Double domination edge removal critical graphs

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Abstract

Let G be a graph without isolated vertices. A set $S \subseteq V(G)$ is a double dominating set if every vertex in $V(G)$ is adjacent to at least two vertices in S. G is said edge removal critical graph with respect to double domination, if the removal of any edge increases the double domination number. In this paper, we first give a necessary and sufficient conditions for $\gamma_{\times 2}$-critical graphs. Then we provide a constructive characterization of $\gamma_{\times 2}-$critical trees.

1 Introduction

In a graph $G = (V(G), E(G))$, the *open neighborhood* of a vertex $v \in V(G)$ is $N_G(v) = N(v) = \{u \in V \mid uv \in E(G)\}$, the *closed neighborhood* is $N_G[v] = N[v] = N(v) \cup \{v\}$ and the *degree* of v, denoted by $\deg_G(v)$, is the size of its open neighborhood. A vertex with degree one in a graph G is called a *pendent vertex* or a *leaf*, and its neighbor is called its *support.* An edge incident to a leaf in a graph G is called a *pendent edge*. We let $S(G), L(G)$ be the set of support vertices and leaves of G, respectively. If $A \subseteq V(G)$, then $G[A]$ is the graph induced by the vertex set A. The *diameter* $\mathrm{diam}(G)$ of a graph G is the maximum distance over all pairs of vertices of G. We denote by K_n the *complete graph* of order n, and by $K_{m,n}$ the *complete bipartite graph* with partite sets X and Y such that $|X| = m$ and $|Y| = n$. A star of order $n+1$ is $K_{1,n}$. A subdivided star $K^*_{1,n}$ is the graph obtained by subdividing each edge of a star $K_{1,n}$ once. A graph is k-regular if all its vertices have degree k. The path and the cycle on n vertices are denoted by P_n and C_n, respectively.

A subset S of vertices of $V(G)$ is a *dominating set* of G if every vertex in $V(G) - S$ is adjacent to a vertex in S, and $S \subseteq V$ is a *double dominating set* of G, abbreviated *DDS*, if every vertex in $V - S$ has at least two neighbors in S and every vertex of S has a neighbor in S. The *double domination number* $\gamma_{\times 2}(G)$ is the minimum cardinality of a double dominating set of G. A double dominating set of G with minimum cardinality is called a $\gamma_{\times 2}(G)$-set. Double domination was introduced by Harary and Haynes [4] and is studied for example in [1, 3, 4]. For a comprehensive survey of domination in graphs and its variations, see [5, 6].

Given a graph, a new graph can be obtained by removing or adding an edge. The study of the effects of such modifictions have been considered for several domination parameters. Note that Sumner and Blitch [7] were the first introducing *edge removal critical graphs* for the domination number. For a survey we cite [5] (Chapter 5). In this paper we study the effects on increasing double domination number when an edge is deleted.

2 Preliminary results

We begin by giving a straightforward property of double dominating sets.

Remark 1 *Every DDS of a graph contains all its leaves and support vertices.*

Next we show that the removal of a non-pendent edge of a graph G can increase the double domination number of G by at most two.

Theorem 1 *Let G be a graph without isolated vertices. Then $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)+2$ for every non-pendent edge $e \in E(G)$.*

Proof. Let $e = xy$ be a non-pendent edge. Clearly every $\gamma_{\times 2}(G-e)$-set is a DDS of G and so $\gamma_{\times 2}(G) \leq \gamma_{\times 2}(G-e)$. Now let S be a $\gamma_{\times 2}(G)$-set. If $S \cap \{x, y\} = \emptyset$, then S is a DDS of $G-e$ and hence $\gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)$. Assume now, without loss of generality, that $S \cap \{x, y\} = \{y\}$. Then since x has two neighbors in S, $S \cup \{x\}$ is a DDS of $G-e$ implying that $\gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)+1$. Finally assume that $\{x, y\} \subseteq S$. We examine three cases.

If each x and y has degree at least two in $G[S]$, then since e is a non pendent edge, S remains a DDS of $G-e$ and so $\gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)$. Assume that both x and y are pendent vertices in $G[S]$. Since $e = xy$ is a non-pendent edge each of x and y has a neighbor in $V-S$. Let $x', y' \in V-S$ be a the neighbors of x and y, respectively. Then $S \cup \{x', y'\}$ is a DDS of $G-e$ and so $\gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)+2$. Finally, assume without loss of generality, that x is a vertex of degree one in $G[S]$ and y has degree at least two in $G[S]$. Since xy is a non-pendent edge, let $x' \in V-S$ be any neighbor of x. Then $S \cup \{x'\}$ is a DDS of $G-e$ and so $\gamma_{\times 2}(G-e) \leq \gamma_{\times 2}(G)+1$. ■

A graph G is said to be edge removal critical (ER-critical) with respect to double domination or just $\gamma_{\times 2}$-critical, if for every edge $e \in E(G)$, $\gamma_{\times 2}(G-e) > \gamma_{\times 2}(G)$. If $G-e$ contains isolated vertices, then we set that $\gamma_{\times 2}(G-e) = +\infty$. Thus nontrivial stars are $\gamma_{\times 2}$-critical. Let $X_G \subset E(G)$ be the set of non-pendent edges in G. Clearly if $X_G = \emptyset$, then $\gamma_{\times 2}(G-e) = +\infty$ and so G is a $\gamma_{\times 2}$-critical graph.

