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Lagrange multipliers method for variational inequalities of the second kind

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DEDICATION

I dedicate this modest work to my loving parents who are my greatest inspiration and my support in life

who never stop giving of themselves in countless ways,

To my brothers Mohammed , Lotfi and Abdo

To my sisters Sabrine and Maria

To all my family "Kessal"

To all my teachers

To all my friends who encourage and support me.

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Notations and Conventions

➤ V , Y: a Hilbert spaces.

➤ K: nonempty closed and convex subset of V.

➤ Λ : closed, convex,bounded subset of Y that contains 0_Y.

➤ $(.,.)_V$: the Inner product.

➤$\|.\|_V$:The norm in V.

➤ $0_V, 0_Y$: the zeros element of V and Y respectively.

➤ ∇: gradient operator.

➤ $\rightarrow$: Strong convergence.

➤ $\rightharpoonup$: Weak convergence . .

INTRODUCTION

The theory of variational inequalities represents in fact a very natural generalization of the theory of nonlinear boundary value problems arising in mechanics, physics, an engineering science.

The first problem involving a variational inequalities posed by Signorini [14] and solved by Fichera [6], where it was the first papers of the theory. Later by Stampacchia [15], Lions and Stampacchia [10] and Brezis[3, 4], this subject has developed after that in several directions using by applied mathematicians as Mosco [13], Glowinski [7], Lions and Tremolieres [8]. A several methods were used to prove the existence of the solution for these inequalities, such as Lagrange multipliers method (named after Joseph Louis Lagrange), which we chose apply in this memory.

We will focus our attention in this work to study the existence and uniqueness of the solution of variational inequalities of the second kind and qusivariational inequalities by using Lagrange multipliers method, the solvability of this inequalities is established by using saddle-point theory and a fixed- point technique,

This memory divided into three chapters, in **the first chapter** we represent a mathematical preliminaries which will be used in the next chapters. In **the second chapter** we present a basic existence and uniqueness result for elliptic variational inequalities, and in **the last chapter** we state Lagrange multipliers method for solving a nonlinear variational inequalities of the second kind in Hilbert space, before this we state regularization of this inequalities, then we provide the convergence result. Moreover we study the solvability of a generalizations of variational inequalities so called quasi-variational inequalities.

CHAPTER 1

MATHEMATICAL PRELIMINARIES

We present in this chapter preliminary material from functional analysis that will be used in subsequent chapters.

1.1 REMINDERS

Let $(\mathbf{X}, (.,.)_\mathbf{X})$ be inner product space and let f be a function on X with values in $\mathbb{R}$. The effective domain of f is the set

$$\mathbf{D(f) = \{u \in X : f(u) < \infty\}}.$$

Definition 1.1.1 We say that the function f is proper if $\mathrm{D(f)} \neq \emptyset$, that is, there exists $u \in X$such that $f(u) < +\infty$.

Definition 1.1.2 The function $\mathbf{f} : \mathbf{X} \to \mathbb{R}$ is convex if the inequality :

$$\mathbf{f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)}$$

hold for every $\lambda \in [\mathbf{0}, \mathbf{1}]$,and all $\mathbf{x}, \mathbf{y} \in \mathbf{X}$.

Definition 1.1.3 The function $\mathbf{f} : \mathbf{X} \to \mathbb{R}$ is strictly convex if the inequality :

$$\mathbf{f}(\lambda\mathbf{x}+(\mathbf{1}-\lambda)\mathbf{y})<\lambda\mathbf{f}(\mathbf{x})+(\mathbf{1}-\lambda)\mathbf{f}(\mathbf{y})$$

hold for every $\lambda \in]\mathbf{0},\mathbf{1}[$,and all $\mathbf{x},\mathbf{y}\in\mathbf{X}$.

Definition 1.1.4 Let $(\mathbf{X},\|.\|_{\mathbf{X}})$ be a normed space. A subset $A\subset X$ is bounded if there exists $\mathrm{M}>\mathbf{0}$ such that $\|\mathbf{u}\|_{\mathbf{X}}\leq\mathrm{M}$ for all $\mathbf{u}\in\mathbf{A}$. A sequence $\{\mathbf{u_n}\}\subset\mathbf{X}$ is bounded if there exists $\mathbf{M}>\mathbf{0}$ such that $\|u_n\|_X\leq M$ for all $n\in\mathbb{N}$.

Definition 1.1.5 Let $\mathbf{X}$ be a normed space. A sequence $\{\mathbf{u_n}\}\subset\mathbf{X}$ is said to converge strongly to $\mathbf{u}\in\mathbf{X}$ if

$$\|\mathbf{u_n}-\mathbf{u}\|_{\mathbf{X}}\to\mathbf{0}\quad\text{as}\quad\mathbf{n}\to\infty$$

In this case, u is called the " strong limit" of the sequence $\{\mathbf{u_n}\}$ and we write

$$\mathbf{u}=\lim_{\mathbf{n}\to\infty}\mathbf{u_n}\quad\text{or}\quad\mathbf{u_n}\to\mathbf{u}\quad\text{in }\mathbf{X}$$

Definition 1.1.6 Let $\mathbf{X}$ be a normed space. A sequence $\{u_n\}\subset X$ is said to converge weakly to $\mathbf{u}\in\mathbf{X}$ if for every $\mathbf{T}\in\mathbf{X}'$,

$$\mathbf{T}(\mathbf{u_n})\to\mathbf{T}(\mathbf{u})\quad\text{as}\quad\mathbf{n}\to\infty$$

In this case, u is called the "weak limit" of $\{u_n\}$and we write $u_n\rightharpoonup u$ in X.

Definition 1.1.7 Let X be a normed space. A subset $B\subset X$ is said to be weakly closed if it contains the limits of all weakly convergent sequences $\{u_n\}\subset B$.

Clearly, every weakly closed subset of X is closed .

Definition 1.1.8 Let$X$ be a linear space. A subset $K\subset X$ is said to be convex if it has the property

$$u,v\in K,\Longrightarrow(1-t)u+tv\in K\quad\forall t\in[0,1].$$

Definition 1.1.9 We say that the functional $j:V\to\mathbb{R}\cup\{+\infty\}$ is upper semi-continuous (u.s.c)if it satisfies the following equivalent conditions:

- $\forall\alpha\in\mathbb{R},\quad\{u\in V,J(u)\geq\alpha\}$ is closed
- $\forall u^*\in V,\quad\limsup_{u\to u^*}j(u)\leq j(u^*)$

Definition 1.1.10 We say that the functional $j : V \to \mathbb{R} \cup \{+\infty\}$ is lower semi-continuous (l.s.c) if it satisfies the following equivalent conditions:

- $\forall \alpha \in \mathbb{R}, \quad \{u \in V, J(u) \leq \alpha\}$ is closed
- $\forall u^* \in V, \quad \liminf_{u \to u^*} j(u) \geq j(u^*)$

Corollary 1.1.11 Let $j : V \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semi-continuous functional (for example continue) for strong topology, if $\{v_n\}$ is a sequence of V weakly convergent to v then

$$j(v) \leq \lim_{n \to +\infty} \inf j(v_n)$$

Definition 1.1.12 Let $(V, \langle ., . \rangle_V, \|.\|_V)$ be Hilbert space $\forall u, v \in V$, The operator $A : V \to V$ is said to be:

(i) monotone ,if:

$$\langle Au - Av, u - v \rangle_V \geq 0$$

(ii) strictly monotone, if

$$\langle Au - Av, u - v \rangle_V > 0$$

(iii) Strongly monotone ,if there exist a constant $\alpha_1 > 0$ such that:

$$\langle Au - Av, u - v \rangle_V \geq \alpha_1 \|u - v\|_V^2$$

(iv) Lipschitz continuous, if there exist a constant $\beta > 0$ such that :

$$\|Au - Av\|_V \leq \beta \|u - v\|_V$$

Proposition 1.1.13 Let $(V, (., .)_V)$ be an inner product space and let
$\phi : V \to (-\infty, \infty]$ be a proper convex lower semi-continuous function. Then ϕ is bounded from below by an affine function, i.e., there exists $\alpha \in \mathbb{R}$and $\beta \in \mathbb{R}$ such that $\phi(u) \geq (\alpha, u)_V + \beta \quad \forall u \in V$.

Proposition 1.1.14 Let$(V, (., .)_V)$ be an inner product space and let $a : V \times V \to \mathbb{R}$ be a bilinear symmetric continuous and V-elliptic form. Then the function $v \to a(v, v)$ is convex and lower semi-continuous.

Proof. See [16] ■

Theorem 1.1.15 (Eberlein-Smulyan) If $V$is a reflexive Banach space, then each bounded sequence in V has a weakly convergent subsequence.

Definition 1.1.16 Let $\varphi : V \to \mathbb{R}$ and let $u \in V$ Gâteaux differentiable at u if there exists an element $\nabla\varphi(u) \in V$ such that

$$\lim_{t\to 0}\frac{\varphi(u+tv)-\varphi(u)}{t} = (\nabla\varphi(u), v)_V \quad \forall v \in V.$$

The function $\varphi : V \to \mathbb{R}$ is said to be Gâteaux differentiable if it is Gâteaux differentiable at every point of V.

1.2 The Banach Fixed Point Theorem

Theorem 1.2.1 Let K be a nonempty closed subset of a Banach space $(X, \|.\|_X)$. Assume that $T : K \to K$ is a contraction, i.e., there exists a constant $\alpha \in [0, 1)$ such that

$$\|Tu - Tv\|_X \leq \alpha\|u - v\|_X \;\; \forall u, v \in K$$

Then there exists a unique $u \in K$such that $Tu = u$. A solution $u \in K$ of the operator equation $Tu = u$ is called a fixed point of T in K.

1.3 The Theorem of Stampacchia

Definition 1.3.1 A bilinear form $a : V \times V \to \mathbb{R}$ is said to be

(i) continuous if there is a constant C such that

$$|a(u, v)| \leq C\|u\|\|v\| \quad \forall u, v \in V,$$

(ii) coercive if there is a constant $\alpha > 0$ such that

$$a(v, v) \geq \alpha\|v\|^2 \quad \forall v \in V.$$

Theorem 1.3.2 (Stampacchia) Assume that $a(u, v)$ is a continuous coercive bilinear form on V. Let $K \subset V$ be a nonempty closed and convex subset. Then, given any $\varphi \in V'$, there exists a unique element $u \in K$such that

$$a(u, v - u) \geq (\varphi, v - u) \qquad \forall v \in K$$

Moreover, if a is symmetric, then u is characterized by the property

$$u \in K \quad \text{and} \quad \frac{1}{2}a(u,u) - (\varphi, u) = \min_{v \in K} \left[\frac{1}{2}a(v,v) - (\varphi, v).\right]$$

Proof. see [2] ■

1.4 The Projection Theorem

Theorem 1.4.1 Let K be a nonempty closed convex subset in a Hilbert space V. Then for each $u \in V$ there is a unique element $u^* = P_K u \in K$ such that

$$\|u - u^*\|_V = \min_{v \in K} \|u - v\|_V.$$

The operator $P_K : V \to K$ is called the projection operator onto K.

