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الإهداء

إلى أمي وأبي اللذين سعيا من أجل أن أكون مثابرا وجادا في علمي وعملي،
وحرصا على أن أنال الدرجات العلى في الدنيا والآخرة...

إلى كل من أضاء بعلمه عقل غيره
أو هدى بالجواب الصحيح حيرة سائليه
فأظهر بسماحته تواضع العلماء
وبرحابته سماحة العارفين

إلى محبي العلم والتعلم

أهدي هذا العمل المتواضع.

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Introduction

The theory of variational inequalities plays an important role in the study of both the qualitative and numerical analysis of nonlinear boundary value problems arising in mechanics, physics, and engineering science.
The study of variational inequalities goes back to the work of Lions and Stampacchia [7, 8] in the sixties.

This subject has developed in several directions using new and powerful methods that have led to the solutions of basic and fundamental problems thought to be inaccessible previously. Some of these developments have made mutually enriching contacts with other areas of mathematical and engineering sciences including elasticity, transportation and economics equilibrium, nonlinear programming, operations research.

In 1988 M. Noor is introduced and studied a new class of variational inequalities, which is called general nonlinear variational inequality GVI, and he gave an iterative algorithm for solving this class of variational inequalities, see [13].
GVI means general variational inequalities. It has been shown that general variational inequalities provides a unified, simple, and natural framework to study a wide class of problems, which this study has utilised to conduct our issues.

This memory consists of three chapters:
First chapter: We inserted definitions of the notions and terms that has been used in

the chapter two and three.
Second chapter: We conducted a study the existence and the uniqueness of the solution of variational inequality of the second kind.
Third chapter: we did a study the existence and the uniqueness of GVI. Then, we move to build an Algorithm that depends on the projection theorem. Finally, we proposed a model for GVI.

Chapter 1

Preliminaries

In this chapter we discuss some mathematical concepts that we should know them for use in our theme.

1.1 Reminders

Definition 1 *Let X be a linear space. A subset $K \subset X$ is said to be convex if it has the property*

$$x, y \in K \Longrightarrow \lambda x + (1-\lambda)y \in K \quad \forall \lambda \in [0,1]$$

Definition 2 *Let K be any set in Hilbert space H. The set K is said to be g-convex, if there exists a function $g : H \to H$ such that*

$$g(u) + t(g(v) - g(u)) \in K, \quad \forall u, v \in K, t \in [0,1].$$

Remark 3 *Note that every convex set is g-convex, but the converse is not true.*

Definition 4 *The function $\varphi : X \to (-\infty, \infty]$ is convexe if:*

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y) \tag{1.1}$$

for all $x, y \in X$ and $\lambda \in [0,1]$.

The function φ is strictly convex if the inequality is strict for $x \neq y$ and $\lambda \in (0,1)$.

Remark 5 *We note that if $\varphi, \phi : X \to (-\infty, \infty]$ are convex and $\alpha \geq 0$, then the functions $\varphi + \phi$, $\alpha\varphi$ and sup$\{\varphi, \phi\}$ are also convex.*

Definition 6 *The function $F : K \to H$ is said to be g-convex, if*

$$F(g(u) + t(g(v) - g(u))) \leq (1-t)F(g(u)) + tF(g(v)), \quad \forall u, v \in K.$$

Clearly every convex function is g-convex, but the converse is not true.

Lemma 1 *Let $F : K \to H$ be a differentiable g-convex function. Then $u \in K$ is the minimum of a g-convex function F on K if and only if $u \in K$ satisfies the inequality*

$$(F'(g(v)), g(v) - g(u)), \quad \forall g(v) \in K. \tag{1.2}$$

where F' is the Frechet differential of F at $g(u)$.

Proof. Let $u \in K$ be a minimum of the g-convex function F on K. Then

$$F(g(u)) \leq F(g(v)), \qquad \text{for all} \quad g(v) \in K. \tag{1.3}$$

Since K is a g-convex set, then for all $u, v \in K,\ t \in [0,1],\ g(v_t) = g(u) + t(g(v) - g(u)) \in K$. Setting $g(v) = g(v_t)$ in (1.3), we have

$$\begin{aligned} F(g(u)) &\leq F(g(v) + t(g(v) - g(u))) \\ &\leq F(g(u)) + t(F(g(v) - g(u))). \end{aligned}$$

Dividing the above inequality by t and taking $t \to 0$, we have

$$\langle F'(g(u)), g(v) - g(u) \rangle \geq 0,$$

which is the required result (1.2).
Conversely, let $u \in K, g(u \in K)$ satisfy the inequality (1.2).Since F is a g-convex function, for all $u, v \in K,\ t \in [0,1], g(u) + t(g(v) - g(u)) \in K$, and

$$F(g(u) + t(g(v) - g(u))) \leq (1-t)F(g(u)) + tF(g(v)),$$

which implies that

$$F(g(v)) - Fg((u)) \geq \frac{F(g(u) + t(g(v) - g(u))) - F(g(u))}{t}.$$

Letting $t \to 0$, we have

$$F(g(v)) - Fg((u)) \geq \langle F'(g(u)), g(v) - g(u) \rangle \geq 0, \quad \text{using (1.2)} ,$$

which implies that

$$F(g(u)) \leq F(g(v)), \quad \text{for all } g(v) \in K$$

showing that $u \in K$ is the minimum of F on K in H. ■

Remark 7 *Lemma 1 implies that the g-convex programming problem can be studied via the general variational inequality (3.1) with* $Au = F'(g(u))$.

Definition 8 *(Lower and upper semi-continuity) The function* $\varphi : X \to (-\infty, \infty]$ *is said to be lower semi-countinuous (l.s.c) at* $u \in X$ *if*

$$\varphi(u) \leq \lim_{n \to +\infty} \inf \varphi(u_n)$$

for each sequence $\{u_n\} \subset X$ *converging to* u *in* X.
It is upper semi-continuous (u.s.c) at $u \in X$ *if*

$$\varphi(u) \geq \lim_{n \to +\infty} \sup \varphi(u_n)$$

The function φ *is l.s.c (u.s.c) if it is l.s.c (u.s.c) at every point* $u \in X$.

