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Dedication

To my father and mother, who covered me with their support and vowed unconditional love. You are for me the greatest example of courage and continuous sacrifice, your counsels have been very useful to me, and this humble work testifies my affection, my eternal attachment, and that will always show me your continual affection and blessing.

I will not omit the precious help and support of my father on translations and different helpful language works for the realization of this Thesis. I will never forget the important sympathy of my Brothers, sisters and all my friends.

To all my professors especially, my pedagogical supervisor, I would be vain if I did not enumerate in these few lines your remarkable human and professional qualities, please find here the expression and the testimony of my gratitude for your coaching, your advice, explanations, directions and remarks, this modest work honers and shows you my huge respect. Finally, Hope this thesis will represent the fruit of the efforts and perseverance made during these heard years of study.

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CONTENTS

NOTATIONS

Here below, we will define some notation that will be involved and used within development of this thesis. Some others, will be defined at the mean time of its usage.

➤ $\mathbb{R}^n$ denotes the Euclidean space of ordered N-tuplies of real numbers.

➤ Ω bounded and open subset of $\mathbb{R}^n$.

➤ $\partial\Omega$ the boundary of Ω.

➤ $D(\Omega)$ the space of infinitely smooth functions with a compact support in Ω.

➤ V real Hilbert space with scalar product $(\cdot\,,\cdot)$ and associated norm $\|\cdot\|$.

➤ V' the dual space of V.

➤ $\rightarrow$ strong convergence.

➤ $\rightharpoonup$ weak convergence.

➤ $\overset{*}{\rightharpoonup}$ weak star convergence.

➤ ψ the obstacle.

INTRODUCTION

Variational inequality theory has been fastly developed since 1967 introduced by Lions and Stampacchia [17] who successfully treated a coercive variational inequality. After the fundamental work of Lions and Stampacchia, the theory of variational inequalities was studied by many researchers (e.g. Brezis ([6], [5]), Browder ([7], [8]), Kinderlehrer [14], Duvaut and Lions ([11]) and others) and became an important subject in non-linear analysis.

It plays an important role in mechanics, partial differential equations, control theory, game theory, optimizations and so on.

In this thesis, our subject of concern is the existence and uniqueness of the solution of the evolutionary variational inequalities of the first kind.

This work is organized as follows:

In the first chapter, we will recall essential tools for our study.

In the second chapter, we will study the existence, uniqueness and approximation of the solutions of elliptic variational inequalities of the first kind.

In the third chapter, we will study the existence, uniqueness of the solutions of evolutionary variational inequalities of the first kind using penalty and elliptic regularization methods.

CHAPTER 1

PRELIMINARIES

This chapter recalls some basic notions and the main mathematical results of the functional analysis which will be used throughout this work. Most of the results are stated without proofs, as they are standard and can be found in many references.

1.1 FUNCTIONAL SPACES

Let Ω be a regular bounded open subset of $\mathbb{R}^n$. We denote by $D(\Omega)$ the set of indefinitely differentiable functions with compact support in Ω.

1.1.1 Lebesgue spaces

Definition 1.1 *Let (Ω, μ) be a measure space, the space $\mathsf{L}^1(\Omega, \mu)$, or simply $\mathsf{L}^1(\Omega)$ consists of all measurable functions on Ω that satisfy*

$$\|u\|_{\mathsf{L}^1} = \|u\|_1 = \int_\Omega |u(x)|\, d\mu < \infty.$$

Definition 1.2 *Let* $p \in \mathbb{R}$ *with* $1 < p < \infty$*, we set*

$$\mathsf{L^p} = \left\{ u : \Omega \to \mathbb{R},\ \ u\ \ is\ \ measurable\ \ and\ \ |u(x)|^p \in \mathsf{L}^1(\Omega) \right\},$$

with

$$\|u\|_{\mathsf{L^p}} = \|u\|_p = \left[\int_\Omega |u(x)|^p \, d\mu \right]^{\frac{1}{p}}.$$

Definition 1.3 *We set*

$$\mathsf{L}^\infty = \left\{ u : \Omega \to \mathbb{R}, \ \middle| \ \begin{array}{l} u\ \ is\ \ measurable\ \ and\ \ there\ \ is\ \ a\ \ constant\ \ C \\ such\ \ that\ \ |u(x)| \leq C\ \ a.e\ \ on\ \ \Omega \end{array} \right\},$$

with

$$\|u\|_{\mathsf{L}^\infty} = \|u\|_\infty = ess \sup_{x\in\Omega} |u(x)|.$$

Definition 1.4 *Let* $p \in \mathbb{R}$ *with* $1 \leq p < \infty$*, we set*

$$L^p(0,T;X) = \left\{ u : [0,T] \to X,\ \ u\ \ is\ \ measurable\ \ and\ \ \int_0^T \|u(t)\|_X^p \, dt < \infty \right\}.$$

with

$$\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p \, dt \right)^{\frac{1}{p}}.$$

Definition 1.5 *We set*

$$L^\infty(0,T;X) = \{ u : [0,T] \to X,\ \ u\ \ is\ \ measurable\ \ and\ \ \exists C > 0\, \|u(t)\|_X < C\ \ a.e.\ \ in\ \ t \}.$$

with

$$\|u\|_{L^\infty(0,T;X)} = \inf \{ C > 0,\ \ \|u(t)\|_X < C\ \ a.e.\ \ in\ \ t \}.$$

Theorem 1.6 (Basic properties of Lebesgue spaces)

(i) $L^p(\Omega)$ *is a vector space and* $\|\cdot\|_p$ *is a norm for any* p, $1 \le p \le \infty$.

(ii) *The spaces* $L^p(\Omega)$ *are Banach spaces (complete normed spaces).*

(iii) *The space* $L^2(\Omega)$ *becomes a Hilbert spaces with the inner product*

$$(u, v) = \int_\Omega u(x)v(x)dx, \qquad \|u\|_{L^2} = \|u\|_2 = (u, u)^{\frac{1}{2}}.$$

(iv) $L^p(\Omega)$ *is reflexive for any* p, $1 < p < \infty$.

(v) $L^p(\Omega)$ *is separable for any* p, $1 \le p < \infty$.

Proof. See [4]. ■

1.1.2 Sobolev spaces

Definition 1.7 *Let* $u \in L^1(\Omega)$ *and* $\alpha \in \mathbb{N}$. *The function* u *is said to have a weak derivative* $D^\alpha u$, *if there exists a function* $v \in L^1(\Omega)$ *such that:*

$$\int_\Omega u D^\alpha \varphi = (-1)^{|\alpha|} \int_\Omega v\varphi, \qquad \varphi \in D(\Omega).$$

Where we use the standard multi-index notation $\alpha = (\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n)$ *with* $\alpha_i \ge 0$ *an integer,*

$$|\alpha| = \sum_{i=1}^{n} \alpha_i \quad \textit{and} \qquad D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}.$$

We denote $D^\alpha u = v$.

Definition 1.8 *Let p be a real number with $1 \leq p \leq \infty$. The Sobolev space $W^{m,p}(\Omega)$ is defined to be:*

$$W^{m,p}(\Omega) = \{u \in L^p(\Omega) \mid D^\alpha u \in L^p(\Omega), \quad |\alpha| \leq m\}.$$

The space $W^{m,p}(\Omega)$ equipped with the norm

$$\|u\|_{W^{m,p}} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_p.$$

Definition 1.9 *In the special case where $p = 2$, we define the Hilbert-Sobolev space $H^k(\Omega) = W^{m,2}(\Omega)$*

$$for \;\; k \in \mathbb{N} \quad H^k(\Omega) = \left\{u \in L^2(\Omega) \mid D^\alpha u \in L^2(\Omega), \quad |\alpha| \leq k\right\}.$$

The space $H^k(\Omega)$ is equipped with the inner product

$$\langle u, v \rangle_{H^k} = \sum_{|\alpha| \leq k} \int_\Omega D^\alpha u D^\alpha v dx,$$

and the norm

$$\|u\|_{H^k} = \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|_2.$$

Definition 1.10 *We set*

$$H_0^1(\Omega) = \left\{u \in H^1(\Omega), \quad u_{|\partial\Omega} = 0\right\}.$$

1.2 General Theorems and Definitions

Theorem 1.11 (Rellich-Kondrachov) *Let* $\Omega \subset \mathbb{R}^n$ *be a Lipschitz domain,* $m \in \mathbb{N}$ *and* $1 \leq p \leq \infty$*. Then, the following mappings are compact embeddings:*

(i) $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad 1 \leq q \leq p^*, \quad \frac{1}{p^*} = \frac{1}{p} - \frac{m}{d}, \quad if \;\; m < \frac{d}{p},$

(ii) $W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), \qquad q \in [1, \infty), \quad if \;\; m = \frac{d}{p},$

(iii) $W^{m,p}(\Omega) \hookrightarrow C^0\left(\bar{\Omega}\right), \quad if \;\; m > \frac{d}{p}.$

Proof. See [1]. ■

Theorem 1.12 (Riesz representation theorem) *Let* V *be a Hilbert space, for all* $f \in V'$*, there exists a unique element* $\tilde{f} \in V$ *such that*

$$f(v) = (\tilde{f}, v) \qquad \forall v \in V.$$

In addition, we have

$$\|f\|'_V = \left\|\tilde{f}\right\|_V .$$

Proof. See [4]. ■

Theorem 1.13 (Projection Theorem) *Suppose that* V *is a Hilbert space and* $K \subset V$ *be a closed convex subset of* V*. Then for any* $x \in V$ *there exists a unique* $y \in K$ *such that*

$$\|x - y\| = \inf_{z \in K} \|x - z\| .$$

Moreover, x *is characterized by the property*

$$x \in K \;\; and \;\; (x - y, z - y) \leq 0 \qquad \forall z \in K.$$

The above element y *is called the projection of* x *onto* K *and is denoted by*

$$y = P_K x.$$

Proof. See [4]. ■

Proposition 1.14 *Let* $K \subset V$ *be a non empty closed convex set. Then* P_K *is a contraction, i.e.,*

$$\|P_K x_1 - P_K x_2\| \leq \|x_1 - x_2\| \qquad \forall x_1,\ x_2 \in V.$$

Proof. See [4]. ■

Theorem 1.15 (Banach fixed-point theorem) *Let* $(V, \|\cdot\|)$ *be a Banach space, and let* K *be a nonempty closed subset of* V. *Suppose that the operator* $T : K \to K$ *is a contraction, i.e. there exists a constant* $C \in [0,1)$ *such that*

$$\|Tu - Tv\|_V \leq C \|u - v\|_V \qquad \forall u,\ v \in K.$$

Then T *has a unique fixed point,* $Tu = u$.

Proof. See [2]. ■

Lemma 1 (Gronwall Lemma) *Let* y *and* g *be non negative integrable functions and* C *a non negative constant. If*

$$y(t) \leq C + \int_0^t g(s)y(s)ds \qquad for \ \ t \geq 0,$$

then

$$y(t) \leq C \exp\left(\int_0^t g(s)\, ds \right) \qquad for \ \ t \geq 0.$$

Proof. See [15] ■

Definition 1.16 *Let* X *be a normed linear space and let* X' *denote its dual. Let* $u_n,\ u \in X$.