The element $u^* = P_K u \in K$ is called the projection of u on K and is characterized by the inequality

$$(u^* - u, u^* - v) \geq 0 \quad \forall v \in K.$$

Proof. see [2] ■

1.5 Saddle point theorem

Definition 1.5.1 Let A and B be nonempty sets. A pair $(u, \lambda) \in A \times B$ is said to be a saddle point of a functional $\mathcal{L} : A \times B \to \mathbb{R}$ if and only if

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda) \quad \text{for all}\, v \in A, \mu \in B.$$

Theorem 1.5.2 Let $(V, (.,.)_V, \|.\|_V), (Y, (.,.)_Y,\ \|.\|_Y)$ be two Hilbert space and let $A \subseteq V, B \subseteq Y$ be nonempty, closed, convex subsets. Assume that a real functional $\mathcal{L} : A \times B \to \mathbb{R}$ satisfies the following conditions

a) $v \mapsto \mathcal{L}(v, \mu)$ is convex and lower semi-continuous for all $\mu \in B$.

b) $\mu \mapsto \mathcal{L}(v, \mu)$ is concave and upper semi-continuous for all $v \in A$.

Moreover, assume that

c) A is bounded or $\lim\limits_{\|v\|_V \to \infty, v \in A} \mathcal{L}(v, \mu_0) = \infty$ for some $\mu_0 \in B$.

d) B is bounded or $\liminf_{\|\mu\|_Y \to \infty, \mu \in B} \inf_{v \in A} .\mathcal{L}(v, \mu) = -\infty$

Then the functional $\mathcal{L}$ has at least one saddle point.

For the more details on the saddle point theory see at [1, 5, 9] .

Chapter 2

Elliptic Variational inequalities

In this chapter we study the existence and uniqueness of solutions of elliptic variational inequalities of the second kind and quasivariationl inequalities.

2.1 Variational inequalities of the second kind

Let $a : V \times V \to \mathbb{R}$ and let $j : V \to (-\infty, \infty]$. Given $f \in V$ we consider the following problem

$$\begin{cases} \text{Find} \quad u \in V. \\ \\ a(u, v-u) + j(v) - j(u) \geq (f, v-u)_V \quad \forall v \in V \end{cases} \tag{2.1}$$

An inequality of the form (2.1) is called an *elliptic variational inequality of the second kind.* we consider the following assumptions:

Assumption 2.1.1 $a : V \times V \to \mathbb{R}$ is a bilinear symmetric form and

(a) there exists $M > 0$ such that

$$|a(u, v)| \leq M\|v\|_V\|u\|_V \quad \forall u, v \in V.$$

(b) there exists $m > 0$ such that

$$a(v,v) \geq m\|v\|_V^2 \quad \forall v \in V.$$

Assumption 2.1.2 $j : V \to (-\infty, \infty]$ is a proper convex l.s.c. function.

2.1.1 Existence and Uniqueness Result

✦ Symmetric case

Theorem 2.1.3 *[16]* Let V be a Hilbert space and assume that assumptions 2.1.1, 2.1.2 hold. Then, for each $f \in V$, the elliptic variational inequality (2.1) has a unique solution. Moreover, the solution depends Lipschitz continuously on f.

In order to prove this theorem, for all $f \in V$, we define the function
$J : V \to (-\infty, \infty]$ by the formula

$$J(v) = \frac{1}{2}a(v,v) + j(v) - (f,v)_V. \tag{2.2}$$

Firstly, we have the following equivalence result.

Lemma 2.1.4 Let $f \in V$. An element $u \in V$ is solution of the variational inequality (2.1) if and only if u is a minimizer of J on V, that is,

$$J(v) \geq J(u) \quad \forall v \in V. \tag{2.3}$$

Proof. Assume that u is a solution of (2.1). Using the properties of the form **a**, we have

$$J(v) - J(u) = a(u, v-u) + j(v) - j(u) - (f, v-u)_V + \frac{1}{2}a(u-v, u-v),$$

for every $v \in V$. Therefore, by using (2.1) and assumption 2.1.1(b) we obtain

$$J(v) - J(u) \geq 0 \quad \forall v \in V$$

Then u is a solution of the minimization problem (2.3). Conversely, assume that (2.3) holds and let $v \in V,\ t \in [0,1]$. We have

$$J(u + t(v-u)) \geq J(u)$$

and, using the convexity of j and the properties of **a**, we deduce that

$$ta(u, v-u) + \frac{t^2}{2}a(u-v, u-v) + t(j(v) - j(u)) \geq t(f, v-u)_V.$$

Dividing by t and passing to the limit as $t \to 0^+$ to conclude that u satisfies the variational inequality (2.1). ■

secondly, we have the following existence and uniqueness result.

Lemma 2.1.5 Let $f \in V$ then there exists a unique element $u \in V$ satisfies the minimization Problem (2.3.)

Proof.

- **Existence** Let $l_f = \inf_{v \in V} J(v)$. Since $j : V \to (-\infty, \infty]$ is a proper function, we deduce that $J : V \to (-\infty, \infty]$ is a proper function and, therefore,

$$l_f < \infty \tag{2.4}$$

let $\{u_n\} \in V$ a sequence of elements of V such that

$$J(u_n) \to l_f \text{ as } n \to \infty \tag{2.5}$$

Proposition 1.1.13 implies that there exist $\alpha \in V$ an $\beta \in \mathbb{R}$ such that

$$j(u_n) \geq (\alpha, u_n)_V + \beta \;\; \forall n \in \mathbb{N}$$

and using assumption 2.1.1(b), we obtain

$$J(u_n) \geq \frac{m}{2}\|u_n\|_V^2 - \|\alpha\|_V\|u_n\|_V - \|f\|_V\|u_n\|_V + \beta \quad \forall n \in \mathbb{N}. \tag{2.6}$$

The previous inequality implies that $\{u_n\}$ is a bounded sequence in V, Indeed, in the opposite case there exists a subsequence $\{u_{n_k}\} \subset \{u_n\}$ such that $\|u_{n_k}\|_V \to \infty$ as $k \to \infty$, then (2.6) implies that

$$J(u_{n_k}) \to \infty \quad \text{as } k \to \infty. \tag{2.7}$$

By (2.5) and (2.7) to see that $l_f = \infty$, which contradicts (2.4).

We use Theorem 1.1.15 of (Eberlein-Smulyan) to show that there exists an element $u \in V$ and a subsequence $\{u_{n_p}\} \subset \{u_n\}$ such that

$$u_{n_p} \rightharpoonup u \quad \text{in } V \quad \text{as } p \to \infty. \tag{2.8}$$

In addition, by Proposition 1.1.14 we deduce that J is is a convex lower semi-continuous function on V and, therefor (2.8) implies that

$$\liminf_{p\to\infty} J(u_{n_p}) \geq J(u). \tag{2.9}$$

By (2.5) and (2.9) we deduce that $l_f \geq J(u)$, using the definition of l_f then we have $l_f = \inf_{v\in V} J(v)$ which implies $J(u) = l_f$, then we conclude that the element $u \in V$ satisfies the minimization problem (2.3).

- **Uniqueness** To prove the uniqueness of the solution of (2.3) we use the assumptions 2.1.1 and 2.1.2, assume that u_1, u_2 two solution of (2.3) such that $u_1 \neq u_2$, then we have

$$\frac{1}{2}J(u_1)+\frac{1}{2}J(u_2)-J(\frac{u_1+u_2}{2}) = \frac{1}{8}a(u_1-u_2,u_1-u_2)+\frac{1}{2}j(u_1)+\frac{1}{2}j(u_2)-j(\frac{u_1+u_2}{2})$$

We use assumption 2.1.1 and by the convexity of j, since $u_1 \neq u_2$ we obtain that

$$\frac{1}{2}J(u_1)+\frac{1}{2}J(u_2) > J(\frac{u_1+u_2}{2})$$

Recall that the minimization Problem(2.3) has unique solution which implies

$$J(u_1) = J(u_2) = \inf_{v\in V} J(v)$$

then the previous inequality lead to follows

$$\inf_{v\in V} J(v) > J(\frac{u_1+u_2}{2})$$

which is a contradiction, We deduce that $u_1 = u_2$.

The proof of theorem 2.1.3 is straightforward consequence of Lemmas 2.1.4,2.1.5, then we conclude that the variational inequality (2.1) has unique solution $u \in V$.

Now we prove that this solution is depends Lipschitz continuously on f, let u_1, u_2 are the solutions of the inequality (2.1) for $f = f_1$ and $f = f_2$, respectively. we have $j(u_1) < \infty$, $j(u_2) < \infty$ and moreover

$$a(u_1, v-u_1) + j(v) - j(u_1) \geq (f, v-u_1)_V \quad \forall v \in V,$$

$$a(u_2, v-u_2) + j(v) - j(u_2) \geq (f, v-u_2)_V \quad \forall v \in V.$$

We take $v = u_1$ in the first inequality, $v = u_2$ in the second one, and add the corresponding inequalities to obtain

$$a(u_1 - u_2, u_1 - u_2) \leq (f_1 - f_2, u_1 - u_2)_V.$$

Using the assumption 2.1.1(b) and Cauchy-Schwarz inequality, we obtain

$$\|u_1 - u_2\|_V \leq \frac{1}{m}\|f_1 - f_2\|_V.$$

Then we conclude that the solution of inequality (2.1) is depends Lipschitz continuously on f, with Lipschitz constant $L \leq \dfrac{1}{m}$.

■

✦Non-symmetric case

We can prove the existence and uniqueness of solution of inequality (2.1) the case when a is *non-symmetric* bilinear form.

Theorem 2.1.6 We keep the same assumptions 2.1.1 with a is *non-symmetric* bilinear form, and 2.1.2, then for each $f \in V$ Problem (2.1) has unique solution.

Proof.

① **Uniqueness** suppose that u_1, u_2 solution of (2.1) to obtain

$$a(u_1, v - u_1) + j(v) - j(u_1) \geq (f, v - u_1)_V \quad \forall v \in V, \tag{2.10}$$

$$a(u_2, v - u_2) + j(v) - j(u_2) \geq (f, v - u_2)_V \quad \forall v \in V. \tag{2.11}$$

we take $v = u_2$ in the first inequality and $v = u_1$ in the second one, then add the corresponding inequalities to find

$$m\|u_1 - u_2\|_V^2 \leq a(u_1 - u_2, u_1 - u_2) \leq 0$$

$\Longrightarrow u_1 = u_2$ then we have the uniqueness.