Definition 9 *Let* H *be a reaal Hilbert space,* $\forall u, v \in H$, *the operator* $A : H \to H$ *is said to be:*

1. *g-monotone, if*

$$(Au - Av, g(u) - g(v)) \geq 0.$$

2. *g-pseudomonotone, if*

$$(Au, g(v) - g(u)) \geq 0 \Longrightarrow (Av, g(v) - g(u)) \geq 0.$$

3. *g-strongly monotone, if there exist a constant* $\alpha > 0$ *such that*

$$(Au - Av, g(u) - g(v)) \geq \alpha \|g(u) - g(v)\|^2.$$

4. *g-cocoercive or g-inverse strongly monotone, if there exists* $\beta > 0$ *such that*

$$(Au - Av, g(u) - g(v)) \geq \beta \|Au - Av\|^2, \quad \forall u, v \in H.$$

5. *L-Lipschitz continuous, if there exists a positive real number L such that*

$$\|Au - Av\| \leq L\|u - v\|.$$

If $0 < L < 1$*, then it is a contraction with constant L.*

6. *hemicontinuous, if the mapping* $t \in [0, 1] \Longrightarrow (A(u + t(v - u)), v - u)$ *is continuous.*

Remark 10 *We remark that if* $z = u$*, then g-partially relaxed strongly monotonicity is equivalent to monotonicity. It is well known that cocoercivity implies partially relaxed strongly monotonicity, but the converse is not true.*
This shows that the partially relaxed strongly monotonicity is a weaker condition than cocoercivity.

Definition 11 *Let* K *be a nonempty closed convex subset of a real Hilbert space* H*.*

1. *A mapping* $S : K \to K$ *is called nonexpansive if*

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in K.$$

2. *A mapping $S : K \to K$ is called k-strict pseudo contractive mapping, if there exists a constant $0 \leq k < 1$ such that*

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + k\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in K.$$

Observe that if S is a k-strictly pseudocontractive mapping, then $T = I - S$ is an α-inverse strongly monotone operator with $\alpha = \dfrac{1-k}{2}$.

Theorem 12 (Banach's Fixed Point Theorem) *[4] Let $(V, \|.\|_V)$ be a Banach space, and let K be a nonempty closed subset of V. Suppose that the operator $A : K \longrightarrow K$ is a contraction, i.e. there exists a constant $c \in [0, 1)$ such that*

$$\|A(u) - A(v)\|_V \leq c\|u - v\|_V.$$

Then A has a unique fixed point.

Theorem 13 (Brouwer's Fixed Point Theorem) *Let K be a nonempty, convex, compact subset of a finite dimensional normed linear space V. If the operator $T : K \to K$ is continuous, then T has a fixed point, i.e. there exists $u \in K$ such that $T(u) = u$.*

Theorem 14 (Schauder's Fixed Point Theorem) *[4] Let V be a Banach space, and let $K \subset V$ be a nonempty, convex, compact subset. If the operator $T : K \to K$ is continuous, then T has a fixed point.*

1.2 The Riesz Representation Theorem

Theorem 15 *[5] Let H be a Hilbert space and let $l \in H^{'}$ (dual of H). Then there is a unique $u \in H$ such that*

$$l(v) = (u, v), \quad \forall v \in H$$

Moreover

$$\|l\|_{H'} = \|u\|_H$$

Theorem 16 *Let H be a real Hilbert space with its inner product noted $(.)$.*

Let K be a nonempty closed convex subset of H.

If $a(u, v)$ is a bilinear form which is

continuous on $H \times H$: $\exists c > 0 \ \forall u, v \in H, \ \mid a(u, v) \mid \leq c\|u\|\|v\|$,

coercivity on H : $\exists \alpha > 0 \ \forall u \in H \ \ a(u, u) \geq \alpha\|u\|^2$,

If $L(.)$ is a linear continuous form on H.

Under these conditions, then there exists a unique u of K such that

$$\forall v \in K \quad a(u, v - u) \geq L(v - u)$$

In addition, if a is symetric, then u is the unique element of K that minimizes the functional $J : H \longrightarrow \mathbb{R}$ defined by $J(v) = \frac{1}{2}a(v, v) - L(v)$ for all v of K:

$$\exists! u \in K \quad J(v) = \min_{v \in K} J(u)$$

Proof. See [1] ■

1.3 The Projection Theorem

Theorem 17 *Let $K \subset H$ be a nonempty closed convex. Then for each $u \in H$, there is a unique element $u_0 = P_K u \in K$ such that*

$$\|u - u_0\| \ = \ \min_{v \in K}\|u - v\|$$

Proof. See [5] ■

The operator $P_K : H \to K$ is called the projection operator onto K. The element $u_0 = P_K u$ is called the projection of u on K, and is characterized by the property

$$(u - u_0, u_0 - v) \geq 0, \ \forall v \in K. \tag{1.4}$$

We know that P_K is a nonexpansive, monotone and satisfies

$$(u-v, P_K u - P_K v) \geq \|P_K u - P_K v\|^2, \ \forall u, v \in H.$$

$$\|P_K u - u_0\|^2 \leq \|u-u_0\|^2 - \|u - P_K u\|^2.$$

In the context of the variational inequality problem, this implies that

$$u \in VI(K, A) \ (3.5) \iff u = P_K(u - \lambda A u), \ \forall \lambda > 0.$$

1.4 Hartman-Stampacchia's Theorem

Theorem 18 *Let K be a nonempty compact convex subset of a finite dimensional space V. If we suppose that $A : K \to V$ is a continuous mapping and $\varphi : K \to (-\infty, +\infty]$ is a proper l.s.c convex function, then there exists at least one $u \in K$ such that*

$$(Au, v-u) + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K.$$

Proof. See [4] ■

Lemma 2 *[5] Suppose that $j : K \to \overline{\mathbb{R}}$ is proper convex and l.s.c and $a(.,.) H \times H \to \mathbb{R}$ is continuous. in addition, that the form a is positive. Then, the variational inequality (2.3) is equivalent to*

$$\begin{cases} Find\ u \in K \\ a(v, v-u) + j(v) + j(u) \geq (f, v-u)\ \forall v \in K \end{cases} \tag{1.5}$$

Theorem 19 (Eberlein–Smulyan Theorem) *Let V be a reflexive Banach space. Then any bounded sequence in V contains a weakly convergent subsequence.*

Proof. See [4] ■

Corollary 20 *Any nonempty, bounded, and weakly closed subset in a reflexive Banach space is weakly compact.*

Chapter 2

Variational Inequalities

2.1 Variational Inequalities of the first kind

Let an operator $A : H \longrightarrow H$, a subset $K \subset H$ and an element $f \in H$, we consider the problem of finding an element u such that

$$(Au, v - u) \geq (f, v - u) \ \forall v \in K. \tag{2.1}$$

An inequality of the form (2.1) is called an elliptic variational inequality of the first kind.