(i) *We say that* u_n *converges strongly or converges in norm to* u *and we write* $u_n \to u$ *if*

$$\lim_{n \to \infty} \|u - u_n\| = 0.$$

(ii) *We say that* u_n *converges weakly to* u *and we write* $u_n \rightharpoonup u$ *if*

$$\forall \mu \in X', \qquad \lim_{n\to\infty} \langle u_n, \mu \rangle = \langle u, \mu \rangle .$$

Definition 1.17 *Let* X *be a normed linear space and let* X' *denote its dual. A sequence* $u_n \subset X'$ *is said to converge weakly* to* $u \in X'$ *if*

$$\langle u_n, v \rangle = \langle u, v \rangle \quad as \;\; n \to \infty, \qquad \forall v \in X.$$

In this case, u *is called the weak* limit of* u_n *and we write* $u_n \overset{*}{\rightharpoonup} u$ *in* X'.

Theorem 1.18 (Eberlein-Smulyan) *If* X *is a reflexive Banach space and the sequence* $\{u_n\} \subset X$ *is bounded*

$$\|u_n\|_X \leq C,$$

then we can find a subsequence $\{u_{nk}\} \subset X$ *and an element* $u \in X$ *such that:*

$$u_{nk} \rightharpoonup u \qquad in \;\; X.$$

Furthermore, it can be proved that if the limit u *is independent of the subsequence extracted, then the whole sequence* $\{u_n\}$ *converges weakly to* u.

Proof. See [4]. ■

Theorem 1.19 (Boundedness of weakly converging sequences) *Suppose* $1 \leq p < \infty$ *and* $u_n \rightharpoonup u$ $in\ L^p(\Omega)$ $\left(u_n \overset{*}{\rightharpoonup} u \;\; in\ L^\infty(\Omega) \;\; if \;\; p = \infty\right)$. *Then*

(i) u_n *is bounded in* $L^p(\Omega)$.

(ii) $\|u\|_{L^p(\Omega)} \leq \lim_{n\to\infty} \inf \|u_n\|_{L^p(\Omega)}$.

Proof. See [12]. ■

Theorem 1.20 (First Green's formula) *For all* $u \in H^2(\Omega)$ *and* $v \in H^1(\Omega)$

$$\int_\Omega \nabla u(x) \cdot \nabla v(x) dx = \int_{\partial\Omega} \frac{\partial u}{\partial n}(s) v(s) ds - \int_\Omega \Delta u(x) v(x) dx.$$

Proposition 1.21 (Cauchy-Schwarz inequality.) *Let* $(H, (\cdot\,,\cdot))$ *be an inner product space. Define* $\|.\| = (\cdot\,,\cdot)^{\frac{1}{2}}$*. Then, for every* $u, v \in H$

$$|(u,v)| \leq \|u\| \, \|v\| \, .$$

Proof. See [4]. ■

Definition 1.22 *Let* $(V, \langle\cdot\,,\cdot\rangle)$ *be a n-dimensional euclidean vector space and* $T : V \to V$ *a linear operator. We will call the adjoint of* T*, the linear operator* $T^* : V \to V$ *such that:*

$$\langle Tu, w\rangle = \langle u, T^* w\rangle \, , \qquad \forall v, \; w \in V.$$

Definition 1.23 *Let* V *be a reflexive Banach space. We call a linear operator* $T : V \to V'$ *monotone if for all* u *and* v *in* V

$$(Tu - Tv, u - v) \geq 0.$$

Definition 1.24 *A bilinear form* $a : V \times V \to \mathbb{R}$ *is said to be*

(i) *continuous if there is a constant* C *such that*

$$|a(u,v)| \leq C \, \|u\|_V \, \|v\|_V \qquad \forall u, \; v \in V,$$

(ii) V*-elliptic if there is a constant* $\alpha > 0$ *such that*

$$a(v,v) \geq \alpha \, \|v\|_V^2 \qquad \forall v \in V.$$

CHAPTER 2

ELLIPTIC VARIATIONAL INEQUALITIES

In this chapter, we shall restrict our attention to the study of the existence, uniqueness and approximation of the solutions of elliptic variational inequalities of the first kind.

2.1 ELLIPTIC VARIATIONAL INEQUALITY OF THE FIRST KIND

Notation 2.1

➤ $a(\cdot\,,\cdot\,) : V \times V \mapsto \mathbb{R}$ *is a bilinear, continuous and* V*-elliptic mapping on* $V \times V$.

➤ $L(\cdot\,) : \mathsf{V} \mapsto \mathsf{R}$ *is a continuous, linear functional on* V.

➤ K *is a closed, convex, non-empty subset of* V.

Definition 2.2 *We call elliptic variational inequality of the first kind Any inequality defined by:*

$$\left\{ \begin{array}{l} Find \ \ u \in \mathsf{K} \ \ such \ \ that \\ a(u, v-u) \geq L(v-u) \qquad \forall v \in \mathsf{K}. \end{array} \right. \tag{2.1}$$

2.2 EXISTENCE AND UNIQUENESS RESULTS

Theorem 2.3 (LIONS-STAMPACCHIA) *If* $a(\cdot,\cdot):\mathsf{V}\times\mathsf{V}\mapsto\mathsf{R}$ *is a bilinear, continuous and cœrcive form on a Hilbert space* V*,* $L(\cdot):\mathsf{V}\mapsto\mathsf{R}$ *is a continuous, linear functional on* V *and* K *is a closed, convex, non-empty subset of* V *then the problem (2.1) has one and only one solution.*

Proof.

❶ **Uniqueness:**

Let u_1 and u_2 be solutions of (2.1). We have then:

$$a(u_1, v-u_1) \geq L(v-u_1) \qquad \forall v\in\mathsf{K}, \tag{2.2}$$

$$a(u_2, v-u_2) \geq L(v-u_2) \qquad \forall v\in\mathsf{K}. \tag{2.3}$$

Choosing $v = u_2$ in (2.2) and $v = u_1$ in (2.3) and adding the corresponding inequalities, we obtain:

$$a(u_1-u_2, u_1-u_2) \leq 0, \tag{2.4}$$

by using the V-ellipticity of $a(\cdot,\cdot)$, we get:

$$\alpha\left\|u_1-u_2\right\|_V \leq 0.$$

Which implies

$$u_1 = u_2.$$

❷ **Existence:**

we will reduce the problem (2.1) to a fixed point problem . By the Riesz representation theorem for Hilbert spaces there exist $A:V\to V$ *and* $f\in V$ such that:

$$(Au,v) = a(u,v) \qquad \forall u,v\in V,$$

$$L(v) = (f,v) \qquad \forall v\in V.$$

Then the problem (2.1) is equivalent to finding $u\in V$ such that:

$$\begin{cases} (u-[u-\rho(Au-f)], v-u) \leq 0 & \forall v\in K \\ u\in K, \quad \rho>0. \end{cases} \tag{2.5}$$

This is equivalent to finding u such that:

$$u = P_K(u - \rho(Au - f)) \qquad \forall \rho > 0.$$

Consider the map $T : V \to V$ defined by $T(u) = P_K(u - \rho(Au - f))$.
Let $u_1, u_2 \in V,$ then since P_K is a contraction we have:

$$\|T(u_1) - T(u_2)\|^2 \leq \|u_1 - u_2\|^2 + \rho^2 \|A(u_1 - u_2)\|^2 - 2\rho\alpha \|u_1 - u_2\|^2 .$$

Hence we have

$$\|T(u_1) - T(u_2)\|^2 \leq (1 - 2\rho\alpha + \rho^2 \|A\|^2) \|u_1 - u_2\|^2 .$$

Thus T is a strict contraction mapping if $0 < \rho < \dfrac{2\alpha}{\|A\|^2}$. By taking ρ in this range we have a unique solution for the fixed point problem which implies the existence of a solution for (2.1). ■

Remark 2.4 *If $K = V$ then the problem (2.1) reduce to the classical variational equation*

$$\begin{cases} Find \;\; u \in V \;\; such \;\; that \\ a(u, v) = L(v) \qquad \forall v \in V. \end{cases}$$

Proposition 2.5 *If $a(\cdot, \cdot)$ is symmetric then the variational inequality (2.1) is equivalent to the following minimization problem:*

$$\begin{cases} Find \;\; u \in \mathsf{K} \;\; such \;\; that \\ J(u) \leq J(v) \qquad \forall v \in \mathsf{K}, \end{cases} \tag{2.6}$$

with

$$J(v) = \frac{1}{2} a(v, v) - L(v).$$

Proof.

❶ **(2.6) $\implies$ (2.1)**:

If $u \in K$ is a solution of the minimization problem (2.6), then we have:

$$J(u) \leq J(v),$$

hence

$$\frac{1}{2}a(u,u) - L(u) \leq \frac{1}{2}a(v,v) - L(v),$$

we put $v = (1-t)u + tv \quad \forall t \in [0,1]$, then:

$$\frac{1}{2}a(u,u) - L(u) \leq \frac{1}{2}a((1-t)u + tv, (1-t)u + tv) - L((1-t)u + tv),$$

$$\leq \frac{1}{2}\left[(1-t)^2 a(u,u) + t^2 a(v,v) + 2t(1-t)a(u,v)\right] - (1-t)L(u) - tL(v),$$

which implies

$$\frac{t^2}{2}a(u-v,u-v) + ta(u,v-u) \geq tL(v-u).$$

Therefore, dividing the above inequality by t and passing to the limit with $t \to 0$ we obtain

$$a(u,v-u) \geq L(v-u) \qquad \forall v \in K.$$

❷ **(2.1) $\implies$ (2.6)**:

If u satisfies (2.1), then:

$$J(v) - J(u) = a(u,v-u) - L(v-u) + \frac{1}{2}a(u-v,u-v),$$

for every $v \in K$. Therefore

$$J(v) - J(u) \geq 0 \qquad \forall v \in K,$$

which shows that u is a solution of the minimization problem (2.6). ∎

2.3 APPROXIMATION OF THE VARIATIONAL INEQUALITY

2.3.1 Approximation of V and K

Let $\{V_h\}_{h>0}$ be a family of finite dimensional closed subspaces of V, and $\{K_h\}_{h>0}$ A family of closed non-empty convex subsets of V with $K_h \subset V_h \ \forall h$, such that $\{K_h\}_h$ satisfies the following two conditions:

(i) $\forall v \in K, \ \exists\, v_h = r_h v \in K_h$ such that $v_h \to v$ in V,

(ii) $\forall v_h \in K_h$ if $v_h \rightharpoonup v$ then $v \in K$.

2.3.2 Approximation of (2.1)

The problem (2.1) is approximated by:

$$\begin{cases} Find \ \ u_h \in \mathsf{K_h} \ \ such \ \ that \\ \\ a(u_h, v_h - u_h) \geq L(v_h - u_h) \qquad \forall v_h \in \mathsf{K_h}. \end{cases} \tag{2.7}$$

Theorem 2.6 *(2.7) has a unique solution.*

Proof. In Theorem 2.3, taking V to be V_h and K to be K_h, we have the result. ■

2.4 CONVERGENCE RESULTS

Theorem 2.7 *With the above assumptions on K and $\{K_h\}_h$, we have*

$$\lim_{h \to 0} \|u_h - u\|_V = 0,$$

with u_h the solution of (2.7) and u the solution of (2.1).

Proof. For proving this kind of convergence result, we usually divide the proof into three parts. First we obtain a priori estimates for $\{u_h\}_h$, then the weak convergence of $\{u_h\}_h$, and finally with the help of the weak convergence, we will prove strong convergence.