The following Properties are straightforward.

Remark 2 *If G is a $\gamma_{\times 2}$-critical graph, then no two support vertices are adjacent.*

Proposition 1 *Let G be a graph obtained from a subdivided star $K^*_{1,r}$ $(r \geq 2)$ of center y by adding an edge from y to any vertex x of a nontrivial graph G'. Then G is not a $\gamma_{\times 2}$-critical graph.*

Proof. Assume that G is $\gamma_{\times 2}$-critical. Let u_i for $1 \leq i \leq r$ be the support vertices of the subdivided star $K^*_{1,r}$ with center y and let S be any $\gamma_{\times 2}(G)$-set. By Remark 1 each u_i belongs to S. If $y \in S$, then removing any edge yu_i does not increase the double domination number. Thus $y \notin S$ and hence S is a DDS of $G - xy$ implying that $\gamma_{\times 2}(G - e) \leq \gamma_{\times 2}(G)$, a contradiction. It follows that G is not a $\gamma_{\times 2}$-critical graph. ■

Next we give a necessary and a sufficient condition for a graph to be $\gamma_{\times 2}$-critical.

Theorem 2 *G is a $\gamma_{\times 2}$-critical graph if and only if for every $\gamma_{\times 2}(G)$-set S the following conditions hold.*

i) Each component in $G[S]$ is a star.

ii) $V - S$ is an independent set.

iii) Every vertex of $V - S$ has degree two.

Proof. Assume that G is a $\gamma_{\times 2}$-critical graph and let S be any $\gamma_{\times 2}(G)$-set. Observe that $G[S]$ contains no cycle for otherwise removing any edge on the cycle does not increase the double domination number, a contradiction. Thus $G[S]$ is a forest. If $G[S]$ contains a component with diamter at least three, then there exists an edge on the diametrical path of such a component whose removal does not increase the double domination number, a contradiction too. Since $G[S]$ does not contains isolated vertices, every component of $G[S]$ has diameter at most two, that is a star. Now assume that $V - S$ contains two adjacent vertices x, y. Then S remains a DDS for $G - xy$ and so $\gamma_{\times 2}(G - xy) \leq \gamma_{\times 2}(G)$, a contradiction. It follows that $V - S$ is an independent set. Finally assume that a vertex $x \in V - S$ has degree at least three. By item (ii) $N(x) \subset S$, and hence removing any edge incident to x does not increase the double domination number, a contradiction.

Conversely, suppose that for every $\gamma_{\times 2}(G)$-set conditions (i), (ii) and (iii) are satisfied. Assume that G is not $\gamma_{\times 2}$-critical and let uv be an edge of X_G for which $\gamma_{\times 2}(G - uv) = \gamma_{\times 2}(G)$. Let D be a $\gamma_{\times 2}(G - uv)$-set. Clearly D is a DDS of G and since $\gamma_{\times 2}(G - uv) = \gamma_{\times 2}(G)$, D is also $\gamma_{\times 2}(G)$-set. If $\{u, v\}$ $\cap D = \emptyset$, then D is a $\gamma_{\times 2}(G)$-set and $V - D$ is not an independent set in G. Thus D contains at least one of u or v. Assume that $\{u, v\} \subset D$. Then u has a neighbor, say $x \neq v$, in D and likewise, v has a neighbor, say $y \neq u$, in D, with possibly $x = y$. Then D is a $\gamma_{\times 2}(G)$-set such that $\{u, v, x, y\}$ induces in $G[D]$ either a cycle C_3, C_4 or a path P_4, a contradiction. Thus, without loss of generality, assume that $u \in D$ and $v \notin D$. Then v is dominated at least twice by D in $G - uv$ but then condition (iii) does not hold for D in G since v would have at least three neighbors. In any case D is a $\gamma_{\times 2}(G)$-set for which conditions (i), (ii) and (iii) are not all satisfied. It follows that G is $\gamma_{\times 2}$-critical. ■

As immediate consequence to Theorem 2 we have the following two corollaries.

Corollary 1 *If G is a $\gamma_{\times 2}$-critical graph, then every $\gamma_{\times 2}(G)$-set contains all vertices of degree at least three.*

Corollary 2 *If G is a graph with minimum degree at least three, then G is not $\gamma_{\times 2}$-critical.*

The following observation will be useful for the proof of the next result.