② **Existence**

For each $u \in V$ and $\rho > 0$ we consider the auxiliary problem which defined as below:

$$\begin{cases} \text{Find} \quad w \in V. \\ (w, v - w) + \rho j(v) - \rho j(w) \geq -\rho(a(u, v - w) - (f, v - w)) + (u, v - w) \ \forall v \in V \end{cases} \tag{2.12}$$

Problem (2.12) has unique solution (follows from Theorem of Weierstrass). we define the operator

$$T_\rho : V \to V \quad T_\rho(u) = w$$

such that w is solution of (2.12), we prove that T_ρ has unique fixed point, it suffices to prove that T_ρ is strictly contracting .

Let $u_1, u_2 \in V$ and w_i solution of (2.12) for $T_\rho(u_i) = w_i \ i = 1, 2.$

$$(w_1, v - w_1) + \rho j(v) - \rho j(w_1) \geq -\rho a(u_1, v - w_1) + \rho(f, v - w_1) + (u_1 - v - w_1), \quad \textbf{(2.13)}$$

$$(w_2, v - w_2) + \rho j(v) - \rho j(w_2) \geq -\rho a(u_2, v - w_2) + \rho(f, v - w_2) + (u_2 - v - w_2). \quad \textbf{(2.14)}$$

We put $v = w_2$ in the first inequality, $v = w_1$ in the second one and by adding these inequalities we obtain

$$\begin{aligned} \|T_\rho(u_1) - T_\rho(u_2)\|_V^2 &= \|w_1 - w_2\|_V^2 \\ &\leq ((I - \rho A)(u_1 - u_2), w_1 - w_2)_V \\ &\leq \|I - \rho A\| \|u_1 - u_2\| \|w_1 - w_2\| \end{aligned}$$

$$\Longrightarrow \|T_\rho(u_1) - T_\rho(u_2)\|_V \leq \|I - \rho A\| \|u_1 - u_2\| \|w_1 - w_2\|$$

we can prove that $\exists \rho > 0$ such that $\|I - \rho A\| < 1$

$$\begin{aligned} \|(I - \rho A)v\|^2 &= ((I - \rho A)v, (I - \rho A)v) \\ &= (v, v) - 2\rho(Av, v) + \rho^2(Av, Av) \\ &\leq \|v\|^2 - 2\rho(Av, v) + \rho^2(Av, Av) \\ &\leq \|v\|^2 - 2\rho m\|v\|^2 + \rho^2\|A\|^2\|v\|^2 \\ &\leq (1 - 2\rho m + \rho^2\|A\|^2)\|v\|^2 \end{aligned}$$

by asumption 2.1.1(b) we have $-2\rho(Av, v) \leq -2\rho m\|v\|^2$

If $\rho \in \left]0, \dfrac{2m}{\|A\|^2}\right[\Longrightarrow 1 - 2m\rho + \rho^2\|A\|^2 < 1$

then we have $\|I - \rho A\| < 1$ which implies T_ρ is strictly contracting and hence has a unique fixed point.

We deduce that $T_\rho u = u = w$, this u is a solution of (2.1), then we conclude the existence part.

■

Remark 2.1.7 An example of an elliptic variational inequality of the second kind is a simplified version of the friction problem in elasticity (see [8]). Let $f \in L^2(\Omega)$ ($\Omega \subset \mathbb{R}^2$) denote the axial component of the body forces, $g > 0$ the given friction bound and u the unknown displacement field. Then the problem is to find $u \in H^1(\Omega)$ such that

$$(u, v-u)_{H^1(\Omega)} + g\int_\Gamma |v|ds - g\int_\Gamma |u|ds \geq (f, v-u)_{H^1} \quad \forall v \in V.$$

The existence and uniqueness of solution of previous problem follows from Theorem 2.1.6.

Remark 2.1.8 If j is the indicator function of a nonempty, convex, and closed subset $K \subset V$, which define by

$$I_K(v) = \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{if } v \notin K \end{cases}$$

Then we consider a particular case of inequality (2.1), shows at follows

$$a(u, v-u) \geq (f, v-u)_V \quad \forall v \in K. \tag{2.15}$$

this inequality is called *elliptic variational inequality of the first kind*.

Corollary 2.1.9 Let V be a Hilbert space, let $K \subset V$ be a nonempty, convex, and closed subset, and the assumption 2.1.1. Then for each $f \in V$ there exists a unique element $u \in K$ such that

$$a(u, v-u) \geq (f, v-u)_V \quad \forall v \in K$$

2.2 Quasivariational inequalities

We study in this section some generalization of inequality (2.1), which called *Quasivariational inequality* has the follows form

$$\begin{cases} \text{Find} \quad u \in V. \\ \\ a(u, v-u) + j(u,v) - j(u,u) \geq (f, v-u)_V \quad \forall v \in V \end{cases} \tag{2.16}$$

Keeping in mind Assumption 2.1.1 and considering the following assumption

Assumption 2.2.1 $j : V \times V \to \mathbb{R}$

(a) for all $\eta \in V,\ j(\eta, .)$ is convex and l.s.c. on V.

(b) there exist $\alpha \geq 0$ such that

$$j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_1) - j(\eta_2, v_2) \leq \alpha \|\eta_1 - \eta_2\|_V \|v_1 - v_2\|_V \quad \forall \eta_1, \eta_2, v_1, v_2 \in V$$

2.2.1 Existence and uniqueness results

Theorem 2.2.2 *[16]* If Assumptions 2.1.1 and 2.2.1 hold true, assume moreover, that $m > \alpha$. then, for each $f \in V$, the quasivariational inequality (2.16) has a unique solution.

Proof.

♦ **Existence**

For every $\eta \in V$ we consider the auxiliary problem of finding $u_\eta \in V$, which solves the variational inequality .

$$a(u_\eta, v - u_\eta) + j(\eta, v) - j(\eta, u_\eta) \geq (f, v - u_\eta)_V \quad \forall v \in V. \tag{2.17}$$

By Theorem 2.1.3, for each $\eta \in V$, problem (2.17) has unique solution.

We define the operator

$$T : V \to V, \qquad T\eta = u_\eta \ \ \forall \eta \in V.$$

We prove that T has a fixed point, it suffices to prove that T is strictly contracting.

Let $\eta_1, \eta_2 \in V$ and let u_i solution of (2.17) for $\eta = \eta_i$,i.e., $u_i = u_{\eta i},\ i = 1, 2.$ we have

$$a(u_1, v - u_1) + j(\eta_1, v) - j(\eta_1, u_1) \geq (f, v - u_1)_V \qquad \forall v \in V, \tag{2.18}$$

$$a(u_2, v - u_2) + j(\eta_2, v) - j(\eta_2, u_2) \geq (f, v - u_2)_V \qquad \forall v \in V, \tag{2.19}$$

we put $v = u_2$ in (2.18), and $v = u_1$ in (2.19), and add the resulting inequalities to obtain

$$a(u_1 - u_2, u_1 - u_2) \leq j(\eta_1, u_2) - j(\eta_1, u_1) + j(\eta_2, u_1) - j(\eta_2, u_2). \tag{2.20}$$

Using assumption 2.1.1(b) and 2.2.1 (b) in (2.20) we find

$$\|u_1 - u_2\|_V \leq \frac{\alpha}{m} \|\eta_1 - \eta_2\|_V.$$

We have from above $m > \alpha$,then the previous inequality shows that T is a contraction on V and by Banach's fixed point theorem it has a unique fixed point .

Denoting by η^* the fixed point of the operator T, then $\eta^* = T\eta^* = u_{\eta^*}$, which implies that the solution of auxiliary problem (2.17) is a solution of quasivariational inequality (2.16).

♦**Uniqueness**

if u is solution of (2.16) we denote $\eta = u$, then u is solution of auxiliary problem 2.17, we know that this inequality has unique solution u_η implies that $u = u_\eta$ then $\eta = u_\eta$ and by the definition of T we deduce that $T\eta = \eta$, Since T has a unique fixed point denoted by η^* we obtain $\eta = \eta^*$ implies that $u = u_{\eta^*}$ then we conclude that the inequality 2.16 has a unique solution . ■

CHAPTER 3

LAGRANGE MULTIPLIERS METHOD

We concentrate our attention in this chapter to study the solvability of nonlinear variational inequalities of the second kind and the quasivariational inequalities by using method of Lagrange multipliers in Hilbert space V.

3.1 VARIATIONAL INEQUALITIES OF THE SECOND KIND

Let $(\mathbf{V}, (.,.)_{\mathbf{V}}, \| \,.\, \|_{\mathbf{V}})$ be Hilbert space. We consider the operator $\mathbf{A} : \mathbf{V} \longrightarrow \mathbf{V}$ which not necessary linear such that A be a :

- ***strongly monotone***
  ($\exists m_A > 0$ such that $(Au - Av, u - v)_V \geq m_A \|u - v\|_V^2 \;\forall u, v \in V$).
- ***Lipschitz continuous***
  ($\exists L_A > 0$ such that $\|Au - Av\|_V \leq L_A \|u - v\|_V \;\forall u, v \in V$).

and let $\mathbf{j} : \mathbf{V} \longrightarrow [0, +\infty)$ is a proper convex lower semi-continuous functional. Given $f \in V$, we consider the following problem

$$\begin{cases} \text{Find} \quad \mathbf{u} \in \mathbf{V} \quad \text{such that,} \\ \\ (\mathbf{Au}, \mathbf{v} - \mathbf{u})_{\mathbf{V}} + \mathbf{j}(\mathbf{v}) - \mathbf{j}(\mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V} \end{cases} \tag{3.1}$$

An inequality of the form (3.1) is called a *variational inequality of the second kind*.

3.1.1 Regularization

The function j in general is not Gâteaux differentiable, but it can be approached by a family of Gâteaux differentiable functions , then assume that there exists a family of functions $(j_\rho)_{\rho>0}$ such that : For every $\rho > 0$ a function j_ρ is proper convex and Gâteaux differentiable l.s.c.

We consider the following problem

$$\begin{cases} \text{Find} \quad \mathbf{u}_\rho \in \mathbf{V} \quad \text{such that,} \\ \\ (\mathbf{A}\mathbf{u}_\rho, \mathbf{v}-\mathbf{u}_\rho)_\mathbf{V} + \mathbf{j}_\rho(\mathbf{v}) - \mathbf{j}_\rho(\mathbf{u}_\rho) \geq (\mathbf{f}, \mathbf{v}-\mathbf{u}_\rho)_\mathbf{V} \quad \forall \mathbf{v} \in \mathbf{V} \end{cases} \tag{3.2}$$

Since j and j_ρ are a propers convexes l.s.c functions. it follows that $j(u) < \infty$ and $j_\rho(u_\rho) < \infty$.

The variational inequality (3.2) considered a ***regularization*** of the variational inequality (3.1).

Consider now the following assumptions.

