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Stabilized finite element method for the elliptic Cauchy problem

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Dedication

I dedicate this thesis to

In memory of my dead father ♡ MOHAMED ♡

for giving me life and the joy of living. Your good education, your counsels and your blessings have never failed, may Allah grant you Jannah Firdaus (amen).

My dear mom ♡ FATMA ♡

who has always loved me unconditionally and whose good examples have taught me to work hard for the things that I aspire to achieve.

♡ My beloved brothers and sisters. ♡

♡ The people who paved our way of science and knowledge

All my teachers distinguished. ♡

♡ All my friends, thank you for your understanding and encouragement in many moments. Your friendship makes my life a wonderful experience. I can't list all the names here, but you are always on my mind. ♡

D. MESSAOUDI.

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Notations

- $\frac{\partial^k}{\partial x_i^k}$: The k^{th} partial derivative.
- $|\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbb{R}^n$.
- I : Identity operator.
- ∇ : The gradient operator.
- Δ : The Laplace operator.
- $\Delta^2 = \Delta\Delta$: The biharmonic operator.
- $\partial_n = n \cdot \nabla$: The derivative according to the normal.
- $\partial\Omega$: The boundary of Ω.
- $\bar{\Omega}$: The closure of Ω.
- $\mathring{\Omega}$: The interior of Ω.
- $\|f\|_{L^2(\Omega)}^2 = \int_\Omega |f(x)|^2 dx$.
- H^s : The usual Sobolev space.

- $C_0^\infty = \mathcal{D}$: The test functions space.
- $\|u\|_{m,\Omega} = \left(\sum_{\alpha\leq m} \|D^\alpha u\|^2_{L^2(\Omega)}\right)^{1/2}$.
- $|u|_{m,\Omega} = \left(\sum_{\alpha=m} \|D^\alpha u\|^2_{0,\Omega}\right)^{1/2}$.
- H_0^s : The closure of $\mathcal{D}$ in H^s.
- H^{-s} : The dual space of H_0^s.
- $\mathrm{div}(u)$: The divergence of u.
- $B(x_0, r) := \{x \in \mathbb{R}^n; |x - x_0| < r\}$.
- $\mathrm{dist}(x, \Omega)$: The distance between x and Ω.
- W' : The dual space of W.
- $[\![u]\!] = u_+ - u_-$: The jump of u.
- $\{u\} = (u_+ + u_-)/2$: The average of u.
- $\left[\!\!\left[\frac{\partial u}{\partial n}\right]\!\!\right] = (\nabla u_+ + \nabla u_-) \cdot n$.
- $[\![\partial_t u]\!]$: The jump of the tangential derivative of u.

Introduction

Cauchy problems for elliptic operators are encountered in many practical applications such as electrocardiography (ECG) and plasma physics. In addition, Cauchy problems play an important role in inverse problems, (see [11]).

The ill-posedness of the Cauchy problem was first pointed out by J. Hadamard in his lectures at Yale University, 1923 (see [15]) who proved that it is ill-posed in the case of linear second order elliptic equations.

The most popular regularizing methods for ill-posed problems are the Tikhonov regularization or a so-called quasi reversibility method and the stabilized finite-element methods. Stabilized finite element methods have emerged as an efficient and reliable tool for the design of computational methods for ill-posed elliptic problems, which are representing a general technique for the regularization of the standard Galerkin method in order to improve its stability properties, (see [2] and [5]).

In this work we will study the ill-posed elliptic Cauchy problem and present a stabilized conforming and nonconforming finite element methods to solve it. This thesis structured as follows:

In the first chapter, we recall basic definitions for elliptic Cauchy problem and we introduce Hadamard's concept of well-posedness, and we describe the methods for solving an ill-posed problem, this chapter is based on the references [13], [15], [16], [23], [25], [26].

In the second chapter, we show some inequalities to prove the uniqueness of elliptic Cauchy problem and we consider the elliptic Cauchy problem for the Laplace operator, and we introduce the conditional stability estimates for ill-posed problems.

In the third chapter, we consider the stabilized finite element method, a conforming finite element method in section one and in section two we present Crouzeix-Raviart non-conforming finite element method.

In the last chapter, we present numerical examples for stabilized finite element method which was introduced in the third chapter, by using `FreeFEM++`.

Chapter 1

Generalities

1.1 Some preliminaries of elliptic Cauchy problem

Definition 1.1.1

The principal part of the differential operator

$$\begin{aligned} L \equiv L(x,D) : &= \sum_{i=0}^{m} \sum_{|\alpha|=i} a_{(\alpha_1,\alpha_2,\cdots,\alpha_n)}(x) D_1^{\alpha_1} D_2^{\alpha_2} \cdots D_n^{\alpha_n} \\ &= \sum_{|\alpha|\le m} a_\alpha(x) D^\alpha \qquad \text{on } \Omega \subset \mathbb{R}^n \end{aligned}$$

is

$$L_P(x,D) = \sum_{|\alpha|=m} a_\alpha(x) D^\alpha.$$

The coefficients $a_\alpha(x)$ are complex-valued functions of x, and $D_i^{\alpha_i},\ D^\alpha$ are partial differential operators defined by

$$D_i^{\alpha_i} = \frac{\partial^{\alpha_i}}{\partial x_i^{\alpha_i}} \quad \text{and} \quad D^\alpha = \prod_{i=1}^{n} D_i^{\alpha_i} = \frac{\partial^{\alpha_1+\alpha_2+\cdots+\alpha_n}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_n^{\alpha_n}}.$$

Remark 1.1.2

The principal part is also called the principal symbol.

Definition 1.1.3

A characteristic form of L at $x \in \Omega$ is the homogeneous polynomial of degree m on $\mathbb{R}^n$ defined by

$$\chi(x,\xi) = L_P(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \qquad (\xi \in \mathbb{R}^n)$$

Definition 1.1.4

The vector $\xi \neq 0$ is called **characteristic** for L at x if $\chi(x,\xi) = 0$, and the set of all such ξ is called the characteristic variety of L at x and is denoted by $\mathrm{char}_x(L)$:

$$\mathrm{char}_x(L) = \{\xi \neq 0, \quad \chi(x,\xi) = 0\}$$

Definition 1.1.5

We say L is elliptic at x if

$$\mathrm{char}_x(L) = \emptyset,$$

and elliptic on Ω if it is elliptic for each $x \in \Omega$.

Example 1.1

Consider the second-order partial differential operator in two variables

$$L(x,D) = \sum_{|\alpha|\leq 2} a_\alpha(x) D^\alpha$$

$$\alpha = (\alpha_1,\alpha_2) \quad \alpha_i = 0,1,2 \quad |\alpha| = \alpha_1 + \alpha_2$$

then

$$\begin{aligned} L(x,D) = {} & a_{(0,0)}(x)D^{(0,0)} + a_{(1,0)}(x)D^{(1,0)} + a_{(0,1)}(x)D^{(0,1)} \\ & + a_{(2,0)}(x)D^{(2,0)} + a_{(1,1)}(x)D^{(1,1)} + a_{(0,2)}(x)D^{(0,2)}. \end{aligned}$$

Where

$$D^{(0,0)} = I, \qquad D^{(1,0)} = \frac{\partial}{\partial x_1}, \qquad D^{(0,1)} = \frac{\partial}{\partial x_2}$$

$$D^{(1,1)} = \frac{\partial^2}{\partial x_1 \partial x_2}, \qquad D^{(2,0)} = \frac{\partial^2}{\partial x_1^2}, \qquad D^{(0,2)} = \frac{\partial^2}{\partial x_2^2}$$

and thus, L becomes

$$L(x_1, x_2, D) = a_{(2,0)}(x_1, x_2)\frac{\partial^2}{\partial x_1^2} + a_{(1,1)}(x_1, x_2)\frac{\partial^2}{\partial x_1 \partial x_2} + a_{(0,2)}(x_1, x_2)\frac{\partial^2}{\partial x_2^2}$$
$$+ a_{(1,0)}(x_1, x_2)\frac{\partial}{\partial x_1} + a_{(0,1)}(x_1, x_2)\frac{\partial}{\partial x_2} + a_{(0,0)}(x_1, x_2).$$

Let $\xi \in \mathbb{R}^2 \setminus \{0\} \quad (\xi = (\xi_1, \xi_2))$, then

$$\chi(x_1, x_2, \xi) = L_P(x_1, x_2, \xi) = a_{(2,0)}(x_1, x_2)\xi_1^2 + a_{(1,1)}(x_1, x_2)\xi_1\xi_2 + a_{(0,2)}(x_1, x_2)\xi_2^2$$

$L(x_1, x_2, D)$ is elliptic at a point $(x_1, x_2) \in \Omega \subset \mathbb{R}^2$ if $\chi(x_1, x_2, \xi) \neq 0$.

Definition 1.1.6

A hypersurface S is called characteristic for L at $x \in S$ if the normal vector $v(x)$ to S at x is in $\mathrm{char}_x(L)$, otherwise S is called non-characteristic.

1.1.1 The General Cauchy Problem

The Cauchy problem

Consider the linear partial differential equation of order m

$$L(x, D)u = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u(x) = f(x) \qquad \text{on } \mathbb{R}^n \tag{1.1}$$

Let S be a given hypersurface in $\mathbb{R}^n$ and let $n = n(x)$ denote the unit vector normal to S at x (S is non-characteristic for L). Suppose that on S the values of u and all of its directional derivatives in the direction n of order up to $m-1$ are given, moreover

$$u|_S = \varphi_0, \quad \left.\frac{\partial u}{\partial n}\right|_S = \varphi_1, \cdots, \quad \left.\frac{\partial^{m-1} u}{\partial n^{m-1}}\right|_S = \varphi_{m-1}, \tag{1.2}$$

The Cauchy[1] problem for the differential operator $L(x, D)$ with the Cauchy data $(\varphi_0, \varphi_1, \cdots, \varphi_{m-1})$ on S consists in finding a solution u of equation (1.1) defined in a domain Ω containing S and satisfying conditions (1.2) on S.

The surface S is called the initial surface of the problem and the conditions (1.2) are called the initial conditions. The given functions $\varphi_0, \varphi_1, \cdots, \varphi_{m-1}$ which are defined on S are called the initial data.

[1] Augustin-Louis Cauchy (1789-1857) French mathematician.

Definition 1.1.7

The surface S in $\mathbb{R}^n$ is said to be analytic if it is described by an equation of the form

$$F(x_1, \cdots, x_n) = 0$$

where F is an analytic function.

The Cauchy-Kowalewsky Theorem and Holmgren's Theorem

Theorem 1.1.8 (Cauchy-Kowalewsky)

Let S be an analytic hypersurface of $\mathbb{R}^n$ and $L(x, D)$ is an operator with analytic coefficients in some neighbourhood of a point $x_0 \in S$ (S is non-characteristic for L). We take $f, \varphi_j \quad j = 0, \cdots, m-1$ which are analytic in the neighbourhood of x_0. Then the Cauchy problem (1.1)-(1.2) has a solution $u(x)$ which is defined and analytic in a neighborhood of x_0, and this solution is unique in the class of analytic functions.

Proof.

See [9, p 330]. ■

Remark 1.1.9

The Cauchy-Kowalewsky[2] theorem is a theorem of fundamental importance in the theory of partial differential equations. However its practical usefulness is often limited by the stringent requirement that the initial data and the right-hand side of the equation must be analytic and by the fact that it asserts the existence and uniqueness of the solution only in a (possibly very small) neighborhood of a point.

Remark 1.1.10

The Cauchy-Kowalewky theorem provides uniqueness of the solution just in the restricted class of analytic functions. A priori, there may be other non-analytical solutions.

Theorem 1.1.11 (Holmgren)

The Cauchy problem with the coefficients of L are analytic, given on an analytic non-characteristic hypersurface S, has at most one solution in a neighbourhood of S.

[2] Sofya Vasilyevna Kowalewsky (1850-1891) Russian mathematician.

Proof.

See [25, p 34]. ■

1.2 Well posed and ill posed problems

In general, it is impossible to find explicit expressions of all solutions of all PDEs. In the absence of explicit solutions, we need to seek methods to prove existence of solutions of PDEs and discuss properties of these solutions. A given PDE may not have solutions at all or may have many solutions, when it has many solutions, we intend to find side conditions to pick the most reasonable solutions. So Hadamard[3] introduced the notion of well-posed problem in the beginning of the twentieth century, (see [19]).

Definition 1.2.1

A problem is called well-posed in the sense of Hadamard if:

1. there exists a solution to the problem (existence),
2. there is at most one solution to the problem (uniqueness),
3. the solution depends continuously on the data (stability).

Remark 1.2.2

The third condition means that small variations on the data imply small variations on the solution.

Definition 1.2.3

An ill-posed problem is a problem that does not satisfy at least one of the well-posedness conditions.

Remark 1.2.4

In the theory of ill posed problems, the main attention is focused on the third condition.

[3] Jacques-Salomon Hadamard (1865-1963) French mathematician.

1.2.1 Examples of well-posed problems

1. Let us now consider the Laplace's equation with Dirichlet boundary conditions

$$\begin{cases} -\Delta u = 0 \quad \text{in} \quad \Omega := \mathbb{R} \times (0,1), \\ u(x,0) = g_n(x), \quad u(x,1) = 0 \quad \forall x \in \mathbb{R}, \end{cases} \tag{1.3}$$

with $g_n(x) := \dfrac{1}{n}\sin(nx)$, for any $x \in \mathbb{R}$, this problem has a unique solution (by Theorem (1.1.8)). Then the solution is

$$u_n(x,y) = \frac{1}{n(1-e^{2n})}\sin(nx)(e^{ny} - e^{n(2-y)})$$

vanishes uniformly as $n \longrightarrow \infty$.

And thus, the problem (1.3) is well-posed.

2. Let Ω be a bounded domain of $\mathbb{R}^n$ and f, g be sufficiently regular functions. The Dirichlet problem for the operator $-\Delta + \lambda I$:

find $u : \bar{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u + \lambda u = f \quad & \text{in} \quad \Omega, \\ \quad u = g, & \text{on} \quad \partial\Omega \end{cases} \tag{1.4}$$

is well-posed for any $\lambda \geq 0$.

The Neumann problem for the operator $-\Delta + \lambda I$ is well-posed for $\lambda > 0$, (See (1.6)).

3. Let us consider the Cauchy problem

$$\begin{cases} \Delta^2 u = 0 \quad \text{in} \quad \Omega, \\ \partial_n^{j-1} u = f_j, \quad j = 1, 2 \;\; \text{on} \quad \partial\Omega, \end{cases} \tag{1.5}$$

with f_j real analytic on $\partial\Omega$ for $j = 1, 2$.

This problem is well-posed.

1.2.2 Examples of ill-posed problems

1. Let Ω be a bounded domain of $\mathbb{R}^n$ and $f : \Omega \to \mathbb{R}$, $g : \partial\Omega \to \mathbb{R}$ be prescribed functions. We search for $u : \bar{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = f \quad & \text{in} \quad \Omega, \\ \quad \partial_n u = g, & \text{on} \quad \partial\Omega \end{cases} \tag{1.6}$$

this problem is ill-posed, because if u is a solution then $u + C$ is also a solution for any constant C, thus the problem (1.6) has an infinity of solutions.

2. (Hadamard's example) [15]

   Let us consider the Cauchy problem for the Laplace equation

$$\begin{cases} \Delta u_\varepsilon(x, y) = 0, & \forall\ (x, y) \in \mathbb{R} \times \mathbb{R}_+ \\ u_\varepsilon(x, 0) = 0, & \text{for every } x \in \mathbb{R} \\ \dfrac{\partial u_\varepsilon}{\partial y}(x, 0) = \varepsilon \sin(x/\varepsilon) & \text{for every } x \in \mathbb{R} \end{cases} \tag{1.7}$$

   this problem satisfy the assumption of the Cauchy-Kowalewsky theorem. This theorem implies that there exists a unique solution to the problem (1.7) in the class of analytic functions.

   The solution to the problem (1.7) can be obtained using separation of variables as follow:

   we put $u_\varepsilon(x, y) = f(x)g(y)$, we then get

$$\frac{\partial^2 u_\varepsilon}{\partial x^2} = f''(x)g(y)\ , \quad \frac{\partial^2 u_\varepsilon}{\partial y^2} = f(x)g''(y)$$

   implies this

$$\Delta u_\varepsilon = \frac{\partial^2 u_\varepsilon}{\partial x^2} + \frac{\partial^2 u_\varepsilon}{\partial y^2} = f''(x)g(y) + f(x)g''(y) = 0$$

   So

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} = C$$

   if $C \geq 0$ we find the trivial solution, then $C < 0$ and we may write $C = -r^2 < 0$, so we have

$$\frac{f''(x)}{f(x)} = -\frac{g''(y)}{g(y)} = -r^2$$

   then

$$f''(x) + r^2 f(x) = 0\ , \quad g''(y) - r^2 g(y) = 0$$

   thus

$$f(x) = C_1 \cos(rx) + C_2 \sin(rx) \ \text{ and } \ g(y) = C_3 \cosh(ry) + C_4 \sinh(ry)$$

and therefore

$$u_\varepsilon(x,y) = (C_1\cos(rx) + C_2\sin(rx))(C_3\cosh(ry) + C_4\sinh(ry))$$

According to the boundary conditions, we would find

$$u_\varepsilon(x,y) = \varepsilon^2\sin(x/\varepsilon)\sinh(y/\varepsilon)$$

Otherwise, a solution to

$$\begin{cases} \Delta u_\varepsilon(x,y) = 0, & \forall\ (x,y) \in \mathbb{R}\times\mathbb{R}_+ \\ u_\varepsilon(x,0) = 0, & \text{for every } x \in \mathbb{R} \\ \dfrac{\partial u_\varepsilon}{\partial y}(x,0) = 0 & \text{for every } x \in \mathbb{R} \end{cases}$$

is $u_\varepsilon(x,y) \equiv 0$. Note that

$$\lim_{\varepsilon\to 0}\left|\varepsilon\sin\left(\frac{x}{\varepsilon}\right) - 0\right| = \lim_{\varepsilon\to 0}\left|\varepsilon\sin\left(\frac{x}{\varepsilon}\right)\right| = 0$$

Nevertheless, for any fixed $y > 0$

$$\lim_{\varepsilon\to 0}|u_\varepsilon(x,y) - 0| = \lim_{\varepsilon\to 0}|u_\varepsilon(x,y)| = +\infty$$

Thus a very small change in the initial data results in a large change in the solution. Therefore, the requirement that the solution depends continuously on the data does not hold.

We can see it by taking different values for ε at $y = 1$ and $x \in [0,\pi]$, then we obtain:

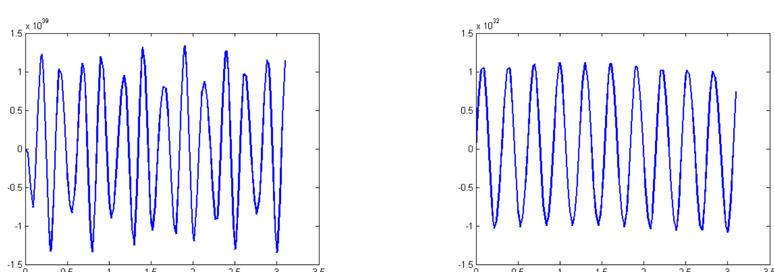


Figure 1.1: The solution for $\varepsilon = 0.010$ (left) and for $\varepsilon = 0.012$ (right)

3. Consider the following Cauchy problem for the Helmholtz equation (see [22]) in the rectangle $\Omega = (0,a) \times (0,b)$:

$$\begin{cases} \Delta u(x,y) + k^2 u(x,y) = 0, & \forall\ (x,y) \in \Omega \\ u(x,0) = f(x), & 0 \le x \le a \\ \dfrac{\partial u}{\partial y}(x,0) = g(x) & 0 \le x \le a \\ u(0,y) = u(a,y) = 0 & 0 \le y \le b \end{cases} \tag{1.8}$$

where k is the wave number and $f, g \in L^2(0,a)$. The solution to this problem can be obtained using separation of variables in the form

$$u(x,y) = \sum_{n=0}^{+\infty} \sin\left(\frac{n\pi}{a}x\right)\left(A_n \cosh(\lambda_n y) + \lambda_n^{-1} B_n \sinh(\lambda_n y)\right)$$

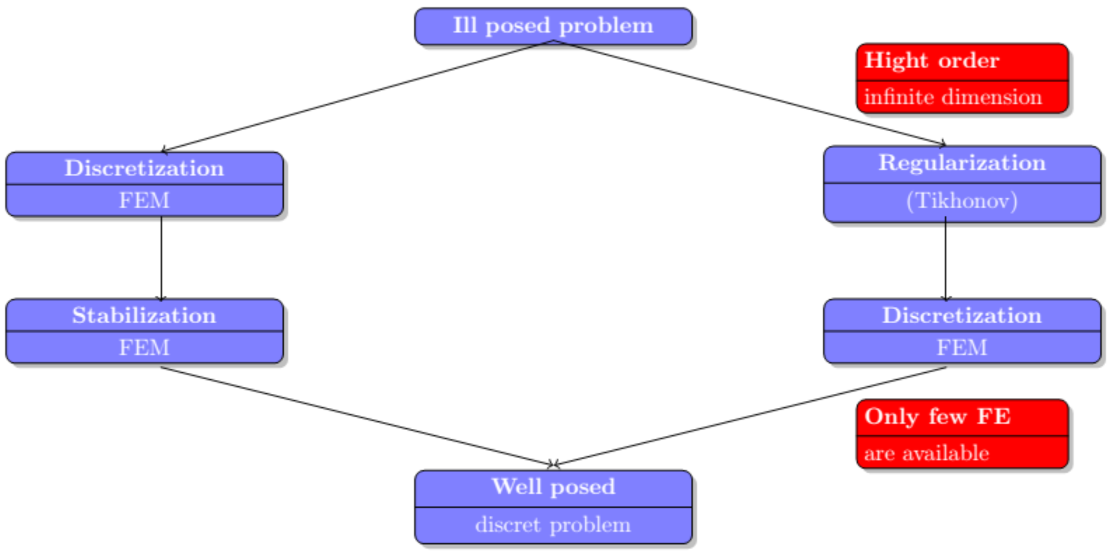
where $\lambda_n = \sqrt{a^{-2}n^2\pi^2 - k^2}$ and the coefficients A_n and B_n are given by

$$A_n = \frac{2}{a}\int_0^a f(x) \sin\frac{n\pi}{a}x\ dx \quad \text{and} \quad B_n = \frac{2}{a}\int_0^a g(x) \sin\frac{n\pi}{a}x\ dx.$$

Since the estimate $\|u\|_{L^2(\Omega)} \le C(\|f\|_{L^2(0,a)} + \|g\|_{L^2(0,a)})$ can not hold in general, the requirement that the solution depends continuously on the data does not hold. This estimate can not hold for any reasonable choice of f and g (as in the above example).