Remark 3 *1)If $n \geq 3$, then $\gamma_{\times 2}(C_n) = \left\lceil \frac{2n}{3} \right\rceil$.*

2)If $n \geq 2$, then $\gamma_{\times 2}(P_n) = \begin{cases} 2n/3 + 1 & \text{if } n \equiv 0 (\bmod 3) \\ 2\lceil n/3 \rceil & \text{otherwise} \end{cases}$

Proposition 2 *The only $\gamma_{\times 2}$-critical k-regular graphs with $k \geq 2$ are the cycles C_n with $n \equiv 0, 1 \pmod 3$.*

Proof. Assume that G is a k-regular $\gamma_{\times 2}$-critical graph. By Corollary 2 $k \leq 2$ and it follows that $k = 2$, that G is a cycle. Using Remark 3 it is a simple exercice to see that the order of G must satisfy $n \equiv 0, 1 \pmod 3$. ■

3 $\gamma_{\times 2}$-critical trees

For ease of presentation, we next consider rooted trees. For a vertex v in a (rooted) tree T, we let $C(v)$ and $D(v)$ denote the set of children and descendants, respectively, of v, and we define $D[v] = D(v) \cup \{v\}$. The maximal subtree at v is the subtree of T induced by $D[v]$, and is denoted by T_v. Also, a vertex of degree at least three in T is called a *branch vertex*, and we denote by $B(T)$ the set of such vertices.

Remark 4 *If T is the tree obtained from a tree T' by attaching a vertex to a support vertex, then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$.*

Lemma 1 *Let T be a tree obtained from a nontrivial tree T' by adding k ($k \geq 1$) paths $P_3 = a_i b_i c_i$ attached by edges $c_i x$ for every i, at a vertex x of T' which belongs to some $\gamma_{\times 2}(T')$-set, then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$.*

Proof. Let S' be a $\gamma_{\times 2}(T')$-set that contains x. S' can be extended to a DDS of T by adding the vertices a_i, b_i for every i, and so $\gamma_{\times 2}(T) \leqslant \gamma_{\times 2}(T') + 2k$. Now let D be a $\gamma_{\times 2}(T)$-set. By Remark 1, D contains a_i, b_i for every i. If D contains three vertices from $\{c_1, c_2, ..., c_k\}$, say c_1, c_2, c_3, then $x \notin D$ and so $\{x\} \cup D - \{c_1, c_2\}$ is a DDS smaller than D, a contradiction. Thus every $\gamma_{\times 2}(T)$-set contains at most two vertices from $\{c_1, c_2, ..., c_k\}$. Now, without loss of generality, we can assume that $D \cap \{c_1, c_2, ..., c_k\} = \emptyset$ (else we replace such vertices by x or/and a neighbor of x in T'). Hence $x \in D$ to double dominate every c_i, implying that $\gamma_{\times 2}(T') \leqslant \gamma_{\times 2}(T) - 2k$. It follows that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$. ■

Remark 5 *If T is a tree obtained from a tree T' by attaching a new vertex x to a pendent vertex u whose support vertex v is adjacent to at least one pendent path of order three, then $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$.*

Proof. Let $x_i y_i z_i$ with $1 \leq i \leq k$ be k pendent paths P_3 attached to v by the vertices x_i and S any $\gamma_{\times 2}(T)$-set. Since every $\gamma_{\times 2}(T')$-set can be extended to a DDS of T by adding the set $\{x\}$, $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T') + 1$. On the other hand, without loss of generality, we may assume that $v \in S$ (if $v \notin S$, then by minimality, $k = 1$ and $x_1 \in S$, and so we can replace x_1 by v in S), hence the set $S' = S \cap T'$ is a DDS of T' and so $\gamma_{\times 2}(T') \leq \gamma_{\times 2}(T) - 1$. It follows that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$. ■

In order to characterize $\gamma_{\times 2}$-critical trees, we define the family of all trees $\mathcal{F}$ that can be obtained from a sequence $T_1, T_2, ..., T_j$ ($j \geqslant 1$) of trees such that T_1 is a star $K_{1,r}$ with $r \geq 1$, $T = T_j$, and if $j \geqslant 2$, T_{i+1} can be obtained recursively from T_i by one of the operations listed below.

- **Operation $\mathcal{O}_1$:** Add a new vertex and join it by an edge to any support vertex of T_i.
- **Operation $\mathcal{O}_2$:** Add a path P_3 and join by an edge a leaf of P_3 to a support vertex of T_i.
- **Operation $\mathcal{O}_3$:** Add k ($k \geq 1$) paths P_3 and join by edges a leaf of each path P_3 to the same leaf of T_i.
- **Operation $\mathcal{O}_4$:** Add a new vertex u and join it by an edge to a leaf v of T_i whose support neighbor x has degree $k + 2$ and is adjacent to $k \geq 1$ pendent paths P_3 such that every vertex in $N(x) - \{v\}$ has degree two and does not belong to any $\gamma_{\times 2}(T_i)$-set.