Assumption 3.1.1 there exist $F : \mathbb{R}^+ \to \mathbb{R}^+$ such that

(a) $|j_\rho(v) - j(v)| \leq F(\rho) \quad \forall v \in V,$ for each $\quad \rho > 0.$

(b) $\lim\limits_{\rho \to 0} F(\rho) = 0.$

Assumption 3.1.2 for each $\rho > 0$

(a) $j_\rho(v) \geq 0, \quad \forall v \in V$ and $j_\rho(0_V) = 0.$

(b) $j_\rho(v) \to j(v)$ as $\rho \to 0, \forall \quad v \in V.$

(c) For each sequence $\{v_\rho\} \subset V$, such that $v_\rho \rightharpoonup v \in V$

as $\rho \to 0$ one has $\liminf\limits_{\rho \to 0} j_\rho(v_\rho) \geq j(v).$

3.1.2 Convergence result

Theorem 3.1.3 *[16]* Under the assumption (3.1.1) or (3.1.2), the solution u_ρ of problem (3.2) converges to the solution u of problem (3.1), i.e.,

$$\lim_{\rho\to 0}\|\mathbf{u}-\mathbf{u}_\rho\|_{\mathbf{V}} = \mathbf{0} \tag{3.3}$$

Proof.

Let $\rho > 0$ and assume that assumption 3.1.1 hold. We take $v = u_\rho$ in inequality (3.1) and $v = u$ in the inequality (3.2) to see that

$$(Au, u_\rho - u)_V + j(u_\rho) - j(u) \geq (f, u_\rho - u)_V \tag{3.4}$$

$$(Au_\rho, u - u_\rho)_V + j_\rho(u) - j_\rho(u_\rho) \geq (f, u - u_\rho)_V \tag{3.5}$$

and add the corresponding inequalities to obtain

$$\begin{aligned}(Au_\rho - Au, u_\rho - u)_V &\leq j(u_\rho) - j(u) + j_\rho(u) - j_\rho(u_\rho)\\ &\leq |j_\rho(u_\rho) - j(u_\rho)| + |j_\rho(u) - j(u)|.\end{aligned}$$

Since the operator A is strongly monotone and by using assumption 3.1.1(a) on j_ρ to find that

$$m_A\|u_\rho - u\|_V^2 \leq (Au_\rho - Au, u_\rho - u)_V \leq 2F(\rho) \quad \Rightarrow m_A\|u_\rho - u\|_V^2 \leq 2F(\rho) \tag{3.6}$$

Using the previous inequality combined with assumption 3.1.1(b) to show that

$$\lim_{\rho\to 0}\|u_\rho - u\|_V^2 \leq 0$$

then we conclude the convergence result (3.3).

(by assumption(3.1.2))

i) A priori estimate

Assume that the assumption 3.1.2 holds, and let $\rho > 0$. We choose $v = 0_V$ in inequality (3.2) and use assumption 3.1.2 (a) to find that

$$(Au_\rho, u_\rho)_V \leq (f, u_\rho)_V. \tag{3.7}$$

Since A is strongly monotone, it follows that

$$(Au_\rho - Av, u_\rho - v)_V \geq m_A \|u_\rho - v\|_V^2 \quad \forall v \in V. \tag{3.8}$$

we choose $v = 0_V$ in (3.8) to obtain

$$(Au_\rho - A0_V, u_\rho)_V \geq m_A \|u_\rho\|_V^2$$

it follows now from (3.7) and(3.8) that :

$$m_A\|u_\rho\|_V^2 \leq (f, u_\rho)_V - (A0_V, u_\rho) \leq \|f\|_V\|u_\rho\|_V + \|A0_V\|_V\|u_\rho\|_V$$

$$\Longrightarrow \quad \|u_\rho\|_V \leq C. \tag{3.9}$$

ii) Weak convergence

We use (3.9) ,which shows that $\{u_\rho\}$ is bounded sequence in V . Therefore, by Theorem 1.1.15 of (*Eberlein-Smulyan*) it follows that there exist a subsequence $\{u_{\rho'}\}$ of $\{u_\rho\}$ and an element $u \in V$ such that

$$u_{\rho'} \rightharpoonup u \quad \text{in } V \text{ as } \rho' \to 0. \tag{3.10}$$

Recall that A Lipschitz continuous, then we have

$$(Au_{\rho'}, v) \to (Au, v); \quad (f, v - u_{\rho'}) \longrightarrow (f, v - u) \tag{3.11}$$

and from the positivity of (Av,v) (A strongly monotone),we find that

$$(Au_{\rho'} - Au, u_{\rho'} - u)_V \geq m_A\|u_{\rho'} - u\|_V^2 \geq 0$$

which implies that

$$(Au_{\rho'}, u_{\rho'}) - (Au_{\rho'}, u) - (Au, u_{\rho'}) + (Au, u) \geq 0$$

from (3.11) equivalently

$$(Au_{\rho'}, u_{\rho'}) \geq (Au_{\rho'}, u) + (Au, u_{\rho'}) - (Au, u)$$

Then from (3.11) we find that

$$\liminf_{\rho \to 0}(Au_{\rho'}, u_{\rho'}) \geq (Au, u) \tag{3.12}$$

We use inequality (3.2) to obtain

$$(Au_{\rho'}, v)_V + j_{\rho'}(v) \geq (f, v - u_{\rho'})_V + (Au_{\rho'}, u_{\rho'})_V + j_{\rho'}(u_{\rho'}) \quad \forall v \in V$$

implies that

$$(Au_{\rho'}, u_{\rho'})_V + j_{\rho'}(u_{\rho'}) \leq j_{\rho'}(v) - (f, v - u_{\rho'})_V + (Au_{\rho'}, v)_V \quad \forall v \in V$$

then we pass to the lower limit as $\rho' \to 0$ and use (3.12) ,and assumption 3.1.2(b)(c); we find that

$$\begin{aligned} (Au, u)_V + j(u) &\leq \liminf_{\rho \to 0}[(Au_{\rho'}, u_{\rho'}) + j_{\rho'}(u_{\rho'})] \\ &\leq j(v) - (f, v - u) + (Au, v) \end{aligned}$$

then we obtain

$$(Au, v)_V + j(v) \geq (f, v - u)_V + (Au, u)_V + j(u) \quad \forall v \in V$$

The previous inequality shows that **u** satisfies (3.1), we deduce that **u** is a solution of problem (3.1).

We conclude that every $\{u_{\rho'}\}$ converges weakly to **u**, which implies that the whole sequence $\{u_\rho\}$ converges weakly to **u**,

$$\mathbf{u}_\rho \rightharpoonup \mathbf{u} \quad \text{in } V \quad \text{as } \rho \to 0 \tag{3.13}$$

iii) Strong convergence

We consider $\rho > 0$ and take $v = u$ in (3.2) to obtain

$$(Au_\rho, u_\rho - u)_V \leq (f, u_\rho - u)_V + j_\rho(u) - j_\rho(u_\rho)$$

Which implies that

$$(Au_\rho - Au, u_\rho - u)_V \leq (f, u_\rho - u)_V + j_\rho(u) - j_\rho(u_\rho) + (Au, u - u_\rho)$$

Since A is strongly monotone, then we find that

$$m_A \|u_\rho - u\|^2 \leq (f, u_\rho - u)_V + j_\rho(u) - j_\rho(u_\rho) + (Au, u - u_\rho) \tag{3.14}$$

It follows from assumption3.1.2(b) that

$$\begin{aligned}\limsup_{\rho\to 0}(j_\rho(u) - j_\rho(u_\rho)) &= \limsup_{\rho\to 0}(j_\rho(u) - j(u) + j(u) - j_\rho(u_\rho)) \\ &= \limsup_{\rho\to 0}(j(u) - j_\rho(u_\rho)) \\ &= j(u) - \liminf_{\rho\to 0} j_\rho(u_\rho)\end{aligned}$$

we have $\mathbf{u}_\rho \rightharpoonup \mathbf{u}$ in V as $\rho \to 0$, and using assumption 3.1.2 (c), to find that

$$\limsup_{\rho\to 0}(j_\rho(u) - j_\rho(u_\rho)) \leq 0 \tag{3.15}$$

by using the weak convergence (3.13) and (3.14),(3.15) to see that

$$\limsup_{\rho\to 0}\|u_\rho - u\|_V^2 \leq 0$$

which implies that

$$\lim_{\rho\to 0}\|u_\rho - u\|_V = 0$$

Then we conclude that the strong convergence holds .

■

3.2 Lagrange Multipliers Method for Variational Inequalities of the Second kind

Firstly, we define the method of Lagrange multipliers as following:

The ***Method of Lagrange Multipliers*** is a useful way to determine the minimum or maximum of a function of several variables. It is an alternative to the method of substitution and works particularly well for non-linear constraints.

3.2.1 Mixed variational formulation

We consider the *mixed variational formulation* of the variational inequality (3.2):

$$\begin{cases} \text{Find} \quad (\mathbf{u}_\rho, \lambda) \in \mathbf{V} \times \mathbf{\Lambda}. \\ (\mathbf{A}\mathbf{u}_\rho, \mathbf{v} - \mathbf{u}_\rho)_\mathbf{V} + \mathbf{b}(\mathbf{v} - \mathbf{u}_\rho, \lambda) + \mathbf{j}_\rho(\mathbf{v}) - \mathbf{j}_\rho(\mathbf{u}_\rho) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}_\rho)_\mathbf{V} \quad \text{for all} \quad \mathbf{v} \in \mathbf{V} \\ \mathbf{b}(\mathbf{u}_\rho, \mu - \lambda) \leq \mathbf{0} \quad \text{for all} \quad \mu \in \mathbf{\Lambda} \end{cases} \tag{3.16}$$

We make the following assumptions

Assumption 3.2.1 $b : V \times \Lambda$ is a bilinear form such that :

1. $\exists M > 0 : |b(v,\mu)| \leq M\|v\|_V\|\mu\|_Y$ for all $v \in V, \mu \in \Lambda,$
2. $\exists \alpha > 0 : \inf_{\mu\in\Lambda,\mu\neq 0_Y} \sup_{v\in V, v\neq 0_V} \dfrac{b(v,\mu)}{\|v\|_V \|\mu\|_Y} \geq \alpha$

Assumption 3.2.2 Let $(Y, (.,.)_Y, \|.\|_Y)$ be real Hilbert space such that Λ is closed convex bounded subset of Y that contains 0_Y.

Assumption 3.2.3 j_ρ is a convex lower semi-continuous functional.In addition , there exist $m_1, m_2 > 0$ such that, for all $v \in V$, we have

$$m_1\|v\|_V^2 \geq j_\rho(v) \geq m_2\|v\|_V^2$$

.