What are the methods for solving an ill-posed problem?

In general to solve an ill-posed problems there is two techniques as in this schema



For the regularisation or Tikhonov regularization methods we refer the reader to reference [7] by using the quasi-reversibility method.
Routhly speaking, the quasi-reversibility method was proposed to solve the Cauchy problem for elliptic equations, it consists of transforming the ill-posed second-order initial problem into a family (depending on a small parameter ε) of fourth-order problems.
We consider the following problem for Laplace's equation. We seek to determine u satisfying

$$\begin{cases} \Delta u = 0, & \text{in } \mathit{\Omega} \\ u = \psi_0 & \text{on } \mathit{\Gamma} \\ \partial_n u = \psi_1 & \text{on } \mathit{\Gamma} \end{cases} \tag{1.9}$$

Where $\Omega \subset \mathbb{R}^n \quad (n = 2, 3)$ and $\mathit{\Gamma} \subset \partial\Omega,\ \psi_0 \in H^{\frac{1}{2}}(\mathit{\Gamma})$ and $\psi_1 \in H^{-\frac{1}{2}}(\mathit{\Gamma})$.
In the method of quasi-reversibility we replace the problem (1.9) by the following:

find u_ε such that

$$\begin{cases} \Delta(\Delta u_\varepsilon) - \varepsilon\Delta u_\varepsilon + \varepsilon u_\varepsilon = 0, & \text{in } \mathit{\Omega} \\ u_\varepsilon = \psi_0 & \text{on } \mathit{\Gamma} \\ \partial_n u_\varepsilon = \psi_1 & \text{on } \mathit{\Gamma} \end{cases} \tag{1.10}$$

where ε is a small positive parameter. For more detail see [7], [8] and [20]. The main point of this method is that (1.10) is well-posed problem and that u_ε converge to u as ε tends to 0.

Remark 1.2.5 In (1.10)

▶ There is an additional parameter ε.

▶ We have fourth-order problem.

So in our study we will use the other method (discretization-stabilization).

Chapter 2

Conditional stability for the ill-posed elliptic Cauchy problem

The main objective of this chapter is to explain and to prove the conditional stability for the ill-posed Cauchy problem for Laplace operator.

2.1 Some inequalities

Let us now focus on some inequalities which can be applied to the estimation of stability. There are two families of such inequalities:

- Carleman[1] estimates,
- Three-spheres inequalities.

Both types have been successfully used in the study of stability, and they are strictly intertwined, in fact three-spheres inequalities can be deduced by Carleman estimates. (see [1])

[1]Torsten Carleman (1892-1949) Swedish mathematician.

2.1.1 Carleman estimates

Lemma 2.1.1 Let $w \in H^2_{loc}(\mathbb{R}^N), (N \geq 2)$ such that [2]

$$\text{supp } w \subset \bar{D}_{\epsilon,r_0} \;=\; \{x \in \mathbb{R}^N \;;\; \epsilon \leq |x| \leq r_0\}$$

we then have

$$\int_{D_{\epsilon,r_0}} x \cdot \Delta w \nabla w \; dx \;=\; (\frac{N}{2} - 1) \int_{D_{\epsilon,r_0}} |\nabla w|^2 \; dx \tag{2.1}$$

$$\int_{D_{\epsilon,r_0}} x \cdot w \nabla w \; \frac{dx}{|x|^\gamma} \;=\; \frac{\gamma - N}{2} \int_{D_{\epsilon,r_0}} |w|^2 \; \frac{dx}{|x|^\gamma} \tag{2.2}$$

Proof. By density it is enough to take $w \in \mathcal{D}(D_{\epsilon,r_0})$, let ϕ be a function defined by

$$\phi(x) = x \; \frac{|\nabla w|^2}{2}$$

it is clear that

$$\int_{\partial D_{\epsilon,r_0}} \phi \cdot n \; ds \;=\; 0$$

by Stokes formula we get

$$\int_{D_{\epsilon,r_0}} \text{div}(\phi) \; dx \;=\int_{\partial D_{\epsilon,r_0}} \phi \cdot n \; ds \;=\; 0 \tag{2.3}$$

From this

$$\begin{aligned} \int_{D_{\epsilon,r_0}} \text{div}(\phi) \; dx \;&=\; \int_{D_{\epsilon,r_0}} \left(\frac{N}{2} |\nabla w|^2 + x \cdot \nabla \left(\frac{|\nabla w|^2}{2} \right) \right) \; dx \\ &=\; \int_{D_{\epsilon,r_0}} \left(\frac{N}{2} |\nabla w|^2 + x \cdot D^2 w \nabla w \right) \; dx = \; 0 \end{aligned} \tag{2.4}$$

Otherwise

$$\int_{D_{\epsilon,r_0}} x \cdot D^2 w \nabla w \; dx \;=\; \sum_{i=1}^N \int_{D_{\epsilon,r_0}} x_i \sum_{j=1}^N \frac{\partial^2 w}{\partial x_i \partial x_j} \frac{\partial w}{\partial x_j} \; dx$$

[2] supp $v := \overline{\{x \in \mathbb{R}^N, \; v(x) \neq 0\}}$.

and

$$\begin{aligned}
\sum_{j=1}^{N}\int_{D_{\epsilon,r_0}} x_i\frac{\partial^2 w}{\partial x_i\partial x_j}\frac{\partial w}{\partial x_j}\ dx &= \int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\frac{\partial^2 w}{\partial x_i^2}\ dx\ +\ \sum_{j\neq i}\int_{D_{\epsilon,r_0}} x_i\frac{\partial^2 w}{\partial x_i\partial x_j}\frac{\partial w}{\partial x_j}\ dx \\
&= \int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\frac{\partial^2 w}{\partial x_i^2}\ dx\ -\ \sum_{j\neq i}\int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\frac{\partial^2 w}{\partial x_j^2}\ dx \\
&= 2\int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\frac{\partial^2 w}{\partial x_i^2}\ dx\ -\ \sum_{j=1}^{N}\int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\frac{\partial^2 w}{\partial x_j^2}\ dx \\
&= 2\int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\frac{\partial^2 w}{\partial x_i^2}\ dx\ -\ \int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\Delta w\ dx
\end{aligned}$$

then

$$\begin{aligned}
\int_{D_{\epsilon,r_0}} x\cdot D^2 w\nabla w\ dx &= \sum_{i=1}^{N}\left(2\int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\frac{\partial^2 w}{\partial x_i^2}\ dx\ -\ \int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\Delta w\ dx\right) \\
&= 2\sum_{i=1}^{N}\int_{D_{\epsilon,r_0}} x_i\frac{\partial w}{\partial x_i}\frac{\partial^2 w}{\partial x_i^2}\ dx\ -\ \int_{D_{\epsilon,r_0}} x\cdot\nabla w\Delta w\ dx \\
&= -\int_{D_{\epsilon,r_0}} |\nabla w|^2\ dx\ -\ \int_{D_{\epsilon,r_0}} x\cdot\nabla w\Delta w\ dx \qquad (2.5)
\end{aligned}$$

by (2.4) and (2.5), we find that

$$\int_{D_{\epsilon,r_0}} x\cdot\Delta w\nabla w\ dx\ =\ (\frac{N}{2}-1)\int_{D_{\epsilon,r_0}} |\nabla w|^2\ dx$$

as required.

To get the second formula 2.2, we use 2.3 with

$$\phi(x) = x\ \frac{w^2}{2|x|^\gamma}$$

and therefore

$$\begin{aligned}
\int_{D_{\epsilon,r_0}} \mathrm{div}(\phi)\ dx &= \int_{D_{\epsilon,r_0}} \left(\frac{N}{2|x|^\gamma}w^2 + x\cdot\nabla\left(\frac{w^2}{2|x|^\gamma}\right)\right)\ dx \\
&= \frac{N}{2}\int_{D_{\epsilon,r_0}} \frac{w^2}{|x|^\gamma}\ dx\ +\ \int_{D_{\epsilon,r_0}} x\cdot\nabla\left(\frac{w^2}{2|x|^\gamma}\right)\ dx\ =\ 0. \qquad (2.6)
\end{aligned}$$

Otherwise

$$\int_{D_{\epsilon,r_0}} x\cdot\nabla\left(\frac{w^2}{2|x|^\gamma}\right)\,dx \;=\; \int_{D_{\epsilon,r_0}} x\cdot\frac{\nabla(w^2)}{2|x|^\gamma}\,dx \;+\; \int_{D_{\epsilon,r_0}} x\cdot\frac{w^2}{2}\nabla(|x|^{-\gamma})\,dx$$
$$= \int_{D_{\epsilon,r_0}} x\cdot w\nabla w\frac{dx}{|x|^\gamma} \;+\; \int_{D_{\epsilon,r_0}} x\cdot\frac{-\gamma w^2}{2|x|^{\gamma+2}}x\,dx$$
$$= \int_{D_{\epsilon,r_0}} x\cdot w\nabla w\frac{dx}{|x|^\gamma} \;-\; \frac{\gamma}{2}\int_{D_{\epsilon,r_0}} w^2\frac{dx}{|x|^\gamma} \tag{2.7}$$

by (2.6) and (2.7), we find the required formula. ■

Theorem 2.1.2 (Carleman estimates) [14]

Let $u\in H^2_{loc}(\mathbb{R}^N)(N\geq 2)$ satisfying:

$$\text{supp } u\subset \bar{D}_{\epsilon,r_0} \;=\; \{x\in\mathbb{R}^N\;;\;\epsilon\leq|x|\leq r_0\}$$

with $0<\epsilon<r_0\leq 1$. Then there exists a constant $\beta_0>0$ and there exists C_0, depending on β_0 only, then for every $\beta,\ (\beta>\beta_0)$ we have

$$\int_{D_{\epsilon,r_0}} |u|^2\exp(\frac{2}{|x|^\beta})\,\frac{dx}{|x|^{2\beta+2}} \;\leq\; \frac{C_0}{\beta^4}\int_{D_{\epsilon,r_0}} |\Delta u|^2\exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\,dx \tag{2.8}$$

Proof. Let $\psi_\beta(x)=\exp(\frac{1}{|x|^\beta})$ and $\phi_\beta(x)=\exp(-\frac{1}{|x|^\beta})$. We define a function w by

$$w=\psi_\beta u,$$

then $u=\phi_\beta w$. Note that:

$$\Delta u=\phi_\beta\Delta w+2\nabla\phi_\beta\cdot\nabla w+\Delta\phi_\beta w$$

with $\nabla\phi_\beta=\frac{\beta x}{|x|^{\beta+2}}\,\phi_\beta$ and $\Delta\phi_\beta=\frac{\beta}{|x|^{\beta+2}}(N-2-\beta+\frac{\beta}{|x|^\beta})\,\phi_\beta$, then

$$\begin{aligned}|\Delta u|^2 &= |\phi_\beta\Delta w+2\nabla\phi_\beta\cdot\nabla w+\Delta\phi_\beta w|^2\\ &= \phi_\beta^2|\Delta w|^2+4|\nabla\phi_\beta\cdot\nabla w|^2+(\Delta\phi_\beta)^2|w|^2+2\phi_\beta\Delta\phi_\beta\Delta w w\\ &\quad+4\phi_\beta\nabla\phi_\beta\cdot\Delta w\nabla w+4\Delta\phi_\beta\nabla\phi_\beta\cdot w\nabla w.\end{aligned}$$

In particular, we have:

$$|\Delta u|^2\geq 4\frac{\beta\phi_\beta^2}{|x|^{\beta+2}}x\cdot\Delta w\nabla w+4\frac{\beta^2\phi_\beta^2}{|x|^{2\beta+4}}\left(N-2-\beta+\frac{\beta}{|x|^\beta}\right)x\cdot w\nabla w$$

so

$$
\begin{aligned}
|\Delta u|^2\psi_\beta^2|x|^{\beta+2} &\geq 4\beta x\cdot\Delta w\nabla w \\
&+\frac{4\beta^2}{|x|^{\beta+2}}(N-2-\beta)x\cdot w\nabla w+\frac{4\beta^3}{|x|^{2\beta+2}}x\cdot w\nabla w.
\end{aligned}
$$

According to density, we assume that $u\in\mathcal{D}(D_{\epsilon,r_0})$.

Integrating over D_{ϵ,r_0} we obtain

$$
\begin{aligned}
\int_{D_{\epsilon,r_0}}|\Delta u|^2\psi_\beta^2|x|^{\beta+2}\,dx &\geq 4\beta\int_{D_{\epsilon,r_0}}x\cdot\Delta w\nabla w\,dx \\
&+4\beta^2(N-2-\beta)\int_{D_{\epsilon,r_0}}x\cdot w\nabla w\,\frac{dx}{|x|^{\beta+2}}+4\beta^3\int_{D_{\epsilon,r_0}}x\cdot w\nabla w\,\frac{dx}{|x|^{2\beta+2}}.
\end{aligned}
$$

Since $w=\psi_\beta u$, we can deduce by Lemma 2.1.1 that

$$
\begin{aligned}
\int_{D_{\epsilon,r_0}}|\Delta u|^2\psi_\beta^2|x|^{\beta+2}\,dx &\geq 2\beta(N-2)\int_{D_{\epsilon,r_0}}|\nabla(\psi_\beta u)|^2 \\
&\quad+2\beta^3(2\beta+2-N)\int_{D_{\epsilon,r_0}}|u|^2\psi_\beta^2\frac{dx}{|x|^{2\beta+2}} \\
&\quad-2\beta^2(N-2-\beta)^2\int_{D_{\epsilon,r_0}}|u|^2\psi_\beta^2\frac{dx}{|x|^{\beta+2}} \\
&\geq 2\beta^3(2\beta+2-N)\int_{D_{\epsilon,r_0}}|u|^2\psi_\beta^2\frac{dx}{|x|^{2\beta+2}} \\
&\quad-2\beta^2(N-2-\beta)^2\int_{D_{\epsilon,r_0}}|u|^2\psi_\beta^2\frac{dx}{|x|^{\beta+2}}\quad(\text{ because } N\geq 2)
\end{aligned}
$$

As supp $u\subset\bar{D}_{\epsilon,r_0}$ and $|x|\longmapsto|x|^\beta$ a increasing function, then

$$
\int_{D_{\epsilon,r_0}}|u|^2\psi_\beta^2\frac{dx}{|x|^{\beta+2}}\leq r_0^\beta\int_{D_{\epsilon,r_0}}|u|^2\psi_\beta^2\frac{dx}{|x|^{2\beta+2}}
$$

Therefore

$$
\int_{D_{\epsilon,r_0}}|\Delta u|^2\psi_\beta^2|x|^{\beta+2}\,dx\geq\beta^4F(\beta)\int_{D_{\epsilon,r_0}}|u|^2\psi_\beta^2\frac{dx}{|x|^{2\beta+2}}
$$

where

$$
F(\beta)\;=\;\frac{4\beta^4-2\beta^3(N-2)-2r_0^\beta\beta^2(N-2-\beta)^2}{\beta^4}
$$

for $r_0<1$, we get $\lim\limits_{\beta\to+\infty}F(\beta)=4$. Hence we take $\beta>\beta_0$ with $F(\beta_0)\geq 1/C_0$ to obtain the result. ■

2.1.2 The uniqueness in Cauchy's problem for elliptic equations

We can generalize the Holmgren theorem in the case of constant coefficients and the Cauchy-Kowalewsky theorem in the case of the analytic coefficients, to prove the uniqueness of solution for the elliptic Cauchy problem by using of Carleman estimates.

Lemma 2.1.3 Let $\epsilon < r_0/2 < 1/2$ and $u \in H^2_{loc}(\mathbb{R}^N)$, such that $u = 0$ in $B(0,\epsilon)$. We assume that

$$|\Delta u| \leq C|u| \text{ almost everywhere in } B(0,r_0).$$

Then $u \equiv 0$ almost everywhere in $B(0,r_0/2)$.

Proof. Let φ cut-off function belong to $C_0^\infty(\mathbb{R}^N)$, $supp\,\varphi \subset B(0,r_0)$ and $\varphi(x) = 1$ in $B(0,r_0/2)$. By application of the Carleman estimates to $\varphi\, u$, we get

$$\int_{B(0,r_0)} \frac{|\varphi|^2|u|^2}{|x|^{2\beta+2}} \exp(\frac{2}{|x|^\beta})\, dx \;\leq\; \frac{C_0}{\beta^4}\int_{B(0,r_0)} |\Delta(\varphi u)|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\, dx$$

In particular

$$\begin{aligned}
\int_{B(0,r_0/2)} \frac{|u|^2}{|x|^{2\beta+2}} \exp(\frac{2}{|x|^\beta})\, dx \;&\leq\; \frac{C_0}{\beta^4}\int_{B(0,r_0)} |\Delta(\varphi u)|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\, dx \\
&= \frac{C_0}{\beta^4}\int_{B(0,r_0/2)} |\Delta u|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\, dx \\
&\quad + \frac{C_0}{\beta^4}\int_{D_{r_0/2,r_0}} |\Delta(\varphi u)|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\, dx
\end{aligned}$$

since $|\Delta u| \leq C|u|$, we have

$$\begin{aligned}
\int_{B(0,r_0/2)} \frac{|u|^2}{|x|^{2\beta+2}} \exp(\frac{2}{|x|^\beta})\, dx \;&\leq\; \frac{C^2C_0}{\beta^4}\int_{B(0,r_0/2)} |u|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\, dx \\
&\quad + \frac{C_0}{\beta^4}\int_{D_{r_0/2,r_0}} |\Delta(\varphi u)|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\, dx \\
&\leq \frac{C^2C_0}{\beta^4}(\frac{r_0}{2})^{3\beta+4}\int_{B(0,r_0/2)} |u|^2 \exp(\frac{2}{|x|^\beta})\; \frac{dx}{|x|^{2\beta+2}} \\
&\quad + \frac{C_0}{\beta^4} r_0^{3\beta+4}\int_{D_{r_0/2,r_0}} |\Delta(\varphi u)|^2 \exp(\frac{2}{|x|^\beta})\; \frac{dx}{|x|^{2\beta+2}}
\end{aligned}$$

we choose β so large that $C^2C_0(r_0/2)^{3\beta+4}/\beta^4 < 1$, to obtain

$$\left(1-\frac{C^2C_0}{\beta^4}(\frac{r_0}{2})^{3\beta+4}\right)\int_{B(0,r_0/2)}|u|^2\exp(\frac{2}{|x|^\beta})\ \frac{dx}{|x|^{2\beta+2}} \le \frac{C_0}{\beta^4}r_0^{3\beta+4}\int_{D_{r_0/2,r_0}}|\Delta(\varphi u)|^2\exp(\frac{2}{|x|^\beta})\ \frac{dx}{|x|^{2\beta+2}}$$

and $|x| \longmapsto \exp(2/|x|^\beta)/|x|^{2\beta+2}$ a decreasing function, then

$$\begin{aligned}\int_{B(0,r_0/2)}|u|^2\exp(\frac{2}{|x|^\beta})\ \frac{dx}{|x|^{2\beta+2}} &\ \ge\ C_1\int_{B(0,r_0/2)}|u|^2\ dx\\ \int_{D_{r_0/2,r_0}}|\Delta(\varphi u)|^2\exp(\frac{2}{|x|^\beta})\ \frac{dx}{|x|^{2\beta+2}} &\ \le\ C_1\int_{D_{r_0/2,r_0}}|\Delta(\varphi u)|^2\ dx\end{aligned}$$

where $C_1 = \exp(2^{\beta+1}/r_0^\beta)(2/r_0)^{2\beta+2}$. So

$$\int_{B(0,r_0/2)}|u|^2\ dx\ \le\ \frac{C_0r_0^{3\beta+4}}{\beta^4 - C^2C_0(r_0/2)^{3\beta+4}}\int_{D_{r_0/2,r_0}}|\Delta(\varphi u)|^2\ dx$$

letting $\beta \longrightarrow +\infty$, we obtain

$$\int_{B(0,r_0/2)}|u|^2\ dx\ = 0.$$

Namely that $u \equiv 0$ in $B(0, r_0/2)$. ■

Theorem 2.1.4 Let $\mathcal{O}$ be a connected open set in $\mathbb{R}^N$ and $u \in H^2_{loc}(\mathbb{R}^N)$ satisfying

i. $\exists\ x_0 \in \mathcal{O},\ \exists\ \epsilon > 0,\ \ u(x) = 0$ in $B(x_0, \epsilon)$

ii. $\exists\ C > 0,$ for almost all $x \in \mathcal{O},\ \ |\Delta u| \le C|u|$

then $u(x) = 0,\ \ \forall\ x \in \mathcal{O}$.

