Note on b-colorings in Harary graphs

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Abstract

A b-coloring is a coloring of the vertices of a graph such that each color class contains a vertex that has a neighbor in all other color classes. The b-chromatic number $b(G)$ is the largest integer k such that G admits a b-coloring with k colors. In this note, according to the values taken by the order n of a graph, we determine exact values or bounds for the b-chromatic number of $H_{2m,n}$ which is the Harary graph $H_{k,n}$ when k is even. Therefore our result improves the result concerning the b-chromatic of p-th power graphs of cycles and give a negative answer to the open problem of Effantin and Kheddouci.

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1 Introduction

A proper coloring of a graph $G = (V, E)$ is a mapping c from V to the set of positive integers (colors) such that any two adjacent vertices are mapped to different colors. Each set of vertices colored with one color is a stable set of vertices or color class of G, so a coloring is a partition of V into stable sets. The smallest number k for which G admits a coloring with k colors is the chromatic number $\chi(G)$ of G.

A *b-coloring* is a proper coloring such that every color class i contains at least one vertex that has a neighbor in all the other classes. Any such vertex will be called a *b-dominating vertex* of color i. The *b-chromatic number* $b(G)$ is the largest integer k such that G admits a b-coloring with k colors.

The motivation of this special coloring is as follow. Let c be an arbitrary proper coloring of G and suppose we want to decrease the number of colors by recoloring all the vertices of a given color class X with other colors that is by putting the vertices of X in other color class. Then this is possible if and only if no vertex of X is a b-dominating vertex. In other words, one color can be recuperated by recoloring each vertex of some fixed color class if and only if the coloring c is not a b-coloring.

The open neighborhood of a vertex $v \in V$ is $N(v) = \{u \in V \mid uv \in E\}$, i.e, the set of all vertices adjacent with v. The closed neighborhoods of v is $N[v] = N(v) \cup \{v\}$. The degree of a vertex v of G is $d(v) = |N(v)|$. By $\Delta(G)$ we denote the maximum degree of G. Let $\Delta(G)$ be the maximum degree in G, and let $m(G)$ be the largest integer k such that G has k vertices of degree at least $k-1$. It is easy to see that every graph G satisfies

$$b(G) \leq m(G) \leq \Delta(G) + 1$$

(the first inequality follows from the fact that if G has any b-coloring with k colors then it has k vertices of degree at least $k-1$; the second inequality follows from the definition of $m(G)$). Irving and Manlove [10, 18] proved that every tree T has b-chromatic number $b(T)$ equal to either $m(T)$ or $m(T) - 1$, and their proof is a polynomial-time algorithm that computes the value of $b(T)$. On the other hand, Kratochvíl, Tuza and Voigt [17] proved that it is NP-complete to decide if $b(G) = m(G)$, even when restricted to the class of connected bipartite graphs such that $m(G) = \Delta(G)+1$. These NP-completeness results have incited searchers to establish bounds on the b-chromatic number in general or to find exact or approximate values for subclasses of graphs (see: [2, 3, 4, 6, 5, 7, 8, 9, 11, 12, 13, 14, 15, 17, 16]).

For $2 \leq k < n$, the Harary graph $H_{k,n}$ on n vertices is defined by West [19] as follows: Place n vertices around a circle, equally spaced. If k is even, $H_{k,n}$ is formed by making each vertex adjacent to the nearest $\dfrac{k}{2}$ vertices in each direction around the circle. If k is odd and n is even, $H_{k,n}$ is formed by making each vertex adjacent to the nearest $\dfrac{(k-1)}{2}$ vertices in each direction around the circle and to the diametrically opposite vertex. In both cases, $H_{k,n}$ is k-regular. If both k and n are odd, $H_{k,n}$ is constructed as follows. It has vertex $v_0, v_1, ..., v_{n-1}$ and is constructed from $H_{k-1,n}$ by adding edges joining vertex v_i to vertex $v_{i+\frac{(n-1)}{2}}$ for $0 \leq i \leq \dfrac{(n-1)}{2}$.