Lemma 3.2.4 Denoting by ∇j_ρ the Gâteaux differential of j_ρ then Problem (3.16) is equivalent to the following problem:

$$\begin{cases} \text{Find} \quad (\mathbf{u}_\rho, \lambda) \in \mathbf{V} \times \mathbf{\Lambda}. \\ (\mathbf{A}\mathbf{u}_\rho, \mathbf{v})_\mathbf{V} + (\nabla \mathbf{j}_\rho(\mathbf{u}_\rho), \mathbf{v})_\mathbf{V} + \mathbf{b}(\mathbf{v}, \lambda) = (\mathbf{f}, \mathbf{v})_\mathbf{V} \quad \text{for all} \quad \mathbf{v} \in \mathbf{V} \\ \mathbf{b}(\mathbf{u}_\rho, \mu - \lambda) \leq \mathbf{0} \quad \text{for all} \quad \mu \in \mathbf{\Lambda} \end{cases} \tag{3.17}$$

Proof. We replacing v by $u_\rho \pm tw$ in (3.16), with an arbitrary $w \in V$ and $t > 0$, we obtain the following inequality :

$$\begin{cases} \text{Find} \quad (u_\rho, \lambda) \in V \times \Lambda. \\ \pm t(Au_\rho, w)_V \pm tb(w,\lambda) + j_\rho(u_\rho \pm tw) - j_\rho(u_\rho) \geq \pm t(f,w)_V \quad \text{for all} \quad w \in V \\ b(u_\rho, \mu - \lambda) \leq 0 \quad \text{for all} \quad \mu \in \Lambda \end{cases} \tag{3.18}$$

Dividing by t and passing to the limit as $t \to 0$, we obtain two inequalities that lead to

$$\begin{cases} \text{Find} \quad (u_\rho, \lambda) \in V \times \Lambda. \\ \pm(Au_\rho, w)_V \pm b(w,\lambda) + \lim_{\rho\to 0} \dfrac{j_\rho(u_\rho \pm tw) - j_\rho(u_\rho)}{t} \geq \pm(f,w)_V \quad \text{for all} \quad w \in V \\ b(u_\rho, \mu - \lambda) \leq 0 \quad \text{for all} \quad \mu \in \Lambda \end{cases} \tag{3.19}$$

The first inequality in (3.19) implies that

$$\pm(Au_\rho, w)_V \pm b(w,\lambda) \pm (\nabla j_\rho(u_\rho), w)_V \geq \pm(f,w)_V \quad \text{for all} \quad w \in V$$

Then we deduce Problem(3.17). ■

We define a new assumption for ∇j_ρ

Assumption 3.2.5 we consider the following conditions

1. $\exists m > 0 : (\nabla j_\rho(u_\rho) - \nabla j_\rho(v), u_\rho - v)_V \geq m\|u_\rho - v\|_V^2 \quad u_\rho, v \in V.$

2. $\exists L > 0 : \|\nabla j_\rho(u_\rho) - \nabla j_\rho(v)\|_V \leq L\|u_\rho - v\|_V \quad u_\rho, v \in V.$

Denoting $\nabla j_\rho + A$ by $\tilde{A}$, then the problem (3.17) become as following

$$\begin{cases} \text{Find} \quad (u_\rho, \lambda) \in V \times \Lambda & \text{such that,} \\ (\tilde{A}u_\rho, v)_V + b(v,\lambda) = (f,v)_V & \text{for all} \quad v \in V \\ b(u_\rho, \mu - \lambda) \leq 0 & \text{for all} \quad \mu \in \Lambda \end{cases} \tag{3.20}$$

3.2.2 Existence and Uniqueness of mixed variational formulation

Theorem 3.2.6 *[13]* If assumptions (3.2.1),(3.2.2),(3.2.3), and (3.2.5) hold true, then Problem (3.20) has unique solution $(u_\rho, \lambda) \in V \times \Lambda$.

The proof of theorem (3.2.6) is divided into several steps.

Firstly, for arbitrarily fixed $\eta \in V$, we consider the ***auxiliary problem*** of finding $(u_\rho^\eta, \lambda_\eta) \in V \times \Lambda$ such that.

$$(u_\rho^\eta, v)_V + \frac{m_A + m}{2(L_A + L)^2} b(v, \lambda_\eta) = \left(\frac{m_A + m}{2(L_A + L)^2} f - \frac{m_A + m}{2(L_A + L)^2}\tilde{A}\eta + \eta, v\right)_V \quad \forall \quad v \in V \tag{3.21}$$

$$b(u_\rho^\eta, \mu - \lambda_\eta) \leq 0 \qquad \forall \quad \mu \in \Lambda. \tag{3.22}$$

We are interested in determining the solution of *auxiliary problem* using saddle-point theory applied to the ***Lagrangian*** $\mathcal{L}_\eta : V \times \Lambda \to \mathbb{R}$

$$\mathcal{L}_\eta(v,\mu) = \frac{1}{2}(v,v)_V - \left(\frac{m_A+m}{2(L_A+L)^2}f - \frac{m_A+m}{2(L_A+L)^2}\tilde{A}\eta + \eta, v\right)_V + \frac{m_A+m}{2(L_A+L)^2}b(v,\mu) \tag{3.23}$$

we consider the technical lemma of saddle point:

Lemma 3.2.7 If the auxiliary Problem has a solution $(u_\rho^\eta, \lambda_\eta) \in V \times \Lambda$, then this solution is saddle point of the functional $\mathcal{L}_\eta$. Conversely, if a functional $\mathcal{L}_\eta$ has a saddle point then this saddle point is a solution of auxiliary Problem .

Proof. Assume that $(u_\rho^\eta, \lambda_\eta) \in V \times \Lambda$ is a solution of auxiliary Problem , and we prove it is a saddle point of $\mathcal{L}_\eta$, Then we use the equality (3.21) to prove that $\mathcal{L}_\eta(u_\rho^\eta, \mu) \le \mathcal{L}_\eta(u_\rho^\eta, \lambda_\eta)$ for all $\mu \in \Lambda$,Whereas

$$\mathcal{L}_\eta(u_\rho^\eta, \mu) - \mathcal{L}_\eta(u_\rho^\eta, \lambda_\eta) = \frac{m_A+m}{2(L_A+L)^2}b(u_\rho^\eta, \mu - \lambda^\eta) \le 0$$

which implies that

$$\mathcal{L}_\eta(u_\rho^\eta, \mu) \le \mathcal{L}_\eta(u_\rho^\eta, \lambda_\eta)$$

Moreover, by using (3.23) and (3.21) we find,

$$\begin{aligned}
\mathcal{L}_\eta(u_\rho^\eta, \lambda_\eta) - \mathcal{L}_\eta(v, \lambda_\eta) &= \frac{1}{2}(u_\rho^\eta, u_\rho^\eta)_V - \frac{1}{2}(v,v)_V + \frac{m_A+m}{2(L_A+L)^2}b(u_\rho^\eta - v, \lambda_\eta) \\
&- \left(\frac{m_A+m}{2(L_A+L)^2}f - \frac{m_A+m}{2(L_A+L)^2}\tilde{A}\eta + \eta, u_\rho^\eta - v\right)_V \\
&= \frac{1}{2}(u_\rho^\eta, u_\rho^\eta)_V - \frac{1}{2}(v,v)_V - (u_\rho^\eta, u_\rho^\eta - v)_V \\
&= -\frac{1}{2}\|u_\rho^\eta - v\|_V^2 \le 0
\end{aligned}$$

Therefore, $\mathcal{L}_\eta(u_\rho^\eta, \lambda_\eta) \le \mathcal{L}_\eta(v, \lambda_\eta)$ for all $v \in V$.

Conversely, let $(u_\rho^\eta, \lambda_\eta) \in V \times \Lambda$ be a saddle point of the *Lagrangian* $\mathcal{L}_\eta$. Then we have

$$\mathcal{L}_\eta(u_\rho^\eta, \mu) \le \mathcal{L}_\eta(u_\rho^\eta, \lambda_\eta), \text{ for all } \mu \in \Lambda,$$

implies (3.22) . Furthermore, we assume that

$$\mathcal{L}_\eta(u_\rho^\eta, \lambda_\eta) \le \mathcal{L}_\eta(w, \lambda_\eta), \text{ for all } w \in V,$$

and we prove (3.21). Using again (3.23), we deduce that, for all $w \in V$,

$$\frac{1}{2}(u^{\eta}_{\rho}, u^{\eta}_{\rho})_V - \frac{1}{2}(w,w)_V - (\frac{m_A+m}{2(L_A+L)^2}f - \frac{m_A+m}{2(L_A+L)^2}\tilde{A}\eta + \eta, u^{\eta}_{\rho} - w)_V +$$

$$\frac{m_A+m}{2(L_A+L)^2)}b(u^{\eta}_{\rho} - w, \lambda_\eta) \leq 0.$$

Replacing w by $u^{\eta}_{\rho} \pm tv$, with an arbitrary $v \in V$ and $t > 0$, we obtain the following inequality:

$$\pm t(u^{\eta}_{\rho}, v) - \frac{t^2}{2}(v,v)_V \pm t\left(\frac{m_A+m}{2(L_A+L)^2}f - \frac{m_A+m}{2(L_A+L)^2}\tilde{A}\eta + \eta, v\right)_V \pm \frac{m_A+m}{2(L_A+L)^2}tb(v,\lambda_\eta) \leq 0.$$

Dividing by t and passing to the limit as $t \to 0$, we obtain two inequalities that lead to (3.21). ■

Existence and uniqueness of the auxiliary problem

Lemma 3.2.8 The auxiliary Problem has a unique solution $(u^{\eta}_{\rho}, \lambda_\eta) \in V \times \Lambda$.

Proof.

① <u>**Existence**</u>

We prove that the map $v \mapsto \mathcal{L}_\eta(v,\mu)$ is convex and lower semi-continuous for all $\mu \in \Lambda$. Then firstly we prove that

$$\mathcal{L}_\eta(tv + (1-t)u, \mu) \leq t\mathcal{L}_\eta(v,\mu) + (1-t)\mathcal{L}_\eta(u,\mu) \quad t \in [0,1]\ v, u \in V$$

we have

$$\begin{aligned}\mathcal{L}_\eta(tv + (1-t)u, \mu) - t\mathcal{L}_\eta(v,\mu) - (1-t)\mathcal{L}_\eta(u,\mu) &= \frac{t(1-t)}{2}(v,v)_V + t(1-t)(u,v)_V \\ &- \frac{t(1-t)}{2}(u,u)_V\end{aligned}$$

Dividing by t and passing to the limit as $t \to 0$, we obtain

$$-\frac{1}{2}(u,u)_V - \frac{1}{2}(v,v)_V + (v,u)_V = -\frac{1}{2}\|u-v\|^2_V \leq 0.$$

Then we deduce that the map $v \mapsto \mathcal{L}_\eta(v,\mu)$ is convex, and since the Lagrangian $\mathcal{L}_\eta$ is continuous then, it is l.s.c for all $\mu \in \Lambda$.