❶ **A priori estimates for u_h:**

We will now show that there exist two constants C_1 and C_2 independent of h such that

$$\|u_h\|_V^2 \leq C_1 \|u_h\| + C_2 \qquad \forall h. \tag{2.8}$$

Since u_h is the solution of (2.7), we have

$$a(u_h, v_h - u_h) \geq L(v_h - u_h) \qquad \forall v_h \in K_h,$$
$$a(u_h, u_h) \leq a(u_h, v_h) - L(v_h - u_h).$$

By continuity of $a(\cdot\,,\cdot)$ and $L(\cdot)$, we get

$$a(u_h, u_h) \leq \|A\| \, \|u_h\| \, \|v_h\| + \|L\| \, (\|v_h\| + \|u_h\|), \qquad \forall v_h \in K_h.$$

Using the V-ellipticity of $a(\cdot\,,\cdot)$, we get

$$\alpha \, \|u_h\|^2 \leq \|A\| \, \|u_h\| \, \|v_h\| + \|L\| \, (\|v_h\| + \|u_h\|), \qquad \forall v_h \in K_h. \tag{2.9}$$

By condition (i) on K_h we have $r_h v_0 \to v_0$ strongly in V and hence $\|v_h\|$ is uniformly bounded by a constant m. Hence (2.9) can be written as

$$\|u_h\|^2 \leq \frac{1}{\alpha}[(m \, \|A\| + \|L\|) \, \|u_h\| + \|L\| \, m] = C_1 \, \|u_h\| + C_2,$$

$$\Longrightarrow \qquad \|u_h\|^2 \leq C_1 \, \|u_h\| + C_2.$$

where $C_1 = \dfrac{1}{\alpha}(m \, \|A\| + \|L\|)$ and $C_2 = \dfrac{m}{\alpha} \, \|L\|$. Without loss of generality we assume $C_2 > 1$, then (2.8) implies

$$\|u_h\| \leq C \qquad \forall h.$$

❷ **Weak convergence of $\{u_h\}_h$:**

The relation (2.8) implies that u_h is uniformly bounded. Hence we can extract a subsequence, also denoted by $\{u_h\}$ such that u_h converges to $\bar{u}$ weakly in V.
By condition (ii) on $\{K_h\}_h$, we have $\bar{u} \in K$. We will prove that $\bar{u}$ is a solution of (2.1). We have

$$a(u_h, u_h) \leq a(u_h, v_h) - L(v_h - u_h), \qquad \forall v_h \in K_{hi}. \tag{2.10}$$

We have also $v_h = r_h v$. Then (2.10) becomes

$$a(u_h, u_h) \leq a(u_h, r_h v) - L(r_h v - u_h). \tag{2.11}$$

Since $r_h v$ converges strongly to v and u_h converges to $\bar{u}$ weakly as $h \to 0,$ taking the limit in (2.11), we obtain

$$\lim_{h\to 0} \inf a(u_h, u_h) \leq a\left(\bar{u}, v\right) - L\left(v - \bar{u}\right), \qquad \forall v \in K. \tag{2.12}$$

Also we have

$$0 \leq a(u_h - \bar{u}, u_h - \bar{u}) \leq a(u_h, u_h) - a(u_h, \bar{u}) - a(\bar{u}, u_h) + a(\bar{u}, \bar{u}),$$

i.e,

$$a(u_h, \bar{u}) + a(\bar{u}, u_h) - a(\bar{u}, \bar{u}) \leq a(u_h, u_h).$$

By taking the limit, we obtain

$$a(\bar{u}, \bar{u}) \leq \lim_{h\to 0} \inf a(u_h, u_h). \tag{2.13}$$

From (2.12) and (2.13), we obtain

$$a(\bar{u}, \bar{u}) \leq \lim_{h\to 0} \inf a(u_h, u_h) \leq a(\bar{u}, v) - L(v - \bar{u}), \qquad \forall v \in K.$$

Therefore we have

$$a(\bar{u}, v - \bar{u}) \geq L(v - \bar{u}), \qquad \forall v \in K, \quad \bar{u} \in K. \tag{2.14}$$

Hence $\bar{u}$ is a solution of (2.1). By Theorem 2.3, the solution of (2.1) is unique and hence $\bar{u} = u$ is the unique solution.

❸ **Strong convergence:**

By V-ellipticity of $a(\cdot\,,\cdot\,)$, we have

$$0 \leq \alpha \left\|u_h - u\right\|^2 \leq a(u_h - u, u_h - u) = a(u_h, u_h) - a(u_h, u) - a(u, u_h) + a(u, u), \tag{2.15}$$

where u_h is the solution of (2.2) and u is the solution of (2.1). Since u_h is the solution of (2.2) and $r_h v \in K_h$ for any $v \in K$, from (2.2) we obtain

$$a(u_h, u_h) \leq a(u_h, r_h v) - L(r_h v - u_h), \qquad \forall v \in K. \tag{2.16}$$

Since $\lim_{h\to 0} u_h = u$ weakly in V and $\lim_{h\to 0} r_h v = v$ strongly in V (by condition (i)), we obtain (2.16) from (2.15), and after taking the limit, $\forall v \in K,$ we have

$$0 \leq \alpha \lim_{h\to 0} \inf \|u_h - u\|^2 \leq \alpha \lim_{h\to 0} \sup \|u_h - u\|^2 \leq a(u, v-u) - L(v-u). \tag{2.17}$$

Taking $v = u$ in (2.17) we obtain

$$\lim_{h\to 0} \|u_h - u\| = 0.$$

■

CHAPTER 3

EVOLUTIONARY VARIATIONAL INEQUALITIES

In this chapter we will consider a class of evolutionary variational inequalities. We will indicate sufficient conditions in order to have the existence, uniqueness and regularity results of the solution. The existence of the solution is obtained by using a penalty method. Finally, we will study the estimation of the penalization error.

3.1 THE CLASSIC PROBLEM

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 1$. We shall put :

$$\mathsf{Q} = \Omega \times]0, T[\quad , \quad \Sigma = \partial\Omega \times]0, T[\quad , \quad 0 < T < +\infty, \quad \mathbf{V} = \mathsf{H}_0^1(\Omega)\,, \;\; \mathbf{H} = \mathsf{L}^2(\Omega)\,.$$

We consider the functions $a_{ij}(x,t)\,,\ a_j(x,t)\,,\ a_o(x,t)\,,\ i,j=1,\dots,n$ which satisfy:

$$\begin{cases} a_{ij},\ a_j,\ a_o\ \in \mathsf{L}^\infty(\mathsf{Q}), \qquad a_{ij}=a_{ji} \\ \sum\limits_{\substack{i=1\\ j=1}}^{n} a_{ij}\ \xi_i\ \xi_j \geq \alpha \sum\limits_{i=1}^{n} |\xi_i|^2\,, \quad \alpha>0 \ \ in\ \mathsf{Q}, \qquad \forall \xi_i \in \mathbb{R}. \end{cases} \tag{3.1}$$

We also take

$$f,\psi \in \mathsf{L}^2\left(0,T;H\right). \tag{3.2}$$

We are looking for a function $u:\Omega\times]0,+\infty[\to\mathbb{R}$ such that :

$$(P.C)\begin{cases} -\dfrac{\partial u}{\partial t}+\mathbf{A}\left(t\right)u-f\leq 0 \qquad in\ \ \mathsf{Q}, & (3.3)\\ u-\psi\leq 0 \quad in\ \ \mathsf{Q}, & (3.4)\\ \left(-\dfrac{\partial u}{\partial t}+\mathbf{A}\left(t\right)u-f\right)(u-\psi)=0 \qquad in\ \ \mathsf{Q}, & (3.5)\\ u=0 \quad on\ \ \Sigma, \qquad u\left(x,T\right)=\bar{u}\left(x\right) \quad x\in\Omega, & (3.6) \end{cases}$$

the operator A is given by

$$A\left(t\right)u=-\sum_{\substack{i=1\\ j=1}}^{n}\frac{\partial}{\partial x_i}\left(a_{ij}\left(x,t\right)\frac{\partial u}{\partial x_j}\right)+\sum_{j}a_j\left(x,t\right)\frac{\partial u}{\partial x_j}+a_o\left(x,t\right)u. \tag{3.7}$$

Remark 3.1 *The problem (P.C) is backward in time (with initial values at* T*). We could equally well consider initial data at* 0 *as long as we take* $\dfrac{\partial u}{\partial t}$ *instead of* $-\dfrac{\partial u}{\partial t}$.

3.2 STRONG VARIATIONAL INEQUALITY

We introduce:

$$K = \{v \mid v \in V, \quad v(x) \leq \psi(x,t) \quad a.e. \ in \ \Omega\}, \tag{3.8}$$

$$\tilde{\mathcal{K}} = \left\{v \mid v \in L^2(0,T;V), \quad \frac{\partial v}{\partial t} \in L^2(0,T;V'), \quad v(x,t) \leq \psi(x,t) \quad a.e. \ in \ \mathsf{Q}\right\}. \tag{3.9}$$

We shall always assume that:

$$K \neq \emptyset, \tag{3.10}$$

$$\tilde{\mathcal{K}} \neq \emptyset. \tag{3.11}$$

We define a continuous bilinear form on V by:

$$a(t;u,v) = \sum \int_\Omega a_{ij}(x,t) \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx + \sum \int_\Omega a_j(x,t) \frac{\partial u}{\partial x_j}\, v\, dx + \sum \int_\Omega a_o(x,t)\, u\, v\, dx. \tag{3.12}$$ (1)

We say that u is a **'strong solution'** of the evolutionary V.I associated with (P.C), if

$$\begin{cases} u \in \tilde{\mathcal{K}}, \qquad u(x,T) = \bar{u}(x) \quad x \in \Omega, \\ -\left(\frac{\partial u}{\partial t}, v - u(t)\right) + a(t;u(t), v - u(t)) \geq (f(t), v - u(t)) \quad a.e. \ in \ t, \quad \forall\, v \in K. \end{cases} \tag{3.13}$$

Proposition 3.2 *Assume that u is a solution of the problem (P.C), then u satisfies the problem (3.13).*

Proof. Let $v \in K$ a test function, then:

$$\mathbf{v} - \psi \leq \mathbf{0},$$

(1) $(A(t)u,v) = a(t;u,v)$

multiply (3.3) by $(v-\psi)$ and integrate over Ω, we get:

$$\int_{\Omega}\left(-\frac{\partial u}{\partial t}\ +\ A(t)\,u\ -\ f\right)(v-\psi)\,dx \geq 0. \tag{3.14}$$

Furthermore, by integrate (3.5) over Ω we obtain:

$$\int_{\Omega}\left(-\frac{\partial u}{\partial t}+A(t)\,u-f\right)(u-\psi)\,dx = 0, \tag{3.15}$$

by subtracting (3.15) from (3.14), we get:

$$\int_{\Omega}\left(-\frac{\partial u}{\partial t}+A(t)\,u-f\right)(v-u)\,dx \geq 0,$$

$$\Longrightarrow \quad \int_{\Omega}-\frac{\partial u}{\partial t}(v-u)\,dx\ +\ \int_{\Omega}A(t)\,u\,(v-u)-\int_{\Omega}f\,(v-u)\,dx \geq 0. \tag{3.16}$$

On the other hand, we have:

$$\int_{\Omega}\mathbf{A}(t)\,u\,(v-u)\,dx \;=\; \int_{\Omega}-\sum\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial u}{\partial x_j}\right)(v-u)\,dx\ +\ \int_{\Omega}\sum a_j\frac{\partial u}{\partial x_j}(v-u)\,dx$$
$$+\int_{\Omega}a_o u\,(v-u)\,dx,$$

using Green's formula (Theorem 1.20) we get :

$$\int_{\Omega}A(t)\,u\,(v-u)\,dx \;=\; \sum\int_{\Omega}a_{ij}\,\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_i}(v-u)\,dx-\sum\int_{\partial\Omega}a_{ij}\,\frac{\partial u}{\partial n}(v-u)\,ds$$
$$+\ \sum\int_{\Omega}a_j\frac{\partial u}{\partial x_j}(v-u)\,dx+\int_{\Omega}a_o u\,(v-u)\,dx,$$

since $v|_{\partial\Omega} = 0$ and $u|_{\partial\Omega} = 0$ then:

$$\int_\Omega A(t)\,u\,(v-u)\,dx = \sum \int_\Omega a_{ij}\,\frac{\partial u}{\partial x_j}\frac{\partial}{\partial x_i}(v-u)\,dx + \sum \int_\Omega a_j \frac{\partial u}{\partial x_j}(v-u)\,dx + \int_\Omega a_o u\,(v-u)\,dx.$$

Hence, we get (3.12).