Theorem 21 *Let H be a Hilbert space and let $K \subset H$ be a nonempty closed convex* *subset. Assume that $A : K \longrightarrow H$ is a strongly monotone Lipschitz continuous operator. Then, for each $f \in H$ the variational inequality (2.1) has a unique solution.*

Proof. Let $f \in H$ and let $\rho > 0$ be given. We consider the operator $S_\rho : K \longrightarrow K$ defined by

$$S_\rho u = P_K(u - \rho(Au - f)) \; \forall u \in K,$$

where P_K denotes the projection operator on K. Using P_K is nonexpansive, it follows that

$$\|S_\rho u - S_\rho v\| \leq \|(u - v) - \rho(Au - Av)\| \; \forall u, v \in K.$$

Hence

$$\|S_\rho u - S_\rho v\|^2 \leq \|(u - v) - \rho(Au - Av)\|^2$$
$$\leq \|u - v\|^2 + \rho^2\|Au - Av\|^2 - 2\rho(u - v, Au - Av)$$

Using A is Lipshitz continuous with constant M and strongly monotone with constant m

$$\leq \|u - v\|^2 + \rho^2 M^2\|u - v\|^2 - 2\rho m\|u - v\|^2$$

Then, we obtain that

$$\|S_\rho u - S_\rho v\| \leq k(\rho)\|(u - v)\| \; \forall u, v \in K.$$

where $k(\rho) = (1 - 2\rho m + \rho^2 M^2)^{\frac{1}{2}}$ and m, M are the constants of strongly monotone and Lipschitz continuous, respectively. Also, with a convenient choice of ρ we may assume that $k(\rho) \in [0, 1)$. It follows now from Theorem 12 that there exists an element u such that

$$S_\rho u = P_K(u - \rho(Au - f)) = u. \tag{2.2}$$

We now combine (2.2) to see that

$$u \in K, (u, v - u) \geq (u - \rho(Au - f), v - u) \; v \in K.$$

Since $\rho > 0$ we conclude that u satisfies (2.1), which proves the existence part of the theorem.

Next, we consider two solutions u and v of (2.1). It follows that $u \in K$, $v \in K$ and, moreover,

$$(Au, v-u) \geq (f, v-u), \quad (Av, u-v) \geq (f, u-v).$$

We add these inequalities to see that

$$(Au - Av, v-u) \leq 0,$$

then we use assumption A is strongly monotone with m to obtain $u = v$, which proves the uniqueness part.

■

2.2 Variational Inequalities of the second kind

This section contains existence and uniqueness results for the solutions of variational inequalities.

Definition 22 *We consider the following variational inequality of the second kind*

$$\begin{cases} Find\ u \in K \\ a(u, v-u) + j(v) + j(u) \geq (f, v-u)\ \forall v \in K \end{cases} \tag{2.3}$$

Where $a(.,.) : H \times H \longrightarrow \mathbb{R}$ *bilinear continuous form,* $j : H \longrightarrow \overline{\mathbb{R}}$ *a function, and* $f \in H'$.

Theorem 23 *Let* K *be a nonempty closed convex subset of a Hilbert space* H.
If $j : H \longrightarrow \overline{\mathbb{R}}$ *is proper convex and lower semi continuous and* $a(.,.)H \times H \longrightarrow \mathbb{R}$ *is coercive, then there exists a unique solution* $u \in K$ *of the variational inequality (2.3).*

Proof.

- **Uniqueness:**

  Let u_1 and u_2 be two solution of (2.3) then we have

$$a(u_1, v-u_1) + j(v) - j(u_1) \geq (f, v-u_1)\ \forall v \in K \tag{2.4}$$

$$a(u_2, v - u_2) + j(v) - j(u_2) \geq (f, v - u_2) \; \forall v \in K \tag{2.5}$$

Putting u_2 for v in (2.4) and u_1 for v in (2.5) and adding we get:

$$a(u_1, u_2 - u_1) + a(u_2, u_1 - u2) \geq (f, u_2 - u_1) + (f, u_1 - u_2)$$

$$\alpha \|u_1 - u_2\|^2 \leq a(u_1 - u_2, u_1 - u_2) \leq 0$$

$\Longrightarrow u_1 = u_2$.

- **Existence:**

  We defined the auxiliary problem for u fixed in K and $\rho > 0$

$$\begin{cases} \text{Find } w \in K \\ (w, v - w) + \rho j(v) - \rho j(w) \geq -\rho(a(u, v - w) - (f, v - w)) + (u, v - w) \; \forall v \in K \end{cases} \tag{2.6}$$

The problem (2.6) has a unique solution if and only if $w = P_K(u - \rho(Au - f))$, where $a(u, v) = (Au, v)$.
For each ρ we defined the map $T_\rho : u \longmapsto w$, where w is the unique solution (2.6).
We shall that T_ρ has a fixed point.
It is enough to show that T_ρ is uniformly strict contraction mapping i.e
$\|T_\rho(u_1) - T_\rho(u_2)\| \leq C\|u_1 - u_2\| \; \forall u_1, u_2 \in H, \; c < 1$
$\|w_1 - w_2\| \leq C\|u_1 - u_2\|$ such that $w_i = T_\rho(u_1)$, $i = 1, 2$
Then:

$$(w_1, v - w_1) + \rho j(v) - \rho j(w_1) \geq -\rho a(u_1, v - w_1) + \rho(f, v - w_1) + (u_1, v - w_1) \tag{2.7}$$

$$(w_2, v - w_2) + \rho j(v) - \rho j(w_2) \geq -\rho a(u_2, v - w_2) + \rho(f, v - w_2) + (u_2, v - w_2) \tag{2.8}$$