We prove that $\mu \mapsto \mathcal{L}_\eta(v,\mu)$ is concave and upper semi-continuous for all $v \in V$.

then we prove

$$\mathcal{L}_\eta(v, t\mu + (1-t)\lambda) \geq t\mathcal{L}_\eta(v,\mu) + (1-t)\mathcal{L}_\eta(v,\lambda) \quad t \in [0,1]\ \mu, \lambda \in \Lambda \quad \forall v \in V$$

we have

$$\mathcal{L}_\eta(v, t\mu + (1-t)\lambda) - t\mathcal{L}_\eta(v,\mu) + (1-t)\mathcal{L}_\eta(v,\lambda) = 0$$

$\Longrightarrow$

$$\mathcal{L}_\eta(v, t\mu + (1-t)\lambda) = t\mathcal{L}_\eta(v,\mu) + (1-t)\mathcal{L}_\eta(v,\lambda)$$

$\Longrightarrow \mu \mapsto \mathcal{L}_\eta(v,\mu)$ is concave $\forall v \in V$. Let us prove that

$$\lim_{\|v\|_V \to \infty, v \in V} \mathcal{L}_\eta(v, 0_Y) = \infty.$$

We have

$$\mathcal{L}_\eta(v, 0_Y) = \frac{1}{2}(v,v)_V - \left(\frac{m_A + m}{2(L_A+L)^2} f - \frac{m_A+m}{2(L_A+L)^2}\tilde{A}\eta + \eta, v\right)_V + \frac{m_A+m}{2(L_A+L)^2} b(v, 0_Y)$$

$$\geq \frac{1}{2}\|v\|_V^2 - \frac{m_A+m}{2(L_A+L)^2}\|f\|_V\|v\|_V + \frac{m_A+m}{2(L_A+L)^2}\|\tilde{A}\|_V\|\eta\|_V\|v\|_V - \|\eta\|_V\|v\|_V$$

$\Longrightarrow$

$$\lim_{\|v\|_V \to \infty, v \in V} \mathcal{L}_\eta(v, 0_Y) = \infty.$$

Since Λ is bounded, then Theorem 1.5.2 ensures us that the functional $\mathcal{L}_\eta$ has at least one saddle point. Then, by Lemma(3.2.7), we conclude that *the auxiliary problem* has at least one solution .

② **Uniqueness**

Then we prove the uniqueness of the solution of *the auxiliary problem* $(u_\rho^\eta, \lambda_\eta) \in V \times \Lambda$. Let $(u_\rho^{\eta 1}, \lambda_\eta^1)$ and $(u_\rho^{\eta 2}, \lambda_\eta^2)$ be two solution of *auxiliary problem*. Keeping in mind (3.21) ,

$$(u_\rho^{\eta 1}, v)_V + \frac{m_A+m}{2(L_A+L)^2} b(v, \lambda_\eta^1) = \left(\frac{m_A+m}{2(L_A+L)^2} f - \frac{m_A+m}{2(L_A+L)^2}\tilde{A}\eta + \eta, v\right)_V \quad \forall v \in V$$

$$(u_\rho^{\eta 2}, v)_V + \frac{m_A+m}{2(L_A+L)^2} b(v, \lambda_\eta^2) = \left(\frac{m_A+m}{2(L_A+L)^2} f - \frac{m_A+m}{2(L_A+L)^2}\tilde{A}\eta + \eta, v\right)_V \quad \forall v \in V$$

We take $v = u_\rho^{\eta 2}$ in the first equation, $v = u_\rho^{\eta 1}$ in the second one , and add the resulting equalities to obtain

$$(u_\rho^{\eta 1} - u_\rho^{\eta 2}, u_\rho^{\eta 2} - u_\rho^{\eta 1})_V + \frac{m_A+m}{2(L_A+L)^2} b(u_\rho^{\eta 1} - u_\rho^{\eta 2}, \lambda_\eta^2 - \lambda_\eta^1) = 0 \tag{3.24}$$

Using (3.22), we find

$$b(u_\rho^{\eta 1}, \mu - \lambda_\eta^1) \leq 0$$

$$b(u_\rho^{\eta 2}, \mu - \lambda_\eta^2) \leq 0$$

and add the two previous inequalities ,we deduce that

$$b(u_\rho^{\eta 1} - u_\rho^{\eta 2}, \lambda_\eta^2 - \lambda_\eta^1) \leq 0. \tag{3.25}$$

Combining (3.24) and (3.25), we conclude that

$$(u_\rho^{\eta 1} - u_\rho^{\eta 2}, u_\rho^{\eta 1} - u_\rho^{\eta 2})_V = \|u_\rho^{\eta 1} - u_\rho^{\eta 2}\|_V^2 \leq 0.$$

which implies $u_\rho^{\eta 1} = u_\rho^{\eta 2}$. Moreover,

$$b(v, \lambda_\eta^1 - \lambda_\eta^2) = -\frac{2(L_A + L)^2}{m_A + m}(u_\rho^{\eta 1} - u_\rho^{\eta 2}, v)_V \text{ for all } v \in V.$$

By the inf-sup property we have $\exists \alpha > 0$ such that :

$$\sup_{v \in V\, v \neq 0_V} \frac{b(v, \lambda_\eta^2 - \lambda_\eta^1)}{\|v\|_V \left\|\lambda_\eta^2 - \lambda_\eta^1\right\|_Y} \geq \alpha$$

$\Longrightarrow$

$$\alpha \leq \sup_{v \in V\, v \neq 0_V} \frac{b(v, \lambda_\eta^2 - \lambda_\eta^1)}{\|v\|_V \left\|\lambda_\eta^2 - \lambda_\eta^1\right\|_Y} = \sup_{v \in V\, v \neq 0_V} \frac{2(L_A + L)^2}{m_A + m} \frac{(u_\rho^{\eta 1} - u_\rho^{\eta 2}, v)_V}{\|v\|_V \left\|\lambda_\eta^2 - \lambda_\eta^1\right\|_Y}$$

then we obtain

$$\alpha \|\lambda_\eta^1 - \lambda_\eta^2\|_Y \leq \frac{2(L_A + L)^2}{m_A + m}\|u_\rho^{\eta 1} - u_\rho^{\eta 2}\| \leq 0 \quad (\text{we have} \quad u_\rho^{\eta 1} = u_\rho^{\eta 2}).$$

we conclude that $\lambda_\eta^1 = \lambda_\eta^2$.

■

Using the unique solution of auxiliary problem, we define an operator

$$T : V \to V, \qquad T(\eta) := u_\rho^\eta.$$

Lemma 3.2.9 The operator T has a unique fixed point.

Proof. Let us take $\eta, \tilde{\eta} \in V$. Denoting by $(u_\rho^\eta, \lambda_\eta)$ and $(u_\rho^{\tilde{\eta}}, \lambda_{\tilde{\eta}})$ the corresponding solution of auxiliary Problem, and using (3.21),

$$(u_\rho^\eta, v)_V + \frac{m_A + m}{2(L_A + L)^2} b(v, \lambda_\eta) = \left(\frac{m_A + m}{2(L_A + L)^2} f - \frac{m_A + m}{2(L_A + L)^2} \tilde{A}\eta + \eta, v\right)_V, \quad \forall v \in V$$

$$(u_\rho^{\tilde{\eta}}, v)_V + \frac{m_A + m}{2(L_A + L)^2} b(v, \lambda_{\tilde{\eta}}) = \left(\frac{m_A + m}{2(L_A + L)^2} f - \frac{m_A + m}{2(L_A + L)^2} \tilde{A}\tilde{\eta} + \tilde{\eta}, v\right)_V, \quad \forall \quad v \in V$$

We put $v = u_\rho^{\tilde{\eta}}$ in the first equation, $v = u_\rho^{\eta}$ in the second one , and add the resulting inequalities to deduce that

$$\begin{aligned}(u_\rho^\eta - u_\rho^{\tilde{\eta}}, u_\rho^\eta - u_\rho^{\tilde{\eta}})_V &= \frac{m_A + m}{2(L_A + L)^2} b(u_\rho^{\tilde{\eta}} - u_\rho^\eta, \lambda_\eta - \lambda_{\tilde{\eta}}) \\ &+ \left(\eta - \tilde{\eta} + \frac{m_A + m}{2(L_A + L)^2}(\tilde{A}\tilde{\eta} - \tilde{A}\eta), u_\rho^\eta - u_\rho^{\tilde{\eta}}\right)_V.\end{aligned}$$

Taking into account (3.22), we obtain

$$\frac{m_A + m}{2(L_A + L)^2} b(u_\rho^\eta - u_\rho^{\tilde{\eta}}, \lambda_{\tilde{\eta}} - \lambda_\eta) \leq 0$$

Then we deduce

$$\|u_\rho^\eta - u_\rho^{\tilde{\eta}}\|_V \leq \|\eta - \tilde{\eta} - \frac{m_A + m}{2(L_A + L)^2}(\tilde{A}\eta - \tilde{A}\tilde{\eta})\|_V$$

From this last inequality we find

$$\|u_\rho^\eta - u_\rho^{\tilde{\eta}}\|_V^2 \leq \|\eta - \tilde{\eta}\|_V^2 - \frac{m_A + m}{(L_A + L)^2}(\tilde{A}\eta - \tilde{A}\tilde{\eta}, \eta - \tilde{\eta})_V + \frac{(m_A + m)^2}{4(L_A + L)^4}\|\tilde{A}\eta - \tilde{A}\tilde{\eta}\|_V^2 \quad \textbf{(3.26)}$$

We have $\tilde{A} := A + \nabla j_\rho$, since A strongly monotone and continuous Lipschitz, and use the assumption 3.2.5 about ∇j_ρ, we obtain

$$\begin{aligned}(\tilde{A}\eta - \tilde{A}\tilde{\eta}, \eta - \tilde{\eta})_V &= ((A + \nabla j_\rho)\eta - (A + \nabla j_\rho)\tilde{\eta}, \eta - \tilde{\eta})_V \\ &= (A\eta - A\tilde{\eta}, \eta - \tilde{\eta})_V + (\nabla j_\rho(\eta) - \nabla j_\rho(\tilde{\eta}), \eta - \tilde{\eta})_V \\ &\geq (m_A + m)\|\eta - \tilde{\eta}\|_V^2\end{aligned}$$

Moreover, we find

$$\begin{aligned}\|\tilde{A}\eta - \tilde{A}\tilde{\eta}\|_V &= \|(A + \nabla j_\rho)\eta - (A + \nabla j_\rho)\tilde{\eta}\|_V \\ &\leq \|A\eta - A\tilde{\eta}\|_V + \|\nabla j_\rho(\eta) - \nabla j_\rho(\tilde{\eta})\|_V \\ &\leq (L_A + L)\|\eta - \tilde{\eta}\|_V\end{aligned}$$

$\Longrightarrow$

$$\|\tilde{A}\eta - \tilde{A}\tilde{\eta}\|_V^2 \leq (L_A + L)^2\|\eta - \tilde{\eta}\|_V^2$$

then the inequality (3.26) become as follow

$$\|u_\rho^\eta - u_\rho^{\tilde{\eta}}\|_V^2 \leq \left(1 - \frac{3(m_A + m)^2}{4(L_A + L)^2}\right)\|\eta - \tilde{\eta}\|_V^2.$$

Taking into account A strongly monotone ,continuous Lipschitz ,it is straightforward to observe that $m_A \leq L_A$ and $m \leq L$ implies $m_A + m \leq L_A + L$.