Therefore, (3.16) implies:

$$-\left(\frac{\partial u}{\partial t}, v-u\right) + a\,(t; u, v-u) - (f, v-u) \geq 0, \quad ^{(1)} \quad \forall\, v \in K. \tag{3.17}$$

■

Proposition 3.3 *When it is possible to obtain a strong solution which also satisfies*

$$u \in L^2\left(0, T; H^2(\Omega)\right), \tag{3.18}$$

then u *satisfies* $(P.C)$.

Proof.

We have from (3.13):

$$-\left(\frac{\partial u}{\partial t}, v-u(t)\right) + a\,(t; u(t), v-u(t)) \geq (f(t), v-u(t)) \quad a.e. \ \ in \ \ t, \quad \forall\, v \in K,$$

$$\Longrightarrow \quad \left(-\frac{\partial u}{\partial t}, v-u(t)\right) + (A(t)u, v-u(t)) \geq (f(t), v-u(t)) \quad a.e. \ \ in \ \ t, \quad \forall\, v \in K,$$

$$\Longrightarrow \quad \left(-\frac{\partial u}{\partial t} + A(t)u - f, v-u\right) \geq 0 \quad a.e. \ \ in \ \ t, \quad \forall\, v \in K. \tag{3.19}$$

[1] $(\cdot\,,\cdot)$ is the inner product in $L^2(\Omega)$.

Taking $v = u - \psi$, we then deduce

$$\left(-\frac{\partial u}{\partial t} + A(t)u - f, u - \psi - u\right) \geq 0,$$

$$\Longrightarrow \quad \left(-\frac{\partial u}{\partial t} + A(t)u - f, -\psi\right) \geq 0,$$

$$\Longrightarrow \quad \left(-\frac{\partial u}{\partial t} + A(t)u - f, \psi\right) \leq 0,$$

and hence that

$$-\frac{\partial u}{\partial t} + A(t)u - f \leq 0. \tag{3.20}$$

Also if we put $v = \psi$ in (3.19) which is permissible if $\psi \in L^2(0, T; H)$ and (3.18) holds, we deduce that:

$$\left(-\frac{\partial u}{\partial t} + A(t)u - f, \psi - u\right) \geq 0. \tag{3.21}$$

Since $-\dfrac{\partial u}{\partial t} + A(t)u - f \leq 0$ and $\psi - u \geq 0$ [(1)] then

$$\left(-\frac{\partial u}{\partial t} + A(t)u - f, \psi - u\right) \leq 0. \tag{3.22}$$

Hence from (3.20) and (3.22) we obtain

$$\left(-\frac{\partial u}{\partial t} + A(t)u - f, u - \psi\right) = 0 \quad a.e, \tag{3.23}$$

and thus (P.C). ■

3.3 Existence and uniqueness results of the strong solution

Theorem 3.4 (See [3]) *Suppose that we have (3.1), (3.2), (3.11) and* $\dfrac{\partial a_{ij}}{\partial t} \in L^\infty(Q)$ *hold with :*

$$\psi, \frac{\partial \psi}{\partial t} \in L^2\left(, T; H^1(\Omega)\right), \qquad \psi \geq 0 \quad on \;\; \Sigma, \qquad \frac{\partial \psi}{\partial t} = 0 \quad on \;\; \Sigma, \tag{3.24}$$

$$\bar{u} \in V, \qquad \bar{u} \leq \psi(T). \tag{3.25}$$

[(1)] because $u \in \tilde{\mathcal{K}}$

Then the strong problem (3.13) admits a unique solution such that:

$$u \in L^{\infty}(0,T;V). \tag{3.26}$$

Proof.

❶ uniqueness:

Let $u_1,\ u_2$ be tow solutions of (3.13). We have then

$$-\left(\frac{\partial u_1}{\partial t}, v-u_1(t)\right)+a(t;u_1(t),v-u_1(t)) \geq (f(t),v-u_1(t)) \quad \forall v \in V, \tag{3.27}$$

$$-\left(\frac{\partial u_2}{\partial t}, v-u_2(t)\right)+a(t;u_2(t),v-u_2(t)) \geq (f(t),v-u_2(t)) \quad \forall v \in V. \tag{3.28}$$

Choosing $v=u_2$ in (3.27) and $v=u_1$ in (3.28) and adding the corresponding inequalities, we obtain:

$$\left(\frac{\partial}{\partial t}(u_1-u_2), u_1-u_2\right)-a(t;u_1-u_2,u_1-u_2) \geq 0,$$

we put $w=u_1-u_2$ then:

$$-\left(\frac{\partial w}{\partial t}, w\right)+a(t;w,w) \leq 0. \tag{3.29}$$

However, from (3.1) there exists λ such that

$$a(t;v,v)+\lambda\|v\|_H^2 \geq \alpha\|v\|_V^2, \quad \alpha \geq 0, \quad \forall v \in V, \tag{3.30}$$

then the equation (3.29) gives

$$-\frac{1}{2}\frac{d}{dt}\|w\|_H^2+\alpha\|w\|_V^2 \leq \lambda\|w\|_H^2$$

$$\Longrightarrow \qquad \int_t^T -\frac{1}{2}\frac{d}{ds}\|w\|_H^2+\int_t^T \alpha\|w\|_V^2\,ds \leq \int_t^T \lambda\|w\|_H^2\,ds,$$

which implies

$$\frac{1}{2}\|w(t)-w(T)\|_H^2+\int_t^T \alpha\|w\|_V^2\,ds \leq \lambda\int_t^T \|w\|_H^2\,ds,$$

from which we deduce, since $w(T) = u_1(T) - u_2(T) = 0$, that:

$$\frac{1}{2}\|w(t)\|_H^2 + \int_t^T \alpha \|w\|_V^2 \, ds \leq \lambda \int_t^T \|w\|_H^2 \, ds,$$

and therefore in particular, that:

$$\|w(t)\|_H^2 \leq 2\lambda \int_t^T \|w\|_H^2 \, ds, \tag{3.31}$$

using gronwall lemma (Lemma 1) we obtain

$$\|w(t)\|_H^2 \leq 0,$$

from which it follows that $w = 0$, so that $u_1 = u_2$.

❷**Existence:**

To prove the existence of the solution we will use the penalty method that will be developed in the next section. ■

3.4 THE PENALTY METHOD

The idea of penalization consists of approximating (3.13) which is a constrained problem by an unconstrained problem, which expresses the fact that "u belongs to K", is replaced by a penalisation term. The limit of the approximate solution converges to the solution of (3.13).

Specifically, we introduce a penalisation operator β which has the following properties:

$$\begin{cases} \beta : V \to V', \quad \beta \ \ is \ \ Lipschitz \ \ continuous, \\ Ker(\beta) = K, \\ \beta \ \ is \ \ monotone. \end{cases} \tag{3.32}$$

3.4.1 The penalized problem

For $\varepsilon > 0$ we consider the equation

$$\begin{cases} -\dfrac{\partial u_\varepsilon}{\partial t} \ + \ A(t)u_\varepsilon \ + \ \dfrac{1}{\varepsilon}\beta(u_\varepsilon) \ = \ f, \\ u_\varepsilon \in L^2\left(0,T;V\right), \\ u_\varepsilon\left(T\right) = \bar{u}, \end{cases} \tag{3.33}$$

which is the penalised equation associated with (3.13). In variational form (3.33) is written:

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, v\right) + a\left(t; u_\varepsilon, v\right) + \frac{1}{\varepsilon}(\beta(u_\varepsilon), v) = (f, v) \qquad \forall v \in V, \tag{3.34}$$

such that the bilinear form: $a(t; \cdot\,, \cdot\,)$ and the set K are defined in section 3.2. We set:

$$\beta\left(u_\varepsilon\right) = u_\varepsilon - P_K u_\varepsilon = (I - P_K)u_\varepsilon,$$

with: P_K is the projection onto K (theorem 1.13).

For any function v we set: $v = v^+ - v^-$, $v^+ = \max(v, 0)$ and $v^- = \max(-v, 0)$ and get

$$P_K u = u - (u - \psi)^+,$$

and therefore

$$\beta\left(u_\varepsilon\right) = \left(u_\varepsilon - \psi\right)^+,$$

then (3.34) implies:

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, v\right) + a\left(t; u_\varepsilon, v\right) + \frac{1}{\varepsilon}(\left(u_\varepsilon - \psi\right)^+, v) = (f, v) \qquad \forall v \in V, \tag{3.35}$$

We start by proving the followings:

Lemma 2 *Suppose that V is a Hilbert space, The operator β defined by*

$$\beta\left(u\right) = u - P_K u, \qquad with: \ P_K \ is \ the \ projection \ onto \ K \subset V,$$

then β verifies:

$$(\beta(u) - \beta(v), u - v) \geq 0 \ \ (monotony).$$

Proof. We have from the Theorem 1.13:

$$(v - P_K v, w - P_K v) \leq 0 \qquad \forall w \in K,$$

then

$$(v - P_K v, P_K v - w_1) \geq 0 \qquad \forall w_1 \in K, \tag{3.36}$$

$$(u - P_K u, P_K u - w_2) \geq 0 \qquad \forall w_2 \in K. \tag{3.37}$$

Choosing $w_1 = P_K u$ in (3.36) and $w_2 = P_K v$ in (3.37) and adding the corresponding inequalities, we obtain:

$$(\beta(u) - \beta(v), P_K u - P_K v) \geq 0. \tag{3.38}$$

On the other hand, we have:

$$\begin{aligned} & (\beta(u) - \beta(v), \beta(u) - \beta(v)) \geq 0, \\ \Longrightarrow \quad & (\beta(u) - \beta(v), (u - P_K u) - (v - P_K v)) \geq 0, \\ \Longrightarrow \quad & (\beta(u) - \beta(v), u - v) \geq (\beta(u) - \beta(v), P_K u - P_K v)\,. \end{aligned} \tag{3.39}$$

Hence

$$(\beta(u) - \beta(v), u - v) \geq 0 \qquad \forall u, v \in V.$$

■

Theorem 3.5 (See [3]) *Suppose that (3.10) holds along with (3.2),* $\bar{u} \in H$. *There then exists a unique* u_ε *such that*

$$u_\varepsilon \in L^2(0, T; V)\,, \qquad \frac{\partial u_\varepsilon}{\partial t} \in L^2(0, T; V')\,, \tag{3.40}$$

and u_ε *satisfies (3.34).*

Proof.

