Generating efficient solutions with reservation levels in Multiobjective Stochastic Integer Problems

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Abstract. This paper considers a special class of multiobjective stochastic integer linear programming (MOSILP) problems involving random variable coefficients in the constraints. The presumed constraints reliability levels are less than one, then chance constrained programming (CCP) is used to handle with the randomness. It is shown how these problems can be transformed into equivalent multiobjective nonlinear integer programming (EMONLIP) problem when the random variables are independent and normally distributed with mean and variance that are linear in the decision variables. The algorithm developed here is based on the notion of level sets and level curves. It finds the Pareto optimal solutions throughout a linear integer program defined by eigenvalue relaxation.

keywords: Multiple objective programming, Stochastic Programming, Chance constrained programming, nonlinear programming, level sets, level curves.

1 Introduction and background

Much of decision making in the real-world takes place in an environment where the objectives, constraints or parameters are not known precisely (see Lai and Hwang [14] and Liu [15]). Therefore a decision is often made on the basis of vague information or uncertain data.The uncertainty may be interpreted as randomness or fuzziness. The randomness occurring in the multiobjective linear programming (MOLP) problems is categorized as the multiobjective stochastic linear programming (MOSLP) problems. As we have known in the stochastic optimization problems, the coefficients of the problem are assumed as random variables with known distributions in most of cases. The books written by Birge and Louveaux [7], Kall [11], Prékopa [18], Klein Haneveld & Van der Vlerk [12] provide many interesting ideas and useful techniques for tackling the stochastic optimization problems. (MOSLP) models are appropriate when data evolve over time and decisions need to be prior to observing the entire data stream. Then, the way of modeling the (MOSLP) problems and obtaining efficient solutions depends in large part on the nature of available information about the random parameters.

The (MOSLP) models have been developed for a variety of applications, including portfolio selection (Aouni and al. [3], Ballestero and al. [4], Shing and al. [20], Ogryczak [17]), investment planing (Ben Abdelaziz and al. [5]) and electric power generation (Teghem and al. [22]), to mention a few.

Most previous efforts in this field have been devoted to positive decision variables (see, Stancu-Minasian [21] and Caballero [8]. In many situations, however, fractional values of the variables are not physically meaningful. Therefore, modeling with multiobjective stochastic integer linear programming (MOSILP) programs and the development of solution algorithms for such problems are of great interest to management scientists.

Progress has not been substantial on (MOSILP) and the present literature on it is surprisingly thin (see for example, Abbas and Bellahcene [1], Saad and Kittani [19], Teghem [23]). In [1], the Benders decomposition method [6] and four types of cuts are used to develop a generating technique for identifying a compromise solution from a set of available candidates. The stochastic data are treated by a recourse approach to obtain an equivalent deterministic two-stages multiobjective integer linear programming (MOILP) problem and duality proprieties are used to check for feasibility of the recourse function. In [19], a solution algorithm is presented for solving integer linear programming problems involving dependent random parameters in the objective functions and linearly independent random parameters in the constraints. The STRANGE-MOMIX method presented by Teghem in [23] is interactive and based on the generalized Tchebycheff norm to generate efficient solutions.

In this paper we consider a special stochastic model called the multiobjective chance constrained integer linear problem. The random variables are assumed to be normally distributed with mean and variance that are linear in the decision variables. Since problems including randomness are usually transformed into nonlinear programming problems, it is difficult to find a global optimal solution efficiently. Furthermore, since our proposed models are multi-criteria stochastic programming problems, it is almost impossible to solve them directly. We manage to construct efficient solution methods for them using the equivalent transformations to the main problem based on the properties of random variable. Based on the notion of level sets and level curves, the algorithm developed here computes all the Pareto optimal solutions respecting given reservation levels.

The next section describes the considered stochastic model and shows how to convert it into an equivalent multiobjective deterministic nonlinear program. In section 3, we review some basic properties of level sets. Section 4, presents the eigenvalue relaxation of the resulting nonlinear program. The algorithm development is detailed in section 5. We conclude the paper with some considerations on possible future research in this field.