Therefor,

$$0 < 1 - \frac{3(m_A + m)^2}{4(L_A + L)^2}.$$

Consequently

$$\|T\eta - T\tilde{\eta}\|_V \leq \sqrt{1 - \frac{3(m_A + m)^2}{4(L_A + L)^2}}\|\eta - \tilde{\eta}\|_V.$$

Since $1 - (3(m_A + m)^2)/(4(L_A + L)^2) < 1$, then we deduce that the operator T is a contraction and from Banach's fixed-point theorem, it is has a unique fixed point.

■

Let us now prove Theorem (3.2.6).

Proof.

- **Existence** Denoting by η^* the unique fixed point of the operator T, then $\eta^* = T(\eta^*) = u_\rho^{\eta^*}$, we deduce that the solution of *auxiliary problem* with $\eta = \eta^*(u_\rho^{\eta^*}, \lambda_{\eta^*})$ is a solution of Problem (3.20), which concludes the existence part.

- **uniqueness** We assume that Problem 3.20 has two solutions $(u_\rho^1, \lambda^1), (u_\rho^2, \lambda^2)$ we obtain

$$\begin{cases} (\tilde{A}u_\rho^1, v)_V + b(v, \lambda^1) = (f, v)_V \\ b(u_\rho^1, \mu - \lambda^1) \leq 0 \end{cases}$$

$$\begin{cases} (\tilde{A}u_\rho^2, v)_V + b(v, \lambda^2) = (f, v)_V \\ b(u_\rho^2, \mu - \lambda^2) \leq 0 \end{cases}$$

We take $v = u_\rho^2$ in the first problem and $v = u_\rho^1$ in second one, and add the corresponding inequalities to obtain

$$(\tilde{A}u_\rho^1 - \tilde{A}u_\rho^2, u_\rho^2 - u_\rho^1)_V + b(u_\rho^2 - u_\rho^1, \lambda^1 - \lambda^2) = 0,$$

$$b(u_\rho^2 - u_\rho^1, \lambda^1 - \lambda^2) \leq 0,$$

We have $\tilde{A} = A + \nabla j_\rho$, and taking into account A strongly monotone and assumption 3.2.5(1) , we obtain

$$(m_A + m)\|u_\rho^1 - u_\rho^2\|_V \leq (\tilde{A}u_\rho^1 - \tilde{A}u_\rho^2, u_\rho^1 - u_\rho^2)_V \leq 0 \qquad \Longrightarrow u_\rho^1 = u_\rho^2$$

Moreover,

$$(\tilde{A}u_\rho^1 - \tilde{A}u_\rho^2, v)_V + b(v, \lambda^1 - \lambda^2) = 0 \qquad \text{for all} v \in V.$$

By the inf-sup property of the form $b(.,.)$,

$$\alpha \sup_{v\in V, v\neq 0_V} \frac{b(v, \lambda^2 - \lambda^2)}{\|v\|_V \|\lambda^2 - \lambda^2\|_Y} = \sup_{v\in V, v\neq 0_V} \frac{(\tilde{A}u_\rho^1 - \tilde{A}u_\rho^2, v)_V}{\|v\|_V \|\lambda^2 - \lambda^1\|_Y}$$

and since A continuous Lipschitz with Assumption3.2.5(2) to find that

$$\alpha\|\lambda^2 - \lambda^1\|_Y \leq \|\tilde{A}u_\rho^2 - \tilde{A}u_\rho^1\|_V \leq (L_A + L)\|\|u_\rho^2 - u_\rho^1\|_V$$

Then from this we obtain $\lambda^1 = \lambda^2$.

■ We conclude that Problem (3.20) has unique solution $(u_\rho, \lambda) \in V \times \Lambda$.

Corollary 3.2.10 From Theorem 3.2.6 we deduce that problem (3.16) has unique solution $(u_\rho, \lambda) \in V \times \Lambda$ which implies that the variational inequality (3.2) has a solution $u_\rho \in V$.

Corollary 3.2.11 By passing to the limit of $\|u - u_\rho\|$ we obtain a strong convergence which we proved Previously (see theorem 3.1.3) we conclude that the variational inequality (3.1) has a solution $u \in V$.

We can prove the uniqueness of this solution as following:

Let u_1, u_2 solutions of (3.1) the we have

$$(Au_1, v - u_1)_V + j(v) - j(u_1) \geq (f, v - u_1)_V \quad \forall v \in V, \tag{3.27}$$

$$(Au_2, v - u_2)_V + j(v) - j(u_2) \geq (f, v - u_2)_V \quad \forall v \in V. \tag{3.28}$$

We put $v = u_2$ in (3.27), $v = u_1$ in (3.28) and add the corresponding inequalities and since operator A is strongly monotone to see that

$$m_A\|u_1 - u_2\|_V \leq (Au_1 - Au_2, u_1 - u_2)_V \leq 0 \implies u_1 = u_2.$$

then we conclude the proof of the uniqueness.

3.3 Quasivariational Inequalities

Let $(V, (.,.)_V, \|\,.\,\|_V)$ be Hilbert space.In addition, we consider operator $A : V \longrightarrow V$, not necessary linear such that A be a ***strongly monotone*** and ***Lipschitz continuous*** and we have functional $j : V \times V \longrightarrow \mathbb{R}$. Given $f \in V$, we consider the problem of finding an element $u \in V$ such that

$$(P)\begin{cases} \text{Find} \quad \mathbf{u} \in \mathbf{V} \quad \text{such that.} \\ \\ (\mathbf{Au}, \mathbf{v} - \mathbf{u})_{\mathbf{V}} + \mathbf{j}(\mathbf{u}, \mathbf{v}) - \mathbf{j}(\mathbf{u}, \mathbf{u}) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u})_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V} \end{cases} \tag{3.29}$$

An inequality of the form (3.29) is called an *quasivariational inequality of the second kind.* We consider the following assumptions :

Assumption 3.3.1 $j : V \times V \longrightarrow \mathbb{R}$

(a) *for all* $\eta \in V,\ j(\eta, .)$ *is convex and l.s.c. on* V.

(b) *there exist* $\alpha \geq 0$ *such that*

$$j(\eta_1, v_2) - j(\eta_1, v_1) + j(\eta_2, v_2) - j(\eta_2, v_2) \leq \alpha\|\eta_1 - \eta_2\|_V \|v_1 - v_2\|_V \quad \forall \eta_1, \eta_2, v_1, v_2 \in V$$

Theorem 3.3.2 *Let* V *be a Hilbert space and assume that Assumption (3.3.1) hold. Assume ,moreover , that* ${m_A}$[1]$> \alpha$ *.Then , for each* $f \in V$*, the quasivaraitional elliptic inequality (3.29) has a unique solution that depends Lipschitz continuously on* f.

The proof of this theorem is based on several basic steps.

Firstly , for every $\eta \in V$ we consider the auxiliary problem of finding $u_\eta \in V$ which solve the following elliptic variational inequality

$$(P_\eta)\begin{cases} \text{Find} \quad \mathbf{u}_\eta \in \mathbf{V}. \\ \\ (\mathbf{Au}_\eta, \mathbf{v} - \mathbf{u}_\eta)_{\mathbf{V}} + \mathbf{j}(\eta, \mathbf{v}) - \mathbf{j}(\eta, \mathbf{u}_\eta) \geq (\mathbf{f}, \mathbf{v} - \mathbf{u}_\eta)_{\mathbf{V}} \quad \forall \mathbf{v} \in \mathbf{V} \end{cases} \tag{3.30}$$

3.3.1 Regularization

We denote $j(\eta, .) = j_\eta(.)$

For every $\rho > 0$, let j_η^ρ is a regularization of j_η, which Gâteaux differentiable. We consider the Problem of finding an element $u_\eta^\rho \in V$.

[1] m_A constant of strong monotonicity of A.

$$(P_\eta^\rho)\begin{cases} \text{Find} \quad \mathbf{u}_\eta^\rho \in \mathbf{V}. \\ (\mathbf{A}\mathbf{u}_\eta^\rho, \mathbf{v}-\mathbf{u}_\eta^\rho)_\mathbf{V} + \mathbf{j}_\eta^\rho(\mathbf{v}) - \mathbf{j}_\eta^\rho(\mathbf{u}_\eta^\rho) \geq (\mathbf{f}, \mathbf{v}-\mathbf{u}_\eta^\rho)_\mathbf{V} \quad \forall \mathbf{v} \in \mathbf{V} \end{cases} \tag{3.31}$$

Assumption 3.3.3 *there exist* $F : \mathbb{R}^+ \to \mathbb{R}^+$ *such that*

(a) $\left|j_\eta^\rho(v) - j_\eta(v)\right| \leq F(\rho) \quad \forall v \in V,$ *for each* $\quad \rho > 0$

(b) $\lim_{\rho\to 0} F(\rho) = 0$

Assumption 3.3.4 *for each* $\rho > 0$

(a) $j_\eta^\rho(v) \geq 0, \forall v \in V$ *and* $j_\eta^\rho(0_V) = 0$.

(b) $j_\eta^\rho(v) \to j_\eta(v)$*as* $\rho \to 0, \forall v \in V$.

(c) *For each sequence* $\{v_\rho\} \subset V$ *,such that* $v_\rho \rightharpoonup v \in V$

as $\rho \to 0$ *one has* $\liminf_{\rho\to 0} j_\eta^\rho(v_\rho) \geq j(v)$.

3.3.2 Convergence results

Theorem 3.3.5 *Under the assumption (3.3.3) or (3.3.4),the solution* u_η^ρ *of Problem (3.31) converges to the solution* u_η *of Problem (3.30) ,i.e.,*

$$\lim_{\rho\to 0} \|u_\eta - u_\eta^\rho\|_V = 0 \tag{3.32}$$

Proof. The same proof of Theorem(3.1.3). ■

3.3.3 Lagrange Multiplier Method for Quasivariational inequalities

We use Lagrange multiplier method for solving the variational inequality (3.31), then we consider the following mixed variational formulation

$$\begin{cases} \text{Find} \quad (u_\eta^\rho, \lambda) \in V \times \Lambda. \\ (Au_\eta^\rho, v - u_\eta^\rho)_V + b(v - u_\eta^\rho, \lambda) + j_\eta^\rho(v) - j_\eta^\rho(u_\eta^\rho) \geq (f, v - u_\eta^\rho)_V \quad \forall v \in V \\ b(u_\eta^\rho, \mu - \lambda) \leq 0 \qquad \forall \mu \in \Lambda. \end{cases} \tag{3.33}$$

Assumption 3.3.6 $b : V \times \Lambda$ *is a bilinear form such that :*

1. $\exists M > 0 : |b(v,\mu)| \leq M\|v\|_V\|\mu\|_Y$*for all* $v \in V, \mu \in \Lambda,$
2. $\exists \alpha > 0 : \inf_{\mu \in \Lambda, \mu \neq 0_Y} \sup_{v \in V, v \neq 0_V} \frac{b(v,\mu)}{\|v\|_V \ \|\mu\|_Y} \geq \alpha$

Assumption 3.3.7 *Let* $(Y, (.,.)_Y, \|.\|_Y)$ *be real Hilbert space such that* Λ *is closed convex bounded subset of* Y *that contains* 0_Y*.*

Assumption 3.3.8 j^ρ_η *is a convex lower semi-continuous functional.In addition , there exist* $m_1, m_2 > 0$ *such that, for all* $v \in V$*, we have*

$$m_1\|v\|^2_V \geq j^\rho_\eta(v) \geq m_2\|v\|^2_V$$

.