Note that we can determine in polynomial time the vertices that are in no minimum double dominating set of a tree [2].

Now we are ready to characterize $\gamma_{\times 2}$-critical trees.

Theorem 3 *A nontrivial tree T is $\gamma_{\times 2}$-critical if and only if $T \in \mathcal{F}$.*

Proof. We proceed by induction on the order of T. Since stars are $\gamma_{\times 2}$-critical, and by Remark 2, double stars are not $\gamma_{\times 2}$-critical since they have two adjacent support vertices. Hence, assume that T has diameter at least four. The smallest tree of diameter four is the path P_5 and it can be obtained from a star $K_{1,1}$ by Operation $\mathcal{O}_3$, and so T belongs to $\mathcal{F}$. Assume now that $\mathrm{diam}(T) = 4$ and $T \neq P_5$. Let x_1-x_2-x_3-x_4-x_5 be the longest path of T. Clearly x_1 and x_5 are leaves and so x_2 and x_4 are their support vertices, respectively. If $\deg_T(x_3) = 2$, then at least one of x_2 and x_4 is a strong support. Then $T \in \mathcal{F}$ and is obtained from P_5 by using Operation $\mathcal{O}_1$. Now we assume that $\deg_T(x_3) \geq 3$. By Remark 2, x_3 cannot be a support vertex. Thus every neighbor of x_3 is a support vertex but then $\gamma_{\times 2}(T - x_2x_3) \leq \gamma_{\times 2}(T)$, contradicting the fact that T is $\gamma_{\times 2}$-critical. Thus assume that $\mathrm{diam}(T) \geq 5$. The smallest tree of diameter five is the path P_6, which can be obtained from a star $K_{1,2}$ by operation $\mathcal{O}_3$, and so it belongs to $\mathcal{F}$.

Let $n \geq 7$ and assume that every $\gamma_{\times 2}$-critical tree T' of order $n' < n$ is in $\mathcal{F}$. Let T be a $\gamma_{\times 2}$-critical tree of order n and let S be any $\gamma_{\times 2}(T)$-set. If any support vertex, say y, of T is adjacent to two or more leaves, then let T' be the tree obtained from T by removing a leaf adjacent to y. By Remark 4, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$. Clearly $X_{T'} = X_T$ and T' is $\gamma_{\times 2}$-critical. By the inductive hypothesis on T', we have $T' \in \mathcal{F}$. It follows that $T \in \mathcal{F}$ because it is obtained from T' by using Operation $\mathcal{O}_1$. Thus we may assume for the next that every support vertex is adjacent to exactly one leaf.

We now root T at leaf x of a longest path. Let u be a vertex at distance $\mathrm{diam}(T) - 1$ from x on a longest path starting at x, and let r be the child of u on this path. Let w_1, v be the parents of u and w_1, respectively. Since u is a support vertex, $\deg_T(u) = 2$ and so by Remark 2, w_1 cannot be a support vertex. On the other hand if $\deg_T(w_1) \geq 3$, then T_{w_1} is a subdivided star. Thus T is a tree obtained from a tree T' and the subdivided star T_{w_1} of center w_1 by adding the edge w_1v, where $v \in V(T')$. But by Proposition 1 T is not $\gamma_{\times 2}$-critical, a contradiction. Thus $\deg_T(w_1) = 2$. We consider the following two cases.

Case 1. v is a support vertex. Let $T' = T - \{r, u, w_1\}$. Then $X_{T'} = X_T - \{vw_1, w_1u\}$ and by Lemma 1, $\gamma_{\times 2}(T') = \gamma_{\times 2}(T) - 2$. Now let e be any edge of $X_{T'}$. Since T is $\gamma_{\times 2}$-critical, the removal of e produces two trees T_1 and T_2 such that $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) > \gamma_{\times 2}(T)$. Without loss of generality, we can assume that $\{r, u, w_1\} \in T_1$. Thus by Lemma 1, $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 2$ (since $e \in E(T')$, the deletion of the edge e in T' provides on one hand T_1' and on the other hand the tree T_2). It follows that $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) - 2 + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T - e) - 2 > \gamma_{\times 2}(T) - 2 = \gamma_{\times 2}(T')$. Consequently the deletion of every edge of $X_{T'}$ increases the double domination number of T' and so T' is $\gamma_{\times 2}$-critical. By the inductive induction, $T' \in \mathcal{F}$ and so $T \in \mathcal{F}$ since it is obtained from T' by Operation $\mathcal{O}_2$.