We denote by $dist_G(x, y)$ the distance between vertices x and y in G. The p-th power graph G^p with $p \geq 1$ is a graph obtained from G by adding an edge between every pair of vertices x and y with $dist_G(x, y) \leq p$, in particular $G^1 = G$. The p-th power graph of a cycle C_n with $p \geq 1$ which is C_n^p is the the Harary graph $H_{k,n}$ with $k = 2p$. In [5], Effantin and Kheddouci investigate the b-chromatic number of the p-th power graph, so, they determine exact values and bounds for b-chromatic number of the p-th power graph of paths and the p-th power graph of cycles.

In this note, according to the values taken by the order n of a graph, we determine exact values or bounds for the b-chromatic number of $H_{2m,n}$ which is the Harary graph $\mathrm{H}_{k,n}$ when k is even. Therefore our result improves the result in [5], concerning the b-chromatic of p-th power graphs of cycles. Also we give a negative answer to the open problem of Effantin and Kheddouci.

2 Main result

Theorem 1 *Let $H_{2m,n}$ be the Harary graph. Then*

$$b(H_{2m,n}) = \begin{cases} 2m+1 & \text{if } n = 2m+1 \text{ or } n \geq 4m+1 \\ 2m - \left\lfloor \dfrac{4m-n}{3} \right\rfloor & \text{if } \left\lceil \dfrac{5m+3}{2} \right\rceil \leq n \leq 4m \\ \geq n-m-1 & \text{if } 2m+2 \leq n < \left\lceil \dfrac{5m+3}{2} \right\rceil \end{cases}$$

Proof. We distinguish between four cases according to each value of the order of $H_{2m,n}$.

Case 1: $n = 2m+1$. Then $H_{2m,n}$ is a clique of order $2m+1$ and clearly $b(H_{2m,n}) = \chi(H_{2m,n}) = 2m+1$.

Case 2: $n \geq 4m+1$. Since $\Delta(H_{2m,n}) = 2m$, $b(H_{2m,n}) \leq \Delta(H_{2m,n}) + 1 = 2m+1$. To prove equality, we construct a b-coloring with $2m+1$ colors $0, 1, 2, ..., 2m$ as follow. Let $v_0, v_1, ..., v_{n-1}$ be vertices of $H_{2m,n}$ in this order around the circle. First, assign color 0 to v_0. Since $n \geq 4m+1$, we begin by coloring the nearest $4m$ vertices to v_0; $2m$ vertices in each direction around the circle according to the ordering of vertices. Assign color i to v_i; $i = 1, ..., 2m$ and color $i-(n-2m-1)$ to v_i; $i = n-2m, ..., n-1$. The vertices v_i and v_j are adjacent if $i-m \leq j \leq i+m$ where addition is taken modulo n . A vertex v_i and a vertex v_j have the same color if $i = j-(n-2m-1)$ for $i \in \{1, ..., 2m\}$ and $j \in \{n-2m, ..., n-1\}$, so $i-2m-1 \geq j = i+n-2m-1 \geq i+4m+1-2m-1 = i+2m$. Hence two vertices with a same color are not adjacent, which implies that the partial coloring is proper. Also, we can see easily that the vertices $v_i; i = 1, .., m$ and the vertices $v_i; i = n-m, ..., n-1$ with v_0 are b-dominating vertices for this partial proper coloring. Finally, extend this partial proper coloring to a proper coloring of $H_{2m,n}$ as follow. Color the remaining vertices in the whole graph in arbitrary order, assigning to each vertex a color from $\{0, 1, ..., 2m\}$ different from the colors already assigned to its neighbors which is in fact an extension by a standard greedy coloring algorithm. We obtain a b-coloring with $2m+1$ colors in which the vertices $v_0, v_1, ..., v_m, v_{m-n}, ...v_{n-1}$ are b-dominating vertices.

Case 3: $\left\lceil \dfrac{5m+3}{2} \right\rceil \leq n \leq 4m$.