2 Problem statement and structural properties

We consider the multiobjective linear programming problems involving random variable coefficients in the objectives functions formulated as:

$$\begin{array}{c} \text{"}maximize\text{"}\,(u_1, u_2, \ldots, u_p) \\ subject\ to\ \ P_r[C_i^t(w)x \geq u_i] = \alpha_i,\ i = 1, \ldots, p \\ x \in S \end{array} \tag{P1}$$

Where $S = \{x \in R^n \mid Ax \leq b,\ x \geq 0\ and\ integer\}$.
The parameters $u_i,\ i = 1, \ldots, p$, $A \in R^{m \times n}$, $b \in R^m$ and $x \in R^n$ represent deterministic problem data; w is a random vector from the probability space (Ω, Σ, P)

and $C_i(w) \in R^n$ represent stochastic parameters; $P_r\{t\}$ denotes the probability of the event $t \in \Sigma$ under the probability measure P_r ; and $\alpha_i,\ i = 1, \ldots, p$ are probability levels. This model deals with the optimization of upper allowable limits $u_i,\ i = 1, \ldots, p$ for given probabilities $\alpha_i,\ i = 1, \ldots, p$. For instance, this is a situation when expected value of the profit is considered not to be a good measure of criteria. In the following, the basic technique of chance constrained programming (CCP) is used to transform problem (P1) into an equivalent multiobjective nonlinear integer programming (EMONLIP) problem according to the predefined probabilities $\alpha_i,\ i = 1, \ldots, p$.

Assume that each random variable $C_i(w)$ has a multinormal distribution function with mean value vector $\overline{C}_i = (\overline{C}_{i,1}, \overline{C}_{i,2}, \ldots, \overline{C}_{i,n})$ and variance-covariance matrix V_i. Therefore, it is known that $C_i^t(w)x$ has a normal distribution function with mean $\overline{C}_i^t x$ and standard deviation $(x^t V_i x)^{1/2}$. Then the probabilistic constraints can be written as:

$$P_r[C_i^t(w)x \geq u_i] = \alpha_i \Longleftrightarrow \Phi\left(\frac{u_i - \overline{C}_i^t x}{(x^t V_i x)^{1/2}}\right) = 1 - \alpha_i$$
$$\Longleftrightarrow u_i(x) = \overline{C}_i^t x - \Phi^{-1}(\alpha_i)(x^t V_i x)^{1/2}$$

Where $\Phi(\cdot)$ is the distribution function of the standard normal distribution. The (EMOIP) of problem (P1) is shown in problem (P2) :

$$\begin{array}{c} maximize\ u_i(x) = \overline{C}_i^t x - \Phi^{-1}(\alpha_i)(x^t V_i x)^{1/2} \quad ,\ i = 1, \ldots, p \\ subject\ to\ \ x \in S \end{array} \tag{P2}$$

Kataoka [13] is credited for formulating problem (P1) and the development of the (EMONLIP) problem (P2). When $\alpha_i > 0.5$, the objective functions $u_i,\ i = 1, \ldots, p$ are concave. That is because $x^t V_i x$ is convexe for $i = 1, \ldots, p$ (see Ishii [10,p 184]). The values $\Phi^{-1}(\alpha_i)$ which are positive can be obtained from any standard normal distribution table.

3 Multiobjective optimization and level sets

In the following, we will use the concept of Pareto optimality to define the minimization in (P2).

Definition 1. *A solution $x^* \in S$ is called Pareto optimal if and only if there is no $x \in S$ such that $u_i(x) \geq u_i(x^*)$, $i = 1, \ldots, p$ and $u_i(x) > u_i(x^*)$ for at least one $i \in \{1, \ldots, p\}$. The set of all Pareto optimal solutions is denoted by $Spar$. If x^* is Pareto optimal then $u(x^*) = (u_1(x^*), \ldots, u_p(x^*))$ is called efficient.*

Independent of the properties of the objective functions u_i or the constraint set S, Pareto optimal solutions can be characterized geometrically. In order to state this characterization, we introduce the notion of level sets and level curves.