❶ **Uniqueness:**

The uniqueness is an immediate consequence of the monotonicity of the operator β, if in fact u_ε and $\bar{u}_\varepsilon$ are two possible solutions, we obtain from (3.34):

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, v\right) + a\left(t; u_\varepsilon, v\right) + \frac{1}{\varepsilon}(\beta(u_\varepsilon), v) = (f, v) \qquad \forall v \in V, \tag{3.41}$$

$$-\left(\frac{\partial \bar{u}_\varepsilon}{\partial t}, v\right) + a\left(t; \bar{u}_\varepsilon, v\right) + \frac{1}{\varepsilon}(\beta(\bar{u}_\varepsilon), v) = (f, v) \qquad \forall v \in V, \tag{3.42}$$

choosing $v = u_\varepsilon - \bar{u}_\varepsilon$ in **(3.41)** and $v = \bar{u}_\varepsilon - u_\varepsilon$ in **(3.42)** and adding the corresponding inequalities, we obtain:

$$-\left(\frac{\partial}{\partial t}\left(u_\varepsilon - \bar{u}_\varepsilon\right), \left(u_\varepsilon - \bar{u}_\varepsilon\right)\right) + a\left(t; u_\varepsilon - \bar{u}_\varepsilon, u_\varepsilon - \bar{u}_\varepsilon\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon) - \beta(\bar{u}_\varepsilon), u_\varepsilon - \bar{u}_\varepsilon\right) = 0,$$

we put $w = u_\varepsilon - \bar{u}_\varepsilon$ then:

$$-\left(\frac{\partial w}{\partial t}, w\right) + a\left(t; w, w\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon) - \beta(\bar{u}_\varepsilon), u_\varepsilon - \bar{u}_\varepsilon\right) = 0,$$

so that, since $(\beta(u_\varepsilon) - \beta(\bar{u}_\varepsilon), u_\varepsilon - \bar{u}_\varepsilon) \geq 0$ we have:

$$-\left(\frac{\partial w}{\partial t}, w\right) + a\left(t; w, w\right) \leq 0. \tag{3.43}$$

However, from (3.30) we have:

$$-\frac{1}{2}\frac{d}{dt}\left\|w\right\|_H^2 + \alpha\left\|w\right\|_V^2 \leq \lambda\left\|w\right\|_H^2$$

$$\Longrightarrow \qquad \int_t^T -\frac{1}{2}\frac{d}{ds}\left\|w\right\|_H^2 + \int_t^T \alpha\left\|w\right\|_V^2 ds \leq \int_t^T \lambda\left\|w\right\|_H^2 ds,$$

which implies

$$\frac{1}{2}\left\|w\left(t\right) - w\left(T\right)\right\|_H^2 + \int_t^T \alpha\left\|w\right\|_V^2 ds \leq \lambda\int_t^T \left\|w\right\|_H^2 ds,$$

from which we deduce, since $w(T) = u_\varepsilon(T) - \bar{u}_\varepsilon(T) = 0$, that:

$$\frac{1}{2}\|w(t)\|_H^2 + \int_t^T \alpha \|w\|_V^2 \, ds \le \lambda \int_t^T \|w\|_H^2 \, ds,$$

and therefore, in particular, that:

$$\|w(t)\|_H^2 \le 2\lambda \int_t^T \|w\|_H^2 \, ds, \tag{3.44}$$

using gronwall lemma we obtain

$$\|w(t)\|_H^2 \le 0,$$

from which it follows that $w = 0$, so that $u_\varepsilon = \bar{u}_\varepsilon$.

❷**Existence:**

To prove the existence of the solution we will use the elliptic regularization method that will be developed in the next section. ■

3.5 THE ELLIPTIC REGULARISATION METHOD

The concept of this method, is regularizing the evolutionary equation (or inequality) by an elliptic equation (or inequality) already solved. Then we pass to the limit.

3.5.1 Elliptic regularised equation of (3.33)

In (3.34) we put:

$$\begin{cases} L = -\dfrac{\partial}{\partial t}, & (3.45) \\ \mathcal{V} = \left\{ v | \;\; v \in L^2(0,T;V), \;\; \dfrac{\partial v}{\partial t} \in L^2(0,T;H), \;\; v(T) = 0 \right\}, & (3.46) \end{cases}$$

then we get:

$$Lu_\varepsilon + A(t)u_\varepsilon + \frac{1}{\varepsilon}\beta(u_\varepsilon) = f. \tag{3.47}$$

since:

$$L^* = -L,$$

then the regularization of (3.47) will be:

$$\gamma L^* L u_{\varepsilon\gamma} \ + \ Lu_{\varepsilon\gamma} \ + \ A(u_{\varepsilon\gamma}) \ + \ \frac{1}{\varepsilon}\beta(u_{\varepsilon\gamma}) = f. \tag{3.48}$$

Therefore, for $\gamma > 0$, we seek $u_{\varepsilon\gamma}$, a solution of :

$$(P_{\varepsilon\gamma}) \begin{cases} -\gamma \dfrac{\partial^2 u_{\varepsilon\gamma}}{\partial t^2} - \dfrac{\partial u_{\varepsilon\gamma}}{\partial t} + A(t)u_{\varepsilon\gamma} + \dfrac{1}{\varepsilon}\beta(u_{\varepsilon\gamma}) = f, & (3.49) \\ u_{\varepsilon\gamma} \in L^2(0,T;V), \quad \dfrac{\partial u_{\varepsilon\gamma}}{\partial t} \in L^2(0,T;H), & (3.50) \\ u_{\varepsilon\gamma}(T) = \bar{u}_\gamma, & (3.51) \\ \dfrac{\partial u_{\varepsilon\gamma}}{\partial t}(0) = 0, & (3.52) \end{cases}$$

where $\bar{u}_\gamma \in V$, $\bar{u}_\gamma \to \bar{u} \;\; in \;\; H \;\; as \;\; \gamma \to 0.$

The problem $(P_{\varepsilon\gamma})$ is an elliptic problem (hence, the terminology: $(P_{\varepsilon\gamma})$ is called an ”elliptic regularised equation” of (3.47)), and is a simple variant of the stationary problem treated in [3].

3.5.2 The variational formulation of $(P_{\varepsilon\gamma})$

Let us assume that $\bar{u}_\gamma = 0$, the variational formulation of $(P_{\varepsilon\gamma})$ is then as follows:

$$\phi\left(u_{\varepsilon\gamma}, v\right) + \int_0^T \frac{1}{\varepsilon}\left(\beta(u_{\varepsilon\gamma}), v\right) dt = \int_0^T (f, v)\, dt \qquad \forall v \in \mathcal{V}. \tag{3.53}$$

Such that the bilinear form :

$$u, v \to \phi(u,v) = \int_0^T \left[\gamma\left(\frac{\partial u_{\varepsilon\gamma}}{\partial t}, \frac{\partial v}{\partial t}\right) - \left(\frac{\partial u_{\varepsilon\gamma}}{\partial t}, v\right) + a\left(t; u_{\varepsilon\gamma}, v\right)\right] dt, \tag{3.54}$$

is cœercive on $\mathcal{V}$:

$$\phi(u_{\varepsilon\gamma}, u_{\varepsilon\gamma}) \geq \gamma \int_0^T \left\|\frac{\partial u_{\varepsilon\gamma}}{\partial t}\right\|_H^2 dt + \alpha \int_0^T \left\|u_{\varepsilon\gamma}\right\|_V^2 dt. \tag{3.55}$$

We thus have existence and uniqueness for $u_{\varepsilon\gamma}$ the solution of (3.53), from the following theorem

Theorem 3.6 *Suppose that (3.1), (3.8), (3.10) hold in addition to*

$$a(v,v) \geq \alpha \left\|v\right\|^2, \quad \forall v \in H_0^1(\Omega), \quad \alpha > 0. \text{ [1]}$$

There then exists a unique $u \in K$*, the solution of (3.53).*

Proof. See [3]. ■

3.6 Proof of existence in Theorem 3.5

❶ A priori estimate:

We will now show that there exist an arbitrary constant C independent of γ and ε such that:

$$\left\|u_{\varepsilon\gamma}\right\|_{L^2(0,T;V)} + \sqrt{\gamma}\left\|u'_{\varepsilon\gamma}\right\|_{L^2(0,T;H)} \leq C. \tag{3.56}$$

Since $u_{\varepsilon\gamma}$ is the solution of (3.53), we have:

$$\phi\left(u_{\varepsilon\gamma}, v\right) + \int_0^T \frac{1}{\varepsilon}\left(\beta(u_{\varepsilon\gamma}), v\right) dt = \int_0^T (f,v)\, dt \qquad \forall v \in \mathcal{V},$$

[1] $\|\cdot\| =$ norm in $H_0^1(\Omega)$

putting $v = u_{\varepsilon\gamma}$ we get:

$$\phi\left(u_{\varepsilon\gamma}, u_{\varepsilon\gamma}\right) + \frac{1}{\varepsilon} \int_0^T \left(\beta(u_{\varepsilon\gamma}), u_{\varepsilon\gamma}\right) dt = \int_0^T \left(f, u_{\varepsilon\gamma}\right) dt,$$

$$\Longrightarrow \quad \phi\left(u_{\varepsilon\gamma}, u_{\varepsilon\gamma}\right) - \int_0^T \left(f, u_{\varepsilon\gamma}\right) dt = - \int_0^T \frac{1}{\varepsilon} \left(\beta(u_{\varepsilon\gamma}), u_{\varepsilon\gamma}\right) dt,$$

since $\left(\beta(u_{\varepsilon\gamma}), u_{\varepsilon\gamma}\right) \geq 0$ then:

$$\phi\left(u_{\varepsilon\gamma}, u_{\varepsilon\gamma}\right) - \int_0^T \left(f, u_{\varepsilon\gamma}\right) dt \leq 0,$$

$$\Longrightarrow \quad \phi\left(u_{\varepsilon\gamma}, u_{\varepsilon\gamma}\right) \leq \int_0^T \left(f, u_{\varepsilon\gamma}\right) dt,$$

by the V-ellipticity of $\phi\left(\cdot\, ,\cdot\right)$, we get:

$$\gamma \int_0^T \left\| u'_{\varepsilon\gamma} \right\|_H^2 + \alpha \int_0^T \left\| u_{\varepsilon\gamma} \right\|_V^2 \leq \int_0^T \left(f, u_{\varepsilon\gamma}\right) dt,$$

using Cauchy-Schwarz inequality, we get:

$$\gamma \int_0^T \left\| u'_{\varepsilon\gamma} \right\|_H^2 + \alpha \int_0^T \left\| u_{\varepsilon\gamma} \right\|_V^2 \leq \int_0^T \left\| f \right\|_H \left\| u_{\varepsilon\gamma} \right\|_V dt,$$

$$\Longrightarrow \quad \gamma \left\| u'_{\varepsilon\gamma} \right\|_{L^2(0,T;H)}^2 + \alpha \left\| u_{\varepsilon\gamma} \right\|_{L^2(0,T;V)}^2 \leq \left\| f \right\|_{L^2(0,T;H)} \left\| u_{\varepsilon\gamma} \right\|_{L^2(0,T;V)},$$

$$\Longrightarrow \quad \gamma \left\| u'_{\varepsilon\gamma} \right\|_{L^2(0,T;H)}^2 + \alpha \left\| u_{\varepsilon\gamma} \right\|_{L^2(0,T;V)}^2 \leq C \left\| u_{\varepsilon\gamma} \right\|_{L^2(0,T;V)},$$

which implies:

$$\alpha \left\| u_{\varepsilon\gamma} \right\|^2_{L^2(0,T;V)} \leq C \left\| u_{\varepsilon\gamma} \right\|_{L^2(0,T;V)}, \tag{3.57}$$

$$\gamma \left\| u'_{\varepsilon\gamma} \right\|^2_{L^2(0,T;H)} \leq C \left\| u_{\varepsilon\gamma} \right\|_{L^2(0,T;V)}, \tag{3.58}$$

from (3.57), we have:

$$\left\| u_{\varepsilon\gamma} \right\|_{L^2(0,T;V)} \leq C,$$

hence:

$$\begin{aligned} & \gamma \left\| u'_{\varepsilon\gamma} \right\|^2_{L^2(0,T;H)} \leq C, \\ \Longrightarrow \quad & \sqrt{\gamma} \left\| u'_{\varepsilon\gamma} \right\|_{L^2(0,T;H)} \leq C, \end{aligned}$$

therefore we get (3.56).