Proof. Let $x \in \mathcal{O}$. Since $\mathcal{O}$ is connected, we can find $r_0 < 1$ and a sequence of n points $x_0, x_1, ..., x_n = x$ such that

$$\forall j \le n,\ B(x_j, r_0) \subset \mathcal{O}.$$

Moreover

$$\forall j \le (n-1),\ |x_{j+1} - x_j| < r_0/2$$

since $u = 0$ in a neighborhood of x_0, by Lemma 2.1.3, $u = 0$ in $B(x_0, r_0/2)$ and in particular in a neighborhood of x_1. By recurrence, if $u = 0$ in a neighborhood of x_j, then $u = 0$ in a neighborhood of x_{j+1}. And thus we find the result. ■

Applying Theorem 2.1.4, we obtain the uniqueness of solution for the Cauchy problem associated with the equation

$$\Delta u + Mu = f.$$

Theorem 2.1.5 Let $\mathcal{O}$ be a connected open set in $\mathbb{R}^N$. We assume that there exists $x_0 \in \partial\mathcal{O}$ and $\epsilon > 0$ such that $\Gamma_\epsilon(x_0) = \partial\mathcal{O} \cap B(x_0, \epsilon)$ sufficiently regular.
Let $u \in H^2_{loc}(\mathbb{R}^N)$ be a solution in the sense of distributions to the equation

$$\Delta u + Mu = 0$$

such that

$$u_{|\Gamma_\epsilon(x_0)} = 0 \text{ and } \frac{\partial u}{\partial n}\Big|_{\Gamma_\epsilon(x_0)} = 0$$

then $u \equiv 0$ in $\mathcal{O}$.

Proof. Let $\mathcal{O}_\epsilon = \mathcal{O} \cup B(x_0, \epsilon)$ and u_ϵ be a function defined by

$$u_\epsilon = \begin{cases} u & \text{in} \quad \mathcal{O} \\ 0 & \text{in} \quad \mathcal{O}_\epsilon - \mathcal{O} \end{cases}$$

it is clear that u_ϵ belong to $H^2_{loc}(\mathcal{O}_\epsilon)$. Moreover,

$$\Delta u_\epsilon = \begin{cases} \Delta u & \text{in} \quad \mathcal{O} \\ 0 & \text{in} \quad \mathcal{O}_\epsilon - \mathcal{O} \end{cases}$$

Therefore,

$$\Delta u_\epsilon + u_\epsilon = 0 \text{ in } L^2_{loc}(\mathcal{O}_\epsilon)$$

by Theorem 2.1.4, we deduce that $u_\epsilon \equiv 0$ in $\mathcal{O}_\epsilon$. ■

2.1.3 A three-spheres inequality

Theorem 2.1.6 (Homogeneous equation)
Let u be a solution of $\Delta u + Mu = 0$ in $B(0, R)$ and let $C_0,\ 0 < C_0 \leq 1$ for every r_1, r_2, r_3 with

$$0 < 2r_0 = r_1 < r_2 < r_3 < R_0, \quad R_0 = \min\{R, C_0\rho_0\}$$

then $\exists\ C > 0,\ \alpha \in (0, 1)$ such that

$$\|u\|_{L^2(B_{r_2})} \leq C\|u\|^{\alpha}_{L^2(B_{r_1})}\|u\|^{1-\alpha}_{L^2(B_{r_3})} \tag{2.9}$$

Proof. [17] Let $\varphi \in C_0^\infty(r_3)$ defined by

$$\varphi(x) = \begin{cases} 1 & \text{in} \quad \mathrm{I}_2 \\ 0 & \text{in} \quad \mathrm{I}_0 \cup \mathrm{I}_4 \end{cases}$$

such that

$$\mathrm{I}_1 = \left\{ r_0 \le |x| < \frac{3}{2} r_0 \right\}, \quad \mathrm{I}_2 = \left\{ \frac{3}{2} r_0 \le |x| < \frac{1}{2} r_3 \right\}, \quad \mathrm{I}_3 = \left\{ \frac{1}{2} r_3 \le |x| \le \frac{3}{4} r_3 \right\}$$

$$\mathrm{I}_0 = \{ 0 < |x| < r_0 \}, \quad \mathrm{I}_4 = \left\{ \frac{3}{4} r_3 < |x| \le r_3 \right\}.$$

We assume that

$\bullet |\nabla \varphi| + r_0 |\Delta \varphi| \le c/r_0$ on $[0, 3/2 r_0] = \mathrm{I}_0 \cup \mathrm{I}_1$,

$\bullet |\nabla \varphi| + r_3 |\Delta \varphi| \le c/r_3$ on I_3.

By application of the Carleman estimates to φu, we get

$$\begin{aligned} \int |\varphi|^2 |u|^2 \exp(\frac{2}{|x|^\beta}) \ \frac{dx}{|x|^{2\beta+2}} \ &\le \ \frac{C_0}{\beta^4} \int |\Delta(\varphi u)|^2 \exp(\frac{2}{|x|^\beta}) |x|^{\beta+2} \ dx \\ \Longrightarrow \int_{\mathrm{I}_2} |u|^2 \exp(\frac{2}{|x|^\beta}) \ \frac{dx}{|x|^{2\beta+2}} \ &\le \ \frac{C_0}{\beta^4} \int_{\mathrm{I}_2} |\Delta u|^2 \exp(\frac{2}{|x|^\beta}) |x|^{\beta+2} \ dx \ + \frac{1}{\beta^4} J \\ \Longrightarrow \beta^4 \int_{\mathrm{I}_2} |u|^2 \exp(\frac{2}{|x|^\beta}) \ \frac{dx}{|x|^{2\beta+2}} \ &\le \ C_0 \int_{\mathrm{I}_2} |\Delta u|^2 \exp(\frac{2}{|x|^\beta}) |x|^{\beta+2} \ dx \ + J \end{aligned}$$

with

$$J = C_0 \int_{\mathrm{I}_1 \cup \mathrm{I}_3} |\Delta(\varphi u)|^2 \exp(\frac{2}{|x|^\beta}) |x|^{\beta+2} \ dx$$

then

$$\begin{aligned} \beta^4 \int_{\mathrm{I}_2} |u|^2 \exp(\frac{2}{|x|^\beta}) \ \frac{dx}{|x|^{2\beta+2}} \ &\le \ C_0 M^2 \int_{\mathrm{I}_2} |u|^2 \exp(\frac{2}{|x|^\beta}) |x|^{\beta+2} \ dx \ + J \\ &\le C_0 M^2 (1/2 r_3)^{3\beta+4} \int_{\mathrm{I}_2} |u|^2 \exp(\frac{2}{|x|^\beta}) \ \frac{dx}{|x|^{2\beta+2}} + J \end{aligned}$$

If we choose $\beta^4 > C_0 M^2 (1/2 r_3)^{3\beta+4}$, then

$$\int_{\mathrm{I}_2} |u|^2 \exp(\frac{2}{|x|^\beta}) \ \frac{dx}{|x|^{2\beta+2}} \le J$$

Next we estimate J :

we have

$$\begin{aligned}|\Delta(\varphi u)| &\leq |\Delta\varphi||u| + 2|\nabla\varphi||\nabla u| + |\Delta u||\varphi| \\ &\leq |\Delta\varphi||u| + 2|\nabla\varphi||\nabla u| + M|u|\end{aligned}$$

then

$$\begin{cases}|\Delta(\varphi u)| \leq (M + c/r_0^2)|u| + 2c/r_0|\nabla u| \quad \text{in } \mathrm{I}_1, \\ \\ |\Delta(\varphi u)| \leq (M + c/r_3^2)|u| + 2c/r_3|\nabla u| \quad \text{in } \mathrm{I}_3.\end{cases}$$

And so

$$\begin{aligned}J &\leq C_0 \int_{\mathrm{I}_1} \big((M + c/r_0^2)|u| + 2c/r_0|\nabla u|\big)^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\ dx \\ &\quad + C_0 \int_{\mathrm{I}_3} \big((M + c/r_3^2)|u| + 2c/r_3|\nabla u|\big)^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\ dx \\ &\leq C_0 \int_{\mathrm{I}_1} \big((M + c/r_0^2)^2|u|^2 + c/r_0^2|\nabla u|^2\big) \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\ dx \\ &\quad + C_0 \int_{\mathrm{I}_3} \big((M + c/r_3^2)^2|u|^2 + 2c/r_3^2|\nabla u|^2\big) \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\ dx \\ &\leq C_0(M^2 + c/r_0^4) \int_{\mathrm{I}_1} |u|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\ dx\ + CC_0/r_0^2 \int_{\mathrm{I}_1} |\nabla u|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\ dx \\ &\quad + C_0(M^2 + c/r_3^4) \int_{\mathrm{I}_3} |u|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\ dx\ + CC_0/r_3^2 \int_{\mathrm{I}_3} |\nabla u|^2 \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}\ dx \\ &\leq C_0(M^2 + c/r_0^4)m(r_0) \int_{\mathrm{I}_1} |u|^2\ dx\ + CC_0/r_0^2 m(r_0) \int_{\mathrm{I}_1} |\nabla u|^2\ dx \\ &\quad + C_0(M^2 + c/r_3^4)m(r_3/2) \int_{\mathrm{I}_3} |u|^2\ dx\ + CC_0/r_3^2 m(r_3/2) \int_{\mathrm{I}_3} |\nabla u|^2\ dx\end{aligned}$$

where $m(x) = \exp(\frac{2}{|x|^\beta})|x|^{\beta+2}$.

Let $\mathrm{I}_5 = \{x \in \mathrm{I}_2 : |x| \leq r_2\}$, and therefore

$$\int_{\mathrm{I}_5} |u|^2 \exp(\frac{2}{|x|^\beta})\ \frac{dx}{|x|^{2\beta+2}} \leq \int_{\mathrm{I}_2} |u|^2 \exp(\frac{2}{|x|^\beta})\ \frac{dx}{|x|^{2\beta+2}} \leq J$$

$$\Longrightarrow \exp(\frac{2}{r_2^\beta})/r_2^{2\beta+2} \int_{\mathrm{I}_5} |u|^2\ dx\ \leq \int_{\mathrm{I}_5} |u|^2 \exp(\frac{2}{|x|^\beta})\ \frac{dx}{|x|^{2\beta+2}}\ \leq J$$

$$\Longrightarrow \int_{\mathrm{I}_5} |u|^2 \ dx \ \leq h(r_2)J, \quad \text{with } h(r_2) = r_2^{2\beta+2}/\exp(\frac{2}{r_2^\beta})$$

then

$$\int_{\mathrm{I}_5} |u|^2 \ dx \ \leq h(r_2)C_0(M^2 + c/r_0^4)m(r_0)\int_{\mathrm{I}_1} |u|^2 \ dx + h(r_2)CC_0/r_0^2 m(r_0)\int_{\mathrm{I}_1} |\nabla u|^2 \ dx$$
$$+ h(r_2)C_0(M^2 + c/r_3^4)m(r_3/2)\int_{\mathrm{I}_3} |u|^2 \ dx \ + h(r_2)CC_0/r_3^2 m(r_3/2)\int_{\mathrm{I}_3} |\nabla u|^2 \ dx \tag{2.10}$$

by Caccippoli estimates, we obtain

$$\int_{\mathrm{I}_1} |\nabla u|^2 \ dx \ \leq (M + c/r_0^2) \int\limits_{B_{2r_0}\setminus B_{r_0/2}} |u|^2 \ dx$$

$$\int_{\mathrm{I}_3} |\nabla u|^2 \ dx \ \leq (M + c/r_3^2) \int\limits_{B_{r_3}\setminus B_{r_3/4}} |u|^2 \ dx.$$

We add $\int\limits_{|x|<3/2r_0} |u|^2 \ dx$ to (2.10), and we use Caccippoli inequality to obtain:

$$\begin{aligned}
\int\limits_{|x|<r_2} |u|^2 \ dx \ \leq\ & \frac{r_2^{2\beta+2}}{\exp(\frac{2}{r_2^\beta})} C_0(M^2 + C/r_0^4)\exp(\frac{2}{r_0^\beta})r_0^{\beta+2}\int_{\mathrm{I}_1} |u|^2 \ dx \\
& + \frac{r_2^{2\beta+2}}{\exp(\frac{2}{r_2^\beta})} CC_0/r_0^2\exp(\frac{2}{r_0^\beta})r_0^{\beta+2}(M^2 + C/r_0^2) \int\limits_{B_{2r_0}\setminus B_{r_0/2}} |u|^2 \ dx \\
& + \frac{r_2^{2\beta+2}}{\exp(\frac{2}{r_2^\beta})} C_0(M^2 + C/r_3^4)\exp(\frac{2^{\beta+1}}{r_3^\beta})r_3^{\beta+2}/2^{\beta+2}\int_{\mathrm{I}_3} |u|^2 \ dx \\
& + \frac{r_2^{2\beta+2}}{\exp(\frac{2}{r_2^\beta})} CC_0/r_3^2\exp(\frac{2^{\beta+1}}{r_3^\beta})r_3^{\beta+2}/2^{\beta+2}(M + C/r_3^2) \int\limits_{B_{r_3}\setminus B_{r_3/4}} |u|^2 \ dx \\
\leq\ & r_2^{2\beta}CC_0\left(M^2r_0^2 + \frac{1}{r_0^2}\right)r_2^2\exp(\frac{2}{r_0^\beta})r_0^\beta\int_{B_{2r_0}} |u|^2 \ dx \\
& + r_2^{2\beta}CC_0\left(M^2r_3^2 + \frac{1}{r_3^2}\right)r_2^2\exp(\frac{2^{\beta+1}}{r_3^\beta})r_3^\beta\int_{B_{r_3}} |u|^2 \ dx
\end{aligned}$$

We define $a, \ n_1, \ n_3$ by

$$a^2 = CC_0, \ n_1^2 = \left(M^2r_0^2 + \frac{1}{r_0^2}\right)r_2^2\|u\|_{L^2(B_{2r_0})}, \ n_3^2 = \left(M^2r_3^2 + \frac{1}{r_3^2}\right)r_2^2\|u\|_{L^2(B_{r_3})}$$

we then have

$$\begin{aligned}
\|u\|_{L^2(B_{r_2})} &\leq r_2^{2\beta} a^2 n_1^2 \exp(\frac{2}{r_0^\beta}) r_0^\beta + r_2^{2\beta} a^2 n_3^2 \exp(\frac{2}{r_0^\beta}) r_0^\beta \\
&\leq 2 r_2^{2\beta} r_0^\beta a^2 n_3^2 \exp(\frac{2}{r_0}) \\
&\leq 2A^2 r_2^{2\beta} r_0^\beta n_3^2 \\
&\leq 2A^2 r_2^{4\beta} n_3^2
\end{aligned}$$

we choose

$$\beta = \frac{\alpha}{2\log r_2} \log(n_1/n_3)$$

and therefore

$$\begin{aligned}
\|u\|_{L^2(B_{r_2})} &\leq 2A^2 r_2^{\frac{4\alpha}{2\log r_2}\log(n_1/n_3)} n_3^2 \\
&= 2A^2 r_2^{\frac{\log(n_1/n_3)^{2\alpha}}{\log r_2}} n_3^2 \\
&= 2A^2 (n_1/n_3)^{2\alpha} n_3^2 \\
&= 2A^2 n_1^{2\alpha} n_3^{2(1-\alpha)}
\end{aligned}$$

where

$$\frac{1}{\alpha} = 1 + \frac{\log r_1}{\log r_2}$$

then $\alpha \in (0, 1)$. ■

Theorem 2.1.7 (Complete equation)

Let u be a solution of $\Delta u + Mu = f$ in $B(0, R)$ and if the hypothesis given in Theorem (2.1.6) hold, then for every r_1, r_2, r_3 with

$$0 < 2r_0 = r_1 < r_2 < r_3 < R_0,$$

$\exists\ C > 0,\ \alpha\ \in (0, 1)$ such that

$$\|u\|_{L^2(B_{r_2})} \ \leq\ C(\|u\|_{L^2(B_{r_1})} + \varepsilon)^\alpha (\|u\|_{L^2(B_{r_3})} + \varepsilon)^{1-\alpha} \tag{2.11}$$

Proof. Let us consider the unique solution u_0 to

$$\begin{cases} \Delta u_0 + M u_0 = f & \text{in } B_R \\ u_0 = 0 & \text{on } \partial B_R \end{cases}$$

we have that

$$\|u_0\|_{L^2(B_R)} \leq C\|f\|_{L^2(\mathbb{R}^n)}.$$

Note that

$$\begin{aligned} \Delta(u - u_0) + M(u - u_0) &= \Delta u - \Delta u_0 + Mu - Mu_0 \\ &= (\Delta u + Mu) - (\Delta u_0 + Mu_0) \\ &= f - f = 0 \end{aligned}$$

then $u - u_0$ satisfies the hypotheses of Theorem 2.1.6. And thus, we find the required formula. ■

2.1.4 Propagation of smallness

Theorem 2.1.8 (Propagation of smallness in the interior) [1]
Let Ω be a bounded connected open set in $\mathbb{R}^n$ with boundary $\partial\Omega$ of Lipschitz class with constants $\rho_0,\ M_0,$ (see Appendix) and let $B_{r_0}(x_0) \subset \Omega$ be a fixed ball. Let C_0 be as in the thesis of Theorem 2.1.6. Let $h,\ 0 < h < \min\{2C_0\rho_0, \frac{r_0}{2}\}$, be fixed and let $G \subset \Omega$ be a connected open set such that $\mathrm{dist}(G, \partial\Omega) \geq h$ and $B_{\frac{r_0}{2}}(x_0) \subset G$.

Let $u \in H^1_{loc}(\Omega)$ be a solution to the equation $\Delta u + Mu = f,$ in Ω. Let us assume that

$$\exists\, \eta > 0,\ E_0 > 0 \ \text{ st } \ \|u\|_{L^2(B_{r_0}(x_0))} \leq \eta, \quad \|u\|_{L^2(\Omega)} \leq E_0.$$

Then

$$\|u\|_{L^2(G)} \leq C(\eta + \varepsilon)^{\delta}(E_0 + \varepsilon)^{1-\delta}, \tag{2.12}$$

where

$$C = C_1 \left(\frac{|\Omega|}{h^n}\right)^{\frac{1}{2}}$$

and

$$\delta \geq \alpha^{\frac{C_2|\Omega|}{h^n}}$$

with $C_1 > 0$ and $\alpha,\ 0 < \alpha < 1$.