Definition 2. *Let $\beta_i \in R$ for $i = 1, \dots, p$*
1. The set $L^i_{\geq}(\beta_i) = \{x \in S \mid u_i(x) \geq \beta_i\}$ is called the level set of u_i with respect to the level β_i
2. The set $L^i_{=}(\beta_i) = \{x \in S \mid u_i(x) = \beta_i\}$ is called the level curve of u_i with respect to the β_i.

The following characterization of Pareto optimal solutions by level sets and level curves was given by Ehrgott and al. [9].

Lemma 1. *Let $x^* \in S$. Then x^* is Pareto optimal if and only if*

$$\bigcap_{i=1}^{p} L^i_{\geq}(u_i(x^*)) = \bigcap_{i=1}^{p} L^i_{=}(u_i(x^*))$$

i.e. x^* is Pareto optimal if and only if the intersection of all p level sets of u_i with respect to levels $u_i(x^*)$ is equal to the intersection of the level curves of $u_i,\ i = 1, \dots, p$ with respect to the same levels.

Because we will use the result of Lemma 1 throughout the paper the following notation will be convenient.
For $\beta \in R^p$ let

$$S(\beta) = \{x \in S \mid u_i(x) \geq \beta_i,\ i = 1, \dots, p\} = \bigcap_{i=1}^{p} L^i_{\geq}(\beta_i)$$

Correspondingly, $S(\beta)_{par}$ will denote the Pareto set of $S(\beta)$.

Note that the range of values that efficient points can reach is given by a lower and upper bound on the efficient set defined by the ideal and the nadir point of the multiobjective programming (P2). The ideal point $u^I = (u^I_1, u^I_2, \dots, u^I_p)$ is given by $u^I_i = max\ u_i(x)$ and the nadir point $u^N = (u^N_1, u^N_2, \dots, u^N_p)$ is given by $u^N_i = min\ u_i(x)$. With the nadir point, we can choose $\beta_i = u^N_i$ for $i = 1, \dots, p$ as lower bounds. The major difficulty in this choice is the need for a nonlinear algorithm. This article introduces an eigenvalue relaxation to define two linear functions as lower and upper bounds for the nonlinear objective function u_i.

4 The eigenvalue relaxation

In order to define the eigenvalue relaxation problem, we state some of well known results regarding the eigenvalues of symmetric positive definite matrices. The proofs of these results can be found in [16].

Proposition 1. *If V is an n by n symmetric positive definite matrix, then its eigenvalues are real and positive.*

Proposition 2. *If V is an n by n symmetric positive definite matrix, σ_1 and σ_n are its smallest and largest eigenvalue respectively, then*

$$\sigma_1 x^t x \leq x^t V x \leq \sigma_n x^t x \quad \forall x \in R^n$$

Knowing that a and b are two nonnegative numbers, the following inequality will hold:

$$(a+b)^{1/2} < a^{1/2} + b^{1/2} \tag{1}$$

According to a property of inequality (1), the linear function $\sum_{i=1}^{p} (\sigma_n^i)^{1/2} x_j$ may be used as an upper bound for the nonlinear term $(x^t V_i x)^{1/2}$ appearing in (P2). This can be stated as :

$$(x^t V_i x)^{1/2} < \sum_{i=1}^{p} (\sigma_n^i)^{1/2} x_j \tag{2}$$

For generalization purposes and for having both the upper and lower bound linear functions to correspond with the nonlinear term of problem (P2), we define inequality (3) as shown below :

$$-\sum_{i=1}^{p} (\sigma_n^i)^{1/2} x_j < -(x^t V_i x)^{1/2} < \sum_{i=1}^{p} (\sigma_n^i)^{1/2} x_j \tag{3}$$

By multiplication of the positive constant $\Phi^{-1}(\alpha_i)$ by all terms of inequality (3) and addition of a linear function of all decision variables, such as $\overline{C}_i^t x$, to those terms, inequality (4) would be obtained:

$$\overline{C}_i^t x - \Phi^{-1}(\alpha_i) \sum_{i=1}^{p} (\sigma_n^i)^{1/2} x_j \leq \overline{C}_i^t x - \Phi^{-1}(\alpha_i)(x^t V_i x)^{1/2} \leq \overline{C}_i^t x + \Phi^{-1}(\alpha_i) \sum_{i=1}^{p} (\sigma_n^i)^{1/2} x_j \tag{4}$$