❷ **Weak convergence:**

The relation (3.56) implies that $u_{\varepsilon\gamma}$ and $u'_{\varepsilon\gamma}$ are uniformly bounded.

Hence according to the theorem 1.18 we can extract a subsequence, also denoted by $u_{\varepsilon\gamma}$ such that when $\gamma \to 0$:

$$\begin{cases} u_{\varepsilon\gamma} \rightharpoonup u_\varepsilon \quad in \;\; L^2(0,T;V), \\ \dfrac{\partial u_{\varepsilon\gamma}}{\partial t} \rightharpoonup \dfrac{\partial u_\varepsilon}{\partial t} \quad in \;\; L^2(0,T;H). \end{cases}$$

Since the injection from $V \to H$ is compact (theorem 1.11), we thus have:

$$u_{\varepsilon\gamma} \to u_\varepsilon \quad in \;\; L^2(0,T;H),$$

and we can immediately proceed to the limit in γ in (3.53); we therefore obtain that u_ε is a solution of

$$\int_0^T \left[-\left(\frac{\partial u_\varepsilon}{\partial t}, v \right) + a(t; u_\varepsilon, v) + \frac{1}{\varepsilon} (\beta(u_\varepsilon), v) \right] dt = \int_0^T (f, v) \qquad \forall v \in \mathcal{V},$$

from which we deduce :

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, v\right) + a\left(t; u_\varepsilon, v\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), v\right) = (f, v) \qquad a.e \quad \forall v \in V.$$

Then the problem (3.33) admits a unique solution.

3.7 Proof of existence in Theorem 3.4

❶ **A priori estimate (I):**

We will show that there exist an arbitrary constant C independent of ε such that:

$$\|u_\varepsilon\|_{L^\infty(0,T;V)} \;+\; \frac{1}{\sqrt{\varepsilon}}\left\|(u_\varepsilon - \psi)^+\right\|_{L^2(0,T;H)} \leq C. \tag{3.59}$$

Since the constant C in (3.56) is independent of ε (and of γ), we have

$$\|u_\varepsilon\|_{L^2(0,T;V)} \leq C. \tag{3.60}$$

Since u_ε is the solution of (3.34), we have:

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, v\right) + a\left(t; u_\varepsilon, v\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), v\right) = (f, v) \qquad \forall v \in V,$$

we take $v_0 \in K$ and we take the inner product of (3.34) with $v = u_\varepsilon - v_0$ we get:

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon - v_0\right) + a\left(t; u_\varepsilon, u_\varepsilon - v_0\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), u_\varepsilon - v_0\right) = \left(f, u_\varepsilon - v_0\right),$$

$$\Longrightarrow \; -\left(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon - v_0\right) + a\left(t; u_\varepsilon, u_\varepsilon - v_0\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), u_\varepsilon - \psi + \psi - v_0\right) = \left(f, u_\varepsilon - v_0\right),$$

$$\Longrightarrow \; -\left(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon - v_0\right) + a\left(t; u_\varepsilon, u_\varepsilon - v_0\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), u_\varepsilon - \psi\right) - \left(f, u_\varepsilon - v_0\right) = -\frac{1}{\varepsilon}\left(\beta(u_\varepsilon), \psi - v_0\right),$$

since $(\psi - v_0) \geq 0$ then:

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon - v_0\right) + a\left(t; u_\varepsilon, u_\varepsilon - v_0\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), u_\varepsilon - \psi\right) - \left(f, u_\varepsilon - v_0\right) \leq 0,$$

$$\Longrightarrow \quad -\left(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon\right) + \left(\frac{\partial u_\varepsilon}{\partial t}, v_0\right) + a\left(t; u_\varepsilon, u_\varepsilon\right) - a\left(t; u_\varepsilon, v_0\right) + \frac{1}{\varepsilon}\left\|(u_\varepsilon - \psi)\right\|^+ - \left(f, u_\varepsilon - v_0\right) \leq 0,$$

$$\Longrightarrow \quad -\left(\frac{\partial u_\varepsilon}{\partial t}, u_\varepsilon\right) + a\left(t; u_\varepsilon, u_\varepsilon\right) + \frac{1}{\varepsilon}\left\|(u_\varepsilon - \psi)^+\right\|_H^2 \leq -\left(\frac{\partial u_\varepsilon}{\partial t}, v_0\right) + a\left(t; u_\varepsilon, v_0\right) - \left(f, u_\varepsilon - v_0\right),$$

integrate over (t,T) we get:

$$-\frac{1}{2}\int_t^T \frac{d}{ds}\|u_\varepsilon\|_H^2 + \int_t^T a(s;u_\varepsilon,u_\varepsilon)\,ds + \int_t^T \frac{1}{\varepsilon}\left\|(u_\varepsilon-\psi)^+\right\|_H^2 ds$$
$$\leq \int_t^T \left[-\left(\frac{\partial u_\varepsilon}{\partial s}, v_0\right) + a(s;u_\varepsilon,v_0) - (f, u_\varepsilon - v_0)\right] ds,$$

so that

$$-\frac{1}{2}\|u_\varepsilon(T)\|_H^2 + \frac{1}{2}\|u_\varepsilon(t)\|_H^2\, ds + \int_t^T a(s;u_\varepsilon,u_\varepsilon)\,ds + \int_t^T \frac{1}{\varepsilon}\left\|(u_\varepsilon-\psi)^+\right\|_H^2 ds$$
$$\leq \int_t^T \left[-\left(\frac{\partial u_\varepsilon}{\partial s}, v_0\right) + a(s;u_\varepsilon,v_0) - (f, u_\varepsilon - v_0)\right] ds,$$

which implies, since $u_\varepsilon(T) = \bar{u}(x)$:

$$\frac{1}{2}\|u_\varepsilon(t)\|_H^2 + \int_t^T a(s;u_\varepsilon,u_\varepsilon)\,ds + \int_t^T \frac{1}{\varepsilon}\left\|(u_\varepsilon-\psi)^+\right\|_H^2 ds$$
$$\leq \frac{1}{2}\|\bar{u}\|_H^2\, ds + \int_t^T \left[-\left(\frac{\partial u_\varepsilon}{\partial s}, v_0\right) + a(s;u_\varepsilon,v_0) - (f, u_\varepsilon - v_0)\right] ds.$$

Using Cauchy-Schwarz inequality and the V-ellipticity of $a(t;\cdot,\cdot)$, we obtain:

$$\|u_\varepsilon(t)\|_H^2 + \int_t^T \frac{1}{\varepsilon}\left\|(u_\varepsilon-\psi)^+\right\|_H^2 ds \leq C_1 + C_2\|u_\varepsilon(t)\|_H,$$

which implies:

$$\|u_\varepsilon(t)\|_H^2 \ \leq \ C_1 + C_2 \|u_\varepsilon(t)\|_H \,, \tag{3.61}$$

$$\int_t^T \frac{1}{\varepsilon} \left\|(u_\varepsilon - \psi)^+\right\|_H^2 ds \ \leq \ C_1 + C_2 \|u_\varepsilon(t)\|_H \,. \tag{3.62}$$

Without loss of generality we assume $C_1 > 1$, then (3.61) implies:

$$\begin{aligned} & \|u_\varepsilon\|_H \leq C, \\ \Longrightarrow \quad & \|u_\varepsilon\|_{L^\infty(0,T;H)} \leq C, \end{aligned}$$

hence:

$$\int_t^T \frac{1}{\varepsilon} \left\|(u_\varepsilon - \psi)^+\right\|_H^2 ds \ \leq \ C,$$

$$\Longrightarrow \qquad \frac{1}{\sqrt{\varepsilon}} \left\|(u_\varepsilon - \psi)^+\right\|_{L^2(0,T;H)} \ \leq \ C.$$

Therefore we get (3.59).

❶ **A priori estimate (II):**

We will show that there exist an arbitrary constant C independent of ε such that:

$$\|u'_\varepsilon\|_{L^\infty(0,T;H)} \leq C. \tag{3.63}$$

We put:

$$a'(t;u,v) = \sum \int_\Omega \frac{\partial a_{ij}}{\partial t} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx + \sum \int_\Omega \frac{\partial a_j}{\partial t} \frac{\partial u}{\partial x_j}\, v\, dx + \sum \int_\Omega \frac{\partial a_o}{\partial t}\, u\, v\, dx,$$

$$(A'(t)u,v) = a'(t;u,v) \quad if \ \ v \in D(\Omega)\,.$$

In (3.34) we replace v by $u'_\varepsilon - \psi'$ (where $u'_\varepsilon = \dfrac{\partial u_\varepsilon}{\partial t}$, $\psi' = \dfrac{\partial \psi}{\partial t}$); this is permissible since $\dfrac{\partial \psi}{\partial t} = 0$ on Σ. We have:

$$- \left(u'_\varepsilon, u'_\varepsilon - \psi'\right) + a\left(t; u_\varepsilon, u'_\varepsilon - \psi'\right) + \frac{1}{\varepsilon}\left(\beta\left(u_\varepsilon\right), \left(u_\varepsilon - \psi\right)'\right) = \left(f, u'_\varepsilon - \psi'\right),$$

$$\Longrightarrow \quad - \left(u'_\varepsilon, u'_\varepsilon\right) + \left(u'_\varepsilon, \psi'\right) + a\left(t; u_\varepsilon, u'_\varepsilon\right) - a\left(t; u_\varepsilon, \psi'\right) + \frac{1}{\varepsilon}\left(\beta\left(u_\varepsilon\right), \left(u_\varepsilon - \psi\right)'\right) = \left(f, u'_\varepsilon - \psi'\right),$$

$$\Longrightarrow \quad - \left\|u'_\varepsilon\right\|_H^2 + a\left(t; u_\varepsilon, u'_\varepsilon\right) + \frac{1}{\varepsilon}\left(\beta\left(u_\varepsilon\right), \left(u_\varepsilon - \psi\right)'\right) = \left(f, u'_\varepsilon - \psi'\right) - \left(u'_\varepsilon, \psi'\right) + a\left(t; u_\varepsilon, \psi'\right). \tag{3.64}$$