Proof. We shall need uniform three-spheres inequality in a domain slightly larger than G. We can fix radii $r_1,\ r_2,\ r_3$ as follows

$$r_3 = \frac{h}{2}, \quad r_2 = \frac{r_3}{5K} = \frac{h}{10K}, \quad r_1 = \frac{1}{3}r_2 = \frac{h}{30K}, \qquad K \in (0,1).$$

With such a choice the inequality (2.11) applies with $C \geq 1$ and $\alpha,\ 0 < \alpha < 1$.
Let us consider the set G^{r_1} as following

$$G^{r_1} = \left\{ x \in \mathbb{R}^n \mid \operatorname{dist}(x, \overline{G}) < r_1 \right\}.$$

We have that G^{r_1} is a connected open set containing G such that

$$\begin{aligned} h &\leq \operatorname{dist}(G, \partial\Omega) \\ &\leq \operatorname{dist}(G, G^{r_1}) + \operatorname{dist}(G^{r_1}, \partial\Omega) \\ &\leq \operatorname{dist}(G^{r_1}, \partial\Omega) + r_1 \end{aligned}$$

which implies that

$$\operatorname{dist}(G^{r_1}, \partial\Omega) \geq h - r_1 > h - r_3 = h - \frac{h}{2} = \frac{h}{2} = r_3.$$

Since G^{r_1} is a connected open set in $\mathbb{R}^n$, then G^{r_1} is path-connected. And therefore, for every $y \in G^{r_1}$, there exists a continuous path $\gamma : [0,1] \to G^{r_1}$ such that $\gamma(0) = x_0,\ \gamma(1) = y$. Let us define $0 = t_0 < t_1 < \cdots < t_N = 1$, according to the following rule. We set

$$t_{k+1} = \max\{t \mid |\gamma(t) - x_k| = 2r_1\} \text{ if } |x_k - y| > 2r_1, \tag{2.13}$$

otherwise we stop the process and set $N = k+1,\ t_N = 1$ and $|x_{k+1} - x_k| = 2r_1$. Since $r_2 = 3r_1$ we have that $B_{r_1}(x_{k+1}) \subset B_{r_2}(x_k)$ and therefore, by (2.11)

$$\|u\|_{L^2(B_{r_1}(x_{k+1}))} + \varepsilon \ \leq \ C \left(\|u\|_{L^2(B_{r_2}(x_k))} + \varepsilon \right)^{\alpha} (E_0 + \varepsilon)^{1-\alpha}$$

for $k = 0, \cdots, N-1$, where $C \geq 1$ and $\alpha,\ 0 < \alpha < 1$. Denoting

$$m_k = \frac{\|u\|_{L^2(B_{r_2}(x_k))} + \varepsilon}{E_0 + \varepsilon},$$

we then have

$$m_{k+1} \leq C m_k^{\alpha}, \quad \text{for } k = 0, \cdots, N-1$$

and thus

$$m_N \leq C m_{N-1}^{\alpha} \leq C^{1+\alpha} m_{N-2}^{\alpha^2} \leq \cdots \leq \widetilde{C} m_0^{\delta},$$

where $\widetilde{C} = C^{1+\alpha+\cdots+\alpha^N}$ and $\delta = \alpha^N$. Hence we have obtained

$$\|u\|_{L^2(B_{r_1}(y))} \ \leq\ \widetilde{C}\left(\|u\|_{L^2(B_{r_1}(x_0))} + \varepsilon\right)^{\delta}(E_0+\varepsilon)^{1-\delta} \tag{2.14}$$

it is clear that $1+\alpha+\cdots+\alpha^N \leq \dfrac{1}{1-\alpha}$ and $B_{r_1}(x_0), \cdots, B_{r_1}(x_{N-1})$ are pairwise disjoint, by contradiction, we assume that

$$\exists\ z \in B_{r_1}(x_k) \cap B_{r_1}(x_{k+1}) \Longrightarrow \begin{cases} |x_k - z| < r_1 \\ |x_{k+1} - z| < r_1 \end{cases}$$

$$\Longrightarrow |x_{k+1} - z - x_k + z| \leq |x_k - z| + |x_{k+1} - z| < 2r_1 \Longrightarrow |x_{k+1} - x_k| < 2r_1$$

which is a contradiction with (2.13). Then $B_{r_1}(x_k) \cap B_{r_1}(x_{k+1}) = \emptyset \quad \forall\ k = 0, \cdots, N-1$. We have that

$$\widetilde{C} \leq C^{\frac{1}{1-\alpha}} \ \text{ and } \ \delta \geq \alpha^{\frac{C_2|\Omega|}{h^n}} \ \left(\text{because } C \geq 1 \text{ and } N \leq \frac{C_2|\Omega|}{h^n}\right).$$

Let us tessellate $\mathbb{R}^n$ with internally non-overlapping closed cubes of side $l = \dfrac{2r_1}{\sqrt{n}}$ and let $Q_j,\ j = 1, \cdots, J$, be those cubes which intersect G. Clearly, any such cube is contained in a ball of radius r_1 and center $\omega_j \in G^{r_1}$ and $J \leq \dfrac{n^{\frac{n}{2}}|\Omega|}{2^n r_1^n}$. Therefore, from (2.14), we have

$$\begin{aligned}
\int_G u^2 \leq \sum_{j=1}^{J}\int_{Q_j} u^2 \leq \sum_{j=1}^{J}\int_{B_{r_1}(\omega_j)} u^2 &\leq J\widetilde{C}^2\rho_0^n\left(\|u\|_{L^2(B_{r_1}(x_0))} + \varepsilon\right)^{2\delta}(E_0+\varepsilon)^{2(1-\delta)} \\
&\leq J\widetilde{C}^2\rho_0^n\left(\|u\|_{L^2(B_{r_0}(x_0))} + \varepsilon\right)^{2\delta}(E_0+\varepsilon)^{2(1-\delta)} \\
&\leq J\widetilde{C}^2\rho_0^n(\eta+\varepsilon)^{2\delta}(E_0+\varepsilon)^{2(1-\delta)}.
\end{aligned}$$

This completes the proof. ■

Remark 2.1.9

We can use Theorem 2.1.8 for proving Theorem 2.1.7 for every $0 < r_1 < r_2 < r_3$.

Theorem 2.1.10 (Global propagation of smallness)

Let Ω be a bounded connected open set in $\mathbb{R}^n$ with boundary $\partial\Omega$ of Lipschitz class with constants $\rho_0,\ M_0$. Let $u \in H^1(\Omega)$ be a solution to the equation

$$\Delta u + Mu = f, \text{ in } \Omega.$$

Let $B_{r_0}(x_0) \subset \Omega$ and let us assume that

$$\exists\, \eta > 0,\ E > 0 \ \text{ st } \ \|u\|_{L^2(B_{r_0}(x_0))} \ \leq \ \eta, \quad \|u\|_{H^1(\Omega)} \ \leq \ E.$$

Then, we have

$$\|u\|_{L^2(\Omega)} \leq (E+\varepsilon)\omega\left(\frac{\eta+\varepsilon}{E+\varepsilon}\right),$$

where

$$\omega(t) \leq \frac{C}{\left(\log \dfrac{1}{t}\right)^{\mu}}, \quad \text{for } t < 1.$$

With $C > 0$ and $\mu,\ 0 < \mu < 1$.

Proof.

See [1]. ■

2.2 The elliptic Cauchy problem for the Laplace operator

The Cauchy problem for the Laplace operator is one of the main examples of ill-posed problems. One can pick up the harmonic functions with arbitrarily small Cauchy data on a piece of the domain boundary, which will be arbitrarily large in the domain (as in the famous example of Hadamard 1.7).

2.2.1 Classical problem

We consider the following linear elliptic Cauchy problem: find $u : \Omega \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \partial_n u = \psi & \text{on } \Gamma_N \end{cases} \tag{2.15}$$

where Ω be a convex polygonal (polyhedral) domain in $\mathbb{R}^d, d = 2, 3$, $\Gamma_D, \Gamma_N \subset \partial\Omega$ denotes a simply connected parts of the boundary and $f \in L^2(\Omega)$, $\psi \in H^{\frac{1}{2}}(\Gamma_N)$.
We put $\Gamma = \Gamma_D = \Gamma_N$.
As it is mentioned above, it is well-known that this problem is ill-posed in the sense of Hadamard.

Lemma 2.2.1
There is at most one solution $u \in H^2(\Omega)$ which satisfies (2.15).

Proof.
It is a consequence of the uniqueness Theorem (2.1.5) with $M = 0$. ■

2.2.2 A variational setting

Let us introduce the spaces V and W by

$$V := \{v \in H^1(\Omega) : v|_{\Gamma} = 0\}, \quad W := \{v \in H^1(\Omega) : v|_{\Gamma'} = 0\}$$

where $\Gamma' := \partial\Omega \setminus \Gamma$. Eq.(2.15) may be cast in the abstract weak formulation

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u,w) = l(w) \quad \forall w \in W \end{cases} \tag{2.16}$$

where $a : V \times W \mapsto \mathbb{R}$ and $l : W \mapsto \mathbb{R}$ are a bilinear and a linear form, given by

$$a(u,w) := \int_\Omega \nabla u \cdot \nabla w \text{ dx}, \text{ and } l(w) := \int_\Omega fw \text{ dx} + \int_\Gamma \psi w \text{ ds}$$

Since $u \notin W$ coercivity fails and inf-sup stability does not hold either in general (see [6]), so the variational problem is ill-posed in general.

The lake of coerciveness makes it worthless to write down the minimization problem related to the variational problem.

2.3 Conditional stability

There is a rich literature on conditional stability estimates for ill-posed problems, which used for the derivation of error estimates, without relying on the Lax-Milgram Lemma or the Babuska-Brezzi Theorem.

The estimates are conditional, in the sense that they only hold under the condition that the exact solution exists in some Sobolev space V and it satisfies some a priori estimates with respect to the norm of the considered spaces, hence we assume that the linear form $l(w)$ is such that the problem (2.16) admits a unique solution $u \in V$.

Definition 2.3.1 (Conditional stability)

We say that a solution u of (2.16) satisfies the conditional stability if for some sufficiently small $\epsilon > 0$, there hods

$$\|l\|_{W'} \leq \epsilon \text{ in (2.16) then } |j(u)| \leq \Xi(\epsilon). \tag{2.17}$$

where $j : V \mapsto \mathbb{R}$ and $\Xi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ be a continuous, monotone increasing function, with

$$\lim_{x \longrightarrow 0^+} \Xi(x) = 0.$$

2.3.1 Interior and global estimates

Theorem 2.3.2 [1] If (2.16) admits a unique solution $u \in H^1(\Omega)$, a conditional stability of the form (2.17), with $0 < \epsilon < 1$, holds for

$$j(u) := \|u\|_{L^2(\omega)},\ \omega \subset \Omega : \mathrm{dist}(\omega, \partial\Omega) =: d_{\omega,\partial\Omega} > 0$$

with

$$\Xi(x) := C_{u\varsigma} x^{\varsigma},\ C_{u\varsigma} > 0,\ \varsigma := \varsigma(d_{\omega,\partial\Omega}) \in (0,1) \tag{2.18}$$

and for

$$j(u) := \|u\|_{L^2(\Omega)},\ \text{with } \Xi(x) := C_u(|\log(x)| + C)^{-\varsigma}. \tag{2.19}$$

with $C_u,\ C > 0,\ \varsigma \in (0,1)$.

Proof.

It suffices to apply Theorem 2.1.8 and Theorem 2.1.10. ■

Remark 2.3.3

▶ The constant $C_{u\varsigma}$ in (2.18) grows monotonically in $\|u\|_{L^2(\Omega)}$ (from Theorem 2.1.8)

▶ The constant C_u in (2.19) grows monotonically in $\|u\|_{H^1(\Omega)}$. (from Theorem 2.1.10)

Chapter 3

Finite elements approximation of the ill-posed problem (2.16)

3.1 Conforming finite elements

3.1.1 A Finite element discretization

Let $\mathcal{T}_h$ be a shape regular, conforming, subdivision of Ω, the family of meshes $\{\mathcal{T}_h\}_h$ is indexed by the mesh parameter $h := \max(\mathrm{diam}(T))$. $\mathcal{T}_h$ is a finite set of triangles such that:

1. $\forall\, T \in \mathcal{T}_h, \quad T \subset \overline{\Omega}$
2. $\forall\, T_1, T_2 \in \mathcal{T}_h,\ T_1 \neq T_2 \Longrightarrow \mathring{T}_1 \cap \mathring{T}_2 = \emptyset$
3. $\bigcup_{T \in \mathcal{T}_h} T = \overline{\Omega}$.

Let $\mathcal{F}_I$ be the set of interior faces in $\mathcal{T}_h$ and $\mathcal{F}_\Gamma$, $\mathcal{F}_{\Gamma'}$ the set of element faces of $\mathcal{T}_h$ whose interior intersects Γ and Γ' respectively. We assume that $\mathcal{F}_\Gamma \cap \mathcal{F}_{\Gamma'} = \emptyset$.

Let X_h^1 the space of continuous, affine functions. We approximate the spaces V, W by the following finite element spaces

$$V_h := V \cap X_h^1 \text{ and } W_h := W \cap X_h^1.$$

Then the discrete problem is given by

$$\begin{cases} \text{find } u_h \in V_h \text{ such that} \\ a_h(u_h, w_h) = l_h(w_h) \quad \forall w_h \in W_h \end{cases} \tag{3.1}$$

where the forms $a_h(\cdot,\cdot)$ and $l_h(\cdot)$ are discrete realisations of $a(\cdot,\cdot)$ and $l(\cdot)$. The discrete problem can be written as follow

$$\begin{cases} \text{find } u_h := \sum_{j=1}^{N_1} u_j \varphi_j \in V_h \text{ such that} \\ a_h(u_h, \phi_i) = l_h(\phi_i) \quad i = 1, \cdots, N_2 \end{cases} \tag{3.2}$$

where the $\{\varphi_i\}$ and $\{\phi_i\}$ are suitable bases for V_h and W_h respectively, and $N_1 = \dim(V_h)$, $N_2 = \dim(W_h)$. The problem (3.2) may be written as the linear system

$$AU = L,$$

where A is an $N_1 \times N_2$ matrix, with coefficients $A_{ij} := a_h(\varphi_i, \phi_j)$ and

$$U = (u_1, \cdots, u_{N_1})^T, \; L = (l_h(\phi_1), \cdots, l_h(\phi_{N_2}))^T.$$

Observe that since we have not assumed $N_1 = N_2$ this system may not be square, but even if it is, it may have zero eigenvalues. Possibly implies

1. non-uniqueness: $\exists\, \tilde{U} \in \mathbb{R}^{N_1} \setminus \{0\}$ such that $A\tilde{U} = 0$;

2. non-existence: $\exists\, L \in \mathbb{R}^{N_1}$ such that $L \notin \mathrm{Im} A$.

Hence, the discrete system may be ill-posed.

Now we define the Lagrangian L (see [5]) by:

$$L(u_h, z_h) = a_h(u_h, z_h) - l_h(z_h) + \frac{1}{2} s_V(u_h - u, u_h - u) - \frac{1}{2} s_W(z_h, z_h)$$

where $s_V(u_h - u, u_h - u)$ and $s_W(z_h, z_h)$ represents a penalty term.

Since (u_h, z_h) is a saddle point of L, then

$$\frac{\partial L}{\partial z_h}(u_h, z_h) = 0$$
$$\frac{\partial L}{\partial u_h}(u_h, z_h) = 0$$

We will need to compute $\frac{\partial L}{\partial z_h}(u_h, z_h)$ and $\frac{\partial L}{\partial u_h}(u_h, z_h)$:

let $(k_1, k_2) \in V_h \times W_h$, we have that

$$\begin{aligned} L(u_h + k_1, z_h + k_2) - L(u_h, z_h) &= a_h(u_h + k_1, z_h + k_2) - l_h(z_h + k_2) \\ &\quad + \frac{1}{2} s_V(u_h + k_1 - u, u_h + k_1 - u) - \frac{1}{2} s_W(z_h + k_2, z_h + k_2) \\ &\quad - a_h(u_h, z_h) + l_h(z_h) - \frac{1}{2} s_V(u_h - u, u_h - u) + \frac{1}{2} s_W(z_h, z_h) \\ &\simeq a_h(u_h, k_2) + a_h(k_1, z_h) - l_h(k_2) + s_V(u_h - u, k_1) - s_W(z_h, k_2) \end{aligned}$$

thus

$$\begin{cases} \dfrac{\partial L}{\partial z_h}(u_h, z_h) = a_h(u_h, w_h) - s_W(z_h, w_h) - l_h(w_h) = 0 \\ \\ \dfrac{\partial L}{\partial u_h}(u_h, z_h) = a_h(v_h, z_h) + s_V(u_h - u, v_h) = 0 \end{cases}$$

We may then write the finite element method:

$$\begin{cases} \text{Find } (u_h, z_h) \in V_h \times W_h \text{ such that} \\ \quad a_h(u_h, w_h) - s_W(z_h, w_h) = l_h(w_h), \quad \forall w_h \in W_h \\ \quad a_h(v_h, z_h) + s_V(u_h, v_h) \ = s_V(u, v_h), \ \forall v_h \in V_h \end{cases} \tag{3.3}$$

A possible choice of stabilization operators for the problem are (see [3])

$$s_V(u_h, v_h) := \sum_{F \in \mathcal{F}_I \cup \mathcal{F}_\Gamma} \int_F h_F [\![\partial_n u_h]\!] [\![\partial_n v_h]\!] \mathrm{ds}, \quad \text{with } h_F := \mathrm{diam}(F) \tag{3.4}$$

$$s_W(z_h, w_h) := a(z_h, w_h) \tag{3.5}$$

Remark 3.1.1

• For this choice $s_V(u, v_h)$ is known if $u \in H^2(\Omega)$ even when u is not known explicitly.

• The Lagrange multiplier z_h is the solution to the adjoint problem and is tends to zero as h goes to zero (see Table 4.1 in Example 4.1).

Lemma 3.1.2

The quantities $(s_V(v_h, v_h))^{1/2}$ and $(s_W(w_h, w_h))^{1/2}$ define norms on V_h and W_h respectively.

Proof.

First it is clear that zero is the only constant function in V_h. Let $v \in V_h$ such that $s_V(v_h, v_h) = 0$. Then

$$[\![\partial_n v_h]\!]_{|e} = 0,\ \forall e \in \mathcal{F}_I \cup \mathcal{F}_\Gamma \implies \nabla v_h \cdot n_e = 0, \forall e \in \mathcal{F}$$

Since $[\![v_h]\!]_{|e} = 0, \forall e \in \mathcal{F}_I \cup \mathcal{F}_\Gamma$ then

$$[\![\partial_t v_h]\!]_{|e} = 0, \forall e \in \mathcal{F}_I \cup \mathcal{F}_\Gamma \implies \nabla v_h \cdot t_e = 0, \forall e \in \mathcal{F}$$

Hence $\nabla v_h = 0$ in $\Omega \implies v_h = Cte \implies v_h = 0$.

For s_W it suffice to observe that $W_h \subset W \subset H_0^1(\Omega)$ then we conclude by Poincaré's inequality. ■

Corollary 3.1.3 s_V and s_W define semi-norms on $H^s(\Omega) + V_h$ and $H^s(\Omega) + W_h$ respectively,

$$\text{for some } s \geq 1, \qquad |v + v_h|_{s_Z} := s_Z(v + v_h, v + v_h)^{\frac{1}{2}},\ \ \forall v \in H^s(\Omega),\ v_h \in Z_h,\ \text{with } Z = V, W. \tag{3.6}$$

Let us now introduce on the space $V_h \times W_h$ the norm:

$$\|(v_h, w_h)\|_h^2 = s_V(v_h, v_h) + s_W(w_h, w_h).$$

and other form of the formulation (3.3), find $(u_h, z_h) \in V_h \times W_h$ such that

$$\mathcal{A}[(u_h, z_h), (v_h, w_h)] = L_h(v_h, w_h) \quad \forall (v_h, w_h) \in V_h \times W_h \tag{3.7}$$

where

$$\mathcal{A}[(u_h, z_h), (v_h, w_h)] := a(u_h, w_h) - s_W(z_h, w_h) + a(v_h, z_h) + s_V(v_h, u_h)$$

and

$$L_h(v_h, w_h) := l_h(w_h) + s_V(u, v_h)$$

Then we have the following stability estimate:

Lemma 3.1.4 Let (u_h, z_h) be a solution of the formulation (3.3), then

$$\|(u_h, z_h)\|_h \leq \sup_{(v_h,w_h)\in V_h\times W_h} \frac{\mathcal{A}((u_h, z_h); (v_h, w_h))}{\|(v_h, w_h)\|_h}$$

Proof.