We take $v = w_2$ and $v = w_1$ respectively in (2.7) and (2.8) we obtain

$$-\|w_1 - w_2\|^2 \geq \rho a(u_1 - u_2, w_1 - w_2) - (u_1 - u_2, w_1 - w_2)$$

$$\Longrightarrow \|w_1 - w_2\|^2 \leq -\rho a(u_1 - u_2, w_1 - w_2) + (u_1 - u_2, w_1 - w_2) \leq$$

$$\leq (-\rho A(u_1 - u_2) + (u_1 - u_2), w_1 - w_2) \leq ((-\rho A + I)(u_1 - u_2), w_1 - w_2) \leq$$

$$\leq \|-\rho A + I\|.\|u_1 - u_2\|\|w_1 - w_2\|$$

$$\Longrightarrow \|w_1 - w_2\| \leq \|-\rho A + I\|.\|u_1 - u_2\|$$

Then $\exists \rho > 0$ such that $\|I - \rho A\| < 1$

$\|(I - \rho A)v\|^2 = (v - \rho Av, v - \rho Av) = (v, v) - 2\rho(Av, v) + \rho^2(Av, Av)$

$\leq \|v\|^2 - 2\rho(Av, v) + \rho^2\|Av\|^2$

We use the coercivity $(Av, v) \geq \alpha\|v\|^2 \Longrightarrow -2\rho(Av, v) \leq -2\rho\alpha\|v\|^2$

then $\|(I - \rho A)v\|^2 \leq \|v\|^2 - 2\rho\alpha\|v\|^2 + \rho^2\|A\|^2.\|v\|^2$

$\leq (1 - 2\rho\alpha + \rho\|A\|^2)\|v\|^2$

if $\rho \in \left]0, \dfrac{2\alpha}{\|A\|^2}\right[\Longrightarrow 1 - 2\alpha\rho + \rho^2\|A\|^2 < 1$

$\Longrightarrow \|I - \rho A\| < 1.$

This proves that is T_ρ uniformly a strict contracting mapping and hence has a unique fixed point u, by Banach fixed point theorem 12.

Hence (2.3) has a unique solution.

■

Variational Inequalities with Operators

Definition 24 *Let H be a Hilbert space with its inner product $(.,.)$, and let $K \subset H$ be a nonempty closed convex subset.*

We call variational inequality of the second kind on the form:

$$\begin{cases} Find\ u \in K\ \ such\ that \\ (Au, v - u) + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K \end{cases} \tag{2.9}$$

where $A : K \to H$ an operator and $\varphi : K \to \mathbb{R}$ a function.

Theorem 25 *Let H be a Hilbert space, and let $K \subset H$ be a nonempty closed convex subset. We suppose a function $\varphi : K \to \mathbb{R}$ is l.s.c convex, and a strongly monotone Libschitz continuous operator $A : K \to H$. Then the variational inequality (2.9) has a unique solution.*

Lemma 3 *Assume that $\varphi : K \to \mathbb{R}$ l.s.c convex function, where K ba a nonempty closed convex subset of H. Then there exists an unique element u such that*

$$(u, v - u) + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K$$

Proof. See [6] ■

Definition 26 *[6] Let H be a Hilbert space, $K \subset H$ be a nonempty closed convex subset and $\varphi : K \to \mathbb{R}$ is convex l.s.c function. Then, for any $f \in H$, the solution u of the variational inequality:*

$$(u, v - u) + \varphi(v) - \varphi(u) \geq (f, v - u), \quad \forall v \in K$$

is called the proximal element of f with respect to φ and it's usually denoted $prox_{\varphi}(f) = u$.

Proof of theorem 25. Let $\rho > 0$ be a parameter to be chosen later. Since $\rho\varphi : K \to \mathbb{R}$ is again a convex lower semicontinuous function, we can define an operator $S_\rho : K \to K$ by

$$S_\rho(v) = prox_{\rho\varphi}(-\rho Av + v) \ \ \forall v \in K$$

Moreover, using $prox_{\varphi}$ is nonexpansive we find that

$$\|S_\rho u - S_\rho v\| \leq \|(u - v) - \rho(Au - Av)\| \ \ \forall u, v \in K.$$

and using the m-strongly monotone and M-Lipschitz continuous of A

$$\|S_\rho u - S_\rho v\|^2 = \|(u - v) - \rho(Au - Av)\|^2 \ \ \forall u, v \in K.$$

$$= \|u - v\|^2 - 2\rho(Au - Av, u - v) + \rho^2\|Au - Av\|^2$$

$$\leq (1 - 2\rho m + \rho^2 M^2)\|u - v\|^2 \ \ \forall u, v \in K$$

we know that $M \geq m$. it is easy to see that if $0 < \rho < \frac{2m}{M^2}$ then

$$0 \leq 1 - 2\rho m + \rho^2 M^2 < 1.$$

Therefore, with this choice of ρ, it follows that

$$\|S_\rho u - S_\rho v\| \leq k(\rho)\|(u - v)\|$$

where $k(\rho) = \sqrt{(1 - 2\rho m + \rho^2 M^2)} \in [0, 1[$. Next, we use the Banach fixed point argument to see that there exists $u \in K$ such that

$$S_\rho u = prox_{\rho\varphi}(-\rho Au + u) = u.$$

Then, by Definition of the proximal operator we obtain

$$(u, v - u) + \rho\varphi(v) - \rho\varphi(u) \geq (-\rho Au + u, v - u) \ \ \forall v \in K,$$