Lemma 3.3.9 *Denoting by* ∇j^ρ_η *the Gâteaux differential of* j^ρ_η *then Problem (3.33) is equivalent to the following Problem :*

$$\begin{cases} \textit{Find} \quad (u^\rho_\eta, \lambda) \in V \times \Lambda. \\ (Au^\rho_\eta, v)_V + (\nabla j^\rho_\eta(u_\rho), v)_V + b(v,\lambda) = (f,v)_V \quad \textit{for all} \quad v \in V \\ b(u^\rho_\eta, \mu - \lambda) \leq 0 \quad \textit{for all} \quad \mu \in \Lambda \end{cases} \tag{3.34}$$

Remark 3.3.10 the previous lemma has the same proof of Lemma (3.2.4).

Let make an additional assumption.

Assumption 3.3.11 we have the following conditions

1. $\exists m > 0 : (\nabla j^\rho_\eta(u^\rho_\eta) - \nabla j^\rho_\eta(v), u^\rho_\eta - v)_V \geq m\|u^\rho_\eta - v\|^2_V \quad u_\rho, v \in V$
2. $\exists L > 0 : \|\nabla j^\rho_\eta(u^\rho_\eta) - \nabla j^\rho_\eta(v)\|_V \leq L\|u^\rho_\eta - v\|_V \quad u^\rho_\eta, v \in V$

We denote $A + j^\rho_\eta$ by $\tilde{A}$, then we get the following problem :

$$\begin{cases} \text{Find} \quad (u^\rho_\eta, \lambda) \in V \times \Lambda. \\ (\tilde{A}u^\rho_\eta, v)_V + b(v,\lambda) = (f,v)_V \quad \text{for all} \quad v \in V \\ b(u^\rho_\eta, \mu - \lambda) \leq 0 \quad \text{for all} \quad \mu \in \Lambda \end{cases} \tag{3.35}$$

3.3.4 Existence and Uniqueness Results

Theorem 3.3.12 If Assumptions (3.3.6), (3.3.7),(3.3.8), and (3.3.11) hold true, then Problem (3.35) has unique solution $(u_\eta^\rho, \lambda) \in V \times \Lambda$.

Proof. The proof of this theorem follows the same pattern as in Theorem3.2.6.
■

using the previous theorem to obtain the following existence and uniqueness result.

Corollary 3.3.13 Since problem (3.33) is equivalent whit (3.35), then for each $\eta \in V$, there exist a unique solution (u_η^ρ, λ) of Problem (3.35) which implies problem (3.31) has a unique solution u_η^ρ .

We use the convergence result $\lim\limits_{\rho\to 0}\|u_\eta - u_\eta^\rho\|_V = 0$ to prove the existence of solution of auxiliary Problem (3.30), then we have the following lemma

Lemma 3.3.14 For each $\eta \in V$, there exist a solution u_η of (3.30).

then we prove the uniqueness of this solution, we assume that Problem (3.30) has two solutions u_η^1 and u_η^2. Using (3.30), we obtain

$$(Au_\eta^1, v - u_\eta^1)_V + j(\eta, v) - j(\eta, u_\eta^1) \geq (f, v - u_\eta^1)_V$$

$$(Au_\eta^2, v - u_\eta^2)_V + j(\eta, v) - j(\eta, u_\eta^2) \geq (f, v - u_\eta^2)_V$$

We take $v = u_\eta^2$ in the first inequality and $v = u_\eta^1$ in the second one and add the corresponding inequalities to obtain

$$(Au_\eta^1 - Au_\eta^2, u_\eta^1 - u_\eta^2) \leq 0$$

Since the operator A is strongly monotone we find

$$m_A\|u_\eta^1 - u_\eta^2\|_V \leq (Au_\eta^1 - Au_\eta^2, u_\eta^1 - u_\eta^2)_V \leq 0$$

hence $u_\eta^1 = u_\eta^2$, then we conclude that (3.30) has a unique solution .

Using the unique solution of auxiliary Problem (3.30), we define un operator

$$S : V \to V, \qquad S(\eta) := u_\eta.$$

Lemma 3.3.15 If $m_A > \alpha$, then S has a unique fixed point $\eta^* \in V$.

Proof. Let $\eta_1, \eta_2 \in V$ and let u_i denote the solution of (3.30) for $\eta = \eta_i$, i.e., $u_i = u_{\eta i}, i = 1, 2$. We have

$$(Au_1, v - u_1)_V + j(\eta_1, v) - j(\eta_1, u_1) \geq (f, v - u_1)_V \quad \forall v \in V$$

$$(Au_2, v - u_2)_V + j(\eta_2, v) - j(\eta_2, u_2) \geq (f, v - u_2)_V \quad \forall v \in V$$

We put $v = u_2$ in the first inequality, $v = u_1$ in the second one, and add the resulting inequalities to obtain

$$(Au_1 - Au_2, u_1 - u_2)_V \leq j(\eta_1, u_2) - j(\eta_1, u_1) + j(\eta_2, u_1) - j(\eta_2, u_2). \tag{3.36}$$

Recall that A strongly monotone and use assumption 3.3.1 (b) in (3.36) to deduce that

$$\|u_1 - u_2\|_V \leq \frac{\alpha}{m_A} \|\eta_1 - \eta_2\|_V.$$

Since $m_A > \alpha$, the operator S is a contraction on V, then from Banach's fixed-point theorem S has unique fixed point. ■

We can now provide the proof of Theorem (3.3.2).

Proof.

- **Existence** Let η^* be the fixed point of the operator S,then we have $\eta^* = S\eta^* = u_{\eta^*}$ and we know that the auxiliary problem (3.30) has unique solution , then we conclude that u_{η^*} is a solution of the quasivariational inequality (3.29) .

- **Uniqueness** We have from the above $m_A > \alpha$ and assume that Problem (3.29) has two solution u_1, u_2 such that $u_1 \neq u_2$ to obtain

  $$(Au_1, v - u_1)_V + j(u_1, v) - j(u_1, u_1) \geq (f, v - u_1)_V \quad \forall v \in V$$

  $$(Au_2, v - u_2)_V + j(u_2, v) - j(u_2, u_2) \geq (f, v - u_2)_V \quad \forall v \in V$$

  We take $v = u_2$ in the first inequality and $v = u_1$ in the second one, and add the resulting inequalities to obtain

  $$(Au_1 - Au_2, u_1 - u_2)_V + j(u_1, u_2) - j(u_1, u_1) + j(u_2, u_1) - j(u_2, u_2) \tag{3.37}$$

  Since A strongly monotone and by assumption 3.3.1(b) we have

  $$m_A \|u_1 - u_2\|^2 \leq \alpha \|u_1 - u_2\|^2 \quad \Longrightarrow m_A \leq \alpha$$

which is a contradiction (recall that $m_A > \alpha$), then we have

$$(1 - \frac{\alpha}{m_A})\|u_1 - u_2\|_V \leq 0 \quad \Longrightarrow u_1 = u_2$$

From this result we conclude that the quasi variational inequality (3.29) has unique solution $u \in V$.

Now we prove that this solution is depends Lipschitz continuously on f, let u_1, u_2 are solutions of the inequality (3.29) for $f = f_1$ and $f = f_2$, respectively. Using the same arguments as we used in the proof of (3.37), we obtain

$$(Au_1, Au_2, u_1 - u_2)_V \leq j(u_1, u_2) - j(u_1, u_1) + j(u_2, u_1) - j(u_2, u_2) + (f_1 - f_2, u_1 - u_2)_V.$$

We use assumption 3.3.1(b) and since A strongly monotone to see that

$$(m_A - \alpha)\|u_1 - u_2\|_V \leq \|f_1 - f_2\|_V$$

Since $m_A > \alpha$, we find that

$$\|u_1 - u_2\|_V \leq \frac{1}{m_A - \alpha}\|f_1 - f_2\|_V.$$

■

CONCLUSION

In this work we point out that by using Lagrange multipliers method can be proved the existence and uniqueness of the solution of variational inequalities of the second kind in Hilbert space. The main idea in this study is to get rid of nonlinear part in variational problem and replacing by Lagrange multiplier λ , By assumptions of regularization we considering an approximate problem and by Lagrange multipliers method we define a mixed variational formulation, the existence of solution of this formulation established by using saddle point theory and fixed point technique, Some generalization of variational inequalities so called Quasi-variational inequalities can be proved there solution existence by apply the same method.

For the perspectives and extensions, it would be interesting to study

✍ Dynamic case of variational inequalities of the second kind.

✍ Existence of solution of parabolic variational inequalities of the second kind .

✍ The existence of solution of variational inequalities of the second kind in Banach frame.

✍ The existence of solution of quasi-variational inequalities in Banach frame.

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ملخص:

قمنا في هذا العمل باستعمال طريقة مضاعفات لاغرانج لإثبات وجود الحل للمتراجحات التغايرية وشبه التغايرية من الصنف الثاني، استخدمنا طريقة التنظيم ، كما قدمنا نتائج عن التقارب، اعتمد وجود الحل بالأساس على نظرية نقطة السرج وتقنية النقطة الصامدة.

الكلمات المفتاحية: طريقة مضاعفات لاغرانج، متراجحات تغايرية من الصنف الثاني، متراجحات شبه تغايرية.

Abstract

In this paper we present Lagrange multipliers method for prove the existence of solution for variational inequalities of the second kind, and quasi-variational inequalities , We used the regularization method which give approximate problem and we prove the convergence , The existence of solution established by using saddle- point theory and fixed-point technique.

Keywords :Lagrange multipliers method, variational inequalities of the second kind, quasi-variational inequalities

Résumé

Dans ce travail nous présentons la méthode de multiplicateurs de Lagrange pour démontré l'existence de solution de inéquations variationnelles de deuxième espèce et inéquations quasi-variationnelles, Nous avons utilisé la méthode de régularisation et nous avons donné résultats de convergence, L'existence d'une solution établie en utilisant la théorie du point de selle et la technique du point fixe.

Mots Clés:la méthode de multiplicateurs de Lagrange, inéquations variationnelles de deuxième espèce, inéquations quasi-variationnelles.