First, we show that $b(H_{2m,n}) \leq 2m - \left\lfloor \dfrac{4m-n}{3} \right\rfloor$. Suppose to the contrary that $H_{2m,n}$ admits a b-coloring with k colors, $k \geq 2m - \left\lfloor \dfrac{4m-n}{3} \right\rfloor + 1$.

Claim 1 *There exists at least one color class with one vertex.*

Proof of Claim 1: Otherwise every color class has at least two vertices, so $n \geq 2k \geq 4m - 2\left\lfloor \dfrac{4m-n}{3} \right\rfloor + 2$ and since $\left\lfloor \dfrac{4m-n}{3} \right\rfloor \leq \dfrac{4m-n}{3}$, $n \geq 4m+6$, a contradiction.$\square$

Let $0, ..., k-1$ be the colors used by a b-coloring of $H_{2m,n}$. Without loss of generality let v_0 be the only vertex with color 0. So, v_0 is a b-dominating vertex of color 0 and there are at least $k-1$ other b-dominating vertices with distinct colors adjacent to v_0.

Let $X = \{x_1, x_2, ..., x_m\}$ and $Y = \{y_1, y_2, ..., y_m\}$ be the neighborhood of v_0 in each direction around the circle in right and left direction of v_0 respectively.

Let x_i (resp. y_j) be the lastest b-dominating vertex in X (resp. Y). Set $A = \{x_k \in X : k \leq i\}$ and $B = \{y_k \in X : k \leq j\}$. Let $Z = V \setminus (\{v_0\} \cup X \cup Y)$ be the set of the non neighborhood of v_0. Let V_{ij} (resp. $\overline{V_{ij}}$) be the set of vertices between x_i and y_j in left (resp. right) direction of x_i around the circle, that is $v_0 \in V_{ij}$ and $v_0 \notin \overline{V_{ij}}$.

If $3m+1 \leq n \leq 4m$, then $|Z| = n-(2m+1) \geq m$; so $\left|\overline{V_{ij}} \cup \{x_i, y_j\}\right| \geq |Z|+2 \geq m+2$. Also we have

$$\begin{aligned}
|A|+|B|+1 \geq k &\geq 2m - \left\lfloor \frac{4m-n}{3} \right\rfloor + 1 \geq 2m - \frac{4m-n}{3} + 1 \\
&\geq \frac{2m+n}{3} + 1 \geq \frac{2m+3m+4}{3} = \frac{5m+4}{3} = m + \frac{2m+4}{3} \\
&\geq m+2,
\end{aligned}$$

then $|V_{ij} \cup \{x_i, y_j\}| \geq m+2$. Hence x_i is not adjacent to y_j.

The lastest b-dominating vertex x_i in A needs at least $k-m$ colors which are assigning to some b-dominating vertices at the end of B, so we need at least $k-m$ distinct vertices with this colors which belong to $V(H_{2m,n}) - (\{v_0\} \cup A \cup B)$ and which are adjacent to x_i. Let A' be the set of this vertices required by x_i. Similarly the lastest b-dominating vertex y_j in B needs at least $k-m$ colors which are assigning to b-dominating vertices at the end of A, so we need at least $k-m$ distinct vertices with this colors which belong to $V(H_{2m,n}) - \{v_0\} \cup A \cup B$ and which are adjacent to y_j. Let B' be the set of vertices required by y_j. Since the colors needed by x_i are in the neighborhood of y_j and the colors needed by y_j are in the neighborhood of x_i, this colors are different, so A' and B' are disjoint. Thus

$$\begin{aligned}
n &\geq |A|+|B|+1+|A'|+|B'| \geq k+2(k-m) = 3k-2m \\
&\geq 3(2m - \left\lfloor \frac{4m-n}{3} \right\rfloor + 1) - 2m = 4m - 3\left\lfloor \frac{4m-n}{3} \right\rfloor + 3 \\
&\geq 4m-4m+n+3 = n+3,
\end{aligned}$$

a contradiction.

Now we suppose that $\left\lceil \frac{5m+3}{2} \right\rceil \leq n \leq 3m$.

