On the unilateral contact between two membranes

NAIMI ABDELOUAHAB

Departement of mathematics

University Kasdi Merbah Ouargla 30000, Algeria

naimimath@gmail.com

Abstract

In this work, we study the a priori error estimates of the unilateral contact between two elastic membranes. first,we analysis of the continuous problem(the well-posedness of the solution),then we study the reduced discrete problem and its a priori analysis ,finally we study the full discrete problem and its a priori analysis .

Keywords: unilateral contact ,a priori analysis , elastic membranes ,discrete problem.

1. Modelization

From the fundamental laws of elasticity ,we write a model for the contact between two membranes. The contact is taken into account according to the following principles:
(i) The two membranes cannot interpenetrate . (ii) Where they are in contact , owing to Newton's action-reaction law , each membrane has an action on the other
An elastic membrane is characterized by its displacement u with respect to its natural configuration which is a two dimensional domain ω. The equilibrium position of the membrane , under the action of a vertical force $\mathcal{F}$ minimizes the potential energy functional:

$$J : v \longrightarrow J(v) = \frac{1}{2}\int_\omega \mu(x)|\nabla v(x)|^2 dx - \int_\omega \mathcal{F}(x)v(x)dx \tag{1.1}$$

The minimization problem reads that's;

$$Find \;\; u \in H_0^1(\omega) \;\; such \;\; that \;\; \forall v \in H_0^1(\omega); J(u) \geqslant J(v) \tag{1.2}$$

or equivalent :

$$\begin{cases} -div(\mu\nabla u) = \mathcal{F} & in\ \omega, \\ u = 0 & on\ \partial\omega \end{cases} \tag{1.3}$$

More details can be found in[1], Chapter I, Section 1.2, for instante. Let us now consider two elastic membranes: The first is fixed on $\partial\omega$ at the height g.Where g is a nonnegative functin, and the second one is fixed at zero. the correspending system of equation reads with obvious notation,

$$\begin{cases} -div(\mu_1\nabla u_1) = \mathcal{F}_1 & in\ w, \\ u = g & on\ \partial\omega \end{cases} \quad \begin{cases} -div(\mu_2\nabla u_2) = \mathcal{F}_2 & in\ w, \\ u = 0 & on\ \partial\omega \end{cases} \tag{1.4}$$

We are interested in the case where the membranes interact. Therefore, if λ represents the action of the second membrane on the first one , we have $\mathcal{F}_1 = f_1 + \lambda$, $\mathcal{F}_2 = f_2 + \lambda$.
where the f_i are external forces. It's follows from the definition of λ that : $\lambda \geq 0$ in ω.
Moreover, clearly the two membranes cannot interpenetate. This yields the conditions $u_1 - u_2 \geq 0$ in ω. Finally, we note that, where the membranes are not in contact . $i;e.,$ where $u_1 - u_2 > 0$, the interaction λ vanishes. This leads to the equation $(u_1 - u_2)\lambda = 0$ in ω.
We are interested in the analysis of this system.

2. Analysis of the continuous problem

Let ω be a bounded open set in $\mathbb{R}^2$ with a Lipschitz-continuous boundary, we consider the following system:

$$\begin{cases} -\mu_1\Delta u_1 - \lambda = f_1 & \text{in } \omega, \\ -\mu_2\Delta u_2 + \lambda = f_2 & in\ \omega, \\ u_1 - u_2 \geq 0, \quad \lambda \geq 0, \quad (u_1 - u_2)\lambda = 0 & in\ \omega, \\ u_1 = g, \quad u_2 = 0 & on\ \partial\omega. \end{cases} \tag{2.1}$$

we consider the following full scales of Sobolev spaces :
$H_+^s(\omega) = \{v \in H^s(\omega); v = g \;\; on \;\; \partial\omega\}; \forall s \geq 0.$. and . . $\varLambda = \{\chi \in L^2(\omega); \chi \geq 0 \;\; a.e\ in \;\; \omega\}.$
So we introduce the following variational problem,for any data $(f_1, f_2) \in H^{-1}(\omega) \times H^{-1}(\omega)$ and $g \in H_+^{\frac{1}{2}}(\partial\omega)$: Find $(u_1, u_2, \lambda) \in H_g^1(\omega) \times H_0^1(\omega) \times \varLambda$ such that

$$\begin{cases} \forall(v_1, v_2) \in H_0^1(w) \times H_0^1(w), \\ \sum_{i=1}^2 \mu_i \int_\omega \nabla u_i(x).\nabla v_i(x)dx - \int_\omega \lambda(x)(v_1 - v_2)(x)dx = \sum_{i=1}^2 \langle f_i, v_i\rangle \\ \forall\chi \in \varLambda, \;\; \int_\omega(\chi - \lambda)(x)(u_1 - u_2)(x)dx \geq 0 \end{cases} \tag{2.2}$$