If we take $v_h = u_h$ and $w_h = -z_h$, we get

$$\begin{aligned}
\sup_{(v_h,w_h)\in V_h\times W_h} \frac{\mathcal{A}((u_h, z_h); (v_h, w_h))}{\|(v_h, w_h)\|_h} &\geq \frac{\mathcal{A}((u_h, z_h); (v_h, w_h))}{\|(v_h, w_h)\|_h} \\
&= \frac{\mathcal{A}((u_h, z_h); (u_h, -z_h))}{\|(u_h, -z_h)\|_h} \\
&= \frac{a(u_h, -z_h) - s_W(z_h, -z_h) + a(u_h, z_h) + s_V(u_h, u_h)}{\|(u_h, -z_h)\|_h} \\
&= \|(u_h, z_h)\|_h.
\end{aligned}$$

■

Theorem 3.1.5 The formulation (3.3) has a unique solution (u_h, z_h).

Proof. To prove unique existence of (u_h, z_h) solution to (3.3) we need to show that there are no zero eigenvalues to the system matrix corresponding to (3.3).
Assume that $l_h(w_h) = s_V(u, v_h) = 0$ in (3.3) then
by Theorem 3.1.4 we get $u_h = 0$, $z_h = 0$ which is a contradiction. ■

Galerkin Orthogonality

Taking the difference of 3.3 and the relation 2.16, with $w = w_h$, we obtain the Galerkin orthogonality

$$a(u_h - u, w_h) - s_W(z_h, w_h) + a(v_h, z_h) + s_V(u_h - u, v_h) = 0, \quad \text{for all} (v_h, w_h) \in V_h \times W_h. \tag{3.8}$$

3.1.2 Some lemma on forms and interpolants

Theorem 3.1.6 (Scott-Zhang) [12]

Let p and l satisfy $1 \leq p \leq \infty$ and

$$\begin{cases} l \geq 1 & \text{if} \quad p = 1 \\ l > \frac{1}{p} & \text{otherwise.} \end{cases}$$

Then, there is c such that the following properties hold

(i) Stability: for all $0 \leq m \leq \min(1, l)$,

$$\forall h > 0, \ \forall v \in W^{l,p}(\Omega), \quad \|\mathcal{SZ}_h v\|_{m,p,\Omega} \leq c \ \|v\|_{l,p,\Omega}$$

(ii) Approximation: provided $l \leq k + 1$, for all $0 \leq m \leq l$,

$$\forall h > 0, \ \forall T \in \mathcal{T}_h, \forall v \in W^{l,p}(V(T)) \quad \|v - \mathcal{SZ}_h v\|_{m,p,T} \leq c \ h_T^{l-m} |v|_{l,p,,V(T)}$$

where $\mathcal{SZ}_h : H^1(\Omega) \to V_h$,

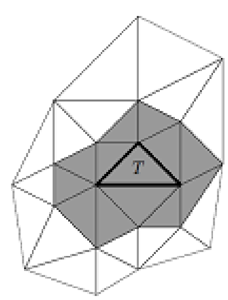


Figure 3.1: The set $V(T)$.

For more detail see [12].

Lemma 3.1.7

The quantities $\|v\|_{*,V} = \|\nabla v\|_{L^2(\Omega)}$ and $\|w\|_{*,W} = \|h^{-1} w\|_{L^2(\Omega)} + (\sum_{F \in \mathcal{F}_I \cup \mathcal{F}_\Gamma} h^{-1} \|w\|^2_{L^2(F)})^{1/2}$ define norms on V and W respectively.

Lemma 3.1.8 The form bilinear $a(\cdot,\cdot)$ satisfies the continuities

$$a(v-\mathcal{S}\mathcal{Z}_h v, w_h) \le \|v-\mathcal{S}\mathcal{Z}_h v\|_{*,V}|w_h|_{s_W}, \quad \forall v\in V,\ w_h\in W_h \tag{3.9}$$

Proof. By the Cauchy-Schwarz inequality, we find

$$\begin{aligned} a(v-\mathcal{S}\mathcal{Z}_h v, w_h) &= \int_\Omega \nabla(v-\mathcal{S}\mathcal{Z}_h v)\cdot\nabla w_h \text{ dx} \\ &\le \left(\int_\Omega |\nabla(v-\mathcal{S}\mathcal{Z}_h v)|^2\right)^{1/2}\left(\int_\Omega |\nabla w_h|^2\right)^{1/2} \\ &= \|v-\mathcal{S}\mathcal{Z}_h v\|_{*,V}(a(w_h,w_h))^{1/2} \\ &= \|v-\mathcal{S}\mathcal{Z}_h v\|_{*,V}(s_W(w_h,w_h))^{1/2} \\ &= \|v-\mathcal{S}\mathcal{Z}_h v\|_{*,V}|w_h|_{s_W} \end{aligned}$$

■

Lemma 3.1.9 For $v\in V$ and $t>0$:

$$|v-\mathcal{S}\mathcal{Z}_h v|_{s_V} + \|v-\mathcal{S}\mathcal{Z}_h v\|_{*,V} \le C_V(v)h^t. \tag{3.10}$$

The factor $C_V(v)>0$ will typically depend on some Sobolev norm of v.

In particular, we have

Corollary 3.1.10
For $u\in H^2(\Omega)$ be a solution of (2.15) there holds

$$|u-\mathcal{S}\mathcal{Z}_h u|_{s_V} + \|u-\mathcal{S}\mathcal{Z}_h u\|_{*,V} \le Ch\|u\|_{H^2(\Omega)}. \tag{3.11}$$

Lemma 3.1.11 There is $C_W>0$ such that

$$\|w-\mathcal{S}\mathcal{Z}_h w\|_{*,W} + \|\mathcal{S}\mathcal{Z}_h w\|_{s_W} \le C_W\|w\|_W, \quad \forall w\in W. \tag{3.12}$$

Proof. Let $w\in W$,

$$\|w-\mathcal{S}\mathcal{Z}_h w\|_{*,W} = \|h^{-1}(w-\mathcal{S}\mathcal{Z}_h w)\|_{L^2(\Omega)} + (\sum_{F\in\mathcal{F}_I\cup\mathcal{F}_\Gamma} h^{-1}\|w-\mathcal{S}\mathcal{Z}_h w\|^2_{L^2(F)})^{1/2}$$

by trace inequality and Theorem 3.1.6 we get

$$\|w - \mathcal{SZ}_h w\|_{*,W} \leq C|w|_{1,\Omega} \leq C\|w\|_{1,\Omega} = C\|w\|_W.$$

$$\|\mathcal{SZ}_h w\|_{s_W} \leq C\|\mathcal{SZ}_h w\|_{W_h} = C\|\mathcal{SZ}_h w\|_{H^1(\Omega)} \leq C\|w\|_{H^1(\Omega)}$$

this complete the proof. ■

Lemma 3.1.12 Let u be a solution of 2.16, then

$$a(u - u_h, w - \mathcal{SZ}_h w) \leq \delta_l(h)\|w\|_W + \|w - \mathcal{SZ}_h w\|_{*,W}|u - u_h|_{s_V}, \quad \forall w \in W. \tag{3.13}$$

With $\delta_l(h)$ only depends on the properties of the interpolant $\mathcal{SZ}_h$ and the data of the problem, and satisfies $\lim\limits_{h \longrightarrow 0} \delta_l(h) = 0$.

Proof.

$$\begin{aligned} a(u - u_h, w - \mathcal{SZ}_h w) &= \int_\Omega \nabla(u - u_h) \cdot \nabla(w - \mathcal{SZ}_h w) \text{ dx} \\ &= \sum_{T \in \mathcal{T}_h} \int_T \nabla(u - u_h) \cdot \nabla(w - \mathcal{SZ}_h w) \text{ dx} \end{aligned}$$

by integration by parts, we obtain

$$\begin{aligned} a(u - u_h, w - \mathcal{SZ}_h w) &= \sum_{T \in \mathcal{T}_h} \left(\int_T -\Delta(u - u_h)(w - \mathcal{SZ}_h w) \text{ dx} + \int_{\partial T} \partial_n(u - u_h)(w - \mathcal{SZ}_h w) \text{ ds} \right) \\ &= (f, w - \mathcal{SZ}_h w)_{L^2(\Omega)} \ + \sum_{F \in \mathcal{F}_I \cup \mathcal{F}_\Gamma} (\llbracket \partial_n(u - u_h) \rrbracket, w - \mathcal{SZ}_h w)_{L^2(F)} \end{aligned}$$

We use the Cauchy-Schwarz inequality, to get

$$\begin{aligned} a(u - u_h, w - \mathcal{SZ}_h w) &\leq \|f\|_{L^2(\Omega)}\|w - \mathcal{SZ}_h w\|_{L^2(\Omega)} \\ &\quad + \left(\sum_{F \in \mathcal{F}_I \cup \mathcal{F}_\Gamma} h|\llbracket \partial_n(u - u_h) \rrbracket|^2 \right)^{1/2} \left(\sum_{F \in \mathcal{F}_I \cup \mathcal{F}_\Gamma} h^{-1}|w - \mathcal{SZ}_h w|^2 \right)^{1/2} \\ &\leq \|f\|_{L^2(\Omega)} h\|w - \mathcal{SZ}_h w\|_{*,W} \\ &\quad + \left(\sum_{F \in \mathcal{F}_I \cup \mathcal{F}_\Gamma} h|\llbracket \partial_n(u - u_h) \rrbracket|^2 \right)^{1/2} \left(\sum_{F \in \mathcal{F}_I \cup \mathcal{F}_\Gamma} h^{-1}|w - \mathcal{SZ}_h w|^2 \right)^{1/2} \end{aligned}$$

By (3.12), we find

$$
\begin{aligned}
a(u-u_h, w-\mathcal{S}\mathcal{Z}_h w) &\leq C_W h\|f\|_{L^2(\Omega)}\|w\|_W \\
&\quad + \left(\sum_{F\in\mathcal{F}_I\cup\mathcal{F}_\Gamma} h|[\![\partial_n(u-u_h)]\!]|^2\right)^{1/2}\left(\sum_{F\in\mathcal{F}_I\cup\mathcal{F}_\Gamma} h^{-1}|w-\mathcal{S}\mathcal{Z}_h w|^2\right)^{1/2} \\
&\leq C_W h\|f\|_{L^2(\Omega)}\|w\|_W + \left(\sum_{F\in\mathcal{F}_I\cup\mathcal{F}_\Gamma} h|[\![\partial_n(u-u_h)]\!]|^2\right)^{1/2} \|w-\mathcal{S}\mathcal{Z}_h w\|_{*,W} \\
&= C_W h\|f\|_{L^2(\Omega)}\|w\|_W + |u-u_h|_{s_V}\|w-\mathcal{S}\mathcal{Z}_h w\|_{*,W} \\
&= \delta_l(h)\|w\|_W + \|w-\mathcal{S}\mathcal{Z}_h w\|_{*,W}|u-u_h|_{s_V}
\end{aligned}
$$

with $\delta(h) = C_W h\|f\|_{L^2(\Omega)}$.

■

3.1.3 Error analysis

A priori error analysis

Lemma 3.1.13 Let u be the solution of (2.16) and (u_h, z_h) the solution of the formulation (3.3) for which (3.6), (3.9) and (3.10) hold. Then

$$|u-u_h|_{s_V} + |z_h|_{s_W} \leq (1+\sqrt{2})C_V(u)h^t.$$

Proof. Let $\xi_h := i_V u - u_h$, by the triangle inequality

$$|u-u_h|_{s_V} \leq |u-i_V u|_{s_V} + |\xi_h|_{s_V} \tag{3.14}$$

and write

$$|\xi_h|_{s_V}^2 + |z_h|_{s_W}^2 = s_V(\xi_h,\xi_h) + a(\xi_h, z_h) - a(\xi_h, z_h) + s_W(z_h, z_h).$$

Using Eq.(3.8) with $v_h = \xi_h$ and $w_h = z_h$ we get

$$-a(\xi_h, z_h) + s_W(z_h, z_h) = a(u_h-u, z_h) + s_V(u_h-u, \xi_h)$$

we then have

$$\begin{aligned}|\xi_h|_{s_V}^2 + |z_h|_{s_W}^2 =& s_V(\xi_h, \xi_h) + a(\xi_h, z_h) + a(u_h - u, z_h) \\ &+ s_V(u_h - u, \xi_h) \\ =& s_V(\xi_h, \xi_h) + a(\xi_h, z_h) - a(\xi_h, z_h) + a(i_V u - u, z_h) \\ &- s_V(\xi_h, \xi_h) + s_V(i_V u - u, \xi_h) \\ =& s_V(i_V u - u, \xi_h) + a(i_V u - u, z_h).\end{aligned}$$

By the Cauchy-Schwarz inequality in the first term

$$s_V(i_V u - u, \xi_h) \leq |i_V u - u|_{s_V} |\xi_h|_{s_V}$$

and the continuity (3.9) in the second,

$$a(i_V u - u, z_h) \leq C \|i_V u - u\|_{*,V} |z_h|_{s_W}$$

Then by (3.10), we may deduce:

$$\begin{aligned}|\xi_h|_{s_V}^2 + |z_h|_{s_W}^2 &\leq |i_V u - u|_{s_V} |\xi_h|_{s_V} + \|i_V u - u\|_{*,V} |z_h|_{s_W} \\ &\leq C_V(u) h^t (|\xi_h|_{s_V}^2 + |z_h|_{s_W}^2)^{\frac{1}{2}}.\end{aligned}$$

Otherwise, we have

$$\left(|\xi_h|_{s_V}^2 + |z_h|_{s_W}^2\right)^{\frac{1}{2}} \geq \frac{1}{\sqrt{2}} \left(|\xi_h|_{s_V} + |z_h|_{s_W}\right).$$

Then

$$|\xi_h|_{s_V} + |z_h|_{s_W} \leq \sqrt{2} C_V(u) h^t$$

and by (3.10) and (3.14), we find that

$$|u - u_h|_{s_V} - |\xi_h|_{s_V} \leq |u - i_V u|_{s_V} \leq C_V(u) h^t.$$

■

A posteriori error analysis

Theorem 3.1.14 Let u be the solution of (2.16) and (u_h, z_h) the solution of the formulation (3.3) for which (3.6), (3.10) hold. Assume that the problem (2.16) has the stability property (2.17). Then

$$|j(u - u_h)| \leq \Xi(\eta(u_h, z_h)) \tag{3.15}$$

where the a posteriori quantity $\eta(u_h, z_h)$ is defined by

$$\eta(u_h, z_h) := \delta_l(h) + C_W(|u - u_h|_{s_V} + |z_h|_{s_W}).$$

Corollary 3.1.15

For sufficiently smooth u there holds:

$$\eta(u_h, z_h) \leq \delta_l(h) + (1 + \sqrt{2})C_W C_V(u)h^t. \tag{3.16}$$

Proof. Directly by Lemma 3.1.13.

■

Proof of Theorem 3.1.14.

Let $e := u - u_h \in V$, for all $w \in W$ we have:

$$\begin{aligned} a(e, w) &= a(e, w - i_W w) + a(e, i_W w) \\ &= a(e, w - i_W w) - s_W(z_h, i_W w) \qquad \text{(By the Galerkin orthogonality)} \\ &= l(w - i_W w) - a(u_h, w - i_W w) - s_W(z_h, i_W w) \end{aligned} \tag{3.17}$$

and we identify $r \in W'$ such that $\forall w \in W$,

$$(r, w)_{W',W} = l(w - i_W w) - a(u_h, w - i_W w) - s_W(z_h, i_W w). \tag{3.18}$$

We have shown that e satisfies equation (2.16) with right-hand side $(r, w)_{W',W}$ (by (3.17)). Now apply the continuity (3.13), Cauchy-Schwarz inequality and the stability (3.12) in the right-hand side of (3.18), leading to:

$$|(r, w)_{W',W}| = |a(e, w - i_W w) - s_W(z_h, i_W w)| \leq (\delta_l(h) + C_W|u - u_h|_{s_V} + C_W|z_h|_{s_W})\|w\|_W.$$

We conclude that

$$\|r\|_{W'} \leq \delta_l(h) + C_W(|u - u_h|_{s_V} + |z_h|_{s_W})$$

and the claim (3.15) follows by assumption (2.17). ■

Corollary 3.1.16

Let $u \in H^2(\Omega)$ be the solution of (2.15) and $u_h,\ z_h$ the solution of (3.3)-(3.4). Then the conclusions of Lemma 3.1.13 and Theorem 3.1.14 hold for $u - u_h,\ z_h$ with $t = 1$ and $j(\cdot),\ \Xi(\cdot)$ given by (2.18) or (2.19).

Proof. We obtain the required by using Corollary 3.1.10. ■

3.2 Nonconforming finite elements

Let us consider the problem 2.15 with $\Gamma_D \neq \Gamma_N$,

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0 & \text{on } \Gamma_D \\ \partial_n u = \psi & \text{on } \Gamma_N \end{cases} \tag{3.19}$$

where Γ_D, Γ_N are two subsets of the boundary $\partial\Omega$, we denote the complement of the Neumann boundary $\Gamma_N' := \partial\Omega \setminus \Gamma_N$. To exclude the well-posed case, we assume that $\partial\Omega \setminus (\Gamma_D \cup \Gamma_N) \neq \emptyset$.

In this case, for the derivation of a weak formulation we introduce the spaces

$$V := \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0\} \ \text{ and } \ W := \{v \in H^1(\Omega) : v|_{\Gamma_N'} = 0\}$$

then we obtain a weak formulation

$$\begin{cases} \text{find } u \in V \text{ such that} \\ a(u, w) = l(w) \ \ \forall w \in W \end{cases} \tag{3.20}$$

where $a : V \times W \mapsto \mathbb{R}$ and $l : W \mapsto \mathbb{R}$ are a bilinear and a linear form, given by

$$a(u, w) := \int_\Omega \nabla u \cdot \nabla w \ \text{dx}, \ \text{and } l(w) := \int_\Omega f w \ \text{dx} + \int_{\Gamma_N} \psi w \ \text{ds}$$

Remark 3.2.1

The same assumptions and the results when $\Gamma_D = \Gamma_N$, are valid in the case of $\Gamma_D \neq \Gamma_N$.

3.2.1 Crouzeix-Raviart nonconforming finite element discretization

Let $\{\mathcal{T}_h\}$ denote a family of shape regular and quasi tessellations of Ω into non-overlapping simplices, such that $\forall\ T_1, T_2 \in \mathcal{T}_h,\ T_1 \neq T_2$

$T_1 \cap T_2$ consists of either the empty set, a common face or edge, a common vertex.

The diameter of $T \in \mathcal{T}_h$ will be denoted h_T and the outward pointing normal n_T. The family $\mathcal{T}_h$ is indexed by $h := \max_{T\in\mathcal{T}_h} (h_T)$.
We denote the set of element faces in $\mathcal{T}_h$ by $\mathcal{F}$ and let $\mathcal{F}_i$ denote the set of interior faces and $\mathcal{F}_\Gamma$ the set of faces in some $\Gamma \subset \partial\Omega$. To each face $F \in \mathcal{F}$ we associate the mesh parameter $h_F := \mathrm{diam}(F)$ and a unit normal vector, n_F. For interior faces its orientation is arbitrary, but fixed. On the boundary $\partial\Omega$ we identify n_F with the outward pointing normal of $\partial\Omega$.

Crouzeix-Raviart finite element space

With the triangulation $\mathcal{T}_h$, we introduce the **Crouzeix-Raviart** finite element space (see [10])

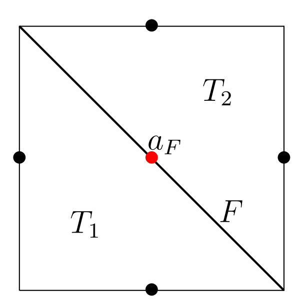
$$V_h^{CR} = \left\{ v_h \in L^2(\Omega) : v_{h|T} \in \mathbb{P}_1(T) \ \ \forall T \in \mathcal{T}_h, \text{ and} \int_F [\![v_h]\!]\ ds = 0 \ \ \forall F \in \mathcal{F}_i \right\}$$

by the condition $\int_F [\![v_h]\!]\ ds = 0$, the space V_h^{CR} can be defined by

$$V_h^{CR} = \begin{pmatrix} v_h \in L^2(\Omega) : v_{h|T} \in \mathbb{P}_1(T) \ \ \forall T \in \mathcal{T}_h \\ v_h \text{ is continuous at midpoint of edges } \mathcal{F}_i \end{pmatrix}$$

Let $F \in \mathcal{F}$, the associated basis function φ_F is defined by

$$\varphi_F(a_E) = \begin{cases} 1 & \text{if } E = F; \\ 0 & \text{if } E \neq F. \end{cases}$$



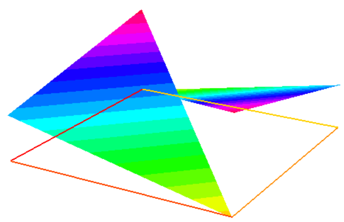


Figure 3.2: Crouzeix-Raviart functions.