Claim 2 *Each set X and Y contains at least $\frac{m+2}{2}$ b-dominating vertices.*

Proof of Claim 2: To see this, assume that X or Y contains at most $\frac{m}{2}$ b-dominating vertices. Then

$$2m - \left\lfloor \frac{4m-n}{3} \right\rfloor + 1 \le k \le \frac{3m}{2} + 1$$

which implies that $n \le \frac{5m}{2}$, a contradiction.□

Claim 3 *All the vertices of $A \cup B$ are b-dominating.*

■

Proof. Proof of Claim 3: First we prove that x_1 is a b-dominating vertex, Suppose that x_1 with the color c_1 is not b-dominating, so in the neighborhood of x_1 there exists some missed color c'_1, which implies that $Y \smallsetminus \{y_m\}$ does not contain colors c'_1 and c_1. Since v_0 is the only b-dominating vertex with his color, the color of y_m must be c'_1. Hence $X \cup Y$ does not contain a b-dominating vertex with the color c_1, a contradiction. Similarly we can prove that y_1 is a b-dominating vertex. Now we suppose that A contains a non b-dominating vertex x_l with the color c_l. Let x_p and x_q; $p < l < q$ the nearest b-dominating vertices in each direction around the circle; in right and left direction of x_l respectively. We denote by F the set of non b-dominating vertices between x_p and x_q; which contains at least x_l. By Claim 2, it is clear that $|F| \le \frac{m-2}{2}$. As x_l is a non b-dominating vertex, so in the neighborhood of x_l there exists some missed color c'_l, which implies that in V there is only one vertex of color c'_l, because the color c'_l does not exist in $N[x_l]$, so it bellow to $M = V \setminus N[x_l]$. Since

$$|M| = |V \setminus N[x_l]| = |V| - |N[x_l]| = n - 2m - 1 \le 3m - 2m - 1 = m - 1,$$

the subgraph $G[M]$ induced by M is a clique. Therefore there is one vertex y_h of color c'_l in $G[M]$ and y_h is a b-dominating vertex, so $y_h \in B$. However x_p and x_q are adjacent to y_h. Then

$$n = |V_{ph}| + \left|\overline{V_{qh}}\right| + |\{x_p, x_q\}| + |F| \le 2m + 1 + |F| \le 2m + 1 + \frac{m-2}{2} = \frac{5m}{2},$$

a contradiction.□

Let B' be the set of b-dominating vertices in B such that no color in B' is repeated in A. Let y_t be the last vertex of Y, whose color does not appear in A. y_t exists, otherwise $k = 1 + |A| \le m + 1$, a contradiction. So y_t is a b-dominating vertex and $y_t \in B'$ and we have

$$|A| + |B'| + |\{v_0\}| \ge k \ge 2m - \left\lfloor \frac{4m-n}{3} \right\rfloor + 1 \ge \frac{2m+n}{3} + 1 \ge \frac{3m}{2} + 2.$$

Claim 4 *x_i is not adjacent to y_t.*

Proof of Claim 4: If x_i is adjacent to y_t, then two cases arise: Assume that $V_{it} \cup \{x_i, y_t\}$ induce a clique, thus $|V_{it} \cup \{x_i, y_t\}| \leq m+1$ (the cardinality maximum of a clique in $H_{2m,n}$ is $m+1$). Since $|V_{it} \cup \{x_i, y_t\}| \geq |A| + |B'| + |\{v_0\}| \geq k$ and $k \geq \dfrac{3m}{2} + 2$, $|V_{it} \cup \{x_i, y_t\}| \geq \dfrac{3m}{2} + 2$, a contradiction. Thus $V_{it} \cup \{x_i, y_t\}$ does not induce a clique, so $\overline{V_{it}} \cup \{x_i, y_t\}$ induce a clique. In this case since every vertex of A is b-domnating, y_t is adjacent to all vertices of A (otherwise it can not have the color of y_t). Hence $H_{2m,n}$ is a clique which contradicts hypothesis. $\square$