Proposition 2.1 *: Problems (2.1) and (2.2) are equivalent*
We introduce the new not empty ($g \geq 0 \Longrightarrow g \in \mathcal{K}_g$) convex set :

$$\mathcal{K}_g = \{(v_1, v_2) \in H_g^1(\omega) \times H_0^1(\omega); v_1 - v_2 \geq 0 \;\; a.e. \;\; in\ \omega\}. \tag{2.3}$$

We then consider the reduced problem : find $(u_1, u_2) \in \mathcal{K}_g$ such that

$$\forall(v_1, v_2) \in \mathcal{K}_g, \sum_{i=1}^2 \mu_i \int_\omega \nabla u_i(x).\nabla(v_i - u_i)(x)dx \geq \sum_{i=1}^2 \langle f_i, v_i - u_i\rangle. \tag{2.4}$$

Lemma 2.2 *: For any solution (u_1, u_2, λ) of (2.2) ,the pair (u_1, u_2) is solution of problem (2.4).*

Proposition 2.3 *: for any data $(f_1, f_2) \in H^{-1}(\omega))^2$ and $g \in H_+^{\frac{1}{2}}(\partial\omega)$, the problem (2.4) has a unique solution. For proving this,we take $v_- = min\{v, 0\}$ in $H^1(\omega)$ and using the Lions-Stampacchia theorem.*

Theorem 2.4 *: for any data $(f_1, f_2) \in H^{-1}(\omega))^2$ and $g \in H_+^{\frac{1}{2}}(\partial\omega)$,the problem (2.2) has a unique solution $(u_1, u_2, \lambda) \in H_g^1(\omega) \times H_0^1(\omega) \times \varLambda$.*

Corollary 2.5 *: for any data$(f_1, f_2) \in L^2(\omega))^2$ and $g \in H_+^{\frac{1}{2}}(\partial\omega)$,the solution (u_1, u_2, λ) of problem (2.2) satisfies*

$$\|u_1\|_{H^1(\omega)} + \|u_2\|_{H^1(\omega)} + \|\lambda\|_{L^2(\omega)} \geq c(\|f_1\|_{L^2(\omega)} + \|f_2\|_{L^2(\omega)} + \|g\|_{H_+^{\frac{1}{2}}(\partial\omega)}) \tag{2.5}$$

3. A priori analysis of the problem

3.1 The reduced discrete problem and its a priori analysis

we that w is a polygonal.Let $\mathcal{T}_h$ be a regular family triangulations of w. We will use the discrete spaces given as:

$$\mathbb{X}_h = \{v_h \in H^1(w); \forall K \in \mathcal{T}_h, v_h|_K \in \mathcal{P}_1\}$$

. $\mathbb{X}_{0h} = \mathbb{X}_h \times H_0^1(w)$ and $\mathbb{X}_{gh} = \mathbb{X}_h \times H_g^1(w)$ then $\mathbb{K}_{gh} = \{(v_1 - v_{2=}) \in \mathbb{X}_{gh}; (v_1 - v_2) \geq 0 \; in \; w\}$.
by the Galerkin method we have the reduced discrete problem which is has a unique solution $(u_{1h}, u_{2h}) \in \mathcal{K}_{gh}$ such that :

$$\forall(v_{1h}, v_{2h}) \in \mathcal{K}_{gh} \;, \sum_{i=1}^2 \mu_i \int_\omega \nabla u_{ih}(x).\nabla(v_{ih} - u_{ih})(x)dx \geq \sum_{i=1}^2 \langle f_i, v_{ih} - u_{ih}\rangle. \tag{3.1}$$

Theorem 3.1 *assume that w is convex , that $(f_1, f_2) \in (L^2(w))^2$ and $g \in H_+^{\frac{3}{2}}(w)$ then we obtain the a priori error estimats between the solution of (u_1, u_2) of problem (2.4) and (u_{1h}, u_{2h}) of problem (3.1) :*

$$\|u_1 - u_2\|_{H^1(w)} + \|u_{1h} - u2h\|_{H^1(w)} \leq Ch\left(\|f_1\|_{L^2(w)} + \|f_2\|_{L^2(w)} + \|g\|_{H^{\frac{3}{2}}(\partial w)}\right). \tag{3.2}$$