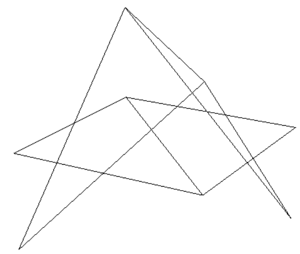


Figure 3.3: Crouzeix-Raviart basis.

The global interpolant operator of Crouzeix-Raviart $\mathcal{I}_h^{CR}$ is defined as follows:

$$\mathcal{I}_h^{CR} : H^1(\Omega) \to V_h^{CR}$$
$$v \mapsto \mathcal{I}_h^{CR}(v)$$
$$\text{with} \qquad \mathcal{I}_h^{CR}(v)(x) := \sum_{F \in \mathcal{F}} v(a_F)\varphi_F(x).$$

and on the reference element, we define the Crouzeix-Raviart interpolant by

$$\hat{\mathcal{I}}_h^{CR}(\hat{v})(\hat{x}) := \sum_{i \leq 3} \hat{N}_i(\hat{v})\hat{\varphi}_i(\hat{x})$$

where

$$\hat{N}_i(\hat{v}) = \frac{1}{|\hat{F}_i|}\int_{\hat{F}_i} \hat{v}\ ds.$$

Let now we define the subspace $V_{h,\Gamma}^{CR}$ of V_h^{CR} by

$$V_{h,\Gamma}^{CR} := \left\{ v_h \in L^2(\Omega) : v_{h|T} \in \mathbb{P}_1(T)\ \ \forall T \in \mathcal{T}_h,\ \text{and} \int_F [v_h]\ ds = 0\ \ \forall F \in \mathcal{F}_i \cup \mathcal{F}_\Gamma \right\}$$

In the nonconforming finite element methods the approximate spaces $V_{h,\Gamma}^{CR}$ are not contained in the spaces V, W. Then the discrete problem consists in finding a function $u_h \in V_{h,\Gamma_D}^{CR}$ such that,

$$\text{for all } w_h \in V_{h,\Gamma_N'}^{CR},\ a_h(u_h, w_h) = l_h(w_h),$$

where the approximate bilinear form $a_h(\cdot,\cdot)$ is defined by

$$a_h(u_h, w_h) = \sum_{T \in \mathcal{T}} \int_T \nabla u_h \cdot \nabla w_h\ \mathrm{dx}$$

the linear form $l(\cdot)$ need not be approximated since the inclusion $V^{CR}_{h,\Gamma'_N} \subset L^2(\Omega)$ holds. In this case, we introduce the norms

$$\|v\|^2_{L^2(\Omega)} := \sum_{T\in\mathcal{T}} \|v\|^2_{L^2(T)} \quad \text{and} \quad \|v\|^2_{L^2(\mathcal{F})} := \sum_{F\in\mathcal{F}} \|v\|^2_{L^2(F)}.$$

As in the above section, we propose the formulation

$$\begin{cases} \text{Find } (u_h, z_h) \in V^{CR}_{h,\Gamma_D} \times V^{CR}_{h,\Gamma'_N} \text{ such that} \\ \quad a_h(u_h, w_h) - s_W(z_h, w_h) = l(w_h), \quad \forall w_h \in V^{CR}_{h,\Gamma'_N} \\ \quad a_h(v_h, z_h) + s_V(u_h, v_h) \ = s_V(u, v_h), \ \forall v_h \in V^{CR}_{h,\Gamma_D} \end{cases} \tag{3.21}$$

A possible choice of stabilization operators for the problem in this case are (see [4])

$$s_W(z_h, w_h) := \sum_{T\in\mathcal{T}} \int_T \gamma_W \nabla z_h \cdot \nabla w_h \ \mathrm{dx} \tag{3.22}$$

$$s_V(u_h, v_h) := \sum_{F\in\mathcal{F}_i\cup\mathcal{F}_{\Gamma_D}} \int_F \gamma_V h_F^{-1} [\![u_h]\!] [\![v_h]\!] \ \mathrm{ds} \tag{3.23}$$

We have assumed $u \in H^2(\Omega)$ then by sobolev embedding $u \in C(\overline{\Omega})$ and thus $s_V(u, v_h) = 0$. So, we can write (3.21) as follow

$$\begin{cases} \text{Find } (u_h, z_h) \in V^{CR}_{h,\Gamma_D} \times V^{CR}_{h,\Gamma'_N} \text{ such that} \\ \quad a_h(u_h, w_h) - s_W(z_h, w_h) = l(w_h), \quad \forall w_h \in V^{CR}_{h,\Gamma'_N} \\ \quad a_h(v_h, z_h) + s_V(u_h, v_h) \ = 0, \qquad \forall v_h \in V^{CR}_{h,\Gamma_D} \end{cases} \tag{3.24}$$

Remark 3.2.2

Since the Dirichlet conditions on V^{CR}_{h,Γ_D} and V^{CR}_{h,Γ'_N} set on different parts of the boundary, we will use the Nitsche's method.

Nitsche's method is a method to incorporate Dirichlet boundary conditions weakly, i.e., without specifying nodal values on the boundary, by impose boundary conditions via penalization, but we introduce new terms may maintain consistency and coercivity of the bilinear form, (see [18]).

Then the problem (3.24) can be written as follows

$$\begin{cases} \text{Find } (u_h, z_h) \in V^{CR}_{h} \times V^{CR}_{h} \text{ such that} \\ \quad a_h(u_h, w_h) - b_h(u_h, w_h) - s_W(z_h, w_h) = l(w_h), \quad \forall w_h \in V^{CR}_{h} \\ \quad a_h(v_h, z_h) - b_h(v_h, z_h) + s_V(u_h, v_h) \ = 0, \qquad \forall v_h \in V^{CR}_{h} \end{cases} \tag{3.25}$$

with the boundary term $b_h(\cdot,\cdot)$ is defined by (see [4])

$$b_h(v_h, w_h) := \sum_{F\in\mathcal{F}_{\partial\Omega}} \left(\int_{F\cap\Gamma_N'} n\cdot\nabla v_h w_h \text{ ds} + \int_{F\cap\Gamma_D} n\cdot\nabla w_h v_h \text{ ds}\right)$$

and we modify the stabilization $s_W(\cdot,\cdot)$ so that the stabilization parameter may be chosen differently in the interior and on the boundary,

$$s_W(z_h, w_h) := \sum_{T\in\mathcal{T}} \int_T \gamma_W \nabla z_h\cdot\nabla w_h \text{ dx} + \sum_{F\in\mathcal{F}_{\Gamma_N'}} \int_F \gamma_{W,bc} h_F^{-1} z_h w_h \text{ ds}$$

Remark 3.2.3

The penalty parameters $\gamma_V, \gamma_W, \gamma_{W,bc}$ are all strictly positive and independent of h.

Remark 3.2.4

If (u_h, z_h) and (v_h, w_h) are restricted to $V^{CR}_{h,\Gamma_D}\times V^{CR}_{h,\Gamma_N'}$ in (3.25) we recover the formulation (3.24), because $V^{CR}_{h,\Gamma_D}\times V^{CR}_{h,\Gamma_N'}$ is in the kernel of the operator $b_h(\cdot,\cdot)$.

Trace and inverse inequalities

Lemma 3.2.5 (Trace inequality) There exists $C_t > 0$ such that for all $v\in H^1(T)$ and all $F\in\mathcal{F}$,

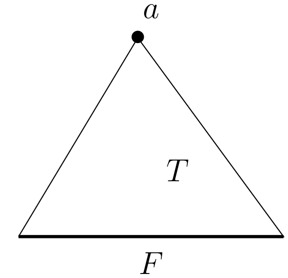
$$\|v\|_{L^2(F)} \le C_t(h_T^{-\frac{1}{2}}\|v\|_{L^2(T)} + h_T^{\frac{1}{2}}\|\nabla v\|_{L^2(T)}).$$

Proof. Let $v\in H^1(T)$ and let $F\in\mathcal{F}$

Let a is the vertex of T opposite to F and let us consider the $\mathbb{R}^2$-valued function

$$\delta(x,y) := \frac{|F|}{2|T|}\begin{pmatrix} x-x_a \\ y-y_a\end{pmatrix}$$

(note that $\delta(x,y)|_F = 1$)



Then by using of the divergence theorem, we find

$$\begin{aligned}
\|v\|^2_{L^2(F)} = \int_F |v|^2 = \int_{\partial T} |v|^2(\delta \cdot n_T) = \int_{\partial T} (|v|^2\delta) \cdot n_T \\
&= \int_T \operatorname{div}(|v|^2\delta) \\
&= \int_T 2\ v\ \delta \cdot \nabla v\ + \int_T |v|^2 \operatorname{div}(\delta) \\
&\leq C_1 h_T \|v\|_{L^2(T)} \|\nabla v\|_{L^2(T)} + C_2 h_T^{-1} \|v\|^2_{L^2(T)} \\
&\leq C_1 h_T \|\nabla v\|^2_{L^2(T)} + C_2 h_T^{-1} \|v\|^2_{L^2(T)}.
\end{aligned}$$

■

Lemma 3.2.6 (Inverse inequality) There exists $C_i > 0$ such that

$$h_T \|\nabla v_h\|_{L^2(T)} + h_T^{\frac{1}{2}} \|v_h\|_{L^2(F)} \leq C_i \|v_h\|_{L^2(T)}, \qquad \forall T \in \mathcal{T}_h, F \in \mathcal{F} \text{ and } \forall v_h \in V_h^{CR}.$$

Proof. First a prove

$$h_T \|\nabla v_h\|_{L^2(T)} \leq C \|v_h\|_{L^2(T)} \tag{3.26}$$

on the reference triangle, we have

$$|\hat{v}_h|_{1,\hat{T}} \leq C \|\hat{v}_h\|_{L^2(\hat{T})} \quad \text{(all norms on finite-dimensional vector spaces are equivalent)}$$

then

$$|v_h|_{1,T} \leq C h_T^{-1} \|v_h\|_{L^2(T)}.$$

By (3.26) and Lemma 3.2.5 we get

$$\begin{aligned}
h_T \|\nabla v_h\|_{L^2(T)} + h_T^{\frac{1}{2}} \|v_h\|_{L^2(F)} &\leq C \|v_h\|_{L^2(T)} + C_t \|v\|_{L^2(T)} + C_t h_T \|\nabla v\|_{L^2(T)} \\
&\leq C_i \|v_h\|_{L^2(T)}.
\end{aligned}$$

■

Lemma 3.2.7 (Discrete trace inequality) For any $v_h \in V_h^{CR}$ there exists C independent of h, such that

$$h_F \|\partial_n v_h\|^2_{L^2(F)} \leq C \|\nabla v_h\|^2_{L^2(T)} \quad \forall T \in \mathcal{T}_h, F \in \mathcal{F}. \tag{3.27}$$

Proof. It is proved by using the equivalence of norms on the reference element. ■

Lemma 3.2.8 (Poincaré inequality for piecewise constant functions)
Let v_h be a piecewise constant function, then there exists $C \geq 0$ such that

$$\|v_h\|_{L^2(\Omega)}^2 \leq C \sum_{F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} h_F^{-1} \|[\![v_h]\!]\|_F^2.$$

Proof.
Let us consider the following auxiliary problem

$$\begin{cases} -\Delta\varphi & = v_h \quad \text{in } \Omega \\ \quad \varphi & = 0 \quad \text{on } \partial\Omega \end{cases}$$

this is a well-posed problem. Moreover, by the elliptic regularity we have

$$\|\varphi\|_{H^2(\Omega)} \leq C\|v_h\|_{L^2(\Omega)}$$

and

$$\begin{aligned} \|v_h\|_{L^2(\Omega)}^2 &= \sum_{T \in \mathcal{T}} \|v_h\|_{L^2(T)}^2 = \sum_{T \in \mathcal{T}} \int_T v_h^2 \\ &= \sum_{T \in \mathcal{T}} \int_T (-\Delta\varphi) v_h = \sum_{T \in \mathcal{T}} \int_{\partial T} \frac{\partial\varphi}{\partial n} v_h \\ &\leq \sum_{F \in \mathcal{F}} \int_F h_F^{\frac{1}{2}} \frac{\partial\varphi}{\partial n} h_F^{-\frac{1}{2}} [\![v_h]\!]. \end{aligned}$$

By Cauchy-Schwarz inequality and normal trace Theorem, we get

$$\begin{aligned} \|v_h\|_{L^2(\Omega)}^2 &\leq C\|\varphi\|_{H^2(\Omega)} \sum_{F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} \int_F h_F^{-\frac{1}{2}} [\![v_h]\!] \\ &\leq C\|v_h\|_{L^2(\Omega)} \sum_{F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} \int_F h_F^{-\frac{1}{2}} [\![v_h]\!] \end{aligned}$$

■

Approximation results of the Crouzeix-Raviart interpolation

Lemma 3.2.9 For any function $v \in H^t(T)$,

$$\|v - \mathcal{I}_h^{CR}(v)\|_{L^2(T)} + h_T |v - \mathcal{I}_h^{CR}(v)|_{H^1(T)} \leq C h_T^t |v|_{H^t(T)}, \qquad t = 1, 2$$

Proof.

1. For $t = 1,$ let $v \in H^1(T)$:

   - $\|v - \mathcal{I}_h^{CR}(v)\|_{L^2(T)} \leq C h_T |v|_{H^1(T)}$?

   by A.3.2, we have

$$\|v - \mathcal{I}_h^{CR}(v)\|^2_{L^2(T)} \leq C|\det B_T| \|\hat{v} - \hat{\mathcal{I}}_h^{CR}(\hat{v})\|^2_{L^2(\hat{T})}$$

   Let $p \in \mathbb{P}_0(\hat{T})$

$$\begin{aligned}
\|\hat{v} - \hat{\mathcal{I}}_h^{CR}(\hat{v})\|^2_{L^2(\hat{T})} &= \|\hat{v} + p - p - \hat{\mathcal{I}}_h^{CR}(\hat{v})\|^2_{L^2(\hat{T})} \\
&= \|\hat{v} + p - \hat{\mathcal{I}}_h^{CR}(\hat{v} + p)\|^2_{L^2(\hat{T})} \\
&\leq 2 \left(\|\hat{v} + p\|^2_{L^2(\hat{T})} + |\hat{\mathcal{I}}_h^{CR}(\hat{v} + p)\|^2_{L^2(\hat{T})} \right) \\
&\leq 2 \left(\|\hat{v} + p\|^2_{H^1(\hat{T})} + \|\hat{\mathcal{I}}_h^{CR}(\hat{v} + p)\|^2_{L^2(\hat{T})} \right) \\
&\leq C \|\hat{v} + p\|^2_{H^1(\hat{T})} \qquad \forall\ p \in \mathbb{P}_0(\hat{T}) \\
&\leq C \inf_{p \in \mathbb{P}_0(\hat{T})} \|\hat{v} + p\|^2_{H^1(\hat{T})} \\
&\leq C\ |\hat{v}|^2_{H^1(\hat{T})} \quad \text{(Deny Lions Lemma)}
\end{aligned}$$

   and thus

$$\begin{aligned}
\|v - \mathcal{I}_h^{CR}(v)\|^2_{L^2(T)} &\leq C|\det B_T| |\hat{v}|^2_{H^1(\hat{T})} \\
&\leq C\ \|B_T\|^2\ |v|^2_{H^1(T)} \\
&= C\ \frac{h_T^2}{\hat{\rho}^2}\ |v|^2_{H^1(T)} \\
&= C\ h_T^2\ |v|^2_{H^1(T)} \quad \text{(as required)}
\end{aligned}$$

   - $|v - \mathcal{I}_h^{CR}(v)|_{H^1(T)} \leq C|v|_{H^1(T)}$?

   we have

$$\begin{aligned}
|v - \mathcal{I}_h^{CR}(v)|^2_{H^1(T)} &\leq C\ \|\det B_T^{-1}\|^2\ |\det B_T|\ |\hat{v} - \hat{\mathcal{I}}_h^{CR}(\hat{v})|^2_{H^1(\hat{T})} \\
&\leq C\ \|\det B_T^{-1}\|^2\ |\det B_T|\ \left(|\hat{v}|_{H^1(\hat{T})} + |\hat{\mathcal{I}}_h^{CR}(\hat{v})|_{H^1(\hat{T})} \right)^2 \\
&\leq C\ \|\det B_T^{-1}\|^2\ |\det B_T|\ \left(|\hat{v}|^2_{H^1(\hat{T})} + |\hat{\mathcal{I}}_h^{CR}(\hat{v})|^2_{H^1(\hat{T})} \right)
\end{aligned}$$

Moreover, we may write

$$\begin{aligned}|\hat{\mathcal{I}}_h^{CR}(\hat{v})|_{H^1(\hat{T})} &= |\hat{\mathcal{I}}_h^{CR}(\hat{v}) - c|_{H^1(\hat{T})} = |\hat{\mathcal{I}}_h^{CR}(\hat{v} - c)|_{H^1(\hat{T})} \\ &= |\sum_{i=1}^{3} \hat{N}_i(\hat{v} - c)\hat{\varphi}_i|_{H^1(\hat{T})} \\ &\leq \sum_{i=1}^{3} |\hat{N}_i(\hat{v} - c)| \ |\hat{\varphi}_i|_{H^1(\hat{T})} \\ &\leq C \ \sum_{i=1}^{3} |\hat{N}_i(\hat{v} - c)|\end{aligned}$$

with

$$\begin{aligned}|\hat{N}_i(\hat{v} - c)| = \left| \frac{1}{|\hat{F}_i|} \int_{\hat{F}_i} (\hat{v} - c) \ ds \right| &\leq |\hat{F}_i|^{-1} \|1\|_{L^2(\hat{F}_i)} \ \|\hat{v} - c\|_{L^2(\hat{F}_i)} = |\hat{F}_i|^{-2} \|\hat{v} - c\|_{L^2(\hat{F}_i)} \\ &\leq C \ \|\hat{v} - c\|_{L^2(\hat{T})}\end{aligned}$$

then

$$|\hat{\mathcal{I}}_h^{CR}(\hat{v})|_{H^1(\hat{T})} \leq C\|\hat{v} - c\|_{L^2(\hat{T})} \leq C|\hat{v}|_{H^1(\hat{T})} \quad \left(\text{by Poincaré-Wirtinger with } \ c = \frac{1}{|\hat{T}|} \int_{\hat{T}} \hat{v}\right)$$

Therefore, we obtain

$$\begin{aligned}|v - \mathcal{I}_h^{CR}(v)|^2_{H^1(T)} &\leq C \ \| \det B_T^{-1}\|^2 \ | \det B_T| \ |\hat{v}|^2_{H^1(\hat{T})} \\ &\leq C \ \| \det B_T^{-1}\|^2 \ \| \det B_T\|^2 \ |v|^2_{H^1(T)} \\ &= C \ |v|^2_{H^1(T)}\end{aligned}$$

2. For $t = 2$ we use the same way.

■

Lemma 3.2.10 For any function $v \in H^t(T)$ there holds

$$h_T^{-\frac{1}{2}} \|v - \mathcal{I}_h^{CR}(v)\|_{L^2(F)} \leq C h_T^{t-1} |v|_{H^t(T)}, \qquad t = 1, 2$$

Proof. Let $v \in H^t(T)$ for $T \in \mathcal{T}_h$, by using of Lemma 3.2.5 we get

$$\begin{aligned}\|v - \mathcal{I}_h^{CR}(v)\|_{L^2(F)} &\leq C_t (h_T^{-\frac{1}{2}} \|v - \mathcal{I}_h^{CR}(v)\|_{L^2(T)} + h_T^{\frac{1}{2}} |v - \mathcal{I}_h^{CR}(v)|_{L^2(T)}) \\ &= C_t \ h_T^{-\frac{1}{2}} (\|v - \mathcal{I}_h^{CR}(v)\|_{L^2(T)} + h_T |v - \mathcal{I}_h^{CR}(v)|_{L^2(T)})\end{aligned}$$

then we obtain the required formula by using Lemma 3.2.9. ■

Lemma 3.2.11 For all $v \in H^2(T)$ and $T \in \mathcal{T}_h$,

$$h_T^{\frac{1}{2}} \|\partial_n(v - \mathcal{I}_h^{CR}(v))\|_{L^2(F)} \leq C h_T |v|_{H^2(T)}$$

Proof.