The lastest b-dominating vertex x_i in A needs at least $k-m$ colors which are assigning to some b-dominating vertices at the end of B', so we need at least $k-m$ distinct vertices with this colors which belong to $V(H_{2m,n}) - (\{v_0\} \cup A \cup B')$ and which are adjacent to x_i. Let A' be the set of this vertices required by x_i. Similarly the lastest b-dominating vertex y_t in B' needs at least $k-m$ colors which are assigning to b-dominating vertices at the end of A, so we need at least $k-m$ distinct vertices with this colors which belong to $V(H_{2m,n}) - \{v_0\} \cup A \cup B'$ and which are adjacent to y_t. Let B'_1 be the set of vertices required by y_t. Since the colors needed by x_i are in the neighborhood of y_t and the colors needed by y_t are in the neighborhood of x_i, this colors are different, so A' and B'_1 are disjoint. Thus

$$
\begin{aligned}
n &\geq |A| + |B'| + |A'| + |B'_1| + 1 \geq k + 2(k-m) = 3k - 2m \\
&\geq 3(2m - \left\lfloor \frac{4m-n}{3} \right\rfloor + 1) - 2m = 4m - 3 \left\lfloor \frac{4m-n}{3} \right\rfloor + 3 \\
&\geq 4m - 4m + n + 3 = n + 3,
\end{aligned}
$$

a contradiction. So in all case, if $\left\lceil \dfrac{5m+3}{2} \right\rceil \leq n \leq 4m$, then $b(H_{2m,n}) \leq 2m - \left\lfloor \dfrac{4m-n}{3} \right\rfloor$.

Now, we give a b-coloring of $H_{2m,n}$ with $2m - \left\lfloor \dfrac{4m-n}{3} \right\rfloor$, when $\left\lceil \dfrac{5m+3}{2} \right\rceil \leq n \leq 4m$. Let $v_1, v_2, ..., v_n$ be vertices of $H_{2m,n}$ in this order around the circle. Set $k = 2m - \left\lfloor \dfrac{4m-n}{3} \right\rfloor$, then $n \leq 2k$, otherwise $n > 2k$ implies that $n > 4m$, a contradiction. Since $n \leq 2k$, we can color all vertices of $H_{2m,n}$ by the following b-coloring, assign color i to v_i; $i = 1, ..., k$ and color $i-(n-k)$ to v_i; $i = k+1, ..., n$, according to the ordering of vertices. The vertices v_i and v_j are adjacent if $i - m \leq j \leq i + m$ where addition is taken modulo $n+1$. A vertex v_i and a vertex v_j have the same color if $i = j - (n-k)$ for $i \in \{1, ..., k\}$ and

$j \in \{k+1, ..., n\}$. Since

$$|j-i| = n-k = n-2m+\left\lfloor \frac{4m-n}{3} \right\rfloor > n-2m+\frac{4m-n}{3}-1$$
$$= \frac{3n-6m+4m-n-3}{3} = \frac{2n-2m-3}{3}$$
$$\geq \frac{2\frac{5m+3}{2}-2m-3}{3} = m,$$

two vertices with a same color are not adjacent, which implies that the coloring is proper. Also, we can see easily that the vertices $v_i; i = 1, .., m+1$ and the vertices $v_i; i = n-k+m+2, ..., n$; with $k \leq m+2$, are b-dominating vertices for this proper coloring.

Case 4: $\boldsymbol{2m+2 \leq n < \left\lceil \frac{5m+3}{2} \right\rceil}$**.**