Case 2: v is not a support vertex. Let $C(v) = \{w_1, w_2, ..., w_k\}$ with $k \geq 1$

. We first assume that $C(v)$ contains no support vertex. If $\deg(v) = 2$, then, without loss of generality, we may assume that $v \in S$ (else replace w_1 by v) and if $\deg(v) \geq 3$, then, by Corollary 1, $v \in S$. Also since every w_i for $i \neq 1$ plays the same role as w_1, every w_i has degree two. Now if $w_1 \in S$, then $\{r, u, w_1, v\}$ induces a path P_4 in S, a contradiction with Theorem 2. Thus S contains no w_i. Now let $T' = T - \bigcup_{1 \leq i \leq k} T_{w_i}$. Thus $X_{T'} = X_T - \{vw_i, w_i c(w_i), yv \text{ for } 1 \leq i \leq k\}$, where y is the parent of v and $c(w_i)$ is the unique child of w_i. Proceeding like in Case 1, it can be seen that T' is $\gamma_{\times 2}$-critical. By our inductive hypothesis, $T' \in \mathcal{F}$ and so $T \in \mathcal{F}$ because it is obtained from T' by Operation $\mathcal{O}_3$.

Now assume that v has a child, say w which is a support vertex. Then such a vertex w is unique for otherwise if $w'' \in C(v)$ is a second support vertex, then S is a DDS of $T - vw''$, implying that $\gamma_{\times 2}(T - vw'') \leq \gamma_{\times 2}(T)$, a contradiction with the fact that T is a $\gamma_{\times 2}$-critical tree. Since $\deg_T(v) \geq 3$, then by Corollary 1, $v \in S$. Let w' be the leaf neighbor of w. By item (i) of Theorem 2, every component of $G[S]$ is a star and so $y \notin S$ and no vertex of $C(v)$ different to w is in S. It follows that $\deg_T(y) = 2$ for otherwise by Corollary 1, y belongs to S. Let $C(v) = \{w, w_1, w_2, ..., w_k\}$ with $k \geq 1$. As mentioned above $\deg_T(w_i) = 2$ for every i. Now let T' be the tree obtained from T by removing the leaf w'. By Remark 5, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$. We shall show now that y does not belong to any $\gamma_{\times 2}(T')$-set. Suppose to the contrary that there is a $\gamma_{\times 2}(T')$-set S' that contains y, and since $v, w \in S'$, the set $S = S' \cup \{w'\}$ is a $\gamma_{\times 2}(T)$-set containing y, v, w and w' and so $G[S]$ contains an induced P_4, a contradiction. Further, it is clear that there is no $\gamma_{\times 2}(T')$-set that contains any vertex w_i for every i.

Clearly, $X_{T'} = X_T - \{vw\}$ and to prove that T' is $\gamma_{\times 2}$-critical, consider any edge e of $X_{T'}$. Since T is $\gamma_{\times 2}$-critical, the removal of e produces two trees T_1 and T_2 such that $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) > \gamma_{\times 2}(T)$.

Subcase 1. $e = vw_i$, for $1 \leq i \leq k$. Then $T_2 = T_{w_i}$ and T_1 contains v. Since $e \in E(T')$, the deletion of the edge e in T' provides on one hand T_1' and on the other hand a tree T_2. We need first to show that $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 1$ and we begin by the case $k = 1$. Since every $\gamma_{\times 2}(T_1')$-set can be extended to a DDS of T_1 by adding w', $\gamma_{\times 2}(T_1) \leq \gamma_{\times 2}(T_1') + 1$ implying that $\gamma_{\times 2}(T_1') \geq \gamma_{\times 2}(T_1) - 1$. Suppose now that $\gamma_{\times 2}(T_1') > \gamma_{\times 2}(T_1) - 1$. This implies that v does not belong to any $\gamma_{\times 2}(T_1)$-set. Assume to the contrary that there exists a $\gamma_{\times 2}(T_1)$-set S that contains v. Then $S_1 = S \cap T_1'$ is a DDS of T_1' and so $\gamma_{\times 2}(T_1') \leq \gamma_{\times 2}(T_1) - 1$, a contradiction. Further we show that in this case, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 3$. Since every $\gamma_{\times 2}(T_1)$-set can be extended to a DDS of T by adding the set $\{w_1, u, r\}$, it follows that $\gamma_{\times 2}(T) \leq \gamma_{\times 2}(T_1) + 3$. Suppose now that $\gamma_{\times 2}(T) < \gamma_{\times 2}(T) + 3$ and let D be any $\gamma_{\times 2}(T)$-set. By corollary 1 and Remark 1, $v, w, w' \in D$, and so $D_1 = D \cap T_1$ is a DDS of T_1 implying that $\gamma_{\times 2}(T_1) \leq \gamma_{\times 2}(T) - 2$ since $w_1 \notin D$ and $u, r \in D$ and so $\gamma_{\times 2}(T) \geq \gamma_{\times 2}(T_1) + 2$. It follows that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 2$, and so D_1 is a $\gamma_{\times 2}(T_1)$-set that contains v, a contradiction. Thus $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 3$. Now let X_1 be any $\gamma_{\times 2}(T_1)$-set. Since $v \notin X_1$ and $\deg_{T_1}(y) = 2$, $y \in X_1$ and so the set $X_2 = X_1 \cup \{w_1, u, r\}$ is a $\gamma_{\times 2}(T)$-set and the set $(X_2 - w_1) \cup \{v\}$ is a $\gamma_{\times 2}(T)$-set that contains a path $P_4 = w', w, v, y$, a contradiction with the fact that T

is $\gamma_{\times 2}$-critical. Consequently $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 1$. For the case $k \geq 2$, by Remark 5, $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 1$. It follows that for any edge $e = vw_i$ we have $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) - 1 + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T - e) - 1 > \gamma_{\times 2}(T) - 1 = \gamma_{\times 2}(T')$.