Since $\rho > 0$, we deduce from the above inequality that u is a solution of the variational inequality (2.9), which proves the existence part of the theorem.
To show the uniqueness, assume there are two solutions $u_1, u_2 \in K$ of the variational inequality (2.9). Then,

$$(Au_1, v - u_1) + \varphi(v) - \varphi(u_1) \geq 0 \ \ \forall v \in K,$$

$$(Au_2, v - u_2) + \varphi(v) - \varphi(u_2) \geq 0 \ \ \forall v \in K,$$

Taking $v = u_2$ in the first inequality, $v = u_1$ in the second one and adding the resulting inequalities, we find that

$$(Au_1 - Au_2, u_1 - u_2) \leq 0.$$

We combine this inequality with A is m-strongly monotone to obtain $u_1 = u_2$, which proves the uniqueness part of Theorem 25. ■

2.3 Variational inequalities in Banach space

• dim $< \infty$

Theorem 27 (Hartman–Stampacchia's Theorem) *Let $(V, \|.\|)$ be a Banach space, and let K be a nonempty compact convex subset of a finite dimensional space V. If we suppose that $A : K \to V$ is a continuous mapping and $\varphi : K \to (-\infty, +\infty]$ is a proper l.s.c convex function, then there exists at least one $u \in K$ such that*

$$(Au, v - u) + \varphi(v) - \varphi(u) \geq 0, \quad \forall v \in K. \tag{2.10}$$

Proof. We consider the proper l.s.c.convex function $\varphi : K \to (-\infty, +\infty]$ defined by

$$\varphi(v) = \begin{cases} j(v) \text{ if } v \in K, \\ +\infty \text{ otherwise.} \end{cases} \tag{2.11}$$

Let the operator $T : K \to K$ be defined by $T(w) = Prox_{\varphi}(w - Aw + f)$, $\forall w \in K$, where $Prox_{\varphi}$ is the proximity operator with respect to φ. We first remark that, from the definition (2.11) of φ and the definition of the proximity operator, it follows that $T(w) \in K$, $\forall w \in K$. Then, by Definition 26, it follows that the inequality (2.10) is equivalent to $u = T(u)$.

The operators A and $Prox_{\varphi}$ are continuous, hence T is itself on the compact convex set K. Hence, from Schauder fixed point Theorem 14 or from Brouwer Theorem 13, it follows that there exists at least one element $u \in K$ such that $u = T(u)$ which completes the proof. ■

• Case K bounded

Theorem 28 *Suppose that $j : K \to \overline{\mathbb{R}}$ is proper convex and l.s.c and $a(.,.)H \times H \to \mathbb{R}$ is continuous. In addition, we assume that the form a is positive and that the closed convex set K is bounded. Then, the set of all solutions of the variational inequality (2.3) is a nonempty, convex, and weakly compact subset of K.*

Proof. From Lemma 2, the set of all solutions of (2.3) is $\chi = \cap_{v \in K} S(v)$ where

$$S(v) = \{u \in K; \quad a(v, v-u) + j(v) - j(u) \geq (f, v-u)\}.$$

The set χ being closed convex, it is weakly closed (i.e., it contains the limits of all weakly convergent sequences $\{v_n\}_n \subset \chi$) in V. On the other hand, as the set K is bounded and weakly closed in V , by Corollary 20, it follows that it is weakly compact. Therefore, we will prove that $\chi \neq \emptyset$ by proving that the family $\{S(v)\}_{v \in K}$ has the finite intersection property, i.e. any finite subcollection $K_Q \subset K$ has nonempty intersection. Let $\{v_1, \ldots, v_q\}$ be a finite part of K and $K_Q = K \cap Q$ where Q is the finite dimensional space spanned by the family $\{v_1, \ldots, v_q\}$. Then, from Hartman-Stampacchia Theorem 27, it follows that there exists a solution $u \in K_Q \subset K$ of the inequality

$$a(u, v-u) + j(v) - j(u) \geq (f, v-u) \quad \forall v \in K_Q,$$

i.e., there exists $u \in S(v)$, $\forall v \in K_Q$, hence $\cap_{v \in K_Q} S(v) \neq \emptyset$. ∎

Proposition 29 *Under the hypotheses of Theorem 28, if the set K is compact, then the set of all solutions of the variational inequality (2.3) forms a nonempty compact convex subset of K.*

Proof. The set χ of all solutions of (2.3) is closed and convex.
In order to prove that it is nonempty, let $T : K \to K$ be the continuous operator defined by $T(w) = Prox_j(w - Aw + f)$, $\forall w \in K$, where the functional φ is defined by (2.11), and $A \in \mathcal{L}(V, V)$ is the operator associated with the bilinear continuous form $a(.,)$, i.e.

$$(Au, v) = a(u, v) \quad \forall u, v \in V \tag{2.12}$$

Hence, by Schauder fixed point Theorem 14, it follows that there exists $u \in K$ such that $u = T(u)$, that is

$$((u - Au + f) - u, v-u) \leq j(v) - j(u) \quad \forall v \in K.$$

and thus, the set of the solutions of (2.3) is nonempty. Moreover, as χ is closed in the compact set K, it follows that χ is compact.
Next, if we refer to the variational inequality of the first kind (2.1), then we have to consider the continuous operator $T : K \to K$ defined by $T(v) = P_K(v - Av + f)$ where P_K :

$V \to K$ is the projection operator on the nonempty closed convex subset K. Therefore, by taking into account the characterization of the projection given by 1.4, fromu $u = T(u)$ it follows

$$(u - (u - Au + f), v - u) \geq 0 \quad \forall v \in K.$$

In fact, it is enough to remark that the projection operator is a particular case of the proximity operator, namely $P_K = Prox_{I_K}$, I_K being the indicator function of K. ■

- **Case K unbounded**

However, the most interesting cases involve unbounded sets K. Existence results are obtained by requiring that the form a is coercive on V (or, V-elliptic), that is, there exists a positive constant α such that:

$$a(u, u) \geq \alpha \|u\|^2 \quad \forall u \in V.$$

Chapter 3

Generalized variational inequalities

3.1 Existence and uniqueness

Let H be a real Hilbert space with its dual H', whose inner product and norm are denoted by $(.,.)$ and $\|.\|$ respectively. Let K be closed convex set in H and $A, g\ H \to H$

be a nonlinear operators. We now consider the problem

$$\begin{cases} \text{Find } \ u \in H, g(u) \in K \ \text{ such that} \\ (Au, g(v) - g(u)) \geq 0, \ \text{for all } g(v) \in K \end{cases} \tag{3.1}$$

which is known as the general nonlinear variational inequality problem.