To show that $b(H_{2m,n}) \geq n-m-1$, we construct a b-coloring with $n-m-1$ colors as follow. Let $v_1, v_2, ..., v_n$ be vertices of $H_{2m,n}$ in this order around the circle. Set $k = n-m-1$, then $n \leq 2k$, otherwise $n > 2k$ implies that $n < 2m+2$, a contradiction. Since $n \leq 2k$, we can color all vertices of $H_{2m,n}$ by the following b-coloring, assign color i to v_i; $i = 1, ..., k$ and color $i-(n-k)$ to v_i; $i = k+1, ..., n$, according to the ordering of vertices. The vertices v_i and v_j are adjacent if $i-m \leq j \leq i+m$ where addition is taken modulo $n+1$. A vertex v_i and a vertex v_j have the same color if $i = j-(n-k)$ for $i \in \{1, ..., k\}$ and $j \in \{k+1, ..., n\}$. Since

$$|j-i| = n-k = n-n+m+1 = m+1,$$

two vertices with a same color are not adjacent, which implies that the coloring is proper. Also, we can see that the vertices $v_i; i = 1, .., m+1$ and the vertices $v_i; i = n-k+m+2, ..., n$; with $k \leq m+2$, are b-dominating vertices for this proper coloring, which completes the proof of Theorem 1. ■

Proposition 2 *Let $H_{2m,2m+3}$ be the Harary graph. Then*

$$n-m-1 \leq b(H_{2m,2m+3}) \leq \left\lfloor \frac{6m+9}{5} \right\rfloor$$

And this bounds are sharp.

Proof. Let c be an arbitrary b-coloring of $H_{2m,2m+3}$. The first inequality leads from Theorem 1. Let $v_0, v_1, ..., v_{2m+2}$ be vertices of $H_{2m,2m+3}$ in this order around the circle. Now we prove the second inequality. Since $|Z| = |V \setminus (\{v_0\} \cup X \cup Y)| = 2$, each color is repeated at most twice. Let k_1 (resp. k_2) be the number of color classes with one vertex (resp. two vertices). By 1-class (resp. 2-class) we denote the color class with one vertex (resp. two vertices). Then $n = k_1 + 2k_2$ and $b = k_1 + k_2 = n - k_2 = 2m+3-k_2$.

If $k_1 = 1$, then $n-1 = 2m+2 = 2k_2$ which implies that $k_2 = m+1$. So $b = n-m-1 = m+2$.

Let $k_1 \geq 3$, (k_1 is odd integer since the order of $H_{2m,2m+3}$ is odd and $2m+3 = k_1 + 2k_2$).

We prove that the two nearest neighbors around the circle of a b-dominating vertex which belongs to an 1-class are b-dominating vertices and everyone is contained in an 2-class. Let v_0 be the vertex which belongs to an 1-class, v_1 and v_{n-1} its nearest neighbors around the circle and v_{m+1}, v_{m+2} its non neighbors with $c(v_{m+1}) = a$ and $c(v_{m+2}) = b$. We must have $c(v_1) = b$ and $c(v_{n-1}) = a$ with v_1 and v_{n-1} b-dominating vertices, because the vertices v_{m+1} and v_{m+2} can not be adjacent to the color of v_0. Therefore two b-dominating vertices where each one is in an 1-class are not consecutive around the circle. Also we prove that between two b-dominating vertices where each one belongs to an 1-class, there exists at least two b-dominating vertices where each one belongs to an 2-class. Assume to the contrary that there exists one exactly b-dominating vertex which belongs to an 2-class. Without loss of generality, let v_0 and v_2 be the b-dominating vertices where each one belongs to an 1-class, so v_1 is a vertex which belongs to an 2-class. It is obvious to verify that this b-coloring is impossible. Hence $k_2 \geq 2k_1$ and since $n = k_1 + 2k_2$, $k_2 \geq \dfrac{2n}{5}$. Consequently $b = 2m + 3 - k_2 \leq \left\lfloor \dfrac{3n}{5} \right\rfloor = \left\lfloor \dfrac{6m+9}{5} \right\rfloor$.

Let c be a b-coloring with $b(H_{2m,2m+3})$ colors (a mapping from V to the set of positive integers (colors)). We give examples which show that the bounds of Proposition 2 are sharp.

For each value of m we have checked the b-coloring given. (In each case the b-dominating vertices are marked by *).