Subcase 2. $e = w_i c(w_i)$, for $1 \leq i \leq k$. Then $T_2 = T_{c(w_i)}$ and T_1 contains v. Since $e \in E(T')$, the deletion of the edge e in T' provides on one hand T_1' and on the other hand the tree T_2. Clearly, $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 1$ and it follows that $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) - 1 + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T - e) - 1 > \gamma_{\times 2}(T) - 1 = \gamma_{\times 2}(T')$.

Subcase 3. $e \neq vw_i$ and $e \neq w_i c(w_i)$. Without loss of generality, suppose that T_1 contains v. Since $e \in E(T')$, the deletion of the edge e in T' provides on one hand T_1' and on the other hand a tree T_2. By Remark 5 $\gamma_{\times 2}(T_1') = \gamma_{\times 2}(T_1) - 1$. Hence $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) - 1 + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T - e) - 1 > \gamma_{\times 2}(T) - 1 = \gamma_{\times 2}(T')$.

Consequently the deletion of every edge of $X_{T'}$ increases the double domination number of T' and so T' is $\gamma_{\times 2}$-critical. By the inductive hypothesis, $T' \in \mathcal{F}$ and so $T \in \mathcal{F}$ since it is obtained from T' by Operation $\mathcal{O}_4$.

Conversely, let $T \in \mathcal{F}$. Then T can be obtained from a sequence $T_1, T_2, ..., T_j$ ($j \geqslant 1$) of trees such that T_1 is a star $K_{1,r}$ with $r \geq 1$ and $T = T_j$, and if $j \geq 2$, then T_{i+1} is obtained from T_i by one of the four operations defined above. Proceed by induction on the length j of the sequence of trees needed to construct T. Suppose $j = 1$. Then T is a star $K_{1,r}$ with $k \geq 1$. So T is $\gamma_{\times 2}$-critical.

Assume that the result holds for all trees in T of length less than j in $\mathcal{F}$, where $j \geq 2$. Let T be a tree of length j in $\mathcal{F}$. Thus, $T \in \mathcal{F}$ can be obtained from a sequence $T_1, T_2, ..., T_j$ of T trees. We denote T_{j-1} simply by T'. Applying the inductive hypothesis, $T' \in \mathcal{F}$ is $\gamma_{\times 2}$-critical. We now consider four possibilities depending on whether T is obtained from T' by operation $\mathcal{O}_1$, $\mathcal{O}_2$, $\mathcal{O}_3$ or $\mathcal{O}_4$.

Case 1: T is obtained from T' by operation $\mathcal{O}_1$.

Let v be a support vertex in T' and let u be the vertex attached to v to obtain the tree T. Clearly, $X_T = X_{T'}$ and by Remark 4, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$. We shall show that T is $\gamma_{\times 2}$-critical. Since T' is $\gamma_{\times 2}$-critical, then the removal of any edge $e \in X_{T'}$ produces two trees T_1' and T_2' with $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2') > \gamma_{\times 2}(T')$. On the other hand, $T - e = T_1 \cup T_2'$ where $T_1 = T_1' \cup \{u\}$, and by Remark 4, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T_1') + 1$. Thus, $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2') = \gamma_{\times 2}(T_1') + 1 + \gamma_{\times 2}(T_2') > \gamma_{\times 2}(T') + 1 = \gamma_{\times 2}(T)$. Hence, T is $\gamma_{\times 2}$-critical.

Case 2: T is obtained from T' by operation $\mathcal{O}_2$.

Let v be a support vertex in T' and let xyz be the path attached to v by x to obtain the tree T. Then by Lemma 1, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2$. To show that T is $\gamma_{\times 2}$-critical, we consider any edge e of the set $X_T = X_{T'} \cup \{vx, xy\}$.

- If e is any edge of $X_{T'}$, then removing e from T' provides the trees T_1' and T_2' such that $v \in T_1'$ with $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2') > \gamma_{\times 2}(T')$, and from T the trees T_1 and T_2' where $T_1 = T_1' \cup \{x, y, z\}$, and by Lemma 1, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T_1') + 2$. Hence, $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2') = \gamma_{\times 2}(T_1') + 2 + \gamma_{\times 2}(T_2') > \gamma_{\times 2}(T') + 2 = \gamma_{\times 2}(T)$.