Example[15]

We consider the third-order obstacle boundary value problem of finding u such that

$$\begin{cases} -u''' \geq f(x) \quad \text{ on } \Omega = [0,1] \\ u \geq \psi(x) \quad \text{ on } \Omega = [0,1] \\ [-u''' - f(x)][u - \psi(x)] = 0 \quad \text{ on } \Omega = [0,1] \\ u(0) = 0, \quad u'(0) = 0, \quad u'(1) = 0. \end{cases} \tag{3.2}$$

where $f(x)$ is a continuous function and $\psi(x)$ is the obstacle function. We study the problem (3.2) in the framework of variational inequality approach. To do so, we first define the set K as

$$K = \{u : \ u \in H_0^2(\Omega) : \ u \geq \psi(x) \quad \text{ on } \Omega\},$$

which is a closed convex set in $H_0^2(\Omega)$, where $H_0^2(\Omega)$ is a Sobolev (Hilbert) space. One can easily show that the energy functional associated with the problem (3.2) is

$$\begin{aligned} I[v] &= -\int_0^1 \Big(\frac{d^3v}{dx^3}\Big)\Big(\frac{dv}{dx}\Big)dx - 2\int_0^1 f(x)\Big(\frac{dv}{dx}\Big)dx, \ \ \forall \frac{dv}{dx} \in K \\ &= \int_0^1 \Big(\frac{d^2v}{dx^2}\Big)^2 dx - 2\int_0^1 f(x)\Big(\frac{dv}{dx}\Big)dx \\ &= (Av, g(v)) - 2(f, g(v)) \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} (Au, g(v)) &= \int_0^1 \Big(\frac{d^2u}{dx^2}\Big)\Big(\frac{d^2v}{dx^2}\Big)dx \\ (f, g(v)) &= \int_0^1 f(x)\Big(\frac{dv}{dx}\Big)dx \end{aligned} \tag{3.4}$$

and $g = \frac{d}{dx}$ is the linear operator.

It is clear that the operator A defined by (3.4) is linear, g-symmetric i.e $(Au, g(v)) = (Ag(v), u)$ and g-positive i.e $(Ag(v), g(v)) \geq 0$. Using the technique of Noor [11], one can easily show that the minimum $u \in H$ of the functional $I[v]$ defined by (3.9) associated with the problem (3.2) on the closed convex-valued set K can be characterized by the inequality of the type

$$(Au, g(v) - g(u)) \geq (f, g(v) - g(u)), \quad \forall g(v) \in K,$$

which is exactly the general variational inequality (3.1).

Special Cases:

1. For $g(u) = u \in K$, then the problem (3.1) is equivalent to finding $u \in K$ such that

$$(Au, v - u) \geq 0, \text{ for all } v \in K \tag{3.5}$$

   This problem is originally due to Stampacchia [8] see also Lions and Stampacchia [7].

2. If $K^* = \{u \in H, (u, v) \geq 0 \text{ for all } v \in K\}$ is a polar cone of the convex cone K in H, then problem (3.1) is equivalent to finding $u \in H$, such that

$$g(u) \in K, \ Au \in K^*, \quad < g(u), Au >= 0, \tag{3.6}$$

   which is known as the **general nonlinear complementarity problem**. Note the symmetry role played by the mapping A and g, since $K = K^* = \mathbb{R}^n_+$.

3. If $K = H$, then problem (3.1) is equivalent to finding $u \in H$ such that

$$(Au, g(u)) = 0; \tag{3.7}$$

   Problem (3.7) is known as the weak formulation of boundary value problems.

Remark 30 *If $N_K(u) = \{w \in H : \ (w, v - u) \leq 0$ for all $v \in K\}$ is a normal cone to the convex set K at u, then the general variational inequality (3.1) is equivalent to finding $u \in H$, $g(u) \in K$ such that*

$$-Au \in N_K(g(u)),$$

which are known as the generalized nonlinear equations.

The auxiliary problem

For a given $u \in H$, $g(u) \in K$, consider the problem of fonding a unique $w \in H$, $g(w) \in K$ such that

$$(\rho Au + g(w) - g(u), g(v) - g(w)) \geq 0 \text{ for all } g(v) \in K, \tag{3.8}$$

where $\rho > 0$ is a constant.
Note that if $w = u$, then w is clearly a solution of the general variational inequality (3.1).

Remark 31 *[11] We note that if the operator g is convex, i.e*

$$g(\lambda u + (1-\lambda)v) \leq \lambda g(u) + (1-\lambda v)g(v) \quad \forall u, v \in K \text{ and } \lambda \in [0,1[,$$

then the auxiliary problem (3.8) is equivalent to finding the minimum of the functional $I[w]$ on the convex set K, where

$$\begin{aligned} I[w] &= \frac{1}{2}(g(w) - g(u), g(w) - g(u)) + (\rho Au, g(w) - g(u)) \\ &= \|g(w) - (g(u) - \rho Au)\|^2. \end{aligned} \tag{3.9}$$

It can be easily shown that the optimal solution of (3.9) is the projection of the point $(g(u) - \rho Au)$ onto the convex set K, that is

$$g(w(u)) = P_K[g(u) - \rho Au],$$

*which is the **fixed-point characterization** of the general variational inequality (3.1), see Lemma 4.*

Lemma 4 *The function $u \in H$, $g(u) \in K$ is a solution of (3.1) if and only if $u \in H$ satisfies the relation*

$$g(u) = P_K[g(u) - \rho Au],$$

where $\rho > 0$ is a constant and g is onto K.