To obtain the required we use normal trace Theorem with Lemme 3.2.9 on the reference element. ■

Lemma 3.2.12 For any $v_h \in V_h^{CR}$ we have

$$h^{-1} \|v_h\|_\Omega \leq c_\tau \left(\sum_{F \in \mathcal{F}} h_F^{-1} \|\bar{v}_h\|^2_{L^2(F)} \right)^{\frac{1}{2}}$$

Proof. It follows by norm equivalence of discrete spaces on the reference element that for all $T \in \mathcal{T}_h$

$$\|\hat{v}_h\|^2_{L^2(\hat{T})} \leq C \sum_{\hat{F} \in \partial \hat{T}} \|\bar{\hat{v}}_h\|^2_{L^2(\hat{F})}$$

then

$$\begin{aligned} h_T^{-2} \|v_h\|^2_{L^2(T)} &\leq C \sum_{F \in \partial T} h_F^{-1} \|\bar{v}_h\|^2_{L^2(F)} \\ \Longrightarrow \quad \sum_{T \in \mathcal{T}} h_T^{-2} \|v_h\|^2_{L^2(T)} &\leq C \sum_{F \in \mathcal{F}} h_F^{-1} \|\bar{v}_h\|^2_{L^2(F)} \\ \Longrightarrow \quad h^{-2} \|v_h\|^2_\Omega &\leq c_\tau \left(\sum_{F \in \mathcal{F}} h_F^{-1} \|\bar{v}_h\|^2_{L^2(F)} \right) \end{aligned}$$

■

3.2.2 Stability estimates

Let us introduce the following compact form of the formulation (3.25),

$$\text{find } (u_h, z_h) \in \mathcal{V}_h := V_h^{CR} \times V_h^{CR} \text{ such that}$$

$$\mathcal{A}_h[(u_h, z_h), (v_h, w_h)] = l(w_h) \quad \forall (v_h, w_h) \in \mathcal{V}_h$$

where

$$\begin{aligned} \mathcal{A}_h[(u_h, z_h), (v_h, w_h)] := a_h(u_h, w_h) - b_h(u_h, w_h) - s_W(z_h, w_h) \\ + a_h(v_h, z_h) - b_h(u_h, w_h) + s_V(v_h, u_h) \end{aligned}$$

Lemma 3.2.13 There exists $c_1, c_2 > 0$ such that

$$c_1 \gamma_W^{\frac{1}{2}} \|w_h\|_{1,\Omega} \leq s_W(w_h, w_h)^{1/2} \leq c_2 \gamma_W^{\frac{1}{2}} \|w_h\|_{1,\Omega}, \quad \forall w_h \in V_{h,\Gamma'_N}^{CR}$$

Proof. To obtain the required, we use the inequality of Poincaré. ■

Let now define the semi-norm

$$|v_h|_{s_V} := s_V(v_h, v_h)^{1/2}, \quad \forall v_h \in V_{h,\Gamma_D}^{CR}$$

and the norm

$$\|w_h\|_{s_W} := s_W(w_h, w_h)^{1/2}, \quad \forall w_h \in V_{h,\Gamma'_N}^{CR}$$

we introduce a mesh-dependent norm

$$|||v_h|||_V := \gamma_V^{\frac{1}{2}} \|h \nabla v_h\|_{L^2(\Omega)} + \gamma_V^{\frac{1}{2}} \|h^{\frac{1}{2}} [\partial_n v_h]\|_{\mathcal{F}_i \cup \mathcal{F}_{\Gamma_N}} + |v_h|_{s_V}. \tag{3.28}$$

Lemma 3.2.14 For all $v \in H^2(\Omega)$,

$$|||v - \mathcal{I}_h^{CR}(v)|||_V \leq C \ \gamma_V^{\frac{1}{2}} \ h \ |v|_{H^2(\Omega)}$$

Proof. It is given by Lemma 3.2.9, Lemma 3.2.10 and Lemma 3.2.11 . ■

We define a norm on $V_h^{CR} \times V_h^{CR}$ by

$$|||(u_h, z_h)||| := |||u_h|||_V + \|z_h\|_{s_W}.$$

Theorem 3.2.15 Assume that $(\gamma_V \gamma_W) \leq 1$. Then there exists a positive C_s independent of γ_V, γ_W and h, such that

$$C_s |||(x_h, y_h)||| \leq \sup_{(v_h, w_h) \in \mathcal{V}_h} \frac{\mathcal{A}_h[(x_h, y_h), (v_h, w_h)]}{|||(v_h, w_h)|||} \tag{3.29}$$

with (x_h, y_h) be a solution of the formulation (3.25).

Proof.

First we calculate the following

$$
\begin{aligned}
a_h(x_h, w_h) &= \sum_{T\in\mathcal{T}} \int_T \nabla x_h \cdot \nabla w_h \ \mathrm{dx} = \sum_{T\in\mathcal{T}} \left(\int_T (-\Delta x_h) w_h \ \mathrm{dx} + \int_{\partial T} \partial_n x_h w_h \ \mathrm{ds} \right) \\
&= \sum_{T\in\mathcal{T}} \int_{\partial T} \partial_n x_h w_h \ \mathrm{ds} \\
&= \sum_{F\in\mathcal{F}} \left(\int_F \partial_n x_h^+ w_h^+ \ \mathrm{ds} + \int_F \partial_n x_h^- w_h^- \ \mathrm{ds} \right) \\
&= \sum_{F\in\mathcal{F}} \int_F [\![\partial_n x_h]\!] \{w_h\} \ \mathrm{ds}
\end{aligned}
$$

Let $\xi_h \in V_{h,\Gamma_N'}^{CR}$ be a function defined by

$$
\forall\ F \in \mathcal{F}_i \cup \mathcal{F}_{\Gamma_N} \qquad \{\xi_h\}|_F := \gamma_V h_F [\![n_F \cdot \nabla x_h]\!]_F .
$$

In (3.25) we take $w_h = \xi_h$ and $v_h = 0$, we then get

$$
\mathcal{A}_h[(x_h, y_h), (0, \xi_h)] = a_h(x_h, \xi_h) - b_h(x_h, \xi_h) - s_W(y_h, \xi_h)
$$

$$
\Longrightarrow \ \mathcal{A}_h[(x_h, y_h), (0, \xi_h)] + b_h(x_h, \xi_h) + s_W(y_h, \xi_h) = a_h(x_h, \xi_h) \tag{3.30}
$$

otherwise, we have

$$
\begin{aligned}
a_h(x_h, \xi_h) &= \sum_{F\in\mathcal{F}} \int_F [\![n_F \cdot \nabla x_h]\!] \{\xi_h\} \ \mathrm{ds} \\
&= \sum_{F\in\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} \gamma_V \int_F h_F [\![n_F \cdot \nabla x_h]\!]^2 \ \mathrm{ds} \\
&= \gamma_V \|h_F^{1/2} [\![n_F \cdot \nabla x_h]\!]\|^2_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} .
\end{aligned} \tag{3.31}
$$

and

$$\begin{aligned}
b_h(x_h,\xi_h) &= \sum_{F\in\mathcal{F}_{\partial\Omega}} \left(\int_{F\cap\Gamma_N'} n\cdot\nabla x_h \xi_h \text{ ds} + \int_{F\cap\Gamma_D} n\cdot\nabla\xi_h x_h \text{ ds} \right) \\
&= \sum_{F\in\mathcal{F}_{\partial\Omega}} \int_{F\cap\Gamma_D} n\cdot\nabla\xi_h x_h \text{ ds} \\
&= \sum_{F\in\mathcal{F}_{\partial\Omega}} \int_{F\cap\Gamma_D} h^{\frac{1}{2}} n\cdot\nabla\xi_h h^{-\frac{1}{2}} x_h \text{ ds} \\
&\leq \left(\sum_{F\in\mathcal{F}_{\partial\Omega}} \|h^{\frac{1}{2}} n\cdot\nabla\xi_h\|^2_{L^2(F)} \right)^{1/2} \|h^{-\frac{1}{2}} x_h\|_{L^2(\Gamma_D)} \\
&\leq C \left(\sum_{T\in\mathcal{T}} \|\nabla\xi_h\|^2_{L^2(T)} \right)^{1/2} \|h^{-\frac{1}{2}} x_h\|_{L^2(\Gamma_D)} \qquad \text{(by Lemma 3.2.7)} \\
&\leq C\ C_i\ \|h^{-1}\xi_h\|_{L^2(\Omega)}\ \|h^{-\frac{1}{2}} x_h\|_{L^2(\Gamma_D)} \\
&\leq C\ C_i\ c_\tau \gamma_V^{\frac{1}{2}} \|h_F^{1/2} [\![n_F\cdot\nabla x_h]\!]\|_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} |x_h|_{s_V} \qquad \text{(by Lemma 3.2.12)}
\end{aligned}$$

By using Young's inequality[1] with $a = (CC_ic_\tau)|x_h|_{s_V}, b = \gamma_V^{\frac{1}{2}}\|h_F^{1/2}[\![n_F\cdot\nabla x_h]\!]\|_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}}$ and $\varepsilon = 2$ we get

$$b_h(x_h,\xi_h) \leq (CC_ic_\tau)^2 |x_h|^2_{s_V} + \frac{1}{4}\gamma_V \|h_F^{1/2}[\![n_F\cdot\nabla x_h]\!]\|^2_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} \tag{3.32}$$

to bound $s_W(\cdot,\cdot)$ we proceed as follow

$$\begin{aligned}
s_W(y_h,\xi_h) &\leq \|y_h\|_{s_W}\|\xi_h\|_{s_W} \\
&\leq \|y_h\|_{s_W} C_i \gamma_W^{\frac{1}{2}} \|h^{-1}\xi_h\|_\Omega \qquad \text{(by Lemma 3.2.6)} \\
&\leq C_i\ c_\tau\ \|y_h\|_{s_W} (\gamma_V\gamma_W)^{\frac{1}{2}} \gamma_V^{\frac{1}{2}} \|h_F^{1/2}[\![n_F\cdot\nabla x_h]\!]\|_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} \\
&\leq (C_ic_\tau)^2 \|y_h\|^2_{s_W} + \frac{1}{4}\gamma_V \|h_F^{1/2}[\![n_F\cdot\nabla x_h]\!]\|^2_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}}
\end{aligned} \tag{3.33}$$

where we used that $(\gamma_V\gamma_W) \leq 1$ and Young's inequality.

Let $\alpha\in\mathbb{R}$,

$$\mathcal{A}_h[(x_h,y_h),(\alpha x_h, -\alpha y_h + \xi_h)] = \mathcal{A}_h[(x_h,y_h),(0,\xi_h)] + \alpha s_W(y_h,y_h) + \alpha s_V(x_h,x_h) \tag{3.34}$$

[1] $ab \leq \dfrac{\varepsilon a^2}{2} + \dfrac{b^2}{2\varepsilon} \qquad \forall a,b\in\mathbb{R}$ and $\varepsilon > 0$.

and from (3.30), (3.31), (3.32) and (3.33) we have

$$\frac{1}{2}\gamma_V \|h_F^{1/2}[\![n_F\cdot\nabla x_h]\!]\|^2_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} - (CC_ic_\tau)^2|x_h|^2_{s_V} - (C_ic_\tau)^2\|y_h\|^2_{s_W} \leq \mathcal{A}_h[(x_h,y_h),(0,\xi_h)]$$

$\Longrightarrow$

$$\frac{1}{2}\gamma_V \|h_F^{1/2}[\![n_F\cdot\nabla x_h]\!]\|^2_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} - (CC_ic_\tau)^2|x_h|^2_{s_V} - (C_ic_\tau)^2\|y_h\|^2_{s_W} + \alpha s_W(y_h,y_h) + \alpha s_V(x_h,x_h) \leq \mathcal{A}_h[(x_h,y_h),(\alpha x_h,-\alpha y_h+\xi_h)]$$

for $\alpha = \frac{1}{2} + (C_ic_\tau)^2\max(1,C^2)$ we obtain

$$\frac{1}{2}\left(|x_h|^2_{s_V} + \|y_h\|^2_{s_W} + \gamma_V\|h_F^{1/2}[\![n_F\cdot\nabla x_h]\!]\|^2_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}}\right) \leq \mathcal{A}_h[(x_h,y_h),(\alpha x_h,-\alpha y_h+\xi_h)] \tag{3.35}$$

To include $\|\nabla x_h\|_{L^2(\Omega)}$ we use Lemma 3.2.8

$$\|\nabla x_h\|^2_{L^2(\Omega)} \leq C\sum_{F\in\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} h_F^{-1}\|[\![\nabla x_h]\!]\|^2_F$$

by decomposing the jump of the gradient and applying Lemma 3.2.6 we find

$$h_F^{-1}\|[\![\nabla x_h]\!]\|^2_F \leq Ch^{-2}(\|h^{\frac{1}{2}}[\![n\cdot\nabla x_h]\!]\|^2_F + \gamma_V^{-1}|x_h|^2_{s_V})$$

then

$$h\gamma_V^{\frac{1}{2}}\|\nabla x_h\|_h \leq C(\gamma_V^{\frac{1}{2}}\|h^{\frac{1}{2}}[\![n\cdot\nabla x_h]\!]\|_{\mathcal{F}_i\cup\mathcal{F}_{\Gamma_N}} + |x_h|_{s_V}). \tag{3.36}$$

By 3.35 and 3.36, we may conclude that there exists $c_0 > 0$ such that

$$c_0|||(x_h,y_h)|||^2 \leq \mathcal{A}_h[(x_h,y_h),(\alpha x_h,-\alpha y_h+\xi_h)] \tag{3.37}$$

otherwise

$$|||(\alpha x_h,-\alpha y_h+\xi_h)||| \leq \alpha|||(x_h,y_h)||| + |||(0,\xi_h)|||$$

and

$$|||(0,\xi_h)||| = \|\xi_h\|_{s_W}$$

we have

$$\begin{aligned}\|\xi_h\|_{s_W} &\leq \gamma_W\frac{1}{2}C_i\|h^{-1}\xi_h\|_\Omega \quad \text{(by Lemma 3.2.6)}\\ &\leq C_ic_\tau(\gamma_W\gamma_V)^{\frac{1}{2}}|||x_h|||_V \quad \text{(by Lemma 3.2.12)}\end{aligned}$$

then

$$|||(\alpha x_h, -\alpha y_h + \xi_h)||| \le (\alpha + C_i c_\tau)|||(x_h, y_h)||| \tag{3.38}$$

by (3.37) and (3.38) we get

$$(c_0/(\alpha + C_i c_\tau))|||(x_h, y_h)||| \le \frac{\mathcal{A}_h[(x_h, y_h), (\alpha x_h, -\alpha y_h + \xi_h)]}{|||(\alpha x_h, -\alpha y_h + \xi_h)|||}$$

then

$$\sup_{(v_h, w_h) \in \mathcal{V}_h} \frac{\mathcal{A}_h[(x_h, y_h), (v_h, w_h)]}{|||(v_h, w_h)|||} \ge \frac{\mathcal{A}_h[(x_h, y_h), (\alpha x_h, -\alpha y_h + \xi_h)]}{|||(\alpha x_h, -\alpha y_h + \xi_h)|||} \ge C_s |||(x_h, y_h)|||$$

with $C_s = c_0/(\alpha + C_i c_\tau)$. ■

Corollary 3.2.16

There exist a unique solution for the formulation (3.25).

Proof. The system matrix corresponding to (3.25) is a square matrix and we only need to show that there are no zero eigenvalues.

Assume that zero is an eigenvalue for the system matrix corresponding to (3.25), then by Theorem 3.2.15 we get

$$\text{For any solution } (u_h, z_h), \qquad C_s |||(u_h, z_h)||| \le \sup_{(v_h, w_h) \in \mathcal{V}_h} \frac{\mathcal{A}_h[(u_h, z_h), (v_h, w_h)]}{|||(v_h, w_h)|||} = 0$$

implying that $(u_h, z_h) = (0, 0)$ which is a contradiction. ■

Chapter 4

Numerical tests

In this chapter, we will present numerical examples for stabilized finite element method which was introduced in chapter 3, by using `FreeFEM++`.

`FreeFEM++` is a Free software to solve PDE using the Finite Element Method.

We choose the following examples as a numerical tests:

Example 4.1 [2]

We solve the Cauchy problem (2.15) on the unit square $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ with

$$\Gamma := \{x \in (0,1), y = 0\} \cup \{x = 1, y \in (0,1)\}$$

and the Neumann data is prescribed, we choose f such that the exact solution is $u(x,y) = 30x(1-x)y(1-y)$. Then the results are as follows

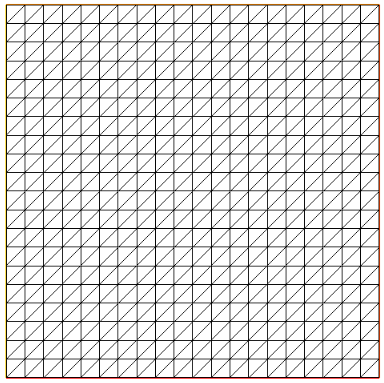


Figure 4.1: Subdivision of Ω.

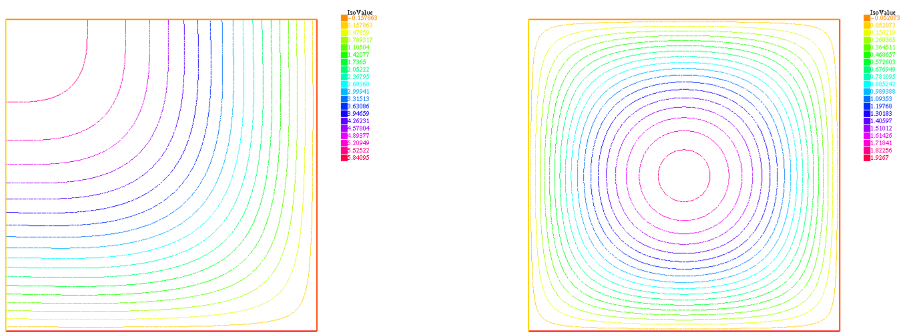


Figure 4.2: The approximate solution before stabilization (left) the exact solution (right).

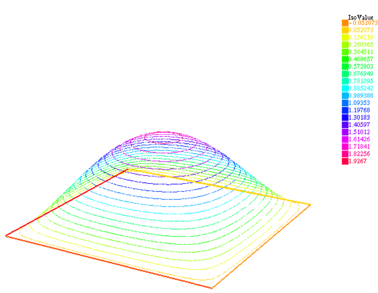


Figure 4.3: The approximate solution after stabilization (d3).

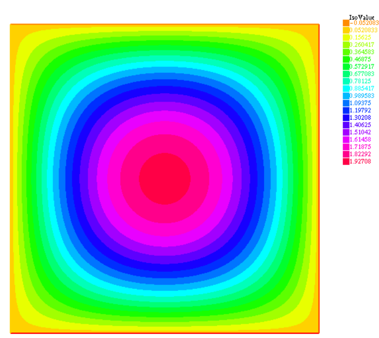


Figure 4.4: The approximate solution after stabilization.

We use unstructured meshes with $2^{(n+3)}$ elements on each side, $n = 0, \cdots, 4$, and we fix the stabilization parameters (0.01) we then get Table 4.1.

n	$\|u-u_h\|_{L^2(\Omega)}$	$\|z_h\|_{H^1(\Omega)}$
0	0.026	0.059
1	0.011	0.041
2	0.005	0.028
3	0.002	0.020
4	0.001	0.014

Table 4.1: The norms of error and z_h under variation of n.

$\gamma_V=\gamma_W$	$\|u-u_h\|_{L^2(\Omega)}$	$\|z_h\|_{H^1(\Omega)}$
0.001	0.0026	0.001
0.01	0.0026	0.019
0.1	0.0026	0.182
0.2	0.0026	0.358
0.5	0.0027	0.888

Table 4.2: The norms of error and z_h under variation of parameters.

We can see the stability of the approximate solution u_h by taking a small perturbation in the data as follows

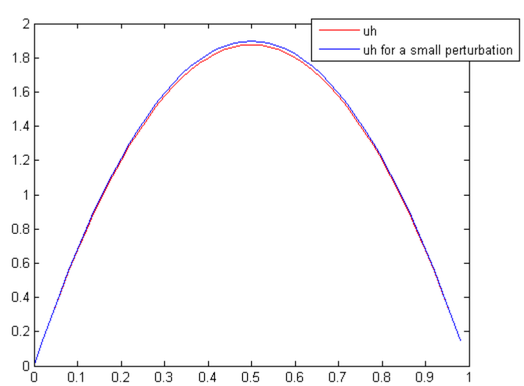


Figure 4.5: $u_h(x,0.5)$ and $u_h(x,0.5)$ for $f=f+0.3,\ \psi=\psi+0.05$.