- If $e = vx$, then deleting e from T produces the trees T_1 and $T_2 = xyz$ with $\gamma_{\times 2}(T_2) = 3$ and $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$. Hence, $\gamma_{\times 2}(T - vx) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$.
- Finally, if $e = xy$, then $T - e = T_1 \cup T_2$ where $T_2 = yz$. Clearly $\gamma_{\times 2}(T_2) = 2$ and $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2 + 1 = \gamma_{\times 2}(T) - 1$. Hence, $\gamma_{\times 2}(T - xy) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$.

Consequently, for any edge e of X_T, $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$. So T is $\gamma_{\times 2}$-critical.

Case 3: T is obtained from T' by operation $\mathcal{O}_3$.

Let v be a leaf of T' and let $x_i y_i z_i$ with $1 \le i \le k$ be the paths attached to v by the vertices x_i to obtain the tree T. By Lemma 1, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 2k$. First we prove that the support vertex u of v is in every $\gamma_{\times 2}(T)$-set. If u remains a support vertex in T, then u belongs to every $\gamma_{\times 2}(T)$-set. Hence we may assume that u is not a support vertex in T and Let D be a $\gamma_{\times 2}(T)$-set not containing u. Then there is a neighbor of u say w in D with its neighbor w', and v is in D with one vertex x_i for $1 \le i \le k$. Without loss of generality, assume that $x_1 \in D$. Then the set $D_1 = (D - \{x_1\}) \cup \{u\}$ is a $\gamma_{\times 2}(T)$-set and so $D' = D_1 \cap T'$ is a $\gamma_{\times 2}(T')$-set containing an induced path $P_4 = w'wuv$, a contradiction. Thus, u belongs to every $\gamma_{\times 2}(T)$-set.

We now consider any edge of the set $X_T = X_{T'} \cup \{uv, vx_i, x_i y_i\}$ with $1 \le i \le k$. Note that considering the edges vx_i or $x_i y_i$ is similar to consider vx_1 or $x_1 y_1$.

- If e is an edge of $X_{T'}$, then the removal of e in T' provides the trees T_1' and T_2' such that $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T_1') + \gamma_{\times 2}(T_2') > \gamma_{\times 2}(T')$. Deleting e from T gives the trees T_1 and T_2' where $T_1 = T_1' \cup \{x_i, y_i, z_i\}$ for $1 \le i \le k$, and by Lemma 1, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T_1') + 2k$. Hence, $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2') = \gamma_{\times 2}(T_1') + 2k + \gamma_{\times 2}(T_2') > \gamma_{\times 2}(T') + 2k = \gamma_{\times 2}(T)$.
- If $e = uv$, then $T - e$ gives the trees T_1 and T_2 such that T_2 is a tree obtained from a star $K_{1,k}$ where each edge is subdivided twice. Clearly, the set $D_2 = \{v, x_1, y_i, z_i\}$ for $1 \le i \le k$ is a $\gamma_{\times 2}(T_2)$-set and so $\gamma_{\times 2}(T_2) = 2k + 2$. Since any $\gamma_{\times 2}(T_1)$-set can be extended to a $\gamma_{\times 2}(T)$-set by adding the set D_2, then $\gamma_{\times 2}(T_1) \ge \gamma_{\times 2}(T) - 2k - 2$. Suppose that D_1 is a $\gamma_{\times 2}(T_1)$-set of cardinality $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2k - 2$. But then $D = D_1 \cup D_2$ is a $\gamma_{\times 2}(T)$-set such that $\{v, x_1, y_1, z_1\} \subset D$. If $u \in D_1$, then the set $D - \{x_1\}$ would be a DDS of cardinality less than $\gamma_{\times 2}(T)$, a contradiction, and if $u \notin S_1$, then D is a $\gamma_{\times 2}(T)$ that does not contain u, a contradiction too. Hence $\gamma_{\times 2}(T_1) \ge \gamma_{\times 2}(T) - 2k - 1$, and so $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) \ge \gamma_{\times 2}(T) - 2k - 1 + 2k + 2 = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$.
- If $e = vx_1$, then $T - e$ is formed by the trees T_1 and $T_2 = x_1 y_1 z_1$ with $\gamma_{\times 2}(T_2) = 3$. If v is pendent in T_1 ($k = 1$), then by Lemma 1, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$. Now if $k \ge 2$, then we can simply see that there exists some $\gamma_{\times 2}(T_1)$-sets that contain v and so by Lemma 1, ${}_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$. Hence, $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$.
- If $e = x_1 y_1$, then $T - e$ is formed by the trees T_1 and $T_2 = y_1 z_1$ such that $\gamma_{\times 2}(T_2) = 2$, and since any $\gamma_{\times 2}(T_1)$-set can be extended to a $\gamma_{\times 2}(T)$-