We put:

$$a_o\left(t; u, v\right) = \textit{ principle part of } a\left(t; u, v\right) = \sum \int_\Omega a_{ij} \frac{\partial u}{\partial x_j} \frac{\partial v}{\partial x_i}\, dx,$$

by putting:

$$r\left(t; u, v\right) = a\left(t; u, v\right) - a_o\left(t; u, v\right),$$

we rewrite (3.64) in the following form:

$$- \left\|u'_\varepsilon\right\|_H^2 + a_o\left(t; u_\varepsilon, u'_\varepsilon\right) + r\left(t; u_\varepsilon, u'_\varepsilon\right) + \frac{1}{2\varepsilon}\frac{d}{dt}\left\|\left(u_\varepsilon - \psi\right)^+\right\|_H^2 = \left(f, u'_\varepsilon - \psi'\right) - \left(u'_\varepsilon, \psi'\right) + a\left(t; u_\varepsilon, \psi'\right). \tag{3.65}$$

However, by virtue of the symmetry of $a_o\left(t; u, v\right)$, we have:

$$a_o\left(t; u_\varepsilon, u'_\varepsilon\right) = \frac{1}{2}\frac{d}{dt} a_o\left(t; u_\varepsilon, u_\varepsilon\right) - \frac{1}{2} a'_o\left(t; u_\varepsilon, u_\varepsilon\right).$$

Hence (3.65) gives:

$$\begin{aligned} & - \left\|u'_\varepsilon\right\|_H^2 + \frac{1}{2}\frac{d}{dt}\left[a_o\left(t; u_\varepsilon, u_\varepsilon\right) + \frac{1}{\varepsilon}\left\|\left(u_\varepsilon - \psi\right)^+\right\|_H^2\right] \\ & = \frac{1}{2} a'_o\left(t; u_\varepsilon, u_\varepsilon\right) - r\left(t; u_\varepsilon, u'_\varepsilon\right) + \left(f, u'_\varepsilon - \psi'\right) - \left(u'_\varepsilon, \psi'\right) + a\left(t; u_\varepsilon, \psi'\right), \end{aligned} \tag{3.66}$$

we then deduce, by integrating over (t,T) and changing the signs, that

$$\int_t^T -\|u'_\varepsilon(s)\|_H^2\,ds \;+\; \int_t^T \frac{1}{2}\frac{d}{ds}\left[a_o\left(t;u_\varepsilon,u_\varepsilon\right) \;+\; \frac{1}{\varepsilon}\left\|(u_\varepsilon-\psi)^+\right\|_H^2\right]ds$$

$$= \int_t^T \left[\frac{1}{2}a'_o\left(t;u_\varepsilon,u_\varepsilon\right) \;-\; r\left(t;u_\varepsilon,u'_\varepsilon\right) \;+\; \left(f,u'_\varepsilon-\psi'\right) \;-\; \left(u'_\varepsilon,\psi'\right) \;+\; a\left(t;u_\varepsilon,\psi'\right)\right]ds, \tag{3.67}$$

which implies

$$\int_t^T -\|u'_\varepsilon(s)\|_H^2\,ds \;+\; \frac{1}{2}\,a_o\left(t;u_\varepsilon(t),u_\varepsilon(t)\right) \;+\; \frac{1}{2\,\varepsilon}\left\|(u_\varepsilon-\psi)^+(t)\right\|_H^2$$

$$= \frac{1}{2}\,a_o\left(t;\bar u,\bar u\right) \;-\; \int_t^T \left[\frac{1}{2}a'_o\left(t;u_\varepsilon,u_\varepsilon\right) \;-\; r\left(t;u_\varepsilon,u'_\varepsilon\right) \;+\; \left(f,u'_\varepsilon-\psi'\right) \;-\; \left(u'_\varepsilon,\psi'\right) \;+\; a\left(t;u_\varepsilon,\psi'\right)\right]ds. \tag{3.68}$$

We note that

$$\left|r\left(t;u_\varepsilon,u'_\varepsilon\right)\right| \le C\,\|u_\varepsilon\|_V\,\|u'_\varepsilon\|_H\,,$$

so that, using Cauchy-Schwarz inequality and the V-ellipticity of $a(t;\cdot,\cdot)$, we deduce from (3.68) that

$$\|u'_\varepsilon\|_H^2 \le C_1 + C_2\,\|u'_\varepsilon\|_H\,. \tag{3.69}$$

Without loss of generality we assume $C_1 > 1$, then (3.69) implies (3.63).

❷ **Weak convergence:**

It results from (3.59) and (3.63) that we can extract a subsequence, also denoted by u_ε such that when $\varepsilon \to 0$:

$$\begin{cases} u_\varepsilon \rightharpoonup u \quad in \;\; L^2(0,T;V), \\ u_\varepsilon \overset{*}{\rightharpoonup} u \quad in \;\; L^\infty(0,T;V), \\ \dfrac{\partial u_\varepsilon}{\partial t} \rightharpoonup \dfrac{\partial u}{\partial t} \quad in \;\; L^2(0,T;H), \\ (u_\varepsilon - \psi)^+ \to 0 \quad in \;\; L^2(0,T;H). \end{cases}$$

Since the injection from $V \to H$ is compact (theorem 1.11), we thus have:

$$u_\varepsilon \to u \quad in \;\; L^2(0,T;H),$$

and since $(u_\varepsilon - \psi)^+ \to 0 \quad in \;\; L^2(0,T;H)$ we have:

$$(u - \psi)^+ = 0,$$

and therefore

$$u \in \tilde{\mathcal{K}}.$$

If $v \in K$, we replace v in (3.34) by $v - u_\varepsilon$, we have:

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, v - u_\varepsilon\right) + a(t; u_\varepsilon, v - u_\varepsilon) - (f, v - u_\varepsilon) = \frac{1}{\varepsilon}\left((v-\psi)^+ - (u_\varepsilon - \psi)^+, v - u_\varepsilon\right) \geq 0,$$

from which we deduce by integrating over (s,t):

$$\int_s^t \left[-\left(\frac{\partial u_\varepsilon}{\partial t}, v - u_\varepsilon\right) + a(\sigma; u_\varepsilon, v) - (f, v - u_\varepsilon)\right] d\sigma \geq \liminf \int_s^t a(\sigma; u_\varepsilon, u_\varepsilon)\, d\sigma,$$

so that we then deduce (since $u_\varepsilon \to u$)

$$\int_s^t \left[-\left(\frac{\partial u}{\partial t}, v - u\right) + a(\sigma; u, v) - (f, v - u)\right] d\sigma \geq \liminf \int_s^t a(\sigma; u_\varepsilon, u_\varepsilon)\, d\sigma$$

$$\geq \int_s^t a(\sigma; u, u)\, d\sigma,$$

hence for any values of s and t, we have

$$\int_s^t \left[-\left(\frac{\partial u}{\partial t}, v-u\right) + a(\sigma; u, v-u) - (f, v-u)\right] d\sigma \geq 0,$$

which implies:

$$-\left(\frac{\partial u}{\partial t}, v-u\right) + a(t; u, v-u) - (f, v-u) \geq 0 \qquad a.e. \ \forall v \in K.$$

Then the problem (3.13) admits a unique solution.

3.8 Estimation of the "penalization error"

We shall now prove the following results:

Theorem 3.7 (See [3]) *The assumptions are those of Theorem 3.4. Suppose also that*

$$A(t)\psi \in L^2(Q). \tag{3.70}$$

Then if u *(resp.* u_ε*) denotes the solution of the V.I obtain in Theorem 3.4 (resp. of the penalized equation) we have:*

$$\|u-u_\varepsilon\|_{L^2(0,T;V)} + \|u-u_\varepsilon\|_{L^\infty(0,T;H)} \leq C\sqrt{\varepsilon}. \tag{3.71}$$

Proof. We take the inner product of the both sides of (3.34) with $(u_\varepsilon-\psi)^+$, this gives:

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, (u_\varepsilon-\psi)^+\right) + a\left(t; u_\varepsilon, (u_\varepsilon-\psi)^+\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), (u_\varepsilon-\psi)^+\right) = \left(f, (u_\varepsilon-\psi)^+\right),$$

then:

$$-\left(\frac{\partial}{\partial t}(u_\varepsilon + \psi - \psi), (u_\varepsilon-\psi)^+\right) + a\left(t; u_\varepsilon + \psi - \psi, (u_\varepsilon-\psi)^+\right) + \frac{1}{\varepsilon}((\beta(u_\varepsilon), (u_\varepsilon-\psi)^+)$$
$$= \left(f, (u_\varepsilon-\psi)^+\right),$$

which implies

$$-\left(\frac{\partial}{\partial t}(u_\varepsilon - \psi)^+, (u_\varepsilon - \psi)^+\right) + a\left(t; (u_\varepsilon - \psi)^+, (u_\varepsilon - \psi)^+\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), (u_\varepsilon - \psi)^+\right)$$
$$= \left(f, (u_\varepsilon - \psi)^+\right) + \left(\frac{\partial\psi}{\partial t}, (u_\varepsilon - \psi)^+\right) - a\left(t; \psi, (u_\varepsilon - \psi)^+\right),$$

so

$$-\left(\frac{\partial}{\partial t}(u_\varepsilon - \psi)^+, (u_\varepsilon - \psi)^+\right) + a\left(t; (u_\varepsilon - \psi)^+, (u_\varepsilon - \psi)^+\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), (u_\varepsilon - \psi)^+\right)$$
$$= \left(f + \frac{\partial\psi}{\partial t} - A\psi, (u_\varepsilon - \psi)^+\right),$$

from which we infer

$$-\frac{1}{2}\frac{d}{dt}\left\|(u_\varepsilon - \psi)^+\right\|_H^2 + a\left(t; (u_\varepsilon - \psi)^+, (u_\varepsilon - \psi)^+\right) + \frac{1}{\varepsilon}\left\|(u_\varepsilon - \psi)^+\right\|_H^2$$
$$= \left(f + \frac{\partial\psi}{\partial t} - A\psi, (u_\varepsilon - \psi)^+\right),$$

by integrating over (t, T), we obtain:

$$-\frac{1}{2}\left\|(u_\varepsilon - \psi)^+(T)\right\|_H^2 + \frac{1}{2}\left\|(u_\varepsilon - \psi)^+(t)\right\|_H^2 + \int_t^T a\left(s; (u_\varepsilon - \psi)^+, (u_\varepsilon - \psi)^+\right) ds$$
$$+ \frac{1}{\varepsilon}\int_t^T \left\|(u_\varepsilon - \psi)^+\right\|_H^2 ds = \int_t^T \left(f + \frac{\partial\psi}{\partial s} - A\psi, (u_\varepsilon - \psi)^+\right) ds,$$

since $(u_\varepsilon - \psi)^+ (T) = 0$, then

$$\frac{1}{2} \left\| (u_\varepsilon - \psi)^+ (t) \right\|_H^2 + \alpha \int_t^T \left\| (u_\varepsilon - \psi)^+ (t) \right\|_V^2 ds + \frac{1}{\varepsilon} \int_t^T \left\| (u_\varepsilon - \psi)^+ \right\|_H^2 ds$$

$$\leq \int_t^T \left\| f + \frac{\partial \psi}{\partial s} - A\psi \right\| \left\| (u_\varepsilon - \psi)^+ \right\|_H ds,$$

we then deduce that

$$\left\| (u_\varepsilon - \psi)^+ (t) \right\|_{L^2(0,T,V)} \leq C\sqrt{\varepsilon},$$

(3.72)

$$\left\| (u_\varepsilon - \psi)^+ (t) \right\|_{L^\infty(0,T,H)} \leq C\sqrt{\varepsilon}.$$

