Szemerédi's Regularity Lemma

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Abstract

This work deals with the famous Regularity Lemma due to E. Szemerédi; our main aim is to give a comprehensive proof of the latter and to discuss some of its applications.

1. Introduction

Szemerédi's regularity lemma is a masterpiece in graph theory. Szemerédi proved a weak version of it in his proof of a conjecture of Erdos aand Turan about the density and arithmetic progressions of natural numbers (1975). Later, in 1978 Szemerédi proved his regularity lemma. This results arises in the proof of many deep results in graph theory and number theory. The precise statement of the lemma is given in Section 3. Informally, the lemma says that every graph can be partitioned into almost equal pieces all behaving in a uniform way. In the last section we discuss the conjecture of Erdos and Turan which was behind the discovery of the regularity lemma.

2. Basic facts on graphs

Definition 2.1 *A graph G is a pair (V, E) where V is a set and E is a family of subsets of V all containing two elements.*

Let $G = (V, E)$ be a graph. The elements of V will be called the vertices of G; we denote it $V(G)$ if we need to keep trace of G. The elements of E will be termed the edges of G (we may write also $E = E(G)$).
If $x, y \in E(G)$, then it is more convenient to write $xy \in E$; and we say simply that xy is an edge of G. Note that xy and yx represent the same edge.

Let $x, y \in V(G)$; we say that x and y are adjacent if xy in an edge of G. The set of the vertices of G adjacent to x will be denoted $\delta(x)$, and called the neighborhood of x; hence

$$\delta(x) = \{y \in V(G) \,|\, xy \in E(G)\}.$$

The cardinality of $\delta(x)$ will be called the degree of x and will be denoted $d(x)$; so $d(x) = |\delta(x)|$.

The graph G is finite if $V(G)$ is finite. It follows in this case that G has at most $\binom{|V|}{2}$ edges.
If G is finite, then $d(x)$ is finite for every vertex x of G. More generally, if $d(x)$ is finite for every vertix x of G, we say that G is a locally finite graph.

In this work we consider only finite graphs.

The following are some basic nice results on graphs.

Proposition 2.2 *let $G = (V, E)$ be a graph; then $\sum_{x \in V} d(x) = 2|E|$.*

Proposition 2.3 *For every graph G, there exist two vertices x and y having the same degree.*

For $X, Y \subseteq V$, we define

$$E(X, Y) = \{xy \in E \,|\, x \in X \text{ and } y \in Y\}.$$

Definition 2.4 *Let $G = (V, E)$ be a graph, and $X, Y \subseteq V$. The density of the $X - Y$ edges is the number $d(X, Y)$ defined by*

$$d(X, Y) = \frac{|E(X, Y)|}{|X||Y|}.$$

Let $\epsilon > 0$ be a real number. We say that the pair (X, Y) is ϵ-uniform if the following condition holds:
For every $X' \subseteq X$ and every $Y' \subseteq Y$ satisfying $|X'| \geqslant \epsilon|X|$ and $|Y'| \geqslant \epsilon|Y|$, we have

$$|d(X, Y) - d(X', Y')| < \epsilon.$$

3. Statement of the main result.

Theorem 3.1 (Szemerédi's Regularity Lemma) *For every positive integer m, and every $0 \leq \epsilon \leq \frac{1}{2}$, there exists an integer M (depending on m and ϵ) such that every graph of order at least m has an ϵ-uniform partition $(C_i)_{0 \leq i \leq k}$, with $m \leq k \leq M$.*

A sketch of the proof.

(a) For each pair (X, Y) of disjoint such that of V, we define the index of regularity $\varrho(X, Y)$ by

$$\varrho(X, Y) = |X||Y|d(X, Y)^2.$$

For a partition $\mathcal{P} = (C_i)$ of V, we define

$$\varrho(\mathcal{P}) = \sum_{X \neq Y} \varrho(X, Y).$$

As $|E(x, y)| \leqslant |X \times Y| = |X||Y|$, we have $d(X, Y) \leqslant 1$ for all (X, Y); thus

$$\varrho(\mathcal{P}) \leqslant \sum_{x, y \in p} |X||Y| \leqslant \frac{n^2}{2}. \qquad (3.1)$$

We assume below that our partitions have components of the same cardinality except one of them (the exceptional component).

(c) We a call a refinement of the partition $\mathcal{P}$ of V every partition Q of V such that $\mathcal{P} \subseteq Q$, that is to say every component of $\mathcal{P}$ lies in Q. One can show that $\varrho(Q) \geqslant \varrho(\rho)$ whenever Q is a refinement of $\mathcal{P}$.

The heart of the proof is to show that if $\mathcal{P}$ is not ϵ-uniform, then there exists a refinement Q whose index is much bigger. More precisely:

(c) Assume $\mathcal{P} = \{C_0, ..., C_k\}$ is not ϵ-uniform, and that $|C_0| \leq (\epsilon - 2^{-k})n$; then there exists a refinement $Q = \{D_0, .., D_l\}$ of V such that

$$|D_0| < |C_0| + \frac{n}{2^k}$$

and

$$\varrho(\mathcal{Q}) - \varrho(\mathcal{P}) \geq \frac{1 - \epsilon}{1 + \epsilon} \epsilon^5 n^2$$

(d) In the last step, we could start with any partition $\mathcal{P} = \{C_0, ..., C_k\}$, where $|C_0| \leqslant K$.
We iterate the step (c);if in every step we find a pofinement which is not ϵ-uniform
Then $\varrho(Q)$ could be made arbitrary large; this contradicts the inequality 3.1.

4. Applications of Szemerédi's Regularity Lemma

Szemerédi's regularity lemma is a basic tool in graph theory, and also plays an important role in additive combinatorics, most notably in proving Szemerédi's theorem on arithmetic progressions (a conjecture of P. Erdös and P. Turán). A noteworthy is that the regularity lemma proved important in the proof of the Green-Tao theorem on arithmetic progressions in prime numbers (T. Tao earned the Fields medal for the latter work (together with other results)).

A conjecture of P. Erdös and P. Turán.

Let $A \subseteq \mathbb{N}$. The upper density of A is defined by

$$\sigma(A) = \limsup_n \frac{a_n}{n},$$

where $a_n = |A \cap \{1, 2, ..., n\}|$.
Recall that for a sequence (x_n) in $\mathbb{R}$, $\limsup_n x_n$ is defined as follows:
first we define the sequence $\bar{x}_n = \sup_{k \geqslant n} x_k$; by definition

$$\lim \bar{x}_n = \limsup_n x_n.$$

Conjecture. *Let $A \subseteq \mathbb{N}$ such that $\sigma(A) > 0$; then A contains arithmetic progressions of arbitrary large length.*

Zsemerédi confirmed the latter conjecture by proving:

Theorem 4.1 *For every integer $k > 2$, and $\epsilon > 0$, there exists an integer n_0 such that: for $n \geq n_0$, if $A \subseteq \{1, 2, ..., n\}$ and $|A| > \epsilon n$, then A contains an arithmetic progression of length k.*

The regularity lemma is used in the proof of the last theorem in a substantial way (actually a much weaker version of the regularity lemma is used).

References

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