Proof. We assume that there exists a solution $u^* \in H$, $g(u^*) \in K$ of (3.1), then

$$(Au^*, g(v) - g(u^*)) \geq 0 \Longleftrightarrow \rho(Au^*, g(v) - g(u^*)) \geq 0, \text{ for all } g(v) \in K$$

$$\Longleftrightarrow (-\rho Au^*, g(u^*) - g(v)) \geq 0 \Longleftrightarrow (g(u^*) - \rho Au^* - g(u^*), g(u^*) - g(v)) \geq 0$$

Hence

$$g(u^*) = P_K[g(u^*) - \rho Au^*].$$

∎

From Lemma 4, we conclude that the problem (3.1) can be transformed into a fixed point problem of solving

$$u = F(u),$$

where

$$F(u) = u - g(u) + P_K[g(u) - \rho Au]. \tag{3.10}$$

Theorem 32 *[10] Let the operators $A, g : H \to H$ be both strongly monotone with constants $\alpha > 0$, $\sigma > 0$ and Lipschitz continuous with constants with $\beta > 0$, $\delta > 0$, respectively. If*

$$|\rho - \frac{\alpha}{\beta^2}| < \frac{\sqrt{\alpha^2 - \beta^2 k(2-k)}}{\beta^2}, \quad \alpha > \beta\sqrt{k(2-k)}, \ k < 1. \tag{3.11}$$

where

$$k = 2(\sqrt{1 - 2\sigma + \delta^2}), \tag{3.12}$$

then there exists a unique solution $u \in H$, $g(u) \in K$ of the general variational inequality (3.1).

Proof. From Lemma 4, it follows that problems (3.10) and (3.1) are equivalent. Thus it is enough to show that the map $F(u)$ has a fixed point.
For all $u, v \in H$,

$$\begin{aligned}
\|F(u) - F(v)\| &= \|u - v - (g(u) - g(v)) + P_K[g(u) - \rho Au] - P_K[g(v) - \rho Av]\| \\
&\leq \|u - v - (g(u) - g(v))\| + \|P_K[g(u) - \rho Au] - P_K[g(v) - \rho Av]\| \\
&\leq \|u - v - (g(u) - g(v))\| + \|g(u) - g(v) - \rho(Au - Av)\| \\
&\leq \|u - v - (g(u) - g(v))\| + \|u - v + g(u) - g(v) - (u - v) - \rho(Au - Av)\| \\
&\leq 2\|u - v - (g(u) - g(v))\| + \|u - v - \rho(Au - Av)\| \qquad (3.13)
\end{aligned}$$

where we have used the fact that the operator P_K is nonexpansive.
Since the operator A is strongly monotone with constant $\alpha > 0$ and Lipschitz continuous with constant $\beta > 0$, it follows that

$$\begin{aligned}\|u - v - \rho(Au - Av)\|^2 &\leq \|u - v\|^2 - 2\rho(Au - Av, u - v) + \rho^2\|Au - Av\|^2 \\ &\leq \|u - v\|^2 - 2\rho\alpha\|u - v\|^2 + \rho^2\beta^2\|u - v\|^2 \\ &\leq (1 - 2\rho\alpha + \rho^2\beta^2)\|u - v\|^2. \quad (3.14)\end{aligned}$$

In a similar way, we have

$$\begin{aligned}\|u - v - (g(u) - g(v))\|^2 &= \|u - v\|^2 + \|g(u) - g(v)\|^2 - 2(u - v, g(u) - g(v)) \\ &\leq \|u - v\|^2 + \delta^2\|u - v\|^2 - 2\sigma\|u - v\|^2 \\ &\leq (1 - 2\sigma + \delta^2)\|u - v\|^2 \quad (3.15)\end{aligned}$$

where $\sigma > 0$ and $\delta > 0$ are the strong monotonicity and Lipschitz continuity constants of the operator g.
From (3.13), (3.14), and (3.18), we have

$$\begin{aligned}\|F(u) - F(v)\| \leq\ & \left(2\sqrt{1 - 2\sigma + \delta^2} + \sqrt{1 - 2\alpha\rho + \beta^2\rho^2}\right)\|u - v\| \\ &= (k + t(\rho))\|u - v\|, \quad \text{from}(3.12)\ . \\ &= \theta\|u - v\|, \quad (3.16)\end{aligned}$$

where

$$t(\rho) = \sqrt{1 - 2\alpha\rho + \rho^2\beta^2}. \quad (3.17)$$

and

$$\theta = k + t(\rho). \quad (3.18)$$

From (3.11), it follows that $\theta < 1$, which implies that the map $F(u)$ has a fixed point, by the Banach fixed point (theorem12) which is the unique solution of (3.1). ■

3.2 Approximation schemes for solving the general variational inequality

Now using the auxiliary principle technique
For a given $u \in H$, compute the approximate solution $\{u_n\}$ by the iterative schemes

$$(\rho Au_n + g(y_n) - g(u_n), g(v) - g(y_n)) \geq 0, \quad \forall g(v) \in K$$

$$(\rho Ay_n + g(w_n) - g(y_n), g(v) - g(w_n)) \geq 0, \quad \forall g(v) \in K$$

$$(\rho Aw_n + g(u_{n+1}) - g(w_n), g(v) - g(u_{n+1})) \geq 0, \quad \forall g(v) \in K$$

where $\rho > 0$.
Using Lemma 4, then can be written

$$g(y_n) = P_K[g(u_n) - \rho Au_n]$$

$$g(w_n) = P_K[g(y_n) - \rho Ay_n]$$

$$g(u_{n+1}) = P_K[g(w_n) - \rho Aw_n]$$

If g is inversible, then

$$g(u_{n+1}) = P_K[I - \rho Ag^{-1}]P_K[I - \rho Ag^{-1}]P_K[I - \rho Ag^{-1}]g(u_n),$$

Algorithm 1

$$y_n = (1 - \gamma_n)u_n + \gamma_n\{u_n - g(u_n) + P_K[g(u_n) - \rho Au_n]\} \tag{3.19}$$

$$w_n = (1 - \beta_n)u_n + \beta_n\{y_n - g(y_n) + P_K[g(y_n) - \rho Ay_n]\} \tag{3.20}$$

$$u_{n+1} = (1 - \alpha_n)u_n + \alpha_n\{w_n - g(w_n) + P_K[g(w_n) - \rho Aw_n]\} \tag{3.21}$$

where $0 \leq \alpha_n,\ \beta_n,\ \gamma_n \leq 1$*; for all* $n \geq 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Remark 33 *1. For $\gamma_n = 0$ in (3.19), Algorithm 1 reduces to the Ishikawa iterative sheme for the generale variational inequality (3.1).*