set by adding the set $\{y_1, z_1\}$ then $\gamma_{\times 2}(T_1) \geq \gamma_{\times 2}(T) - 2$. Suppose that $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T) - 2$, and let S_1 be a $\gamma_{\times 2}(T_1)$-set. By Remark 1, $\{v, x_1\} \subset S_1$, and so $S = S_1 \cup \{y_1, z_1\}$ is a $\gamma_{\times 2}(T)$-set. Now if $u \in S_1$, then the set $S - \{x_1\}$ would be a DDS of cardinality less than $\gamma_{\times 2}(T)$, a contradiction, and if $u \notin S_1$, then S is a $\gamma_{\times 2}(T)$ that does not contain u, a contradiction too. Thus $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) \geq \gamma_{\times 2}(T) - 1 + 2 = \gamma_{\times 2}(T) + 1 > \gamma_{\times 2}(T)$.

Hence, for any edge e of X_T, $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$ and T is $\gamma_{\times 2}$-critical.

Case 4: T is obtained from T' by operation $\mathcal{O}_4$.

Let x be a support vertex of T' and v its leaf-neighbor such that $\deg_{T'}(x) = k + 2$ and let $v_i u_i z_i$ with $1 \leq i \leq k$ be the paths attached to x by the vertices x_i, and y a neighbor of x of degree two such that every vertex in $N_{T'}(x) - \{v\}$ does not belong to any $\gamma_{\times 2}(T')$. We attach to v a new vertex u to obtain the tree T. By Remark 5, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$.

Consider now any edge of the set $X_T = X_{T'} \cup \{xv\}$.

- If $e \in X_{T'} - \{xv_i\}$ with $1 \leq i \leq k$ then deleting e in T' produces two trees T'_1 and T'_2 such that $x \in T'_1$, and so $\gamma_{\times 2}(T' - e) = \gamma_{\times 2}(T'_1) + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T')$. Deleting e in T gives the trees T_1 and T'_2 where $T_1 = T'_1 \cup \{u\}$. By Remark 5, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 1$, and hence $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 1 + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 1 = \gamma_{\times 2}(T)$.

- If $e = xv_i$ with $1 \leq i \leq k$, then let $T' - e = T'_1 \cup T'_2$ such that T'_1 contains x. For the case $k \geq 2$, by Remark 5, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 1$ with $T_1 = T'_1 \cup \{u\}$. Now if $k = 1$, then clearly $\gamma_{\times 2}(T_1) \leq \gamma_{\times 2}(T'_1) + 1$. Suppose that $\gamma_{\times 2}(T_1) < \gamma_{\times 2}(T'_1) + 1$. With the same approach like in subcase 1, it results that x does not belong to any $\gamma_{\times 2}(T_1)$-set and that $\gamma_{\times 2}(T) = \gamma_{\times 2}(T_1) + 3$. Let S_1 be any $\gamma_{\times 2}(T_1)$-set. By Remark 1, $u, v \in S_1$ and since $x \notin S_1$, its neighbor y belongs to S_1 with its neighbor say z, and so the set $S = S_1 \cup \{x, u_1, z_1\}$ is a $\gamma_{\times 2}(T)$-set. Now by Remark 5, $\gamma_{\times 2}(T) = \gamma_{\times 2}(T') + 1$ and since $u, v \in S$, the set $S' = S - \{u\}$ is a $\gamma_{\times 2}(T')$-set that contains the vertices v, x, y and z, and so $G[S']$ induces a path P_4, a contradiction with the fact that T' is $\gamma_{\times 2}$-critical. Consequently, for $k \geq 1$, $\gamma_{\times 2}(T_1) = \gamma_{\times 2}(T'_1) + 1$, and so $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T'_2) = \gamma_{\times 2}(T'_1) + 1 + \gamma_{\times 2}(T'_2) > \gamma_{\times 2}(T') + 1 = \gamma_{\times 2}(T)$.

- If $e = xv$, then let $T - e = T_1 \cup T_2$ such that $T_2 = uv$. Suppose that $\gamma_{\times 2}(T - e) = \gamma_{\times 2}(T_1) + \gamma_{\times 2}(T_2) = \gamma_{\times 2}(T_1) + 2 = \gamma_{\times 2}(T)$ and let S_1 be any $\gamma_{\times 2}(T_1)$-set. Without loss of generality, we may assume that x belongs to S_1 with a vertex from $N_{T_1}(x)$. Then the set $S = S_1 \cup \{v, u\}$ is a $\gamma_{\times 2}(T)$-set, and so the set $S' = S - \{u\}$ is a $\gamma_{\times 2}(T')$-set that contains a neighbor of x, a contradiction. Hence $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$.

Consequently, for any edge e of X_T, $\gamma_{\times 2}(T - e) > \gamma_{\times 2}(T)$. So T is $\gamma_{\times 2}$-critical.

This completes the proof of Theorem 3. ■

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