3.2 construction of the discrete action by local postprocessing

Let $\mathcal{V}_h$ denote the set of elements of $\mathcal{T}_h$ wich not belong to ∂w. we define the Lagrange functions associated with $a \in \mathcal{V}_h$ by $\varphi_a \in \mathbb{X}_{0h}$ satisfis $\varphi_a(a) = 1$ and $\forall a' \in \mathcal{V}_h, a' \neq a, \;\; \varphi_a(a') = 0.$
We denote $\mathcal{T}_h = \{K_h \in \mathcal{T}_h, a \in k_h\}; \;\; h_a = \max_{K \in \mathcal{T}_h}\{diamK\};$
where Δ_a= $supp(\varphi_a)$,
and the nonnegative functions $\chi_a \in L^2(w)$ with asupport in a neighbourhood of a, and define $\mathbb{Y}_h$ space spanned with this functions if χ_a linearly independant so that : $dim\mathbb{Y}_h = dim\mathbb{X}_{0h}$.
the next convex set

$$\Lambda_h = \{\rho_h = \sum_{a \in \mathcal{V}_h} \rho_a\chi_a; \rho_a \geq 0\} \subset \Lambda.$$

we introduce a duality pairing between $\mathbb{Y}_h$ and $\mathbb{X}_h$ by
$\forall\rho_h = \sum_{a \in \mathcal{V}_h} \rho_a\chi_a \in \mathbb{Y}_h$ and $\forall v_h \in \mathbb{X}_{0h}$,

$$\langle\rho_h, v_h\rangle_h = \sum_{a \in \mathcal{V}_h} \rho_a v_h(a) \sum_{K \in \mathcal{T}_h} \int_K \varphi_a(x)dx.$$

with the same notation we have the previous definition;

$$\langle\rho_h, \varphi_a\rangle_h = \rho_a \sum_{K \in \mathcal{T}_h} \int_K \varphi_a(x)dx.$$

then we can define the functions λ_{1h} and $\lambda_{2h} \; in \; \mathbb{Y}_h$, where $v_{1h}, v_{2h} \in \mathbb{X}_{0h}$.

$$\begin{cases} \langle\lambda_{1h}, v_{1h}\rangle_h = \mu_i \int_w \nabla u_{1h}(x).\nabla v_{1h}(x)dx - \langle f_1, v_{1h}\rangle \\ \langle\lambda_{2h}, v_{2h}\rangle_h = -\mu_2 \int_w \nabla u_{2h}(x).\nabla v_{2h}(x)dx + \langle f_2, v_{2h}\rangle \end{cases} \tag{3.3}$$

Proposition 3.2 *the functions λ_{1h} and λ_{2h} are coincide.*
In view of the last proposition, we now write a discrete problem of the problem (2.2) by the Galerkin methode. It reads :find $(u_{1h}, u_{2h}, \lambda_h) \in \mathbb{X}_{gh} \times \mathbb{X}_{0h} \times \Lambda_h$ such that

Proposition 3.3

$$\begin{cases} \forall(v_{1h}, v_{2h}) \in \mathbb{X}_{0h} \times \mathbb{X}_{0h}; \\ \sum_{i=1}^2 \mu_i \int_w \nabla u_{ih}(x).\nabla v_{ih}(x)dx - \langle\lambda_{1h}, v_{1h} - v_{2h}\rangle_h = \sum_{i=1}^2 \langle f_i, v_{ih}\rangle \\ \forall\chi \in \Lambda_h; \quad \langle\lambda_{1h} - \lambda_{2h}, u1h - u_{2h}\rangle_h \geq 0 \end{cases} \tag{3.4}$$

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Lemma 3.4 *: For any solution $(u_{1h}, u_{2h}, \lambda_h)$ of (3.3), the pair (u_{1h}, u_{2h}) is solution of problem (3.1), and the conversely.*

Theorem 3.5 $\forall(f_1, f_2) \in (L^2(w))^2$ *and $g \in H_+^{s+\frac{1}{2}}(\partial w), s \geq 0$, the problem (3.3) has a unique solution $(u_{1h}, u_{2h}, \lambda_h) \in \mathbb{X}_{gh} \times \mathbb{X}_{0h} \times \Lambda_h$.*

Theorem 3.6 *we have the following a priori error estimate of action*

$$\|\lambda - \lambda_h\|_{H^{-1}(\omega)} \leq Ch\left(\|f_1\|_{L^2(\omega)} + \|f_2\|_{L^2(\omega)} + \|g\|_{H^{\frac{3}{2}}(\partial\omega)}\right)$$

4. Conclusion

we propose a standard finite element discrization of the variational formulation constructed by the Galerkin method with Lagrange finite elements. We prove that the discrete problem has a unique sulution and derive optimal a priori error estimates. The discretization of the full problem relies on the reduced discrete problem but is more complex.

5. Perspective

We hope to study the a priori error estimates of the unilateral contact between two plates , together with some numerical experiments.

References

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