2. *For $\gamma_n = 0$ and $\beta_n = 0$, Algorithm 1 is called the Mann iterative method for (3.1).*

3. *For $g = I$, Algorithm 1 reduces to the following Algorithm for variational inequality (3.5):*

$$y_n = (1-\gamma_n)u_n + \gamma_n P_K[u_n - \rho A u_n]$$
$$w_n = (1-\beta_n)u_n + \beta_n P_K[y_n - \rho A y_n]$$
$$u_{n+1} = (1-\alpha_n)u_n + \alpha_n P_K[w_n - \rho A w_n],$$

where $0 \leq \alpha_n,\ \beta_n,\ \gamma_n \leq 1$; for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Theorem 34 *[10] Let the operators A, g satisfy all the assumptions of Theorem 32. If the condition 3.11 holds, then the approximate solution $\{u_n\}$ obtained from Algorithm 1 converges to the exact solution u of the general variational inequality (3.1) strongly in H, for $0 \leq \alpha_n,\ \beta_n,\ \gamma_n \leq 1$; for all $n \geq 0$ and $\sum_{n=0}^{\infty} \alpha_n = \infty$.*

Proof. From Theorem 32, we see that there exists a unique solution $u \in H$ of the general variational inequality (3.1). Let $u \in H$ be the unique solution of (3.1). Then, using Lemma 4, we have

$$\begin{aligned} u &= (1-\alpha_n)u + \alpha_n\{u - g(u) + P_K[g(u) - \rho Au]\} && (3.22)\\ &= (1-\beta_n)u + \beta_n\{u - g(u) + P_K[g(u) - \rho Au]\} && (3.23)\\ &= (1-\gamma_n)u + \gamma_n\{u - g(u) + P_K[g(u) - \rho Au]\} && (3.24)\end{aligned}$$

From (3.19) and (3.22), we have

$$\begin{aligned} \|u_{n+1} - u\| &= \|(1-\alpha_n)(u_n - u) + \alpha_n(w_n - u - (g(w_n) - g(u)))\\ &+ \alpha_n\{P_K[g(w_n) - \rho A w_n] - P_K[g(u) - \rho Au]\}\|\\ &\leq (1-\alpha_n)\|u_n - u\| + 2\alpha_n\|w_n - u - (g(w_n) - g(u))\|\\ &+ \alpha_n\|w_n - u - \rho(Aw_n - Au)\|\\ &\leq (1-\alpha_n)\|u_n - u\| + 2\alpha_n(k + t(\rho))\|w_n - u\|\\ &\qquad\qquad \text{using (3.12) and (3.14) ,}\\ &= (1-\alpha_n)\|u_n - u\| + \alpha_n\theta\|w_n - u\|, \quad \text{using (3.18).} \end{aligned} \tag{3.25}$$

In a similar way, from (3.20) and (3.23), we have

$$\begin{aligned}
\|w_n - u\| &\leq \Big(1-\beta_n\Big)\|u_n - u\| + 2\beta_n\theta\|y_n - u - (g(y_n) - g(u))\| \\
&+ \beta_n\|y_n - u - \rho(Ay_n - Au)\| \\
&\leq \Big(1-\beta_n\Big)\|u_n - u\| + \beta_n(k + t(\rho))\|y_n - u\|, \\
&\qquad\qquad\qquad\qquad \text{using (3.12) and (3.14) ,} \\
&\leq \Big(1-\beta_n\Big)\|u_n - u\| + \beta_n\theta\|y_n - u\| \qquad (3.26)
\end{aligned}$$

and from (3.21) and (3.24), we obtain

$$\begin{aligned}
\|y_n - u\| &\leq \Big(1-\gamma_n\Big)\|u_n - u\| + \gamma_n\theta\|u_n - u\|, \quad \text{using (3.18) ,} \\
&\leq \Big(1-(1-\theta)\gamma_n\Big)\|u_n - u\| \\
&\leq \|u_n - u\|. \qquad (3.27)
\end{aligned}$$

■

Conclusion

The aim of this research is to examine the General Variational inequalities in terms of existence and uniqueness solution, and also classical form that denoted by VI, which has been discussed in chapter two of this study. The result has revealed that there is an existence and uniqueness solution under specific conditions. Then, we move to build an algorithm that depends on the projection method. Finally, we proposed a model for GVI. GVI means general variational inequalities. It has been shown that general variational inequalities provides a unified, simple, and natural framework to study a wide class of problems, which this study has utilised to conduct our issues.

For the perspectives, it would be interesting to find:

1. Weakning the assumtions on α, β, σ, and δ in the main theorem.
2. Extend the study to time dependent problems.
3. Existence of solutions to more general situation as Banach case, quasi-variational inequalities.

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ملخص

قمنا في هذه الأطروحة بدراسة المسائل التغيراتية العامة، بتقديم شروطا على وجود الحل ووحدانيته، مرورا بالمسائل التغيراتية العادية، وبعد ذلك إنتقلنا إلى تكوين خوارزمية تتقارب بقوة نحو الحل الذي يحقق المتراجحة التغيراتية العامة GVI. وفي الأخير قدمنا نموذجا على ذلك.

الكلمات المفتاحية: المسائل التغيراتية؛ التقارب؛ النقطة الثابتة.

Résumé

L'objectif de cette recherche est d'examiner les inéquations variationel générale en termes d'existence et l’unicité de solutions, ainsi que la forme classique décrite par VI, qui a été abordée au chapitre deux de cette étude. Le résultat a révélé qu'il existe une solution unique dans des conditions spécifiques. Ensuite, nous développons un algorithme qui dépend de la méthode de projection. Enfin, nous avons proposé un modèle pour GVI.

Mots clés : Inequations variationel; convergence; point fixe.

Abstract

The aim of this research is to examine the General Variational inequalities in terms of existence and uniqueness solution, and also classical form that denoted by VI, which has been discussed in chapter two of this study. The result has revealed that there is an existence and uniqueness solution under specific conditions. Then, we move to build an algorithm that depends on the projection method. Finally, we proposed a model for GVI.

Key words: Variational inequalities; convergence; fixed point.