On the other hand since

$$\begin{aligned} u - u_\varepsilon &= u - u_\varepsilon + \psi - \psi \\ &= u - \psi - (u_\varepsilon - \psi) \\ &= u - \psi - (u_\varepsilon - \psi)^+ + (u_\varepsilon - \psi)^- , \end{aligned}$$

putting $r_\varepsilon = u - \psi + (u_\varepsilon - \psi)^-$ we get:

$$u - u_\varepsilon = r_\varepsilon - (u_\varepsilon - \psi)^+ . \qquad \textbf{(3.73)}$$

It follows from (3.72) that, in order to prove (3.71), it is sufficient to show that

$$\| r_\varepsilon \|_{L^2(0,T,V)} \leq C\sqrt{\varepsilon},$$

(3.74)

$$\| r_\varepsilon \|_{L^\infty(0,T,H)} \leq C\sqrt{\varepsilon}.$$

In (3.13) we choose to define v by $v = \psi - (u_\varepsilon - \psi)^- \leq \psi$, and $v = r_\varepsilon$ in (3.34) then:

$$-\left(\frac{\partial u}{\partial t}, \psi - (u_\varepsilon - \psi)^- - u \right) + a\left(t; u, \psi - (u_\varepsilon - \psi)^- - u\right) \geq \left(f, \psi - (u_\varepsilon - \psi)^- - u\right),$$

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, r_\varepsilon \right) + a\left(t; u_\varepsilon, r_\varepsilon\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), r_\varepsilon\right) = \left(f, r_\varepsilon\right),$$

which implies

$$-\left(\frac{\partial u}{\partial t}, -r_\varepsilon\right) + a\left(t; u, -r_\varepsilon\right) \geq \left(f, -r_\varepsilon\right), \tag{3.75}$$

$$-\left(\frac{\partial u_\varepsilon}{\partial t}, r_\varepsilon\right) + a\left(t; u_\varepsilon, r_\varepsilon\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), r_\varepsilon\right) = \left(f, r_\varepsilon\right), \tag{3.76}$$

by addition, we get

$$\left(\frac{\partial}{\partial t}\left(u - u_\varepsilon\right), r_\varepsilon\right) + a\left(t; u_\varepsilon - u, r_\varepsilon\right) + \frac{1}{\varepsilon}\left(\beta(u_\varepsilon), r_\varepsilon\right) \geq 0. \tag{3.77}$$

But

$$\begin{aligned}\left(\beta(u_\varepsilon), r_\varepsilon\right) &= \left(\left(u_\varepsilon - \psi\right)^+, r_\varepsilon\right) \\ &= \left(\left(u_\varepsilon - \psi\right)^+, u - \psi + \left(u_\varepsilon - \psi\right)^-\right) \\ &= \left(\left(u_\varepsilon - \psi\right)^+, u - \psi\right) \leq 0,\end{aligned}$$

so that (3.77) gives

$$-\left(\frac{\partial}{\partial t}\left(u - u_\varepsilon\right), r_\varepsilon\right) + a\left(t; u - u_\varepsilon, r_\varepsilon\right) \leq 0, \tag{3.78}$$

and hence using (3.73):

$$-\left(\frac{\partial r_\varepsilon}{\partial t}, r_\varepsilon\right) + a\left(t; r_\varepsilon, r_\varepsilon\right) \leq -\left(\frac{\partial}{\partial t}\left(u_\varepsilon - \psi\right)^+, r_\varepsilon\right) + a\left(t; \left(u_\varepsilon - \psi\right)^+, r_\varepsilon\right). \tag{3.79}$$

We note that $r_\varepsilon(T) = \bar{u} - \psi(T) + \left(\bar{u} - \psi(T)\right)^- = 0$ so that (3.79) gives

$$\begin{aligned}&\frac{1}{2}\left\|r_\varepsilon(t)\right\|_H^2 + \alpha \int_t^T \left\|r_\varepsilon(s)\right\|_V^2 ds \leq \left(\left(u_\varepsilon - \psi\right)^+(t), r_\varepsilon(t)\right) \\ &+ \int_t^T \left(\left(u_\varepsilon - \psi\right)^+, \frac{\partial r_\varepsilon}{\partial s}\right) ds + \int_t^T a\left(s; \left(u_\varepsilon - \psi\right)^+, r_\varepsilon\right) ds .\end{aligned} \tag{3.80}$$

But

$$\left| \int_t^T \left((u_\varepsilon - \psi)^+ , \frac{\partial r_\varepsilon}{\partial s} \right) ds \right| = \left| \int_t^T \left((u_\varepsilon - \psi)^+ , \frac{\partial}{\partial s}(u - \psi) \right) ds \right|$$

$$\leq C\varepsilon \left\| \frac{\partial}{\partial t}(u - \psi) \right\|_{L^2(0,T;H)} \leq C\varepsilon,$$

and hence (3.80) gives

$$\frac{1}{2} \|r_\varepsilon(t)\|_H^2 + \alpha \int_t^T \|r_\varepsilon(s)\|_V^2 \, ds \leq C\sqrt{\varepsilon} \left[\|r_\varepsilon(t)\|_H + \left(\int_t^T \|r_\varepsilon(s)\|_V \right)^{\frac{1}{2}} ds \right] + C\varepsilon. \tag{3.81}$$

So that (3.74) then follows. ■

3.9 WEAK VARIATIONAL INEQUALITY

The existence of a strong solution requires important assumptions, which are not always verified. We will weaken the problem (3.13).
If u is a solution of (3.34) and $v \in \tilde{\mathcal{K}}$ (and not only $v \in K$). let us consider the expression

$$\mathcal{X} = \int_0^T \left[- \left(\frac{\partial v}{\partial t}, v - u \right) + a\,(t; u, v - u) - (f, v - u) \right] dt,$$

we have

$$\mathcal{X} = \int_0^T \left[- \left(\frac{\partial}{\partial t}\,(v + u - u)\,, v - u \right) + a\,(t; u, v - u) - (f, v - u) \right] dt,$$

so

$$\mathcal{X} = \int_0^T \left[-\left(\frac{\partial u}{\partial t}, v - u \right) + a\left(t; u, v - u\right) - \left(f, v - u\right) \right] dt + \int_0^T -\left(\frac{\partial}{\partial t}\left(v - u\right), v - u \right) dt, \tag{3.82}$$

the first integral on the right-hand side of (3.82) is ≥ 0, therefore

$$\mathcal{X} \geq \int_0^T -\left(\frac{\partial}{\partial t}\left(v - u\right), v - u \right),$$

which implies

$$\mathcal{X} \geq \int_0^T -\frac{1}{2}\frac{d}{dt}\left\|v - u\right\|^2_{L^2(Q)},$$

and thus

$$\mathcal{X} + \frac{1}{2}\left\|v(T) - u(T)\right\|^2_{L^2(Q)} \geq \left\|v(0) - u(0)\right\|^2_{L^2(Q)} \geq 0.$$

Replacing $u(T)$ by $\bar{u}$, we are therefore led to the above definition of the weak solution. We say that u is a **'weak solution'** of the evolutionary V.I if

$$\begin{cases} u \in \mathsf{L}^2\left(0, T; V\right), \\ u \leq \psi \quad a.e \;\; on \;\; Q, \\ \int_0^T \left[-\left(\frac{\partial v}{\partial t}, v - u \right) + a\left(t; u, v - u\right) - \left(f, v - u\right) \right] dt + \frac{1}{2}\left\|v(T) - \bar{u}\right\|^2_{L^2(Q)} \geq 0 \quad \forall v \in \mathcal{K}. \end{cases} \tag{3.83}$$

Remark 3.8 *This new inequality no longer involves* $\frac{\partial u}{\partial t}$.

3.10 Existence of the weak solution

Theorem 3.9 (See [3]) *The problem* (3.83) *admits a solution.*

Proof. In section 3.5, we used elliptic regularization to approximate the solution u_ε of the penalized problem. We can use elliptic regularization with regard to the V.I.

We define $\tilde{\mathcal{K}}_0 \subset \tilde{\mathcal{K}}$ by

$$\tilde{\mathcal{K}}_0 = \left\{ v \ \mid \ v \in L^2(0,T;V), \quad \frac{\partial v}{\partial t} \in L^2(0,T;H), \quad v(x,t) \leq \psi(x,t) \ \ in \ \ \mathsf{Q} \right\}, \tag{3.84}$$

and we adopt the assumption that

$$\tilde{\mathcal{K}}_0 \neq \emptyset.$$

For $\gamma > 0$ we seek u_γ a solution of:

$$\begin{cases} u_\gamma \in \tilde{\mathcal{K}}_0, \\ \int\limits_0^T \left[\gamma\left(u'_\gamma, v' - u'_\gamma\right) \ - \ \left(u'_\gamma, v - u_\gamma\right) \ + \ a\left(t; u_\gamma, v - u_\gamma\right) \ - \ \left(f, v - u_\gamma\right)\right] dt \geq 0 \qquad \forall v \in \tilde{\mathcal{K}}_0, \end{cases} \tag{3.85}$$

where, in (3.85):

$$u'_\gamma = \frac{\partial u_\gamma}{\partial t}, \ \ v' = \frac{\partial v}{\partial t} \ \ .$$

If we put

$$b(v,u) = \int\limits_0^T \left[\gamma\left(u'_\gamma, v' - u'_\gamma\right) \ - \ \left(u'_\gamma, v - u_\gamma\right) \ + \ a\left(t; u_\gamma, v - u_\gamma\right) \ - \ \left(f, v - u_\gamma\right)\right] dt,$$

a bilinear form, we see from (3.55) that $b(\cdot\,,\cdot)$ is coercive on $\tilde{\mathcal{K}}_0$, that we can apply the results of section 3.5.

Furthermore, when $\gamma \to 0$, we have (see section 3.6):

$$\left\|u_{\varepsilon\gamma}\right\|_{L^2(0,T;V)} + \sqrt{\gamma}\left\|u'_{\varepsilon\gamma}\right\|_{L^2(0,T;H)} \leq C. \tag{3.86}$$

Without any further assumptions on the coefficients and data, we deduce from (3.86) that we can extract a sequence, also denoted by u_γ, such that

$$\begin{cases} u_\gamma \rightharpoonup w \quad \ in \ \ L^2(0,T;V), \\ \dfrac{\partial u_\gamma}{\partial t} \rightharpoonup \dfrac{\partial w}{\partial t} \quad \ in \ \ L^2(0,T;H). \end{cases}$$

Since the injection from $V \to H$ is compact, we thus have:

$$u_\gamma \to w \qquad in \;\; L^2(0,T;H),$$

and we can immediately proceed to the limit in (3.85); we therefore obtain:

$$\int_0^T \left[- \;(u', v-u) \;+\; a\,(t; u, v-u) \;-\; (f, v-u)\right] dt \geq 0 \qquad \forall v \in \tilde{\mathcal{K}}, \tag{3.87}$$

and since (3.87) implies (3.83) then w is a weak solution of the evolutionary V.I.

■

Conclusion

The variational inequalities studied in this work are evolutionary since they involve the time derivative of the solution.

In this research we reached to the main purpose and conclude that this kind of inequalities has a unique solution and this may be proved by using several methods similar to penalty, elliptic regularization methods we showed during the development of this thesis.

Despite this analysis, the subject of variational inequalities remains open to wide research and perspective such as:

✍ Evolutionary variational inequalities of the second kind.

✍ Hyperbolic variational inequalities.

✍ Quasi variational inequalities.

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