

Conclusion

This is not really a conclusion; as we believe that our ideas can be developed in many directions, and only the lack of time prevented us from including more. This work is still in progress, and we wish to publish some significant results sooner.

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Abstract

In this note, we disscuss the notion of braces, their relevance to the quantum Yang-Baxter equation, and some natural generalizations. This include our introduction of high quantum Yang-Baxter equations, and the notion of diffrence functions and bi-groups.
Key words : Brace, Adjoint group, Radical ring, The quantum Yang-Baxter Equation, YB-like Equation, Cycle set, Difference function and Bi-group.

ملخص

ناقشنا في هذه المذكرة المتواضعة مفهوم الدعامات، وعلاقتها بمعادلات يانج و باكستر، وبعض التعميمات الطبيعية، و التي تتضمن ادخالنا لمفهوم معادلات يانج و باكستر و مفهوم دوال الفرق والزمر المزدوجة.
كلمات مفتاحية : دعامة، مرافق الزمرة، حلقة جذرية، معادلات يانج وباكستر، معادلات يانج وباكستر المميزة، دوال الفرق و الزمرة المزدوجة.

Résumé

Dans ce travail, on traite la notion de Brace, ses relations avec l'équation de Yang-Baxter équation, et des généralisations naturelles de cette notion. Ceci comprend en particulier les équations de Yang-Baxter généralisées et la notion de fonction de différence et b-groupe.

Mot clés: Brace, groupe adjoint, anneau radical, équation quantique de Yang-Baxter, équation de Yang-Baxter généralisée, cycle, fonction de différence et bi-groupe.