1. $m = 1, n = 5, b(H_{2,5}) = n - m - 1 = \left\lfloor \dfrac{6m+9}{5} \right\rfloor = 3$

vertices	v_0^*	v_1^*	v_2^*	v_3	v_4
b-coloring	0	1	2	1	2

2. $m = 6, n = 15, b(H_{12,15}) = n - m = \left\lfloor \dfrac{6m+9}{5} \right\rfloor = 9$

vertices	v_0^*	v_1^*	v_2	v_3	v_4^*	v_5^*	v_6^*	v_7	v_8	v_9^*	v_{10}^*	v_{11}^*
b-coloring	0	1	5	7	2	3	4	8	1	5	6	7

v_{12}	v_{13}	v_{14}^*
2	4	8

3. $m = 11, n = 25, b(H_{22,25}) = n - m + 1 = \left\lfloor \dfrac{6m+9}{5} \right\rfloor = 15$

vertices	v_0^*	v_1^*	v_2	v_3	v_4^*	v_5^*	v_6^*	v_7	v_8	v_9^*	v_{10}^*	v_{11}^*
b-coloring	0	1	8	10	2	3	4	11	13	5	6	7

v_{12}	v_{13}	v_{14}^*	v_{15}^*	v_{16}^*	v_{17}	v_{18}	v_{19}^*	v_{20}^*	v_{21}^*	v_{22}	v_{23}	v_{24}^*
14	1	8	9	10	2	4	11	12	13	5	7	14

4. $m = 16, n = 35, b(H_{32,35}) = n - m + 2 = \left\lfloor \dfrac{6m+9}{5} \right\rfloor = 21$

vertices	v_0^*	v_1^*	v_2	v_3	v_4^*	v_5^*	v_6^*	v_7	v_8	v_9^*	v_{10}^*	v_{11}^*
b-coloring	0	1	11	13	2	3	4	14	16	5	6	7

v_{12}	v_{13}	v_{14}^*	v_{15}^*	v_{16}^*	v_{17}	v_{18}	v_{19}^*	v_{20}^*	v_{21}^*	v_{22}	v_{23}	v_{24}^*
17	19	8	9	10	20	1	11	12	13	2	4	14

v_{25}^*	v_{26}^*	v_{27}	v_{28}	v_{29}^*	v_{30}^*	v_{31}^*	v_{32}	v_{33}	v_{34}^*
15	16	5	7	17	18	19	8	10	20

5. $m = 21, n = 45, b(H_{42,45}) = n - m + 3 = \left\lfloor \dfrac{6m+9}{5} \right\rfloor = 27$

vertices	v_0^*	v_1^*	v_2	v_3	v_4^*	v_5^*	v_6^*	v_7	v_8	v_9^*	v_{10}^*	v_{11}^*
b-coloring	0	1	14	16	2	3	4	17	19	5	6	7

v_{12}	v_{13}	v_{14}^*	v_{15}^*	v_{16}^*	v_{17}	v_{18}	v_{19}^*	v_{20}^*	v_{21}^*	v_{22}	v_{23}	v_{24}^*
20	22	8	9	10	23	25	11	12	13	26	1	14

v_{25}^*	v_{26}^*	v_{27}	v_{28}	v_{29}^*	v_{30}^*	v_{31}^*	v_{32}	v_{33}	v_{34}^*	v_{35}^*	v_{36}^*	v_{37}
15	16	2	4	17	18	19	5	7	20	21	22	8

v_{38}	v_{39}^*	v_{40}^*	v_{41}^*	v_{42}	v_{43}	v_{44}^*
10	23	24	25	11	13	26

6. $m = 26, n = 55, b(H_{52,55}) = n - m + 4 = \left\lfloor \dfrac{6m+9}{5} \right\rfloor = 33.$

$\blacksquare$

By looking into the disposition of the colors assigned to a b-coloring done on the previous examples, it is easy to generalize these examples, it suffices for this to take $m = 5k + 1; k \in IN^*$, then we have $n = 2m + 3 = 10k + 5$ and $b(G) = \dfrac{6m+9}{5} = (n - m - 1) + \dfrac{m-1}{5} = 6k + 3$.

The examples 2-6 given before in the proof of Proposition 2 provide counterexamples to the open problem of Effantin and Kheddouci [5].

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