Another way of exhibiting inequality (4) is shown in (5).

$$f_i(x) \leq u_i(x) \leq g_i(x) \ , \ i = 1, \ldots, p \tag{5}$$

Since inequality (5) would hold for all values of the feasible decision variables, therefore inequality (6) would also hold :

$$f_i^* \leq u_i^* \leq g_i^* \ , \ i = 1, \ldots, p \tag{6}$$

In (6), u_i^* is the optimal value of problem (P3) restricted to the ith objective function and f_i^* and g_i^* are the optimal values of problems (P4) and (P5) that are presented below.

$$\max_{x \in S} f_i(x) = \overline{C}_i^t x - \Phi^{-1}(\alpha_i) \sum_{i=1}^{p} (\sigma_n^i)^{1/2} x_j \tag{P4}$$

and

$$\max_{x \in S} g_i(x) = \overline{C}_i^t x + \Phi^{-1}(\alpha_i) \sum_{i=1}^{p} (\sigma_n^i)^{1/2} x_j \tag{P5}$$

5 Pareto optimal solutions with reservation levels

In this section, we develop a method for the determination of Pareto optimal solutions in the multiobjective (P2) based on the characterization given in Lemma 1. The procedure uses an algorithm which solves the problem of finding a K-best solution in a multiobjective integer programming problem (see for example [2] or [24]). The goal is to find all Pareto optimal solutions of problem (P2) respecting given reservation levels $\beta_i,\ i = 1, \ldots, p$. In other words we want to to compute $S(\beta)_{par}$. Instead of an explicit computation of the intersection of level sets and checking the condition of Lemma 1, we will generate one level set $L_{\geq}(\beta_1)$ (without loss of generality) in order of decreasing values of the corresponding objective function, and then check for each element of this level set if it is also contained in the other level sets and if it dominates or is dominated by a solution found before.

Algorithm

Input: Instance of a (MOSILP) with p criteria, reservation levels $\beta_1, \ldots, \beta_p$.
Output: The set $S(\beta)_{par}$ of all Pareto optimal solutions respecting reservation levels β.

Step 1 : Set $(\beta_1, \beta_2, \ldots, \beta_p) = (f_1^*, f_2^*, \ldots, f_p^*)$.

Step 2 : Let x_1 be the optimal solution of problem $\underset{x \in S}{max}\ g_1(x)$.

If $g_1(x^1) < \beta_1$ then stop $S(\beta)_{par} = \varnothing$
$k = 1$
$S(\beta)_{par} = \left\{x^k\right\}$.

Step 3 : $k = k + 1$
Apply a ranking algorithm to compute the k-best solution x^k of g_1.
If $g_1(x^k) < \beta_1$ then stop $S(\beta)_{par} = \varnothing$.

Step 4 : If $x^k \in L_{\geq}$ for all $i = 2, \ldots, p$ then goto step 5
else goto step 3.

Step 5 : For $1 \leq i \leq k - 1$
If x^kdominates x^i then $S(\beta)_{par} = S(\beta)_{par} \setminus \left\{x^i\right\}$
else if x^i dominates x^k then goto step 3.
else if $g_1(x^k) = u_1(x^i)$ then $S(\beta)_{par} = S(\beta)_{par} \cup \left\{x^k\right\}$ goto step 3.

Step 6 : $S(\beta)_{par} = S(\beta)_{par} \cup \left\{x^k\right\}$
goto step 3.

6 Conclusion

In this paper, we attempted to solve a particular multiobjective stochastic integer linear problem by level sets. The nondominated solutions are determined by solving a linear integer program defined by eigenvalue relaxation. The construction of the lower bounds (levels) is fairly simple. Our method does not require specific mathematical properties to be satisfied by the objectives. It appears—on several examples— that the algorithm performs faster, however further experimental validation of this observation is needed.

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