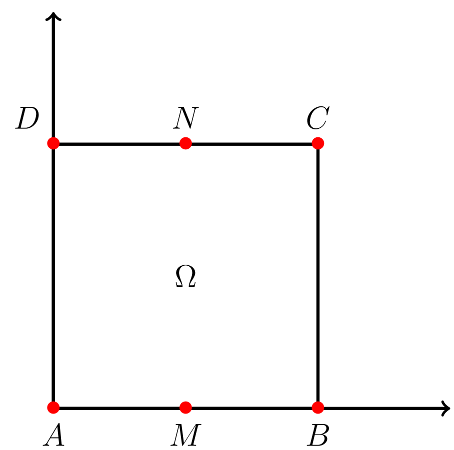
Example 4.2 [7]

Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ in this example we consider the following model problem for the numerical simulation:

$$\begin{cases} \Delta u = 0, & \text{in } \;\Omega \\ u = g_0 & \text{on } \;\Gamma \\ \partial_n u = g_1 & \text{on } \;\Gamma \end{cases} \tag{4.1}$$

this problem has the unique solution $u(x,y) = -yx^2 + y^3/3$.

We denote $A = (0,0)$, $B = (1,0)$, $C = (1,1)$, $D = (0,1)$, $M(\frac{1}{2},0)$ and $N(\frac{1}{2},1)$ in the (x,y) coordinates,



We consider the 3 following cases :

- case 1: $\Gamma = [A,B] \cup [B,C] \cup [C,D]$
- case 2: $\Gamma = [M,B] \cup [B,C] \cup [C,N]$
- case 3: $\Gamma = [B,C]$.

We compute artificial data g_0 and g_1 on Γ from the exact solution u, the results are obtained using finite elements based on P_1 polynomials, on a 20×20 mesh.

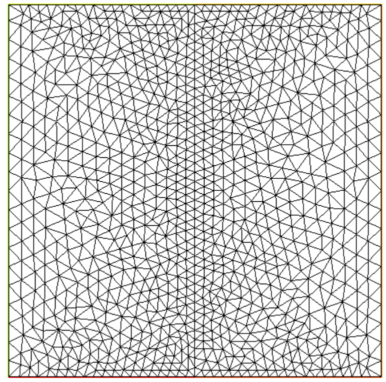


Figure 4.6: Subdivision of Ω.

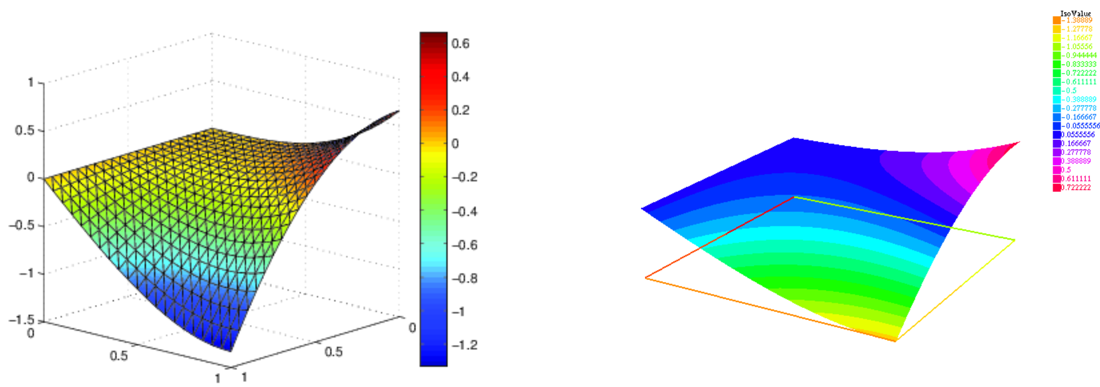


Figure 4.7: The exact solution.

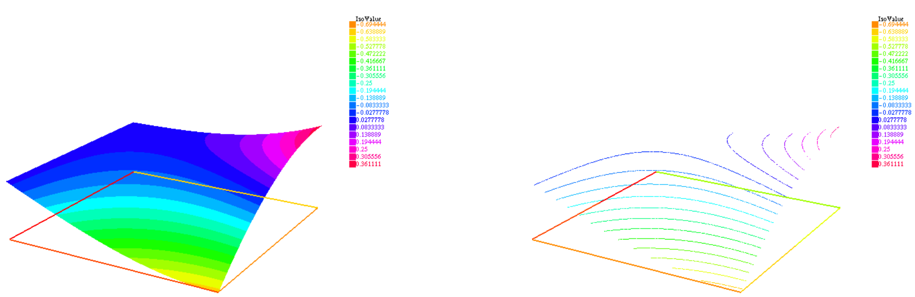


Figure 4.8: The approximate solution case 1 (left), case 2 (right) .

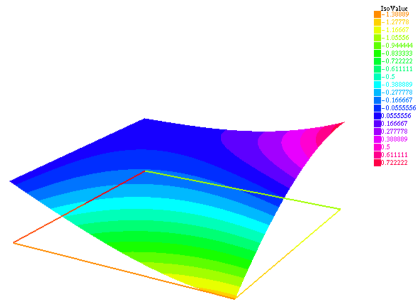


Figure 4.9: The approximate solution case 3.

case	$\|u-u_h\|_{L^2(\Omega)}$	$\|z_h\|_{H^1(\Omega)}$
1	0.196	0.100
2	0.196	0.091
3	0.196	0.082

Table 4.3: The norms of error and z_h in the 3 cases.

Example 4.3 [21]

Let Ω be a square from which a disc is removed:

$$\Omega = [(x,y) \in \mathbb{R}^2; 0 < x < 10; -5 < y < 5] \setminus \Gamma_0$$

with

$$\Gamma_0 = [(x,y) \in \mathbb{R}^2; (x-5)^2 + y^2 \leq 9]$$

In this example we consider the following model problem

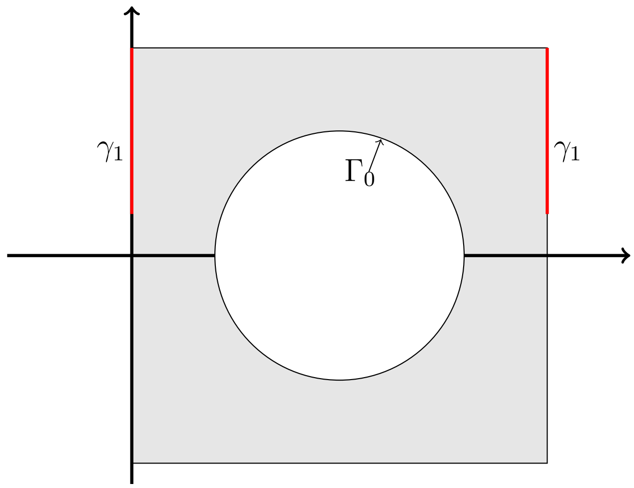
$$\begin{cases} -\Delta u + 0.01u = 0, & \text{in } \Omega \\ u = 0, & \text{on } \gamma_1 \cup \Gamma_0 \\ \partial_n u = -0.5, & \text{on } \Gamma_1 \end{cases} \tag{4.2}$$

where

$$\Gamma_1 = \{x = 0 \cup x = 10 \cup y = \pm 5\}$$

$$\gamma_1 = \{(0, 1 \leq y \leq 5\} \cup \{(10, 1 \leq y \leq 5\}$$

We Note that $\gamma_1 \subset \Gamma_1$



The obtained results are as follow

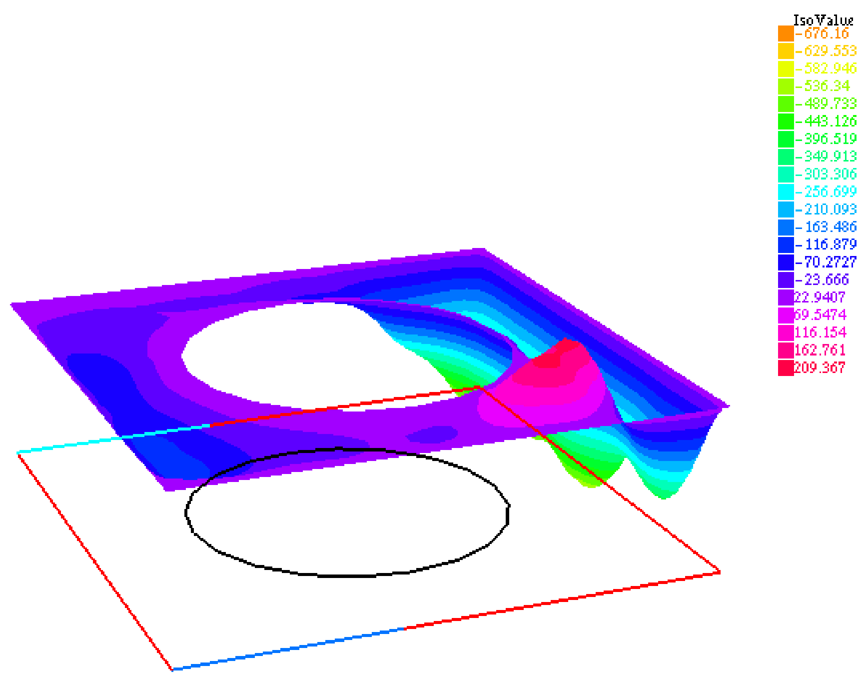


Figure 4.10: The approximate solution.

Example 4.4 (Nonconforming finite element method)

We solve the same example 4.1 but by scheme (3.25) with $\gamma_V = \gamma_W = \gamma_{W,bc} = 0.01$, then we obtain

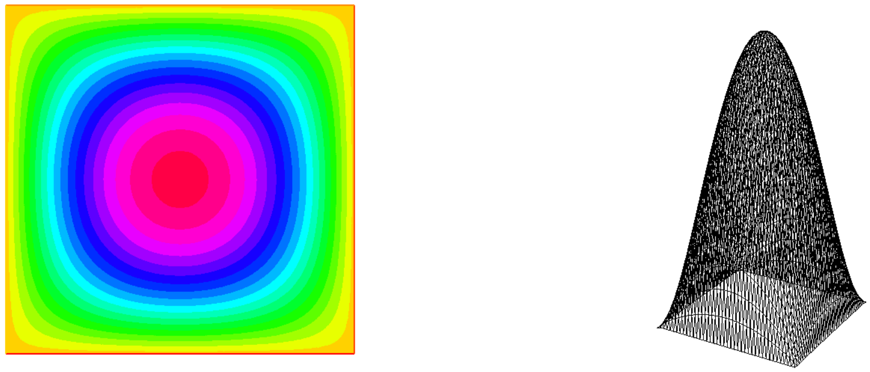


Figure 4.11: The approximate solution.

With a different number of triangles, we get

Number of triangles	$\|u - u_h\|_{L^2(\Omega)}$	$\|z_h\|_{H^1(\Omega)}$	$\|\partial_n(u - u_h)\|_{H^{-\frac{1}{2}}(\Gamma)}$
1800	0.0018	0.0016	0.0580
3200	0.0010	0.0009	0.0359
5000	0.0006	0.0006	0.0249
7200	0.0004	0.0004	0.0185

Table 4.4: The norms of error and z_h, the normal derivative of error.

Conclusion

In this thesis, we have presented stabilized finite element methods for the solution of the ill-posed elliptic Cauchy problem, we prove Carleman estimates type and the three-spheres inequality. By these inequalities, we show that the ill-posed Cauchy problem for Laplace equation has a unique solution and conditional stability results can be obtained.

We have used the stabilized finite element method for the approximation of ill-posed Cauchy problem of Laplace equation which proposed to formulate the problem as a constrained minimization problem that is regularized on the discrete level using tools known from the theory of stabilized finite element methods. And we have got the error estimates without using the Lax-Milgram lemma or the Babushka-Brezzi theorem.

As perspective, we will introduce the following cases to be addressed in the future

- Conforming method with $u \in H^1(\Omega)$.
- Mixed discontinuous Galerkin finite element method.
- Generalized elliptic membrane shells.

Appendix A

Appendix

A.1 The Caccippoli Inequality

The Caccippoli (or Reverse Poincare) inequality bounds similar terms to the Poincare inequalities, but the other way around. The statement is this.

Theorem A.1.1 Let $u : B_{2r} \longrightarrow \mathbb{R}$ satisfy $u\Delta u \geq 0$. Then

$$\int_{B_r} |\nabla u|^2 \leq \frac{4}{r^2} \int_{B_{2r} \setminus B_r} u^2.$$

First prove a Lemma.

Lemma A.1.2

If $u : B_{2r} \longrightarrow \mathbb{R}$ satisfies $u\Delta u \geq 0,$ and $\phi : B_{2r} \longrightarrow \mathbb{R}$ is non-negative with $\phi = 0$ on ∂B_{2r}, then

$$\int_{B_{2r}} \phi^2 |\nabla u|^2 \leq 4 \int_{B_{2r}} |u|^2 |\nabla \phi|^2.$$

Proof. Consider

$$0 \leq \int_{B_{2r}} \phi^2 u \Delta u.$$

Clearly $\int_{\partial B_{2r}} \phi^2 u \nabla u \; dS \; = 0$, so apply Stokes' theorem to get $\int_{B_{2r}} \phi^2 u \Delta u + \int_{B_{2r}} \nabla(\phi^2 u) \cdot \nabla u \; = 0$. From this

$$0 \leq -\int_{B_{2r}} \nabla(\phi^2 u) \cdot \nabla u \; = -2\int_{B_{2r}} \phi u \nabla\phi \cdot \nabla u \; - \; \int_{B_{2r}} \phi^2 |\nabla u|^2,$$

and so

$$\begin{aligned}
\int_{B_{2r}} \phi^2 |\nabla u|^2 \; &\leq \; -2\int_{B_{2r}} \phi u \nabla\phi \cdot \nabla u \\
&\leq \; 2\int_{B_{2r}} \phi |u| |\nabla\phi| |\nabla u| \\
&= \; 2\int_{B_{2r}} \phi |\nabla u| |u| |\nabla\phi| \\
&\leq \; 2\left(\int_{B_{2r}} \phi^2 |\nabla u|^2\right)^{1/2} \left(\int_{B_{2r}} |u|^2 |\nabla\phi|^2\right)^{1/2}. \;\; \text{(By Cauchy-Schwarz inequality)}
\end{aligned}$$

Dividing and squaring then gives

$$\int_{B_{2r}} \phi^2 |\nabla u|^2 \leq 4\int_{B_{2r}} |u|^2 |\nabla\phi|^2.$$

■

To complete the proof of theorem A.1.1 pick

$$\phi(x) \;\; = \;\; \begin{cases} 1 & \text{if } |x| \leq \; r; \\ \frac{2r-|x|}{r} & \text{if } r < |x| \leq \; 2r, \end{cases}$$

so

$$|\nabla\phi(x)| \;\; = \;\; \begin{cases} 0 & \text{on } B_r \; ; \\ \frac{1}{r} & \text{on } B_{2r} \backslash B_r. \end{cases}$$

Substitute this into the lemma to obtain the result, namely

$$\int_{B_r} |\nabla u|^2 \leq \frac{4}{r^2} \int_{B_{2r} \backslash B_r} u^2.$$

A.2 Lipschitz regularity

In the first we shall introduce the following notation

$$B'_r(x') = \{y' \in \mathbb{R}^{n-1} | \; |y' - x'| < r\}, \quad B'_r = B'_r(0),$$

$$\Gamma_{a,b}(x) = \{y = (y', y_n) \in \mathbb{R}^n | \; |y' - x'| < a, \; |y_n - x_n| < b\}, \quad \Gamma_{a,b} = \Gamma_{a,b}(0).$$

Definition A.2.1 [1]

We say that the boundary of Ω is of Lipschitz class with constants $\rho_0,\ M_0 > 0$, if, for any point $p \in \partial\Omega$, there exists a rigid transformation of coordinates under which $p = 0$ and

$$\Omega \cap \Gamma_{\frac{\rho_0}{M_0},\rho_0}(p) = \{x = (x_1, \cdots, x_{n-1}, x_n) = (x', x_n) \in \Gamma_{\frac{\rho_0}{M_0},\rho_0} \mid x_n > Z(x')\},$$

where $Z : B'_{\frac{\rho_0}{M_0}} \rightarrow \mathbb{R}$ is a Lipschitz function satisfying

$$Z(0) = 0, \qquad \|Z\|_{C^{0,1}\left(B'_{\frac{\rho_0}{M_0}}\right)} \leq M_0\rho_0.$$

A.3 Some results about the reference polyhedron

Definition A.3.1

The reference polyhedron $\hat{T}$ is the unit d-simplex, i.e, the triangle of vertices $(0,0),(1,0),(0,1)$ (when $d = 2$), or the tetrahedron of vertices $(0,0,0),(1,0,0),(0,1,0),(0,0,1)$ (when $d = 3$).

Let us write the affine transformation

$$F_T(\hat{x}) = B_T(\hat{x}) + b_T, \qquad \hat{x} \in \hat{T}$$

where B_T is a $(d \times d)$-matrix.

Proposition A.3.2 (Seminorm transformation)

For all integer $m \geq 0$ and all $v \in H^m(T)$, define $\hat{v} := v \circ F_T$. Then $\hat{v} \in H^m(\hat{T})$, and there exists a constant $C = C(m) > 0$ such that

$$|\hat{v}|_{H^m(\hat{T})} \leq C\|B_T\|^m|\det B_T|^{-\frac{1}{2}}|v|_{H^m(T)},$$
$$|v|_{H^m(T)} \leq C\|B_T^{-1}\|^m|\det B_T|^{\frac{1}{2}}|\hat{v}|_{H^T(\hat{T})},$$

where $\|\cdot\|$ is the matrix norm associated to the euclidean norm in $\mathbb{R}^d$. There holds moreover

$$\|B_T\| \leq \frac{h_T}{\hat{\rho}}, \quad \|B_T^{-1}\| \leq \frac{\hat{h}}{\rho_T}$$

with

$$\rho_T := \sup\{\mathrm{diam}(S)|\ S \text{ is a ball contained in } T\}$$

$\hat{h}$ and $\hat{\rho}$ are the diameter and the radius of the ball inscribed in the reference polyhedron $\hat{T}$.

Proof. See [24, p 86/87]. ■

Definition A.3.3

A family of triangulations $\mathcal{T}_h$ is called regular if there exists a constant $\sigma \geq 1$ such that

$$\frac{h_T}{\rho_T} \leq \sigma \qquad \forall h > 0.$$

Lemma A.3.4 (Deny-Lions)

For every $r \geq 0$ there exists a constant $C = C(r, \hat{T})$ such that

$$\inf_{\hat{p} \in \mathbb{P}_r(\hat{T})} \|\hat{p} + \hat{v}\|_{H^{r+1}(\hat{T})} \leq C|\hat{v}|_{H^{r+1}(\hat{T})} \quad \forall \hat{v} \in H^{r+1}(\hat{T}).$$

Proof. See [24, p 88]. ■

Theorem A.3.5 (The Poincaré-Wirtinger inequality)

Assume that $1 \leq p \leq \infty$ and that Ω is a bounded connected open subset of $\mathbb{R}^n$ with a Lipschitz boundary. Then there exists a constant C, depending only on Ω and p, such that for every $u \in W^{1,p}(\Omega)$,

$$\left\| u - \left(\frac{1}{|\Omega|} \int_\Omega u(y)\ dy \right) \right\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)}.$$

Remark A.3.6

On the space $W_0^{1,p}(\Omega)$ we get the Poincaré inequality.

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ملخص

في هذا العمل، نقدم the ill-posed elliptic Cauchy problem. باستخدام تقديرات كارلمان و متراجحة ثلاث سطوح تمكنا من الحصول على نتائج الاستقرار الشرطي. ثم نحلل طريقة العناصر المنتهية المستقرة. وقدمنا أيضا إختبارات عددية التي توضح وتتحقق من صحة طريقتنا.

الكلمات المفتاحية : elliptic Cauchy problem، ill-posed problem ، تقديرات كارلمان، متراجحة ثلاث سطوح ، طريقة العناصر المنتهية المستقرة.

Résumé

Dans ce travail, nous présentons le problème de Cauchy elliptique mal posé. Par l'utilisation d'estimations de Carleman et l'inégalité de trois sphères, nous montrons que les résultats de stabilité conditionnelles peuvent être obtenus. Ensuite, nous analysons les méthodes d'éléments finis stabilisée. Des tests numériques qui illustrent et valider notre approche sont également présenté.

Mots clés : problème de Cauchy elliptique, problème mal posé, estimation de Carleman, inégalité de trois sphères, méthode des éléments finis stabilisée.

Abstract

In this work, we present the ill-posed elliptic Cauchy problem. By the use of Carleman estimates and the three-spheres inequality, we show that conditional stability results can be obtained. Then we analyze the stabilized finite element methods. Numerical tests that illustrate and validate our approach are also presented.

Key words: elliptic Cauchy problem, ill-posed problem, Carleman estimates, three-spheres inequality, stabilized finite element method.